(1)

(a) $X \sim Bin(n, p)$

$$M_X(t) = E(e^{tX}) = \sum_0^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_0^n \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= (1-p+pe^t)^n = (q+pe^t)^n$$

$$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = n(q+pe^t)^{n-1} pe^t\Big|_{t=0} = np$$

$$EX^2 = \mu_2' = \frac{\partial^2}{\partial t^2} M_X(t)\Big|_{t=0}$$

$$= n(n-1)(q+pe^t)^{n-2}(pe^t)^2 + n(q+pe^t)^{n-1} pe^t \Big|_{t=0}$$

$$= n(n-1)p^2 + np$$

$$V(X) = EX^2 - (EX)^2 = n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p) = npq$$

(b) $X \sim NB(r, p)$

$$M_X(t) = E(e^{tX}) = \sum_0^\infty e^{tx} \binom{x+r-1}{x} q^x p^r$$

$$= p^r \sum_0^\infty e^{tx} (-1)^x q^x \binom{-r}{x}$$

$$= p^r \sum_0^\infty \binom{-r}{x} (-qe^t)^x$$

$$= p^r (1-qe^t)^{-r}$$

$$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = p^r(-r)(1-qe^t)^{-r-1}(-qe^t)\Big|_{t=0}$$

$$= p^r(-r)p^{-r-1}(-q) = \frac{rq}{p}$$

$$EX^2 = \frac{\partial^2}{\partial t^2} M_X(t) \Big|_{t=0}$$

$$\frac{\partial^2}{\partial t^2} M_X(t)\Big|_{t=0} = \frac{\partial}{\partial t}\left(r q p^r (1-qe^t)^{-(r+1)} e^t \right)\Big|_{t=0}$$

$$= r q p^r \Big(-(r+1)(1-qe^t)^{-r-2}(-qe^t)\, e^t$$

$$+ (1-qe^t)^{-r-1} e^t \Big) \Big|_{t=0}$$

$$= r q p^r \left((r+1) q\, p^{-r-2} + p^{-r-1} \right)$$

$$= r(r+1) q^2 p^{-2} + r q p^{-1} = EX^2$$

$$V(X) = \frac{r(r+1)q^2}{p^2} + \frac{rq}{p} - \frac{r^2 q^2}{p^2}$$

$$= \frac{\cancel{r^2 q^2} + r q^2 + r p q - \cancel{r^2 q^2}}{p^2} = \frac{rq(q+p)}{p^2} = \frac{rq}{p^2}$$

(c) $X \sim P(\lambda)$

$$M_X(t) = E(e^{tX}) = \sum_0^\infty e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_0^\infty (\lambda e^t)^x / x!$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{-\lambda(1-e^t)}$$

$$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = e^{-\lambda}\left(e^{\lambda e^t} \lambda e^t \right)\Big|_{t=0} = \lambda$$

$$EX^2 = \frac{\partial^2}{\partial t^2} M_X(t)\Big|_{t=0} = \lambda e^{-\lambda}\left(e^{\lambda e^t} \lambda e^t e^t + e^{\lambda e^t} e^t \right)\Big|_{t=0}$$

$$= \lambda e^{-\lambda}(\lambda e^{\lambda} + e^{\lambda}) = \lambda^2 + \lambda$$

(d) $X \sim G(\alpha, \beta)$

$$M_X(t) = E(e^{tX})$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty e^{tx} e^{-x/\beta} x^{\alpha-1} dx$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty e^{-x\left(\frac{1}{\beta}-t\right)} x^{\alpha-1} dx$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \cdot \frac{\Gamma\alpha}{\left(\frac{1}{\beta}-t\right)^\alpha} \quad \text{for } t < \frac{1}{\beta}$$

$$= \frac{1}{(1-\beta t)^\alpha}$$

$$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = (-\alpha)\,(1-\beta t)^{-\alpha-1}(-\beta)\Big|_{t=0} = \alpha\beta$$

$$EX^2 = \frac{\partial^2}{\partial t^2} M_X(t)\Big|_{t=0} = \cdots$$

& get $V(X) = EX^2 - (EX)^2$

(e) Refer to lecture notes

(2) X : # of interview attempts to get 5 interviews.

$$P(X=x) = \binom{x-1}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{x-5} \times \frac{2}{3} \quad ; \; x = 5, 6, \ldots$$

reqd prob $P(X \leq 8) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$$= \binom{4}{4}\left(\frac{2}{3}\right)^5 + \binom{5}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1 + \ldots + \ldots$$

$$= \ldots$$

(3)

P(selecting Box 1) = P(selecting Box 2) = $\frac{1}{2}$

Suppose Box 2 is found empty ~~[illegible]~~, then Box 2 has been chosen $(N+1)^{th}$ times, at this time Box 1 contains K matches iff it has been chosen N−K times.

Choosing Box 2 ≡ success.
choosing Box 1 ≡ failure } Bernoulli trial. $p = \frac{1}{2}$

⇒ Box 2 found empty with K matches left in Box 1
≡ N − K failures preceeding $(N+1)^{th}$ success

$$\text{prob} = \binom{N+(N-K)}{N}\left(\frac{1}{2}\right)^N\left(\frac{1}{2}\right)^{N-K} \cdot \frac{1}{2}$$

$$= \binom{2N-K}{N}\left(\frac{1}{2}\right)^{2N-K+1}$$

Sly Box 1 found empty with k matches in Box 2

$$\text{prob} = \binom{2N-K}{N}\left(\frac{1}{2}\right)^{2N-K+1}$$

$$\Rightarrow \text{reqd prob} = \binom{2N-K}{N}\left(\frac{1}{2}\right)^{2N-K}.$$

(4)

Belt 1

$X \sim Exp$ ~~with~~ with mean $\alpha \sim \frac{1}{\alpha} e^{-x/\alpha}$; $x > 0$

Belt 2 $Y \sim Exp$ with mean $2\alpha \sim \frac{1}{2\alpha} e^{-x/2\alpha}$; $x > 0$

P(system works beyond α)

$$P(X > \alpha \cap Y > \alpha) = P(X > \alpha)\, P(Y > \alpha)$$

$$= \left(\int_\alpha^\infty \frac{1}{\alpha} e^{-x/\alpha}\,dx\right) \times \left(\int_\alpha^\infty \frac{1}{2\alpha} e^{-y/2\alpha}\,dx\right)$$

$$= e^{-1} \times e^{-1/2} = e^{-3/2}$$

(5) (a) $P(X > 5) = P\left(\frac{X-10}{6} > \frac{5-10}{6}\right) = P\left(Z > -\frac{5}{6}\right)$; $Z \sim N(0,1)$

$$= 1 - \Phi(-5/6)$$

$$= 1 - (1 - \Phi(5/6))$$

$$= \Phi(5/6) = .7967$$

(b) $P(4 < X < 16) = P\left(\frac{4-10}{6} < Z < \frac{16-10}{6}\right) = P(-1 < Z < 1)$

$$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1$$

$$= \ldots$$

(c) $P(X < 8) = P\left(Z < \frac{8-10}{6}\right) = \Phi(-\frac{1}{3}) = 1 - \Phi(1/3) = \ldots$

(6)

$$P(X \le 0) = \frac{1}{2} = P(X \ge 0) \Rightarrow \mu = 0$$

$$P(-1.96 \le X \le 1.96) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \le \frac{X}{\sigma} \le \frac{1.96}{\sigma}\right) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \le Z \le \frac{1.96}{\sigma}\right) = 0.95 \; ; Z \sim N(0,1)$$

$$2\Phi\left(\frac{1.96}{\sigma}\right) - 1 = 0.95$$

$$\Phi\left(\frac{1.96}{\sigma}\right) = 0.975$$

$$\Rightarrow \frac{1.96}{\sigma} = \Phi^{-1}(0.975) = 1.96$$

$$\Rightarrow \sigma = 1$$

(7) X : lifetime r.v.

$X \sim N(\mu, \sigma^2)$

$\mu = 1.4 \times 10^6$ hrs

$\sigma = 3 \times 10^5$ hrs

$P(X < 1.8 \times 10^6)$

$$= P\left(\frac{X - 1.4 \times 10^6}{3 \times 10^5} < \frac{0.4 \times 10^6}{3 \times 10^5}\right)$$

$$= P\left(Z < \frac{4}{3}\right) \quad [Z \sim N(0,1)]$$

$$= \Phi\left(\frac{4}{3}\right) = 0.918$$

Y : r.v. denoting # of chips that have lifetime $< 1.8 \times 10^6$ hr

$Y \sim Bin(10, 0.918)$

$$\Rightarrow P(Y \geq 2) = 1 - P(Y < 2)$$

$$= 1 - P(Y=0) - P(Y=1)$$

$$= 1 - \binom{10}{0}(.918)^0(1-.918)^{10} - \binom{10}{1}(0.918)^1(1-.918)^9$$

$= \ldots$

(8) $X \sim N(0,1)$

$\forall\, t > 0 \quad P(|X| \geq t) = 1 - P(|X| < t)$

$= 1 - P(-t < X < t)$

$= 1 - [\Phi(t) - \Phi(-t)]$

$= 1 - [2\Phi(t) - 1]$

or $= 1 - [2(1 - P(X > t)) - 1]$

$= 2 - 2 + 2P(X > t) = 2P(X > t)$

$$P(X > t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2}\, dx$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \frac{x}{t} e^{-x^2/2}\, dx \quad [\, t < x < \infty \,]$$

$y = x^2/2$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{t} \int_{t^2/2}^{\infty} e^{-y}\, dy = \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t}$$

$$\Rightarrow P(|X| \geq t) \leq 2 \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t} = \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}$$

(9)

$X = x$	0	1	2	$\cdots$
$P(X = x)$	p_0	p_1	p_2	$\cdots$

$$\sum_{k=0}^{\infty} (1 - F(k)) = \sum_{k=0}^{\infty} P(X > k) = P(X > 0) + P(X > 1) + P(X > 2) + \cdots$$

$= (p_1 + p_2 + p_3 + \cdots)$

$+ (p_2 + p_3 + \cdots)$

$+ (p_3 + p_4 + \cdots)$

$= p_1 + 2p_2 + 3p_3 + \cdots$

$$= \sum_{i=1}^{\infty} i\, p_i = \sum_{i=0}^{\infty} i\, P(X=i) = E(X)$$

d.f. $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\beta x^2}, & x \geq 0. \end{cases}$

$\beta > 0$

(10) p.d.f. $f(x) = \begin{cases} 2\beta x e^{-\beta x^2}, & x \geq 0 \\ 0 & \text{o/w.} \end{cases}$

$$E(x) = 2\beta \int_0^{\infty} x^2 e^{-\beta x^2} dx$$

$y = x^2$

$$= \beta \int_0^{\infty} y^{1/2} e^{-\beta y} = \beta \cdot \frac{\Gamma 3/2}{\beta^{3/2}} = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}} = \mu.$$

$$E X^2 = 2\beta \int_0^{\infty} x^3 e^{-\beta x^2} dx = \beta \int_0^{\infty} y e^{-\beta y} dy = \beta \frac{\Gamma 2}{\beta^2} = \frac{1}{\beta}$$

$y = x^2$

$$V(x) = E(x^2) - (E X)^2 = \frac{1}{\beta} - \mu^2 = \frac{1}{\beta} - \frac{\pi}{4\beta}$$

median : m_0

$$m_0 \ni F(m_0) = \frac{1}{2} = 1 - F(m_0)$$

i.e. $2\beta \int_0^{m_0} x e^{-\beta x^2} dx = 2\beta \int_{m_0}^{\infty} x e^{-\beta x^2} dx = \frac{1}{2}$

i.e. $1 - e^{-\beta m_0^2} = \frac{1}{2}$

$\Rightarrow m_0 = \ldots$

(11)

$$1-\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_x^{\infty} \frac{1}{y} \left(y e^{-y^2/2} \right) dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{y} \cdot \left(- e^{-y^2/2} \right) \Big|_x^{\infty} - \int_x^{\infty} \left(-\frac{1}{y^2} \right) \left(- e^{-y^2/2} \right) dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{x} e^{-x^2/2} - \underbrace{\int_x^{\infty} \frac{1}{y^2} e^{-y^2/2}\, dy}_{\geq 0} \right]$$

$$\Rightarrow \quad 1-\Phi(x) \leq \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \frac{\phi(x)}{x}.$$

(12)

Mode - pt at which $f(x)$ is max^m.

$$f'(x) = 2\beta\left(x e^{-\beta x^2}(-2\beta x) + e^{-\beta x^2}\right)$$

$$f'(x) = 0 \Rightarrow 2\beta x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{2\beta}}.$$

$$f''(x) = 2\beta \frac{d}{dx}\left(e^{-\beta x^2}(1 - 2\beta x^2)\right)$$

$$= 2\beta\left(e^{-\beta x^2}(-4\beta x) + (1-2\beta x^2)e^{-\beta x^2}(-2\beta x)\right).$$

$$f''(x)\Big|_{x=\frac{1}{\sqrt{2\beta}}} = 2\beta\left(e^{-\frac{1}{2}}\left(-4\sqrt{\beta/2}\right)\right) < 0.$$

$\beta > 0$

$\Rightarrow$ m^*, the mode of the dist^n is at $\frac{1}{\sqrt{2\beta}}$.

$$m^* = \frac{1}{\sqrt{2\beta}}$$

$$\mu = E(X) = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{\beta}} = \left(\frac{\sqrt{\pi}}{2}\right)\sqrt{2}\, m^*$$

i.e. $\mu = \sqrt{\frac{\pi}{2}}\, m^*$.

& $2 m^{*2} - \mu^2 = 2\left(\frac{2}{\pi}\mu^2\right) - \mu^2$

$$= \frac{4}{\pi}\mu^2 - \mu^2 = \frac{4}{\pi}\cdot\frac{\pi}{4}\cdot\frac{1}{\beta} - \mu^2$$

$$= \cancel{\left(\frac{4}{\pi} - 1\right)\mu^2} = \frac{1}{\beta} - \mu^2$$

$$= E X^2 - \mu^2$$

$$= V(X).$$

i.e.

$2 m^{*2} - \mu^2 = \sigma^2$

(13) Let $Y = X - \theta$ & $Z = \theta - X$

(a)

then p.d.f. of Y is $f_Y(y) = f_X(\theta + y) \quad \forall y \in \mathbb{R}$

. . . Z is $f_Z(z) = f_X(\theta - z) \quad \forall z \in \mathbb{R}$

Now $X - \theta \overset{d}{=} \theta - X$ i.e. $Y \overset{d}{=} Z$

$\Leftrightarrow f_Y(x) = f_Z(x) \quad \forall x \in \mathbb{R}$

$\Leftrightarrow f_X(\theta + x) = f_X(\theta - x) \quad \forall x \in \mathbb{R}$

(b) $Y = X - \theta \qquad Z = \theta - X$

$F_Y(x) = P(X \leq x + \theta)$
$= F_X(x + \theta)$
$\forall x \in \mathbb{R}$

$F_Z(x) = P(\theta - X \leq x)$
$= P(X \geq \theta - x)$
$= 1 - F_X(\theta - x)$
$\forall x \in \mathbb{R}$

$Y \overset{d}{=} Z \Leftrightarrow F_Y(x) = F_Z(x) \quad \forall x \in \mathbb{R}$

$\Leftrightarrow F_X(x + \theta) = 1 - F_X(\theta - x) \quad \forall x \in \mathbb{R}$

$\Leftrightarrow F_X(x + \theta) + F_X(\theta - x) = 1$

(c). $X - \theta \overset{d}{=} \theta - X$

$\Rightarrow E(X - \theta) = E(\theta - X)$

$\Rightarrow E(X) = \theta$

Since $X - \theta \overset{d}{=} \theta - X$

$P(X - \theta \leq 0) = P(\theta - X \leq 0)$

$\Leftrightarrow F_X(\theta) = 1 - F_X(\theta)$

i.e. $F_X(\theta) = \frac{1}{2} \Rightarrow \theta$ is the median of X

(14)

(a) $$E\left(\frac{X-\mu}{\sigma}\right)^r = \int_{\mu}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^r \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)} dx$$

$$= \int_{0}^{\infty} t^r e^{-t} dt = \Gamma_{r+1} = r!$$

$r = 1, 2, \ldots$

$$\Rightarrow E(X-\mu)^r = \sigma^r \Gamma_{r+1}$$

(b) $E(X-\mu) = \sigma \Rightarrow E(X) = \mu + \sigma$

$E(X-\mu)^2 = 2\sigma^2$

i.e. $E(X^2 + \mu^2 - 2X\mu) = 2\sigma^2$

i.e. $E(X^2) = 2\sigma^2 - \mu^2 + 2(\mu+\sigma)\mu$

$= 2\sigma^2 - \mu^2 + 2\mu^2 + 2\mu\sigma$

$= 2\sigma^2 + 2\mu\sigma + \mu^2$

(c) let med $= \eta_{1/2}$

$F(\eta_{1/2}) = \frac{1}{2}$

i.e. $1 - e^{-(\eta_{1/2}-\mu)/\sigma} = \frac{1}{2}$

i.e. $\eta_{1/2} = \mu - \sigma \ln\left(\frac{1}{2}\right)$

(15) Let n be the number of indep Bernoulli trials with prob of success p $(0<p<1)$

X: number of successes in n trials

$X \sim B(n, p)$

$$P(X \geq r) = P(\text{at least } r \text{ successes in } n \text{ trials})$$

$$= P\left(\bigcup_{l=0}^{n-r} \{r^{th} \text{ success in } (r+l)^{th} \text{ trial}\}\right)$$

$$= \sum_{l=0}^{n-r} P(r^{th} \text{ success in } (r+l)^{th} \text{ trial})$$

$$= \sum_{l=0}^{n-r} \underbrace{\binom{r+l-1}{r-1} p^{r-1} (1-p)^{l}}_{(r-1) \text{ successes in first } r+l-1 \text{ trials}} \underbrace{p}_{\text{success in } r^{th} \text{ trial}}$$

$$= \sum_{l=0}^{n-r} \binom{r+l-1}{r-1} p^{r} (1-p)^{l}$$

$$= P(Y \leq n-r) \quad ; \; Y \sim NB(r, p)$$

(16) $X \sim P(\lambda)$

$$E \frac{1}{2+X} = \sum_{x=0}^{\infty} (x+2)^{-1} \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(x+1)\lambda^{x}}{(x+2)!} = e^{-\lambda} \sum_{x=2}^{\infty} (x-1) \frac{\lambda^{x-2}}{x!}$$

$$= \frac{1}{\lambda^2} \left(\sum_{x=2}^{\infty} (x-1) \frac{e^{-\lambda} \lambda^{x}}{x!} \right)$$

$$= \frac{1}{\lambda^2} \left[\sum_{x=0}^{\infty} (x-1) \frac{e^{-\lambda} \lambda^{x}}{x!} + e^{-\lambda} \right]$$

$$= \frac{1}{\lambda^2} \left(E(X-1) + e^{-\lambda} \right)$$

$$= \frac{1}{\lambda^2} \left(\lambda - 1 + e^{-\lambda} \right)$$

(17) $X \sim G(n, \theta)$; $Y \sim P(t/\theta)$

To prove that

$$P(X \geq t) = P(Y \leq n-1)$$

i.e. $$\frac{1}{\Gamma n \, \theta^n} \int_t^\infty e^{-x/\theta} x^{n-1} \, dx = \sum_{k=0}^{n-1} \frac{e^{-t/\theta} (t/\theta)^k}{k!}$$

$$\text{l.h.s} = \frac{1}{\Gamma n \, \theta^n} \int_t^\infty e^{-x/\theta} x^{n-1} \, dx \quad (= I_n \text{ (say)})$$

$\left(z = \frac{x}{\theta}\right) \rightarrow$ $$= \frac{1}{(n-1)!} \int_{t/\theta}^\infty e^{-z} z^{n-1} \, dz .$$

Integration by parts $$= \frac{e^{-t/\theta} (t/\theta)^{n-1}}{(n-1)!} + \frac{1}{(n-2)!} \int_{t/\theta}^\infty e^{-z} z^{n-2} \, dz$$

$$= \frac{e^{-t/\theta} (t/\theta)^{n-1}}{(n-1)!} + I_{n-2}$$

$$= P(Y = n-1) + I_{n-2} = P(Y = n-1) + P(Y = n-2) + I_{n-3}$$

$\vdots$ continue integration by parts to get I_1

$$= P(Y = n-1) + P(Y = n-2) + \cdots + \underbrace{I_2}_{P(Y=1)} + \underbrace{I_1}_{P(Y=0)}$$

$$= P(Y \leq n-1)$$