

The Polyhedral Model

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Agenda

Motivation: Dependency Graphs and Transformations

Polydral Analysis: Iteration Domain, Access Relations, Schedule

Presburger Sets, Relations and Associated Operations

Data-Flow Analysis and Dependency Graph

Examples: Checking Validity of Code Transformations

Assignment Exercises

Acknowledgments

The material presented in these slides was taken from the tutorial "Presburger Formulas and Polyhedral Compilation" by Sven Verdoolaege, and associated slides, found online for example at <http://labexcompilation.ens-lyon.fr/wp-content/uploads/2013/02/Sven-slides.pdf> and

https://www.researchgate.net/publication/291352331_Presburger_Formulas_and_Polyhedral_Compilation

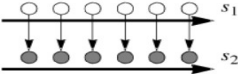

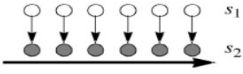
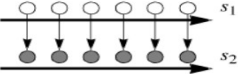
Additionally, we have used material from Andreas Kloeckner's "Languages and Abstractions for High-Performance Scientific Computing" Course, CS598 APK, available online at:

<https://andreask.cs.illinois.edu/cs598apk-f18/notes.pdf#page=214>

Polyhedral model provides a useful framework for reasoning about certain loop-based transformations. Questions to answer:

- How to compute the dependency graph of a loop nest?
- How to represent a code transformation?
- How to prove the legality of such a transformation?

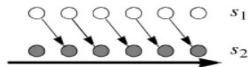
Transformations: Fusion and Fission

SOURCE CODE	PARTITION	TRANSFORMED CODE
<pre>for (i=1; i<=N; i++) Y[i] = Z[i]; /*s1*/ for (j=1; j<=N; j++) X[j] = Y[j]; /*s2*/</pre> 	<p>Fusion</p> $s_1 : p = i$ $s_2 : p = j$	<pre>for (p=1; p<=N; p++){ Y[p] = Z[p]; X[p] = Y[p]; }</pre> 
<pre>for (p=1; p<=N; p++){ Y[p] = Z[p]; X[p] = Y[p]; }</pre> 	<p>Fission</p> $s_1 : i = p$ $s_2 : j = p$	<pre>for (i=1; i<=N; i++) Y[i] = Z[i]; /*s1*/ for (j=1; j<=N; j++) X[j] = Y[j]; /*s2*/</pre> 

[Aho/Ullman/Sethi '07]

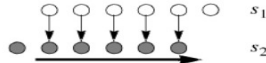
Transformations: Reindexing and Scaling

```
for (i=1; i<=N; i++) {
    Y[i] = Z[i];    /*s1*/
    X[i] = Y[i-1]; /*s2*/
}
```

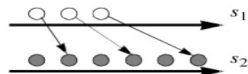


Re-indexing
 $s_1 : p = i$
 $s_2 : p = i - 1$

```
if (N>=1) X[1]=Y[0];
for (p=1; p<=N-1; p++){
    Y[p]=Z[p];
    X[p+1]=Y[p];
}
if (N>=1) Y[N]=Z[N];
```

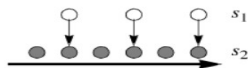


```
for (i=1; i<=N; i++)
    Y[2*i] = Z[2*i]; /*s1*/
for (j=1; j<=2N; j++)
    X[j]=Y[j];      /*s2*/
```



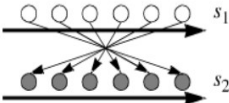
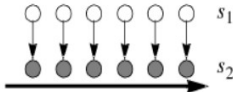
Scaling
 $s_1 : p = 2 * i$
 $(s_2 : p = j)$

```
for (p=1; p<=2*N; p++){
    if (p mod 2 == 0)
        Y[p] = Z[p];
    X[p] = Y[p];
}
```



[Aho/Ullman/Sethi '07]

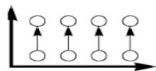
Transformations: Partition

SOURCE CODE	PARTITION	TRANSFORMED CODE
<pre>for (i=0; i<=N; i++) Y[N-i] = Z[i]; /*s1*/ for (j=0; j<=N; j++) X[j] = Y[j]; /*s2*/</pre> 	Reversal $s_1 : p = N - i$ $(s_2 : p = j)$	<pre>for (p=0; p<=N; p++){ Y[p] = Z[N-p]; X[p] = Y[p]; }</pre> 

[Aho/Ullman/Sethi '07]

Transformations: Permutation

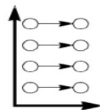
```
for (i=1; i<=N; i++)  
  for (j=0; j<=M; j++)  
    Z[i,j] =  
      Z[i-1,j];
```



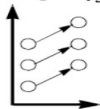
Permutation

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

```
for (p=0; p<=M; p++)  
  for (q=1; q<=N; i++)  
    Z[q,p] = Z[q-1,p]
```



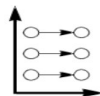
```
for (i=1; i<=N+M-1; i++)  
  for (j=max(1,i+N);  
       j<=min(i,M); j++)  
    Z[i,j] =  
      Z[i-1,j-1];
```



Skewing

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

```
for (p=1; p<=N; p++)  
  for (q=1; q<=M; q++)  
    Z[p,q-p] =  
      Z[p-1,q-p-1]
```



[Aho/Ullman/Sethi '07]

Loop skewing example does not seem quite right (next slide)!

Transformations: Loop Skewing

```
float X[N][N];  
for(int i=1; i<N; i++) {  
    for(int j=1; j < min(i+2, N); j++) {  
        X[i][j] = X[i-1][j] + X[i][j-1];  
    }  
}
```

Change of variables: $p \leftarrow i+j, q \leftarrow j$

```
for(int p=2; p < 2*N-1; p++) {  
    int up_bd = ((p+2)/2) + (p%2);  
    for(int q=max(1,p-N+1); q<min(up_bd,N); q++) {  
        X[p-q][q] = X[p-q-1][q] + X[p-q][q-1];  
    }  
}
```

Polyhedral model provides a useful framework for reasoning about certain loop-based transformations. Questions to answer:

- How to compute the dependency graph of a loop nest?
- How to represent a code transformation?
- How to prove the legality of such a transformation?

Motivation: Dependency Graphs and Transformations

Polydral Analysis: Iteration Domain, Access Relations, Schedule

Presburger Sets, Relations and Associated Operations

Data-Flow Analysis and Dependency Graph

Examples: Checking Validity of Code Transformations

Assignment Exercises

Main Components of Polyhedral Analysis

Key features:

- instance based:
 - ▶ statement instances
 - ▶ array elements
- compact representation
 - ▶ Presburger set and relations ...

Program Representation Uses:

- **Iteration Domain**: the set of all statement instances
- **Access Relations**: maps each statement instance to the array elements accessed (read/written) by that statement instance.
- **Schedule**: maps each statement instance to its execution time. (Execution time is abstractly represented by the total order of iterations in the target loop nest).

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Key features:

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Program Representation Uses:

- **Iteration Domain**: the set of all statement instances
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 - **Schedule**: maps each statement instance to its execution time. (Execution time is abstractly represented by the total order of iterations in the target loop nest).
- ⇒ Compute automatically the **Dependency Graph**: maps the source (statement instance) of a dependence to its sink.
- ⇒ Check automatically the **Validity of a desired transformation**.

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Assignment Exercises

Illustrative Example (Naive)

```
R:   h(A[2]);  
      for(int i=0; i<2; i++)  
          for(int j=0; j<2; j++)  
S:      A[i+j] = f(i,j);  
      for(int k=0; k<2; k++)  
T:      g(A[k], A[0]);
```

- Iteration domain: set of all statement instances:
 $I = \{R[]; S[0,0]; S[0,1]; S[1,0]; S[1,1]; T[0]; T[1]\}$
- Access relation (statement instance accesses array elements):
 $W = \{S[0,0] \rightarrow A[0]; S[0,1] \rightarrow A[1]; S[1,0] \rightarrow A[1]; S[1,1] \rightarrow A[2]\}$
 $R = \{R[] \rightarrow A[2]; T[0] \rightarrow A[0]; T[1] \rightarrow A[1]; T[1] \rightarrow A[0]\}$
- Schedule (total ordering of stmts modeling execution time):
 $S = \{R[] \rightarrow 0; S[0,0] \rightarrow 1; S[0,1] \rightarrow 2; S[1,0] \rightarrow 3; S[1,1] \rightarrow 4;$
 $T[0] \rightarrow 5; T[1] \rightarrow 6\}$

Illustrative Example (Compact)

```
R:   h(A[2]);  
      for(int i=0; i<2; i++)  
          for(int j=0; j<2; j++)  
S:      A[i+j] = f(i,j);  
      for(int k=0; k<2; k++)  
T:      g(A[k], A[0]);
```

- Iteration domain: set of all statement instances:
 $I = \{R[]; S[i,j]: 0 \leq i < 2 \wedge 0 \leq j < 2; T[k]: 0 \leq k < 2\}$
- Access relation (statement instance accesses array elements):
 $W = \{S[i,j] \rightarrow A[i+j]: 0 \leq i < 2 \wedge 0 \leq j < 2\}$
 $R = \{R[] \rightarrow A[2]; T[k] \rightarrow A[0]: 0 \leq k < 2; T[k] \rightarrow A[k]: 0 \leq k < 2\}$
- Schedule (total ordering of stmts modeling execution time):
 $S = \{R[] \rightarrow [0,0,0]; S[i,j] \rightarrow [1,i,j]: 0 \leq i < 2 \wedge 0 \leq j < 2;$
 $T[k] \rightarrow [2,k,0]: 0 \leq k < 2;\}$

Parametric Example: Matrix Matrix Multiplication

```
    for(int i=0; i<M; i++)  
        for(int j=0; j<N; j++) {  
S1:      C[i,j] = 0.0;  
  
          for(int k=0; k<K; k++)  
S2:      C[i,j] = C[i,j] + A[i,k] * B[k,j];  
        }  
    }
```

- **Iteration domain** (set of all statement instances):

Parametric Example: Matrix Matrix Multiplication

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    for(int i=0; i<M; i++)  
        for(int j=0; j<N; j++) {  
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S2:      C[i,j] = C[i,j] + A[i,k] * B[k,j];  
        }
```

- Iteration domain (set of all statement instances):

$$I = \left\{ \begin{array}{ll} S1[i,j] & : 0 \leq i < M \wedge 0 \leq j < N; \\ S2[i,j,k] & : 0 \leq i < M \wedge 0 \leq j < N \wedge 0 \leq k < K \end{array} \right\}$$

- Access relation (R = Read, W = Write):

Parametric Example: Matrix Matrix Multiplication

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    for(int i=0; i<M; i++)  
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```

- Iteration domain (set of all statement instances):

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- Access relation (R = Read, W = Write):

$$W = \{ S1[i,j] \rightarrow C[i,j]; S2[i,j,k] \rightarrow C[i,j] \}$$

$$R = \{ S2[i,j,k] \rightarrow C[i,j]; S2[i,j,k] \rightarrow A[i,k]; S2[i,j,k] \rightarrow B[k,j] \}$$

- Schedule (total ordering of stmts modeling execution time):

Parametric Example: Matrix Matrix Multiplication

```
    for(int i=0; i<M; i++)  
        for(int j=0; j<N; j++) {  
S1:      C[i,j] = 0.0;  
  
        for(int k=0; k<K; k++)  
S2:      C[i,j] = C[i,j] + A[i,k] * B[k,j];  
        }
```

- Iteration domain (set of all statement instances):

$$I = \left\{ \begin{array}{ll} S1[i,j] & : 0 \leq i < M \wedge 0 \leq j < N; \\ S2[i,j,k] & : 0 \leq i < M \wedge 0 \leq j < N \wedge 0 \leq k < K \end{array} \right\}$$

- Access relation (R = Read, W = Write):

$$W = \{ S1[i,j] \rightarrow C[i,j]; S2[i,j,k] \rightarrow C[i,j] \}$$

$$R = \{ S2[i,j,k] \rightarrow C[i,j]; S2[i,j,k] \rightarrow A[i,k]; S2[i,j,k] \rightarrow B[k,j] \}$$

- Schedule (total ordering of stmts modeling execution time):

$$S = \{ S1[i,j] \rightarrow [i,j,0,0]; S2[i,j,k] \rightarrow [i,j,1,k] \}$$

Presburger Sets and Relations

```
R:    h(A[2]);  
      for(int i=0; i<2; i++)  
          for(int j=0; j<2; j++)  
S:          A[i+j] = f(i,j);  
      for(int k=0; k<2; k++)  
T:          g(A[k], A[0]);
```

Examples:

$I = \{R[]; S[i,j]: 0 \leq i < 2 \wedge 0 \leq j < 2; T[k]: 0 \leq k < 2\}$

$R = \{R[] \rightarrow A[2]; T[k] \rightarrow A[0]: 0 \leq k < 2; T[k] \rightarrow A[k]: 0 \leq k < 2\}$

General Form:

- Sets: $\{ S_1[i] : f_1(i); S_2[i] : f_2(i); \dots \}$
with f_k Presburger formulas
 \Rightarrow set of elements of form $S_k[i]$, one for each i satisfying $f_k(i)$.
- Relations: $\{ S_1[i] \rightarrow T_1[j] : f_1(i,j); S_2[i] \rightarrow T_2[j] : f_2(i,j); \dots \}$
 \Rightarrow set of pairs of elements of the form $S_k[i] \rightarrow T_k[j]$.
(Not necessarily single-valued functions.)

Presburger Formulas

Presburger arithmetic allows (quasi-)exact answers/solutions.

- **Language** $\mathcal{L} = \{f_1/r_1, f_2/r_2, \dots, P_1/S_1, P_2/S_2, \dots\}$

f_i function symbol with arity $r_i \geq 0$:

- ▶ addition, subtraction: $+/2, -/2$
- ▶ constant $d/0$, for each integer d
- ▶ integer division: $\lfloor \cdot / d \rfloor / 1$, for a fixed integer $d > 0$
- ▶ set of symbolic constant $c_i/0$

P_i predicate symbol with arity $s_i \geq 0$, e.g., $\leq /2$.

- **Terms** (inductive definition)

- ▶ v is a term if v is a variable
- ▶ $f_i(t_1, \dots, t_{r_i})$ is a term if t_1, \dots, t_{r_i} are terms

- **Formulas** (inductive definition)

true	$F_1 \wedge F_2$ (conjunction)	quantification:
$P_i(t_1, \dots, t_{s_i})$	$F_1 \vee F_2$ (disjunction)	$\exists v : F_1(v)$ (existential)
$t_1 = t_2$	$\neg F_1$ (negation)	$\forall v : F_1(v)$ (universal)

P_i/s_i are predicates, t_j are terms, v variable, F_k are formulas.

Interpretation of Presburger Formulas

- **Domain of Discourse (Universe):** sets of integers in \mathbb{Z}
- **Interpretation** function/predicate symbols \rightarrow functions/predicates
 - ▶ $+/2, -/2$ map to addition and subtraction on integers ...
 - ▶ symbolic constants c_i are “uninterpreted”,
i.e., consider all possible interpretations as integers
- **Truth Values**
 - ▶ **true** is true; $P_i(t_1, \dots, t_{r_i})$ is true if interpretation is true
 - ▶ ...
 - ▶ $\exists v : F(v)$ is true iff $F(d)$ is true for **some** integer d in the universe (\mathbb{Z}).
 - ▶ $\forall v : F(v)$ is true iff $F(d)$ is true for **every** integers d in the universe (\mathbb{Z}).

Syntactic Sugar

Notation: $\bar{i}^n \equiv i_1, \dots, i_n$, and n can be left unspecified.

- $false$ is equal to $\neg true$
- $a \Rightarrow b$ is equal to $\neg a \vee b$
- $S[\bar{i}]$ is equal to $S[\bar{i}] : true$
- $S[i_1, \dots, i_{k-1}, g(i_1, \dots, i_{k-1}), i_{k+1}, \dots, i_n] : f(\bar{i})$ is equal to $S[i_1, \dots, i_{k-1}, i_k, i_{k+1}, \dots, i_n] : i_k = g(i_1, \dots, i_{k-1}) \wedge f(\bar{i})$
e.g., $\{S[i] \rightarrow T(i+1)\}$ is equal to $\{S[i] \rightarrow T(j) : j = i+1\}$
- $a < b$ is equal to $a \leq b - 1 \dots$
- $\{S[i, j] : i, j \geq 0\}$ is equal to $\{S[i, j] : i \geq 0 \wedge j \geq 0\}$
- $\{S[i] : 0 \leq i \leq 10\}$ is equal to $\{S[i] : 0 \leq i \wedge i \leq 10\}$
- $-e$ is equal to $0 - e$
- $n \cdot e$ is equal to $e + \dots + e$, with n positive integer constant
- $a_1, \dots, a_n \prec b_1, \dots, b_n$ equals $\bigvee_{i=1}^n ((\bigwedge_{j=1}^{i-1} a_j = b_j) \wedge a_i < b_i)$
e.g., $\{S[i_1, i_2] \rightarrow T[j_1, j_2] : i_1, i_2 \prec j_1, j_2\}$ equals $\{S[i_1, i_2] \rightarrow T[j_1, j_2] : i_1 < j_1 \vee (i_1 = j_1 \wedge i_2 < j_2)\} \dots$

Examples

- $\{S[i,j] \rightarrow [1,i,j] : 0 \leq i,j < 2; T[k] \rightarrow [2,k,0] : 0 \leq k < 2\}$
is equal to $\{S[0,0] \rightarrow [1,0,0]; S[0,1] \rightarrow [1,0,1]; S[1,0] \rightarrow [1,1,0]; S[1,1] \rightarrow [1,1,1]; T[0] \rightarrow [2,0,0]; T[1] \rightarrow [2,1,0]\}$
- $\{[i] : 0 \leq i \leq 10 \wedge \exists \alpha : i = 2 \cdot \alpha\}$ is equal to $\{[0]; [2]; [4]; [6]; [8]; [10]\}$
- $\{[i] : 0 \leq i \leq 10 \wedge i = 2 \cdot \alpha\}$ is equal to
$$\begin{cases} \{[2 \cdot \alpha]\} & \text{if } 0 \leq \alpha \leq 5 \\ \emptyset & \text{otherwise} \end{cases}$$
- $\{[i] : \forall j : i + j \leq 10\}$ is equal to \emptyset .

Spaces

Recall:

- Sets: $\{ S_1[\bar{i}] : f_1(\bar{i}); S_2[\bar{i}] : f_2(\bar{i}); \dots \}$
- Relations: $\{ S_1[\bar{i}] \rightarrow T_1[\bar{j}] : f_1(\bar{i}, \bar{j}); S_2[\bar{i}] \rightarrow T_2[\bar{j}] : f_2(\bar{i}, \bar{j}); \dots \}$

The identifier (e.g., S_1, S_2, T_1, T_2) together with the dimension, i.e., the number of elements in the subsequent tuple (e.g., \bar{i}, \bar{j}) will be called a **space**.

When we say $S_2[\bar{i}] = T_1[\bar{j}]$ we mean:

- the identifiers S_2 and T_1 are the same, and
- the dimensions of \bar{i} and \bar{j} are the same.

For example: $S[] \neq S[i]; S[a] = S[b]; S[] \neq T[]$.

Operations on Relations

Union: $\{S_1[\bar{i}] \rightarrow T_1[\bar{j}] : f_1(\bar{i}, \bar{j}); \dots\} \cup \{S_2[\bar{i}] \rightarrow T_2[\bar{j}] : f_2(\bar{i}, \bar{j}); \dots\}$
 $\Rightarrow \{S_1[\bar{i}] \rightarrow T_1[\bar{j}] : f_1(\bar{i}, \bar{j}); \dots; S_2[\bar{i}] \rightarrow T_2[\bar{j}] : f_2(\bar{i}, \bar{j}); \dots\}$

Inverse: $R = \{S[\bar{i}] \rightarrow T[\bar{j}] : f(\bar{i}, \bar{j})\} \Rightarrow R^{-1} = \{T[\bar{j}] \rightarrow S[\bar{i}] : f(\bar{i}, \bar{j})\}$

Dom: $R = \{S[\bar{i}] \rightarrow T[\bar{j}] : f(\bar{i}, \bar{j})\} \Rightarrow \text{dom } R = \{S[\bar{i}] : \exists \bar{j} : f(\bar{i}, \bar{j})\}$

Range: $R = \{S[\bar{i}] \rightarrow T[\bar{j}] : f(\bar{i}, \bar{j})\} \Rightarrow \text{ran } R = \{T[\bar{j}] : \exists \bar{i} : f(\bar{i}, \bar{j})\}$

UnivRel: $A = \{S[\bar{i}] : f(\bar{i})\}$ and $B = \{T[\bar{j}] : g(\bar{j})\}$
 $\Rightarrow A \rightarrow B = \{S[\bar{i}] \rightarrow T[\bar{j}] : f(\bar{i}) \wedge g(\bar{j})\}$

Intersect $\{S_1[\bar{i}_1] \rightarrow T_1[\bar{j}_1] : f_1(\bar{i}_1, \bar{j}_1)\} \cap \{S_2[\bar{i}_2] \rightarrow T_2[\bar{j}_2] : f_2(\bar{i}_2, \bar{j}_2)\} \Rightarrow$

$$\begin{cases} \{S_1[\bar{i}] \rightarrow T_1[\bar{j}] : f_1(\bar{i}, \bar{j}) \wedge f_2(\bar{i}, \bar{j})\} & \text{if } S_1[\bar{i}_1] = S_2[\bar{i}_2] \text{ and} \\ & T_1[\bar{j}_1] = T_2[\bar{j}_2] \\ \emptyset & \text{otherwise} \end{cases}$$

Examples: Operations on Relations

```
R:   h(A[2]);  
      for(int i=0; i<2; i++)  
          for(int j=0; j<2; j++)  
S:           A[i+j] = f(i,j);  
      for(int k=0; k<2; k++)  
T:           g(A[k], A[0]);
```

- Access relation (statement instance accesses array elements):

$$W = \{S[i,j] \rightarrow A[i+j] : 0 \leq i < 2 \wedge 0 \leq j < 2\}$$

$$R = \{R[] \rightarrow A[2]; T[k] \rightarrow A[0] : 0 \leq k < 2; T[k] \rightarrow A[k] : 0 \leq k < 2\}$$

- $R \cup W = \{ R[] \rightarrow A[2]; T[k] \rightarrow A[0] : 0 \leq k < 2; T[k] \rightarrow A[k] : 0 \leq k < 2; S[i,j] \rightarrow A[i+j] : 0 \leq i < 2 \wedge 0 \leq j < 2 \}$
- $R^{-1} = \{A[2] \rightarrow R[]; A[0] \rightarrow T[k] : 0 \leq k < 2; A[k] \rightarrow T[k] : 0 \leq k < 2\}$
- $\text{dom } R = \{R[]; T[k] : 0 \leq k < 2\}$
- $\text{ran } R = \{A[k] : 0 \leq k \leq 2\}$
- $\text{dom } R \rightarrow \text{ran } R = \{R[] \rightarrow A[j] : 0 \leq j \leq 2; T[k] \rightarrow A[j] : 0 \leq k < 2 \wedge 0 \leq j \leq 2\}$
- $\{T[k] \rightarrow A[k] : 0 \leq k < 2\} \cap \{T[k] \rightarrow A[0] : 0 \leq k < 2\} = \{T[0] \rightarrow A[0]\}$

Domain/Range Restrictions

Assume $A = \{S_1[i_1] : f(i_1)\}$, $B = \{S_2[i_2] \rightarrow T_2[j_2] : g(i_2, j_2)\}$

- Domain Restrictions: $R \cap_{\text{dom}} S = R \cap (S \rightarrow \text{ran } R)$

$$A \cap_{\text{dom}} B = \begin{cases} \{S_2[i] \rightarrow T_2[j] : f(i) \wedge g(i, j)\}, & \text{if } S_1(i_1) = S_2(i_2) \\ \emptyset & \text{otherwise} \end{cases}$$

- Range Restrictions: $R \cap_{\text{ran}} S = R \cap ((\text{dom } R) \rightarrow S)$

$$B \cap_{\text{ran}} A = \begin{cases} \{S_2[i] \rightarrow T_2[j] : f(i) \wedge g(i, j)\}, & \text{if } S_1(i_1) = T_2(j_2) \\ \emptyset & \text{otherwise} \end{cases}$$

Example:

- $I = \{R[]; S[i, j] : 0 \leq i < 2 \wedge 0 \leq j < 2; T[k] : 0 \leq k < 2\}$
 $S_0 = \{R[] \rightarrow [0, 0, 0]; S[i, j] \rightarrow [1, i, j]; T[k] \rightarrow [2, k, 0]; \}$

$$S = I \cap_{\text{dom}} S_0 = \{ R[] \rightarrow [0, 0, 0]; T[k] \rightarrow [2, k, 0] : 0 \leq k < 2; \\ S[i, j] \rightarrow [1, i, j] : 0 \leq i < 2 \wedge 0 \leq j < 2 \quad \}$$

Relation Difference/Subtraction and Comparisons

$$A = \{S_1[i_1] \rightarrow T_1[j_1] : f(i_1, j_1)\},$$

$$B = \{S_2[i_2] \rightarrow T_2[j_2] : g(i_2, j_2)\}$$

$$A \setminus B = \begin{cases} \{S_1[i] \rightarrow T_1[j] : f(i, j) \wedge \neg g(i, j)\}, & \text{if } S_1(i_1) = S_2(i_2) \text{ and} \\ & T_1(j_1) = T_2(j_2) \\ \{S_1[i] \rightarrow T_1[j] : f(i, j)\} & \text{otherwise} \end{cases}$$

Example:

$$\{T[k] \rightarrow A[k] : 0 \leq k < 2\} \setminus \{T[k] \rightarrow A[0] : 0 \leq k < 2\} = \{T[1] \rightarrow A[1]\}$$

Comparisons:

- emptiness check (if the Preseburger formula reduces to false)
- $A \subseteq B$ is defined as $A \setminus B = \emptyset$
- $A \supseteq B$ is defined as $B \subseteq A$
- $A = B$ is defined as $B \subseteq A \wedge A \subseteq B$
- $A \subset B$ is defined as $A \subseteq B \wedge \neg(A = B)$
- $A \supset B$ is defined as $B \subset A$

Composition of Relations

Composition:

$$A = \{S_1[i_1] \rightarrow T_1[j_1] : f(i_1, j_1)\}, \quad B = \{S_2[i_2] \rightarrow T_2[j_2] : g(i_2, j_2)\}$$

$$B \circ A = \begin{cases} \{S_1[i] \rightarrow T_2[j] : \exists k : f(i, k) \wedge g(k, j)\}, & \text{if } T_1(j_1) = S_2(i_2) \\ \emptyset & \text{otherwise} \end{cases}$$

Example:

$$\text{Write Set: } W = \{S[i, j] \rightarrow A[i + j] : 0 \leq i < 2 \wedge 0 \leq j < 2\}$$

Inverse of Write set (i.e., written array elements to statements):

$$W^{-1} = \{A[a] \rightarrow S[i, j] : a = i + j \wedge 0 \leq i < 2 \wedge 0 \leq j < 2\}$$

Pairs of statement instances that write the same array element:

$$W^{-1} \circ W = \{S[i, j] \rightarrow S[i', j'] : 0 \leq i, j, i', j' < 2 \wedge i + j = i' + j'\} = \\ \{S[0, 0] \rightarrow S[0, 0]; S[0, 1] \rightarrow S[0, 1]; S[1, 0] \rightarrow S[1, 0]; S[1, 1] \rightarrow \\ S[1, 1]; S[0, 1] \rightarrow S[1, 0]; S[1, 0] \rightarrow S[0, 1]; \}$$

Application of a Relation to a Set

Application:

$$A = \{S_1[i_1] : f(i_1)\}, \quad B = \{S_2[i_2] \rightarrow T_2[j_2] : g(i_2, j_2)\}$$

$$B(A) = \begin{cases} \{T_2[j] : \exists i : f(i) \wedge g(i, j)\}, & \text{if } S_1(i_1) = S_2(i_2) \\ \emptyset & \text{otherwise} \end{cases}$$

Example:

Read Set R (Statement instances reading array elements):

$$\{R[] \rightarrow A[2]; T[k] \rightarrow A[0] : 0 \leq k < 2; T[k] \rightarrow A[k] : 0 \leq k < 2\}$$

Instances of T statements:

$$S = \{T[k] : 0 \leq k < 2\}$$

Array elements read by S :

$$R(S) = \{A[k] : 0 \leq k < 2\}$$

Lexicographic Order on Sets

$$A = \{S[\bar{i}] : f(\bar{i})\}, \quad B = \{T[\bar{j}] : g(\bar{j})\}$$

$$A \prec B = \begin{cases} \{S[\bar{i}] \rightarrow S[\bar{j}] : f(\bar{i}) \wedge g(\bar{j}) \wedge \bar{i} \prec \bar{j}\}, & \text{if } S(\bar{i}) = T(\bar{j}) \\ \emptyset & \text{otherwise} \end{cases}$$

Example:

Iteration Domain:

$$I = \{R[]; S[i, j] : 0 \leq i < 2 \wedge 0 \leq j < 2; T[k] : 0 \leq k < 2\}$$

$I \prec I$ lexicographic order on pairs of statement instances:

$$\{S[i, j] \rightarrow S[i', j'] : 0 \leq i, j, i', j' < 2 \wedge i, j \prec i', j'; T[0] \rightarrow T[1]\} =$$

$$\{S[0, 0] \rightarrow S[0, 1]; S[0, 0] \rightarrow S[1, 0]; S[0, 0] \rightarrow S[1, 1]; S[0, 1] \rightarrow S[1, 0]; S[0, 1] \rightarrow S[1, 1]; S[1, 0] \rightarrow S[1, 1]; T[0] \rightarrow T[1]\}$$

Lexicographic Order on Relations

Binary relation on domains reflect lexicographic order of images:

$$A = \{S_1[\bar{i}_1] \rightarrow T_1[\bar{j}_1] : f(\bar{i}_1, \bar{j}_1)\},$$

$$B = \{S_2[\bar{i}_2] \rightarrow T_2[\bar{j}_2] : g(\bar{i}_2, \bar{j}_2)\}$$

$$A \prec B = \begin{cases} \{S_1[\bar{i}_1] \rightarrow S_2[\bar{i}_2] : \exists j_1, j_2 : f(\bar{i}_1, \bar{j}_1) \wedge \\ g(\bar{i}_2, \bar{j}_2) \wedge \bar{j}_1 \prec \bar{j}_2\}, & \text{if } T_1(\bar{j}_1) = T_2(\bar{j}_2) \\ \emptyset & \text{otherwise} \end{cases}$$

Lexicographic Optimizations: Last Write

Binary relation on domains reflect lexicographic order of images:

$$R = \{S[\bar{i}] \rightarrow T[\bar{j}] : f(\bar{i}, \bar{j})\},$$

$$\text{lexmax } R = \{S[\bar{i}] \rightarrow T[\bar{j}] : f(\bar{i}, \bar{j}) \wedge \forall j' : f(i, j') \Rightarrow j \succeq j'\}$$

Example:

$(R \cup W)^{-1}$: statement instances accessing array element

$$\{A[2] \rightarrow R[]; A[a] \rightarrow S[i, j] : a = i + j \wedge 0 \leq i, j < 2; \\ A[0] \rightarrow T[k] : 0 \leq k < 2; A[k] \rightarrow T[k] : 0 \leq k < 2\}$$

$\text{lexmax } (R \cup W)^{-1}$: last instance of statement accessing element

$$\{A[2] \rightarrow R[]; A[a] \rightarrow S[a, 0] : 0 \leq a < 2; A[2] \rightarrow S[1, 1]; \\ A[k] \rightarrow T[1] : 0 \leq k < 2\}$$

Motivation: Dependency Graphs and Transformations

Polydral Analysis: Iteration Domain, Access Relations, Schedule

Presburger Sets, Relations and Associated Operations

Data-Flow Analysis and Dependency Graph

Examples: Checking Validity of Code Transformations

Assignment Exercises

Last-Write Analysis (see file last-write.py)

Given a read from an array element, what was the last write to the same array element before the read?

```
for(int i=0; i<N; i++)  
    for(int j=0; j<N-i; j++)
```

```
F:      A[i+j] = f(A[i+j]);
```

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for(int i=0; i<N; i++)
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S:  X[i] = g(A[i]);
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$$W_1 = \{F[i,j] \rightarrow A[i+j] : 0 \leq i < N \wedge 0 \leq j < N - i\}$$

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- Map each statement instance reading an element to all the statements that have written that element:

$$R = W_1^{-1} \circ R_2 = \{S[i] \rightarrow F[i', i-i'] : 0 \leq i' \leq i < N\}$$

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- Last Write: $\text{lexmax } R = \{S[i] \rightarrow F[i,0] : 0 \leq i < N\}$

Dependency Graph and Code Transformations

Recall: iteration \bar{j} depends on iteration \bar{i} iff:

- \bar{j} is executed before \bar{i} in the original program,
- \bar{i} and \bar{j} may access the same memory location, and
- at least one of those two accesses is a write!

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Dependency Graph Computation:

$$D = ((W^{-1} \circ R) \cup (W^{-1} \circ W) \cup (R^{-1} \circ W)) \cap (S \prec S)$$

W : write-access relation, R : read-access relation,
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W : write-access relation, R : read-access relation,

S : original schedule.

A code transformation corresponds to computing a new schedule S' that executes the same statements in a different order. The transformation is valid if S' respects the dependencies of D :

$$\bar{i} \rightarrow \bar{j} \in D \Rightarrow S'(\bar{i}) \prec S'(\bar{j})$$

Validating New Schedules (see file common.py)

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Safe rescheduling S' iff

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How to implement the test above? Assume S' a new schedule (mapping original statement instances to new time abstraction).

$$T_{src \rightarrow sink} = (S' \circ D) \circ S'^{-1}$$

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- Maps the source stmt to the time of the dependence sink;
- Maps the time of the source stmt to the time of the sink.

$$S'_{desc} = (\text{ran } S') \succeq (\text{ran } S')$$

(S'_{desc} denotes all illegal re-orderings)

Code Transformation is valid if $T_{src \rightarrow sink} \cap S'_{desc} = \emptyset$

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(if for all dependencies, in the new schedule, the time of the source is still smaller than the time of the sink statement!)

Motivation: Dependency Graphs and Transformations

Polydral Analysis: Iteration Domain, Access Relations, Schedule

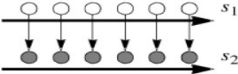

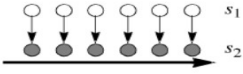
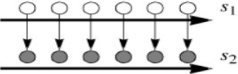
Presburger Sets, Relations and Associated Operations

Data-Flow Analysis and Dependency Graph

Examples: Checking Validity of Code Transformations

Assignment Exercises

Transformations: Fusion and Fission

SOURCE CODE	PARTITION	TRANSFORMED CODE
<pre>for (i=1; i<=N; i++) Y[i] = Z[i]; /*s1*/ for (j=1; j<=N; j++) X[j] = Y[j]; /*s2*/</pre> 	<p>Fusion</p> $s_1 : p = i$ $s_2 : p = j$	<pre>for (p=1; p<=N; p++){ Y[p] = Z[p]; X[p] = Y[p]; }</pre> 
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[Aho/Ullman/Sethi '07]

Fusion Encoding (see fusion.py)

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$$S' = I \cap_{dom} \{S1[i] \rightarrow [i, 1]; S2[i] \rightarrow [i, 2]\}$$

- Check Fusion Safety:

$$\text{checkTimeDepsPreserved}(S', D)$$

Is the fused loop parallel?

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$$S' = I \cap_{dom} \{S1[i] \rightarrow [1, 1]; S2[i] \rightarrow [1, 2]\}$$

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Is it safe to distribute the loop across statements S1 and S2?

```
    for(p=1; p<=N; p++) {  
S1:      Y[p] = f(Z[p]);  
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    ■ Iteration Domain:
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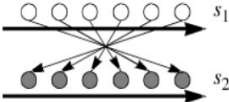
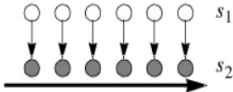
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- Fission Schedule:

$$S' = I \cap_{dom} \{S1[i] \rightarrow [1, i]; S2[j] \rightarrow [2, j]\}$$

- Is Fission Safe? $\text{checkTimeDepsPreserved}(S', D)$

Transformation: Reversal + Fusion

SOURCE CODE	PARTITION	TRANSFORMED CODE
<pre>for (i=0; i<=N; i++) Y[N-i] = Z[i]; /*s1*/ for (j=0; j<=N; j++) X[j] = Y[j]; /*s2*/</pre> 	<p>Reversal</p> $s_1 : p = N - i$ $(s_2 : p = j)$	<pre>for (p=0; p<=N; p++){ Y[p] = Z[N-p]; X[p] = Y[p]; }</pre> 

BUG: should be for (i=0; i<=N; i++)!

[Aho/Ullman/Sethi '07]

Reversal + Fusion Encoding (see fused-rev.py)

- Iteration Domain:

Reversal + Fusion Encoding (see fused-rev.py)

- Iteration Domain:

$$I = \{S1[i] : 1 \leq i \leq N; S2[j] : 1 \leq j \leq N\}$$

- Original Schedule:

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$$I = \{S1[i] : 1 \leq i \leq N; S2[j] : 1 \leq j \leq N\}$$

- Original Schedule:

$$S = I \cap_{dom} \{S1[i] \rightarrow [1, i]; S2[j] \rightarrow [2, j]\}$$

- Read and Write Access Relations:

Reversal + Fusion Encoding (see fused-rev.py)

- Iteration Domain:

$$I = \{S1[i] : 1 \leq i \leq N; S2[j] : 1 \leq j \leq N\}$$

- Original Schedule:

$$S = I \cap_{dom} \{S1[i] \rightarrow [1, i]; S2[j] \rightarrow [2, j]\}$$

- Read and Write Access Relations:

$$W_{access}^{rel} = I \cap_{dom} \{S1[i] \rightarrow Y[N - i]; S2[j] \rightarrow X[j]\}$$

$$R_{access}^{rel} = I \cap_{dom} \{S1[i] \rightarrow Z[i]; S2[j] \rightarrow Y[j]\}$$

- $D = \text{mkDepGraph}(S, R_{access}^{rel}, W_{access}^{rel})$

- Transformed Schedule:

Reversal + Fusion Encoding (see fused-rev.py)

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$$I = \{S1[i] : 1 \leq i \leq N; S2[j] : 1 \leq j \leq N\}$$

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- $D = \text{mkDepGraph}(S, R_{access}^{rel}, W_{access}^{rel})$

- Transformed Schedule:

- ▶ the statements of the first loop are reversed;
- ▶ the two loops are fused, hence $S2[j] \rightarrow [j, 2]$ instead of $S2[j] \rightarrow [2, j]$

$$S' = I \cap_{dom} \{S1[p] \rightarrow [N - p, 1]; S2[j] \rightarrow [j, 2]\}$$

- Is Fission Safe? $\text{checkTimeDepsPreserved}(S', D)$

Transformation: Loop Skewing

```
float X[N][N];  
for(int i=1; i<N; i++) {  
    for(int j=1; j < min(i+2, N); j++) {  
S1:      X[i][j] = X[i-1][j] + X[i][j-1];  
    }  
}
```

Change of variables: $p \leftarrow i+j, q \leftarrow j$

```
for(int p=2; p < 2*N-1; p++) {  
    int up_bd = ((p+2)/2) + (p%2);  
    for(int q=max(1,p-N+1); q<min(up_bd,N); q++)  
S1:      X[p-q][q] = X[p-q-1][q] + X[p-q][q-1];  
    }  
}
```

Encoding Loop Skewing (see loop-skewing.py)

- Iteration Domain:

$$I = \{S1[i,j] : 1 \leq i < N \wedge 1 \leq j < \min(i + 2, N)\}$$

- Original Schedule: $S = I \cap_{dom} \{ S1[i,j] \rightarrow [i,j] \}$
- Read and Write Access Relations:

Encoding Loop Skewing (see loop-skewing.py)

- Iteration Domain:

$$I = \{S1[i, j] : 1 \leq i < N \wedge 1 \leq j < \min(i + 2, N)\}$$

- Original Schedule: $S = I \cap_{dom} \{S1[i, j] \rightarrow [i, j]\}$

- Read and Write Access Relations:

$$W_{access}^{rel} = I \cap_{dom} \{S1[i, j] \rightarrow X[i, j];\}$$

$$R_{access}^{rel} = I \cap_{dom} \{S1[i, j] \rightarrow X[i - 1, j]; S1[i, j] \rightarrow X[i, j - 1]\}$$

- $D = \text{mkDepGraph}(S, R_{access}^{rel}, W_{access}^{rel})$

- Transformed Schedule: $p \leftarrow i+j, q \leftarrow j$

► Original stmt $S1[i, j] = S1[p - q, q]$ is rescheduled to iter $[p, q]$;

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- $D = \text{mkDepGraph}(S, R_{access}^{rel}, W_{access}^{rel})$

- Transformed Schedule: $p \leftarrow i+j, q \leftarrow j$

► Original stmt $S1[i, j] = S1[p - q, q]$ is rescheduled to iter $[p, q]$;

► Hence, $S1[x, q] \rightarrow [x + q, q]$

$$S' = I \cap_{dom} \{S1[x, q] \rightarrow [x + q, q]\}$$

- Is Loop-Skewing Safe? $\text{checkTimeDepsPreserved}(S', D)$

- Inner loop parallel? Try $S'' = I \cap_{dom} \{S1[x, q] \rightarrow [x + q, 1]\}$

Motivation: Dependency Graphs and Transformations

Polydral Analysis: Iteration Domain, Access Relations, Schedule

Presburger Sets, Relations and Associated Operations

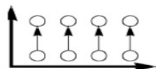
Data-Flow Analysis and Dependency Graph

Examples: Checking Validity of Code Transformations

Assignment Exercises

Exercise: Permutation (Ignore Loop Skewing)

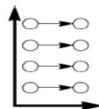
```
for (i=1; i<=N; i++)  
  for (j=0; j<=M; j++)  
    Z[i,j] =  
      Z[i-1,j];
```



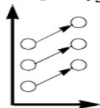
Permutation

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

```
for (p=0; p<=M; p++)  
  for (q=1; q<=N; q++)  
    Z[q,p] = Z[q-1,p]
```



```
for (i=1; i<=N+M-1; i++)  
  for (j=max(1,i+N);  
       j<=min(i,M); j++)  
    Z[i,j] =  
      Z[i-1,j-1];
```



Skewing

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

```
for (p=1; p<=N; p++)  
  for (q=1; q<=M; q++)  
    Z[p,q-p] =  
      Z[p-1,q-p-1]
```

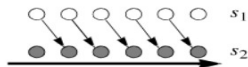


[Aho/Ullman/Sethi '07]

Ignore Loop Skewing

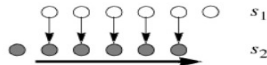
Exercise: Reindexing and Scaling

```
for (i=1; i<=N; i++) {
    Y[i] = Z[i];    /*s1*/
    X[i] = Y[i-1]; /*s2*/
}
```

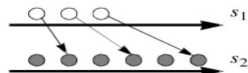


Re-indexing
 $s_1 : p = i$
 $s_2 : p = i - 1$

```
if (N>=1) X[1]=Y[0];
for (p=1; p<=N-1; p++){
    Y[p]=Z[p];
    X[p+1]=Y[p];
}
if (N>=1) Y[N]=Z[N];
```

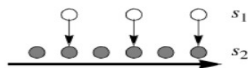


```
for (i=1; i<=N; i++)
    Y[2*i] = Z[2*i]; /*s1*/
for (j=1; j<=2N; j++)
    X[j]=Y[j];      /*s2*/
```



Scaling
 $s_1 : p = 2 * i$
 $(s_2 : p = j)$

```
for (p=1; p<=2*N; p++){
    if (p mod 2 == 0)
        Y[p] = Z[p];
    X[p] = Y[p];
}
```



[Aho/Ullman/Sethi '07]