MSO201: PROBABILITY & STATISTICS
Problem Set #5

Useful data

$\Phi(1/3) = 0.6293, \Phi(5/6) = 0.7967, \Phi(1) = 0.8413, \Phi(4/3) = 0.918, \Phi(1.96) = 0.975$

[1] Find the moment generating function (m.g.f.) of the following distributions: (a) $Bin(n, p)$, (b) $NB(r, p)$, (c) $P(\lambda)$, (d) $G(\alpha, \beta)$ and (e) $N(\mu, \sigma^2)$. Find $E(X)$ and $V(X)$ from the m.g.f.s.

[2] An interviewer is given a list of 8 people whom he can attempt to interview. He is required to interview exactly 5 people. If each person (independently) agrees to be interviewed with probability $2/3$, what is the probability that his list will enable him to complete his task?

[3] A pipe-smoking mathematician carries at all times 2 match boxes, 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly k matches in the other box, $k = 0, 1, \ldots, N$?

[4] A machine contains two belts of different lengths. These have times to failure which are exponentially distributed, with means α and 2α. The machine will stop if either belt fails. The failures of the belts are assumed to be independent. What is the probability that the system performs after time α from the start?

[5] Let X be a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$. Compute (a) $P(X > 5)$, (b) $P(4 < X < 16)$, (c) $P(X < 8)$.

[6] Let $X \sim N(\mu, \sigma^2)$. If $P(X \le 0) = 0.5$ and $P(-1.96 \le X \le 1.96) = 0.95$, find μ and σ^2.

[7] It is assumed that the lifetime of computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu = 1.4 \times 10^6$ and $\sigma = 3 \times 10^5$ hours. What is the approximate probability that a batch of 10 chips will contain at least 2 chips whose lifetime are less than 1.8×10^6 hours?

[8] Let X be a Normal random variable with mean 0 and variance 1, i.e. $N(0,1)$. Prove that

$$P(|X| \geq t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}; \forall\, t > 0.$$

[9] Show that if X is a discrete random variable with values $0,1,2,....$ then $E(X) = \sum_{k=0}^{\infty} (1 - F(k))$, where $F(x)$ is the distribution function of the random variable X.

[10] The cumulative distribution function of a random variable *X* defined over $0 \leq x < \infty$ is $F(x) = 1 - e^{-\beta x^2}, x \geq 0; = 0, x < 0.$ $\beta > 0.$ Find the mean, median and variance of *X*.

[11] Show that for any $x > 0$, $1 - \Phi(x) \leq \frac{\phi(x)}{x}$, where $\Phi(x)$ is the c.d.f. and $\phi(x)$ is the p.d.f. of standard normal distribution.

[12] A point m_0 is said to be mode of a random variable *X*, if the p.m.f. or the p.d.f. of *X* has a maximum at m_0. For the distribution given in problem [10], if m_0 denotes the mode; μ, the mean and σ^2, the variance of the corresponding random variable, then show that $m_0 = \mu\sqrt{2/\pi}$ and $2m_0^2 - \mu^2 = \sigma^2$.

[13] A random variable X is said to be symmetric about a point θ if $X - \theta$ has an identical distribution with $\theta - X$. Suppose X is a continuous random variable, which is symmetric about θ, with distribution function $F_X(.)$ and p.d.f $f_X(.)$.

(a) Prove that distribution of X is symmetric about θ iff $f_X(\theta - x) = f_X(\theta + x)$, for all x.

(b) Prove that distribution of X is symmetric about θ iff $F_X(\theta + x) + F_X(\theta - x) = 1$, for all x.

(c) Prove that $E(X) = median(X) = \theta$.

[14] Let X be a random variable having a 2-parameter exponential distribution with p.d.f.

$$f(x) = \begin{cases} \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, & \text{if } x \geq \mu \\ 0, & \text{otherwise.} \end{cases} \quad \mu \in \mathcal{R}, \sigma > 0.$$

(a) Find $E(X - \mu)^r, r = 1,2, \ldots$

(b) Find $E(X)^r, r = 1,2$

(c) Find the median of X.

[15] Let $X \sim Bin(n,p)$ and $X \sim NB(r,p)$; $0 < p < 1$ and $r \in \{1,2, \ldots, n\}$. Prove that $P(X \geq r) = P(Y \leq n - r)$.

[16] Let $X \sim P(\lambda)$. Find $E((2 + X)^{-1})$.

[17] Let $\theta > 0$ and $t > 0$ and $n \in \{1,2, \ldots\}$. Prove that if $X \sim G(n,\theta)$ and $Y \sim P(t/\theta)$, then $P(X \geq t) = P(Y \leq n - 1)$.