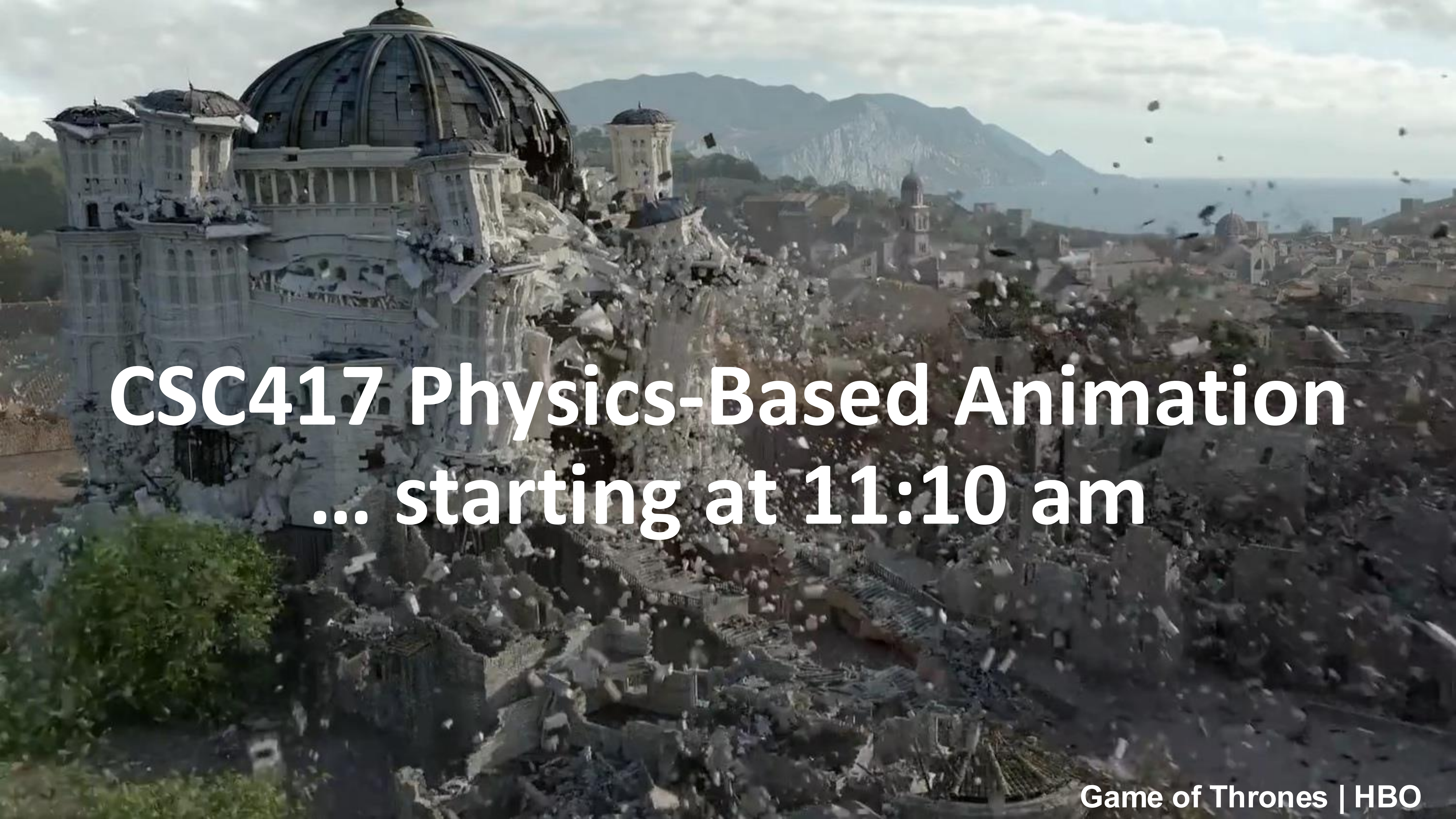
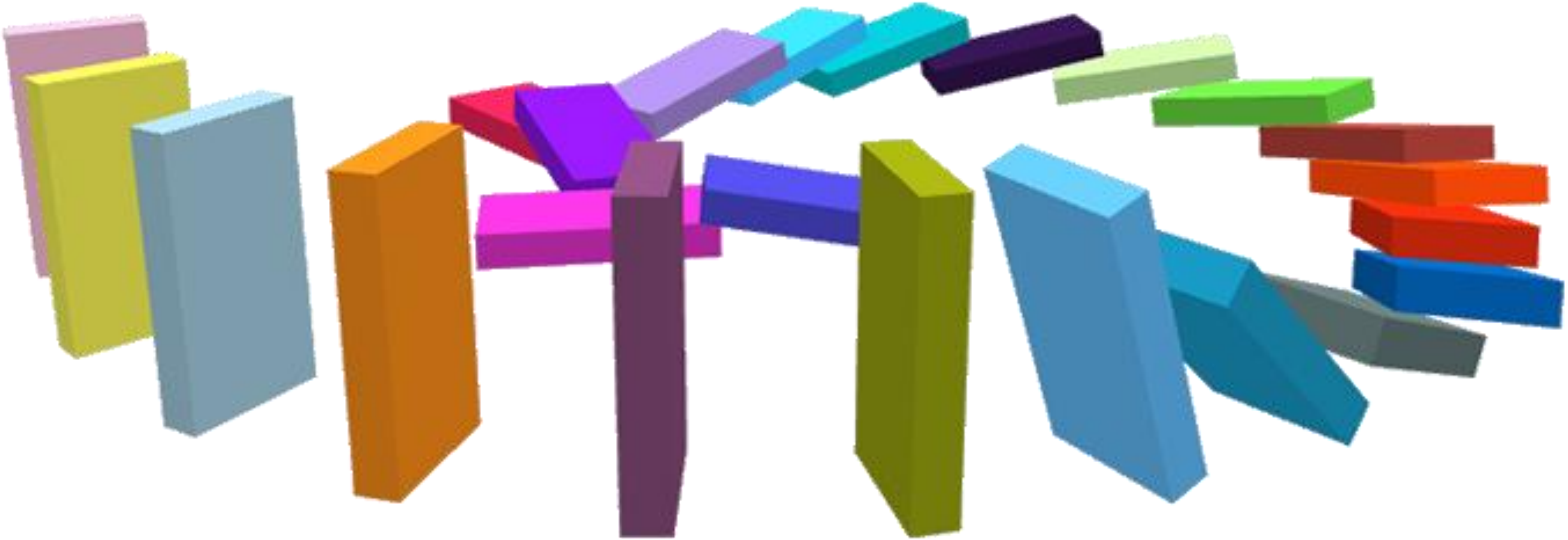


CSC417 Physics-Based Animation
… starting at 11:10 am

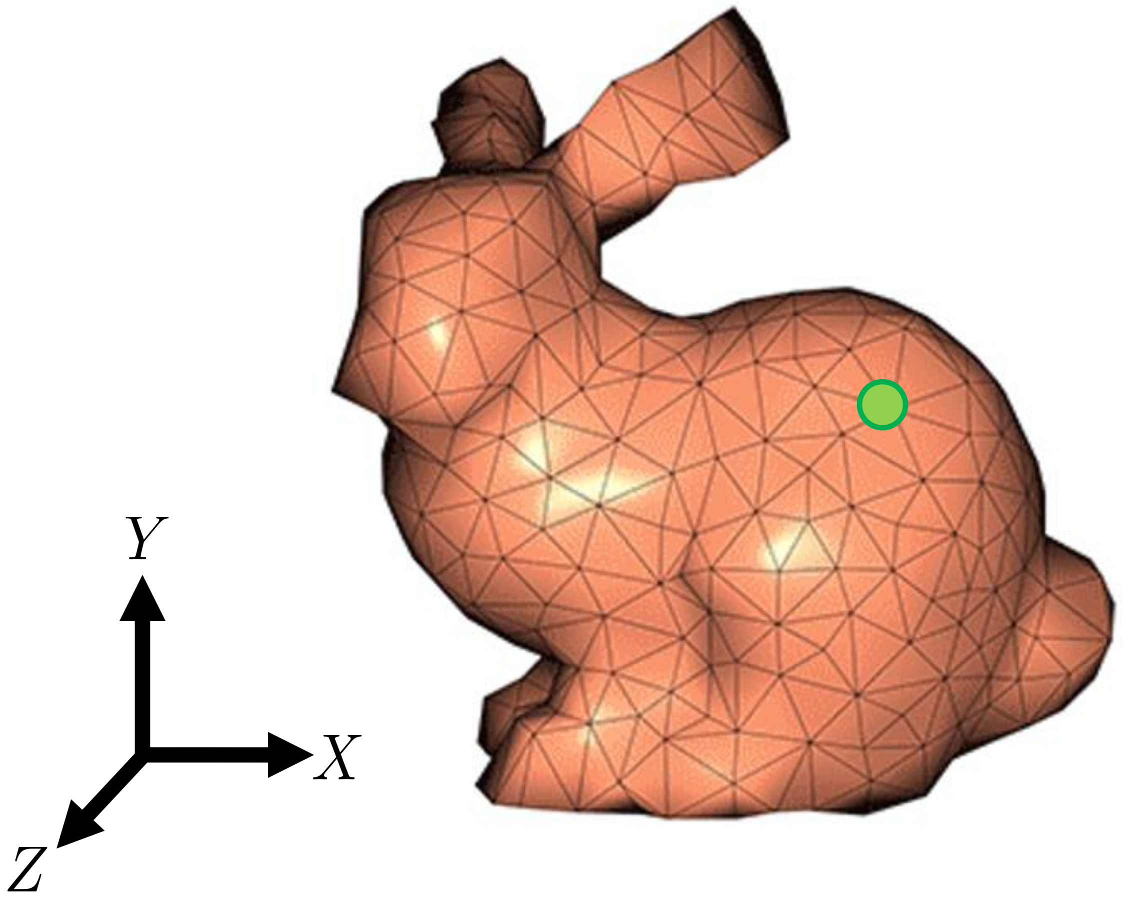
Game of Thrones | HBO

This Video: Rigid Body Simulation with Contact

What Makes an Object Rigid ?



Affine Body Dynamice



Reference (Undeformed) Space

$$\mathbf{x}(\mathbf{X}, t) = \mathrm{J}(\boldsymbol{X})\mathrm{q}(\mathrm{t})$$

Solve using Optimization via Newton's Method

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^{2} V\left(\mathbf{q}^{\mathbf{i+1}}\right)$$

Questions from Previous Lecture ?

Optimization Problem for a single object

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^{2} V\left(\mathbf{q}^{\mathrm{i+1}}\right)$$

What about lots of objects ?

Game of Thrones | HBO

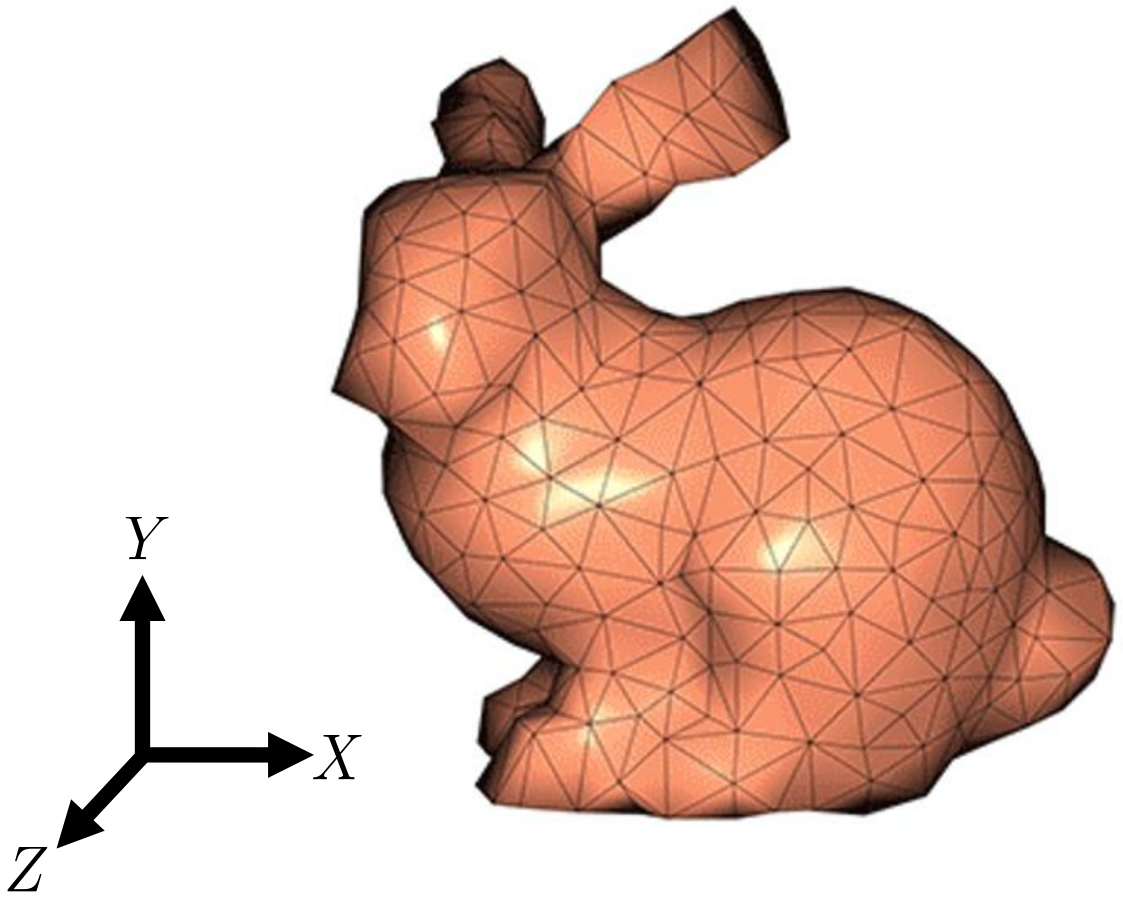
Two Problems with Our Current Approach

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 V\left(\mathbf{q}^{\mathbf{i+1}}\right)$$

Problem 1: Solving this optimization problem only moves one object !!!

Problem 2: There's no term in this optimization that tells it how to handle collisions

Kinetic Energy of an Affine Body



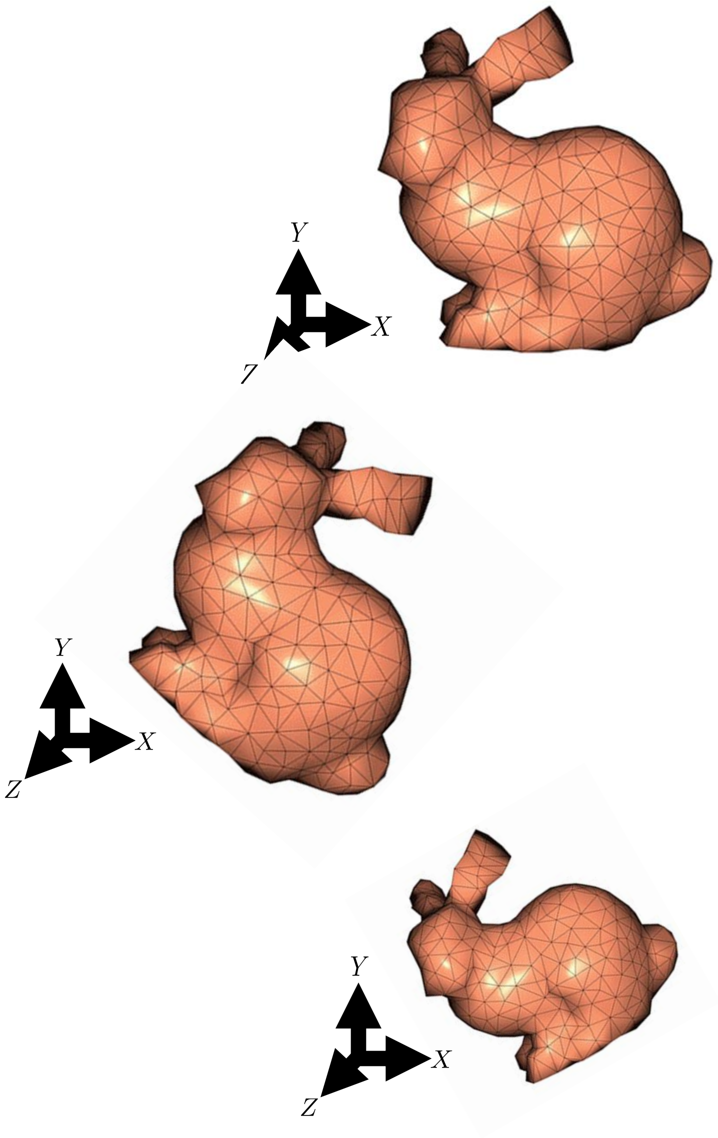
Reference (Undeformed) Space

infinitesimal volume

$$\frac{1}{2}\int_{\Omega}\rho\|\mathbf{v}\left(\mathbf{X}\right)\|_2^2 d\Omega$$

entire rigid body

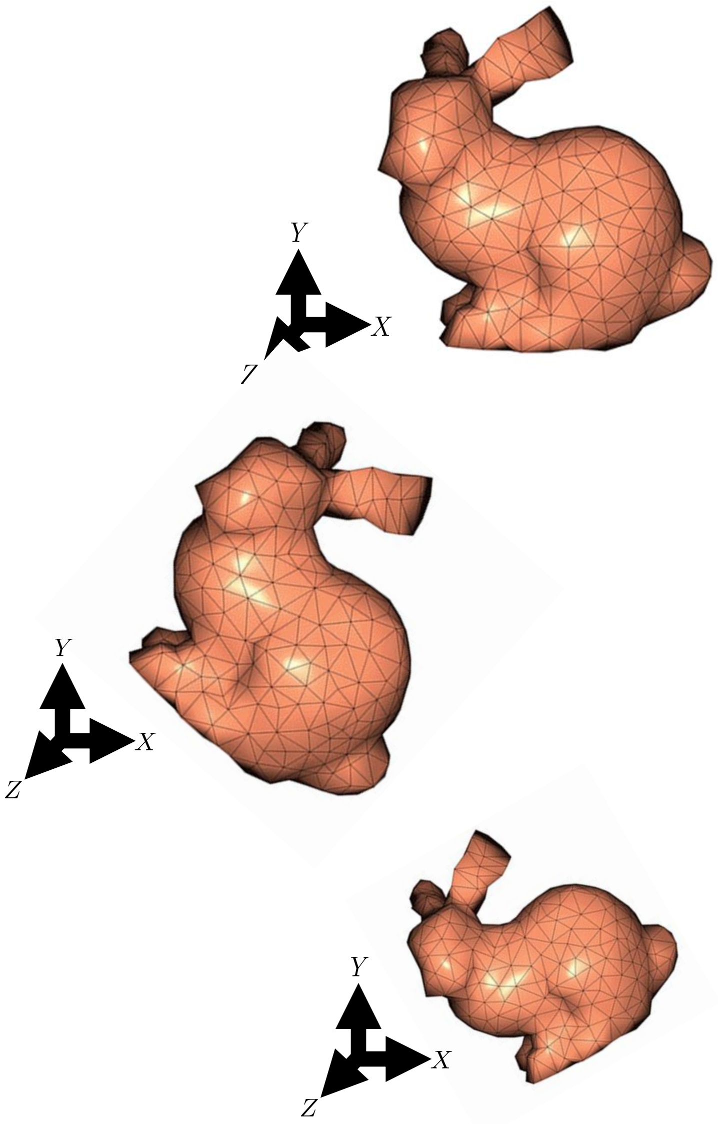
Kinetic Energy of many Affine Bodies



Reference (Undeformed) Spaces

?

Kinetic Energy of many Affine Bodies

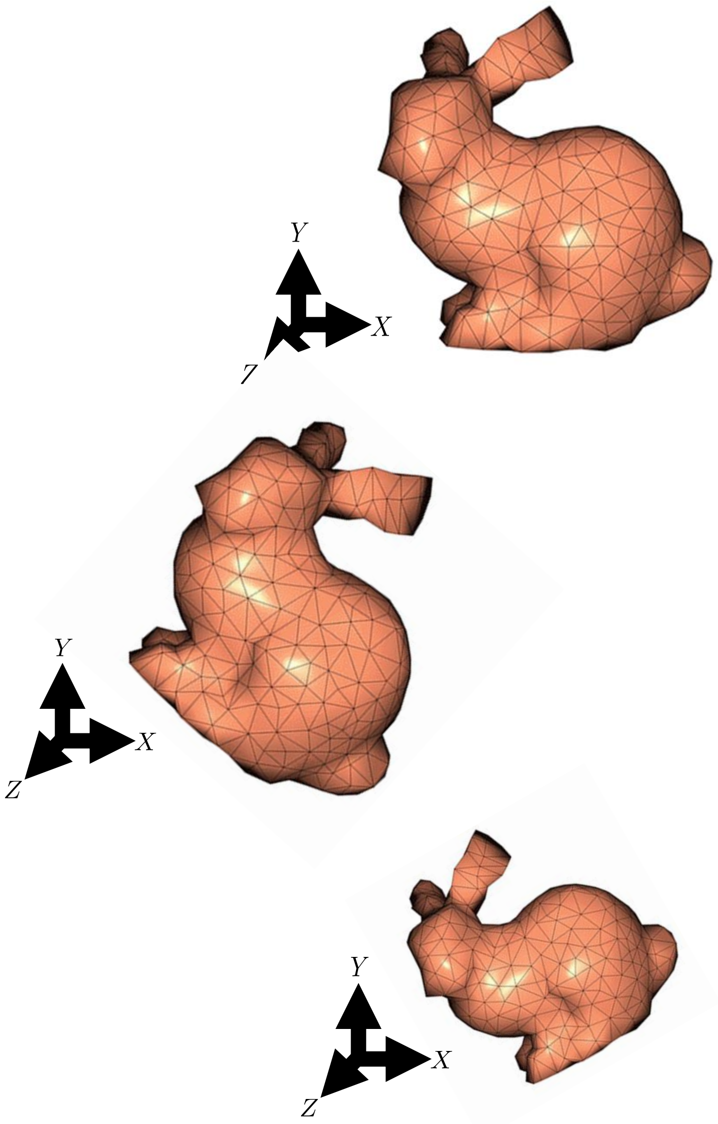


Reference (Undeformed) Spaces

Number of Objects

$$\sum_{i=0}^{N} \frac{1}{2} \int_{\Omega_i} \rho_i ||\mathbf{v}_i(\mathbf{X})|| d\Omega_i$$

Kinetic Energy of many Affine Bodies

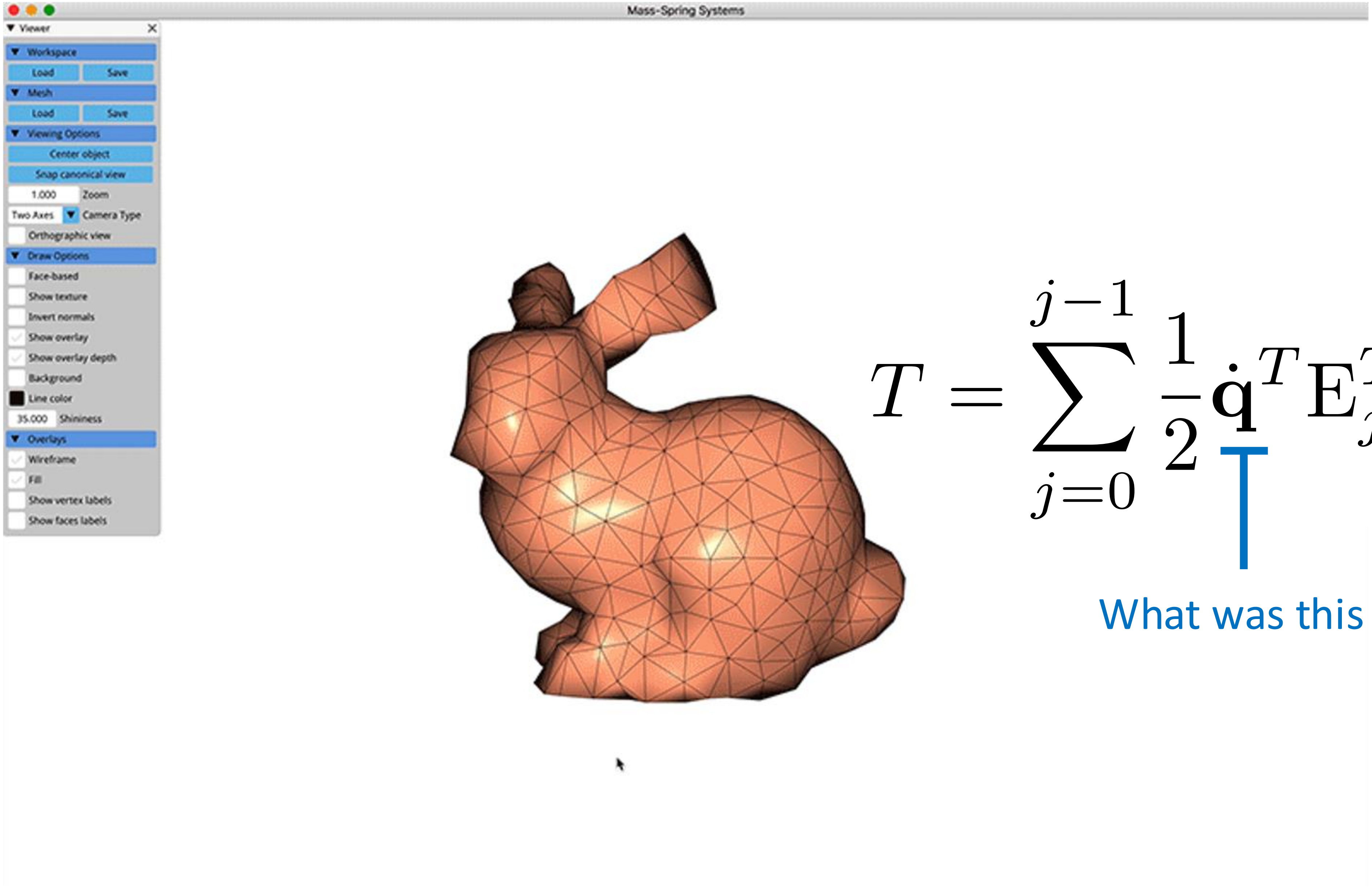


Number of Objects

$$\sum_{i=0}^{N} \frac{1}{2} \dot{\mathbf{q}}_i^T M_i \dot{\mathbf{q}}_i$$

Reference (Undeformed) Spaces

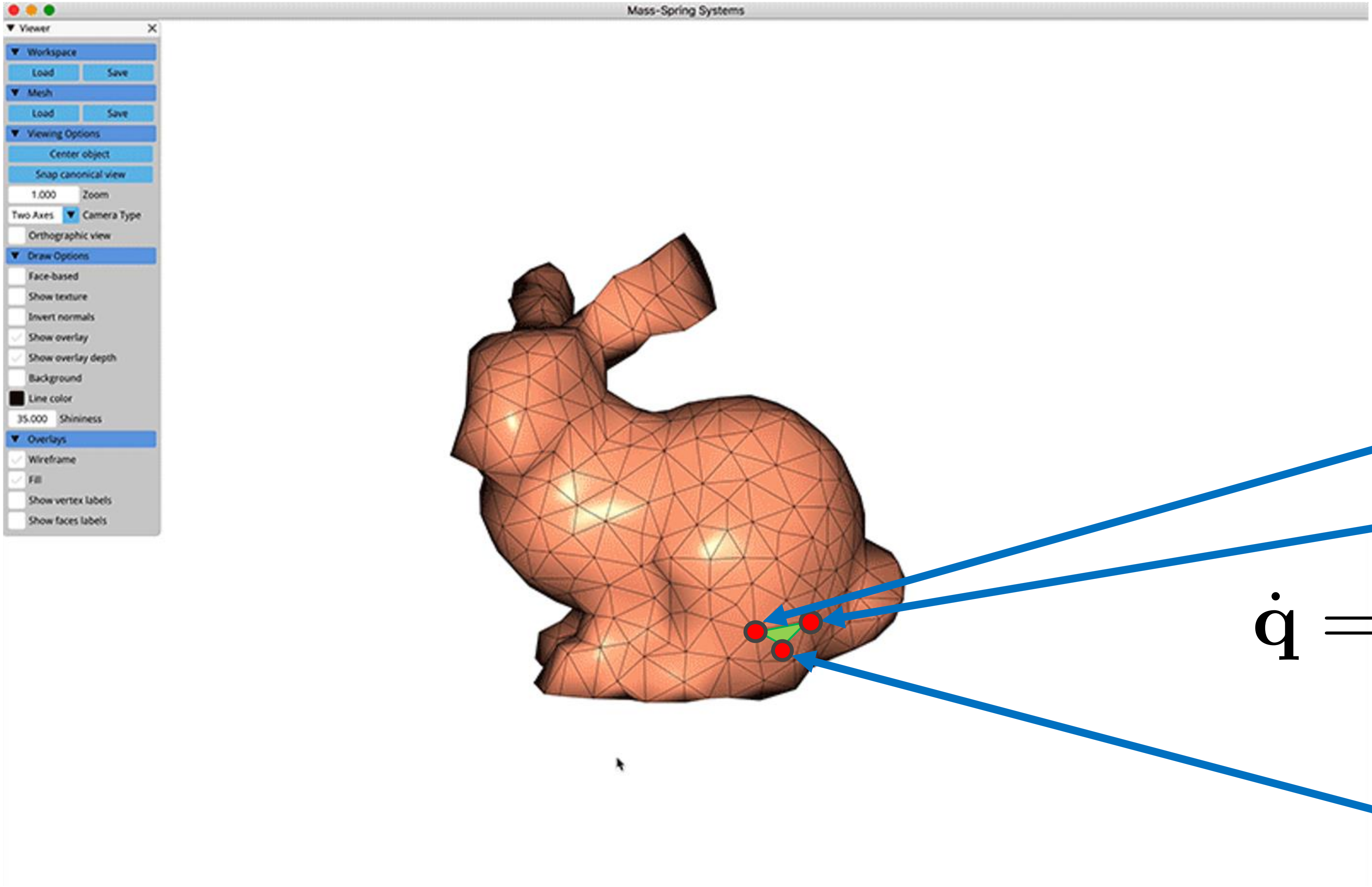
Kinetic Energy for a Bunny using FEM



$$T = \sum_{j=0}^{j-1} \frac{1}{2} \dot{\mathbf{q}}^T \mathrm{E}_j^T \mathrm{M}_j \mathrm{E}_j \dot{\mathbf{q}}$$

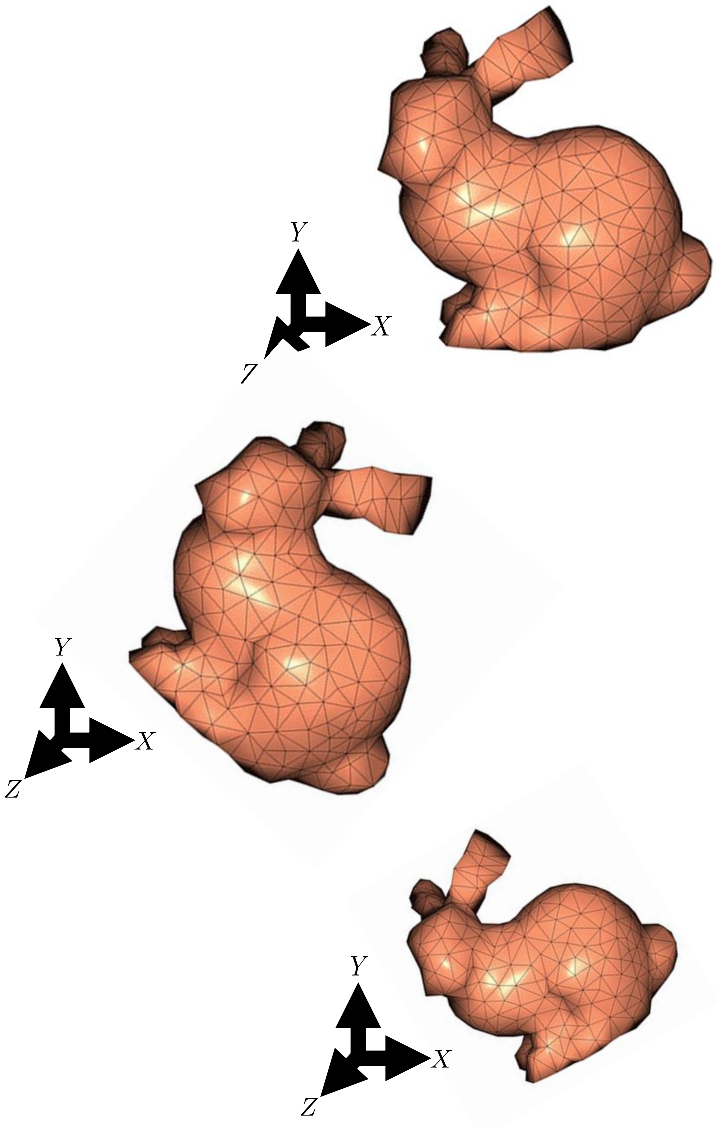
What was this ?

Generalized Coordinates for Bunny FEM



$$\dot{\mathbf{q}} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix}$$

Let’s do the same thing

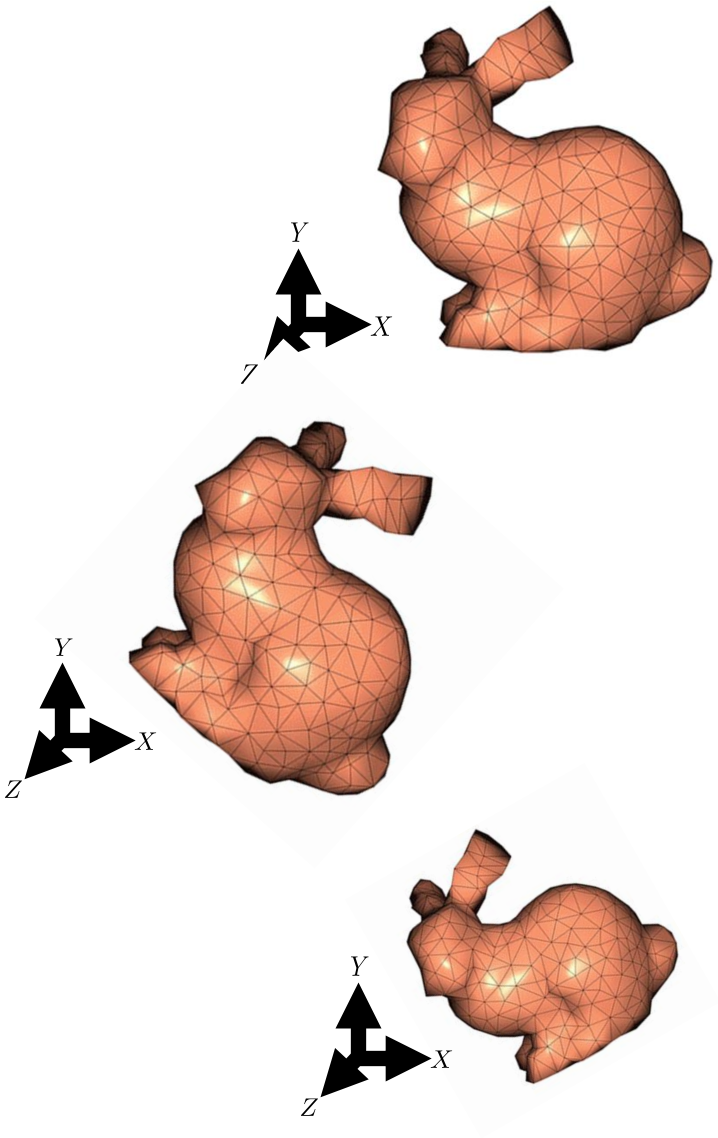


Number of Objects

$$\sum_{i=0}^{N} \frac{1}{2} \dot{\mathbf{q}}_i^T \mathrm{M}_i \dot{\mathbf{q}}_i$$

Reference (Undeformed) Spaces

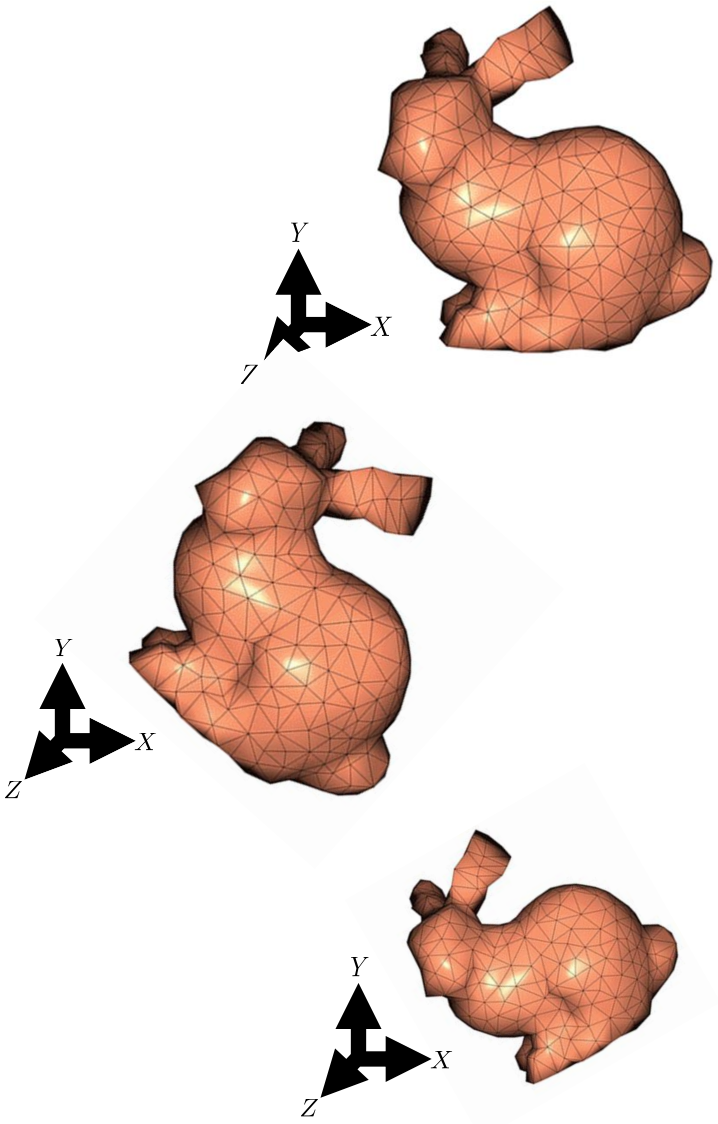
Generalized Velocity for MANY Affine Bodies



$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\mathbf{q}}_0 \\ \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix}$$

Reference (Undeformed) Spaces

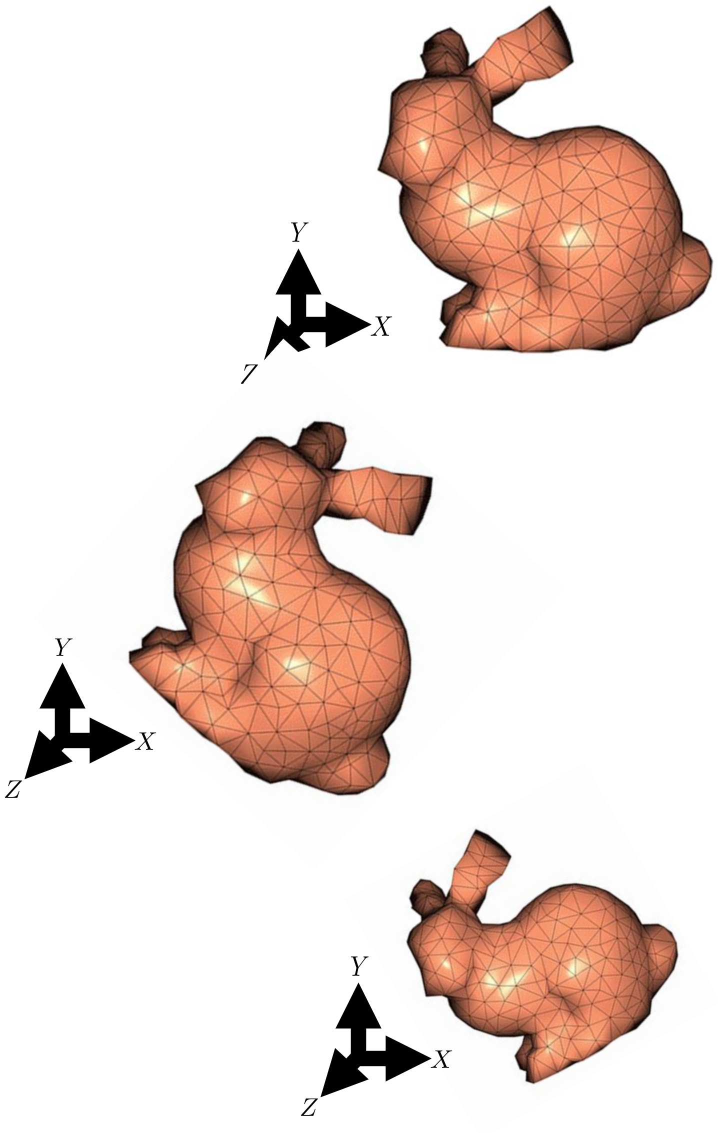
Generalized Coordinates for MANY Affine Bodies



Reference (Undeformed) Spaces

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}$$

Let’s do the same thing

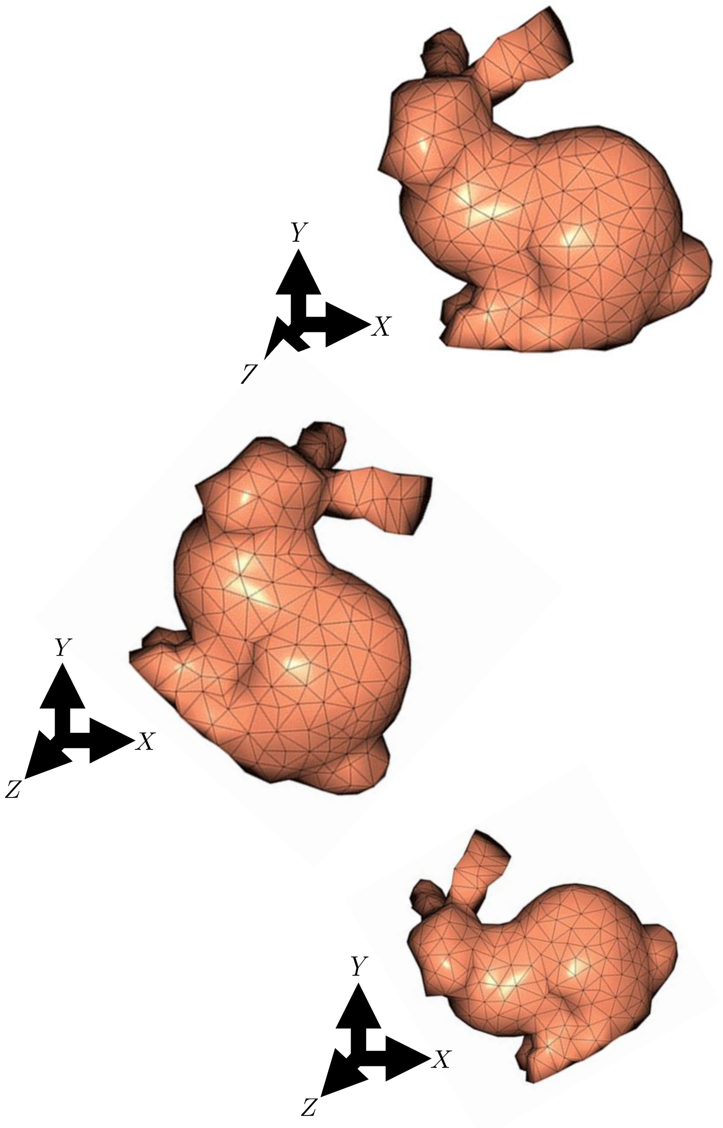


Reference (Undeformed) Spaces

Number of Objects

$$\sum_{i=0}^{N} \frac{1}{2} \dot{\mathbf{q}}^{\mathrm{T}} \mathrm{E_i}^{T} \mathrm{M_i} \mathrm{E_i} \dot{\mathbf{q}}$$

Mass Matrix for Affine Body System

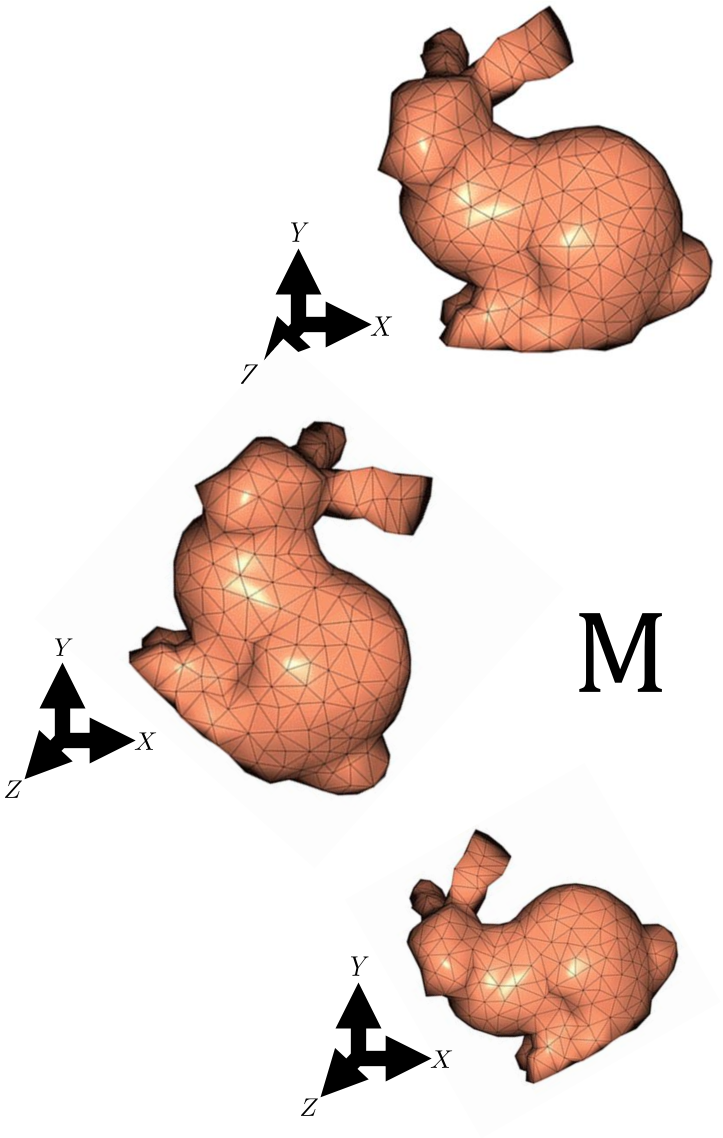


$$\frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\underbrace{\left(\sum_{\mathrm{i}=0}^{\mathrm{N}}\mathrm{E_i}^{T}\mathrm{M_i}\mathrm{E_i}\right)}_{\mathrm{M}}\dot{\mathbf{q}}$$

Mass Matrix

Reference (Undeformed) Spaces

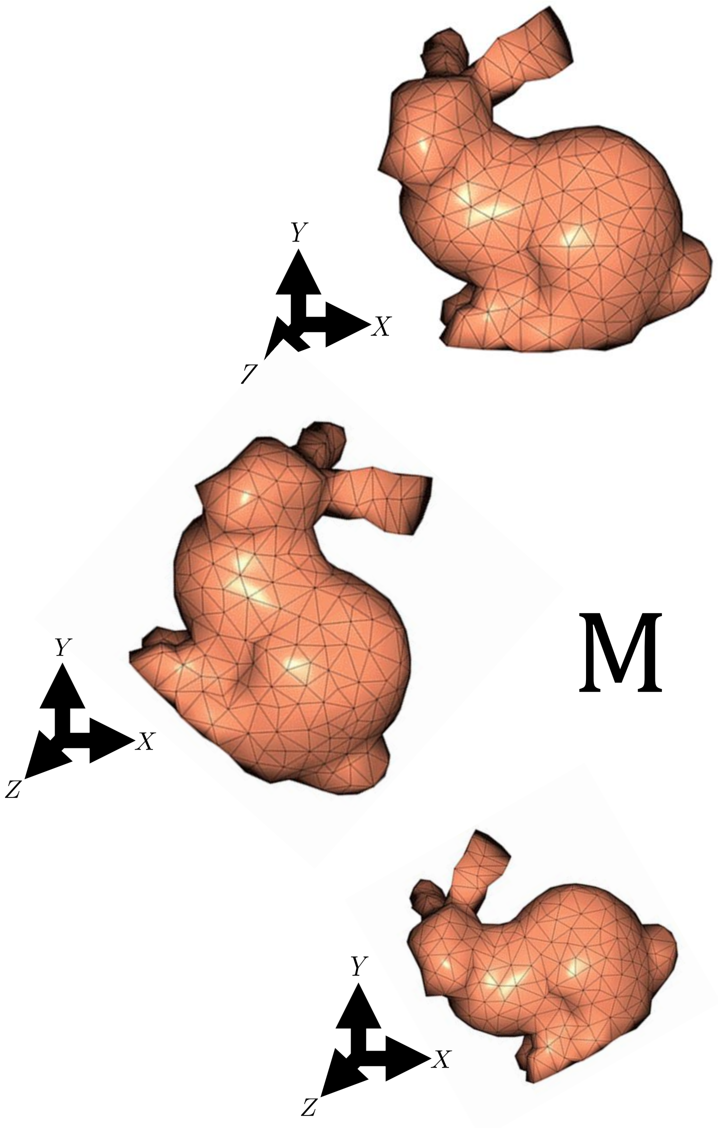
Block Structure of M ?



Reference (Undeformed) Spaces

$$\mathrm{M} \quad \text{Mass Matrix} = \begin{array}{c c} & \begin{array}{ccc} \text{Object 0} & \text{Object 1} & \text{Object 2} \end{array} \\ \begin{array}{c} \text{Object 0} \\ \text{Object 1} \\ \text{Object 2} \end{array} & \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \end{array}$$

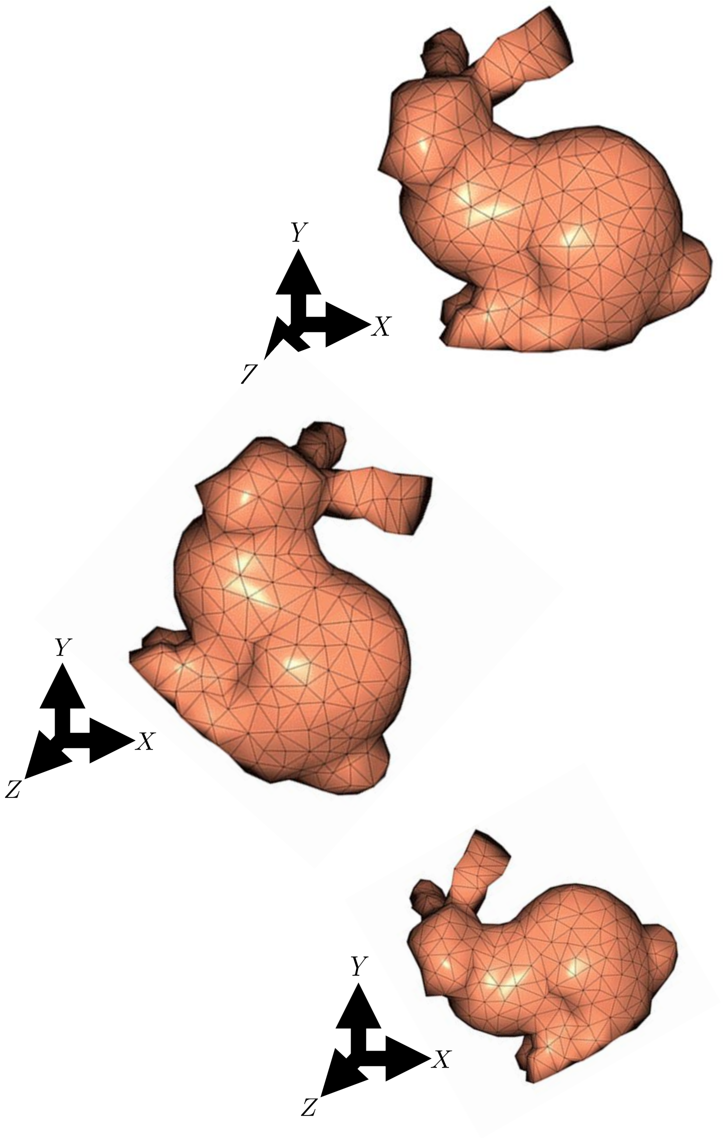
Block Structure of M ?



$$\mathrm{M} \quad \text{Mass Matrix} = \begin{array}{c c} & \begin{array}{ccc} \text{Object 0} & \text{Object 1} & \text{Object 2} \end{array} \\ \begin{array}{c} \text{Object 0} \\ \text{Object 1} \\ \text{Object 2} \end{array} & \begin{bmatrix} M_0 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_2 \end{bmatrix} \end{array}$$

Reference (Undeformed) Spaces

Potential Energy of Affine Body System



?

Reference (Undeformed) Spaces

Optimization Problem for a multi-object system

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^{2} V\left(\mathbf{q}^{\mathbf{i+1}}\right)$$

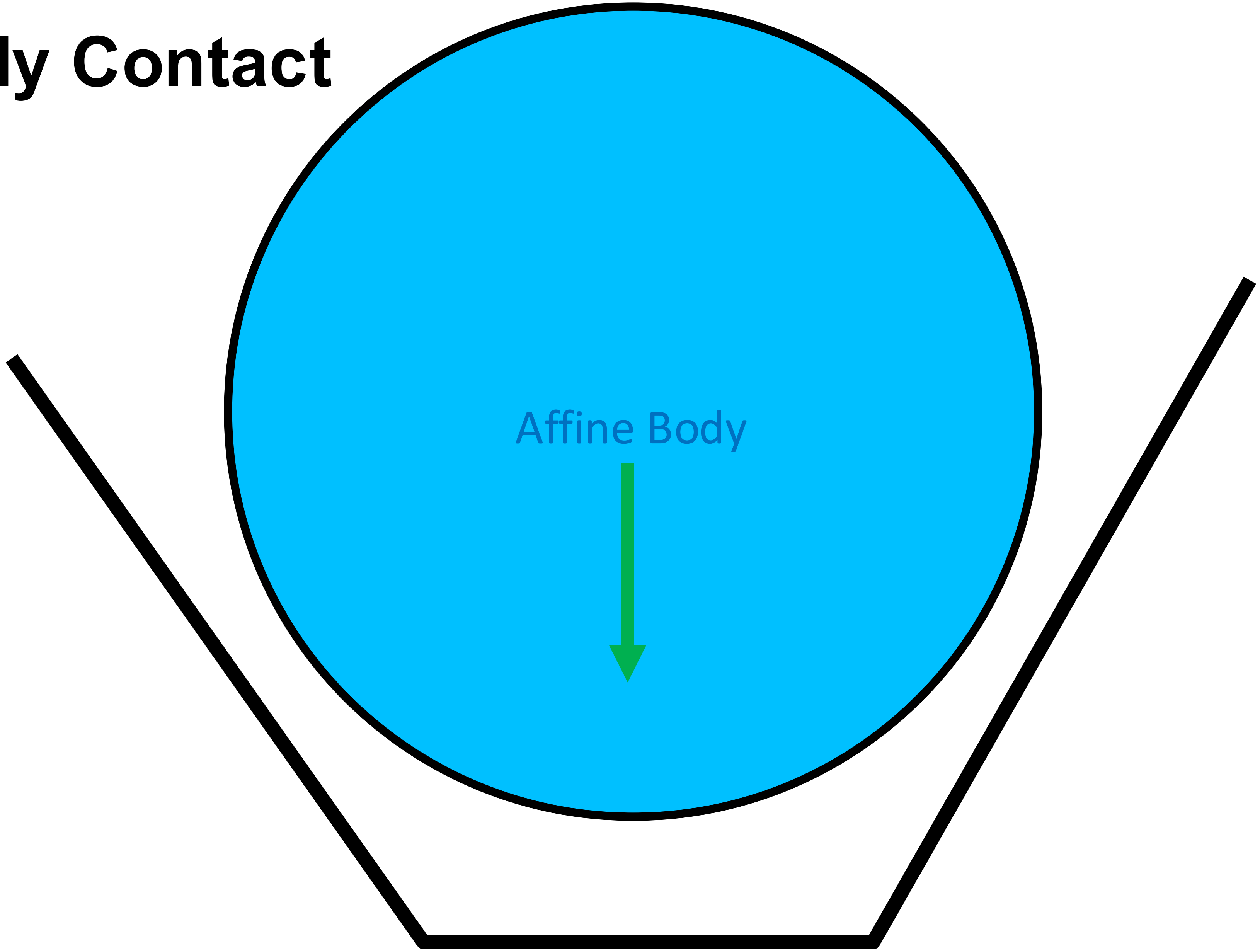
Two Problems with Our Current Approach

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 V\left(\mathbf{q}^{\mathrm{i+1}}\right)$$

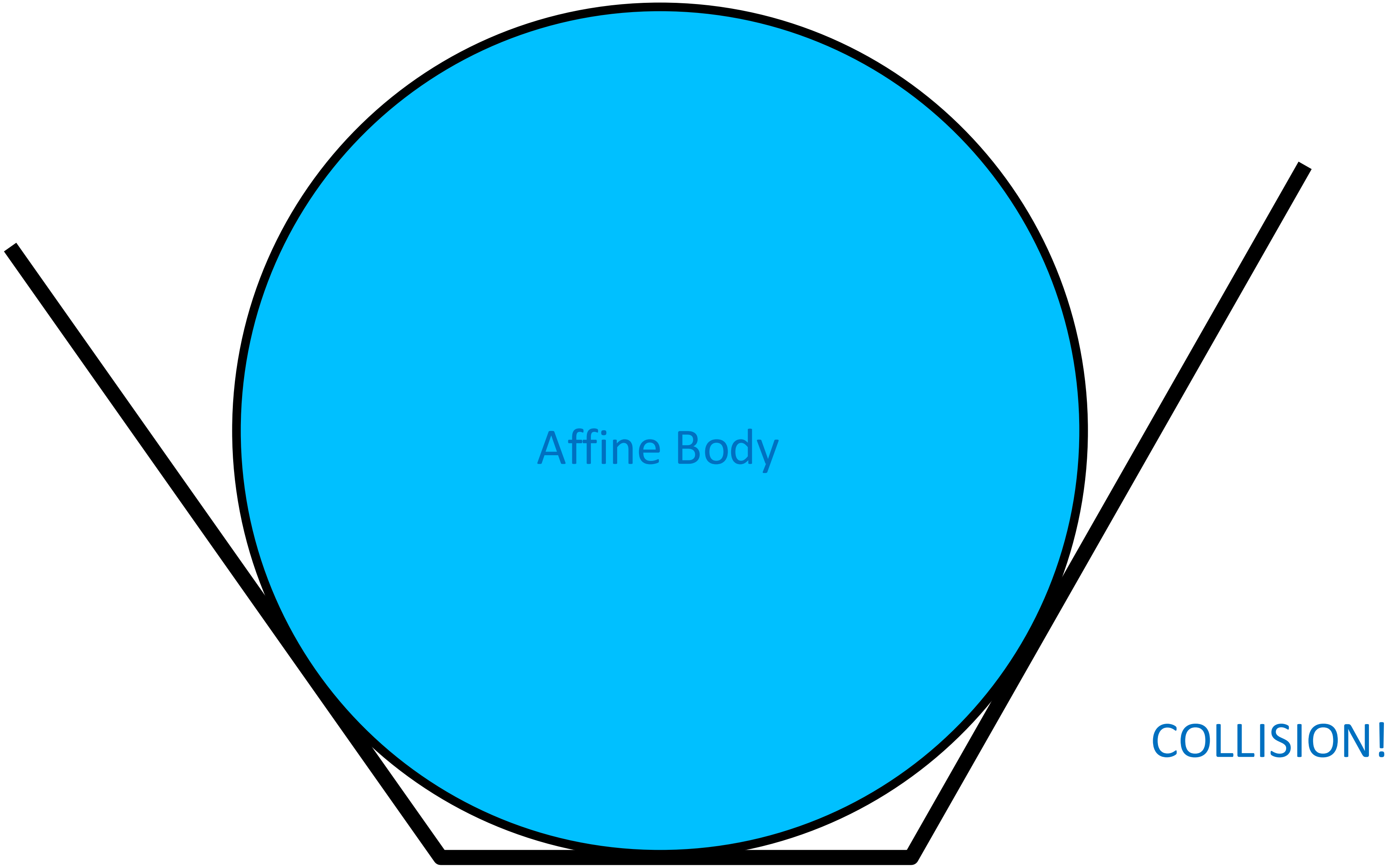
~~Problem 1: Solving this optimization problem only moves one object !!!~~

Problem 2: There's no term in this optimization that tells it how to handle collisions

Affine Body Contact



Affine Body Contact



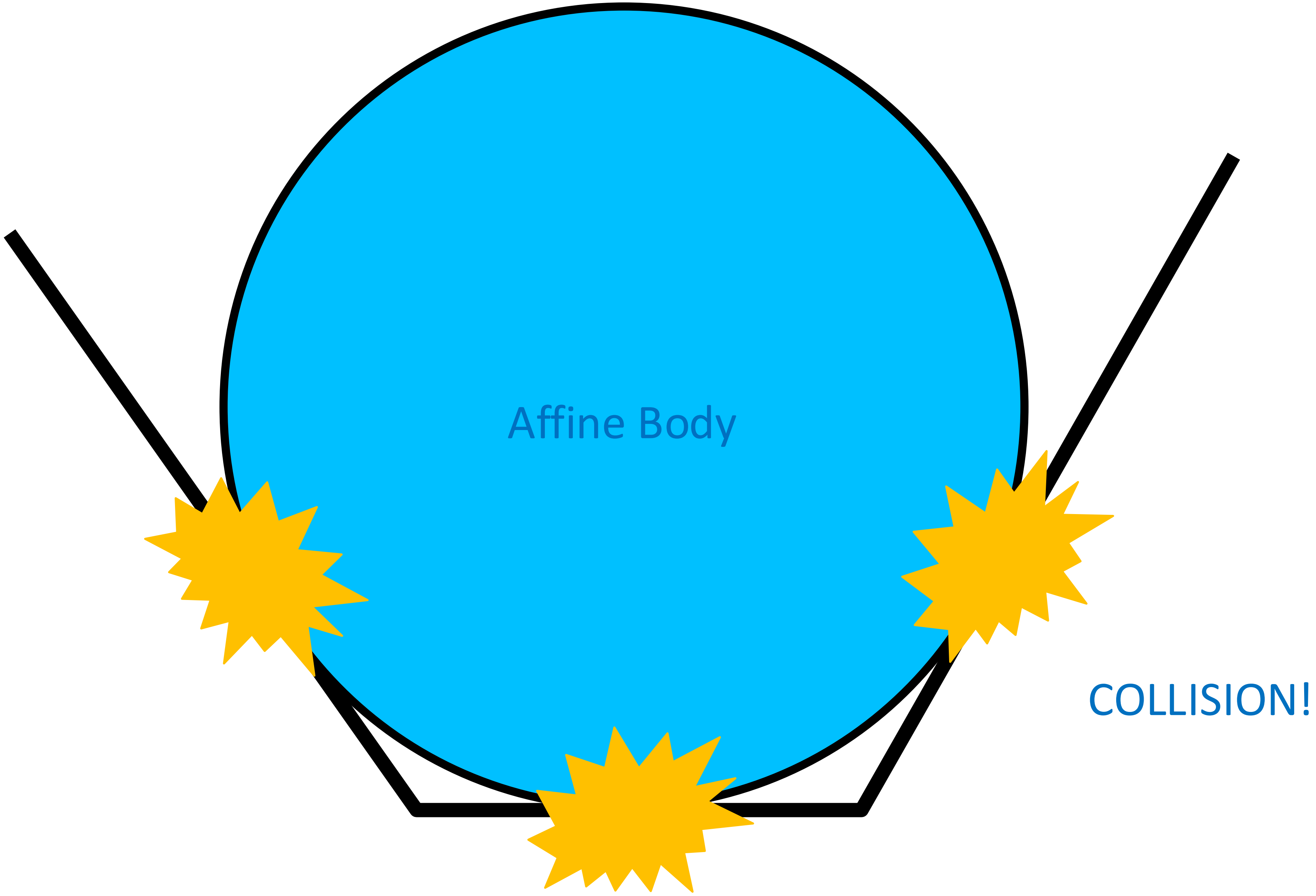
Collisions in Simulation

Two phases detection and response

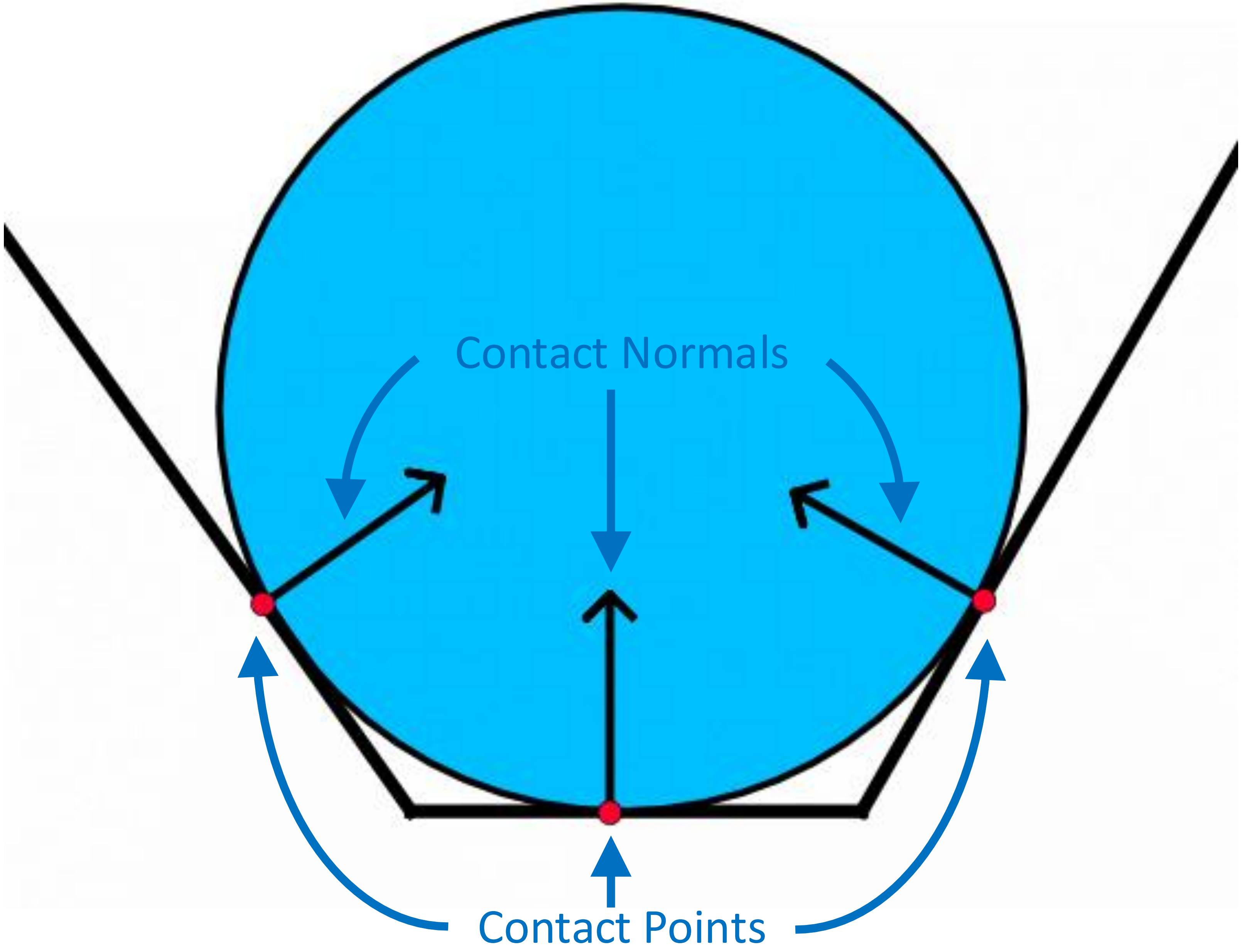
Detection: Did I hit anything ?

Response: I hit something ! What do I do ?

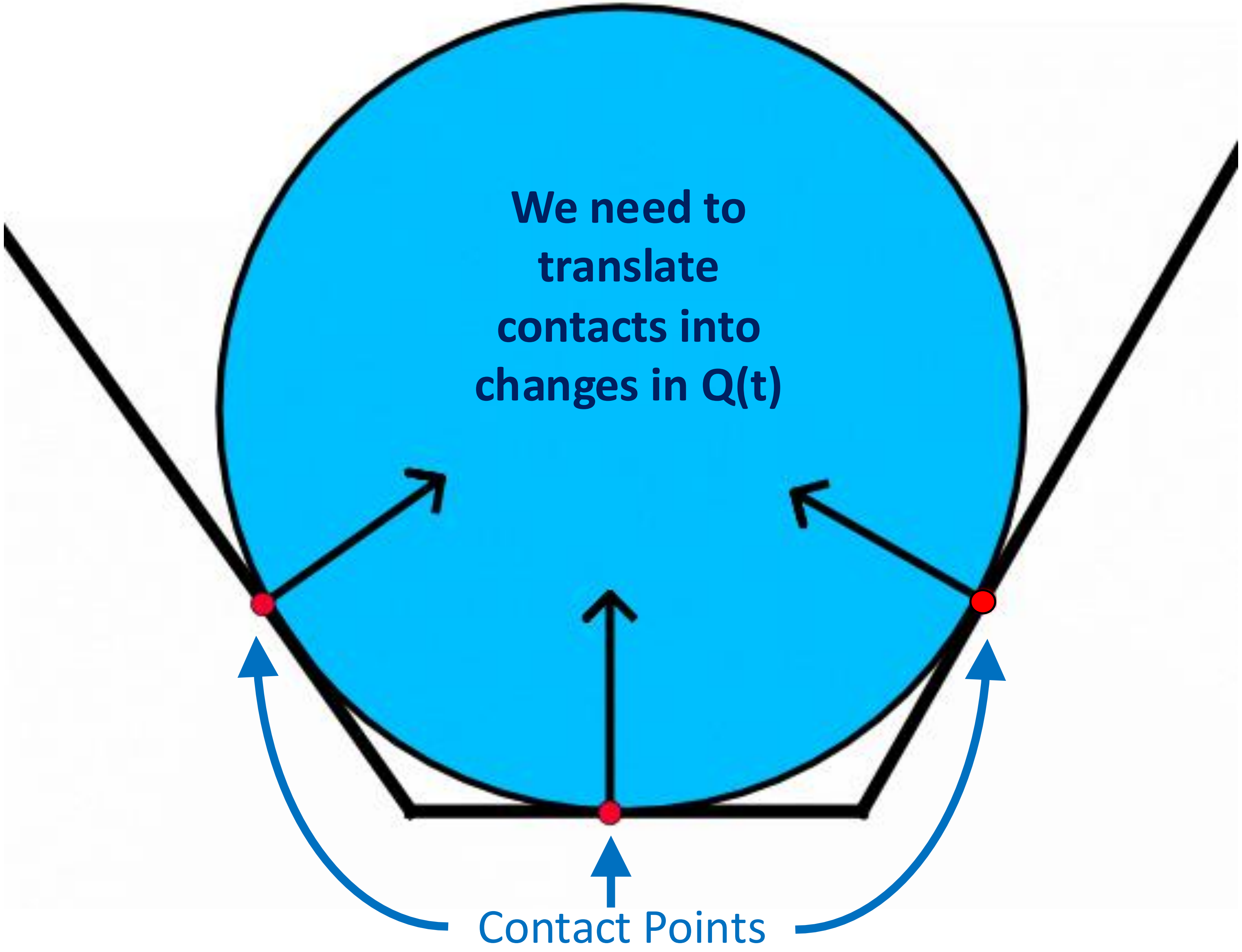
Affine Body Contact



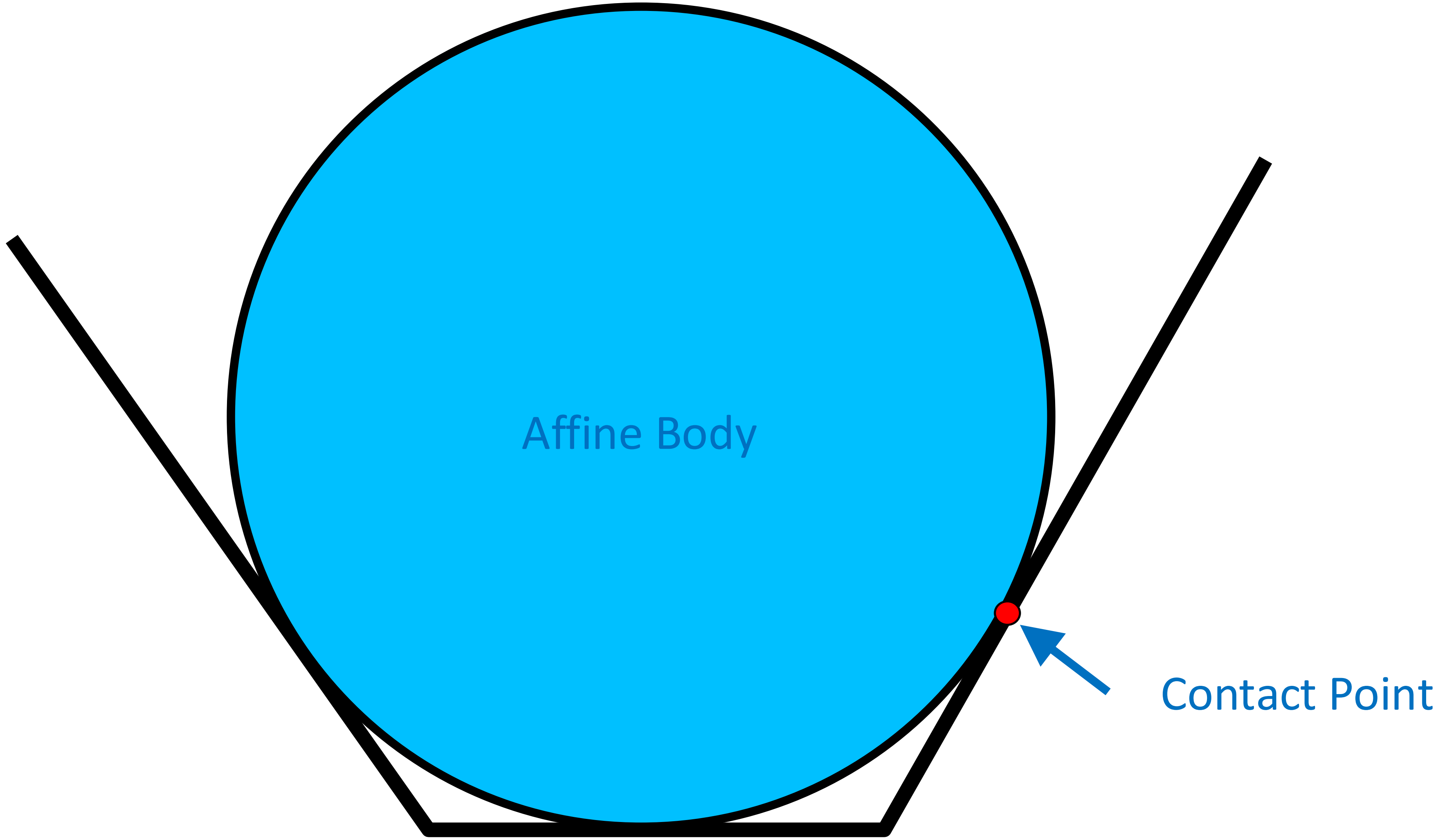
Affine Body Contact



Affine Body Contact

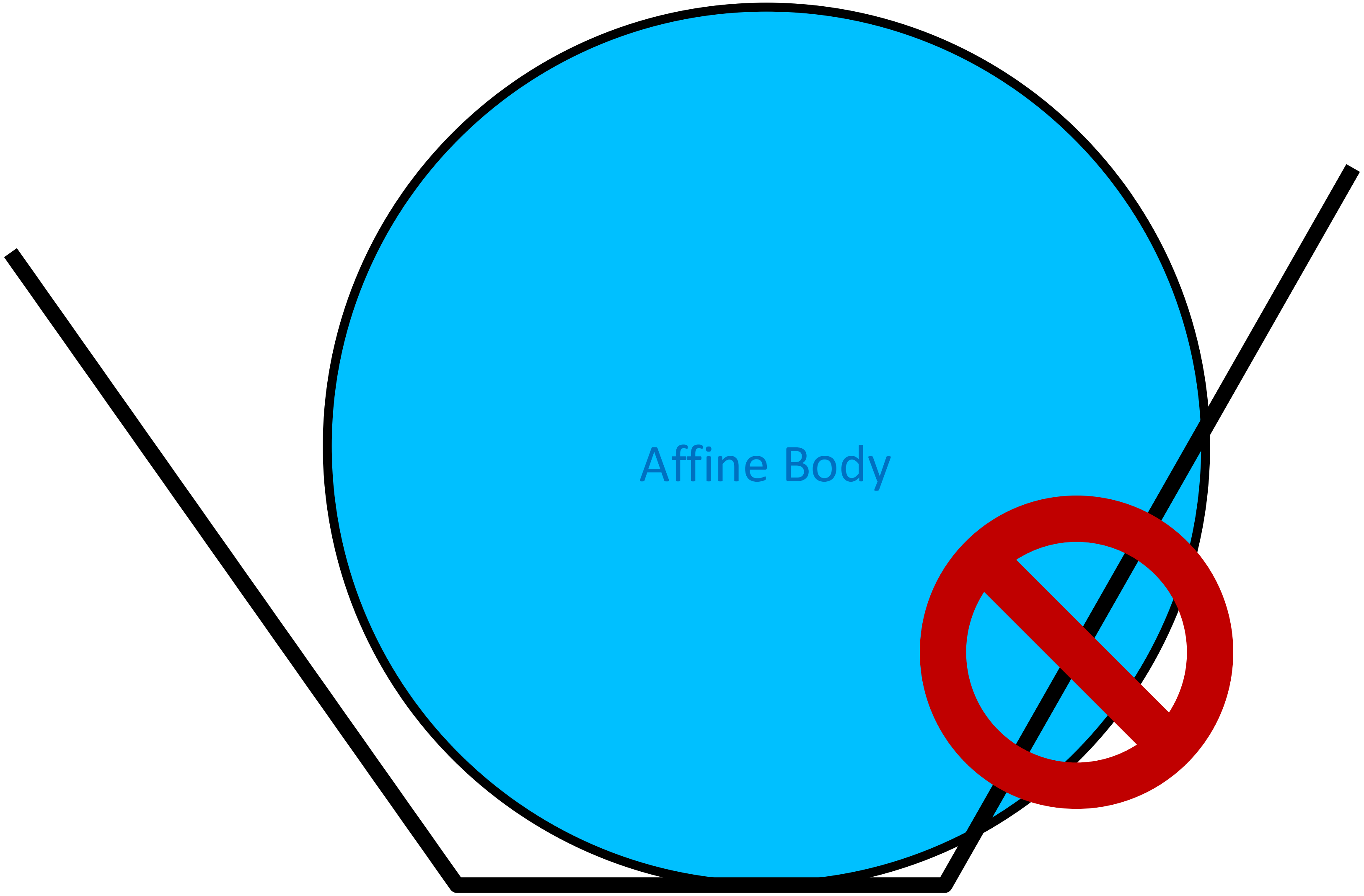


Three Rules of Contact Mechanics



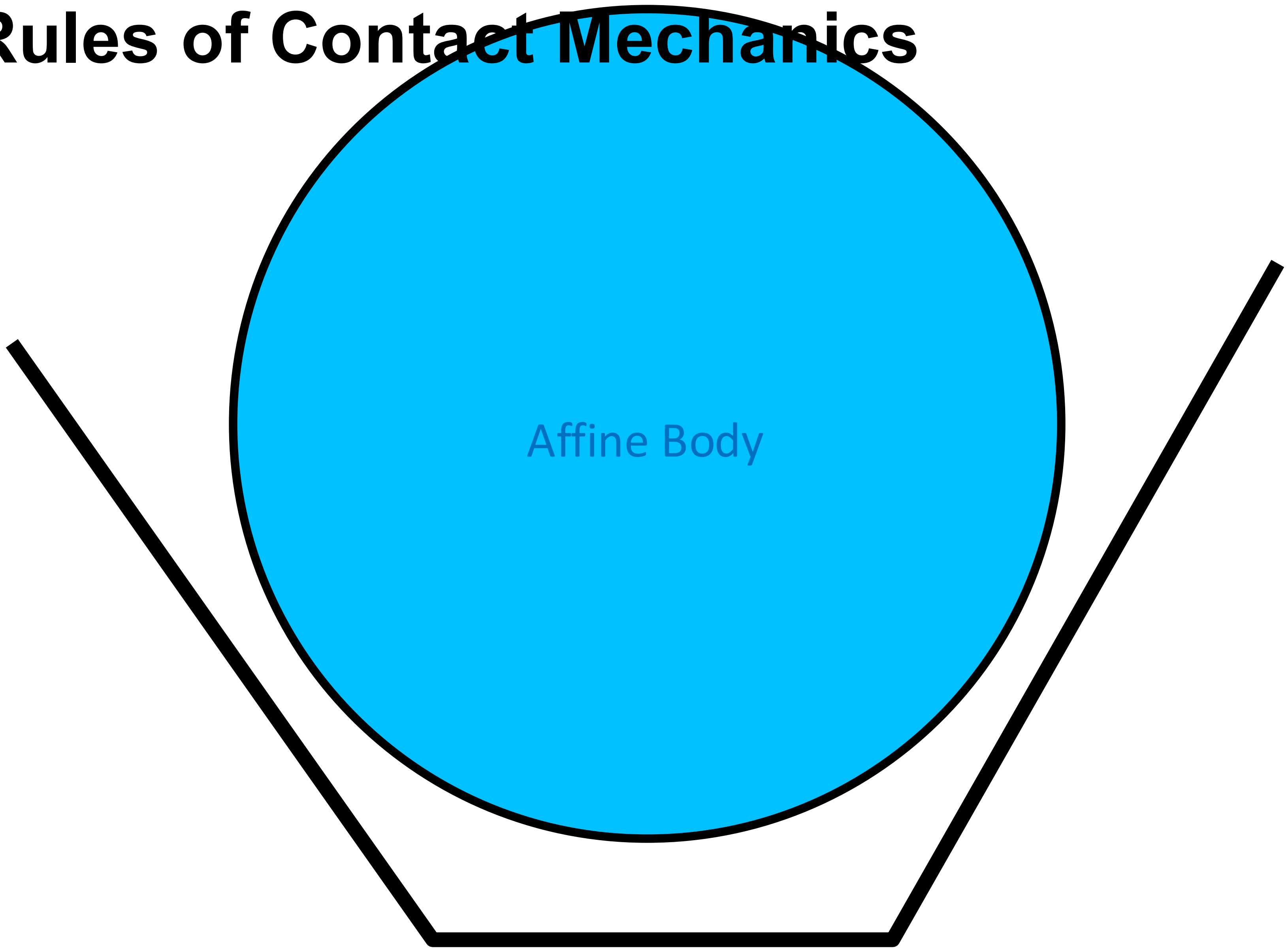
Try to prevent interpenetration at contact point

Three Rules of Contact Mechanics



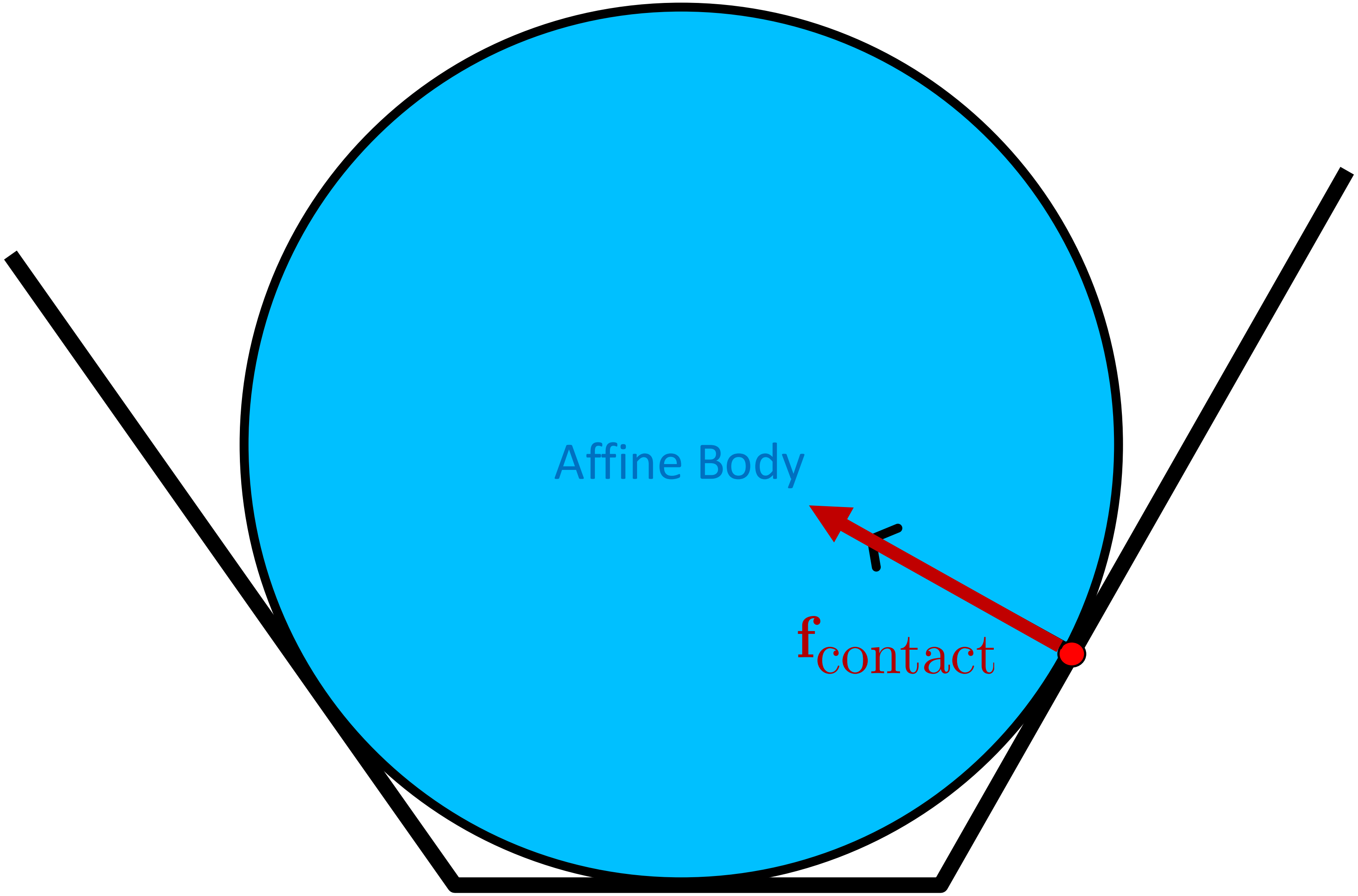
Try to prevent interpenetration at contact point

Three Rules of Contact Mechanics



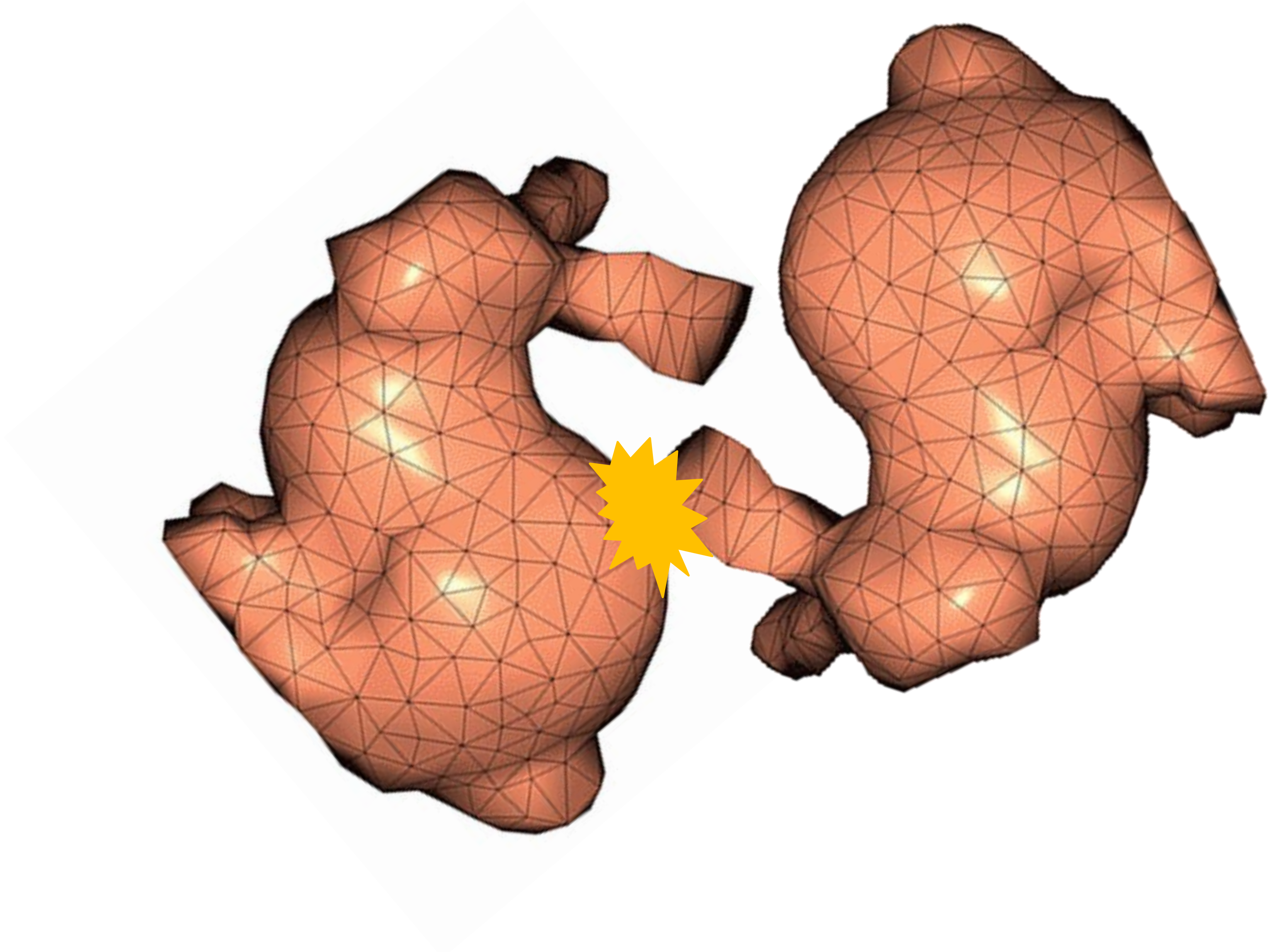
Contact forces can only be applied when objects are in contact

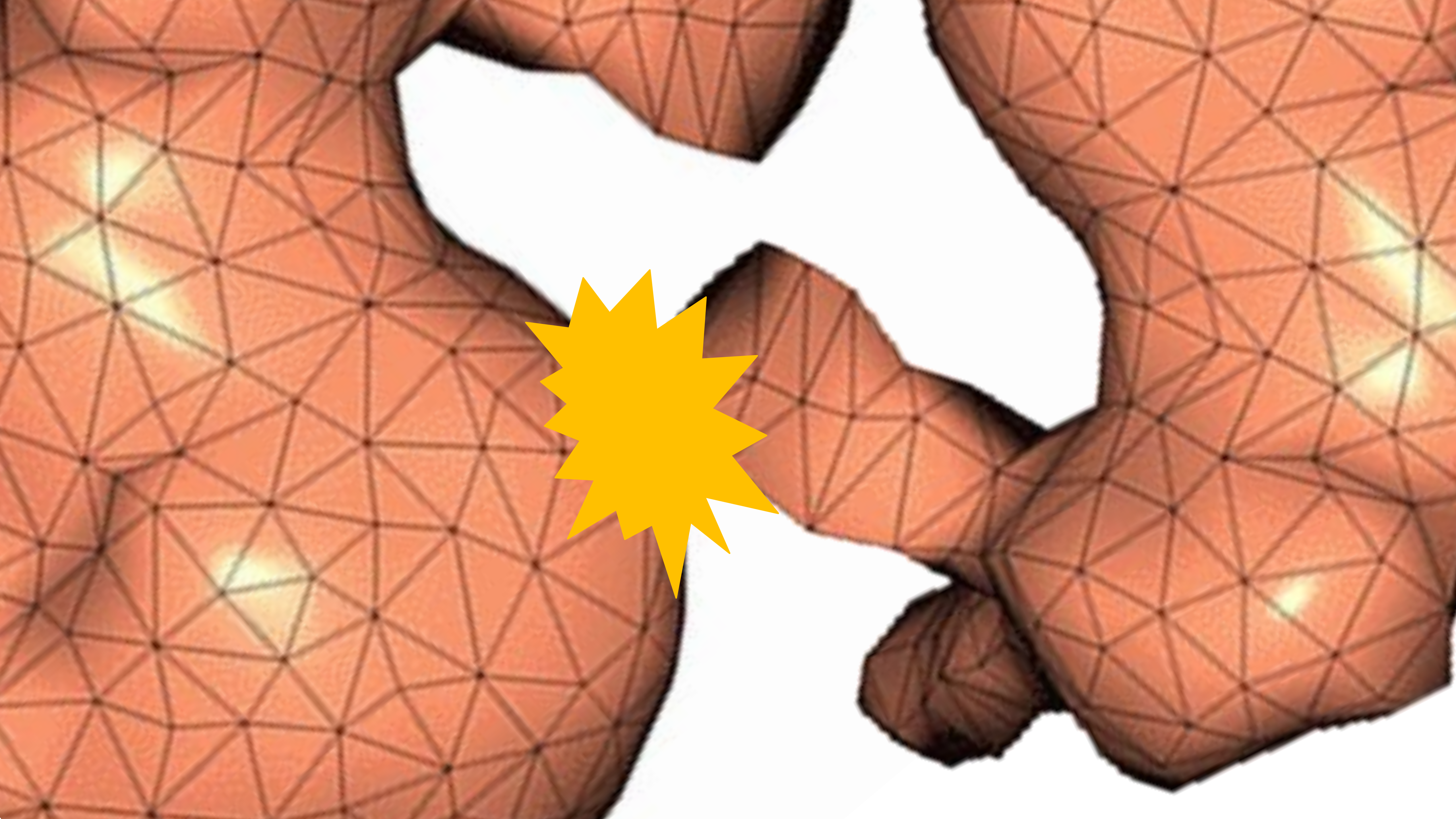
Three Rules of Contact Mechanics

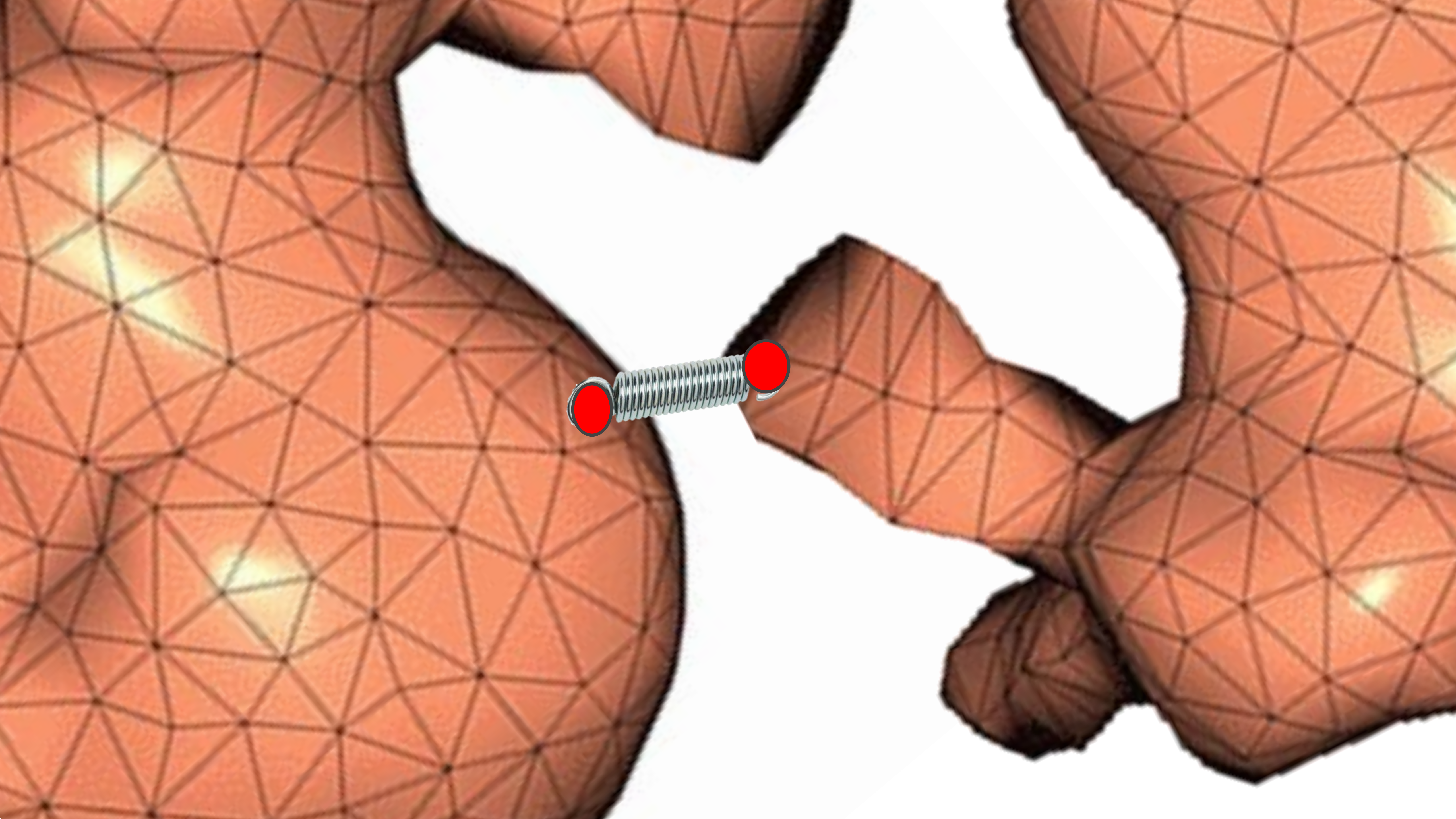


Contact forces “push” objects apart

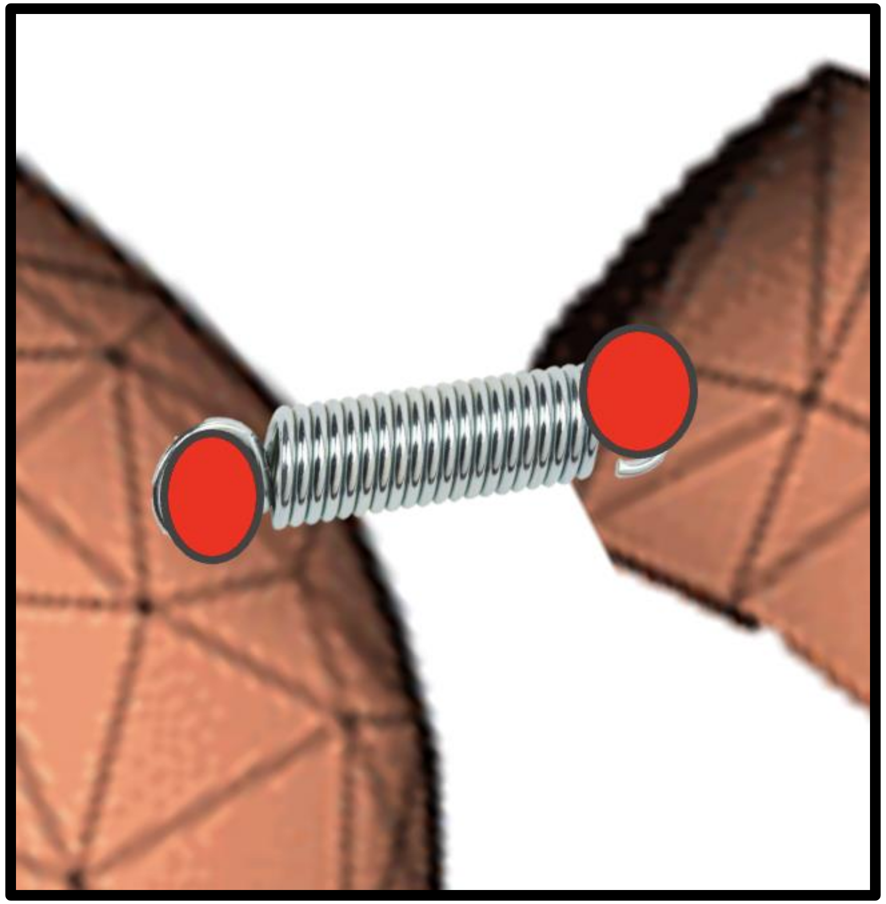
One Approach of Many – Penalty “Springs”

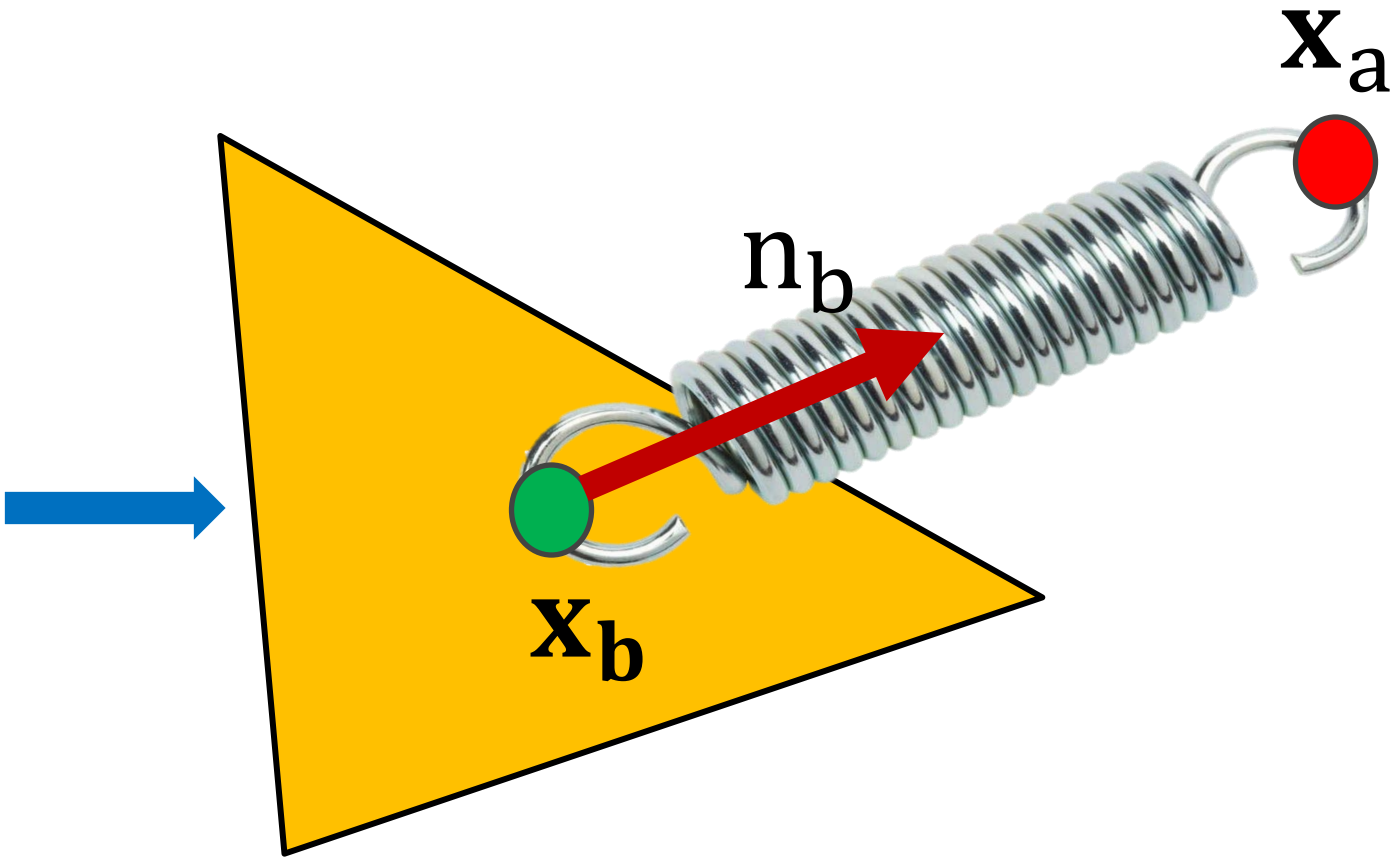




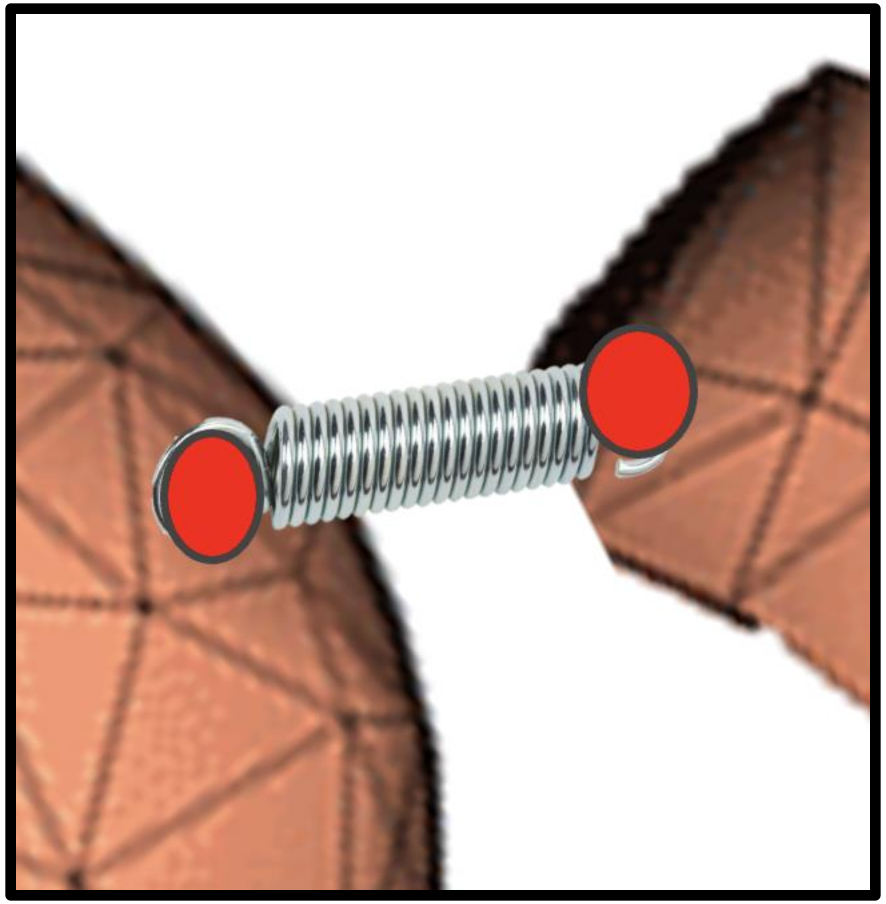


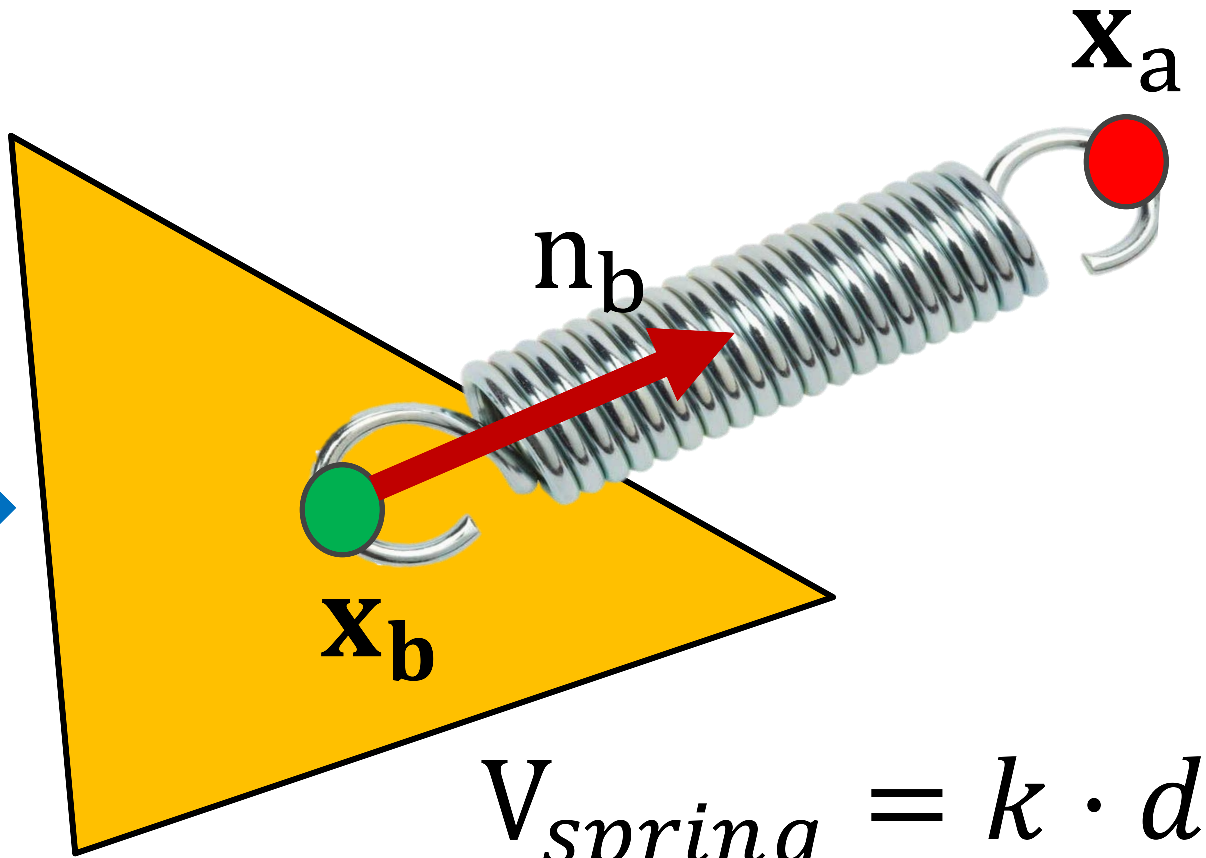
Triangle – Vertex Contacts





Triangle – Vertex Contacts

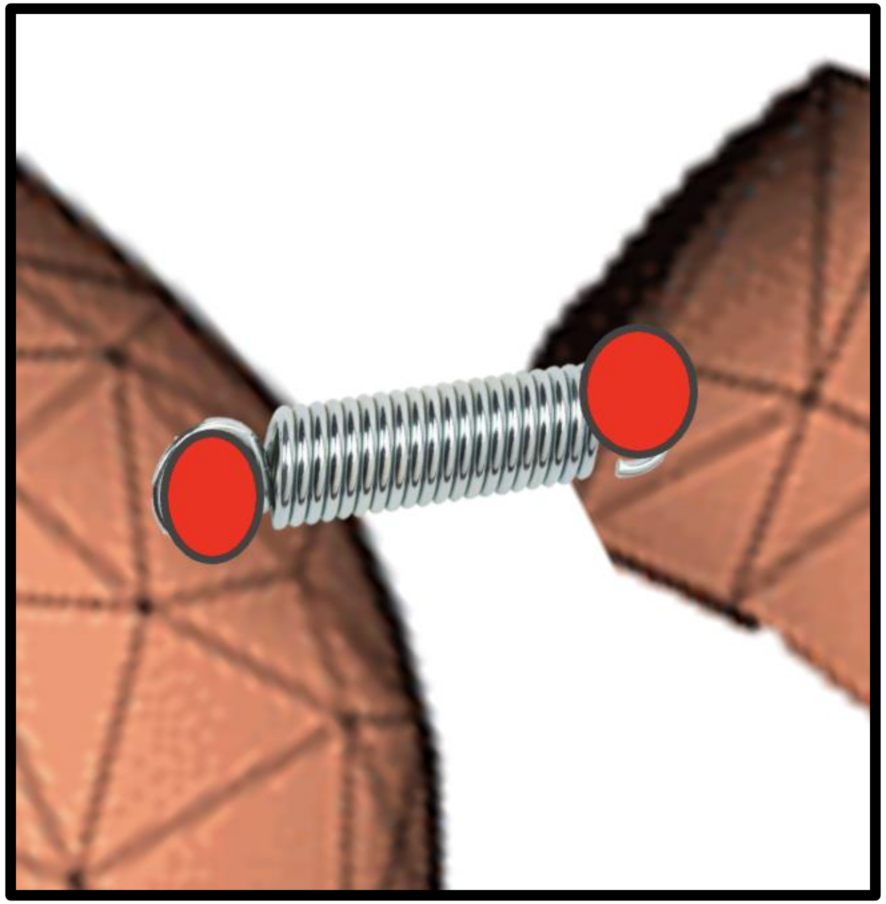


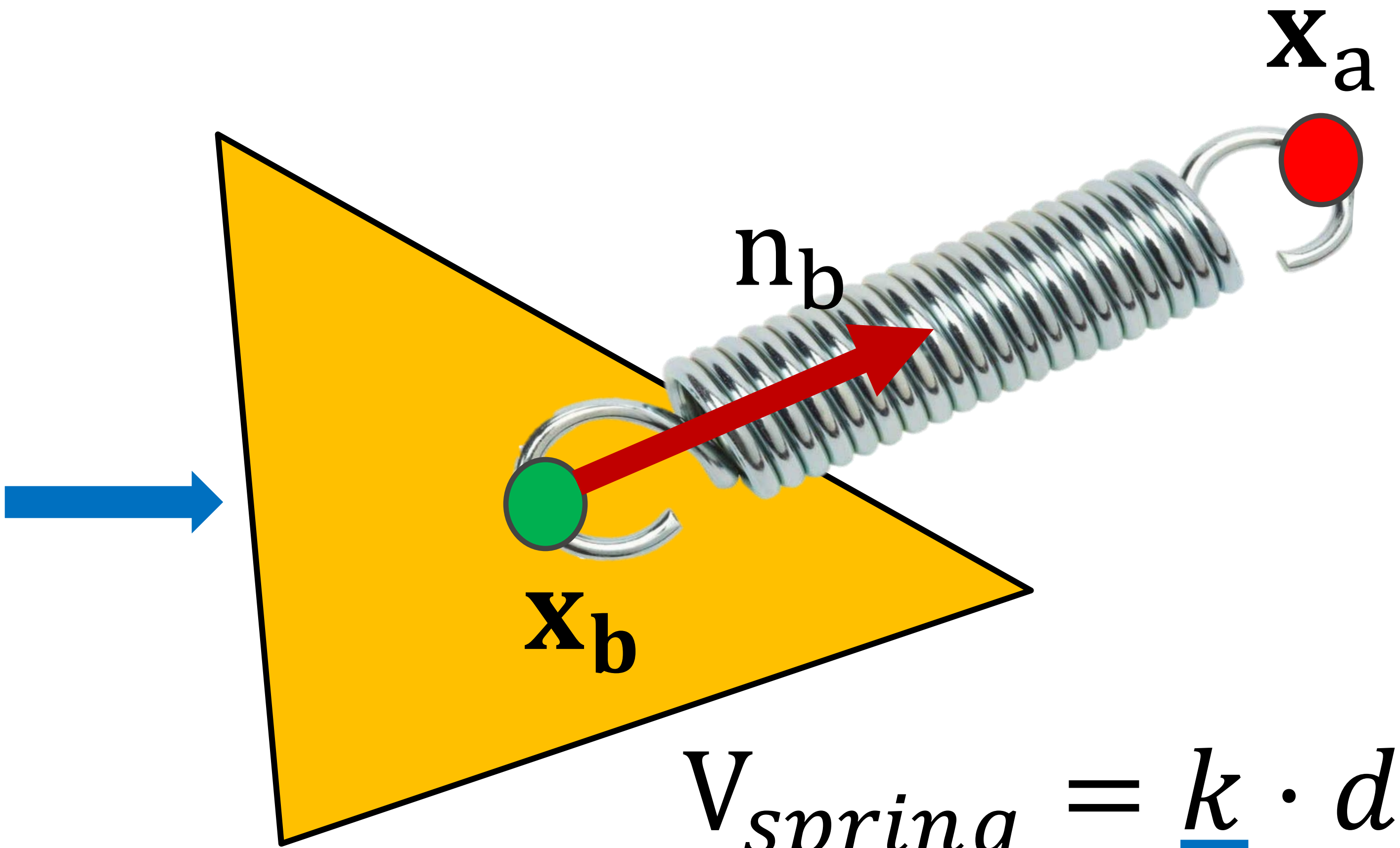


$$\underbrace{V_{spring} = k \cdot d^2}$$

Standard energy form of a zero-rest length spring

Triangle – Vertex Contacts

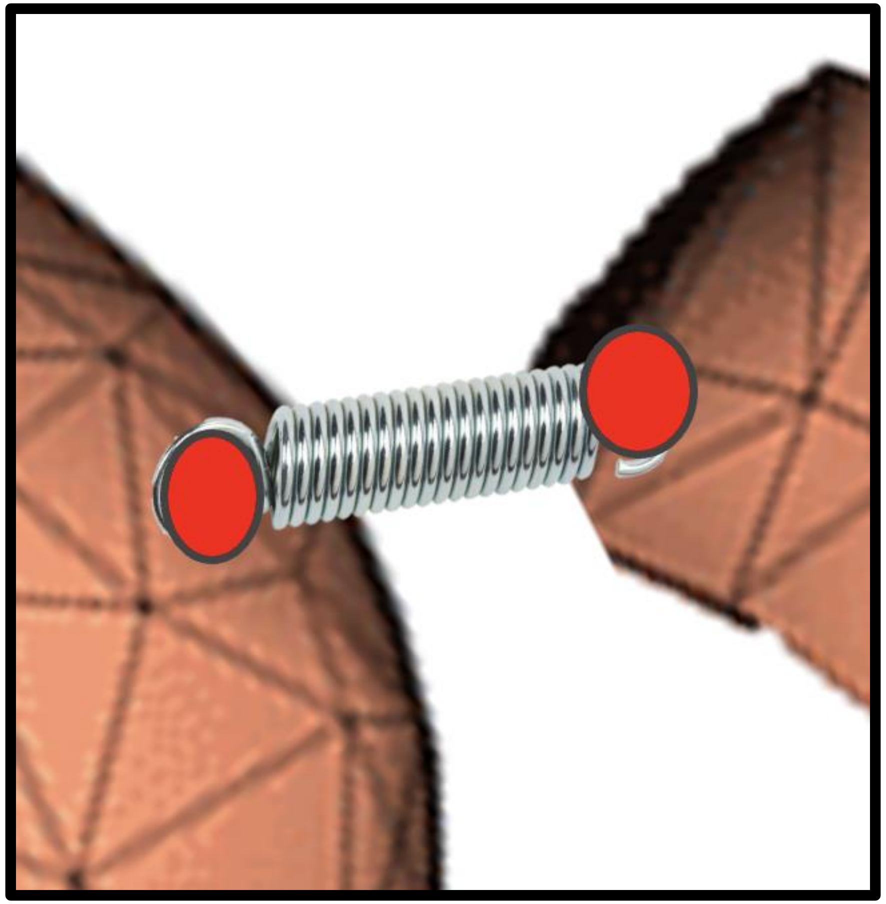


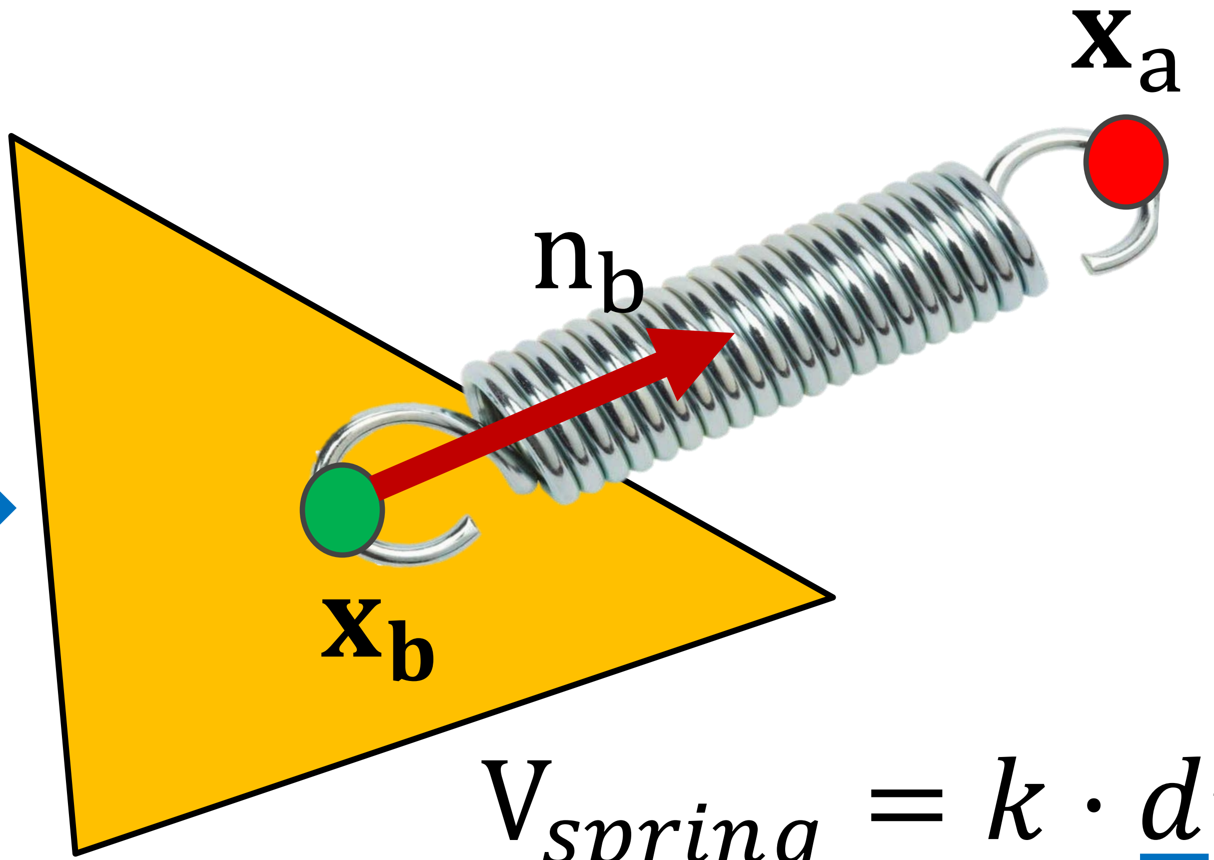


$$V_{spring} = k \cdot d^2$$

Stiffness (user parameter)

Triangle – Vertex Contacts





$$V_{spring} = k \cdot d^2$$

distance ?

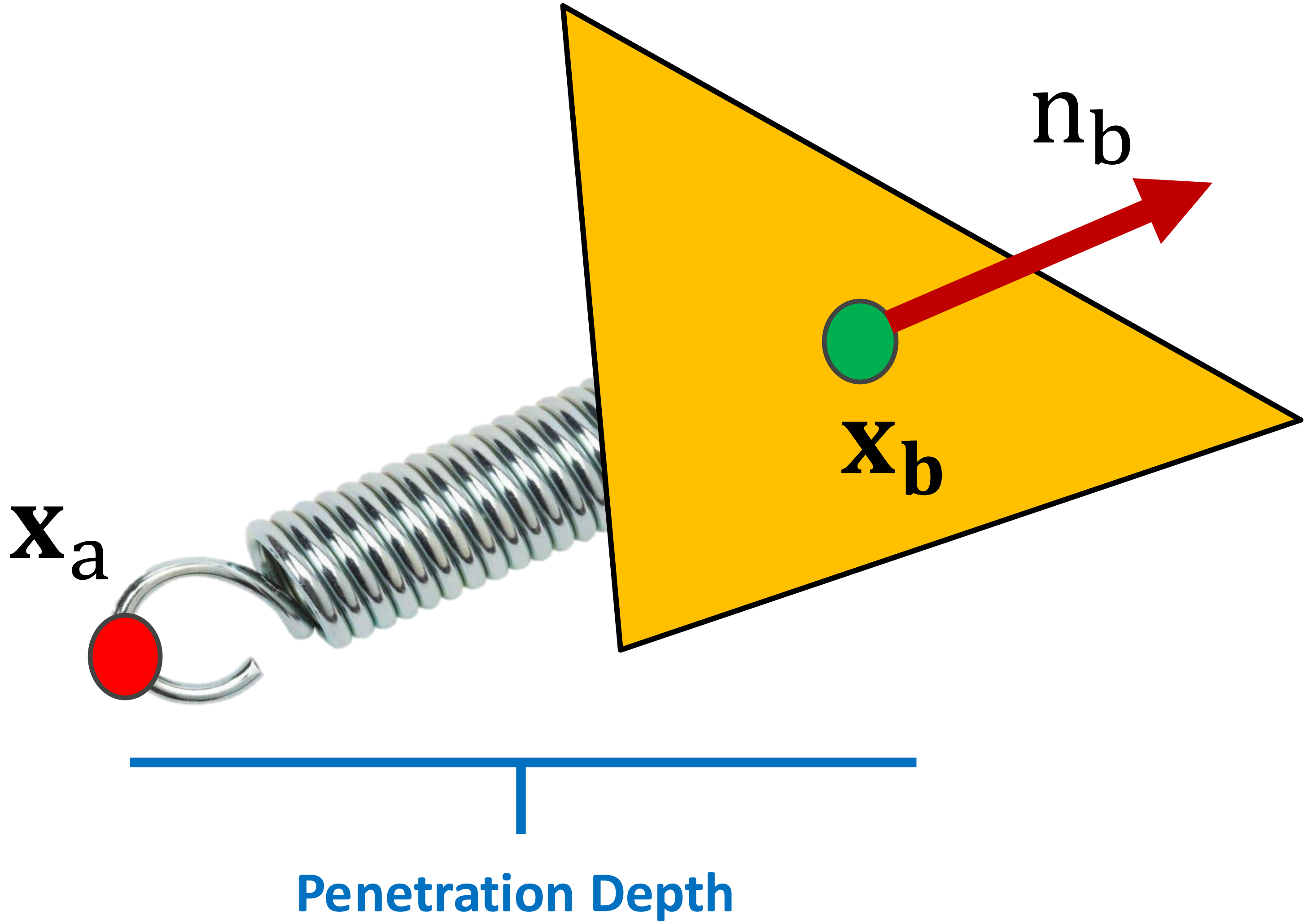
Remember the Rules

1. Contact Forces Prevent Penetration
2. Contact Force Only Push Objects Apart
3. Contact Forces Only Apply when Objects are in Contact

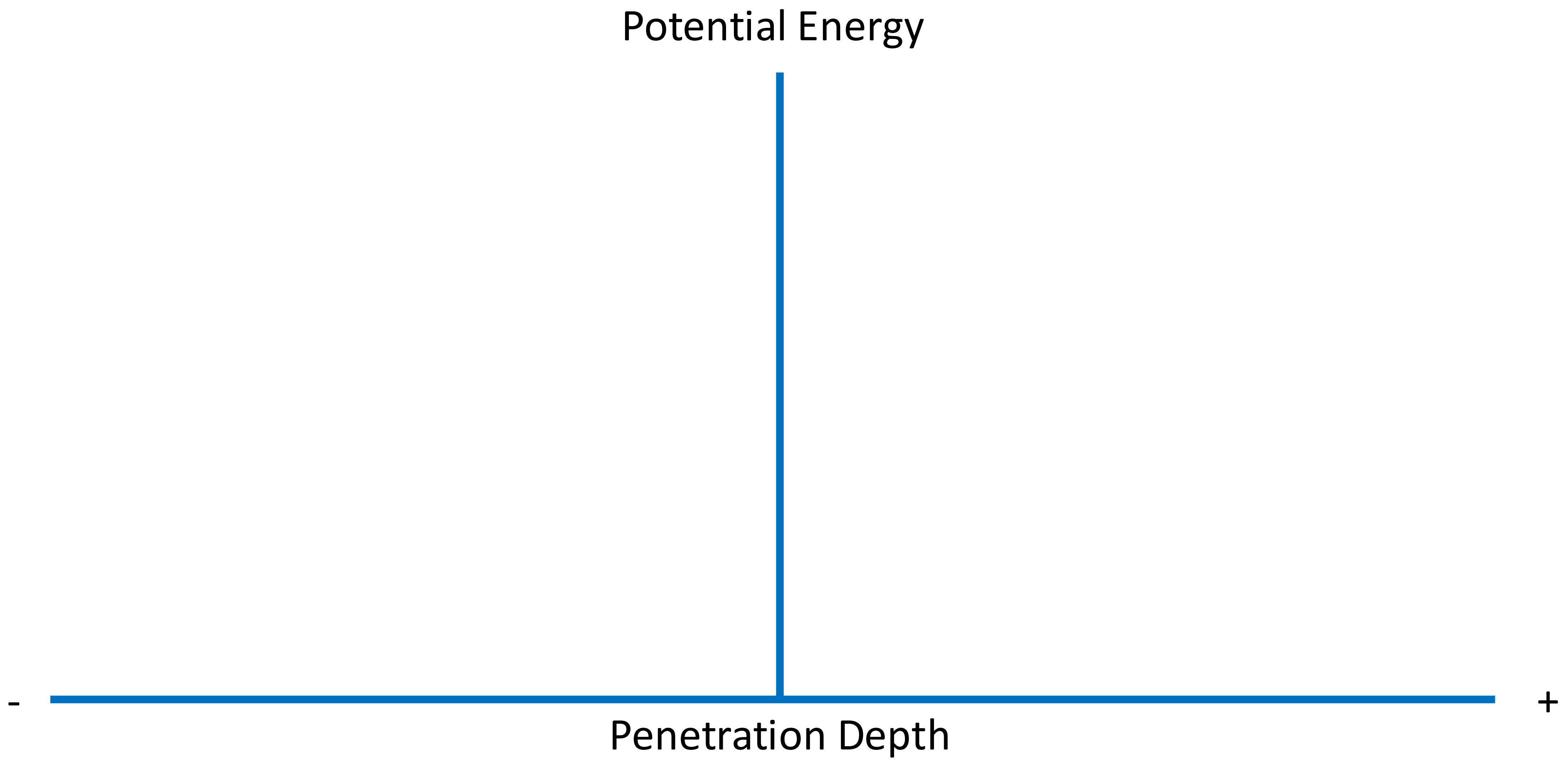
Remember the Rules

1. Contact Forces **UNDO** Penetration
2. Contact Force Only Push Objects Apart
3. Contact Forces Only Apply when Objects Have Penetrated

Triangle – Vertex Contacts



What should this energy look like ?



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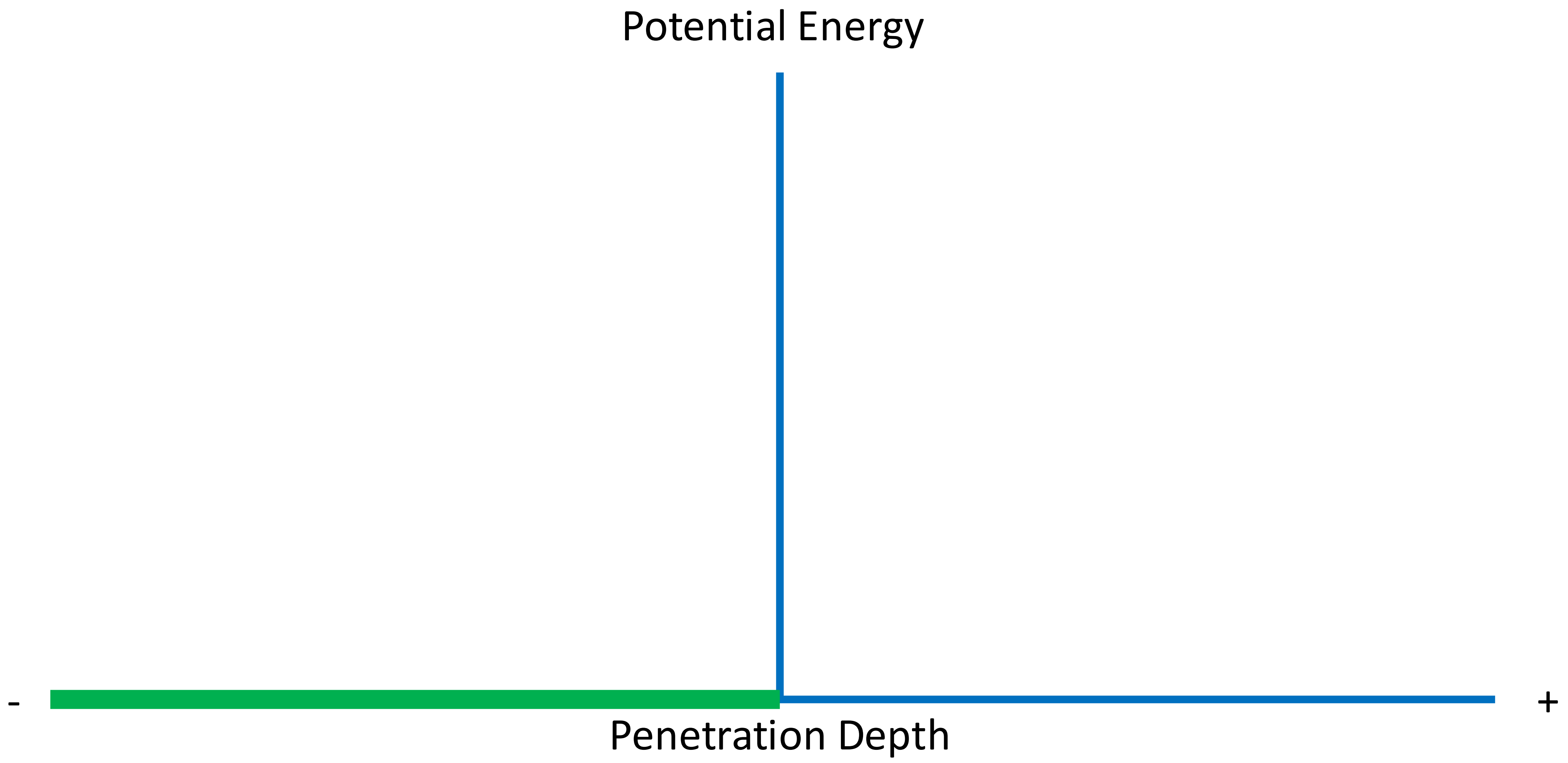

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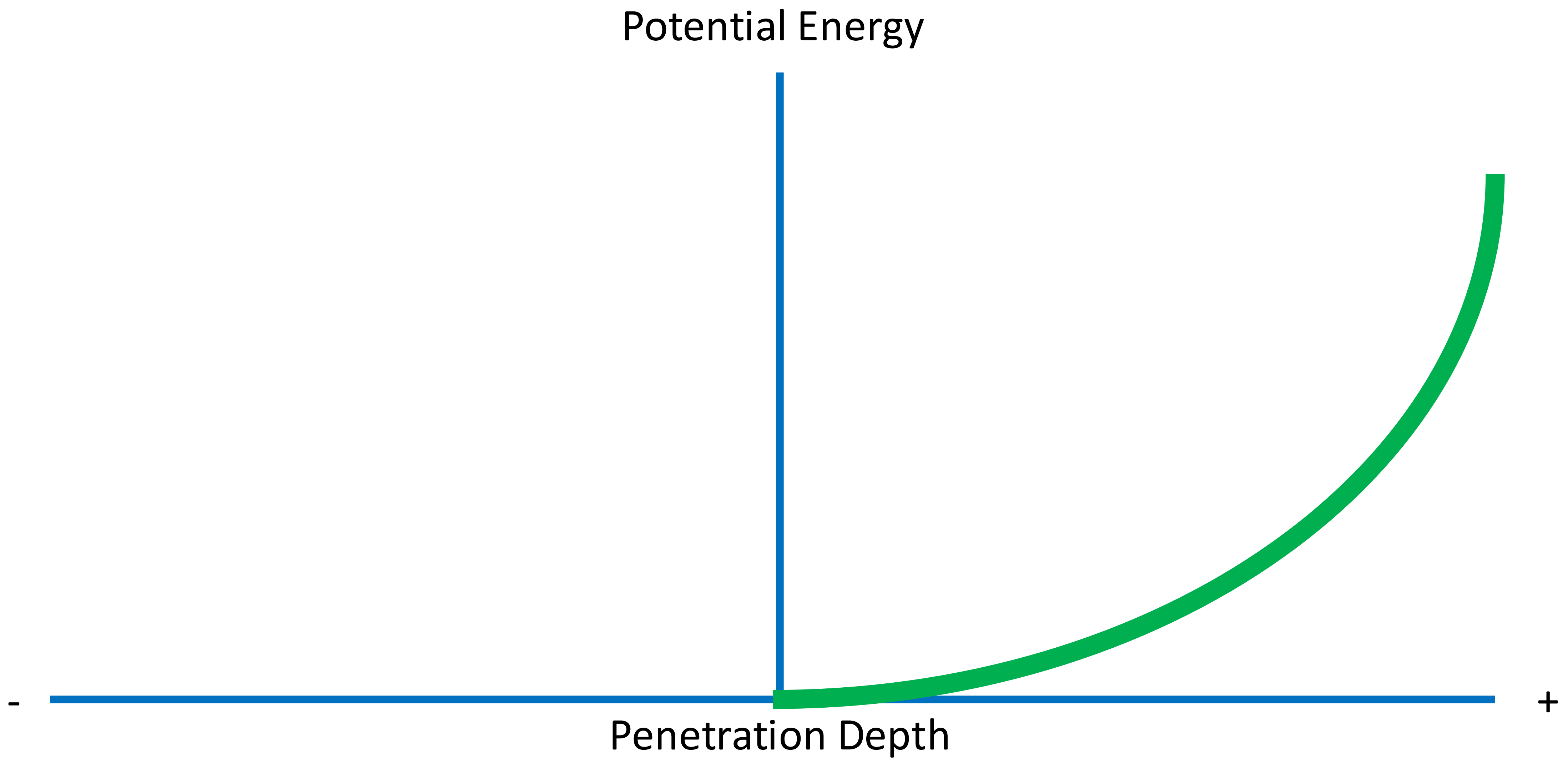
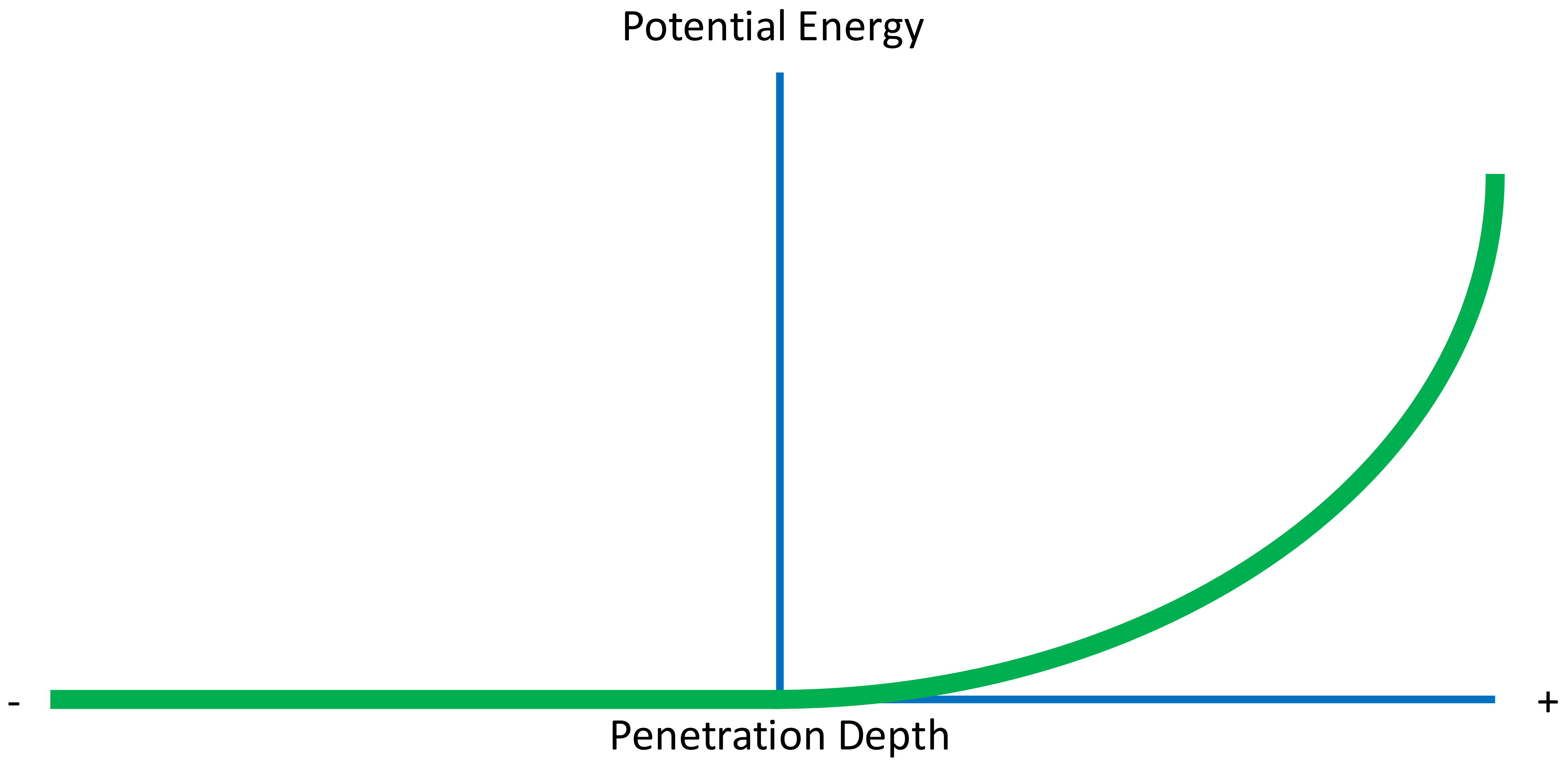
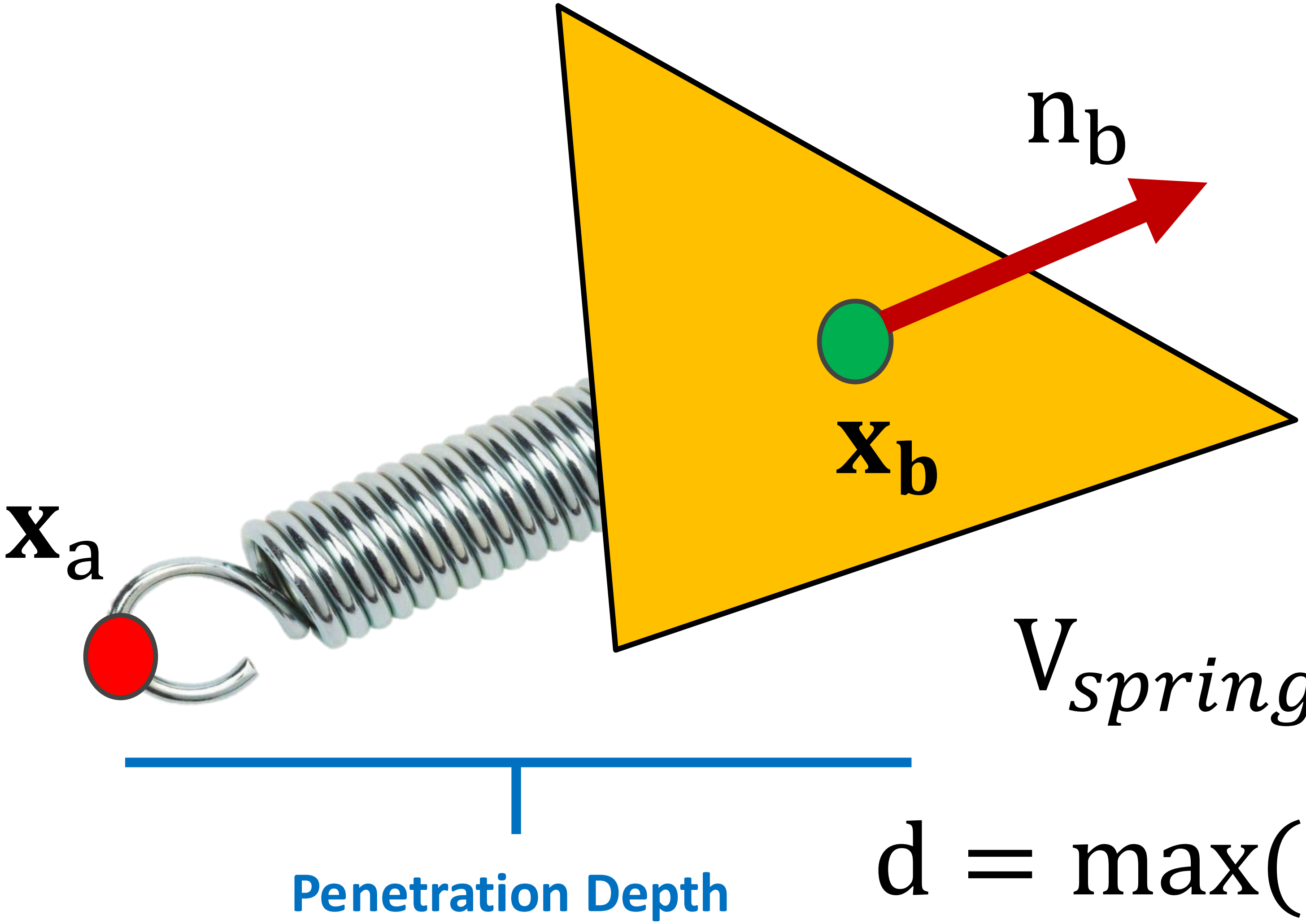


What should this energy look like ?



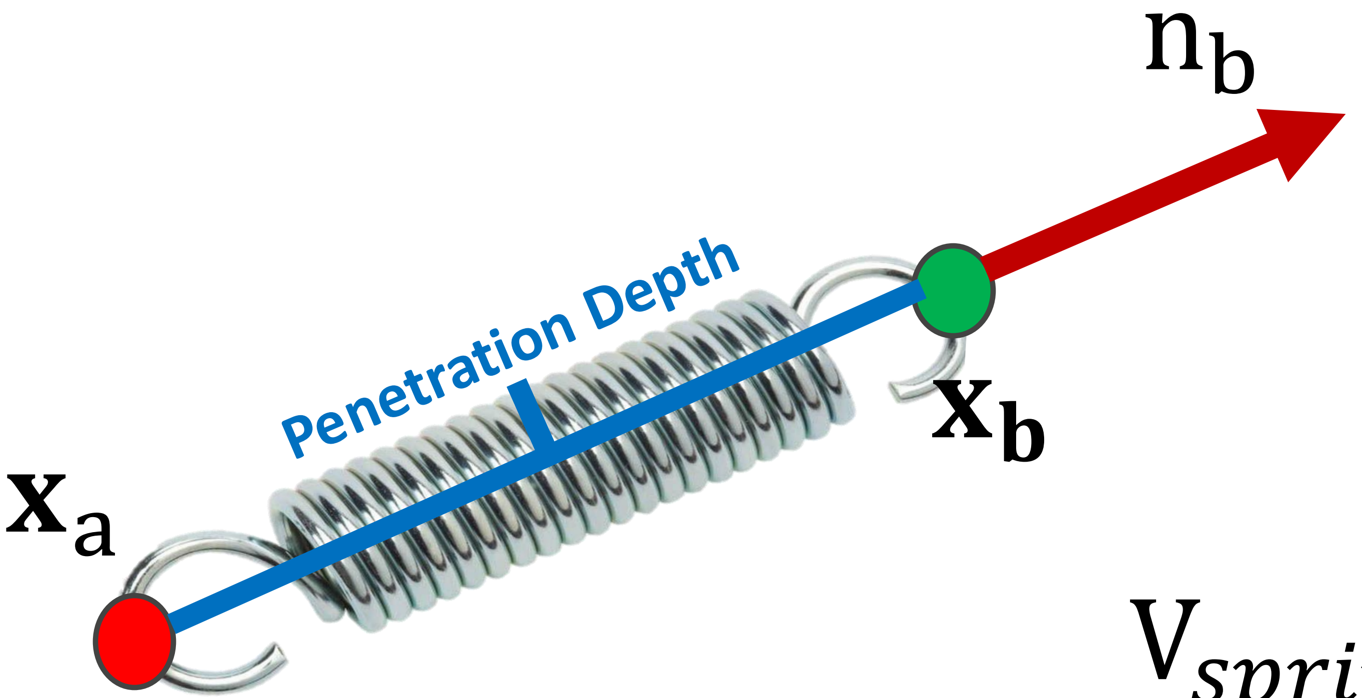
Triangle – Vertex Contacts



$$V_{spring} = k \cdot d^2$$

$$\mathrm{d} = \max(0, \mathrm{depth})$$

Triangle – Vertex Contacts

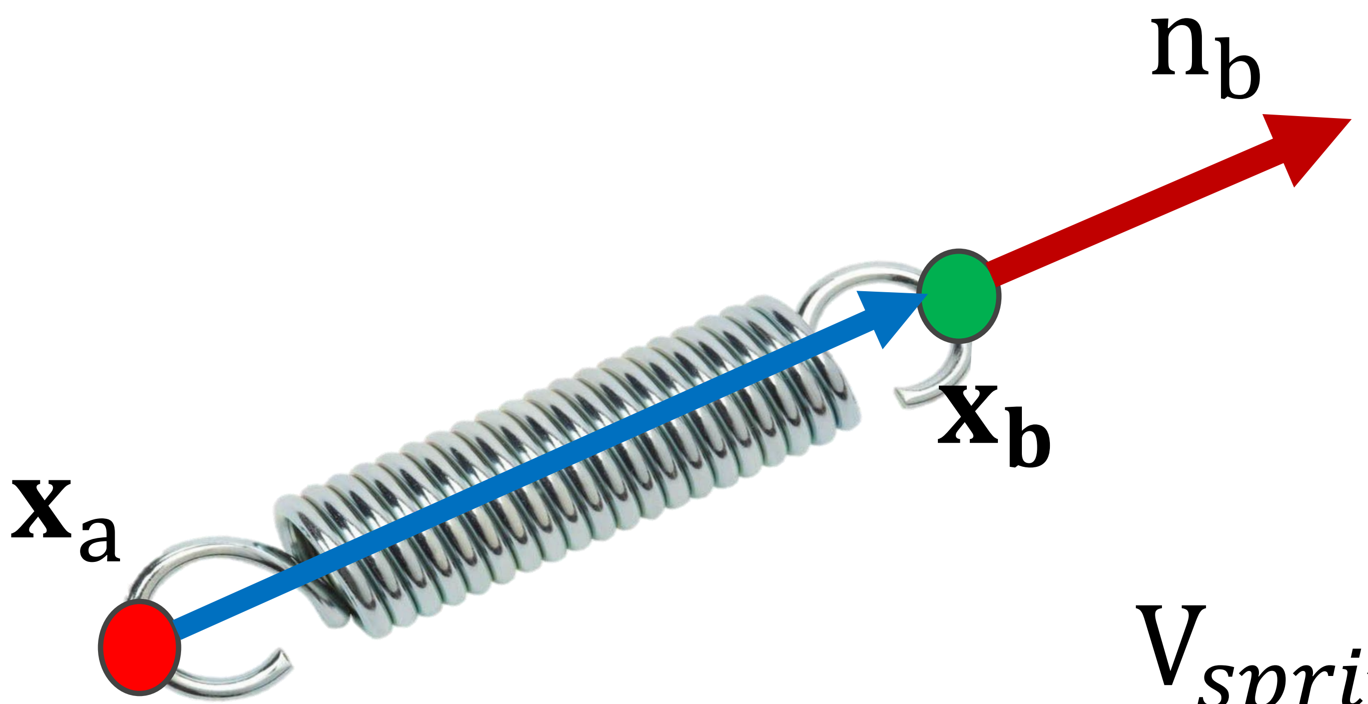


$$\mathrm{V}_{spring} = k \cdot d^2$$

$$\cancel{\mathrm{d} = \max(0, ||\mathbf{x}_\mathbf{b} - \mathbf{x}_\mathrm{a}||_2)}$$

Triangle – Vertex Contacts

What does the normal tell us about the sign of d ?

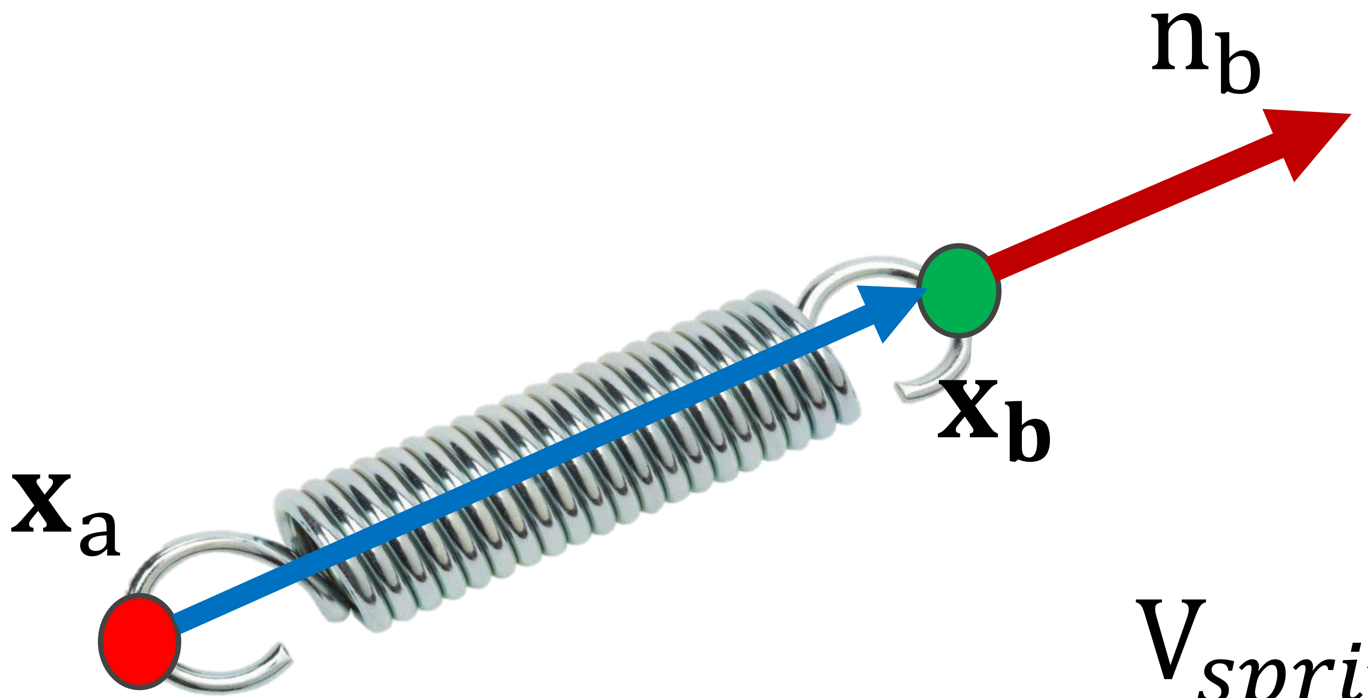


$$V_{spring} = k \cdot d^2$$

$$\cancel{\mathrm{d} = \max(0, ||\mathbf{x_b} - \mathbf{x}_\mathrm{a}||_2\,)}$$

Triangle – Vertex Contacts

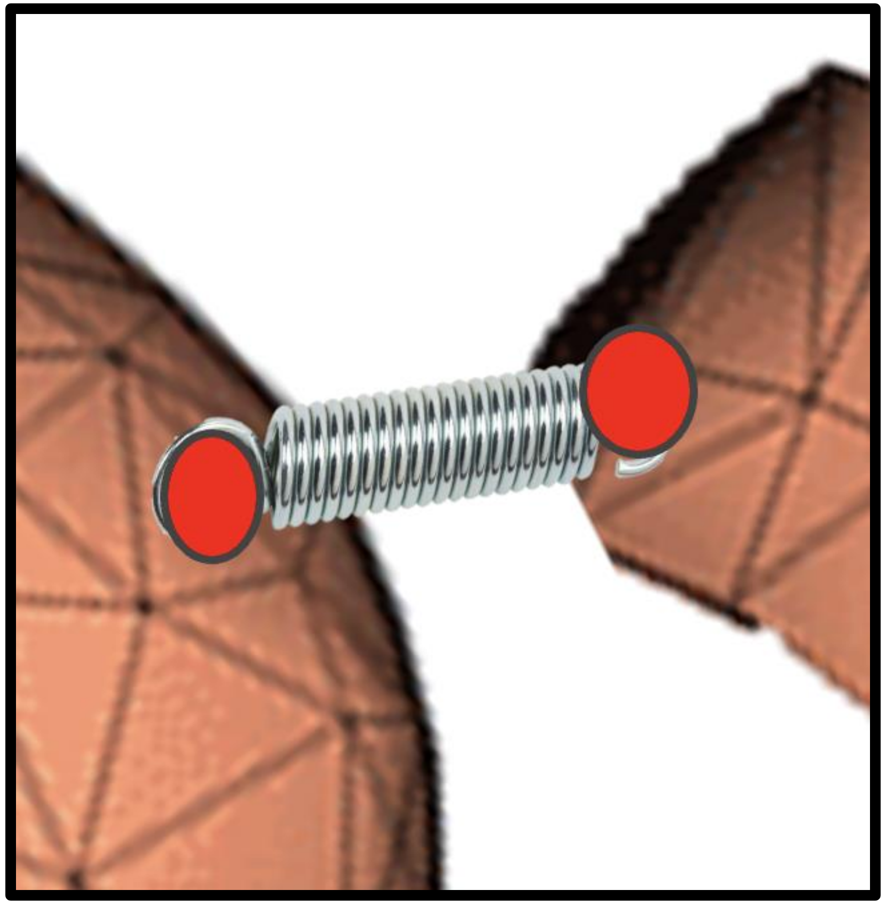
What does the normal tell us about the sign of d ?

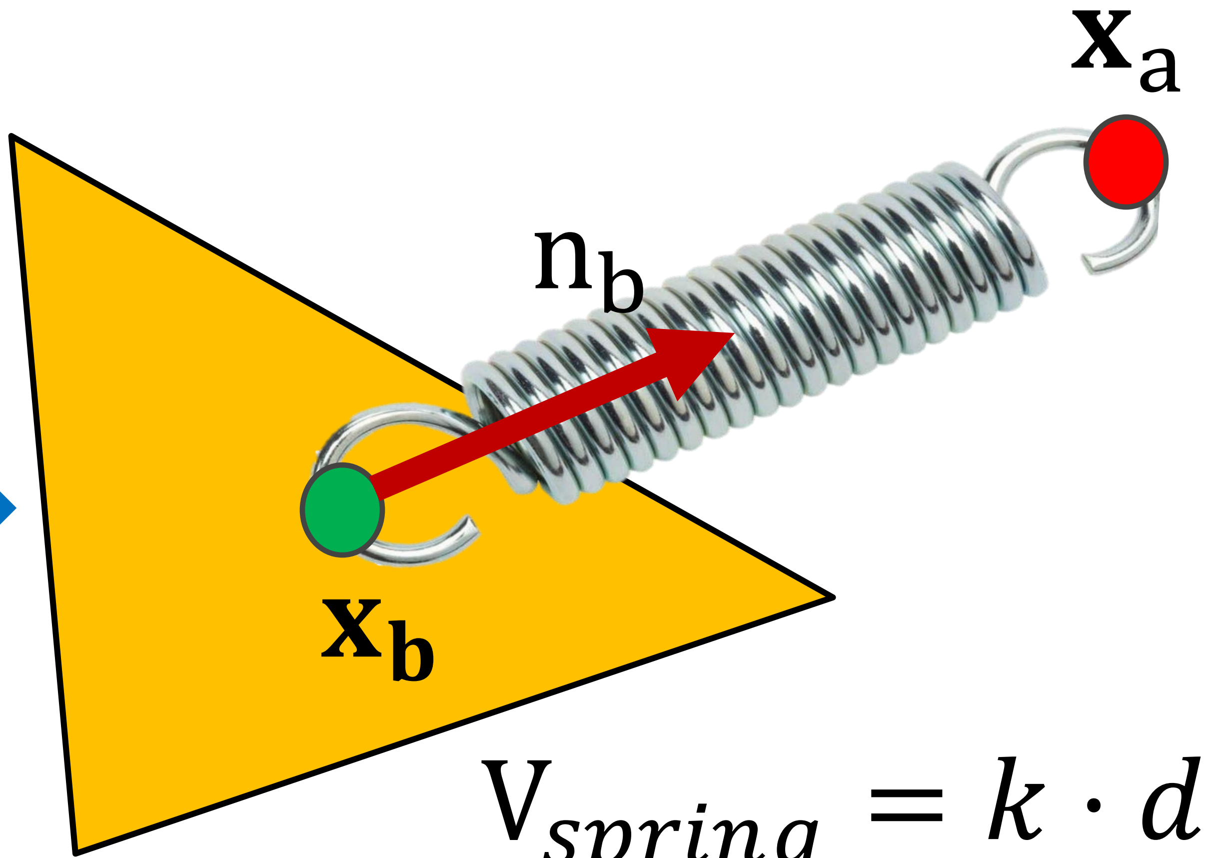


$$V_{spring} = k \cdot d^2$$

$$d = \max(0, (\mathbf{x_b} - \mathbf{x}_a)^T n_b)$$

Triangle – Vertex Contacts

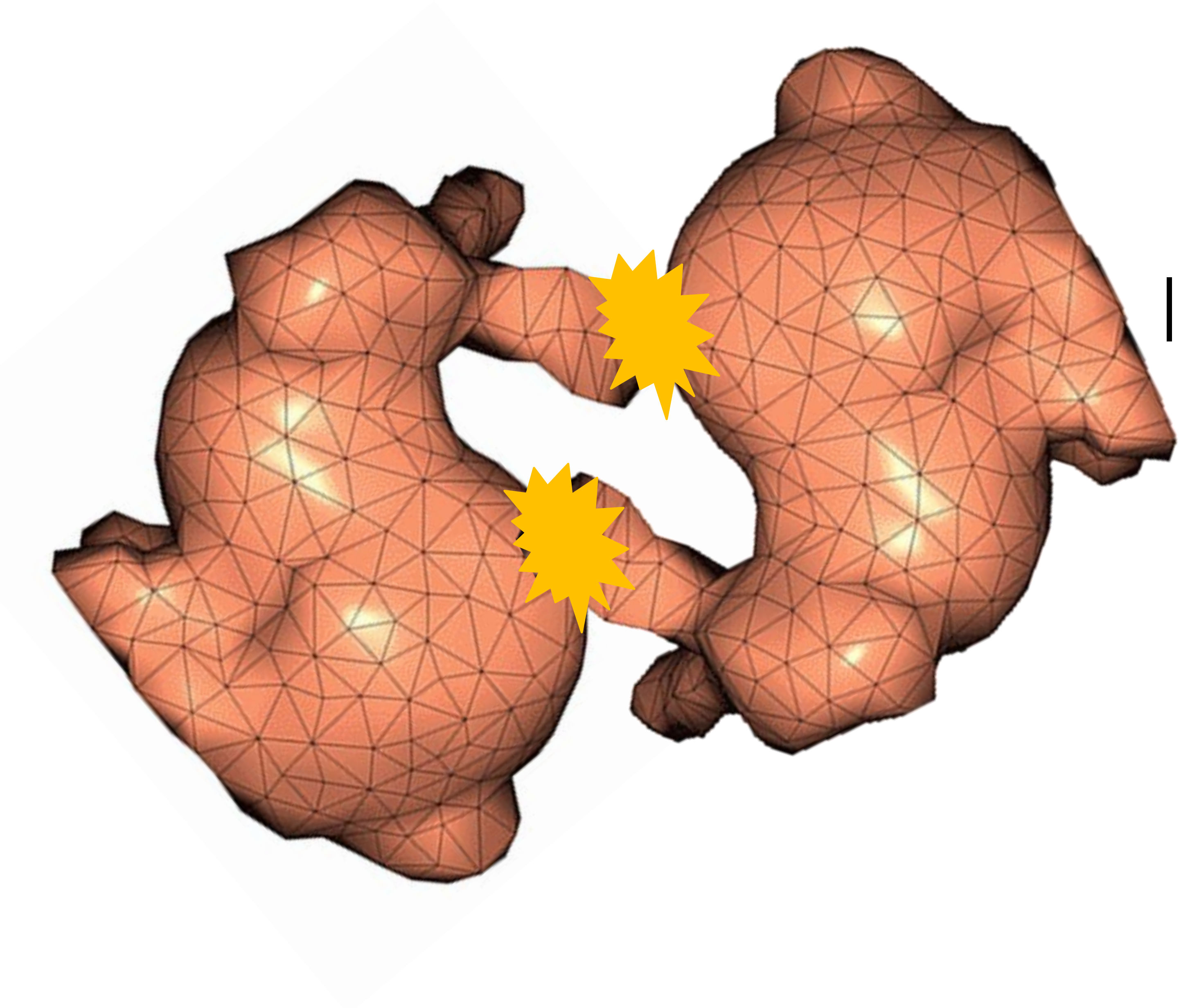




$$V_{spring} = k \cdot d^2$$

$$\mathrm{d} = \max(0, (\mathbf{x_b} - \mathbf{x}_a)^{\mathrm{T}} n_b)$$

Contact Potential Energy



$$\sum_{j=0}^{|\mathrm{Springs}|} V_{spring}(\mathbf{x}(\mathbf{q}))$$

Two Problems with Our Current Approach

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 \underbrace{V\left(\mathbf{q}^{\mathbf{i+1}}\right)}_{\mathrm{V}_{springs} + V_{affine}}$$

~~**Problem 1: Solving this optimization problem only moves one object !!!**~~

~~**Problem 2: There's no term in this optimization that tells it how to handle collisions**~~

Finding Contacts ?

```
list = [] # Empty list of penalty springs
For A in each Object
   For B in each Object
        if A == B
             continue
        else
             For each vertex, v, in A
                  Find triangle, t, with least positive penetration in B
                  Add spring between v  and t  to list
```

Finding Contacts ?

list = [] # Empty list of penalty springs

For *A* in each Object

    For *B* in each Object

        if *A* == *B*

            continue

        else

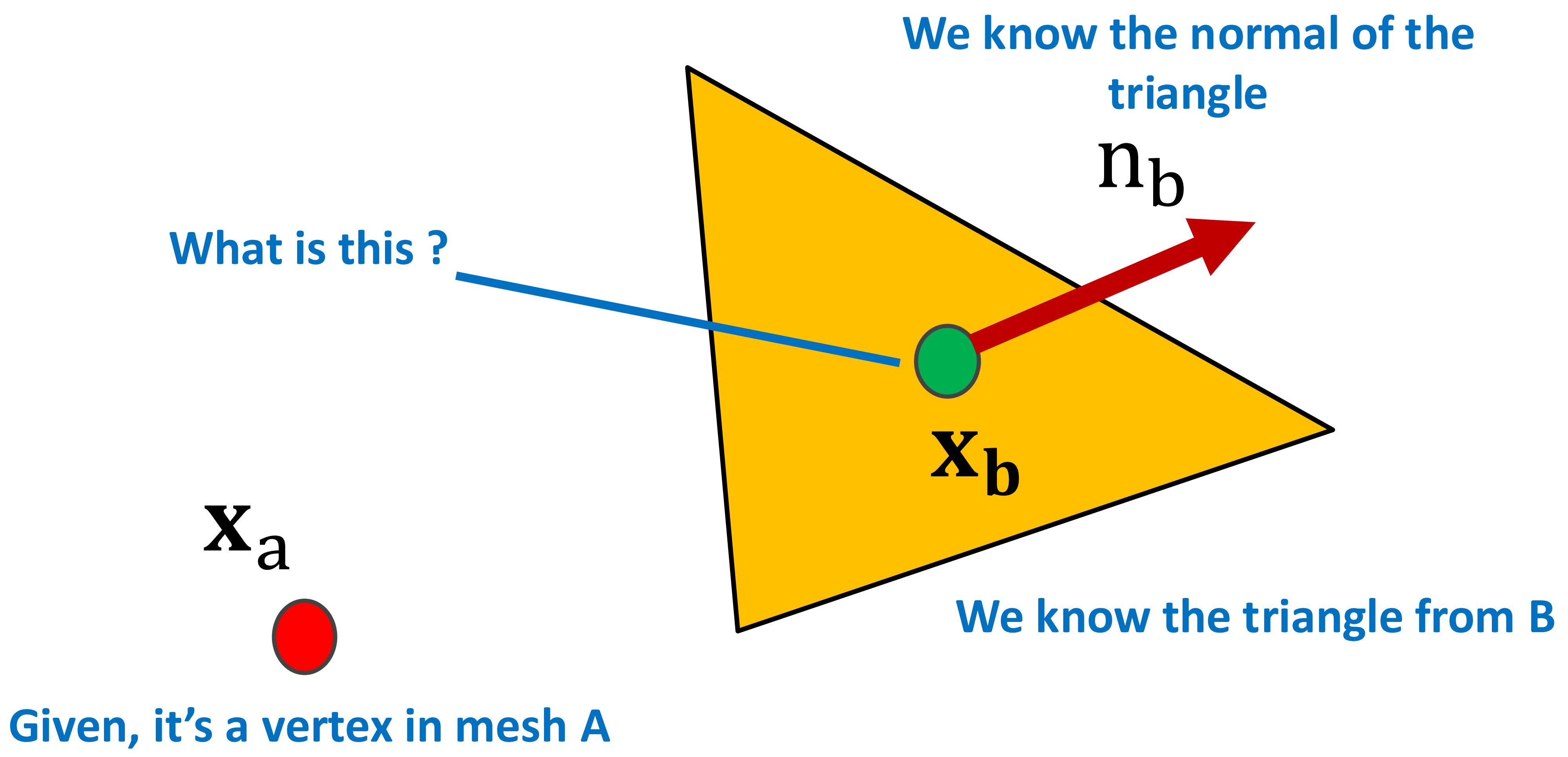
            For each vertex, *v*, in *A*

                Find triangle, *t*, with least positive penetration in B

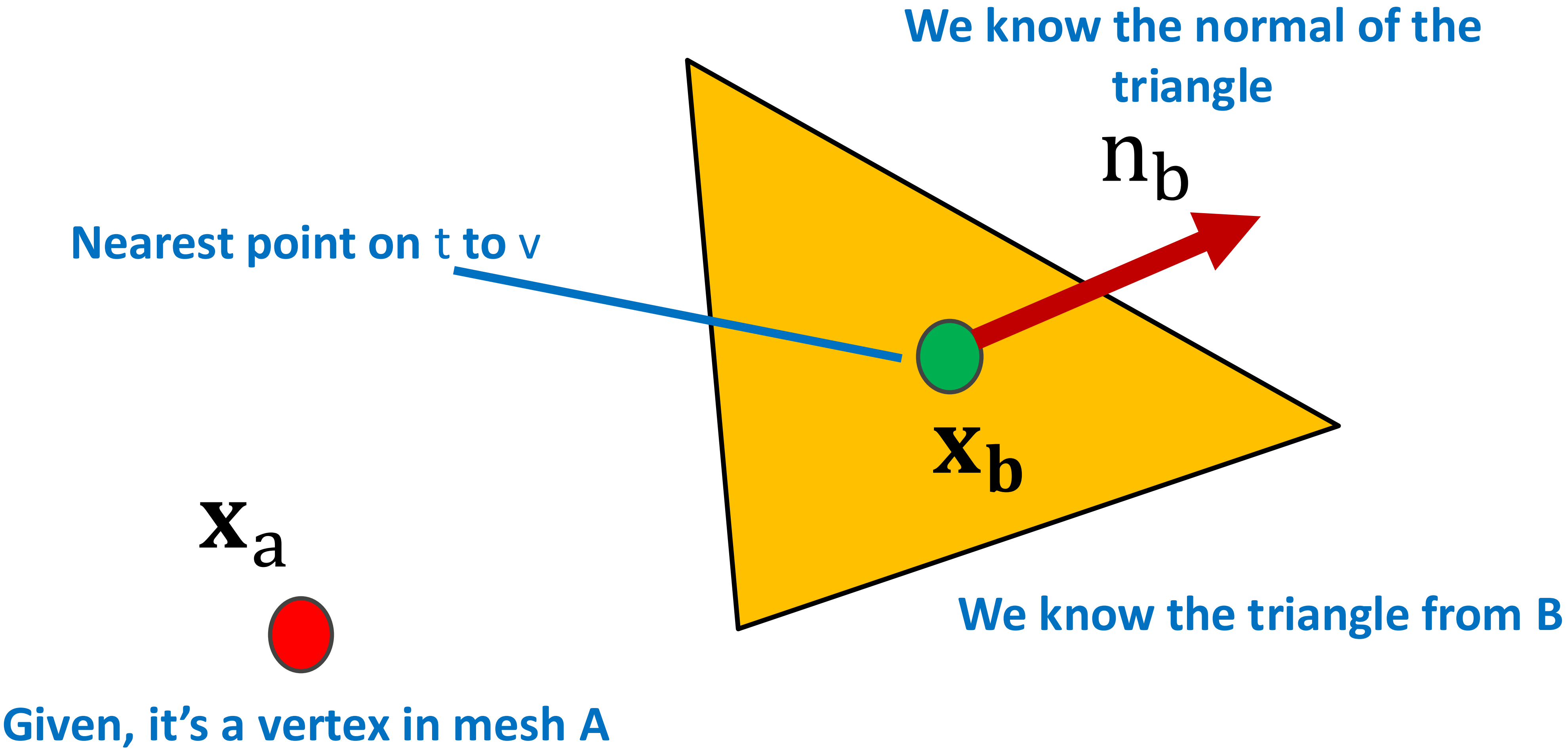
                Add spring between *v* and *t* to *list*

How exactly do we compute this ?

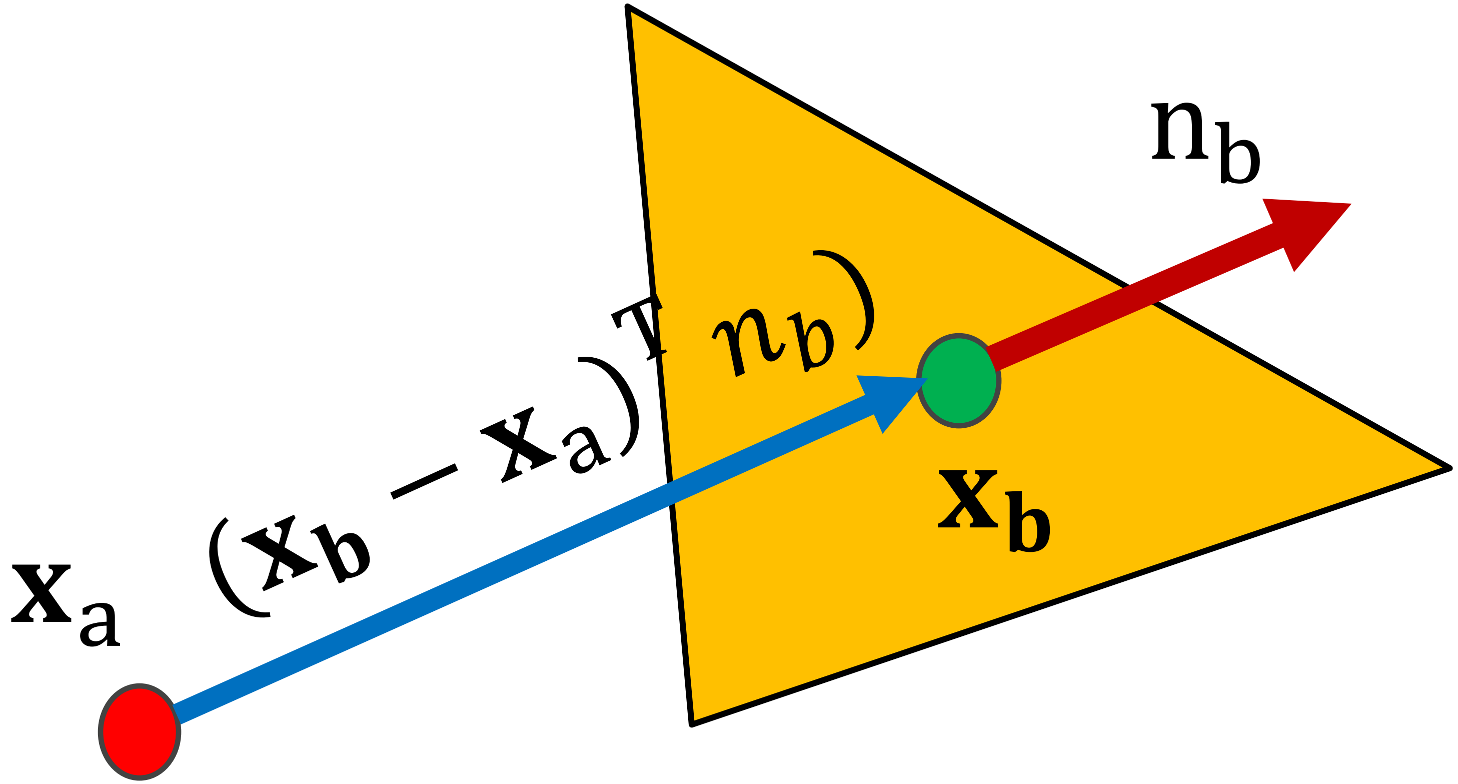
Calculating Penetration Depth For a Single Triangle



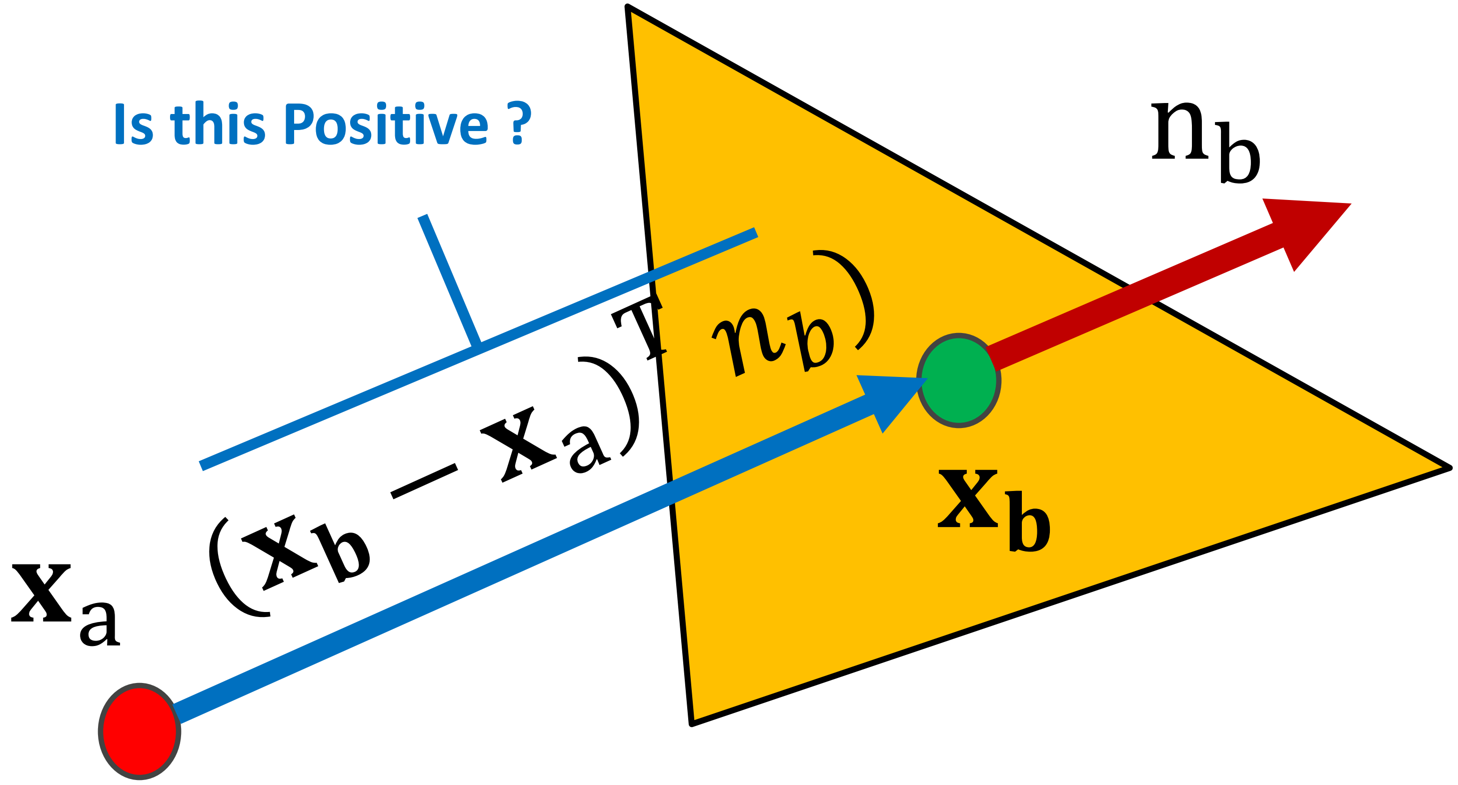
Calculating Penetration Depth For a Single Triangle



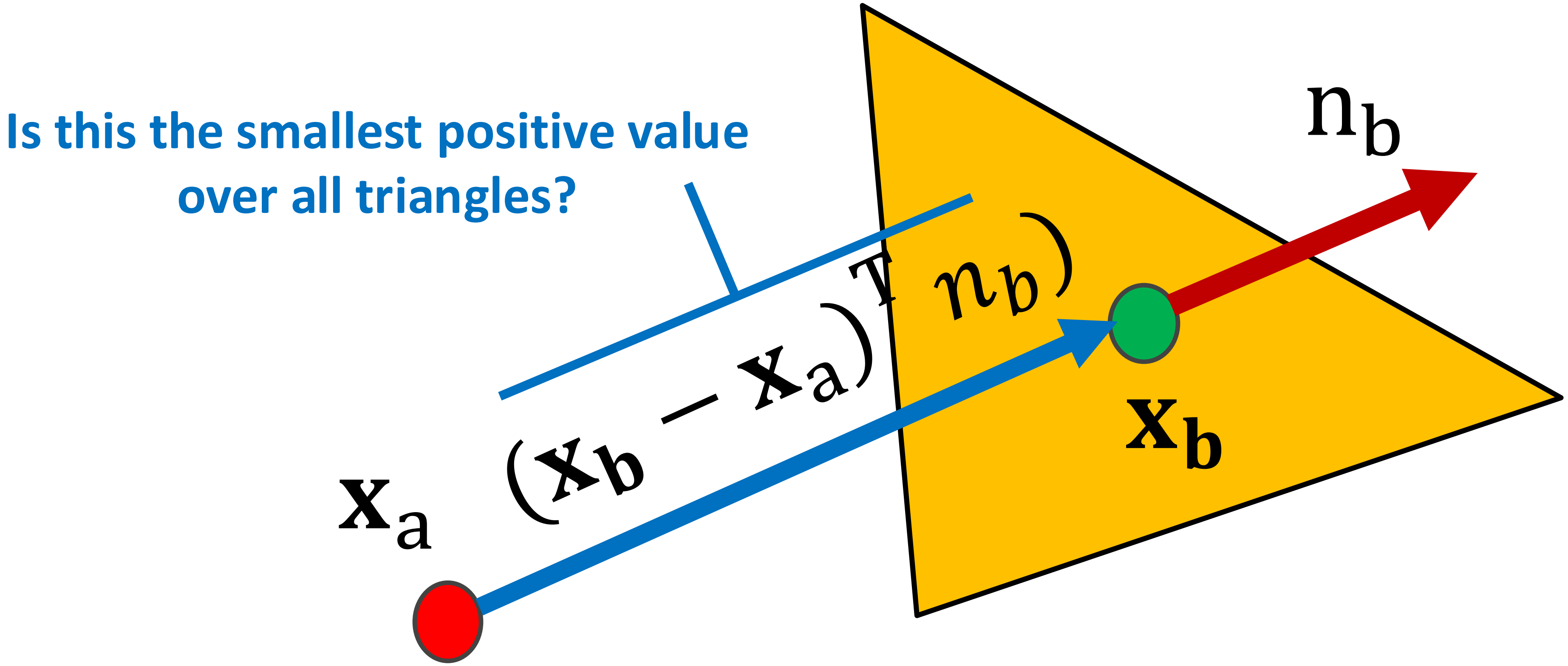
Calculating Penetration Depth For a Single Triangle



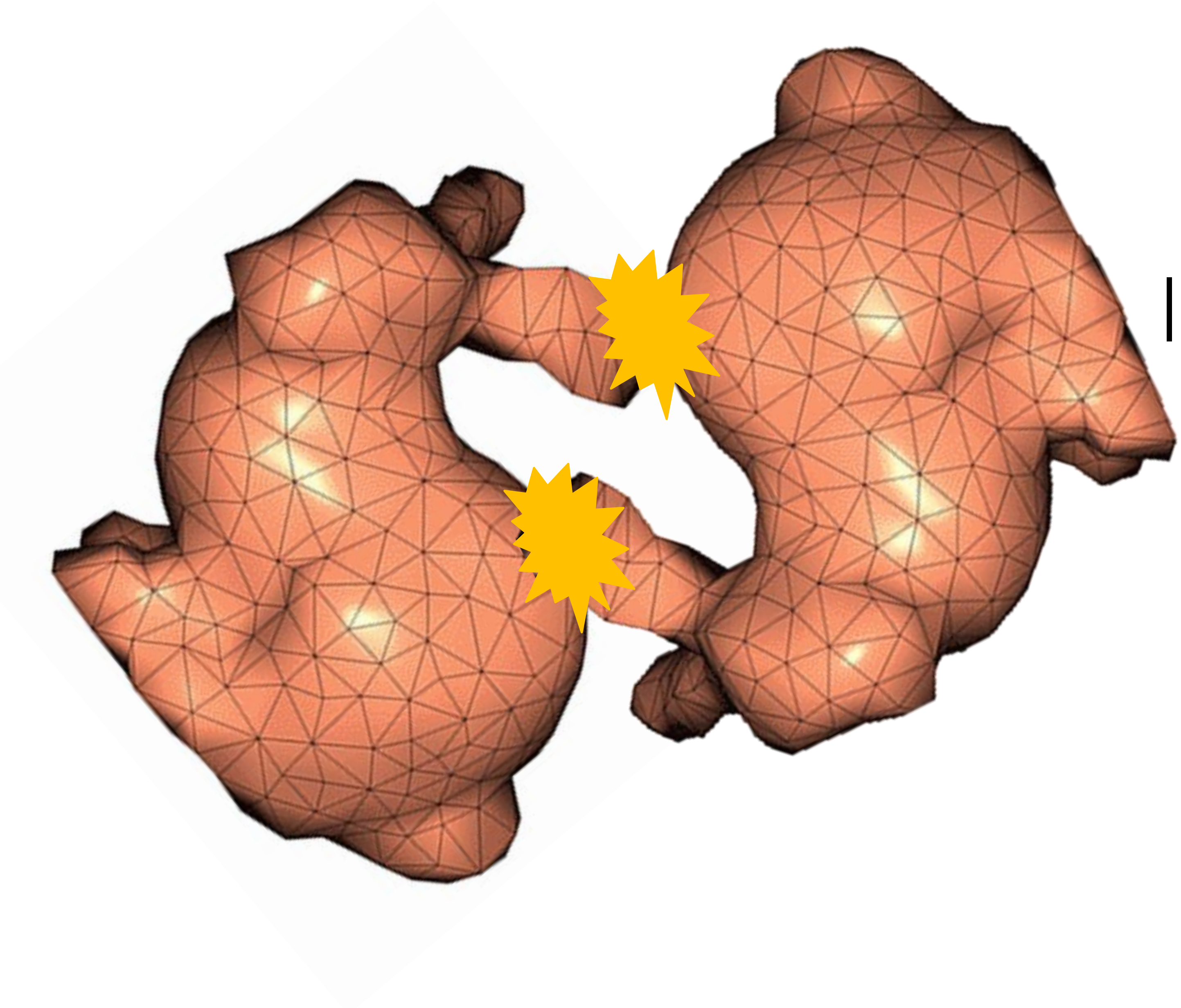
Calculating Penetration Depth For a Mesh



Calculating Penetration Depth For a Mesh



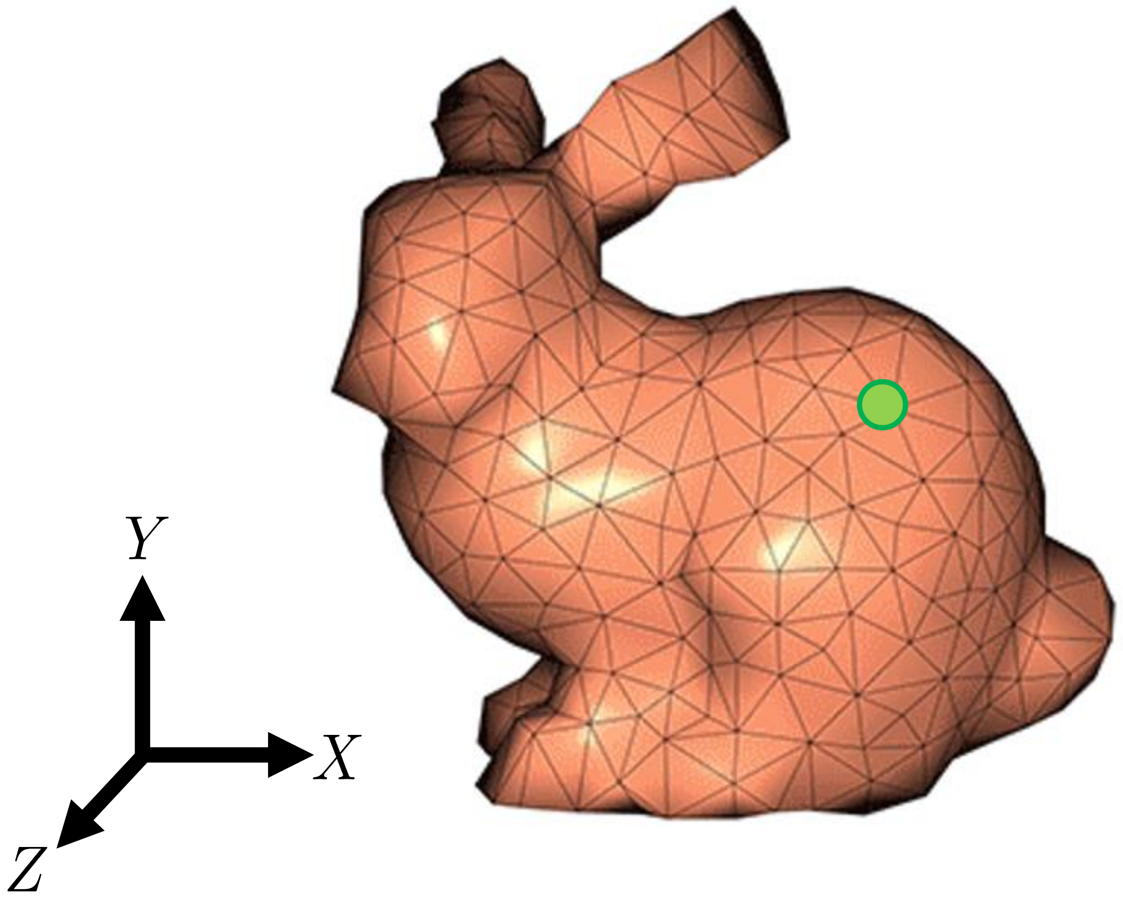
One last thing …



$$\sum_{j=0}^{|\text{Springs}|} V_{spring}(\mathbf{x}(\mathbf{q}))$$

How do I compute this ?

Vectorized Generalized Coordinates

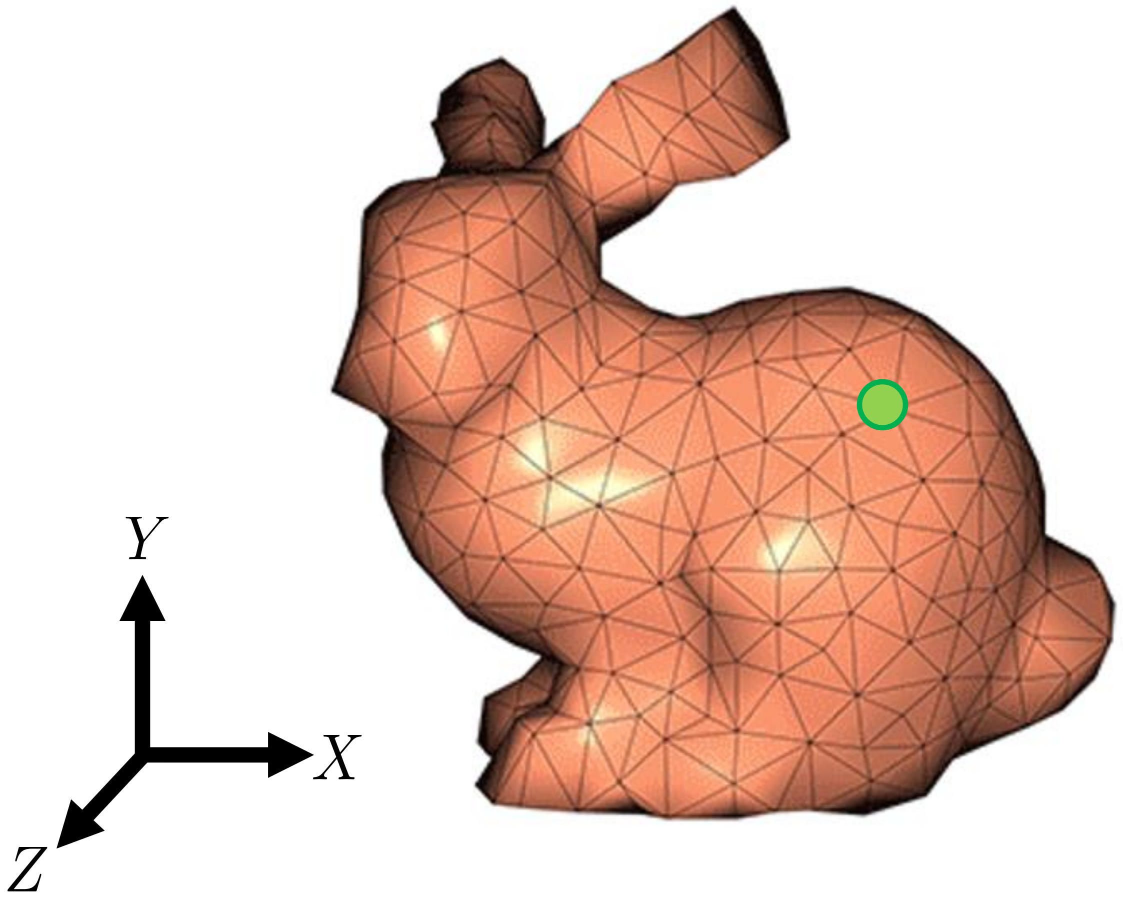


Reference (Undeformed) Space

$$\mathbf{x}(\mathbf{X}, t) = \mathrm{J}(\boldsymbol{X})\mathrm{q}(\mathrm{t})$$

What's the problem?

Vectorized Generalized Coordinates

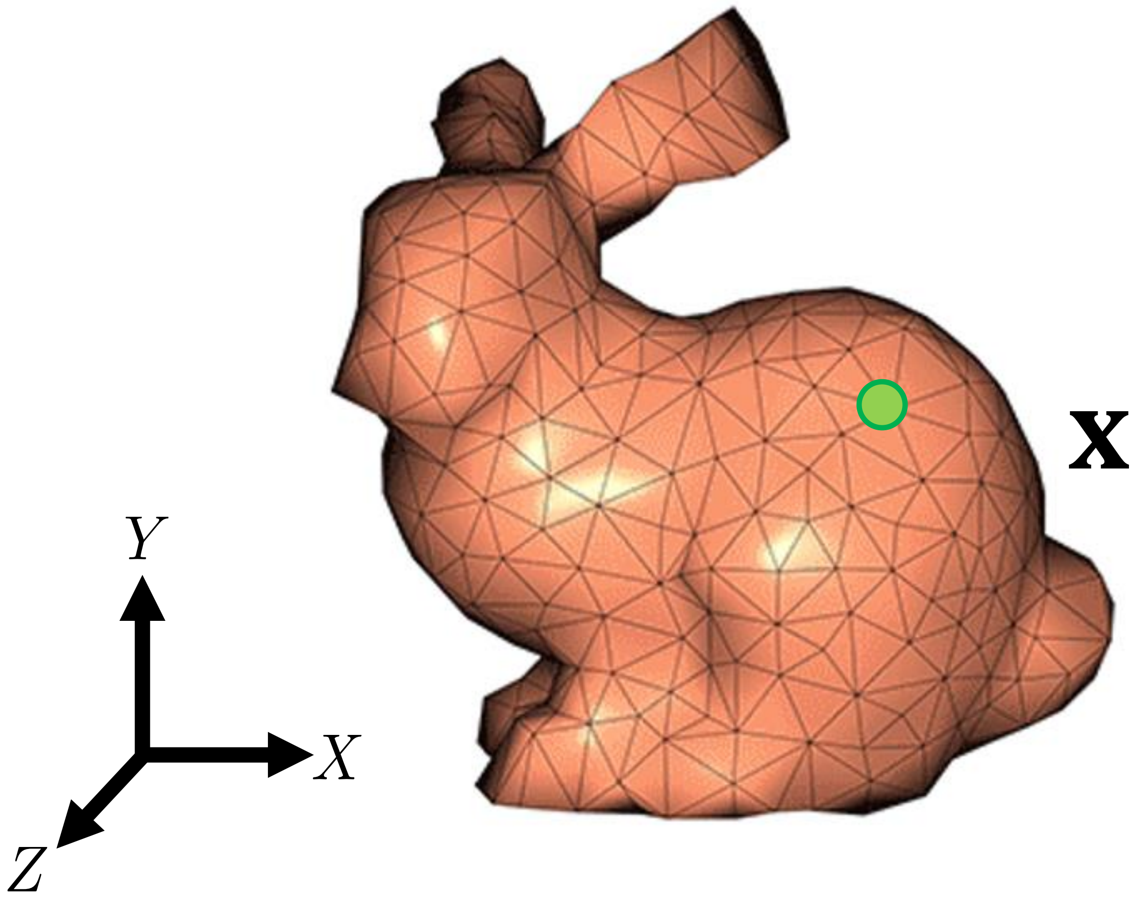


Reference (Undeformed) Space

$$\mathbf{x}(\mathbf{X}, t) = \mathrm{J}(\boldsymbol{X})\mathrm{q}(\mathrm{t})$$

Given x, need to FIND X … grrrrr

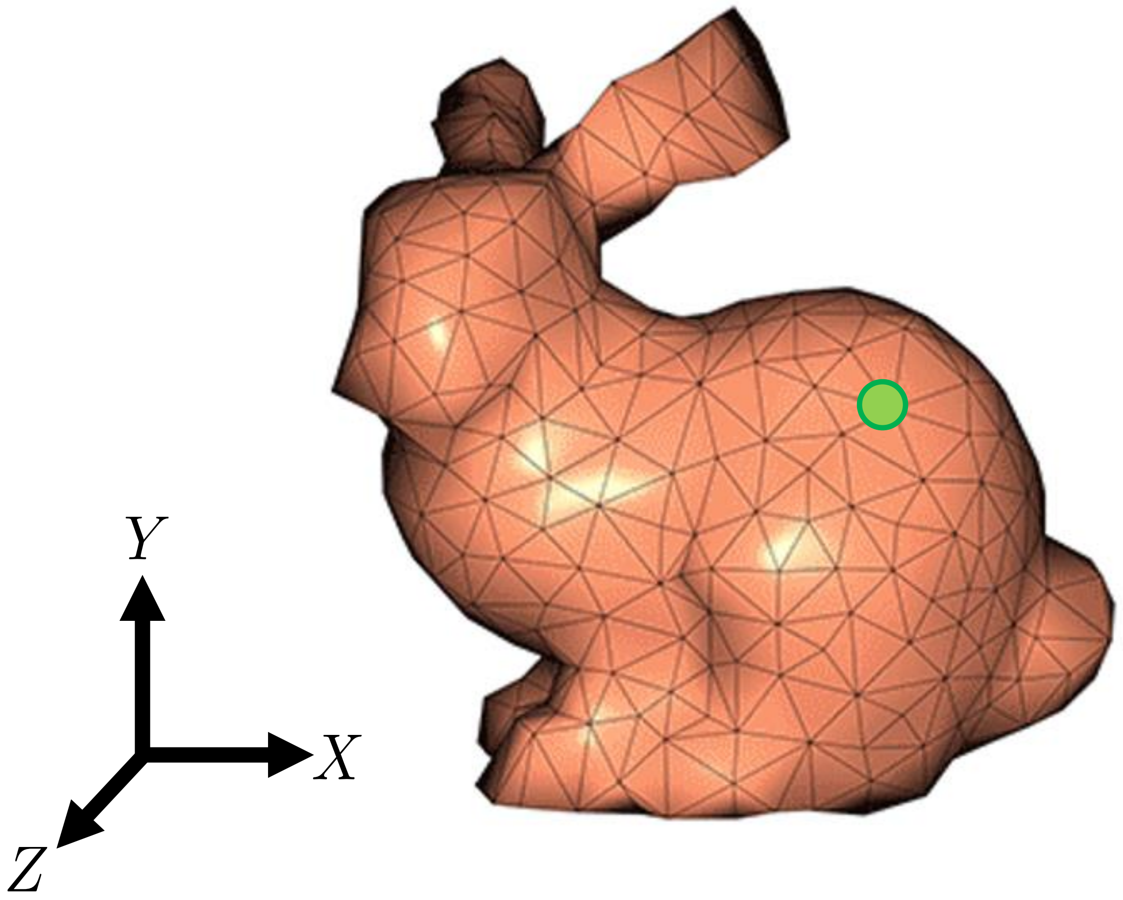
But what is the Deformation Gradient ?



$$\mathbf{x}(\mathbf{X}, t) = \begin{bmatrix} q_0 & q_1 & q_2 \\ q_4 & q_5 & q_6 \\ q_8 & q_9 & q_{10} \end{bmatrix} \mathrm{X} + \begin{bmatrix} q_3 \\ q_7 \\ q_{11} \end{bmatrix}$$

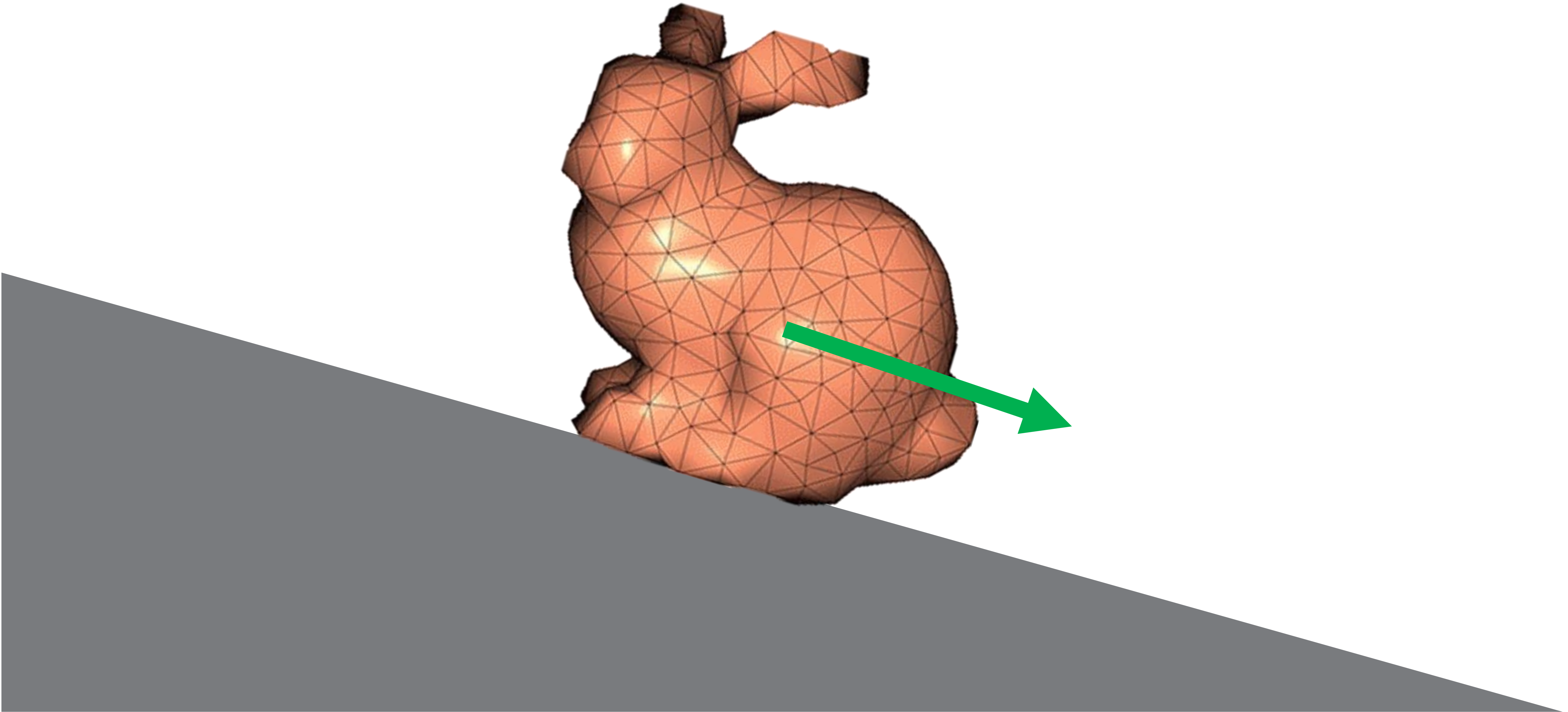
Reference (Undeformed) Space

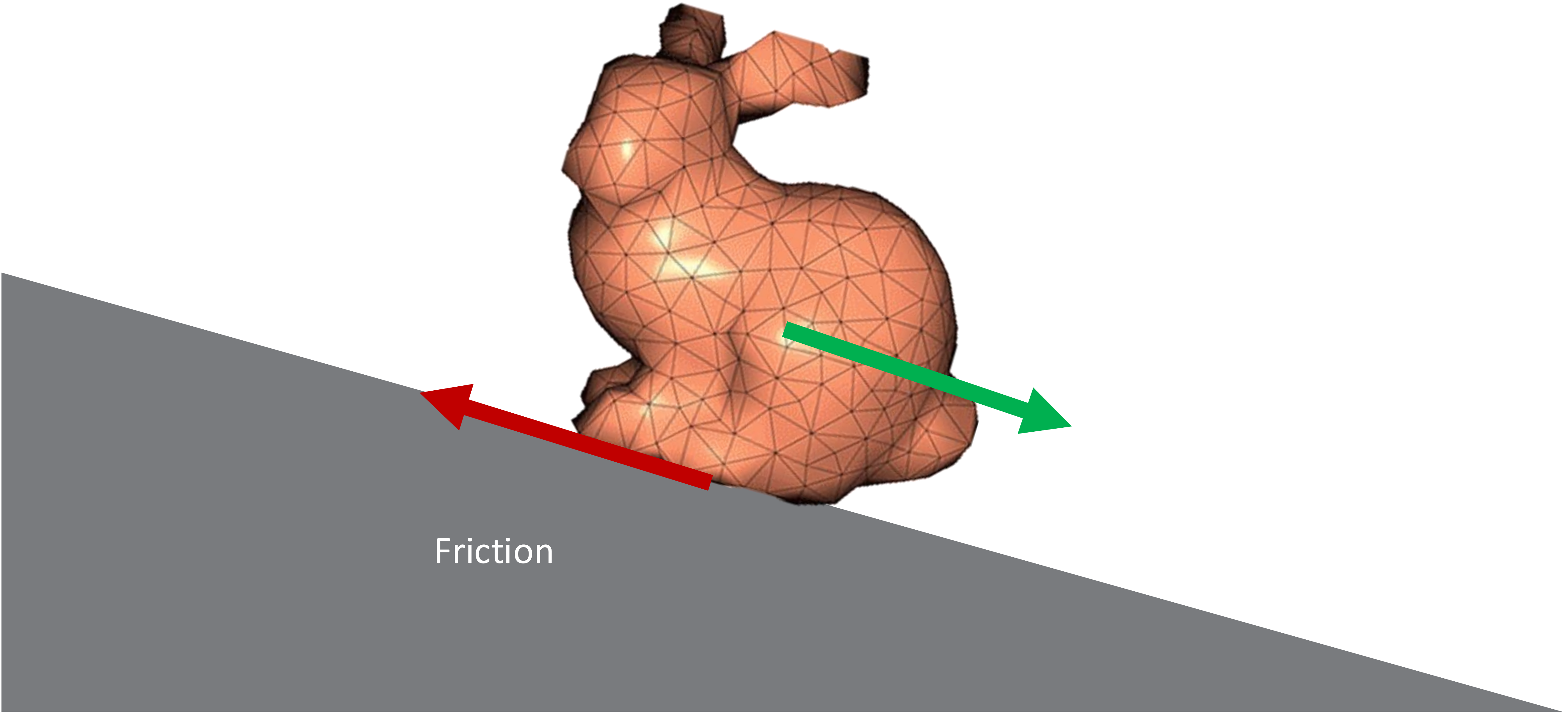
But what is the Deformation Gradient ?

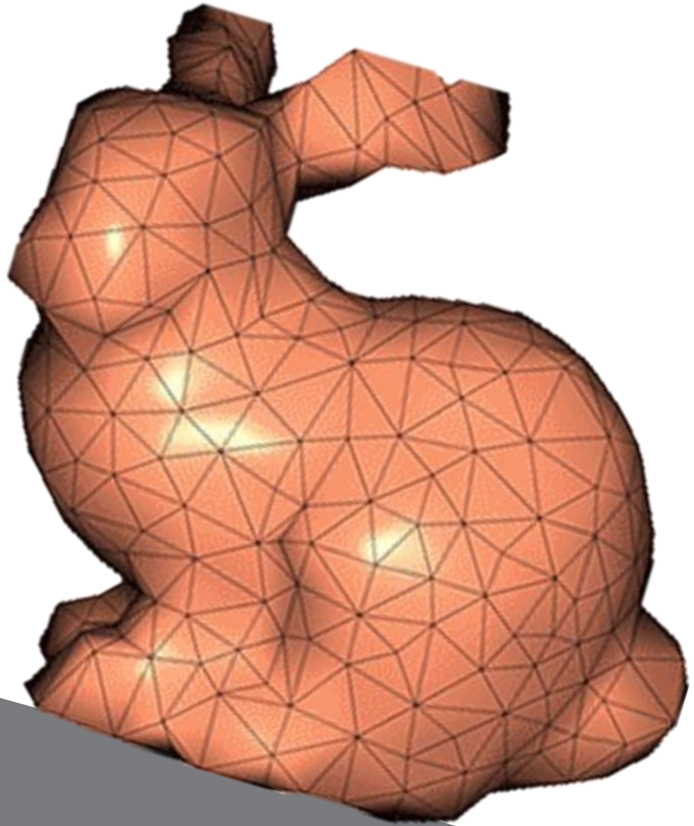


$$\begin{bmatrix} q_0 & q_1 & q_2 \\ q_4 & q_5 & q_6 \\ q_8 & q_9 & q_{10} \end{bmatrix}^{-1} \left(\mathbf{x} - \begin{bmatrix} q_3 \\ q_7 \\ q_{11} \end{bmatrix} \right) = \boldsymbol{X}$$

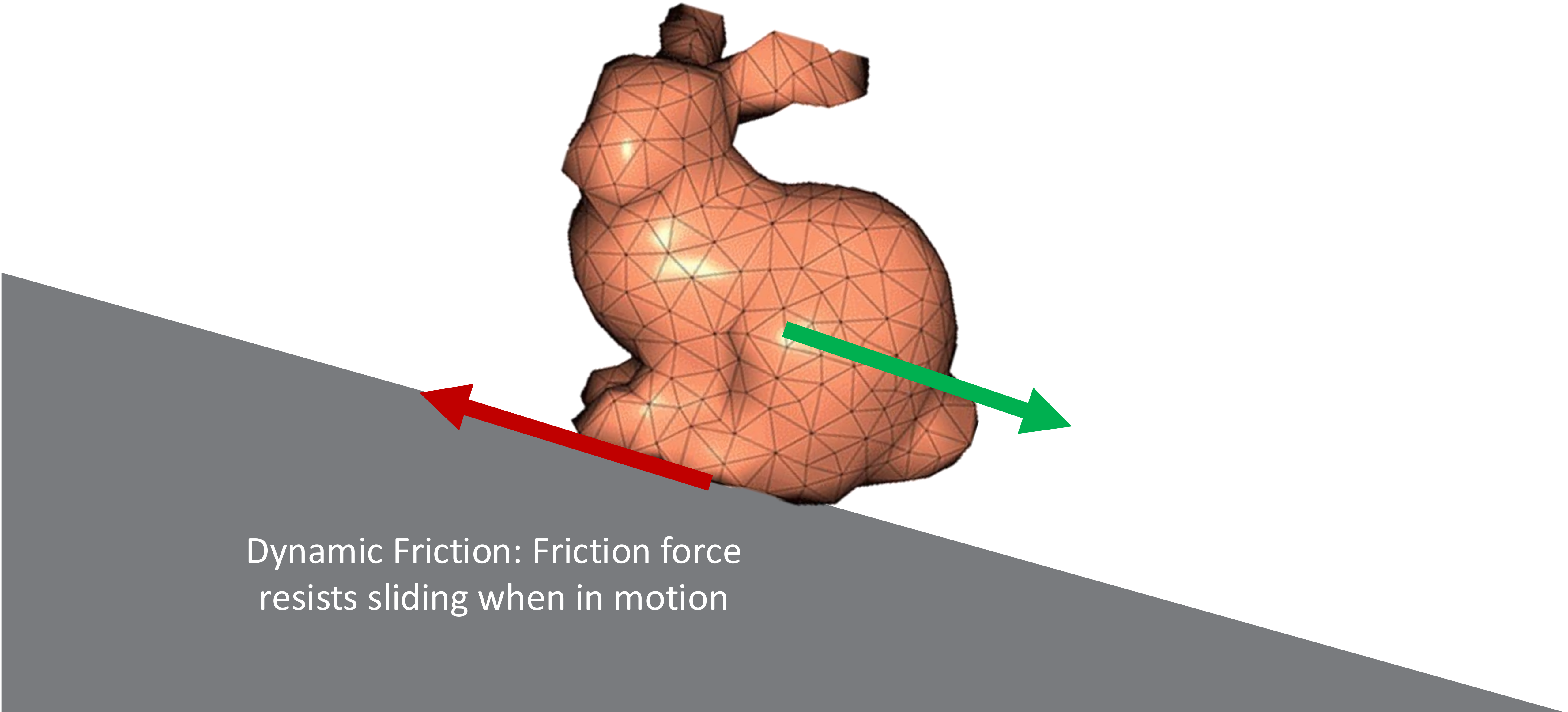
Reference (Undeformed) Space



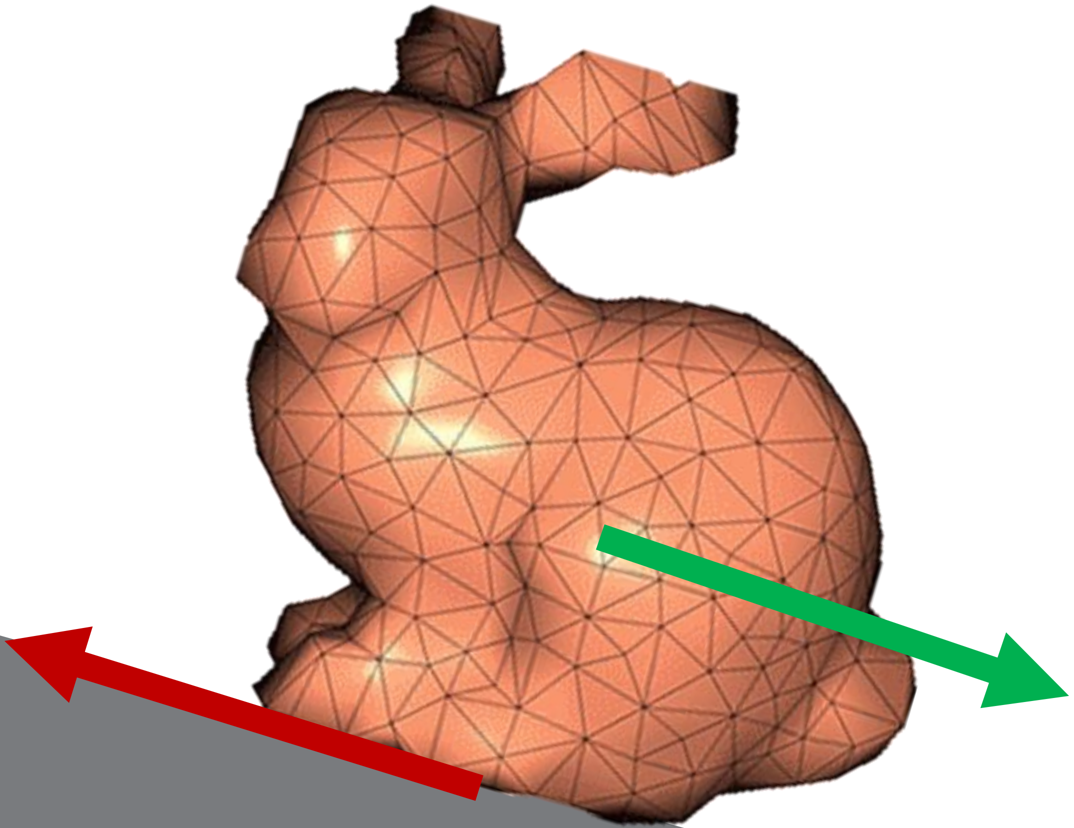

Friction



Static Friction: Holds things still

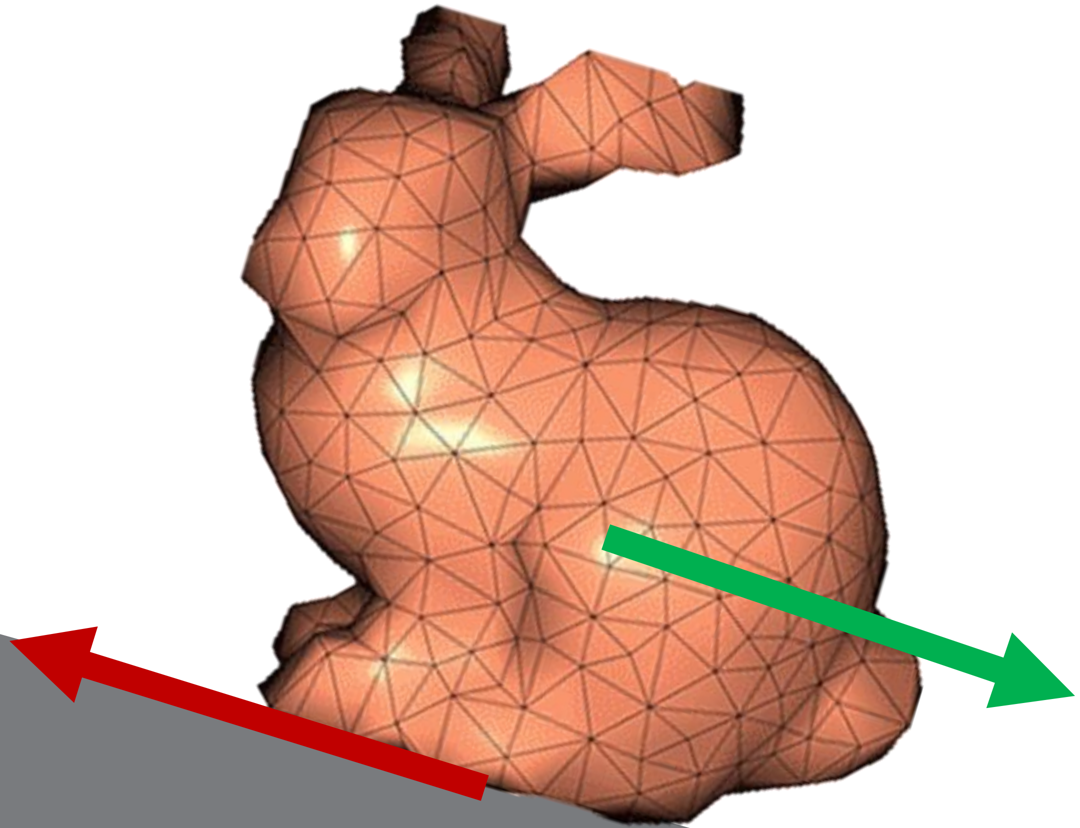


Dynamic Friction: Friction force resists sliding when in motion

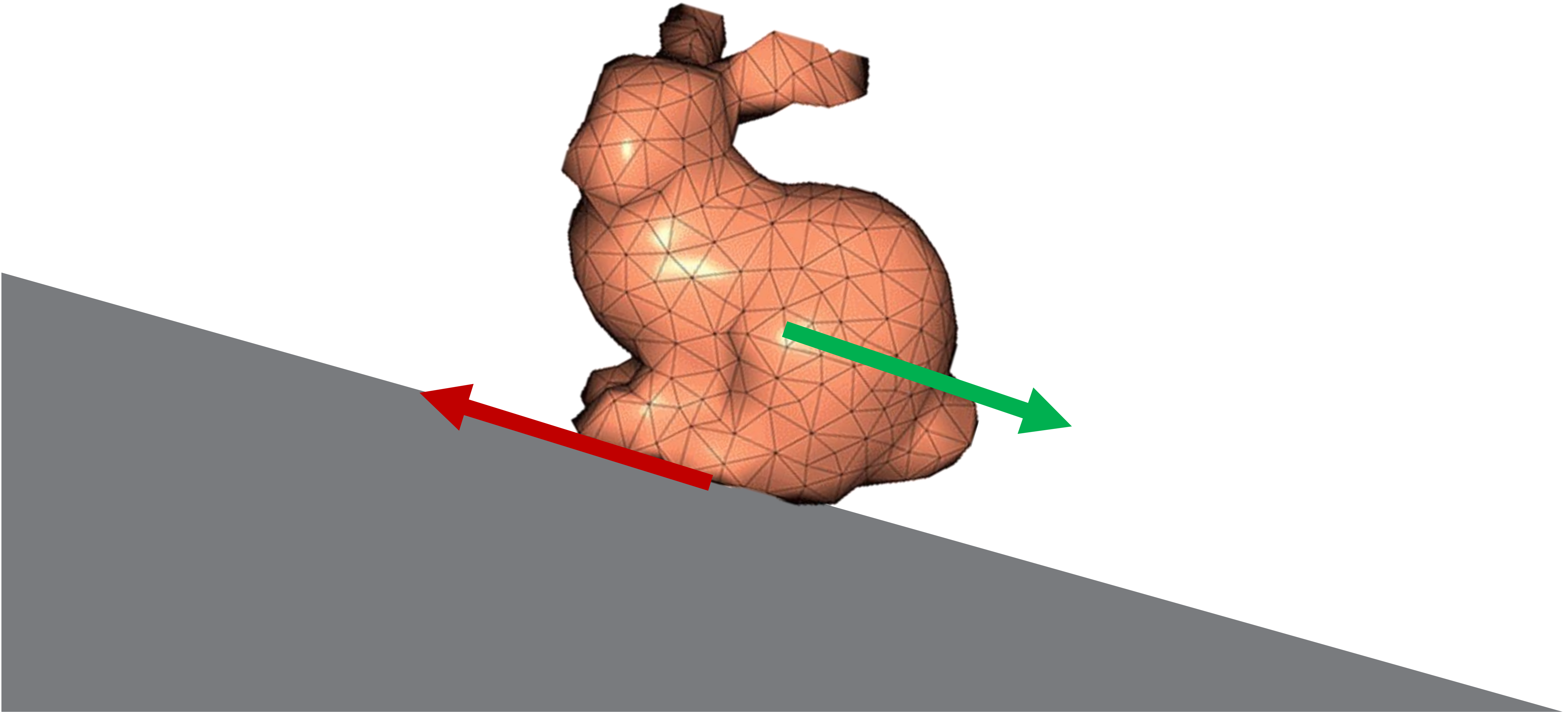


Coloumb's Law: $||\mathbf{f}|| \leq \mu ||\boldsymbol{c}||$

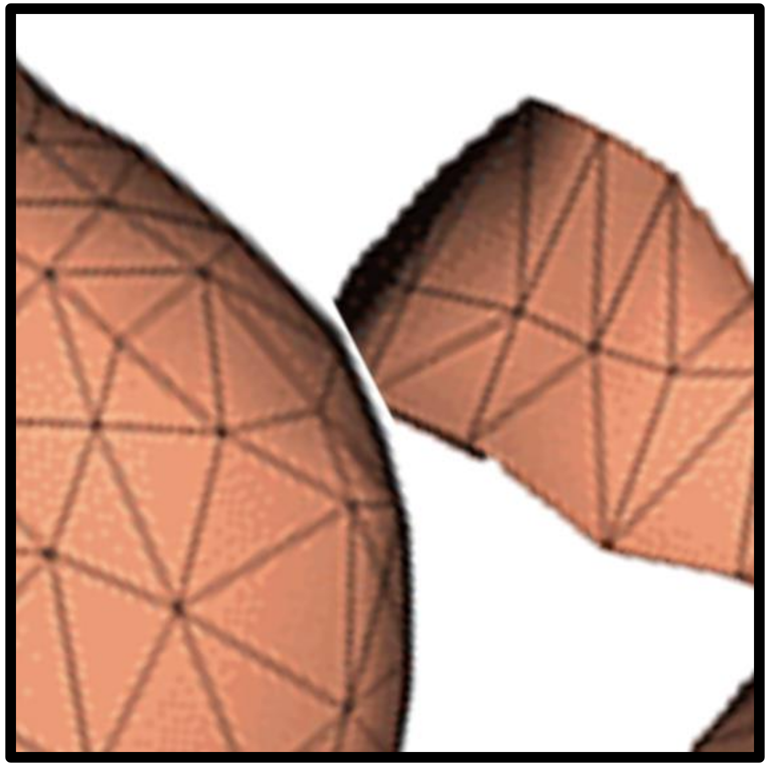
Friction is maximally dissipative

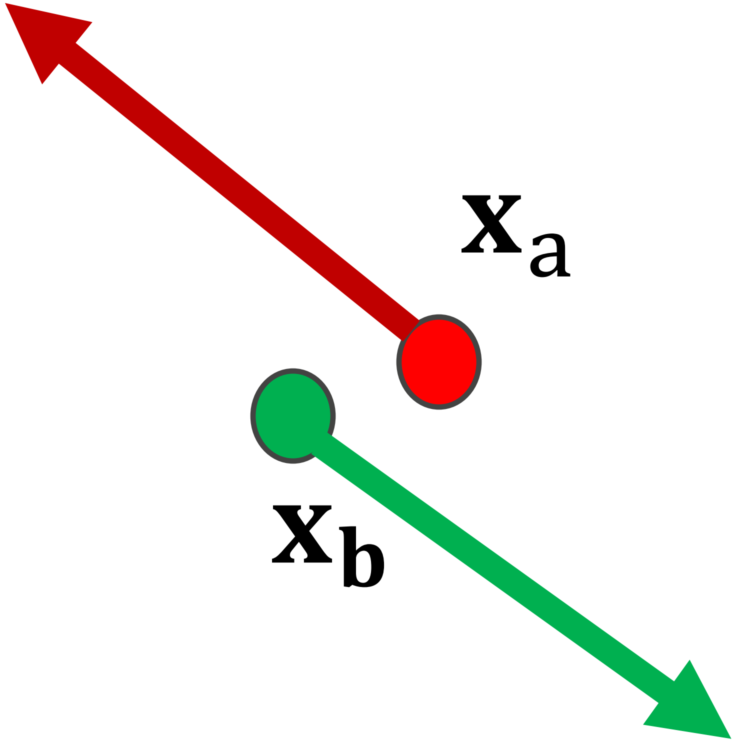


It wants to reduce the kinetic energy in the system as quickly as possible, up to Coloumb’s Law



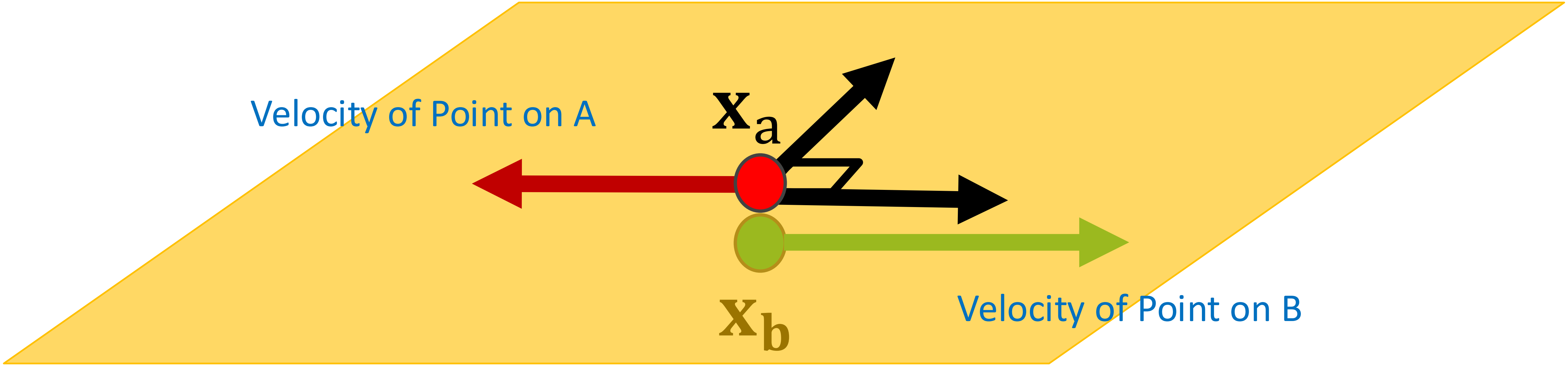
Friction Between Two Objects



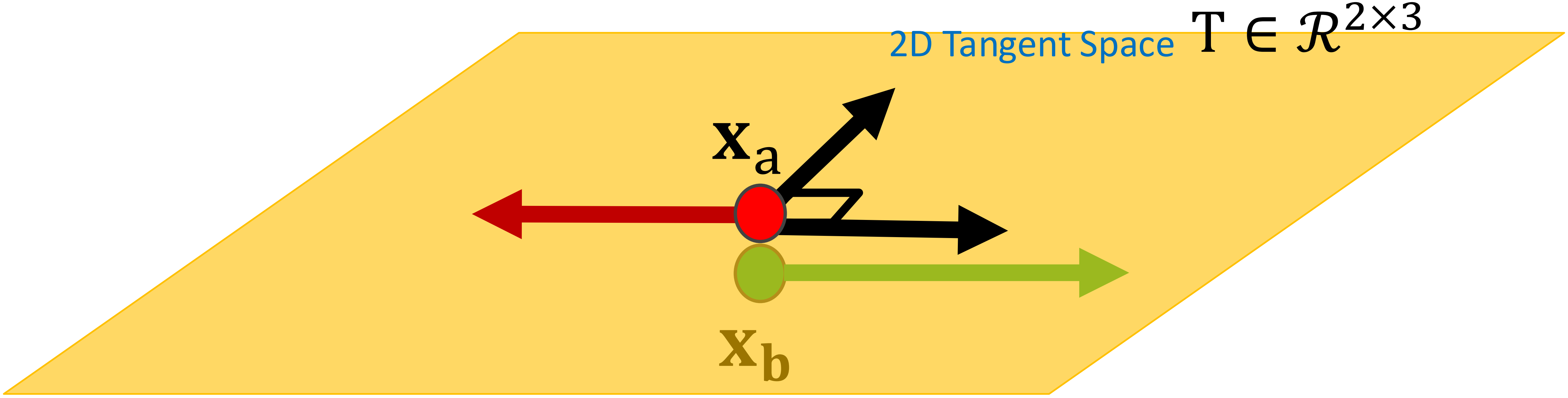


We apply friction between contact points where it opposes relative tangential velocity

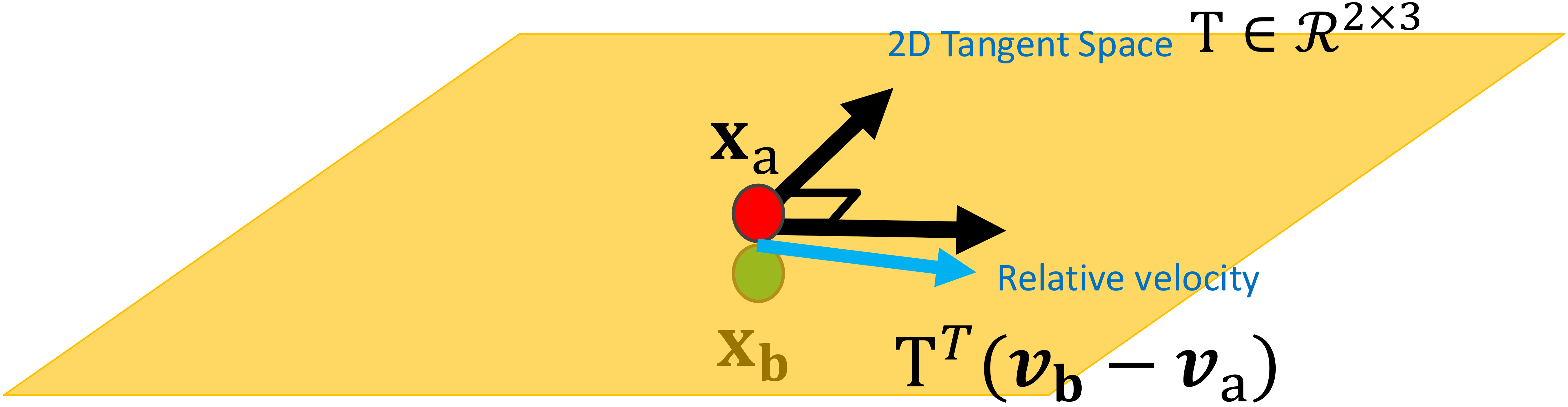
Friction Between Two Objects – the Tangent Space



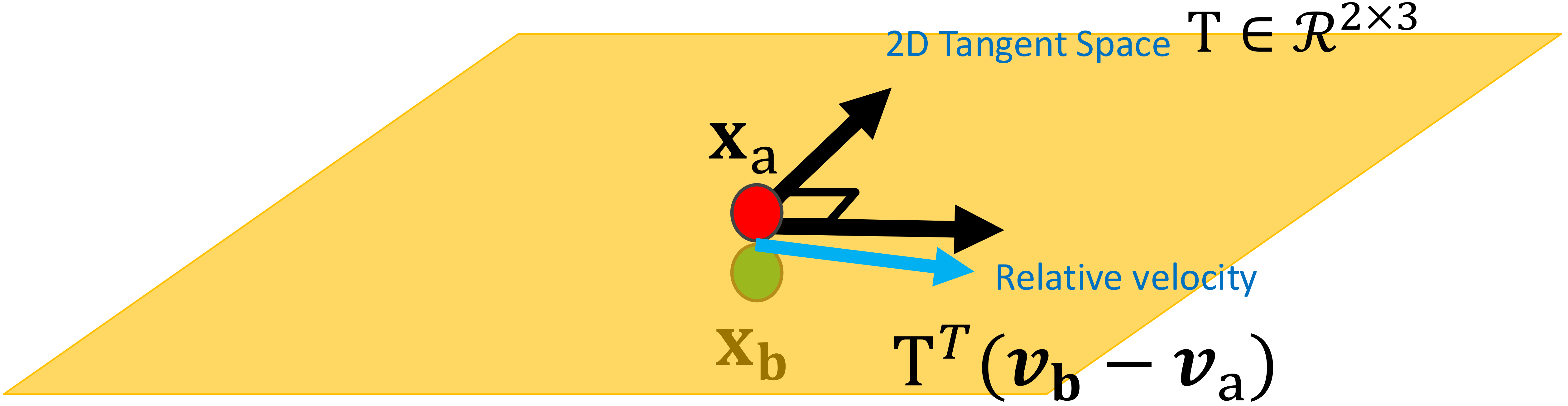
Friction Between Two Objects – the Tangent Space



Friction Between Two Objects – the Tangent Space



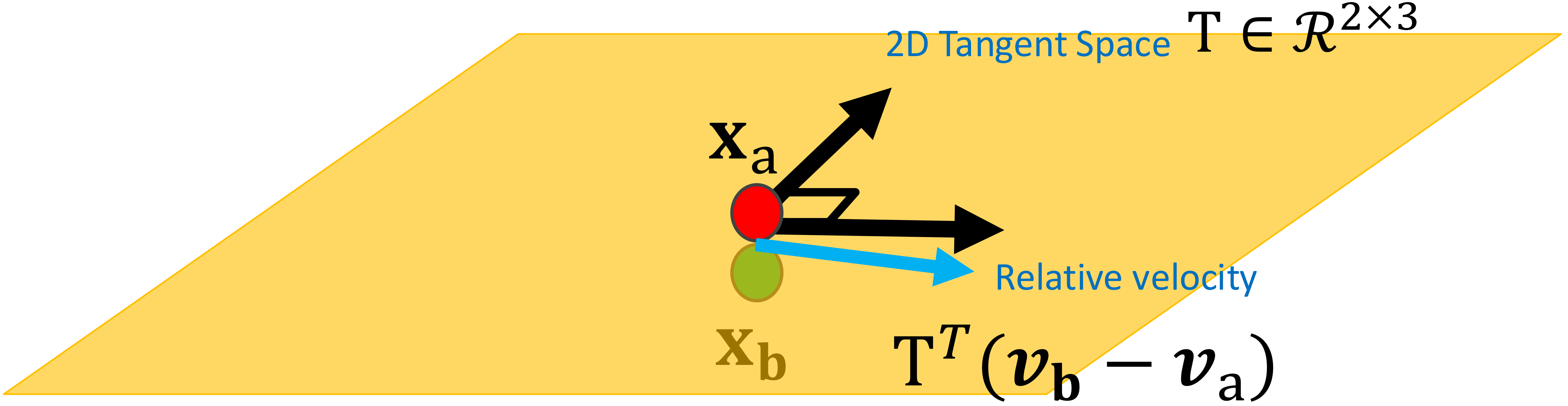
Friction Between Two Objects – the Tangent Space



An approximation:

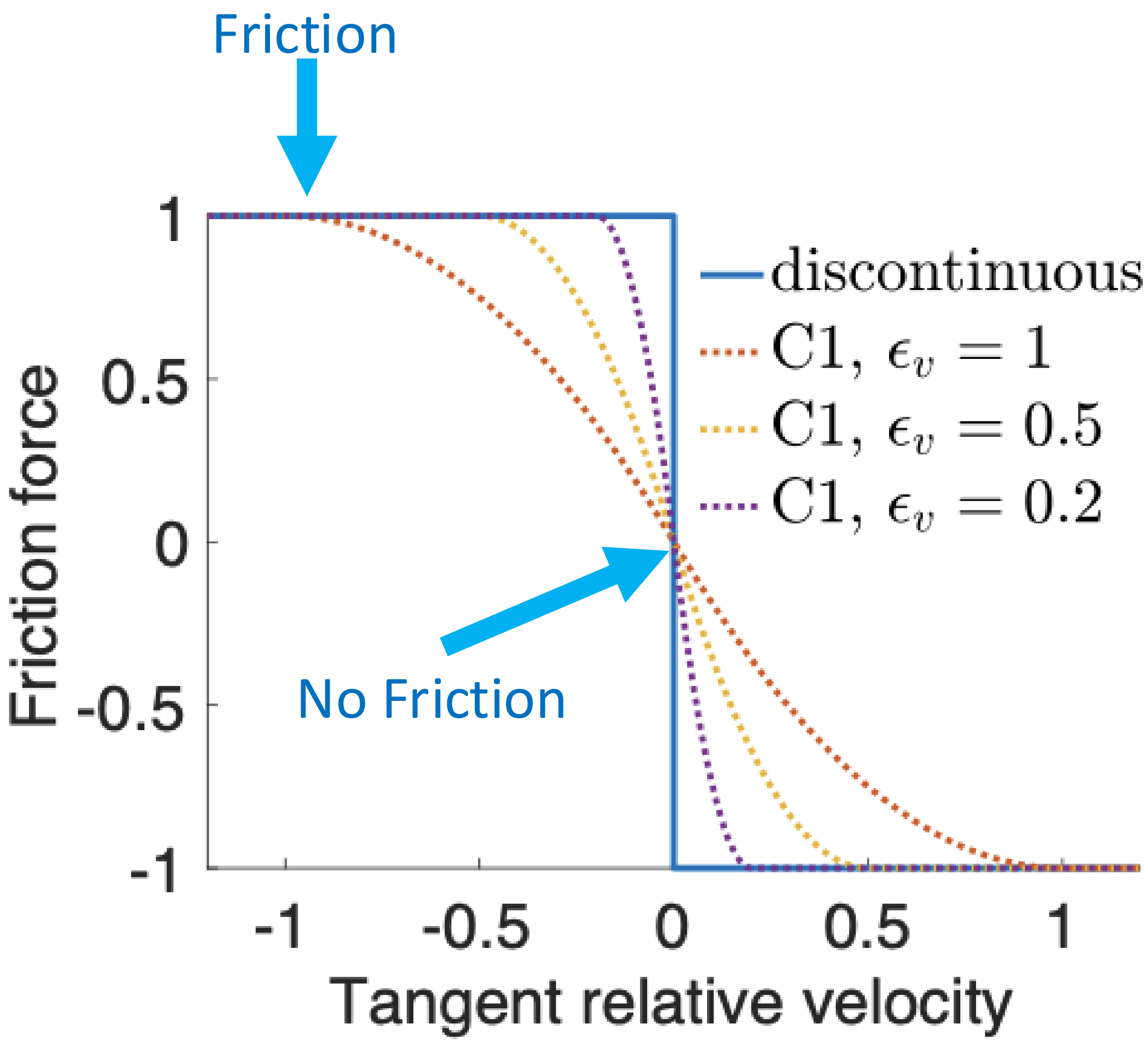
1. **if relative velocity is zero, force of friction is zero**
2. **Otherwise friction opposes relative velocity with coloumb law magnitude.**

Friction Between Two Objects – the Tangent Space



Ideally, we could write this out as an Energy and add it to our implicit integrator !

Introduce a “Threshold” Function



$$f_1(y) = \begin{cases} -\frac{y^2}{\epsilon_v^2 h^2} + \frac{2y}{\epsilon_v h}, & y \in (0, h\epsilon_v) \\ 1, & y \geq h\epsilon_v, \end{cases}$$

Incremental Potential Conact | Li et al

A Simple Friction Spring Energy

$$V_{friction}(\mathbf{q}) = \mu\lambda f_0(||\mathrm{v}_r^{\mathrm{t+1}}(\mathbf{q})||)$$

T only computed at time t

$$\mathrm{v}_r^{\mathrm{t+1}} = \mathrm{T}^T(\boldsymbol{v}_{\mathbf{b}} - \boldsymbol{v}_{\mathrm{a}})$$

$$\lambda^t = ||\mathbf{c}||$$

A Simple Friction Spring Energy

Integral of f_1 wrt magnitude of tangential velocity

$$V_{friction}(\mathbf{q}) = \mu\lambda f_0(||\mathrm{v}_r^{\mathrm{t+1}}(\mathbf{q})||)$$

T only computed at time t

$$\mathrm{v}_r^{\mathrm{t+1}} = \mathrm{T}^T(\boldsymbol{v}_{\mathbf{b}} - \boldsymbol{v}_{\mathrm{a}})$$

$$\lambda^t = ||\mathbf{c}||$$

Multibody AND Contact AND Friction in One Solver

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^{2} V\left(\mathbf{q}^{\mathrm{i+1}}\right)$$

$$\mathrm{V}_{springs} + V_{affine} + \text{V\_\{friction\}}$$

This Video: Rigid Body Simulation with Contact