

## MSO201: PROBABILITY & STATISTICS

### Problem Set #7

- [1] The joint p.d.f. of  $(X, Y)$  is given by  $f(x, y) = \begin{cases} 4xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$

Find the marginal p.d.f.s and verify whether the random variables are independent. Also find  $P(0 < X < 1/2, 1/4 < Y < 1), P(X + Y < 1)$

- [2] If the joint p.d.f. of  $(X, Y)$   $f(x, y) = \begin{cases} e^{-(x+y)} & 0 < x, y < \infty \\ 0 & \text{otherwise,} \end{cases}$

show that  $X$  and  $Y$  are independent.

- [3] If the joint p.d.f. of  $(X, Y)$  is  $f(x, y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & \text{otherwise,} \end{cases}$

show that  $X$  and  $Y$  are not independent.

- [4] Show that the random variables  $X$  and  $Y$  with joint p.d.f.

$$f(x, y) = \begin{cases} 12xy(1-y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

are independent.

- [5] Suppose the joint p.d.f. of  $(X, Y)$  is  $f(x, y) = \begin{cases} cx^2y & 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$

Find (a) the value of the constant  $c$ , (b) the marginal p.d.f.s of  $X$  and  $Y$  and (c)  $P(X + Y \leq 1)$ .

- [6] The joint p.d.f. of  $(X, Y)$  is given by  $f(x, y) = \begin{cases} 6(1-x-y) & x > 0, y > 0, x+y < 1 \\ 0 & \text{otherwise.} \end{cases}$

Find the marginal p.d.f. s of  $X$  and  $Y$  and  $P(2X + 3Y < 1)$ .

- [7] The joint p.d.f. of  $(X, Y)$  is  $f(x, y) = \begin{cases} x+y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$

Find the conditional distribution of  $Y$  given  $X = x, 0 < x < 1$ ; the conditional mean and conditional variance of the conditional distribution.

- [8] Suppose the conditional p.d.f. of  $X$  given  $Y = y$  is  $f(x|y) = \begin{cases} cx/y^2 & 0 < x < y \\ 0 & \text{otherwise.} \end{cases}$

Further, the marginal distribution of  $Y$  is  $g(y) = \begin{cases} dy^4 & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$

(a) Find the constants  $c$  and  $d$ .

(b) The joint p.d.f. of  $(X, Y)$ .

(c)  $P(0.25 < X < 0.5)$  and  $P(0.25 < X < 0.5 | Y = 0.625)$

[9] Let  $f(x)$  and  $g(y)$  be two arbitrary p.d.f.s with corresponding distribution functions  $F(x)$  and  $G(y)$  respectively. Suppose the joint p.d.f. of  $X$  and  $Y$  is given by

$$h(x, y) = f(x)g(y)[1 + \alpha\{2F(x) - 1\}\{2G(y) - 1\}], \quad |\alpha| \leq 1$$

Show that the marginal p.d.f.s of  $X$  and  $Y$  are  $f(x)$  and  $g(y)$ , respectively. Does there exist a value of  $\alpha$  for which the random variables  $X$  and  $Y$  are independent?

[10] Suppose the marginal density of the random variable  $X$  is  $f_X(x) = \begin{cases} 4x(1-x^2), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$

and the conditional density of the random variable  $Y$  given  $X = x$  is

$$f_{Y|X=x}(y|x) = \begin{cases} 2y/(1-x^2), & x < y < 1, 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional p.d.f. of  $X$  given  $Y = y$ ,  $E(X|Y=1/2)$  and  $Var(X|Y=1/2)$ .

[11] The joint p.d.f. of  $(X, Y)$   $f(x, y) = \begin{cases} e^{-(x+y)} & 0 < x, y < \infty \\ 0 & \text{otherwise.} \end{cases}$

Find the joint m.g.f. of  $(X, Y)$  and the m.g.f. of  $Z = X + Y$  and hence  $V(Z)$ .

[12] Derive the joint m.g.f. of  $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$  and using the joint m.g.f find  $\rho(X_1, X_2)$ .

[13] Let the joint p.d.f. of  $(X, Y)$  be  $f(x, y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise,} \end{cases}$

Find the conditional mean and conditional variance of  $X$  given  $Y = y$  and that of  $Y$  given  $X = x$ . Compute further  $\rho(X, Y)$ .

[14] Let  $X, Y$  and  $Z$  be three random variables and  $a$  and  $b$  be two scalar constants. Prove that (a)

$Cov(X, b) = Cov(Y, b) = Cov(Z, b) = 0$ ; (b)  $Cov(X, aY + b) = aCov(X, Y)$ ; (c)

$Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)$ ; (d)  $\rho(X, aY + b) = \rho(X, Y)$  for  $a > 0$ .

[15] Let  $X_1, X_2$  and  $X_3$  be three independent random variables each with a variance  $\sigma^2$ . Define the new random variables

$$W_1 = X_1, \quad W_2 = \frac{\sqrt{3}-1}{2}X_1 + \frac{3-\sqrt{3}}{2}X_2 \quad \text{and} \quad W_3 = (\sqrt{2}-1)X_2 + (2-\sqrt{2})X_3.$$

Find  $\rho(W_1, W_2)$ ,  $\rho(W_1, W_3)$  and  $\rho(W_2, W_3)$ .

[16] Let  $(X, Y) \sim N_2(3, 1, 16, 25, 0.6)$ . Find (a)  $P(3 < Y < 8)$ ; (b)  $P(3 < Y < 8 | X = 7)$ ;

(c)  $P(-3 < X < 3)$  and (d)  $P(-3 < X < 3 | Y = 4)$ .

[17] Let  $(X, Y) \sim N_2(5, 10, 1, 25, \rho)$  with  $\rho > 0$ . If it is given that  $P(4 < Y < 16 | X = 5) = 0.954$

and  $\Phi(2) = 0.977$ , find the value of  $\rho$ .

[18] Let  $X_1, X_2, \dots, X_{20}$  be independent random variables with identical distributions, each with a mean 2 and variance 3. Define  $Y = \sum_{i=1}^{15} X_i$  and  $Z = \sum_{i=11}^{20} X_i$ . Find  $E(Y)$ ,  $E(Z)$ ,  $V(Y)$ ,  $V(Z)$  and  $\rho(Y, Z)$ .

[19] Let  $X$  and  $Y$  be a jointly distributed random variables with  $E(X) = 15$ ,  $E(Y) = 20$ ,  $V(X) = 25$ ,  $V(Y) = 100$  and  $\rho(X, Y) = -0.6$ . Find  $\rho(X - Y, 2X - 3Y)$ .

[20] Suppose that the lifetime of light bulbs of a certain kind follows exponential distribution with p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{50} e^{-x/50} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that among 8 such bulbs, 2 will last less than 40 hours, 3 will last anywhere between 40 and 60 hours, 2 will last anywhere between 60 and 80 hours and 1 will last for more than 80 hours. Find the expected number of bulbs in a lot of 8 bulbs with lifetime between 60 and 80 hours and also the expected number of bulbs in a lot of 8 with lifetime between 60 and 80 hours, given that the number of bulbs with lifetime anywhere between 40 and 60 hours is 2.

[21] Let the random variables  $X$  and  $Y$  have the following joint p.m.f.s

(a)  $P(X = x, Y = y) = 1/3$ , if  $(x, y) \in \{(0, 0), (1, 1), (2, 2)\}$  and 0 otherwise.

(b)  $P(X = x, Y = y) = 1/3$ , if  $(x, y) \in \{(0, 2), (1, 1), (2, 0)\}$  and 0 otherwise.

(c)  $P(X = x, Y = y) = 1/3$ , if  $(x, y) \in \{(0, 0), (1, 1), (2, 0)\}$  and 0 otherwise.

In each of the above cases find the coefficient of correlation between  $X$  and  $Y$ .

[22] The joint p.m.f. of  $(X, Y)$  is

$$P(X = x, Y = y) = xy/10, \text{ if } (x, y) \in \{(1, 1), (2, 1), (2, 2), (3, 1)\} \text{ and 0 otherwise.}$$

Find the joint m.g.f. of  $X$  and  $Y$  and the coefficient of correlation between  $X$  and  $Y$ . Using the joint m.g.f., find the p.m.f.  $Z = X + Y$ .

[23] Let  $M_{X,Y}(u, v)$  denote the joint m.g.f.  $(X, Y)$  and  $\psi(u, v) = \log(M_{X,Y}(u, v))$ .

Show that

$$\left. \frac{\partial \psi(u, v)}{\partial u} \right|_{u=v=0}, \left. \frac{\partial \psi(u, v)}{\partial v} \right|_{u=v=0}, \left. \frac{\partial^2 \psi(u, v)}{\partial u^2} \right|_{u=v=0}, \left. \frac{\partial^2 \psi(u, v)}{\partial v^2} \right|_{u=v=0} \text{ and } \left. \frac{\partial^2 \psi(u, v)}{\partial u \partial v} \right|_{u=v=0} \text{ yields the}$$

means, the variances and the covariance of the two random variables.

[24] The joint probability density function of  $X$  and  $Y$  is given by

$$f_{X,Y}(x, y) = \frac{1}{2} (f_{\rho}(x, y) + f_{-\rho}(x, y)); \quad -\infty < x, y < \infty$$

where,  $f_{\rho}(x, y)$  is the probability density function of  $N_2(0, 0, 1, 1, \rho)$  and  $f_{-\rho}(x, y)$  is the probability density function of  $N_2(0, 0, 1, 1, -\rho)$ . Find the marginal p.d.f.s of  $X$  and  $Y$ , the correlation coefficient between  $X$  and  $Y$ . Are the 2 variables independent?

[25] Let the joint p.d.f. of  $X$  and  $Y$  be given by

$$f_{X,Y}(x,y) = \begin{cases} k, & \text{if } -x < y < x; 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of the constant  $k$  and obtain the conditional expectations  $E(X|Y=y)$  and  $E(Y|X=x)$ . Verify whether the 2 random variables are independent and/or uncorrelated.

[26] The joint moment generating function of  $X$  and  $Y$  is given by

$$M_{X,Y}(s,t) = \left\{ a(e^{s+t} + 1) + b(e^s + e^t) \right\}, \quad a, b > 0; \quad a + b = 1/2.$$

Find the correlation coefficient between  $X$  and  $Y$ .

[27] Let  $X$  and  $Y$  be jointly distributed random variables with

$$E(X) = E(Y) = 0, \quad E(X^2) = E(Y^2) = 2 \quad \text{and} \quad \rho(X, Y) = 1/3$$

Find  $\rho(X/3 + 2Y/3, 2X/3 + Y/3)$ .

[28] Let  $\underline{X} = (X_1, X_2, X_3)' \sim N_3(\underline{0}, \Sigma); \Sigma = \begin{pmatrix} 1 & -0.5 & 0 \\ & 1 & -0.5 \\ & & 1 \end{pmatrix}.$

(a) Verify whether  $X_1 + X_2 + X_3$  and  $X_1 - X_2 - X_3$  are independent.

(b) Find the distribution of  $(X_1 - X_2 - X_3)^2$ .

[29] Let  $\underline{X} = (X_1, X_2, X_3)'$  be distributed as  $N_3(\underline{\mu}, \Sigma)$ ,

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}; \quad -1/2 < \rho < 1.$$

Find the joint distribution of  $(X_1 + X_2, X_1 - X_2)'$ .

[30] Let the joint p.d.f. of  $\underline{X} = (X_1, X_2, X_3, X_4)^T$  be

$$f_{\underline{X}}(\underline{x}) = \frac{\exp(-\underline{x}^T \Sigma_1^{-1} \underline{x} / 2)}{8\pi^2 |\Sigma_1|^{1/2}} + \frac{\exp(-\underline{x}^T \Sigma_2^{-1} \underline{x} / 2)}{8\pi^2 |\Sigma_2|^{1/2}}; \quad \underline{x} \in \Re^4,$$

where,  $\Sigma_1 = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}$ ,  $A_1 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ ;  $\Sigma_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}$ ,  $A_3 = \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$ ;  $|\rho| < 1$ .

(a) Find the joint p.d.f. of  $\underline{X}^{(1)} = (X_1, X_2)^T$ .

(b) Find the joint p.d.f. of  $\underline{X}^{(2)} = (X_3, X_4)^T$ .

(c) Prove or disprove “ $X_3$  and  $X_4$  are uncorrelated but not independent”.