Informal Discussion on Natural Numbers

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How to use natural numbers? That is intuitively easy. We've all learned how to use natural numbers in school. It will be tough to answer, "How to define the natural numbers?". And tougher would be, "How to define the actions we do on natural numbers such as multiplication or addition?" In [Tao, 2009] Terrence Tao has discussed these issues beautifully and I recommend that one should get this book and read Chapter 2 and 3.

Here we discuss one of the methods to define natural numbers.

1 Peano Axioms

A standard way of defining natural numbers is by invoking these axioms. Nonetheless, It will be a good exercise to build your own set of axioms which can define Natural Numbers? If you do that, please let me know. Let's get back to our business. First of all we need some notation so that we can at-least write down what we intend to mean. How to write down a natural number? Although we use standard figures of 0, 1, 2, 3, ... to represent them, here, we do not want to take this for granted. We need some raw material to start. This raw material is the minimum required to build our 'mathematical' structure. For example, if we wish to build the wall, we need bricks. If we are bold enough, we can start from clay and make the bricks by ourselves. Well, I'd like to have superpowers so that I can make silica by myself. All needed are fundamental particles. One can again ask whether I can start from something more fundamental. This is where we are restricted by our imagination. Anyway, to build our structure of natural numbers, we start with two things. Lets assume that someone has given us them, one is the number 0 (we will name it 'zero') and one is an increment operator ⊞ and we will collect anything we can produce using these two 'things' in a ' mathematical basket' of Natural Numbers. We will represent this basket by letter $\mathbb{N}$).

1.1 Operator ⊞

If we apply this operator to any of the natural number, we get the **next** [1] that natural number. So 0⊞ is our next natural number (1 is probably a saner way to write this number). Next in line will be (0⊞)⊞ (i.e. 2) and ((0⊞)⊞)⊞ (i.e. 3) and so on.

[1] If we 'keep' natural numbers on some line then by 'next', we mean a relation R such that for given two number aRb means $bisrighttoa$. Only either of aRb or bRa can be true at a time. See 'well ordering principle' and 'axiom of choice' for more details.

Now we are ready to use these axioms known as Peano Axioms.

Axiom 1. *0 is a natural number.*

No, you can not ask any question at this stage, you mortal earthling!

Axiom 2. *if n is a natural number then $n\boxplus$ is a natural number.*

We start from 0 and apply this axiom once. We get $0\boxplus$, thus born *numero uno* 1 in this big empty world. Apply this axiom again and we get $(0\boxplus)\boxplus$ i.e. 2. Now if someone ask you to prove that 5 is a natural number, then invoke axiom 1 and apply axiom 2 five times and do not forget to laugh at his face. wink, wink.

These two axioms seems to be enough to represent natural numbers but these mathematicians are never satisfied with their creations. They attack each other in the search of **ultimate**. Consider two mathematicians, Superman and Lax Luthor. Superman gave these two axioms and thought that he saved the day. But next day, Lax Luthor came up with his evil arguments. He argues (fortunately anti-Superman Lax Luthor does not fight with mathematician Superman. They only argue.) that as soon as he goes up to some number say 7, next time you invoke axiom 2, I'll wrap back to 0, i.e. $7\boxplus = 0$. What you gonna do about it pretty face? Your two axioms still holds but but natural number set, out basket, $\mathbb{N}$ contains numbers from this series 0 1 2 3 4 5 6 7 0 1 2 3 4 5 6 7 0 1 2 ...

Axiom 3. *0 is not the successor of any natural number, i.e. there is no number n such that $n\boxplus = 0$.*

Luthor now argues, "Ok, fine! $7\boxplus$ is not equal to 0. Let say $7\boxplus$ is now equal to 1 or 2 or 3 etc. This will not contradict with any of your axioms. If $7\boxplus = 2$ then my system will be 0 1 2 3 4 5 6 7 2 3 4 5 6 7 2 3 4 5 6 ....

Axiom 4. *Different natural numbers have different successors; i.e., if n, m are two different natural numbers than $n\boxplus$ and $m\boxplus$ are also different natural numbers. EQUIVALENTLY if $m\boxplus = n\boxplus$ then we must have $m = n$.*

Now what Lax! Now all natural numbers are now distinct. Now $7\boxplus$ can not be equal to 2 because there is another number 1 such that $1\boxplus = 2$. According to this axiom, $7\boxplus$ and $1\boxplus$ must be different. Should Superman think that he is done? Wait! Lax Luthor is not considered one of the top ranked super villain for nothing. He now argues that 0.5, 1.5, 2.5 etc are also natural numbers. He is introducing **rogue elements**. Now its Superman primary duty to keep rogue elements away from human Mathematics. Sure, you can never get 0.5, 1.5 etc if you start building natural numbers from 0 using these axioms. But how Superman will draw this *Lakshman Rekha* to keep these rotten apple out of our basket $\mathbb{N}$. Can Superman get away from it by stating that one can never get 0.5 from 0 using my axioms. But it will be very difficult for Mathematical League (unlike Justice League, our Mathematical League is not biased against Lax Luthor.) to quantify what Superman means by *can never get from 0* without already using the natural numbers which he is trying to defend.

So he went to his 'fortress of solitude' and after discussing it with Brainiac for years, He found the way our of it.

Axiom 5. Principle of Mathematical Induction [2]
Let $P(n)$ be any property pertaining to natural number n. Suppose that $P(0)$ is true, ans suppose that when P(n) is true, $P(n\boxplus)$ is also true. Then $P(n)$ is true.

What Superman exactly means by property is a hard to clarify at this point. For example $P(n)$ might be anything e.g. *n is prime*; *n is odd*; *n solve a given equation*; *n gone crazy!* etc.

Now $P(n)$ is such that $P(0)$ is true, and such that whenever $P(n)$ is true, then $P(n\boxplus)$ is true. Since $P(0)$ is true $P(0\boxplus)$ i.e. $P(1)$ is true. Since $P(1)$ is true so is $P(1\boxplus)$ i.e. $P(2)$ is true and so one. Clearly this will fail with $P(0.5)$. We can use a simple proof given in [Tao, 2009] to keep these rogue elements 0.5, 1.5 etc out of $\mathbb{N}$.

Proof. We will be using integers without defining them. We borrow the definition of integers from somewhere. Lets say P(n) = n "is not a half integer" i.e. an integer plus 0.5. Then P(0) is true. And if P(n) is true, then P(n⊞) is true. Thus this axiom assert that P(n) is true for all natural numbers n, i.e. no natural number can be half integer; and P(0.5) fails to pass this test [Tao, 2009]. □

2 Numbers and Language

At least for anthropologists, it is hard to believe that there could be some people who does not have some idea of natural numbers. People use different sounds (figures) to speak (write) natural numbers. It is well known that some of the animals have some idea or sense of natural numbers. A bird can be seen in stress if one of her eggs is stolen i.e. she knows how to count. Since, language helps us to represent these numbers if came to a be surprise for many that there exists a language in which numbers are missing. Apparently their counting system consists of only three quantity, one, few and many which they represent by sound of `hoi`, `hoiii`, and `hoiiiii`. What would happen of one of them came to know about the idea of infinity and start pronouncing it? Go ahead, read this story published in Nature. Lack of numbers are not the only feature of languages which surprises us.

References

[Tao, 2009] Tao, Terrence. 2009. *Analysis I*. Hindustan Book Agency.

[2]Since this axiom refers not just to variable but also to properties, it should technically be called **axiom schema** rather than an axiom. It is not hard to see that it is a template for producing an infinite number of axioms, rather than being a simple axiom in its own right.[Tao, 2009]