MSO201: PROBABILITY & STATISTICS
Problem Set #3

[1] Find the expected number of throws of a fair die required to obtain a 6.

[2] Consider a sequence of independent coin flips, each of which has a probability p of being heads. Define a random variable X as the length of the run (of either heads or tails) started by the first trial. Find $E(X)$.

[3] Find $E(X)$, if it exists, in the following cases:

(a) X has the p.m.f. $P(X=x)=\begin{cases}\left(x(x+1)\right)^{-1}, & \text{if } x=1,2,\ldots\\ 0, & \text{otherwise.}\end{cases}$

(b) X has the p.d.f. $f(x)=\begin{cases}\left(2x^2\right)^{-1}, & \text{if } |x|>1,\\ 0, & \text{otherwise.}\end{cases}$

(c) X has the p.d.f. $f(x)=\dfrac{1}{\pi}\dfrac{1}{1+x^2}; -\infty<x<\infty.$

[4] Find the mean and variance of the distributions having the following p.d.f. / p.m.f.

(a) $f(x)=a\,x^{a-1}, 0<x<1, a>0.$

(b) $f(x)=1/n; x=1,2,\ldots,n; n>0$ is an integer

(c) $f(x)=\dfrac{3}{2}(x-1)^2; 0<x<2$

[5] Find the mean and variance of the Weibull random variable having the p.d.f.

$$f(x)=\begin{cases}\dfrac{c}{a}\left(\dfrac{x-\mu}{a}\right)^{c-1}\exp\left\{-\left(\dfrac{x-\mu}{a}\right)^{c}\right\} & \text{if } x>\mu\\ 0 & \text{otherwise.}\end{cases}$$

$c>0,\ a>0$ and $\mu\in(-\infty,\infty)$.

[6] A median of a distribution is a value m such that $P(X<m)\le 0.5$ and $P(X\le m)\ge 0.5$, with equality for a continuous distribution. Find the median of the distribution with p.d.f. $f(x)=3x^2, 0<x<1; =0$, otherwise.

[7] Let X be a continuous, nonnegative random variable with d.f. $F(x)$. Show that $E(X)=\int_0^{\infty}\left(1-F(x)\right)dx.$

[8] A target is made of three concentric circles of radii $1/\sqrt{3}$, 1, $\sqrt{3}$ feet. Shots within the inner circle give 4 points, within the next ring 3 points and within the third ring 2 points. Shots outside the target give 0. Let X be the distance of the hit from the centre (in feet) and let the p.d.f. of X be

$$f(x)=\begin{cases}\dfrac{2}{\pi\left(1+x^2\right)} & x>0\\ 0 & \text{otherwise.}\end{cases}$$

What is the expected value of the score in a single shot?

[9] The m.g.f. of a random variable X is given by

$$M_X(t)=\frac{1}{2}e^{-5t}+\frac{1}{6}e^{4t}+\frac{1}{8}e^{5t}+\frac{5}{24}e^{25t}$$

Find the distribution function of the random variable.

[10] Let X be a random variable with $P(X\le 0)=0$ and let $\mu=E(X)$ exists. Show that $P(X\ge 2\mu)\le 0.5$.

[11] Let X be a random variable with $E(X)=3$ and $E\left(X^2\right)=13$, determine a lower bound for $P(-2<X<8)$.

[12] Let X be a random variable with p.m.f.

$$P(X=x)=\begin{cases}1/8 & x=-1,1\\ 6/8 & x=0\\ 0 & \text{otherwise.}\end{cases}$$

Using the p.m.f., show that the bound for Chebychev's inequality cannot be improved.

[13] A communication system consists of n components, each of which will independently function with probability p. The system will be able to operate effectively if at least one-half of its components function.

(a) For what value of p a 5-component system is more likely to operate effectively than a 3-component system?

(b) In general, when is a $(2k+1)$-component system better than a $(2k-1)$-component system?