

# MSO201: PROBABILITY & STATISTICS

## Problem Set #3

- [1] Let  $X$  be a random variable defined on  $(\Omega, \mathcal{F}, \mathcal{P})$ . Show that the following are also random variables;  
 (a)  $|X|$ , (b)  $X^2$  and (c)  $\sqrt{X}$ , given that  $\{X < 0\} = \emptyset$ .

- [2] Let  $\Omega = [0,1]$  and  $\mathcal{F}$  be the Borel  $\sigma$ -field of subsets of  $\Omega$ . Define  $X$  on  $\Omega$  as follows:

$$X(\omega) = \begin{cases} \omega & \text{if } 0 \leq \omega \leq 1/2 \\ \omega - 1/2 & \text{if } 1/2 < \omega \leq 1 \end{cases}$$

Show that  $X$  defined above is a random variable.

- [3] Let  $\Omega = \{1,2,3,4\}$  and  $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2,3,4\}\}$  be a  $\sigma$ -field of subsets of  $\Omega$ . Verify whether  $X(\omega) = \omega + 1; \forall \omega \in \Omega$ , is a random variable with respect to  $\mathcal{F}$ .

- [4] Let a card be selected from an ordinary pack of playing cards. The outcome  $\omega$  is one of these 52 cards. Define  $X$  on  $\Omega$  as:

$$X(\omega) = \begin{cases} 4 & \text{if } \omega \text{ is an ace} \\ 3 & \text{if } \omega \text{ is a king} \\ 2 & \text{if } \omega \text{ is a queen} \\ 1 & \text{if } \omega \text{ is a jack} \\ 0 & \text{otherwise.} \end{cases}$$

Show that  $X$  is a random variable. Further, suppose that  $P(\cdot)$  assigns a probability of  $1/52$  to each outcome  $\omega$ . Derive the distribution function of  $X$ .

- [5] Let  $F(x) = \begin{cases} 0 & \text{if } x < -1 \\ (x+2)/4 & \text{if } -1 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$

Show that  $F(\cdot)$  is a distribution function. Sketch the graph of  $F(x)$  and compute the probabilities  $P(-1/2 < X \leq 1/2)$ ,  $P(X=0)$ ,  $P(X=1)$  and  $P(-1 \leq X < 1)$ . Further, obtain the decomposition  $F(x) = \alpha F_d(x) + (1-\alpha)F_c(x)$ ; where,  $F_d(x)$  and  $F_c(x)$  are purely discrete and purely continuous distribution functions, respectively.

- [6] Which of the following functions is(are) distribution functions?

$$(a) F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1/2 \\ 1, & x > 1/2. \end{cases}; (b) F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & x \geq 0 \end{cases}; (c) F(x) = \begin{cases} 0 & x \leq 1 \\ 1 - 1/x & x > 1. \end{cases}$$

- [7] Let  $F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - \frac{2}{3}e^{-x/3} - \frac{1}{3}e^{-[x/3]} & \text{if } x > 0 \end{cases}$

where,  $[x]$  is the largest integer  $\leq x$ . Show that  $F(\cdot)$  is a distribution function and compute  $P(X > 6)$ ,  $P(X=5)$  and  $P(5 \leq X \leq 8)$ .

- [8] The distribution function of a random variable  $X$  is given by

$$F(x) = \begin{cases} 0, & x < -2, \\ 1/3, & -2 \leq x < 0, \\ 1/2, & 0 \leq x < 5, \\ 1/2 + (x-5)^2/2, & 5 \leq x < 6, \\ 1, & x \geq 6. \end{cases}$$

Find  $P(-2 \leq X < 5)$ ,  $P(0 < X < 5.5)$  and  $P(1.5 < X \leq 5.5 | X > 2)$ .

[9] Prove that if  $F_1(\cdot), \dots, F_n(\cdot)$  are  $n$  distribution functions, then  $F(x) = \sum_{i=1}^n \alpha_i F_i(x)$  is also a

distribution function for any  $(\alpha_1, \dots, \alpha_n)$ , such that  $\alpha_i \geq 0$  and  $\sum_{i=1}^n \alpha_i = 1$ .

[10] Suppose  $F_1$  and  $F_2$  are distribution functions. Verify whether  $G(x) = F_1(x) + F_2(x)$  is also a distribution function.

[11] Find the value of  $\alpha$  and  $k$  so that  $F$  given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \alpha + k e^{-x^2/2} & \text{if } x > 0 \end{cases}$$

is distribution function of a continuous random variable.

[12] Let  $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ (x+2)/8 & \text{if } 0 \leq x < 1 \\ (x^2+2)/8 & \text{if } 1 \leq x < 2 \\ (2x+c)/8 & \text{if } 2 \leq x \leq 3 \\ 1 & \text{if } x > 3. \end{cases}$

Find the value of  $c$  such that  $F$  is a distribution function. Using the obtained value of  $c$ , find the decomposition  $F(x) = \alpha F_d(x) + (1-\alpha)F_c(x)$ ; where,  $F_d(x)$  and  $F_c(x)$  are purely discrete and purely continuous distribution functions, respectively.

[13] Suppose  $F_X$  is the distribution function of a random variable  $X$ . Determine the distribution function of (a)  $X^+$  and (b)  $|X|$ . Where,

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X < 0 \end{cases}$$

[14] The convolution  $F$  of two distribution functions  $F_1$  and  $F_2$  is defined as follows;

$$F(x) = \int_{-\infty}^{\infty} F_1(x-y) dF_2(y); x \in \mathcal{R},$$

and is denoted by  $F = F_1 \star F_2$ . Show that  $F$  is also a distribution function.

[15] Which of the following functions are probability mass functions?

(a)  $f(x) = \begin{cases} (x-2)/2 & \text{if } x=1,2,3,4 \\ 0 & \text{otherwise.} \end{cases}$ ; (b)  $f(x) = \begin{cases} (e^{-\lambda} \lambda^x)/x! & \text{if } x=0,1,2,3,4,\dots \\ 0 & \text{otherwise.} \end{cases}$

where,  $\lambda > 0$ .

$$(c) f(x) = \begin{cases} (e^{-\lambda} \lambda^x) / x! & \text{if } x = 1, 2, 3, 4, \dots \\ 0 & \text{otherwise.} \end{cases}$$

where,  $\lambda > 0$ .

[16] Find the value of the constant  $c$  such that  $f(x) = (1 - c)c^x$ ;  $x = 0, 1, 2, 3, \dots$  defines a probability mass function.

[17] Let  $X$  be a discrete random variable taking values in  $X = \{-3, -2, -1, 0, 1, 2, 3\}$  such that

$$P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3) \quad \text{and}$$

$$P(X < 0) = P(X = 0) = P(X > 0) \text{ Find the distribution function of } X.$$

[18] A battery cell is labeled as good if it works for at least 300 days in a clock, otherwise it is labeled as bad. Three manufacturers,  $A, B$  and  $C$  make cells with probability of making good cells as 0.95, 0.90 and 0.80 respectively. Three identical clocks are selected and cells made by  $A, B$  and  $C$  are used in clock numbers 1, 2 and 3 respectively. Let  $X$  be the total number of clocks working after 300 days. Find the probability mass function of  $X$  and plot the corresponding distribution function.

[19] Prove that the function  $f_\theta(x) = \begin{cases} \theta^2 x e^{-\theta x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

defines a probability density function for  $\theta > 0$ . Find the corresponding distribution function and hence compute  $P(2 < X < 3)$  and  $P(X > 5)$ .

[20] Find the value of the constant  $c$  such that the following function is a probability density function.

$$f_\lambda(x) = \begin{cases} c(x+1)e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where,  $\lambda > 0$ . Obtain the distribution function of the random variable associated with probability density function  $f_\lambda(x)$ .

[21] Show that  $f(x) = \begin{cases} x^2/18 & \text{if } -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

defines a probability density function. Find the corresponding distribution function and hence find

$$P(|X| < 1) \text{ and } P(X^2 < 9)$$