1

$X_1, \dots X_n$ i.i.d. $U(0,\theta)$

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n}{\theta^n} x^{n-1}, & 0<x<\theta \\ 0, & o/w. \end{cases} \qquad F_{X_{(n)}}(x) = (F(x))^n_{\ 0<x<\theta}$$

$$E X_{(n)} = \frac{n}{\theta^n} \int_0^\theta x^n \, dx$$

$$= \frac{n}{n+1} \theta$$

$$E X_{(n)}^2 = \frac{n}{\theta^n} \int_0^\theta x^{n+1} \, dx$$

$$= \frac{n}{n+2} \theta^2$$

Now $P(|X_{(n)} - \theta| \geq \epsilon) \leq \frac{E(X_{(n)} - \theta)^2}{\epsilon^2}$

$\forall \epsilon > 0$

$$= \frac{E X_{(n)}^2 + \theta^2 - 2\theta E(X_{(n)})}{\epsilon^2}$$

$$= \frac{1}{\epsilon^2} \left[\frac{n}{n+2} \theta^2 + \theta^2 - 2\theta^2 \frac{n}{n+1} \right]$$

$$\to 0 \text{ as } n \to \infty$$

$$\Rightarrow X_{(n)} \xrightarrow{p} \theta \text{ as } n \to \infty$$

can also be proved by calculating the exact prob of

$$P(|X_{(n)} - \theta| \geq \epsilon) = 1 - P(|X_{(n)} - \theta| < \epsilon)$$

$$= 1 - P(\theta - \epsilon < X_{(n)} < \epsilon + \theta)$$

$$= 1 - \left(F_{X_{(n)}}(\theta + \epsilon) - F_{X_{(n)}}(\theta - \epsilon)\right)$$

$$\to 0 \text{ as } n \to \infty.$$

$$\Rightarrow \frac{n}{n+1} X_{(n)} \xrightarrow{p} \theta \quad \text{as } n \to \infty$$

i.e. $\frac{n}{n+1} X_{(n)}$ is a consistent estimator of θ

Further, as $X_{(n)} \xrightarrow{p} \theta$ as $n \to \infty$

$$e^{X_{(n)}} \xrightarrow{p} e^{\theta} \quad \text{as } n \to \infty$$

$\Rightarrow e^{X_{(n)}}$ is a consistent estimator of e^{θ}.

(2) $X_1, \ldots, X_n$ i.i.d. $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$$F_X(x) = \int_{\theta - \frac{1}{2}}^{x} dy = (x - \theta + \tfrac{1}{2}) \qquad \theta - \tfrac{1}{2} < x < \theta + \tfrac{1}{2}$$

$$f_{X_{(1)}}(x) = n(1 - F_X(x))^{n-1} f_X(x)$$

i.e. $$f_{X_{(1)}}(x) = \begin{cases} n(\theta - x + \frac{1}{2})^{n-1}, & \theta - \frac{1}{2} < x < \theta + \frac{1}{2} \\ 0, & \text{o/w.} \end{cases}$$

$$E X_{(1)} = n \int_{\theta - \frac{1}{2}}^{\theta + \frac{1}{2}} x(\theta - x + \tfrac{1}{2})^{n-1} dx = \theta + \tfrac{1}{2} - \frac{n}{n+1}$$

$$E X_{(1)}^2 = (\theta + \tfrac{1}{2})^2 + \frac{n}{n+2} - \frac{n}{n+1}(2\theta + 1)$$

(verify).

$$\left(= n \int_{\theta - \frac{1}{2}}^{\theta + \frac{1}{2}} x^2 (\theta - x + \tfrac{1}{2})^{n-1} dx \right)$$

$$P\left[\left|X_{(1)}-\left(\theta-\tfrac{1}{2}\right)\right| \geq \epsilon\right] \leq \frac{E\left(X_{(1)}-\left(\theta-\frac{1}{2}\right)\right)^2}{\epsilon^2}$$

$$\text{r.h.s} = \frac{1}{\epsilon^2}\left[E\left(X_{(1)}^2\right)+\left(\theta-\tfrac{1}{2}\right)^2-2\left(\theta-\tfrac{1}{2}\right)E\left(X_{(1)}\right)\right]$$

$$=\frac{1}{\epsilon^2}\left[\left\{\left(\theta+\tfrac{1}{2}\right)^2+\frac{n}{n+2}-\frac{n}{n+1}(2\theta+1)\right\}+\left(\theta-\tfrac{1}{2}\right)^2 - 2\left(\theta-\tfrac{1}{2}\right)\left(\theta+\tfrac{1}{2}-\frac{n}{n+1}\right)\right]$$

$$\rightarrow \frac{1}{\epsilon^2}\left[\left\{\left(\theta+\tfrac{1}{2}\right)^2+1-(2\theta+1)\right\}+\left(\theta-\tfrac{1}{2}\right)^2-2\left(\theta-\tfrac{1}{2}\right)\left(\theta-\tfrac{1}{2}\right)\right] \quad \text{as } n\to\infty$$

$$=0$$

$$\Rightarrow P\left[\left|X_{(1)}-\left(\theta-\tfrac{1}{2}\right)\right|\geq\epsilon\right]\longrightarrow 0 \quad \text{as } n\to\infty$$

$$\Rightarrow X_{(1)} \xrightarrow{p} \theta-\tfrac{1}{2} \quad -(1)$$

We can similarly prove that

$$X_{(n)} \xrightarrow{p} \theta+\tfrac{1}{2} \quad -(2)$$

Combining (1) & (2), we get.

$$\frac{X_{(1)}+X_{(n)}}{2} \xrightarrow{p} \theta$$

$\Rightarrow \dfrac{X_{(1)}+X_{(n)}}{2}$ is a consistent estimator for θ

Also, $X_{(1)}+\frac{1}{2}$ is a consistent estimator for θ (from (1))

& $X_{(n)}-\frac{1}{2}$ is a consistent estimator for θ (from (2)).

(3)

$X_1, \ldots X_n$ i.i.d. $f_X(x) = \begin{cases} \frac{1}{2}(1+\theta x), & -1 < x < 1 \\ 0, & o/w \end{cases}$

$$E(X) = \frac{1}{2}\int_{-1}^{1}(1+\theta x)\,dx = \frac{\theta}{3}$$

$\Rightarrow X_1, \ldots X_n$ are i.i.d. with $E(X_1) = \frac{\theta}{3}$.

By Khintchine's WLLN

$$\frac{1}{n}\sum_{1}^{n} X_i \xrightarrow{p} E(X_1)$$

i.e. $\bar{X} \xrightarrow{p} \frac{\theta}{3} \Rightarrow 3\bar{X} \xrightarrow{p} \theta$

$\Rightarrow 3\bar{X}$ is a consistent estimator for θ.

(4)

$X_1, \ldots X_n$ are i.i.d. $P(\theta)$

$E(X_i) = \theta \quad \forall i = 1(1)n$

By WLLN $\quad \bar{X}_n \xrightarrow{p} \theta$

$$\Rightarrow g(\bar{X}_n) \xrightarrow{p} g(\theta)$$

$$\Rightarrow \bar{X}_n^3\left(3\sqrt{\bar{X}_n} + \bar{X}_n + 12\right) \xrightarrow{p} \theta^3\left(3\sqrt{\theta} + \theta + 12\right)$$

$\Rightarrow \bar{X}_n^3\left(3\sqrt{\bar{X}_n} + \bar{X}_n + 12\right)$ is a consistent estimator for

$\theta^3\left(3\sqrt{\theta} + \theta + 12\right)$.

(5)

$X_1, \ldots, X_n$ are i.i.d. $G(\alpha, \beta)$

α is known constant and $\beta > 0$ is unknown

By WLLN

$$\frac{1}{n} \sum_{1}^{n} X_i \xrightarrow{p} E(X_1) (= \alpha\beta)$$

i.e. $\frac{1}{n} \sum_{1}^{n} X_i \xrightarrow{p} \alpha\beta$

i.e. $\frac{1}{n\alpha} \sum_{1}^{n} X_i \xrightarrow{p} \beta$

$\Rightarrow \frac{1}{n\alpha} \sum_{i=1}^{n} X_i$ is a consistent estimator for β.

(6)

(a) $$L(\theta) = \frac{e^{-n\theta} \, \theta^{\Sigma x_i}}{\Pi x_i!}$$

log likelihood: $$\lambda(\theta) = -n\theta + \Sigma x_i \log\theta - \log(\Pi x_i!)$$

$$\frac{\partial \lambda(\theta)}{\partial \theta} = -n + \frac{\Sigma x_i}{\theta} = 0$$

$$\Rightarrow \hat{\theta} = \bar{x}$$

$$\left.\frac{\partial^2 \lambda(\theta)}{\partial \theta^2}\right|_{\hat{\theta}} = -\left.\frac{\Sigma x_i}{\theta^2}\right|_{\hat{\theta}} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

(b) $$L(\theta) = \theta^n \left(\frac{1}{\Pi x_i}\right) (\Pi x_i)^\theta \leftarrow \text{int of range (indep of } \theta)$$

$$\lambda(\theta) = n\log\theta + \theta\, \Sigma \log x_i + K \leftarrow \text{indep of } \theta$$

$$\frac{\partial \lambda(\theta)}{\partial \theta} = \frac{n}{\theta} + \Sigma \log x_i = 0$$

$$\Rightarrow \hat{\theta} = -\frac{n}{\Sigma \log x_i}$$

$$\left.\frac{\partial^2 \lambda(\theta)}{\partial \theta^2}\right|_{\hat{\theta}} = -\left.\frac{n}{\theta^2}\right|_{\hat{\theta}} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = -\frac{n}{\Sigma \log X_i}$$

(c) $L(\theta) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i}$ c ← inf on range (indep of θ)

$l(\theta) = -n \log\theta - \frac{1}{\theta}\sum x_i + K$ ← indep of θ

$$\frac{\partial l(\theta)}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum x_i = 0$$

$$\hat{\theta} = \bar{x}$$

$$\left.\frac{\partial^2 l(\theta)}{\partial\theta^2}\right|_{\hat{\theta}} = \left.\frac{n}{\theta^2} - 2\frac{1}{\theta^3}\sum x_i\right|_{\hat{\theta}}$$

$$= \frac{n}{\hat{\theta}^2} - \frac{2}{\hat{\theta}^3} n\hat{\theta} = \frac{n}{\hat{\theta}^2} - \frac{2n}{\hat{\theta}^2} = -\frac{n}{\hat{\theta}^2} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

(d) $L(\theta) = \frac{1}{2^n} e^{-\sum|x_i-\theta|}$

$l(\theta) = K - \sum_1^n |x_i - \theta|$

maximization of $l(\theta)$ (or $L(\theta)$) w.r.t. θ is equiv to

minimization of $\sum_1^n |x_i - \theta|$

$$\Rightarrow \hat{\theta} = \text{median}(x_1, \ldots, x_n)$$

$$\Rightarrow \hat{\theta}_{MLE} = \text{median}(X_1, \ldots, X_n)$$

(7) $L(\underset{\sim}{\theta}) = \dfrac{1}{\theta_2^n} e^{-\frac{1}{\theta_2}\sum(x_i - \theta_1)} \; I(\theta_1, x_{(1)})$

$\underset{\sim}{\theta} = (\theta_1, \theta_2)'$ $\qquad I(a,b) = \begin{cases} 1, & a \le b \\ 0, & \text{o/w} \end{cases}$

$$L(\underset{\sim}{\theta}) = \frac{1}{\theta_2^n} e^{-\frac{1}{\theta_2}\sum x_i} e^{\frac{n\theta_1}{\theta_2}} \; I(\theta_1, x_{(1)})$$

This is a 2-parameter MLE setup.

Realize that for a fixed θ_2, $L(\theta_1, \theta_2)$ is maximized at $\hat{\theta}_1 = x_{(1)}$ (as for fixed θ_2, $L(\theta_1, \theta_2)$ is $\uparrow \theta_1$)

Note that $\hat{\theta}_1 = x_{(1)}$ is indep of fixed level of θ_2

and $\hat{\theta}_1 = x_{(1)}$ would maximise $L(\underset{\sim}{\theta}) \; \forall \; \theta_2$

$$\Rightarrow \hat{\theta}_{1(MLE)} = X_{(1)}$$

Consider now,

$$L(\hat{\theta}_1, \theta_2) = \frac{1}{\theta_2^n} e^{-\frac{1}{\theta_2}\sum(x_i - x_{(1)})}$$

$$l(\hat{\theta}_1, \theta_2) = -n \log \theta_2 - \frac{1}{\theta_2}\sum(x_i - x_{(1)})$$

$$\frac{\partial l(\hat{\theta}_1, \theta_2)}{\partial \theta_2} = -\frac{n}{\theta_2} + \frac{1}{\theta_2^2}\sum(x_i - x_{(1)}) = 0$$

$$\Rightarrow \hat{\theta}_2 = \frac{1}{n}\sum(x_i - x_{(1)})$$

$$\left.\frac{\partial^2 l}{\partial \theta_2^2}\right|_{\hat{\theta}} < 0 \Rightarrow \hat{\theta}_{2(MLE)} = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})$$

(8) $L(\underset{\sim}{\theta}) = \dfrac{\lambda^{n\alpha}}{(\Gamma\alpha)^n} e^{-\lambda \Sigma x_i} \left(\Pi x_i\right)^{\alpha-1}$

$\underset{\sim}{\theta} = (\alpha, \lambda)$

$l(\underset{\sim}{\theta}) = n\alpha \log \lambda - n \log \Gamma\alpha + (\alpha-1)\Sigma \log x_i - \lambda \Sigma x_i$

likelihood eq^{n}s :

$$\frac{\partial \log L}{\partial \lambda} = \frac{n\alpha}{\lambda} - \Sigma x_i$$

$$\frac{\partial \log L}{\partial \alpha} = n \log \lambda - n \frac{(\Gamma\alpha)'}{(\Gamma\alpha)} + \Sigma \log x_i$$

$$\frac{\partial \log L}{\partial \lambda} = 0 \quad \Rightarrow \quad \lambda = \frac{n\alpha}{\Sigma x_i} = \frac{\alpha}{\bar{x}}$$

$\dfrac{\partial \log L}{\partial \alpha} = 0$ gives

$$n \log\left(\frac{\alpha}{\bar{x}}\right) - n \frac{(\Gamma\alpha)'}{(\Gamma\alpha)} + \Sigma \log x_i = 0 \quad \text{---} (*)$$

Solve $(*)$ by numerical method to get $\hat{\alpha}_{MLE}$

$$\& \quad \hat{\lambda}_{MLE} = \frac{\hat{\alpha}_{MLE}}{\bar{X}}$$

(9) $\underset{\sim}{\theta} = (\mu, \sigma)'$

$$L(\underline{\theta}) = \left(\frac{1}{2\sqrt{3}\sigma}\right)^n I_{(\mu-\sqrt{3}\sigma,\, x_{(1)})}\, I_{(x_{(n)},\, \mu+\sqrt{3}\sigma)}$$

Note that $L(\underline{\theta}) = \left(\frac{1}{2\sqrt{3}\sigma}\right)^n$ if $\mu - \sqrt{3}\sigma \le x_{(1)}$ and $x_{(n)} \le \mu + \sqrt{3}\sigma$

$= 0$ o/w

Now $\mu - \sqrt{3}\sigma \le x_{(1)}$ & $x_{(n)} \le \mu + \sqrt{3}\sigma$

$\Rightarrow$ $\mu \le x_{(1)} + \sqrt{3}\sigma$ & $x_{(n)} - \sqrt{3}\sigma \le \mu$

i.e. $x_{(n)} - \sqrt{3}\sigma \le \mu \le x_{(1)} + \sqrt{3}\sigma$

Thus, for a given σ, $L(\underline{\theta})$ is maximized w.r.t μ if

$$\mu \in [x_{(n)} - \sqrt{3}\sigma,\ x_{(1)} + \sqrt{3}\sigma]$$

$\Rightarrow$ Any value of μ in the above interval is an MLE of μ

In particular $\dfrac{(X_{(n)} - \sqrt{3}\sigma) + (X_{(1)} + \sqrt{3}\sigma)}{2} = \dfrac{X_{(n)} + X_{(1)}}{2}$

Choice is not unique clearly

is MLE of μ

Since the MLE of μ is indep of σ, it's MLE of μ $\forall \sigma$

$$\Rightarrow \hat{\mu}_{MLE} = \frac{X_{(n)} + X_{(1)}}{2}$$

Further, $L(\hat{\mu}, \sigma)$ is maximized w.r.t. σ if σ is minimum.

Observe that $\sqrt{3}\sigma \ge \mu - x_{(1)}$ & $\sqrt{3}\sigma \ge x_{(n)} - \mu$

and at the above MLE of μ

$$\sqrt{3}\sigma \ge \frac{x_{(n)} - x_{(1)}}{2}$$

$$\Rightarrow \hat{\sigma}_{MLE} = \frac{X_{(n)} - X_{(1)}}{2\sqrt{3}}$$

(10) $L(\theta) = 1$ if $\theta - \frac{1}{2} \le x_{(1)}$ and $x_{(n)} \le \theta + \frac{1}{2}$

$= 0$ o/w

$\Rightarrow L(\theta)$ is maximized w.r.t. θ if

$$\theta - \frac{1}{2} \le x_{(1)} \;\&\; x_{(n)} \le \theta + \frac{1}{2}$$

i.e. if $x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}$

$\Rightarrow L(\theta)$ is maximized w.r.t. θ $\forall$ values of θ satisfying

$$x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}$$

$\Rightarrow$ Any statistic $U(X_1, \ldots, X_n) \ni$

$x_{(n)} - \frac{1}{2} \le u(x_1, \ldots, x_n) \le x_{(1)} + \frac{1}{2}$ is an MLE of θ

In particular $\frac{X_{(1)} + X_{(n)}}{2}$ (mid pt) is an MLE of θ

In general,

$\alpha \left(X_{(1)} + \frac{1}{2} \right) + (1-\alpha) \left(X_{(n)} - \frac{1}{2} \right) \; \forall \; 0 \le \alpha \le 1$

is an MLE of θ

With $\alpha = \frac{3}{4}$, we get

$\frac{3}{4} \left(X_{(1)} + \frac{1}{2} \right) + \frac{1}{4} \left(X_{(n)} - \frac{1}{2} \right)$ as MLE of θ

(11) X : r.v. denoting lifetime of the component

p.d.f. $f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x > 0 \\ 0, & o/w \end{cases}$

Define the r.v.

$$Y_i = \begin{cases} 1, & \text{if } i^{th} \text{ component has lifetime} < 100 \text{ hrs} \\ 0, & o/w. \end{cases}$$

$$P(Y_i = 1) = P(X < 100) = \frac{1}{\lambda} \int_0^{100} e^{-x/\lambda} dx = (1 - e^{-100/\lambda})$$

$X_1, \ldots, X_n$ are i.i.d.

$\Rightarrow Y_1, \ldots, Y_n$ are i.i.d.

$$Y_i \sim B(1, \underbrace{(1 - e^{-100/\lambda})}_{= \theta, \text{ say}})$$

$$\hat{\theta}_{MLE} = \bar{Y}$$

Note that $\theta = 1 - e^{-100/\lambda}$

$$\Rightarrow e^{-100/\lambda} = 1 - \theta$$

$$\Rightarrow \lambda = -\frac{100}{\log(1-\theta)} = g(\theta)$$

Inv of MLE: MLE of $g(\theta)$ is $g(\hat{\theta}_{MLE})$

$$\Rightarrow \hat{\lambda}_{MLE} = -\frac{100}{\log(1-\hat{\theta}_{MLE})}$$

From data $\bar{x} = \frac{3}{10}$ $\Rightarrow$ ML estimate of λ is $\left(-\frac{100}{\log(7/10)}\right)$

(12)

Let X denote the r.v. denoting number of sales in a day

$$X \sim P(\mu) \quad \mu > 0$$

Define
$$Y_i = \begin{cases} 1, & \text{if 0 sales on day } i \\ 0, & \text{o/w} \end{cases}$$

$$P(Y_i = 1) = P(X = 0) = e^{-\mu}$$

$X_1, \ldots, X_{30}$ i.i.d. $P(\mu)$

$\Rightarrow Y_1, \ldots, Y_{30}$ are i.i.d. $B(1, \underset{\theta \text{ say}}{\underset{\|}{e^{-\mu}}})$

$$\hat{\theta}_{MLE} = \bar{Y}$$

Note that $\theta = e^{-\mu} \Rightarrow \mu = -\log \theta = g(\theta)$

$$\Rightarrow \quad \hat{\mu}_{MLE} = -\log \hat{\theta}_{MLE}.$$

$\Rightarrow$ ML estimate of μ from the data : $\left(-\log(20/30)\right)$

— x —