Abdullah Gul University

Math-301 (Probability & Statistics)

Fall 2022, QUIZ - VI

Name & Surname:
ID Number:

Q 1. (30 pt.)	Let X has the following discrete probability mass function: P(0)=0.2, P(1)=0.5, P(2)=0.3. Then, calculate $E[X^2]$.

SOLUTION:

$$E[g(x)] = \sum_{i=0}^{n=2} g(x_i)f(x_i) = 0^2(0.2) + 1^2(0.5) + 2^2(0.3) = 1.7$$

Q 2.

(30 pt.)

A random variable X, which represents the weight (in ounces) of an article, has density function given by f(x),

$$f(x) = \begin{cases} x - 8 & 8 \leq x \leq 9 \\ 10 - x & 9 < x \leq 10 \\ 0 & otherwise \end{cases}$$

Find the variance of the random variable X.

SOLUTION:

$$Var(X) = E[(X-\mu)^2] = E[X^2] - 2\mu E[X] + E[\mu^2] = E[X^2] - 2\mu\mu + \mu^2 = E[X^2] - \mu^2$$

$$\mu = E[X] = \int xf(x)dx = \int_8^9 x(x-8)dx + \int_9^{10} x(10-x)dx$$

$$\begin{aligned} \mu = E[X] \quad &= \frac{x^3}{3} - 4x^2\Big|_8^9 + 5x^2 - \frac{x^3}{3}\Big|_9^{10} \\ &= \frac{9^3}{3} - 4(9^2) - \frac{8^3}{3} + 4(8^2) + 5(10^2) - \frac{10^3}{3} - 5(9^2) + \frac{9^3}{3} \\ &= 243 - 324 - \frac{512}{3} + 256 + 500 - \frac{1000}{3} - 405 + 243 = 9 \end{aligned}$$

$$E[X^2] = \int x^2 f(x)dx = \int_8^9 x^2(x-8)dx + \int_9^{10} x^2(10-x)dx$$

$$\begin{aligned} E[X^2] &= \frac{x^4}{4} - \frac{8x^3}{3}\Big|_8^9 + \frac{10x^3}{3} - \frac{x^4}{4}\Big|_9^{10} \\ &= \frac{9^4}{4} - \frac{8(9^3)}{3} - \left(\frac{8^4}{4} - \frac{8(8^3)}{3}\right) + \frac{10(10^3)}{3} - \frac{10^4}{4} - \left(\frac{10(9^3)}{3} - \frac{9^4}{4}\right) \\ &= \frac{6561}{4} - 1944 - 1024 + \frac{4096}{3} + \frac{10000}{3} - 2500 - 2430 + \frac{6561}{4} \\ &= 81.16 \end{aligned}$$

$$Var(X) = 81.16 - 9^2 = 0.16$$

Q 3. Let X and Y be jointly continuous random variables with joint PDF below;

(40 pt.)

$$f(x,y) = \begin{cases} 2 & x, y \geq 0, x + y < 1 \\ 0 & otherwise \end{cases}$$

So, find Cov(X,Y).

<u>Hint:</u> You can find the mean of X and Y by considering the marginal distributions of X alone and of Y alone.

SOLUTION:

We know that;

$x \geq 0,$

$y \geq 0,$

$x + y < 1$

Then, we can write that;

$\mathbf{0 \leq y < 1 - x}$

$\mathbf{0 \leq x < 1 - y}$

Then, for $0 \leq x < 1$, we can find the marginal distributions of X alone as follows;

$$g(x) = \int_{0}^{1-x} f(x,y)dy = 2y|_0^{1-x} = 2(1-x) - 2(0) = 2(1-x)$$

Then, for $0 \leq y < 1$, we can find the marginal distributions of Y alone as follows;

$$h(y) = \int_{0}^{1-y} f(x,y)dx = 2x|_0^{1-y} = 2(1-y) - 2(0) = 2(1-y)$$

To calculate the mean of X with g(x);

$$\mu_x = \int_0^1 xg(x) = \int_0^1 2x(1-x)dx = x^2 - \frac{2x^3}{3}\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

To calculate the mean of Y with h(y);

$$\mu_y = \int_0^1 yh(y) = \int_0^1 2y(1-y)dy = y^2 - \frac{2y^3}{3}\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

Now, let us calculate the Cov(X,Y):

$$
\begin{aligned}
Cov(X,Y) = \sigma_{XY} &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)\left(y-\mu_y\right)f(x,y)dxdy \\
&= \int_{0}^{1}\int_{0}^{1-x}\left(x-\frac{1}{3}\right)\left(y-\frac{1}{3}\right)2dxdy \\
&= \int_{0}^{1}\left(x-\frac{1}{3}\right)\left[(1-x)^2-\frac{2(1-x)}{3}\right]dx = \left.\frac{3x^4}{12}-\frac{5x^3}{9}+\frac{7x^2}{18}-\frac{x}{9}\right|_{0}^{1} = \frac{-1}{36}
\end{aligned}
$$