MSO201: PROBABILITY & STATISTICS
Problem Set #8

[1] The joint probability mass function of the random variables X_1 and X_2 is given by

$$P(X_1 = x_1, X_2 = x_2) = \begin{cases} \left(\frac{2}{3}\right)^{x_1+x_2}\left(\frac{1}{3}\right)^{2-x_1-x_2} & \text{if } (x_1,x_2)=(0,0),(0,1),(1,0),(1,1) \\ 0 & \text{otherwise.} \end{cases}$$

- **(a)** Find the joint probability mass function of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.
- **(b)** Find the marginal probability mass functions of Y_1 and Y_2.
- **(c)** Verify whether Y_1 and Y_2 are independent.

[2] Let the joint probability mass function of X_1 and X_2 be

$$P(X_1 = x_1, X_2 = x_2) = \begin{cases} \frac{x_1 x_2}{36} & \text{if } x_1 = 1,2,3; x_2 = 1,2,3; \\ 0 & \text{otherwise.} \end{cases}$$

- **(a)** Find the joint probability mass function of $Y_1 = X_1 X_2$ and $Y_2 = X_2$.
- **(b)** Find the marginal probability mass function of Y_1.
- **(c)** Find the probability mass function of $Z = X_1 + X_2$.

[3] (a) Let $X \sim Bin(n_1, p)$ and $Y \sim Bin(n_2, p)$ be independent random variables. Find the conditional distribution of X given $X + Y = t$, $t \in \{0,1,\dots,n_1+n_2\}$.

(b) Let $X \sim Bin(n_1, 1/2)$ and $Y \sim Bin(n_2, 1/2)$ be independent random variables. Find the distribution of $Y = X_1 - X_2 + n_2$.

[4] Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$ be independent random variables. Find the conditional distribution of X given $X + Y = t$, $t \in \{0,1,\dots\dots\}$.

[5] Let X_1, X_2, X_3 and X_4 be four mutually independent random variables each having probability density function

$$f(x) = \begin{cases} 3(1-x)^2 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density functions of $Y = \min(X_1, X_2, X_3, X_4)$ and $Z = \max(X_1, X_2, X_3, X_4)$.

[6] Suppose $X_1, \dots\dots, X_n$ are n independent random variables, where X_i $(i = 1,\dots,n)$ has the exponential distribution $Exp(\alpha_i)$, with probability density function

$$f_{X_i}(x) = \begin{cases} \alpha_i e^{-\alpha_i x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density functions of $Y = \min(X_1,....,X_n)$ and $Z = \max(X_1,....,X_n)$.

[7] Let X and Y be the respective arrival times of two friends A and B who agree to meet at a spot and wait for the other only for t minutes. Supposing that X and Y are i.i.d. $Exp(\lambda)$. Show that the probability of A and B meeting each other is $1-e^{-\lambda t}$.

[8] Let X_1 and X_2 be i.i.d. $U(0,1)$. Define two new random variables as $Y_1 = X_1 + X_2$ and $Y_2 = X_2 - X_1$. Find the joint probability density function of Y_1 and Y_2 and also the marginal probability density functions of Y_1 and Y_2.

[9] Let X and Y be i.i.d. $N(0,1)$. Find the probability density function of $Z = X/Y$.

[10] Let X and Y be independent random variables with probability density functions

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha_1 - 1}}{\overline{|\alpha_1}\, \theta^{\alpha_1}} e^{-x/\theta} & x > 0 \\ 0 & \text{otherwise;} \end{cases} \qquad f_Y(y) = \begin{cases} \dfrac{y^{\alpha_2 - 1}}{\overline{|\alpha_2}\, \theta^{\alpha_2}} e^{-y/\theta} & y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distributions of $U = X + Y$ and $V = X/(X+Y)$ and show that they are independently distributed.

[11] Let X and Y be i.i.d. random variables with common probability density function

$$f(x) = \begin{cases} \dfrac{c}{1+x^4} & -\infty < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where, c is a normalizing constant. Find the probability density function of $Z = X/Y$.

[12] Let X and Y be i.i.d. $N(0,1)$, Define the random variables R and Θ by $X = R\cos\Theta, Y = R\sin\Theta.$

(a) Show that R and Θ are independent with $R^2/2 \sim Exp(1)$ and $\Theta \sim U(0,2\pi)$

(b) Show that $X^2 + Y^2$ and X/Y are independently distributed.

[13] Let U_1 and U_2 be i.i.d. $U(0,1)$ random variables. Show that

$$X_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2) \text{ and } X_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$$

are i.i.d. $N(0,1)$ random variables.

[14] Let X_1, X_2 and X_3 be i.i.d. with probability density function

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density function of Y_1, Y_2, Y_3; where

$$Y_1 = \frac{X_1}{X_1 + X_2}; Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}; Y_3 = X_1 + X_2 + X_3.$$

[15] Let X_1, X_2 and X_3 be three mutually independent chi-square random variables with n_1, n_2 and n_3 degrees of freedom respectively; i.e. $X_1 \sim \chi^2_{n_1}, X_2 \sim \chi^2_{n_2}$ and $X_3 \sim \chi^2_{n_3}$ and they are independent.

(a) Show that $Y_1 = X_1/X_2$ and $Y_2 = X_1 + X_2$ are independent and that Y_2 is chi-square random variable with $n_1 + n_2$ degrees of freedom.

(b) Find the probability density functions of

$$Z_1 = \frac{X_1/n_1}{X_2/n_2} \text{ and } Z_2 = \frac{X_3/n_3}{(X_1 + X_2)/(n_1 + n_2)}$$

[16] Let X and Y be independent random variables such that $X \sim N(0,1)$ and $Y \sim \chi^2_n$. Find the probability density function of

$$T = \frac{X}{\sqrt{Y/n}}.$$

[17] Let $X_1, \ldots., X_n$ be a random sample from $N(0,1)$ distribution. Find the m.g.f. of $Y = \sum_{i=1}^{n} X_i^2$ and identify its distribution. Further, suppose X_{n+1} is another random sample from $N(0,1)$ independent of $X_1, \ldots., X_n$. Derive the distribution of $\left(X_{n+1}/\sqrt{Y/n}\right)$.

[18] X and Y are i.i.d. random variables each having geometric distribution with the following p.m.f.

$$P(X = x) = \begin{cases} (1-p)^x p, & x = 0,1,\ldots. \\ 0, & \text{otherwise.} \end{cases}$$

Identify the distribution of $X \mid X + Y$. Further find the p.m.f. of $Z = \min(X, Y)$.