Problem Set # 9

① $P(|X_n - c| \geq \epsilon) \leq \dfrac{E(X_n - c)^2}{\epsilon^2}$ by chebyshev's ineq

$= E(X_n - EX_n + EX_n - c)^2 / \epsilon^2$

$= \dfrac{E(X_n - EX_n)^2 + (EX_n - c)^2}{\epsilon^2}$

Since $EX_n \to c$ and $VX_n \to 0$ as $n \to \infty$

$$\frac{E(X_n - EX_n)^2 + (EX_n - c)^2}{\epsilon^2} = \frac{V(X_n) + (EX_n - c)^2}{\epsilon^2} \longrightarrow 0 \text{ as } n \to \infty \ \forall \epsilon > 0.$$

$\Rightarrow P(|X_n - c| \geq \epsilon) \to 0$ as $n \to \infty$ $\forall \epsilon > 0$

$\Rightarrow X_n \xrightarrow{p} c.$

② $S_n = \sum_{i=1}^{n} X_i$; Take $a_n = \sum \mu_i$ and $b_n = n$

$$\frac{S_n - a_n}{b_n} = \frac{\sum (X_i) - \sum \mu_i}{n}$$

$$P\left(\left|\frac{S_n - a_n}{b_n}\right| \geq \epsilon\right) = P\left(\left|\frac{\sum X_i - \sum \mu_i}{n}\right| \geq \epsilon\right)$$

$$\leq \frac{E(\sum X_i - \sum \mu_i)^2}{n^2 \epsilon^2}$$

$$= \frac{E(\sum X_i - E(\sum X_i))^2}{n^2 \epsilon^2}$$

$$= \frac{V(\sum X_i)}{n^2 \epsilon^2} \longrightarrow 0 \text{ as } n \to \infty \ \forall \epsilon > 0.$$

from the given condition

$$\Rightarrow \quad \frac{S_n - a_n}{b_n} \xrightarrow{p} 0$$

$\Rightarrow$ WLLN holds for $\{X_n\}$.

Furthermore,
$$\frac{S_n - a_n}{b_n} = \frac{\sum X_i}{n} - \frac{\sum \mu_i}{n}$$
$$= \bar{X}_n - \bar{\mu}_n \xrightarrow{p} 0$$

③ $X_1, \ldots X_n$ i.i.d $U(0,1)$

$Y_n = \min(X_1, \ldots X_n)$; $Z_n = \max(X_1, \ldots X_n)$

d.f. $F_{Y_n}(y) = P(\min(X_1, \ldots X_n) \le y)$
$$= 1 - P(\min(X_1, \ldots X_n) > y)$$
$$= 1 - (1 - F_X(y))^n$$

p.d.f. $f_{Y_n}(y) = n(1 - F_X(y))^{n-1} f_X(y)$; $0 < y < 1$

i.e. $f_{Y_n}(y) = \begin{cases} n(1-y)^{n-1} & ; \quad 0 < y < 1 \\ 0 & \text{o/w.} \end{cases}$

$$P(|Y_n| > \epsilon) \le \frac{E Y_n^2}{\epsilon^2}$$

Now $E Y_n^2 = n \int_0^1 y^2 (1-y)^{n-1} dy = n \int_0^1 (1-x)^2 x^{n-1} dx$

$$= n \int_0^1 (1 + x^2 - 2x) x^{n-1} dx = n\left(\frac{1}{n} + \frac{1}{n+2} - \frac{2}{n+1}\right)$$
$$\longrightarrow 0 \text{ as } n \to \infty$$

$\Rightarrow \frac{E Y_n^2}{\epsilon^2} \longrightarrow 0$ as $n \to \infty$ $\forall \epsilon > 0$. however small.

$$\Rightarrow P(|Y_n| > \epsilon) \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow Y_n \xrightarrow{p} 0$$

p.d.f. of Z_n : $f_{Z_n}(z) = \begin{cases} n z^{n-1}, & 0 < z < 1 \\ 0, & \text{o/w} \end{cases}$

$$P(|Z_n - 1| > \epsilon) \le \frac{E(Z_n-1)^2}{\epsilon^2} = \frac{E Z_n^2 + 1 - 2E(Z_n)}{\epsilon^2}$$

$$E Z_n = n \int_0^1 z^n \, dz = \frac{n}{n+1}$$

$$E Z_n^2 = n \int_0^1 z^{n+1} \, dz = \frac{n}{n+2}$$

$$\Rightarrow \frac{E(Z_n-1)^2}{\epsilon^2} = \frac{1}{\epsilon^2}\left(\frac{n}{n+2} + 1 - 2\frac{n}{n+1}\right) \longrightarrow 0 \quad \text{as } n \to \infty \quad \forall \epsilon > 0 \text{ fixed.}$$

$$\Rightarrow P(|Z_n - 1| > \epsilon) \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow Z_n - 1 \xrightarrow{p} 0 \quad \text{i.e. } Z_n \xrightarrow{p} 1$$

$$\Rightarrow Z_n^2 \xrightarrow{p} 1 \quad \left(\begin{array}{l}\text{If } X_n \xrightarrow{p} X \\ \text{for } g \text{ cont} \\ \quad g(X_n) \xrightarrow{p} g(X)\end{array}\right).$$

Since $Y_n \xrightarrow{p} 0$ & $Z_n^2 \xrightarrow{p} 1$

$$Y_n \cdot Z_n^2 \xrightarrow{p} 0 \quad \left(\begin{array}{l}\text{If } X_n \xrightarrow{p} X \text{ \& } Y_n \xrightarrow{p} Y \\ \text{then } X_n Y_n \xrightarrow{p} XY\end{array}\right).$$

(4) $X_1, \ldots X_n$ i.i.d. $N(0,1)$

WLLN $\Rightarrow$ $\frac{1}{n}\sum X_i \xrightarrow{p} EX_1 = 0$

(Khintchime's WLLN) i.e. $\bar{X}_n \xrightarrow{p} 0$

$$S_n^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2$$

Note that $X_1^2, X_2^2, \ldots X_n^2$ are i.i.d with $E(X_1^2) = 1$

WLLN $\Rightarrow$ $\frac{1}{n}\sum_1^n X_i^2 \xrightarrow{p} EX_1^2 = 1$

(Khintchime's WLLN)

Since $\bar{X}_n \xrightarrow{p} 0 \Rightarrow \bar{X}_n^2 \xrightarrow{p} 0.$

$$\Rightarrow S_n^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2 \xrightarrow{p} 1 - 0 = 1$$

$$\Rightarrow S_n^{-1} \xrightarrow{p} 1$$

Since $\bar{X}_n \xrightarrow{p} 0$ & $S_n^{-1} \xrightarrow{p} 1$

$$\bar{X}_n S_n^{-1} \xrightarrow{p} 0 \left(= 0/1\right)$$

(5) $Y_n \sim Bin(n, p)$

$$P\left(\left|\frac{Y_n}{n} - p\right| > \epsilon\right) \le \frac{E\left(\frac{Y_n}{n} - p\right)^2}{\epsilon^2} = \frac{E(Y_n - np)^2}{n^2\epsilon^2}$$

$$= \frac{npq}{n^2\epsilon^2} = \frac{pq}{n\epsilon^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \frac{Y_n}{n} \xrightarrow{p} p$$

$$\Rightarrow \left(1 - \frac{Y_n}{n}\right) \xrightarrow{p} p.$$

(6) $E(X_n) = 0$; $V(X_n) = E X_n^2 = \frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{2} = \sqrt{n}$

$E\bar{X}_n = 0$; $V(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} \sqrt{i} \le \frac{n\sqrt{n}}{n^2} \to 0$ as $n \to \infty$

$\Rightarrow P(|\bar{X}_n - 0| > \epsilon) \le \frac{E\bar{X}_n^2}{\epsilon^2} = \frac{V\bar{X}_n}{\epsilon^2} \to 0$ as $n \to \infty \;\forall \epsilon > 0$

$\Rightarrow \bar{X}_n \xrightarrow{p} 0$.

(7) (a) $Y_n = \frac{2}{n(n+1)} \sum_{i=1}^{n} i X_i$

$E Y_n = \frac{2}{n(n+1)} \sum_{1}^{n} i \mu = \mu$

$V Y_n = \frac{4}{n^2(n+1)^2} \sum_{1}^{n} i^2 \sigma^2 = \frac{4\sigma^2}{n^2(n+1)^2} \cdot \frac{n(n+1)(2n+1)}{6} \to 0$ as $n \to \infty$

$\Rightarrow P(|Y_n - \mu| > \epsilon) \le \frac{E(Y_n - \mu)^2}{\epsilon^2} = \frac{V(Y_n)}{\epsilon^2} \to 0$ as $n \to \infty$

$\Rightarrow Y_n \xrightarrow{p} \mu$.

(b) Sly as in (a)

(8) $X_1, \ldots X_n$ i.i.d $U(0,1)$

Let $T_i = -\log X_i \quad i = 1(1)n$

$T_i \sim Exp(0,1) \qquad f_{T_i}(t) = \begin{cases} e^{-t} & t > 0 \\ 0, & o/w. \end{cases}$

$E T_i = 1$

$T_1, \ldots T_n$ are i.i.d. $Exp(0,1)$ with $E T_i = 1$

$Z_n = \left(\prod_{1}^{n} X_i\right)^{1/n}$

$-\log Z_n = \frac{1}{n} \sum_{1}^{n} (-\log X_i) = \frac{1}{n} \sum_{1}^{n} T_i$

WLLN $\Rightarrow \frac{1}{n} \sum_{1}^{n} T_i \xrightarrow{p} E(T_1) = 1$

i.e. $-\log Z_n \xrightarrow{p} 1$

$\Rightarrow Z_n \xrightarrow{p} e^{-1}$.

(9) $E X_i = \mu_i$, $V(X_i) = \sigma_i^2$

If $\sum_1^n \sigma_i^2 \to \infty$, take $a_n = \sum_1^n \mu_i$

& $b_n = \sum_1^n \sigma_i^2$

then
$$P\left(\left|\frac{S_n - a_n}{b_n}\right| > \epsilon\right) = P\left(\left|\frac{1}{\sum \sigma_i^2} \sum (X_i - \mu_i)\right| > \epsilon\right)$$
$$\leq \frac{E\left(\sum (X_i - \mu_i)\right)^2}{\left(\sum \sigma_i^2\right)^2 \epsilon^2}$$
$$= \frac{\sum \sigma_i^2}{\epsilon^2 \left(\sum \sigma_i^2\right)^2} = \frac{1}{\epsilon^2 \sum \sigma_i^2} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \frac{S_n - a_n}{b_n} \xrightarrow{p} 0$$

i.e. WLLN holds for $\{X_n\}$.

(10) $X_1 \ldots X_n$ i.i.d. $U(0,1)$ with $E X_i = \frac{1}{2}$

W.LLN $\Rightarrow$ $\frac{1}{n} \sum X_i \xrightarrow{p} E X_1 = \frac{1}{2}$

i.e. $\bar{X}_n \xrightarrow{p} \frac{1}{2}$

(11)

$$F_{X_n}(x) = P(X_n \le x)$$

$$= P\left(\frac{X_n - \frac{1}{n}}{\sqrt{1-\frac{1}{n}}} \le \frac{x - \frac{1}{n}}{\sqrt{1-\frac{1}{n}}} \right)$$

$$= \Phi\left(\frac{x - \frac{1}{n}}{\sqrt{1-\frac{1}{n}}} \right) \to \Phi(x) \quad \text{as } n \to \infty$$

$$\Rightarrow X_n \xrightarrow{L} X \sim N(0,1)$$

Alt :

$$M_{X_n}(t) = \exp\left(\frac{t}{n} + \frac{t^2}{2}\left(1 - \frac{1}{n}\right) \right)$$

$$\longrightarrow e^{t^2/2} ; \text{ m.g.f of } N(0,1).$$

as $n \to \infty$

$$\Rightarrow X_n \xrightarrow{L} X \sim N(0,1)$$

(12) $S_n = \sum_{1}^{n} X_i$

$E S_n = np$; $V(S_n) = npq$

$$P\left(\left| \frac{S_n}{n} - p \right| \ge \epsilon \right) \le \frac{E(S_n - np)^2}{n^2 \epsilon^2}$$

$$= \frac{np(1-p)}{n^2 \epsilon^2} \le \frac{1}{4n\epsilon^2}$$

From the given condition

$$P\left(\left| \frac{S_n}{n} - p \right| \ge 0.01 \right) \le 0.01$$

$$\Rightarrow \quad n \ge \frac{1}{0.04\,(0.01)^2} \ldots$$

(13) $f_{X_n}(x) = \begin{cases} \frac{1}{\Gamma n} e^{-x} x^{n-1}, & x > 0 \\ 0, & o/w \end{cases}$

m.g.f. of X_n ; $M_{X_n}(t) = \frac{1}{\Gamma n} \int_0^\infty e^{tx} e^{-x} x^{n-1} dx$

$= \frac{1}{\Gamma n} \int_0^\infty e^{-x(1-t)} x^{n-1} dx = (1-t)^{-n}$

m.g.f. of $Y_n = \frac{X_n}{n}$; $M_{Y_n}(t) = E(e^{\frac{t}{n} X_n}) = (1 - \frac{t}{n})^{-n}$

$\rightarrow e^t$ as $n \to \infty$

$\Rightarrow Y_n \xrightarrow{L} Y \ni P(Y=1) = 1$

(14) Let $Y_i = (X_i - \mu)^2$

$E Y_i = E(X_i - \mu)^2 = \sigma^2$

$V Y_i = E((X_i - \mu)^2 - \sigma^2)^2 = E(X_i - \mu)^4 - \sigma^4 = 1$

$Y_1, \ldots, Y_n$ i.i.d. with $E Y_i = \sigma^2$, $V Y_i = 1$ $\forall i$

$S_n = \sum_1^n Y_i$ $\quad E S_n = n\sigma^2$; $V S_n = n$

By CLT $\frac{S_n - E S_n}{\sqrt{V S_n}} \xrightarrow{L} N(0,1)$

i.e. $\frac{(X_1-\mu)^2 + \cdots + (X_n - \mu)^2 - n\sigma^2}{\sqrt{n}} \xrightarrow{L} N(0,1)$.

Now $P\left(\sigma^2 - \frac{1}{\sqrt{n}} \le \frac{\sum_1^n (X_i - \mu)^2}{n} \le \sigma^2 + \frac{1}{\sqrt{n}}\right)$

$= P\left(-1 \le \frac{\sum (X_i - \mu)^2 - n\sigma^2}{\sqrt{n}} \le 1\right)$

$\xrightarrow[\text{as } n \to \infty]{} \Phi(1) - \Phi(-1) = 2\Phi(1) - 1$

by CLT

(15) $X_1, \ldots, X_n$ i.i.d. $E X_i = \mu$; $V(X_i) = \sigma^2 \ \forall i$

By CLT
$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{L} N(0,1) \quad \text{as } n \to \infty \quad \text{— (i)}$$

$$S_n^2 = \frac{1}{n} \Sigma (X_i - \bar{X})^2 = \frac{1}{n} \Sigma X_i^2 - \bar{X}^2$$

By WLLN, $\frac{1}{n} \Sigma X_i^2 \xrightarrow{p} \sigma^2 + \mu^2 (= E X_1^2)$

& $\frac{1}{n} \Sigma X_i (= \bar{X}) \xrightarrow{p} \mu \Rightarrow \bar{X}^2 \xrightarrow{p} \mu^2$

$\Rightarrow S_n^2 \xrightarrow{p} \sigma^2 \Rightarrow S_n \xrightarrow{p} \sigma$ $\Big\}$ as $n \to \infty$

i.e. $\frac{S_n}{\sigma} \xrightarrow{p} 1$ as $n \to \infty$ — (ii)

Applying Slutsky's lemma (on (i) & (ii)),
$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \Big/ \frac{S_n}{\sigma} \xrightarrow{L} N(0,1)$$

i.e. $$\frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \xrightarrow{L} N(0,1)$$

(16). $X_1, \ldots, X_{72}$ i.i.d. (r.s) from $f(x) = \begin{cases} 1/x^2, & x \geq 1 \\ 0, & o/w \end{cases}$

Define $Y_i = \begin{cases} 1, & \text{If } X_i < 3 \\ 0, & o/w \end{cases}$

$$P(Y_i = 1) = P(X_i < 3) = \int_1^3 \frac{1}{x^2} dx = \frac{2}{3} = \theta, \text{ say}$$

$Y_1, \ldots, Y_{72}$ are i.i.d. $B(1, \theta)$

$$Y = \sum_1^{72} Y_i \sim B\left(72, \theta = \frac{2}{3}\right)$$

CLT $\Rightarrow$ $$\frac{Y - 72 \times \frac{2}{3}}{\sqrt{72 \times \frac{2}{3} \times \frac{1}{3}}} \underset{\text{for large } n}{\overset{\text{approximately}}{\sim}} Z \sim N(0,1)$$

i.e. $\frac{Y-48}{4} \overset{\text{approx}}{\sim} N(0,1)$

$$P(Y>50) = 1-P(Y\le 50) \quad \text{continuity correction}$$
$$= 1-P(Y\le 50.5)$$
$$= 1-P\left(\frac{Y-48}{4} \le \frac{50.5-48}{4}\right)$$
$$\approx 1-\Phi\left(\frac{2.5}{4}\right) = \cdots$$

(17) $X_1, \ldots, X_{100}$ i.i.d. $P(3)$

$EX_i = 3 \,;\, VX_i = 3 \quad \forall i$

$Y = \sum_1^{100} X_i \sim P(300) \;;\; E(Y) = 300 \,;\, V(Y) = 300$

CLT $\Rightarrow$ $\frac{Y-300}{10\sqrt{3}} \left(= \frac{S_n - ES_n}{\sqrt{VS_n}}\right) \overset{\text{approx}}{\sim} N(0,1)$

continuity correction

$$P(100 \le Y \le 200) = P(99.5 \le Y \le 200.5)$$
$$= P\left(\frac{99.5-300}{10\sqrt{3}} \le \frac{Y-300}{10\sqrt{3}} \le \frac{200.5-300}{10\sqrt{3}}\right)$$
$$\approx \Phi\left(\frac{200.5-300}{10\sqrt{3}}\right) - \Phi\left(\frac{99.5-300}{10\sqrt{3}}\right) = \cdots$$

(18) $f_{X_n}(x) = \begin{cases} \frac{1}{\Gamma\alpha\, \alpha^p} e^{-x/\alpha} x^{p-1}, & x > 0 \\ 0, & o/w \end{cases}$

i.e. $X_n \sim \text{Gamma}(\alpha, p)$

$$E X_n = \alpha p \;;\; V X_n = \alpha^2 p$$
$$= 8 \qquad\qquad = 16$$

By CLT

$$\frac{\sqrt{n}\,(\bar{X}_n - \alpha p)}{\sqrt{\alpha^2 p}} \xrightarrow{L} N(0,1) \quad \text{as } n \to \infty$$

i.e. $2(\bar{X}_{64} - 8) \overset{\text{approx}}{\sim} N(0,1)$

$$P(7 < \bar{X}_{64} < 9) = P\big(2(7-8) < 2(\bar{X}_{64} - 8) < 2(9-8)\big)$$
$$= P(-2 < 2(\bar{X}_{64} - 8) < 2)$$
$$\approx \Phi(2) - \Phi(-2) = 2\,\Phi(2) - 1$$

(19) $X_1, \ldots X_n$ i.i.d. $U(0,2)$

$$E X_i = \frac{1}{2}\int_0^2 x\,dx = 1$$
$$E X_i^2 = \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3} \;;\; V(X_i) = \frac{1}{3}$$

By CLT $\quad \sqrt{n}\,(\bar{X}_n - 1) \xrightarrow{L} N\left(0, \frac{1}{3}\right)$