MSO201: PROBABILITY & STATISTICS
Problem Set #9

[1] Let $\{X_n\}$be a sequence of random variables with $E(X_n) \to c$ and $V(X_n) \to 0$ as $n \to \infty$. Show that $X_n \xrightarrow{p} c$.

[2] Let $\{X_n\}$be a sequence of random variables with $E(X_n) = \mu_n$ and finite variance such that $\frac{1}{n^2} V\left(\sum_{i=1}^{n} X_i\right) \to 0$ as $n \to \infty$. Show that WLLN holds and $\bar{X}_n - \bar{\mu}_n \xrightarrow{\mathcal{L}} 0$.

[3] Let $X_1, X_2, \ldots X_n$ be a random sample from $U(0,1)$. Let $Y_n = \min(X_1, \ldots, X_n)$ and $Z_n = \max(X_1, \ldots, X_n)$. Show that (a) $\sqrt{Y_n} \xrightarrow{p} 0$, (b) $Z_n^2 \xrightarrow{p} 1$ and (c) $Y_n^2 Z_n^2 \xrightarrow{p} 0$.

[4] Let $X_1, X_2, \ldots X_n$ be i.i.d. $N(0,1)$. Show that $\bar{X}_n / S_n \xrightarrow{p} 0$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$.

[5] Suppose $Y_n \sim Bin(n, p)$, show that $(1 - Y_n/n) \xrightarrow{p} 1 - p$.

[6] Let $\{X_n\}$be a sequence of independent random variables with

$$P(X_n = x) = \begin{cases} 1/2, & x = -n^{1/4},\ n^{1/4} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\bar{X}_n \xrightarrow{p} 0$.

[7] Let $\{X_n\}$be a sequence of i.i.d. random variables with mean μ and finite variance. Show that

(a) $\frac{2}{n(n+1)}\sum_{i=1}^{n} i\, X_i \xrightarrow{p} \mu$

(b) $\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2\, X_i \xrightarrow{p} \mu$

[8] Let $\{X_n\}$be a sequence of i.i.d. random variables with $U(0,1)$ distribution and $Z_n = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$. Show that $Z_n \xrightarrow{p} e^{-1}$.

[9] Let $\{X_n\}$be a sequence of uncorrelated random variables with $E(X_n) = \mu_n$ and $V(X_n) = \sigma_n^2$. Show that if $\sum_{i=1}^{n} \sigma_i^2 \to \infty$ as $n \to \infty$, then WLLN holds for $\{X_n\}$.

[10] Let $\{X_n\}$be a sequence of i.i.d. random variables with $U(0,1)$ distribution. Find c such that $\bar{X}_n \xrightarrow{p} c$.

[11] Let $\{X_n\}$ be a sequence of $N\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$. Show that $X_n \xrightarrow{L} Z$, where $Z \sim N(0,1)$.

[12] Let $X_1, X_2, \ldots X_n$ be i.i.d. $B(1, p)$, $S_n = \sum_{i=1}^{n} X_i$. Find n which would guarantee $P\left(\left|\frac{S_n}{n} - p\right| \geq 0.01\right) \leq 0.01$, no matter whatever the unknown p may be.

[13] The p.d.f. of X_n is given by

$$f_n(x) = \begin{cases} \frac{1}{\Gamma n} e^{-x} x^{n-1} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the limiting distribution of $Y_n = X_n / n$.

Problems on CLT (to be discussed during lecture class scheduled on March 21)

[14] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $E(X_i) = \mu$, $Var(X_i) = \sigma^2$ and $E(X_i - \mu)^4 = \sigma^4 + 1$. Find $\lim_{n \to \infty} P\left[\sigma^2 - \frac{1}{\sqrt{n}} \leq \frac{(X_1 - \mu)^2 + \ldots + (X_n - \mu)^2}{n} \leq \sigma^2 + \frac{1}{\sqrt{n}}\right]$.

[15] Let $X_1, \ldots, X_n$ be i.i.d. from a distribution with mean μ and finite variance σ^2. Prove that $\frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{S_n} \xrightarrow{\text{L}} Z$, where $Z \sim N(0,1)$.

[16] The p.d.f. of a random variable X is

$$f(x) = \begin{cases} 1/x^2 & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Consider a random sample of size 72 from the distribution having the above p.d.f. Compute, approximately, the probability that more than 50 of these observations are less than 3.

[17] Let $X_1, \ldots, X_{100}$ be i.i.d. from $\text{Poisson}(3)$ distribution and let $Y = \sum_{i=1}^{100} X_i$. Using CLT, find an approximate value of $P(100 \leq Y \leq 200)$.

[18] Let $\bar{X}$ denote the mean of a random sample of size 64 from the Gamma distribution with density

$$f_n(x) = \begin{cases} \frac{1}{\Gamma p \alpha^p} e^{-x/\alpha} x^{p-1} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

With $\alpha = 2, p = 4$. Compute the approximate value of $P(7 < \bar{X} < 9)$.

[19] $X_1, \ldots, X_n$ is a random sample from $U(0, 2)$. Let $Y_n = \bar{X}_n$, show that $\sqrt{n}(Y_n - 1) \xrightarrow{\mathcal{L}} N(0, 1/3)$.