Problem Set # 11

(1)

(a) jt p.d.f

$$f_\alpha(\underset{\sim}{x}) = \frac{1}{\alpha^n} e^{-\frac{1}{\alpha}\sum_1^n x_i}$$

$\forall\, \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_\alpha(\underset{\sim}{x})}{f_\alpha(\underset{\sim}{y})} = e^{-\frac{1}{\alpha}\left(\sum_1^n x_i - \sum y_i\right)}$$

indep of α iff $\sum x_i = \sum y_i$

$\Rightarrow T(\underline{X}) = \sum_{i=1}^n X_i$ is minimal suff stat

(b) jt p.d.f

$$f_\beta(\underset{\sim}{x}) = \begin{cases} e^{-\sum_1^n x_i + n\beta}, & x_1, \ldots, x_n > \beta \\ 0, & o/w \end{cases}$$

$$= \begin{cases} e^{-\sum_1^n x_i + n\beta}, & x_{(1)} > \beta \\ 0 & o/w \end{cases}$$

i.e. $f_\beta(\underline{x}) = e^{-\sum_1^n x_i} e^{n\beta} I(\beta, x_{(1)})$

$\forall\, \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$ $$\frac{f_\beta(\underline{x})}{f_\beta(\underline{y})} = e^{-\sum_1^n x_i + \sum_1^n y_i} \frac{I(\beta, x_{(1)})}{I(\beta, y_{(1)})}$$

indep of β iff $x_{(1)} = y_{(1)}$

$\Rightarrow T(\underline{X}) = X_{(1)}$ is minimal sufficient statistic

(c) jt p.d.f is

$$f_{\alpha,\beta}(\underset{\sim}{x}) = \frac{1}{\alpha^n} \exp\left(-\frac{1}{\alpha}\sum_1^n x_i + \frac{n\beta}{\alpha}\right) I(\beta, x_{(1)})$$

$\forall \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_{\alpha,\beta}(\underset{\sim}{x})}{f_{\alpha,\beta}(\underset{\sim}{y})} = \frac{\exp\left(-\frac{1}{\alpha}\sum_1^n x_i + \frac{n\beta}{\alpha}\right) I(\beta, x_{(1)})}{\exp\left(-\frac{1}{\alpha}\sum y_i + \frac{n\beta}{\alpha}\right) I(\beta, y_{(1)})}$$

$$= e^{-\frac{1}{\alpha}\left(\sum_1^n x_i - \sum y_i\right)} \frac{I(\beta, x_{(1)})}{I(\beta, y_{(1)})}$$

indep of (α, β) iff $\sum_1^n x_i = \sum_1^n y_i$ and $x_{(1)} = y_{(1)}$

$\Rightarrow T(\underset{\sim}{X}) = \left(\sum X_i, X_{(1)}\right)$ is jointly minimal suff for (α, β)

(d) jt p.d.f

$$f_{\mu,\sigma}(\underset{\sim}{x}) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \left(\prod_1^n \frac{1}{x_i}\right) \exp\left(-\frac{1}{2\sigma^2}\left(\sum_1^n (\log x_i)^2 + n\mu^2 - 2\mu\sum_1^n \log x_i\right)\right)$$

$\forall \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_{\mu,\sigma}(\underset{\sim}{x})}{f_{\mu,\sigma}(\underset{\sim}{y})} = \left(\prod_{i=1}^n \left(\frac{y_i}{x_i}\right)\right) \exp\left(-\frac{1}{2\sigma^2}\left(\sum(\log x_i)^2 - \sum(\log y_i)^2\right) + \frac{\mu}{\sigma^2}\left(\sum \log x_i - \sum \log y_i\right)\right)$$

$$\Rightarrow \frac{f_{\mu,\sigma}(\underset{\sim}{x})}{f_{\mu,\sigma}(\underset{\sim}{y})} \text{ is indep of } (\mu,\sigma) \text{ iff}$$

$$\sum_1^n \log x_i = \sum_1^n \log y_i \text{ and } \sum_1^n (\log x_i)^2 = \sum_1^n (\log y_i)^2$$

jointly

$$\Rightarrow T(\underset{\sim}{X}) = \left(\sum_1^n \log X_i \, , \, \sum_1^n (\log X_i)^2 \right) \text{ is a minimal suff for } (\mu,\sigma)$$

(e)

$$f_\theta(x) = \begin{cases} \frac{1}{\theta}, & |x| < \theta/2 \\ 0, & o/w \end{cases}$$

jt p.d.f.

$$f_\theta(\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n}, & |x_i| < \theta/2 \text{ for } i = 1, \ldots, n \\ 0, & o/w \end{cases}$$

$$= \begin{cases} \frac{1}{\theta^n}, & \max_i |x_i| < \theta/2 \\ 0, & o/w \end{cases}$$

i.e. $f_\theta(\underset{\sim}{x}) = \frac{1}{\theta^n} I\left(\max_i |x_i|, \theta/2 \right)$

$\forall \underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_\theta(\underset{\sim}{x})}{f_\theta(\underset{\sim}{y})} = \frac{I\left(\max_i |x_i|, \theta/2 \right)}{I\left(\max_i |y_i|, \theta/2 \right)} \text{ indep of } \theta \text{ iff}$$

$$\max_i |x_i| = \max_i |y_i|$$

$$\Rightarrow T(\underset{\sim}{X}) = \max_i |X_i| \text{ is minimal suff statistic}$$

(f) jt p.d.f.

$$f_{\alpha,\beta}(\underset{\sim}{x}) = \left(\frac{\Gamma\alpha+\beta}{\Gamma\alpha\ \Gamma\beta}\right)^n \left(\prod_1^n x_i\right)^{\alpha-1} \left(\prod_{i=1}^n (1-x_i)\right)^{\beta-1}$$

$\forall\ \underset{\sim}{x}, \underset{\sim}{y} \in \mathfrak{X}$

$$\frac{f_{\alpha,\beta}(\underset{\sim}{x})}{f_{\alpha,\beta}(\underset{\sim}{y})} = \left(\frac{\prod_1^n x_i}{\prod_1^n y_i}\right)^{\alpha-1} \left(\frac{\prod_1^n (1-x_i)}{\prod_1^n (1-y_i)}\right)^{\beta-1}$$

indep of (α,β) iff $\prod_1^n x_i = \prod_1^n y_i$

and $\prod_1^n (1-x_i) = \prod_1^n (1-y_i)$

$\Rightarrow T(\underset{\sim}{X}) = \left(\prod_1^n X_i,\ \prod_1^n (1-X_i)\right)$ is jointly minimal

suff for (α,β)

(2) <u>completeness of $T = \sum_1^n X_i$</u>

p.d.f. of $P(\theta)$ $\qquad f_\theta(x) = \dfrac{e^{-\theta}\,\theta^x}{x!}$

i.e. $f_\theta(x) = e^{-\theta}\, e^{\log\theta^x} \cdot \dfrac{1}{x!}$

$$= \underbrace{\left(\frac{1}{x!}\right)}_{h(x)} \exp\Big(\underbrace{x}_{T(x)} \underbrace{\log\theta}_{\eta} - \underbrace{\theta}_{B(\theta)}\Big)$$

$\Rightarrow$ $P(\theta)$ is 1-parameter exponential family

Natural parameter space : $\{\eta : \eta \in \mathbb{R}\}$

So the natural parameter space associated with the 1-par exponential family representation contains open-intervals

$\Rightarrow$ the 1-parameter exponential family is of full rank

$\Rightarrow$ $\sum_{i=1}^{n} X_i$ is complete sufficient

(a) $g(\theta) = \theta$

$E(\bar{X}) = \theta$

$\bar{X}$ is a function of complete sufficient statistic

$\Rightarrow$ $\bar{X}$ is UMVUE of θ

(b) $g(\theta) = e^{-\theta}$

(difficult to guess u.e. of $g(\theta)$ which is a f^n of complete suff stat, hence we use Rao-Blackwell approach)

$$\delta_0(\underline{X}) = \begin{cases} 1, & X_1 = 0 \\ 0, & o/w \end{cases} \qquad E\,\delta_0(\underline{X}) = e^{-\theta}$$

i.e. δ_0 is u.e. of $e^{-\theta}$

Rao-Blackwellise δ_0:

$\eta(T) = E(\delta_0 \mid T)$

$$E(\delta_0 \mid T = t) = P(X_1 = 0 \mid T = t)$$

$$= \frac{P(X_1 = 0, \sum_{2}^{n} X_i = t)}{P(T = t)}$$

$$= \frac{(\cancel{e^{-\theta}})\left(\cancel{e^{-(n-1)\theta}}((n-1)\cancel{\theta})^t / \cancel{t!}\right)}{\cancel{e^{-n\theta}}(n\cancel{\theta})^t / \cancel{t!}} = \left(\frac{n-1}{n}\right)^t$$

$\eta(T) = \left(\frac{n-1}{n}\right)^T$ is u.e. based on complete suff stat T

$\Rightarrow \eta(T)$ is the UMVUE

(c) $g(\theta) = e^{-\theta}(1+\theta) = e^{-\theta} + \theta e^{-\theta}$

$$\delta_0(\underline{x}) = \begin{cases} 1, & x_1 \le 1 \\ 0, & \text{o/w} \end{cases} \qquad \begin{aligned} E\,\delta_0(\underline{x}) &= P(X_1 \le 1) \\ &= P(X_1 = 0 \cup X_1 = 1) \\ &= e^{-\theta} + \theta e^{-\theta} = g(\theta) \end{aligned}$$

$\Rightarrow \delta_0$ is u.e. of $e^{-\theta}(1+\theta)$

Rao-Blackwellisation of δ_0:

$$\eta(T) = E(\delta_0 | T) = P(X_1 \le 1 | T)$$

Now $P(X_1 \le 1 | T = t) = \dfrac{P(X_1 \le 1,\ \sum_1^n X_i = t)}{P(T=t)}$

$$= \frac{P(\{X_1 = 0 \cup X_1 = 1\} \cap \sum_1^n X_i = t)}{P(T=t)}$$

$$= \frac{P(X_1 = 0,\ \sum_1^n X_i = t) + P(X_1 = 1,\ \sum_1^n X_i = t)}{P(T=t)}$$

$$= \frac{P(X_1 = 0,\ \sum_2^n X_i = t) + P(X_1 = 1,\ \sum_2^n X_i = t-1)}{P(T=t)}$$

$$= \frac{P(X_1 = 0)\, P(\sum_2^n X_i = t) + P(X_1 = 1)\, P(\sum_2^n X_i = t-1)}{P(T=t)}$$

$$= \frac{\cancel{e^{-\theta}}\left(e^{-(n-1)\theta}((n-1)\theta)^t/t!\right) + \cancel{\theta}\,\cancel{e^{-\theta}}\left(e^{-(n-1)\theta}((n-1)\theta)^{t-1}/(t-1)!\right)}{\cancel{e^{-n\theta}}(n\theta)^t/t!}$$

$$= \left(\frac{n-1}{n}\right)^t \left(1 + \frac{t}{n-1}\right)$$

$\Rightarrow \eta(T) = \left(\frac{n-1}{n}\right)^T \left(1 + \frac{T}{n-1}\right)$ is u.e. of $g(\theta)$ based on complete suff stat

$\Rightarrow \left(\frac{n-1}{n}\right)^T \left(1 + \frac{T}{n-1}\right)$ is UMVUE of $g(\theta) = e^{-\theta}(1+\theta)$

(3) $X_1, \ldots, X_n$ r.s. from $B(1, \theta)$; $\theta \in (0,1) = \Theta$

$T = \sum_{i=1}^{n} X_i$ is minimal suff ; $T \sim B(n, \theta)$

p.m.f. of $X_i \sim B(1, \theta)$

$$f_\theta(x) = \theta^x (1-\theta)^{1-x}$$

$$= \exp\left(x \log \frac{\theta}{1-\theta} + \log(1-\theta)\right)$$

with $h(x) = 1$, $\eta(\theta) = \log \frac{\theta}{1-\theta}$, $T(x) = x$ & $\beta(\theta) = -\log(1-\theta)$

this is 1-param exponential family distⁿ

Natural parameter space : $\{\eta : \eta \in \mathbb{R}\}$

Natural parameter space contains open intervals

$\Rightarrow$ The 1-parameter expo family distⁿ is of full rank (complete)

$\Rightarrow T(\underline{X}) = \sum_1^n X_i$ is complete sufficient statistic

(a) $g(\theta) = \theta$

$$E\left(\frac{T}{n}\right) = \theta$$

$\frac{T}{n}$ is an u.e. based on complete suff stat

$\Rightarrow \frac{T}{n}$ is UMVUE of θ.

(b) $g(\theta) = \theta^4$

$$\delta_0(\underline{X}) = \begin{cases} 1, & X_1=1, X_2=1, X_3=1, X_4=1 \\ 0, & o/w \end{cases}$$

$$E\,\delta_0(\underline{X}) = P(X_1=1, X_2=1, X_3=1, X_4=1) = \theta^4$$

$\Rightarrow \delta_0(\underline{X})$ is u.e. of θ^4

Rao-Blackwellization of δ_0 : $\eta(T) = E(\delta_0 \mid T)$

$$E(\delta_0 \mid T=t) = P(X_1=1, X_2=1, X_3=1, X_4=1 \mid T=t)$$

$$= \frac{P(X_1=1, X_2=1, X_3=1, X_4=1, \sum_5^n X_i = t-4)}{P(T=t)}$$

$$= \frac{P(X_1=1)P(X_2=1)P(X_3=1)P(X_4=1)\ P(\sum_5^n X_i = t-4)}{P(T=t)}$$

$$= \frac{\cancel{\theta}.\cancel{\theta}.\cancel{\theta}.\cancel{\theta}\left(\binom{n-4}{t-4}\cancel{\theta^{t-4}}\cancel{(1-\theta)^{n-4-(t-4)}}\right)}{\binom{n}{t}\cancel{\theta^{t}}\cancel{(1-\theta)^{n-t}}}$$

$$= \frac{\binom{n-4}{t-4}}{\binom{n}{t}} = \frac{t(t-1)(t-2)(t-3)}{n(n-1)(n-2)(n-3)}$$

$\eta(T) = \dfrac{T(T-1)(T-2)(T-3)}{n(n-1)(n-2)(n-3)}$ is an u.e. based on complete suff stat and hence is UMVUE of θ^4

(c) $g(\theta) = \theta(1-\theta)^2$

$$\delta_0(\underline{X}) = \begin{cases} 1, & X_1=1, X_2=0, X_3=0 \\ 0, & o/w \end{cases}$$

$$E\,\delta_0(\underline{X}) = \theta(1-\theta)^2$$

i.e. $\delta_0(\underline{x})$ is u.e. based on $(X_1, \ldots X_n)$

Rao - Blackwellization of $\delta_0(\underline{X})$: $\eta(T) = E(\delta_0 | T)$

$$E(\delta_0(\underline{X}) | T=t) = P(X_1=1, X_2=0, X_3=0 | T=t)$$

$$= \frac{P(X_1=1, X_2=0, X_3=0, \sum_4^n X_i = t-1)}{P(T=t)}$$

$$= \frac{\cancel{\theta}\,\cancel{(1-\theta)}\,\cancel{(1-\theta)}\left(\binom{n-3}{t-1}\cancel{\theta^{t-1}}\cancel{(1-\theta)}^{n-3-(t-1)}\right)}{\binom{n}{t}\cancel{\theta^t}\cancel{(1-\theta)}^{n-t}}$$

$$= \frac{\binom{n-3}{t-1}}{\binom{n}{t}} = \frac{t(n-t)(n-t-1)}{n(n-1)(n-2)}$$

$\Rightarrow$ $\eta(T) = \dfrac{T(n-T)(n-T-1)}{n(n-1)(n-2)}$ is u.e. based on complete suff stat and hence is UMVUE for $\theta(1-\theta)^2$.

(4) $X_1, \ldots X_n$ r.s. from $f_\theta(x) = \begin{cases} e^{-x+\theta}, & x > \theta \\ 0, & o/w. \end{cases}$

i.e. $f_\theta(x) = e^{-x+\theta} I(\theta, x)$

Can not be expressed in s-parameter exponential distn form

Proof of completeness of minimal suff statistic using direct approach.

$T = X_{(1)}$ is minimal suff statistic.

p.d.f. of T

$$f_T(t) = \begin{cases} n e^{-n(t-\theta)}, & t > \theta \\ 0, & o/w \end{cases}$$

Now $E\, g(T) = 0 \quad \forall \theta \in \Theta$

$$\Rightarrow \int_\theta^\infty g(t)\, n e^{-nt} e^{n\theta}\, dt = 0 \quad \forall \theta \in \Theta$$

i.e. $$\int_\theta^\infty g(t)\, e^{-nt}\, dt = 0 \quad \forall \theta \in \Theta$$

Differentiating w.r.t. θ, we get

$$g(\theta) e^{-n\theta} = 0 \quad \forall \theta \in \Theta$$

$$\Rightarrow g(\theta) = 0 \quad \forall \theta$$

i.e $$g(\theta) = 0 \quad \forall\ 0 < \theta < t$$

$$\Rightarrow g(t) = 0 \quad \forall\ \theta < t$$

i.e. $g(t) = 0$ w.p. 1 (a.e.)
(range of t is (θ, ∞))

$\Rightarrow T = X_{(1)}$ is complete suff statistic

$g(\theta) = \theta^2,$

$$E X_{(1)} = \int_{\theta}^{\infty} t \, n e^{-n(t-\theta)} \, dt$$

with $y = t - \theta$

$$= n \int_{0}^{\infty} (y+\theta) e^{-ny} \, dy$$

$$= n \left[\frac{\Gamma 2}{n^2} + \theta \frac{1}{n} \right] = \theta + \frac{1}{n}$$

$$\Rightarrow E\left(X_{(1)} - \frac{1}{n} \right) = \theta$$

$\Rightarrow X_{(1)} - \frac{1}{n}$ is u.e. of θ based on complete suff stat

$\Rightarrow X_{(1)} - \frac{1}{n}$ is UMVUE for θ

Further,

$$E X_{(1)}^2 = n \int_{\theta}^{\infty} t^2 e^{-n(t-\theta)} \, dt$$

$$= n \int_{0}^{\infty} (y+\theta)^2 e^{-ny} \, dy$$

$$= n \int_{0}^{\infty} (y^2 + \theta^2 + 2\theta y) e^{-ny} \, dy$$

$$= n \left[\frac{\Gamma 3}{n^3} + \theta^2 \frac{1}{n} + 2\theta \frac{\Gamma 2}{n^2} \right]$$

$$= \frac{2}{n^2} + \theta^2 + \frac{2\theta}{n}$$

$$\Rightarrow E\left(X_{(1)}^2 \right) = \frac{2}{n^2} + \theta^2 + \frac{2}{n} E\left(X_{(1)} - \frac{1}{n} \right)$$

$$\Rightarrow E\left(X_{(1)}^2 - \cancel{\frac{2}{n^2}} - \frac{2}{n}\left(X_{(1)} - \cancel{\frac{1}{n}} \right) \right) = \theta^2$$

i.e. $E\left(X_{(1)}^2 - \frac{2}{n} X_{(1)} \right) = \theta^2$

$X_{(1)}^2 - \frac{2}{n} X_{(1)}$ is u.e. of θ^2 based on complete suff

stat $X_{(1)}$

$\Rightarrow X_{(1)}^2 - \frac{2}{n} X_{(1)}$ is UMVUE of θ^2

(5) $X_1, \ldots, X_n$ r.s. from $U(0, \theta)$; $\theta > 0$

$T = X_{(n)}$ is minimal suff statistic

$$f_T(t) = \begin{cases} \frac{n}{\theta^n} t^{n-1}, & 0 < t < \theta \\ 0, & \text{o/w} \end{cases}$$

Note that $U(0, \theta)$ does not confirm to exponential family dist^n representation.

Direct approach to prove completeness:

$$E(g(T)) = 0 \quad \forall \theta \in \Theta$$

$$\Rightarrow \int_0^\theta g(t) \frac{n}{\theta^n} t^{n-1} dt = 0 \quad \forall \theta \in \Theta$$

i.e. $$\int_0^\theta g(t) t^{n-1} dt = 0 \quad \forall \theta \in \Theta$$

Differentiating both sides w.r.t. θ, we get

$$g(\theta) \theta^{n-1} = 0 \quad \forall \theta \in \Theta$$

i.e. $g(\theta) = 0 \quad \forall \theta > 0$

i.e. $g(t) = 0 \quad \forall 0 < t < \theta$

$\Rightarrow g(t) = 0$ w.p.1 (a.e.).

$\Rightarrow T = X_{(n)}$ is complete sufficient statistic

$g(\theta) = \theta^2$

$$E\, T^2 = \frac{n}{\theta^n} \int_0^\theta t^2\, t^{n-1}\, dt$$

$$= \frac{n}{\theta^n} \int_0^\theta t^{n+1}\, dt = \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2}$$

$$= \frac{n}{n+2}\, \theta^2$$

$$\Rightarrow E\left(\frac{n+2}{n} T^2 \right) = \theta^2$$

$\frac{n+2}{n} T^2$ is u.e. of θ^2 based on complete suff stat

$\Rightarrow \frac{n+2}{n} X_{(n)}$ is UMVUE for θ^2.

(6) $X_1, \ldots, X_n$ r.s. from $G(2, \theta)$ $\quad \theta > 0$

p.d.f.

$$f_\theta(x) = \begin{cases} \frac{1}{\sqrt{2}\, \theta^2} e^{-x/\theta} x, & x > 0 \\ 0, & o/w. \end{cases}$$

Note that

$$f_\theta(x) = x \exp\left(-\frac{1}{\theta} x - 2\log\theta\right)$$

with $h(x) = x$; $\eta(\theta) = -\frac{1}{\theta}$; $T(x) = x$, $\beta(\theta) = 2\log\theta$

the above is 1-parameter exponential family distn

Natural parameter space is

$$\{\eta : \eta < 0\} \quad (\{\eta(\theta) : \theta \in (0,\infty)\})$$

The above obviously contains open intervals

$\Rightarrow$ The above is 1-parameter exponential family with full rank

$\Rightarrow \quad T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$ is complete

Thus $\sum_{i=1}^{n} X_i$ is complete suff stat

Note that

$$E X_i = \frac{1}{\theta^2} \int_0^\infty x^2 e^{-x/\theta} dx = \frac{\sqrt{3}\, \theta^3}{\theta^2} = 2\theta$$

$$\Rightarrow \quad E\left(\sum_{i=1}^{n} X_i\right) = 2n\theta$$

$\Rightarrow \quad \sum_{1}^{n} X_i / 2n$ is u.e. of θ based on complete suff stat

and hence is UMVUE of θ.

(7) $X_1, \ldots, X_n$ r.s. from $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$$f_\theta(\underset{\sim}{x}) = I(\theta - \tfrac{1}{2}, x_{(1)})\, I(x_{(n)}, \theta + \tfrac{1}{2})$$

$T(\underset{\sim}{X}) = (X_{(1)}, X_{(n)})$ is minimal sufficient stat

Note that distⁿ of $X_{(n)} - X_{(1)}$ is indep of θ. as

$$P(X_{(n)} - X_{(1)} \le x) = P((X_{(n)} - \theta) - (X_{(1)} - \theta) \le x)$$

$$= P\left(\max_i (X_i - \theta) - \min_i (X_i - \theta) \le x \right)$$

$$\underbrace{X_i - \theta \sim U\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)}_{\text{dist}^n \text{ indep of } \theta}$$

$\Rightarrow \min_i (X_i - \theta)$: dist^n is indep of θ

& $\max_i (X_i - \theta)$: dist^n is indep of θ

$\Rightarrow$ dist^n of $\max_i (X_i - \theta) - \min_i (X_i - \theta)$ is indep of θ

i.e. dist^n of $X_{(n)} - X_{(1)}$ is indep of θ

$$\Rightarrow E\left(X_{(n)} - X_{(1)}\right) = a \leftarrow \text{indep of } \theta$$

$$\Rightarrow E\left(X_{(n)} - X_{(1)} - a\right) = 0 \quad \forall \theta \in \Theta$$

$$\not\Rightarrow X_{(n)} - X_{(1)} = a \quad \text{w.p. } 1$$

(in fact $P(\underbrace{X_{(n)} - X_{(1)}}_{\text{cont r.v.}} = a) = 0 \ \forall a$)

$\Rightarrow T(\underline{X}) = (X_{(1)}, X_{(n)})$ is NOT Complete

Alt approach:

One can show that

$$E\left(X_{(n)} - X_{(1)}\right) = \frac{n-1}{n+1}$$ by direct calculation using p.d.f of $X_{(1)}$ & $X_{(n)}$

$$\Rightarrow E\left(X_{(n)} - X_{(1)} - \frac{n-1}{n+1}\right) = 0 \quad \forall \theta \in \Theta$$

$$\not\Rightarrow X_{(n)} - X_{(1)} = \frac{n-1}{n+1} \text{ a.e.}$$ $\Rightarrow (X_{(1)}, X_{(n)})$ is NOT Complete

(8) $X_1, \ldots, X_n$ r. s. from ~~N(0,θ)~~, $N(0, \theta)$

p.d.f. $f_\theta(x) = \frac{1}{\sqrt{2\pi}\sqrt{\theta}} e^{-\frac{1}{2\theta} x^2}$

The above is 1-parameter exponential family of full rank

$\Rightarrow T = \sum_{i=1}^{n} X_i^2$ is complete suff stat

Note that $\frac{T}{\theta} \sim \chi^2_n$

$\Rightarrow E\left(\frac{T}{\theta}\right) = n \qquad V\left(\frac{T}{\theta}\right) = 2n$

$E(T) = n\theta \qquad V(T) = 2n\theta^2$

$\Rightarrow ET^2 = V(T) + (ET)^2 = 2n\theta^2 + n^2\theta^2$

$= \theta^2 n(n+2).$

$\Rightarrow E\left(\frac{T^2}{n(n+2)}\right) = \theta^2$

$\frac{T^2}{n(n+2)}$ is u.e. of θ^2 based on complete suff stat and

hence is UMVUE for θ^2.

(9) $X_1, \ldots, X_n$ r.s. from $N(\mu, \theta)$

(a) μ is known

p.d.f. $\frac{1}{\sqrt{2\pi}\sqrt{\theta}} e^{-\frac{1}{2\theta}(x-\mu)^2}$ Known

The above is 1-parameter exp family distⁿ of full rank

$\Rightarrow T = \sum_{i=1}^{n} (X_i - \mu)^2$ is complete suff stat

$$\frac{T}{\theta} \sim \chi^2_n$$

$$E\left(\frac{T}{\theta}\right) = n$$

$\Rightarrow \sum_1^n (X_i - \mu)^2 / n$ is u.e. of θ based on complete suff stat and hence is UMVUE for θ, when μ is known

(b) <u>μ & θ both unknown</u> : estimand θ

p.d.f. $\frac{1}{\sqrt{2\pi}\sqrt{\theta}} \exp\left(-\frac{1}{2\theta}(x^2 + \mu^2 - 2\mu x)\right)$

$$f_{\mu,\theta}(x) = \underbrace{\frac{1}{\sqrt{2\pi}}}_{h(x)} \exp\Big(\underbrace{-\frac{1}{2\theta}}_{\eta_1} \overset{T_1}{x^2} + \underbrace{\frac{\mu}{\theta}}_{\eta_2} \overset{T_2}{x} - \frac{\mu^2}{2\theta} - \frac{1}{2}\log\theta\Big)$$

The above is a 2-parameter exponential family distn of full rank

(Natural parameter space : $\{(\eta_1, \eta_2) : \eta_1 < 0, \eta_2 \in \mathbb{R}\}$

The natural parameter space contains 2-dimensional open rectangles and hence the 2-param expo family is of full rank)

$$\Rightarrow T(\underline{X}) = \left(\sum_1^n X_i, \sum X_i^2\right)$$

$$\Leftrightarrow \left(\bar{X},\ s^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2\right) \text{ is}$$

Complete suff stat for (μ, θ)

Now $\frac{(n-1)s^2}{\theta} \sim \chi^2_{n-1}$

$$\Rightarrow E\left(\frac{\Sigma(x_i-\bar{x})^2}{\theta}\right) = n-1$$

i.e. $E\left(\frac{\Sigma(x_i-\bar{x})^2}{n-1}\right) = \theta$

$\frac{\Sigma(x_i-\bar{x})^2}{n-1}$ is u.e. of θ based on complete suff stat

and hence is UMVUE of θ

~~(10)~~ (c) <u>μ & θ both unknown</u> : estimand is $\delta \ni$

$$\underset{\text{given const}}{p} = P(X \le \delta)$$

Note that $p = P(X \le \delta) = P\left(\frac{X-\mu}{\sqrt{\theta}} \le \frac{\delta-\mu}{\sqrt{\theta}}\right) = \Phi\left(\frac{\delta-\mu}{\sqrt{\theta}}\right)$

$$\Rightarrow \frac{\delta-\mu}{\sqrt{\theta}} = \Phi^{-1}(p)$$

$$\Rightarrow \delta = \mu + \sqrt{\theta}\ \underbrace{\Phi^{-1}(p)}_{\text{known fixed const}} = f(\mu,\theta) \leftarrow \text{estimand}$$

$\left(\Sigma X_i, \Sigma X_i^2\right) \Leftrightarrow \left(\bar{X}, \frac{1}{n-1}\Sigma(X_i-\bar{X})^2 = s^2\right)$ is complete suff statistic

$\Rightarrow \bar{X}$ is UMVUE of μ

$$Y = \frac{(n-1)s^2}{\theta} \sim \chi^2_{n-1}$$

For $Y \sim \chi^2_{n-1}$ $\quad (r = n-1 \text{ say})$

$$E\, Y^{\frac{1}{2}} = \frac{1}{2^{r/2}\,\Gamma\frac{r}{2}} \int_0^\infty y^{1/2}\, e^{-y/2}\, y^{\frac{r}{2}-1}\, dy$$

$$= \frac{1}{2^{r/2}\,\Gamma\frac{r}{2}} \int_0^\infty e^{-y/2}\, y^{\frac{r+1}{2}-1}\, dy$$

$$= \frac{\Gamma\frac{r+1}{2}\; 2^{\frac{r+1}{2}}}{2^{r/2}\;\Gamma\frac{r}{2}} = \frac{\Gamma\frac{r+1}{2}\; 2^{1/2}}{\Gamma\frac{r}{2}}$$

i.e. $E\left(\frac{\sqrt{n-1}}{\sqrt{\theta}}\, s\right) = \frac{\Gamma\frac{n}{2}\;\sqrt{2}}{\Gamma\frac{n-1}{2}}$

$$\Rightarrow E\left(\frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}} \cdot \sqrt{\frac{n-1}{2}}\; s\right) = \sqrt{\theta}$$

$\Rightarrow \frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}} \sqrt{\frac{n-1}{2}}\; s$ is UMVUE of $\sqrt{\theta}$

$\Rightarrow \bar{X} + \frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}} \sqrt{\frac{n-1}{2}}\; s\; \Phi^{-1}(p)$ is UMVUE of δ

$(\delta = \mu + \sqrt{\theta}\; \Phi^{-1}(p))$

(10) T_1, T_2, T_3 all are u.e. of θ (trivial to verify)

Among the 3, T_1 is UMVUE and hence is the preferred estimator

(11) $X_1, \ldots, X_n$ r.s. from $N(\mu, \sigma^2)$

$\mu \in \mathbb{R}, \sigma > 0$

$\left(\sum_1^n X_i, \sum X_i^2\right) \Leftrightarrow \left(\bar{X}, s^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2\right)$ is

complete sufficient statistic

$$\left.\begin{array}{l}\bar{X} \sim N(\mu, \sigma^2/n) \\ \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}\end{array}\right\rangle \text{indep}$$

$\underline{g(\mu,\sigma^2) = \mu^2}$

$$E(\bar{X}^2) = V(\bar{X}) + (E\bar{X})^2$$

$$E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$$

i.e. $E(\bar{X}^2) = \frac{1}{n} E(s^2) + \mu^2$

$$\Rightarrow E\left(\bar{X}^2 - \frac{1}{n} s^2\right) = \mu^2$$

$\left(\bar{X}^2 - \frac{1}{n} s^2\right)$ is u.e. of μ^2 based on complete sufficient statistic and hence is UMVUE for μ^2

$g(\mu, \sigma^2) = \mu + \sigma$

$E(\bar{X}) = \mu$

$$E\left(\frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}} \frac{1}{\sqrt{2}} \left(\Sigma (X_i - \bar{X})^2 \right)^{1/2} \right) = \sigma \quad (\text{from } 9(c))$$

$$\Rightarrow E\left(\underbrace{\bar{X} + \frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}\sqrt{2}} \left(\Sigma (X_i - \bar{X})^2 \right)^{1/2}} \right) = \mu + \sigma$$

u.e. of $(\mu+\sigma)$ based on complete suff stat and hence is UMVUE of $\mu+\sigma$.

(12) $x_1, \ldots, x_n$ i.i.d. $N(\mu, \sigma^2)$

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$\log f = K - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\frac{\partial \log f}{\partial \mu} = \frac{(x-\mu)}{\sigma^2} ; \quad \frac{\partial^2 \log f}{\partial \mu^2} = -\frac{1}{\sigma^2}$$

$$-E\left(\frac{\partial^2 \log f}{\partial \mu^2}\right) = \frac{1}{\sigma^2} = I(\mu)$$

CRLB for an u.e. for $\mu = \dfrac{\sigma^2}{n}$.

Since $V(\bar{X}) = \dfrac{\sigma^2}{n}$; $\bar{X}$ attains CRLB.

$$\frac{\partial \log f}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(x-\mu)^2$$

$$\frac{\partial^2 \log f}{\partial (\sigma^2)^2} = \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6}$$

$$I(\sigma^2) = -E\left(\frac{\partial^2 \log f}{\partial (\sigma^2)^2}\right) = -\frac{1}{2\sigma^4} + \frac{1}{\sigma^4} = \frac{1}{2\sigma^4}$$

CRLB for an u.e. for $\sigma^2 = \dfrac{2\sigma^4}{n}$.

Now $s^2 = \dfrac{1}{n-1}\sum_1^n (X_i - \bar{X})^2$ is UMVUE for σ^2 with

$$V(s^2) = \frac{2\sigma^4}{n-1} > \text{CRLB}$$

Since UMVUE is the unbiased est with lowest variance in the class of all unbiased estimators, CRLB can't be attained by any unbiased estimator of σ^2.

(13)

$$f(x|\beta) = \frac{1}{\Gamma\alpha \ \beta^\alpha} e^{-x/\beta} x^{\alpha-1}; \quad x > 0$$

$$\log f = -\log\Gamma\alpha - \alpha\log\beta - \frac{x}{\beta} + (\alpha-1)\log x$$

$$\frac{\partial \log f}{\partial\beta} = -\frac{\alpha}{\beta} + \frac{x}{\beta^2}$$

$$\frac{\partial^2 \log f}{\partial\beta^2} = \frac{\alpha}{\beta^2} - 2\frac{x}{\beta^3}.$$

$$I(\beta) = -E\left(\frac{\partial^2 \log f}{\partial\beta^2}\right) = -\frac{\alpha}{\beta^2} + 2\frac{\alpha\beta}{\beta^3} = \frac{\alpha}{\beta^2}.$$

$\Rightarrow$ CRLB for u.e. of β : $\dfrac{1}{n.\frac{\alpha}{\beta^2}} = \dfrac{\beta^2}{n\alpha}$.

(14) $X_1, \ldots X_n$ i.i.d. $P(\theta)$

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}$$

$$\log f(x|\theta) = -\theta + x\log\theta - \log x!$$

$$\frac{\partial \log f}{\partial\theta} = -1 + \frac{x}{\theta}; \quad \frac{\partial^2 \log f}{\partial\theta^2} = -\frac{x}{\theta^2}.$$

$$I(\theta) = -E\left(\frac{\partial^2 \log f}{\partial\theta^2}\right) = \frac{1}{\theta}$$

CRLB for any u.e. of θ : $\dfrac{1}{n\frac{1}{\theta}} = \dfrac{\theta}{n}$

CRLB for any u.e. of $g(\theta) = \theta^2$: $\dfrac{(2\theta)^2}{\frac{n}{\theta}} = \dfrac{4\theta^3}{n}$

CRLB for any u.e. of $g(\theta) = e^{-\theta}$: $\dfrac{(-e^{-\theta})^2}{\frac{n}{\theta}} = \dfrac{\theta e^{-2\theta}}{n}$.

15)

$X_1, \ldots, X_n$ i.i.d $B(1,\theta)$

$$f(x|\theta) = \theta^x (1-\theta)^{1-x}$$

$$\log f(x|\theta) = x \log \theta + (1-x) \log(1-\theta)$$

$$\frac{\partial \log f}{\partial \theta} = \frac{x}{\theta} + \frac{(1-x)}{1-\theta}(-1) = \frac{x}{\theta(1-\theta)} - \frac{1}{1-\theta}.$$

$$I(\theta) = E\left(\frac{\partial \log f}{\partial \theta}\right)^2 = V\left(\frac{\partial \log f}{\partial \theta}\right) = \frac{\theta(1-\theta)}{(\theta(1-\theta))^2} = \frac{1}{\theta(1-\theta)}.$$

CRLB for u.e. of θ^4: $$\frac{(4\theta^3)^2}{n \cdot \frac{1}{\theta(1-\theta)}} = \frac{16\,\theta^7(1-\theta)}{n}$$

CRLB for u.e. of $\theta(1-\theta)$: $$\frac{(1-2\theta)^2}{n \cdot \frac{1}{\theta(1-\theta)}} = \frac{(1-2\theta)^2\,\theta(1-\theta)}{n}.$$