MSO201: PROBABILITY & STATISTICS
Problem Set #11

[1] Find minimal sufficient statistic based on a random sample $X_1,...,X_n$ in each of the following cases

(a) $f_\alpha(x)=\begin{cases}\frac{1}{\alpha}\exp\left(-\frac{x}{\alpha}\right) & \text{if } x>0\\ 0 & \text{otherwise.}\end{cases}\quad \alpha>0.$

(b) $f_\beta(x)=\begin{cases}\exp\left(-(x-\beta)\right) & \text{if } x>\beta\\ 0 & \text{otherwise.}\end{cases}\quad \beta\in\mathbb{R}.$

(c) $f_{\alpha,\beta}(x)=\begin{cases}\frac{1}{\alpha}\exp\left(-\frac{(x-\beta)}{\alpha}\right) & \text{if } x>\beta\\ 0 & \text{otherwise.}\end{cases}\quad \alpha>0,\beta\in\mathbb{R}.$

(d) $f_{\mu,\sigma}(x)=\begin{cases}\frac{1}{x\,\sigma\,\sqrt{2\pi}}\exp\left(-\frac{(\log x_i-\mu)^2}{2\sigma^2}\right) & \text{if } x>0\\ 0 & \text{otherwise.}\end{cases}\quad \mu\in\mathbb{R};\sigma>0.$

(e) $f_\theta(x)=\begin{cases}\frac{1}{\theta} & -\theta/2\le x\le\theta/2\\ 0 & \text{otherwise}\end{cases}\quad \theta>0.$

(f) $f(x)=\begin{cases}\frac{\overline{|\alpha+\beta}}{\overline{|\alpha}\,\overline{|\beta}}x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1\\ 0 & \text{otherwise.}\end{cases}\quad \alpha>0,\beta>0.$

[2] Let $X_1,...,X_n$ be a random sample from $P(\theta),\theta\in(0,\infty)$. Show that $T=\sum_{i=1}^{n}X_i$ is complete sufficient statistic. Find the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of the following parametric functions: (a) $g(\theta)=\theta$, (b) $g(\theta)=e^{-\theta}$ and (c) $g(\theta)=e^{-\theta}(1+\theta)$.

[3] Suppose $X_1,...,X_n$ be a random sample from $B(1,\theta),\theta\in(0,1)$. Show that $T=\sum_{i=1}^{n}X_i$ is complete sufficient statistic and hence find the UMVUE for each of the following parametric functions: (a) $g(\theta)=\theta$, (b) $g(\theta)=\theta^4$ and (c) $g(\theta)=\theta(1-\theta)^2$.

[4] Let $X_1,...,X_n$ be a random sample from $Exp(\theta,1)$, i.e.

$$f(x \mid \theta)=\begin{cases} e^{-(x-\theta)} & \text{if } x>\theta \\ 0 & \text{otherwise} \end{cases}$$

Show that $T = X_{(1)} = \min\{X_1, \ldots, X_n\}$ is a complete sufficient statistic and hence find the UMVUE of $g(\theta)=\theta.$

[5] $X_1, \ldots, X_n$ is a random sample from $U(0,\theta), \theta>0.$ Show that $T = X_{(n)} = \max\{X_1, \ldots, X_n\}$ is a complete sufficient statistic and find the UMVUE of $g(\theta) = \theta^2$.

[6] $X_1, \ldots, X_n$ is a random sample from $Gamma(2,\theta), \theta>0,$ i.e.

$$f(x \mid \theta)=\begin{cases} \frac{1}{\Gamma 2\, \theta^2} e^{-x/\theta} x & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $T=\sum_{i=1}^{n} X_i$ is complete sufficient statistic and find the UMVUE of $\theta.$

[7] Let $X_1, \ldots, X_n$ be a random sample from $U(\theta-1/2, \theta+1/2).$ Show that the minimal sufficient statistic is not complete.

[8] Let $X_1, \ldots, X_n$ be a random sample from $N(0, \theta).$ Find the UMVUE of $\theta^2.$

[9] Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \theta).$ Find the UMVUE of (a) θ when μ is known, (b) θ when μ is not known and (c) δ such that $P(X \le \delta) = p; p$ is a known fixed constant, both μ and θ are unknown parameters.

[10] Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta), \theta>0.$ Of the following three estimators given below, which one would you prefer and why?

$$T_1(\underset{\sim}{X})=\frac{n+1}{n} X_{(n)}, T_2(\underset{\sim}{X})=2\bar{X} \text{ and } T_3(\underset{\sim}{X})=X_{(1)}+X_{(n)}.$$

[11] $X_1, \ldots, X_n$ be a random sample from $N(\mu,\sigma^2), \mu \in \Re, \sigma \in \Re^+.$ Assuming completeness of the associated minimal sufficient statistic find the UMVUE of μ^2 and $\mu+\sigma.$

[12] $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ distribution. Find the Cramer-Rao Lower Bounds (CRLB) on the variances of unbiased estimators of μ and $\sigma^2.$ Can you find unbiased estimators μ and σ^2 whose variances attain the respective CRLB?

[13] $X_1, \ldots, X_n$ is a random sample from $Gamma(\alpha,\beta)$

$$f(x \mid \alpha, \beta) = \begin{cases} \dfrac{1}{\Gamma\alpha\, \beta^{\alpha}} e^{-x/\beta} x^{\alpha-1} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

α is assumed to be known. Find the Fisher Information $I(\beta)$ and the CRLB on the variances of unbiased estimators of β.

[14] $X_1, \ldots, X_n$ be a random sample from $P(\theta), \theta \in (0, \infty)$. Find the CRLB on the variances of unbiased estimators of the following estimands: (a) $g(\theta) = \theta$, (b) $g(\theta) = \theta^2$ and (c) $g(\theta) = e^{-\theta}$.

[15] Suppose $X_1, \ldots, X_n$ be a random sample from $B(1, \theta), \theta \in (0, 1)$. Find the CRLB on the variances of unbiased estimators of the following estimands: (a) $g(\theta) = \theta^4$ (b) $g(\theta) = \theta(1-\theta)$.