MSO201: PROBABILITY & STATISTICS
Problem Set # 10

[1] Let $X_1, X_2, ... X_n$ be a random sample from an exponential distribution with p.d.f.

$$f_X(x) = \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right); x > 0$$

Show that $\bar{X} = \sum_{i=1}^{n} X_i / n$ is an unbiased estimator of β.

[2] Let $X_1, X_2, ... X_n$ be a random sample from $U(0, \theta)$; $\theta > 0$. Show that $\frac{n+1}{n} X_{(n)}$

and $2\bar{X}$ are both unbiased estimators of θ.

[3] Let $X_1, X_2, ... X_n$ be a random sample from an exponential distribution with p.d.f.

$$f(x) = \beta \exp(-\beta x); x > 0$$

Show that $\bar{X}$ is an unbiased estimator of $1/\beta$.

[4] Let $X_1, X_2, ... X_n$ be a random sample from $N(\theta, \theta^2)$, $\theta > 0$. Show that

$$\left(\sum_{i=1}^{n} X_i\right)^2 / n(n+1) \text{ and } \sum_{i=1}^{n} X_i^2 / 2n \text{ are both unbiased estimators of } \theta^2.$$

[5] Let $X_1, X_2, ... X_n$ be a random sample from $P(\theta)$; $\theta > 0$. Find an unbiased estimator of $\theta e^{-2\theta}$.

[6] Let $X_1, X_2, ... X_n$ be a random sample from $B(1, \theta); 0 \le \theta \le 1$.

    (a) Show that the estimator $T(\underset{\sim}{X}) = \dfrac{\frac{1}{2}\sqrt{n} + \sum_{i=1}^{n} X_i}{n + \sqrt{n}}$ is not unbiased θ?

    (b) Show that $\lim_{n \to \infty} E(T(\underset{\sim}{X})) = \theta$.

    (An estimator satisfying the condition in (b) is said to be unbiased in the limit)

[7] $X_1, ..., X_n$ be a random sample from $N(\mu, \sigma^2), \mu \in \mathfrak{R}, \sigma \in \mathfrak{R}^+$. Find unbiased estimators of μ/σ^2 and μ/σ.

[8] Let $X_1, X_2, ... X_n$ be a random sample from $B(1, \theta); 0 \le \theta \le 1$. Find an unbiased estimator of $\theta^2(1-\theta)$.

[9] Using Neyman Fisher Factorization Theorem, find a sufficient based on a random sample $X_1, X_2, ... X_n$ from each of the following distributions

(a) $f_\alpha(x) = \begin{cases} \dfrac{1}{\alpha}\exp\left(-\dfrac{x}{\alpha}\right) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$

(b) $f_\beta(x) = \begin{cases} \exp(-(x-\beta)) & \text{if } x > \beta \\ 0 & \text{otherwise.} \end{cases}$

(c) $f_{\alpha,\beta}(x) = \begin{cases} \dfrac{1}{\alpha}\exp\left(-\dfrac{(x-\beta)}{\alpha}\right) & \text{if } x > \beta \\ 0 & \text{otherwise.} \end{cases}$

(d) $f_{\mu,\sigma}(x) = \begin{cases} \dfrac{1}{x\,\sigma\,\sqrt{2\pi}}\exp\left(-\dfrac{(\log x_i - \mu)^2}{2\sigma^2}\right) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$

(e) $f_\theta(x) = \begin{cases} \dfrac{1}{\theta} & -\theta/2 \le x \le \theta/2 \\ 0 & \text{otherwise} \end{cases}$

[10] Let X_1 and X_2 be independent random samples with densities $f_1(x_1) = \theta e^{-\theta x_1}$ and $f_2(x_2) = 2\theta e^{-2\theta x_2}$ as the respective p.d.f.s where $\theta > 0$ is an unknown parameter and $0 < x_1, x_2 < \infty$. Using Neyman Fisher Factorization Theorem find a sufficient statistic for θ.

[11] Let $X_1, \ldots, X_n$ be a random sample with densities
$$f_{X_i}(x) = \begin{cases} \exp(i\theta - x) & \text{if } x \ge i\theta \\ 0 & \text{otherwise.} \end{cases}$$
Using Neyman Fisher Factorization Theorem find a sufficient statistic for θ.

[12] Let $X_1, X_2, \ldots X_n$ be a random sample from a $Beta(\alpha, \beta)$ distribution $(\alpha > 0, \beta > 0)$ with p.d.f.
$$f(x) = \begin{cases} \dfrac{\overline{|\alpha+\beta}}{\overline{|\alpha}\,\overline{|\beta}}\, x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that

(a) $\prod\limits_{i=1}^{n} X_i$ is sufficient for α if β is known to be a given constant.

(b) $\prod\limits_{i=1}^{n}(1-X_i)$ is sufficient for β if α is known to be a given constant.

(c) $\left(\prod\limits_{i=1}^{n} X_i, \prod\limits_{i=1}^{n}(1-X_i)\right)$ is jointly sufficient for (α, β) if both the parameters are unknown.

[13] Let T and T^* be two statistic such that $T = \psi\left(T^*\right)$. Show that if T is sufficient then T^* is also sufficient.

[14] $X_1,..., X_n$ be a random sample from $U\left(\theta - 1/2, \theta + 1/2\right), \theta \in \mathfrak{R}$. Find a sufficient statistic for θ.

[15] Let $X_1,..., X_n$ be independent random variables with $X_i\ (i = 1, 2,..., n)$ having the probability density function

$$f_i(x_i) = \begin{cases} i\theta e^{-i\theta x_i} & x_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a sufficient statistic for θ.