2 A: The Shallow Water Equations

2.1 Surface motions on shallow water

Consider two-dimensional (x-z) motions on a nonrotating, shallow body of water, of uniform density ρ, as shown in Fig. 1 below. The flow is assumed

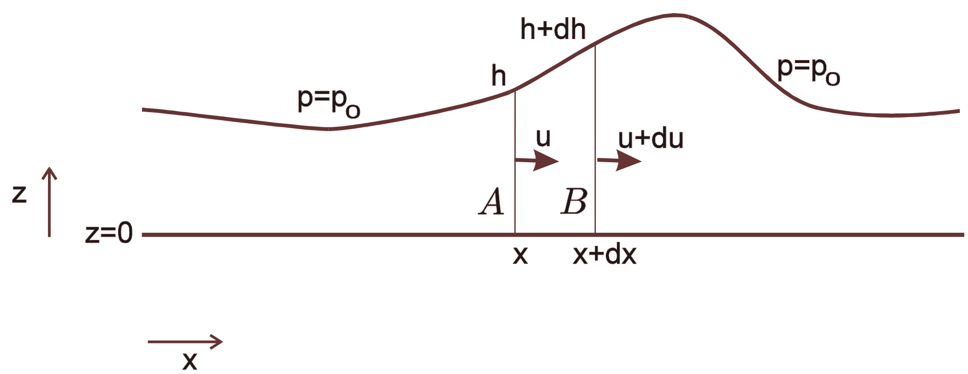

Figure 1: The shallow water system.

to be inviscid and independent of the spatial dimension y (into the paper). We shall *assume* that the water is so shallow that the flow velocity $u(x,t)$ is constant with depth. (We'll see later under what conditions this is reasonable; for now, let's just assume it to be true.) At the free surface, located at height $z = h(x,t)$, pressure is equal to atmospheric pressure p_0, assumed constant and uniform.

Consider the volume of water bounded by the vertical surfaces A and B in the figure. These surfaces are located at x and $x + dx$ respectively. The mass of this volume, per unit length in y, is just $dm = \rho h\, dx$. Now, mass cannot be created or destroyed within the volume, so the only way it can change is because of the **fluxes** of mass across the interfaces A and B. Consider Fig. 2. Since the velocity at A is u, in a time interval dt all the fluid between A' and A passes across A, where the distance between A' and A is $dx = u\, dt$. Thus, the area (i.e., the volume per unit length in y) passing across A in this time is $hu\, dt$, and so the mass (per unit length in y) is $\rho hu\, dt$. Therefore the **mass flux**—the mass crossing A per unit time, per unit length in y—is $\rho u(x)h(x)$. The mass flux across interface B is $\rho u(x + dx)h(x + dx)$ (directed toward positive x, out of the volume). Therefore the rate of accumulation of

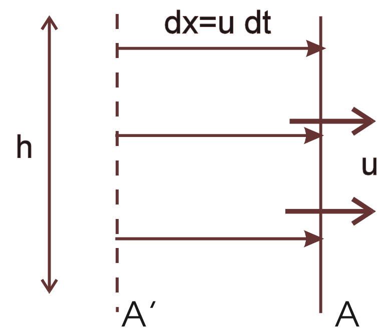

Figure 2: Illustrating the flux of mass across the interface A.

mass (per unit length in y) within the volume defined by AB is

$$\begin{aligned}\frac{\partial m}{\partial t} &= \rho u(x)h(x) - \rho u(x+dx)h(x+dx)\\ &= -\rho\frac{\partial(uh)}{\partial x}\,dx\ .\end{aligned}$$

Since $m = \rho h\,dx$, the factors of $\rho\,dx$ cancel, leaving us with

$$\frac{\partial h}{\partial t} = -\frac{\partial(uh)}{\partial x}\ .$$

Differentiating the RHS by parts and rearranging, we arrive at the **equation of continuity**:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = -h\frac{\partial u}{\partial x}\ . \tag{1}$$

This equation expresses the local rate of change of surface height in terms of two contributions:

(i) by advection of height $-u\partial h/\partial x$

(ii) by **volume convergence** $-h\partial u/\partial x$.

These two effects are depicted (both in a sense to increase h locally) in Fig. 3.

Now, in a similar way, consider the momentum balance of the water in the volume. We shall need to know the distribution of pressure p within the water. To do this, we use the principle of **hydrostatic balance**, which

Figure 3: Contributions to $\partial h/\partial t$.

states that the pressure increases with depth according to the overhead mass per unit area. Specifically (see Fig. 1), the pressure at any depth $h-z$ below the surface is related to surface pressure by

$$p(z,t) = p_0 + \int_z^h \rho g \; dz = p_0 + \rho g(h-z) \; , \tag{2}$$

where g is the acceleration due to gravity (and both ρ and g are constants). The second term on the RHS of (2) simply represents the mass of water per unit area above level z. Newton's law of motion applied to the volume gives

$$m\frac{du}{dt} = F \; ,$$

where F is the net force (per unit length in y) applied to the volume. Since we are assuming friction to be negligible, the only such forces acting are pressure forces, which are as depicted in Fig. 4[1]. That acting on the volume across interface A (tending to accelerate the volume in the positive x direction) is equal to a force, per unit length in y, of $F_1 = \int_0^h p(x,z) \; dz$; that acting across interface B (tending to accelerate the volume in the negative x direction) is $F_2 = \int_0^h p(x+dx,z) \; dz$. However, there is a third component of the net force acting on the free surface, represented in the figure as F_s. Atmospheric pressure exerts a force, normal to the surface, of $p_0 dl$ per unit length in y,

[1]We are in fact neglecting here one contribution to the force felt at the surface, that due to surface tension. Surface tension effects are negligible for motions of large horizontal scale (typically a few cm.), so this analysis is restricted to these large scales. Small-scale motions for which surface tension effects are important are known as *capillary waves*.

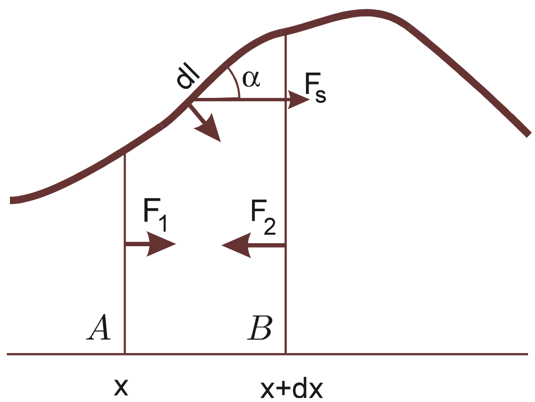

Figure 4: Forces acting on the fluid volume.

where dl is the volume's width along the surface. Because the surface is tilted, this force has a nonzero component $p_0 dl \sin\alpha$ acting in the positive x-direction, where α is the angle of the interface. Since $dl = dx/\cos\alpha$, this contribution to the x-force is

$$F_s = p_0 \frac{\partial h}{\partial x} dx$$

(since $\tan\alpha = \partial h/\partial x$). Therefore the net force on the volume, per unit length in y, is

$$F = p_0 \frac{\partial h}{\partial x} dx + \int_0^h p(x,z)\, dz - \int_0^h p(x+dx,z)\, dz \quad .$$

But, from (2), we have

$$\begin{aligned} \int_0^h p\, dz &= \int_0^h p_0\, dz + \rho g \int_0^h (h-z)\, dz \ , \\ &= p_0 h + \frac{1}{2}\rho g h^2 \ . \end{aligned}$$

So

$$\int_0^h p(x,z)\,dz - \int_0^h p(x+dx,z)\,dz = p_0 h(x) - p_0 h(x+dx) + \frac{1}{2}\rho g h^2(x) - \frac{1}{2}\rho g h^2(x+dx) = -p_0 \frac{\partial h}{\partial x} dx - \rho g h \frac{\partial h}{\partial x} dx \ .$$

Therefore the acceleration of the volume is given by

$$m\frac{du}{dt} = F = -\rho g h \frac{\partial h}{\partial x} dx \ .$$

Note that this is independent of surface pressure p_0 (the terms involving it have cancelled): the net force on the volume is entirely due to the pressure gradients within the water which, because of hydrostatic balance, are entirely due to gradients in surface height. Now, using our expression $m = \rho h\ dx$, the preceding equation gives us (cancelling the factors $\rho h\ dx$)

$$\frac{du}{dt} = -g\frac{\partial h}{\partial x}$$

Here the derivative d/dt is the *material derivative*—this tells us how the velocity of the marked volume changes *as it moves around.* We need to convert this into a form that tells us how u changes in fixed coordinates. Since $u = u(x,t) \equiv dx/dt$, we simply apply the chain rule to write

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

and thus to write our **equation of motion** in final form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} \ . \tag{3}$$

Like (1), this links the local rate of change of velocity to two terms:

(i) the pressure gradient term and

(ii) the *advection of momentum.*

The two equations (1) and (3) give us two *predictive* equations in the two unknowns $u(x,t)$ and $h(x,t)$, and so in principle tell us all we need to know to determine how this system will evolve, given initial and boundary conditions. The equations are *nonlinear* (through the advective terms) and so, despite their simplicity, have complex properties in general.

2.2 With rotation

It is straightforward to extend these equations to two horizontal dimensions (x, y) with vector velocity $\mathbf{u} = (u, v)$, and to add rotation we simply add the Coriolis term to momentum equation (3), when our set of equations becomes, in vector form,

$$\begin{aligned} \frac{d\mathbf{u}}{dt} - f\hat{\mathbf{z}} \times \mathbf{u} &= -g\boldsymbol{\nabla}h \\ \frac{dh}{dt} &= -h\boldsymbol{\nabla} \cdot \mathbf{u} \end{aligned}$$

where f is the Coriolis parameter. Alternatively, in component form,

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} &= -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \ . \end{aligned}$$