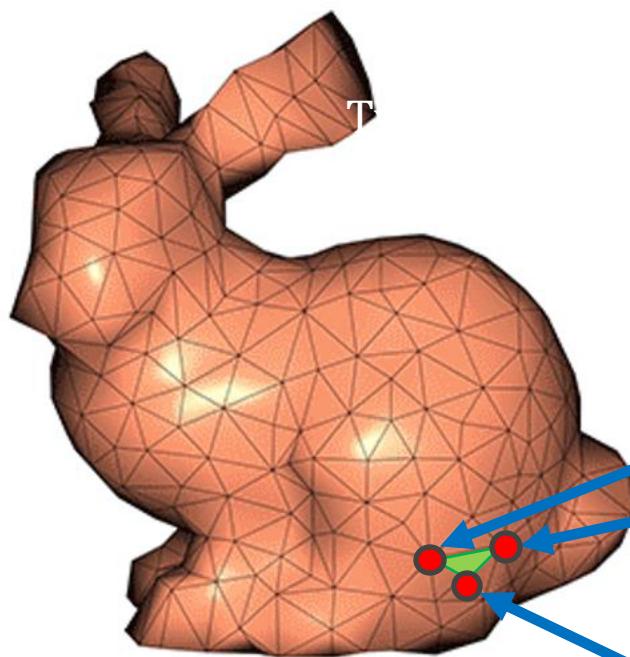
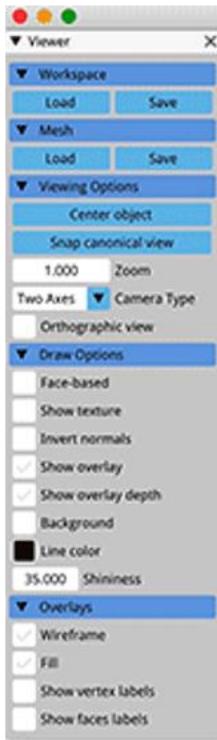


CSC417 Physics-Based Animation

36M verts, 124M tets
avg frame time: 7.2s
max: 7.8s

Vertex Block Descent|Chen et al.

Last Week: Reduced-Order Methods



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} : q(0) + U_r$$

Generalized
Coordinates

Questions from Previous Lecture ?

This Week: Fast Simulation via Coordinate Descent



48M verts, 151M tets

avg frame time: 14.4s, max: 15.6s

We Solve This Every Time Step

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

Gradient of what equals this ? Let's guess, then check

Newton's Method

Choose an initial guess

$$i = 0$$

$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{H}^i \mathbf{d} = -\mathbf{g}^i \quad \text{Solve linear system to get } \mathbf{d}$$

Choose α using line search

Use search direction to update current guess

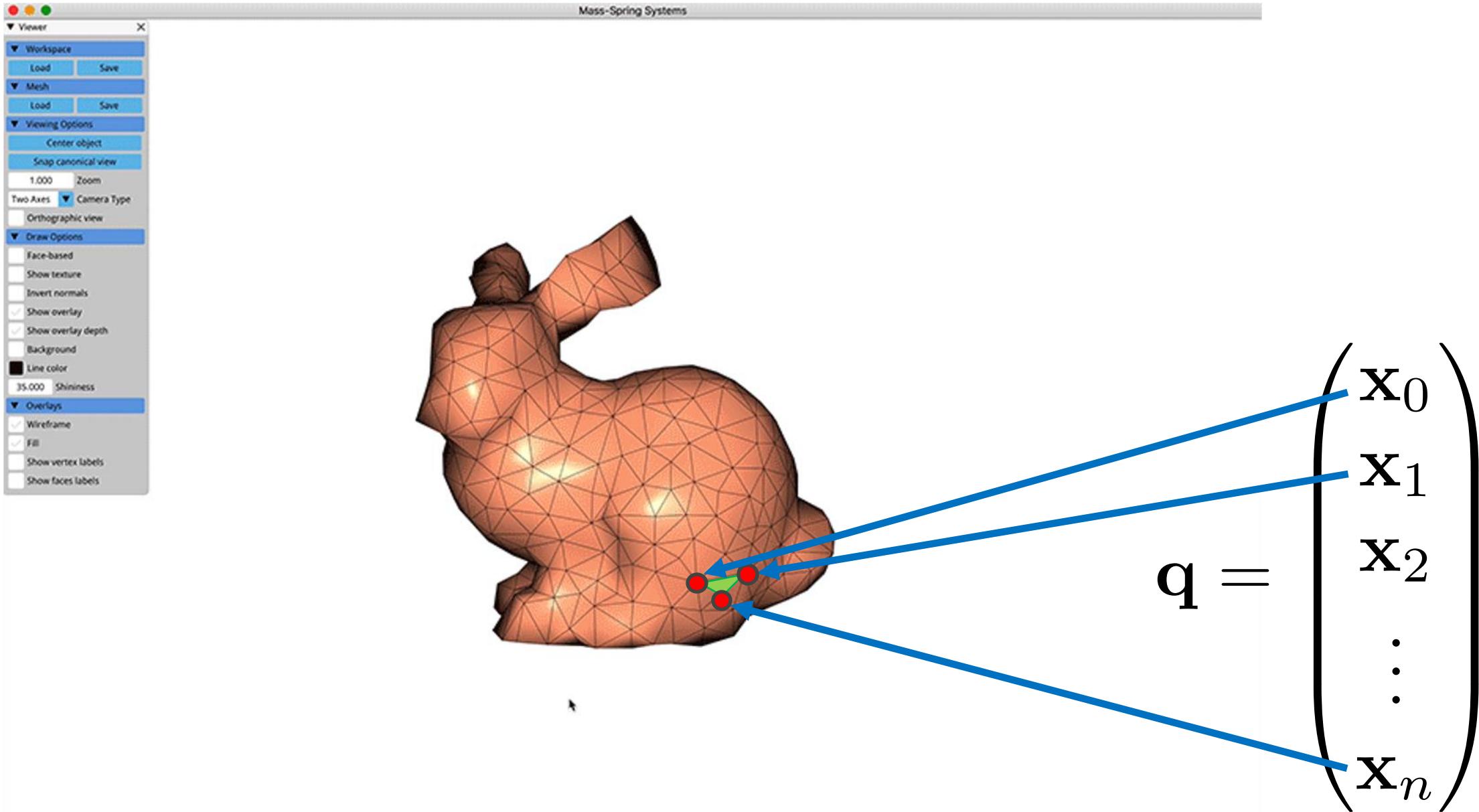
$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

$$i = i + 1$$

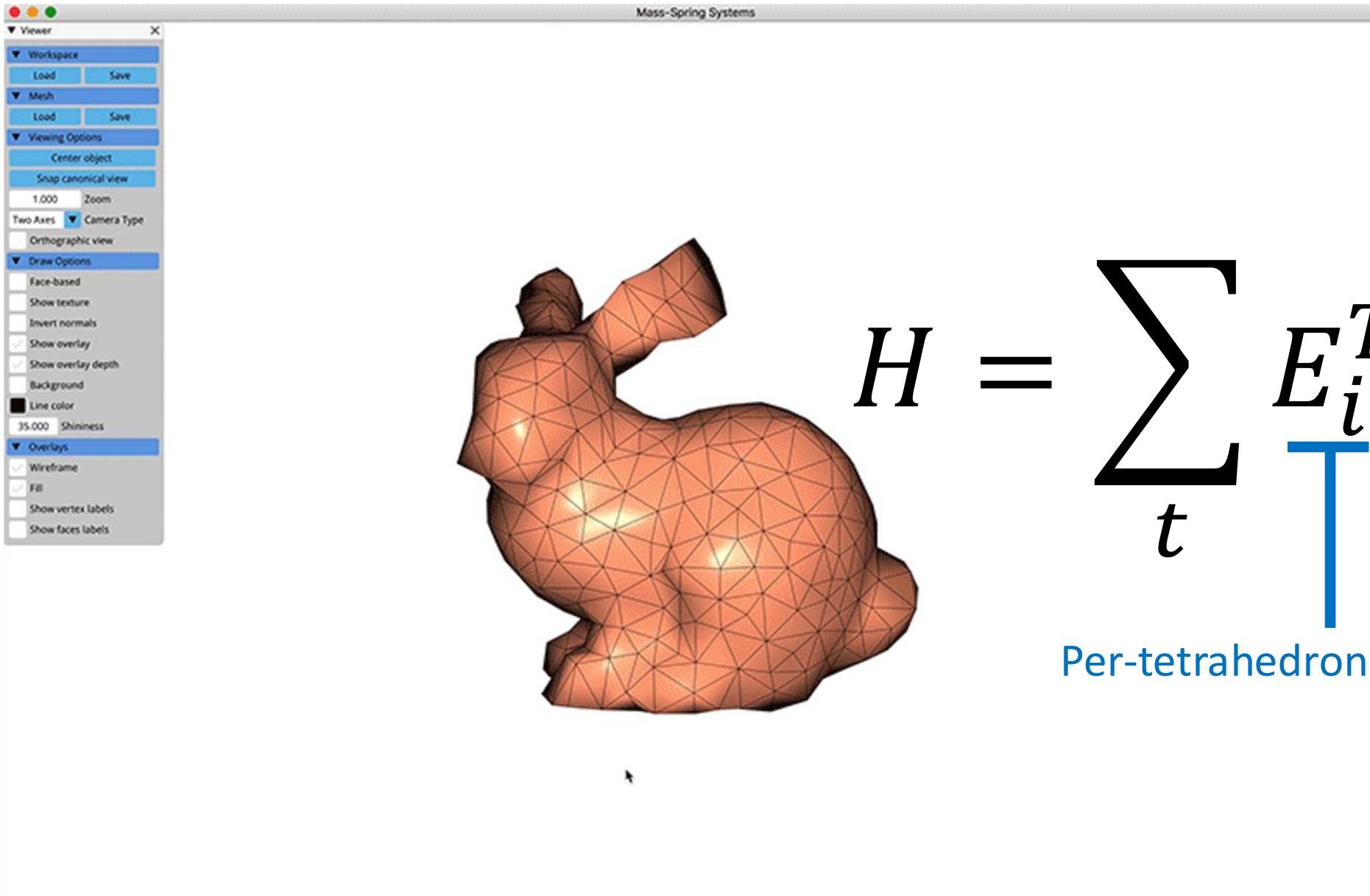
Repeat until converged



Spatial Discretization -- Finite Elements



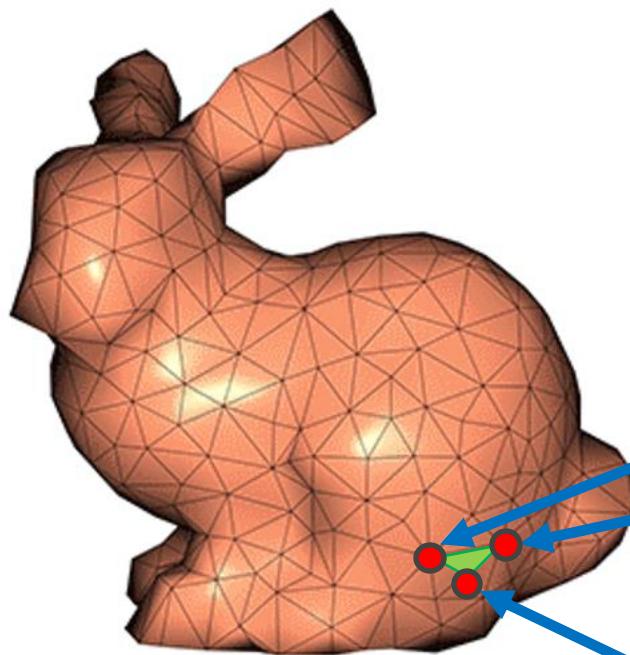
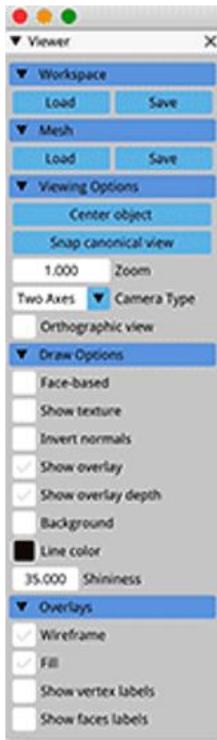
Assembly still visits every element 😞



$$H = \sum_t E_i^T H_i E_i$$

Per-tetrahedron Hessian

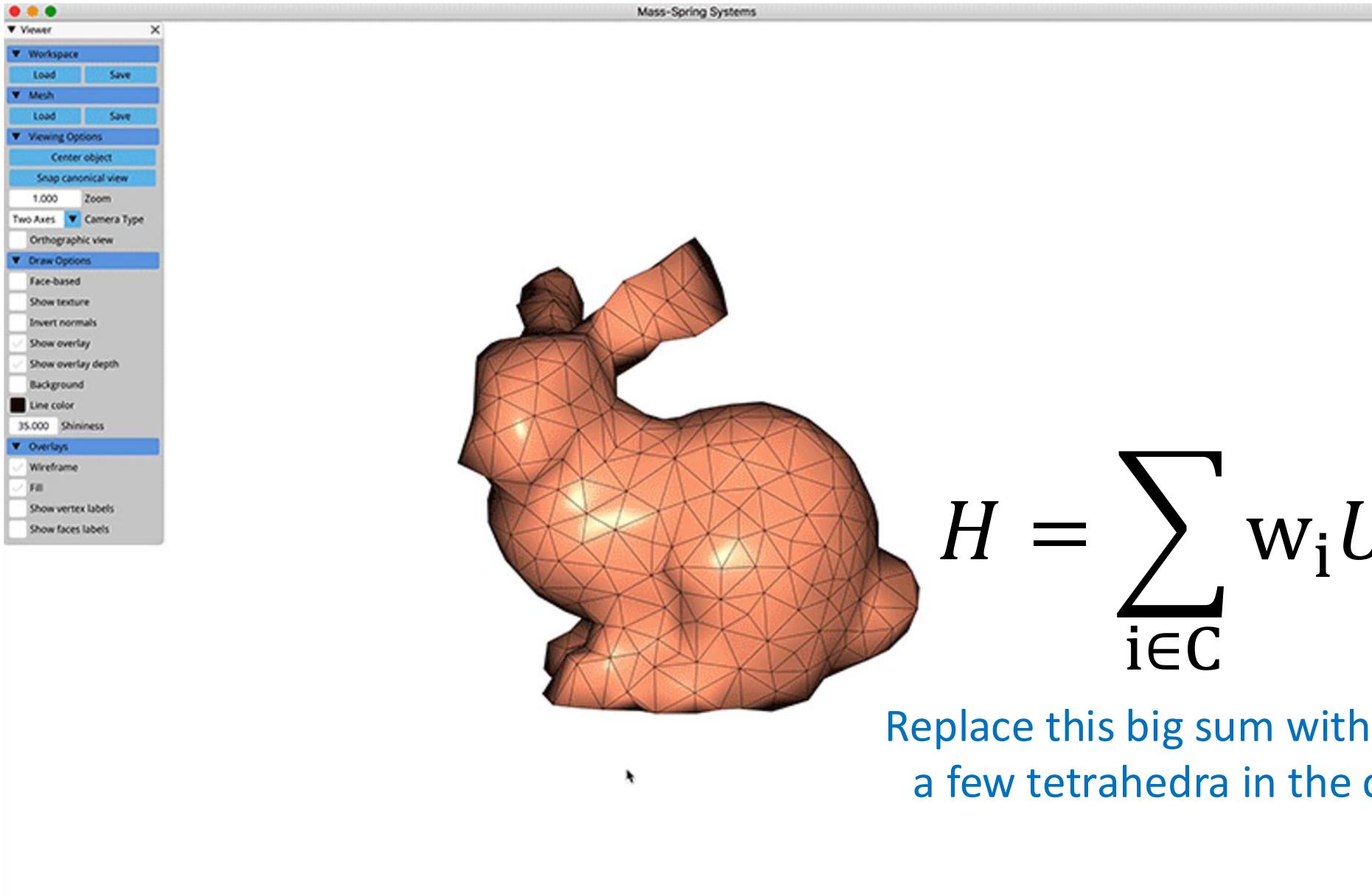
Last Week: Reduced-Order Methods



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} : q(0) + U_r$$

Generalized
Coordinates

Optimal Quadrature



$$H = \sum_{i \in C} w_i U^T H_i U$$

Replace this big sum with a small sum over a few tetrahedra in the cubature set C .

Another way to go fast

1. Avoid reducing the simulation space
2. Split up problem in a way that allows us to exploit parallelism and avoid building large matrices

Re-thinking Energy Minimization

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$


Global Potential Energy

Global “Inertial” Energy

Newton's Method

Choose an initial guess

$$i = 0$$

$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{H}^i \mathbf{d} = -\mathbf{g}^i$$

Large Global Sparse System

Choose α using line search

Use search direction to update current guess

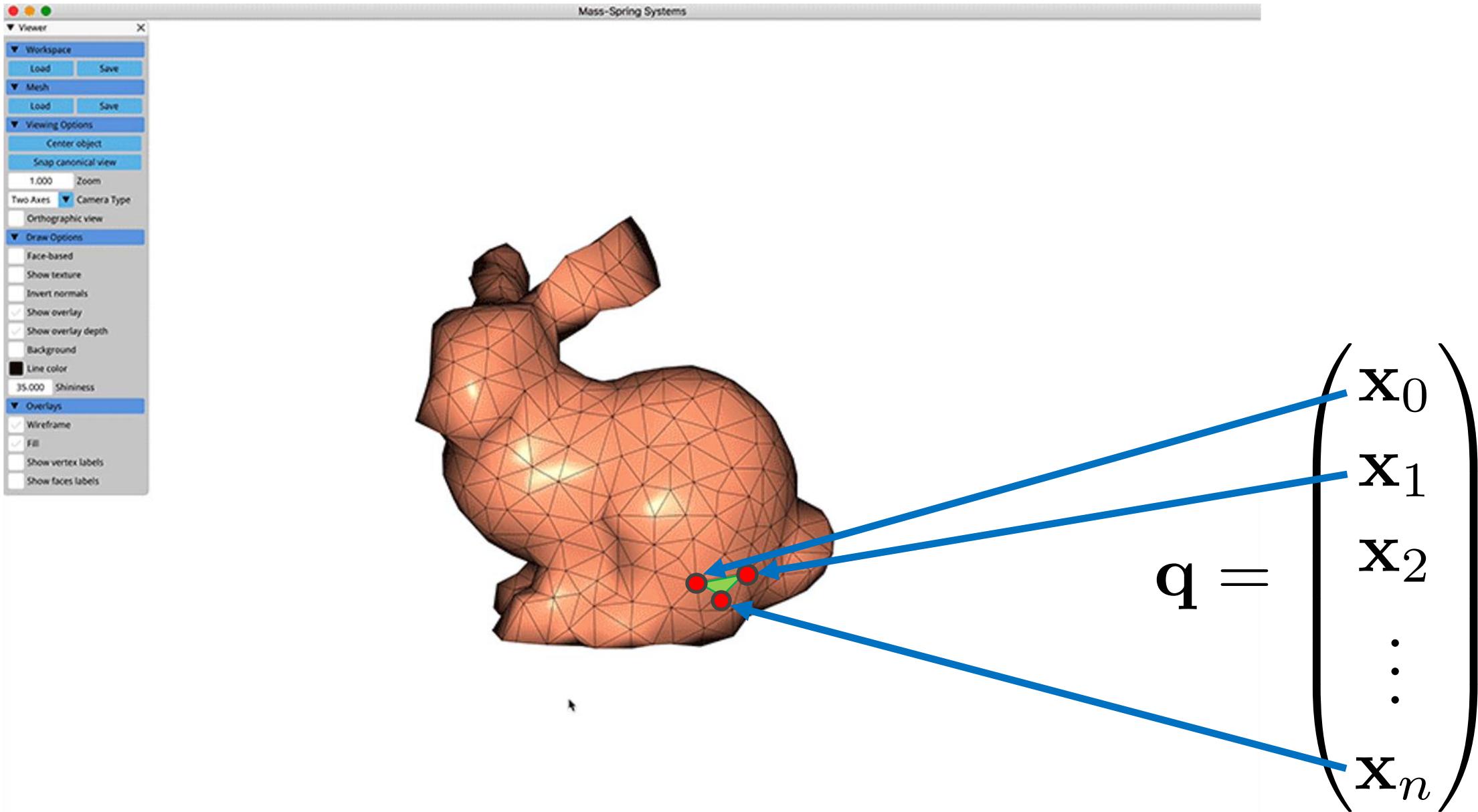
$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

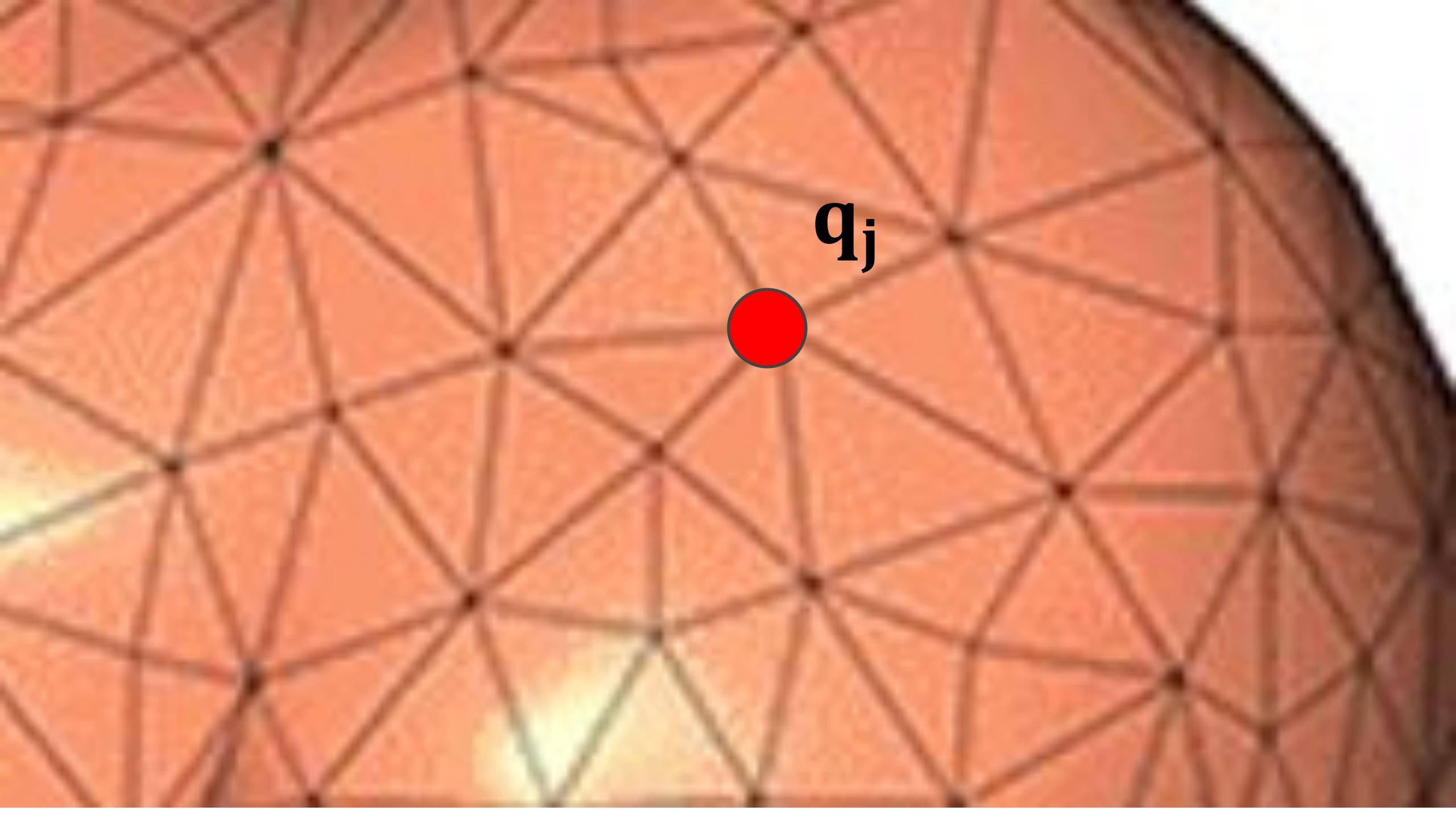
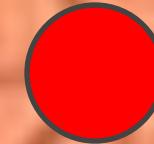
$$i = i + 1$$

Repeat until converged

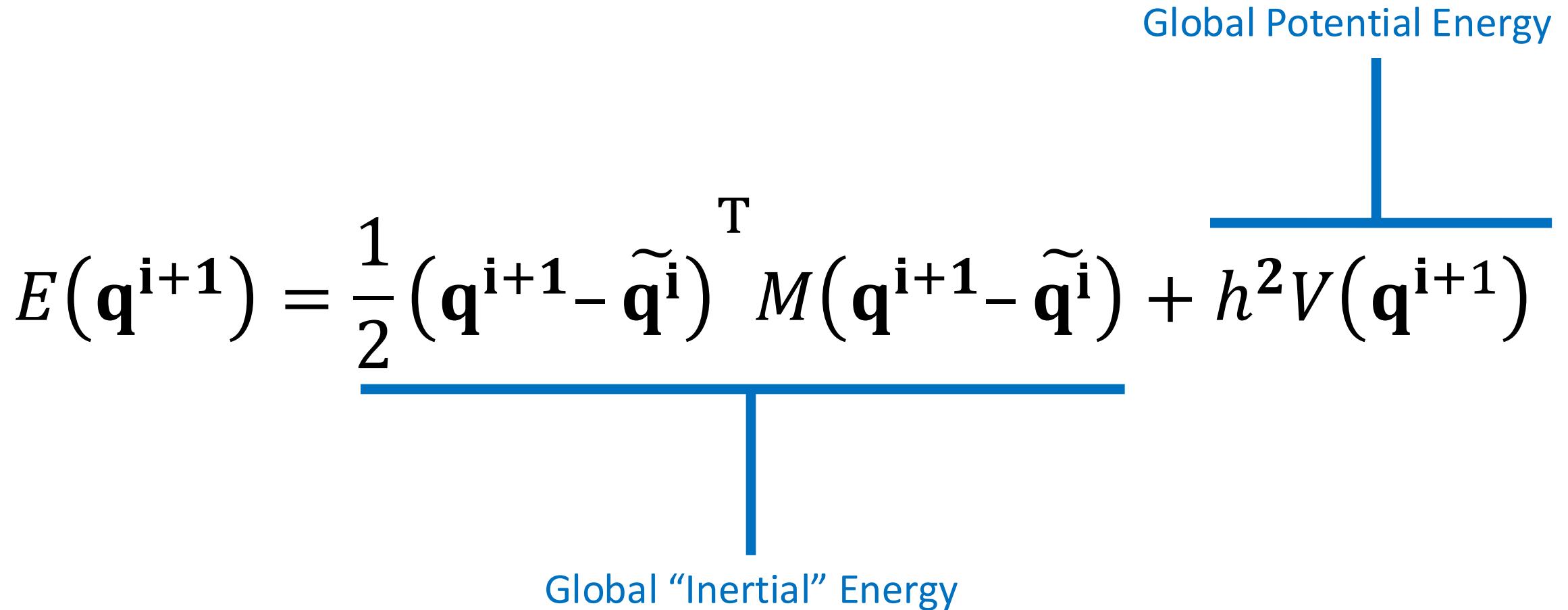


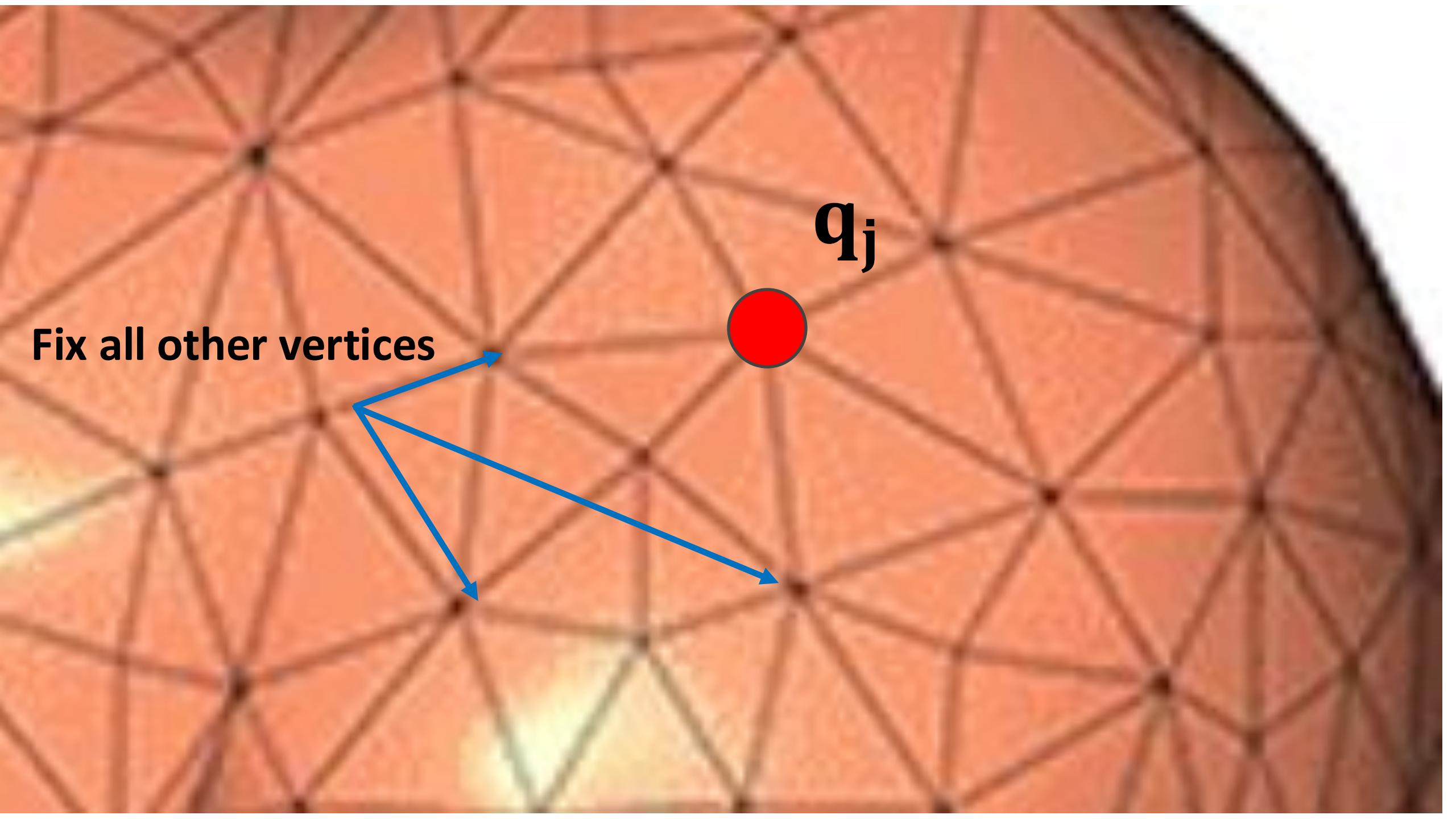
Coordinate Descent



 q_j 

What happens if we just minimize wrt q_j ?

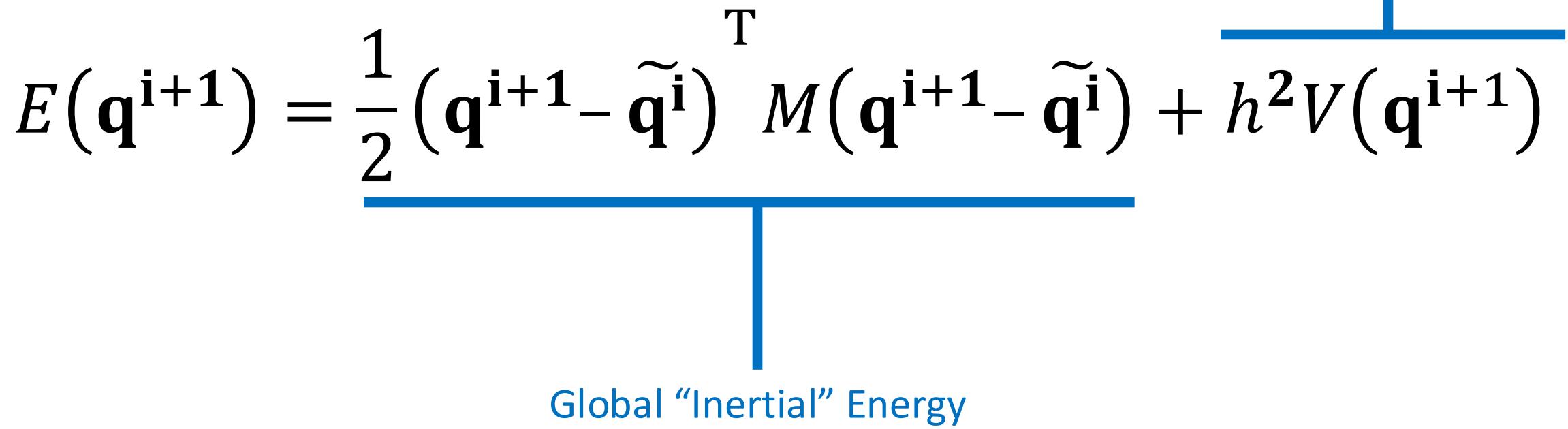




Fix all other vertices

q_j

What happens if we just minimize wrt q_j ?



Inertial Energy Structure

$$\frac{1}{2}(\mathbf{q}_0^{i+1} - \tilde{\mathbf{q}}_0^i, \mathbf{q}_1^{i+1} - \tilde{\mathbf{q}}_1^i, \dots)$$

| Vertex 0 | Vertex 1 | Vertex 2 | |
|----------|----------|----------|----------|
| Vertex 0 | M_{00} | M_{01} | M_{02} |
| Vertex 1 | ... | M_{11} | M_{12} |
| Vertex 2 | ... | ... | M_{22} |

Diagram illustrating the Inertial Energy Structure. The structure is represented as a 3x3 grid of cells, each containing a value M_{ij} where $i, j \in \{0, 1, 2\}$. The rows and columns are labeled "Vertex 0", "Vertex 1", and "Vertex 2". The first row contains M_{00} , M_{01} , and M_{02} . The second row contains "..." (ellipsis), M_{11} , and M_{12} . The third row contains "..." (ellipsis), "..." (ellipsis), and M_{22} . A large bracket on the right side groups the entire grid, and a smaller bracket on the left side groups the first two columns. To the right of the grid, a set of three vectors is shown in parentheses: $(\mathbf{q}_0^{i+1} - \tilde{\mathbf{q}}_0^i, \mathbf{q}_1^{i+1} - \tilde{\mathbf{q}}_1^i, \dots)$.

Inertial Energy Structure

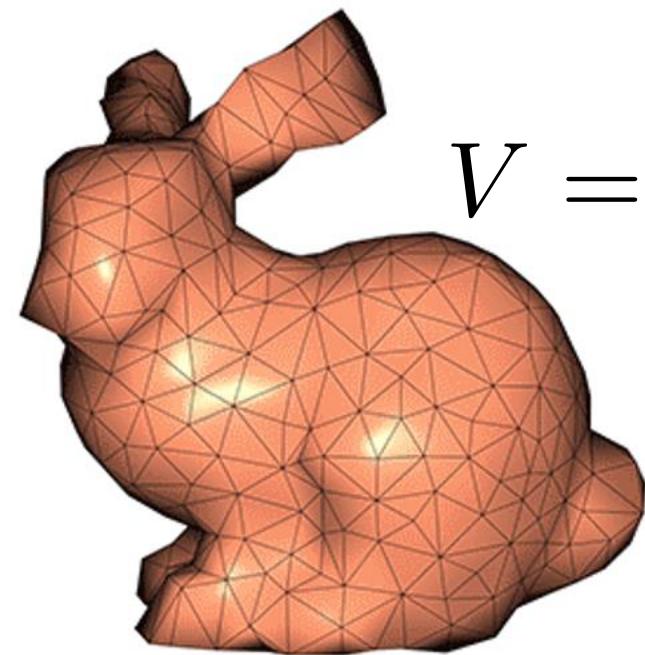
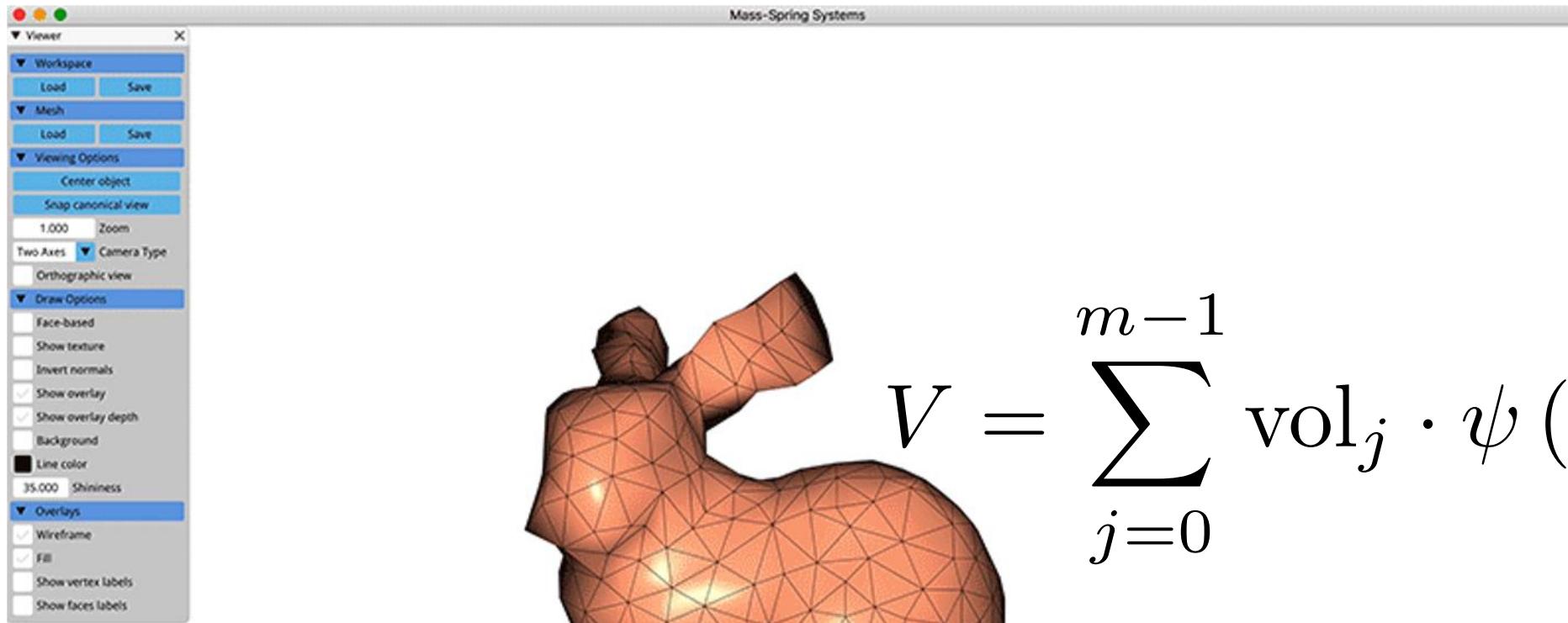
$$\frac{1}{2} (\mathbf{q}_0^{i+1} - \tilde{\mathbf{q}}_0^i)^T M_{00} (\mathbf{q}_0^{i+1} - \tilde{\mathbf{q}}_0^i)$$

$$+ (\mathbf{q}_1^{i+1} - \tilde{\mathbf{q}}_1^i)^T M_{01} (\mathbf{q}_0^{i+1} - \tilde{\mathbf{q}}_0^i)$$

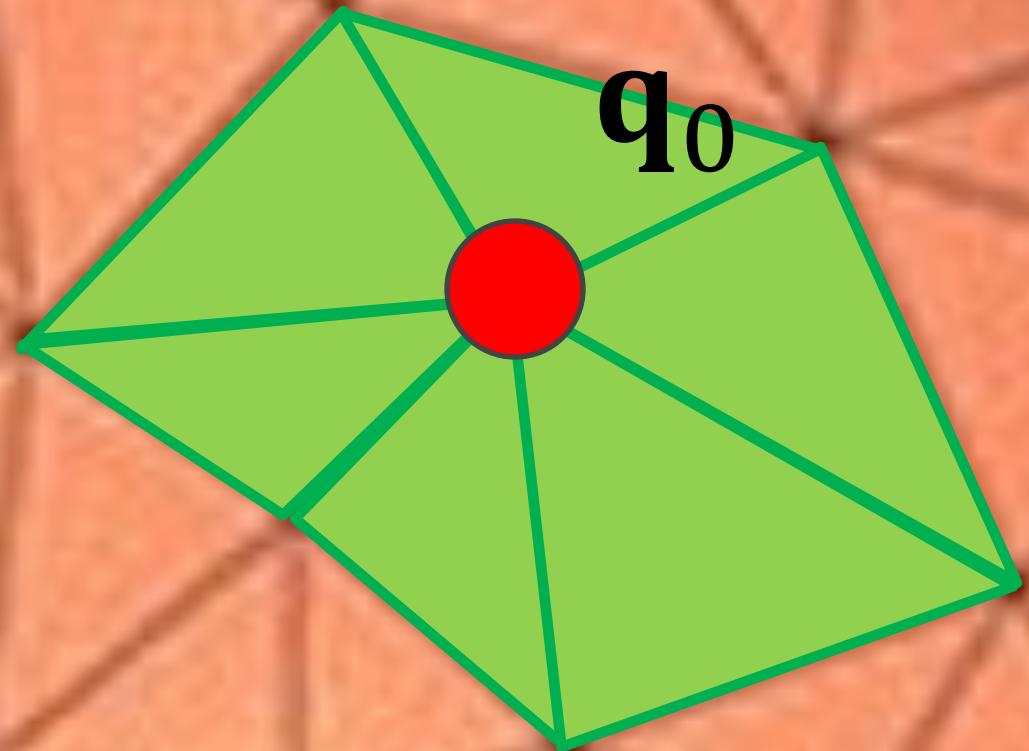
$$+ (\mathbf{q}_2^{i+1} - \tilde{\mathbf{q}}_2^i)^T M_{02} (\mathbf{q}_0^{i+1} - \tilde{\mathbf{q}}_0^i)$$

+ Bunch of terms that don't depend on \mathbf{q}_0

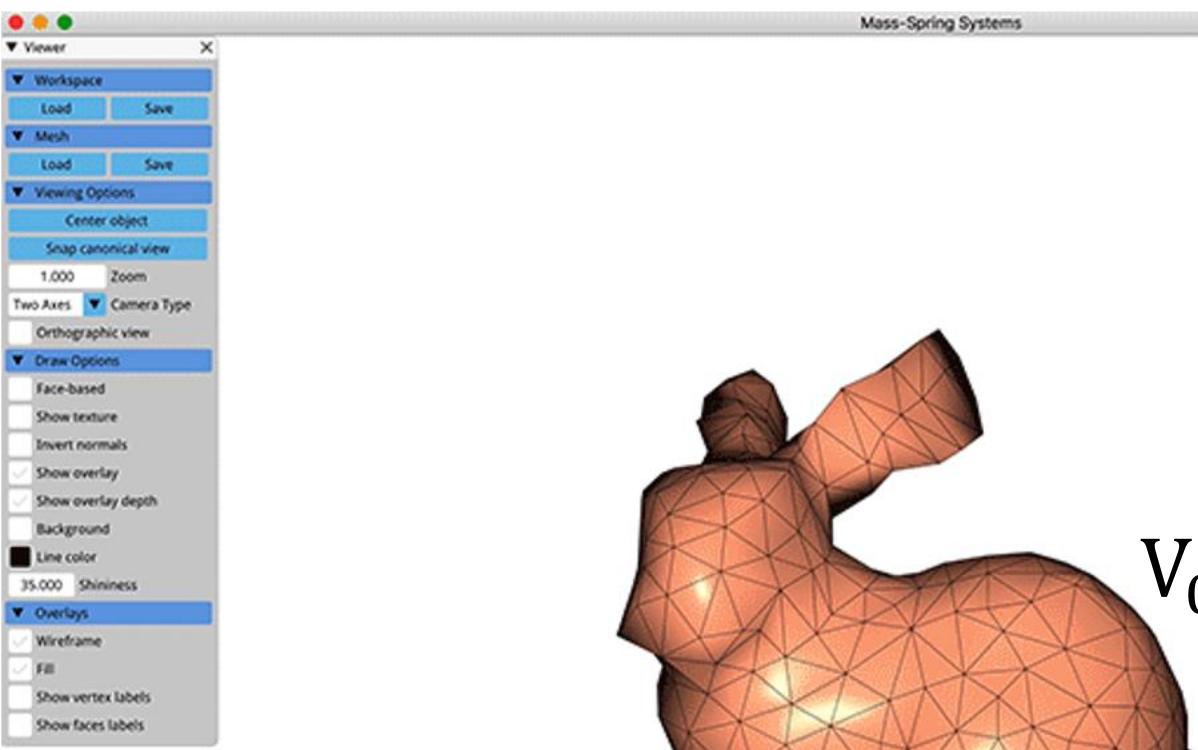
Potential Energy Structure (Way back in Lectures 2 and 3)



$$V = \sum_{j=0}^{m-1} \text{vol}_j \cdot \psi(F_j(E_j \mathbf{q}))$$



Potential Energy Structure (Way back in Lectures 2 and 3)



$$V_0(\mathbf{q}_0^{i+1}) = \sum_{\substack{i \in \mathcal{N} \\ \text{T}}} vol_i \psi(F_i(\mathbf{q}_0^{i+1}, \dots))$$

Tetrahedra containing \mathbf{q}_0



Other vertices of the tetrahedron

For a single vertex

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

T

Momentum from other degrees-of-freedom

Inertial Energy Structure

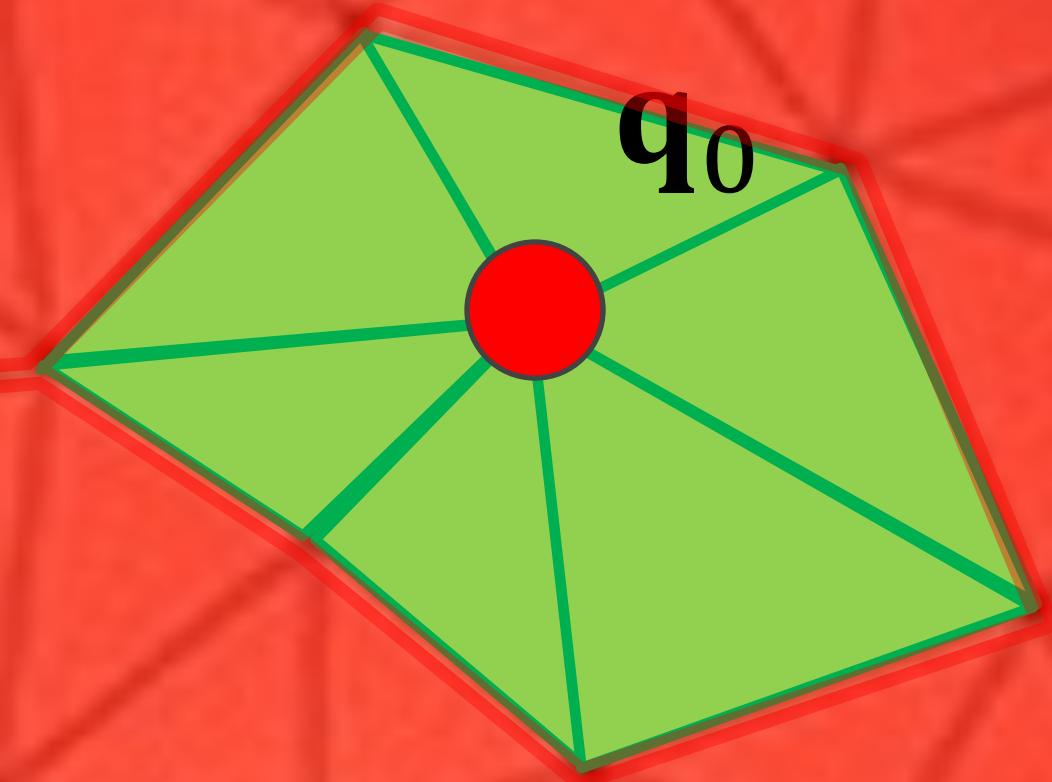
$$p_0 = \begin{matrix} \text{Vertex 0} \\ M_{01} & M_{02} \end{matrix} \left(\begin{matrix} q_1^{i+1} - \tilde{q}_1^i \\ q_2^{i+1} - \tilde{q}_2^i \\ \dots \end{matrix} \right)$$

For a single vertex

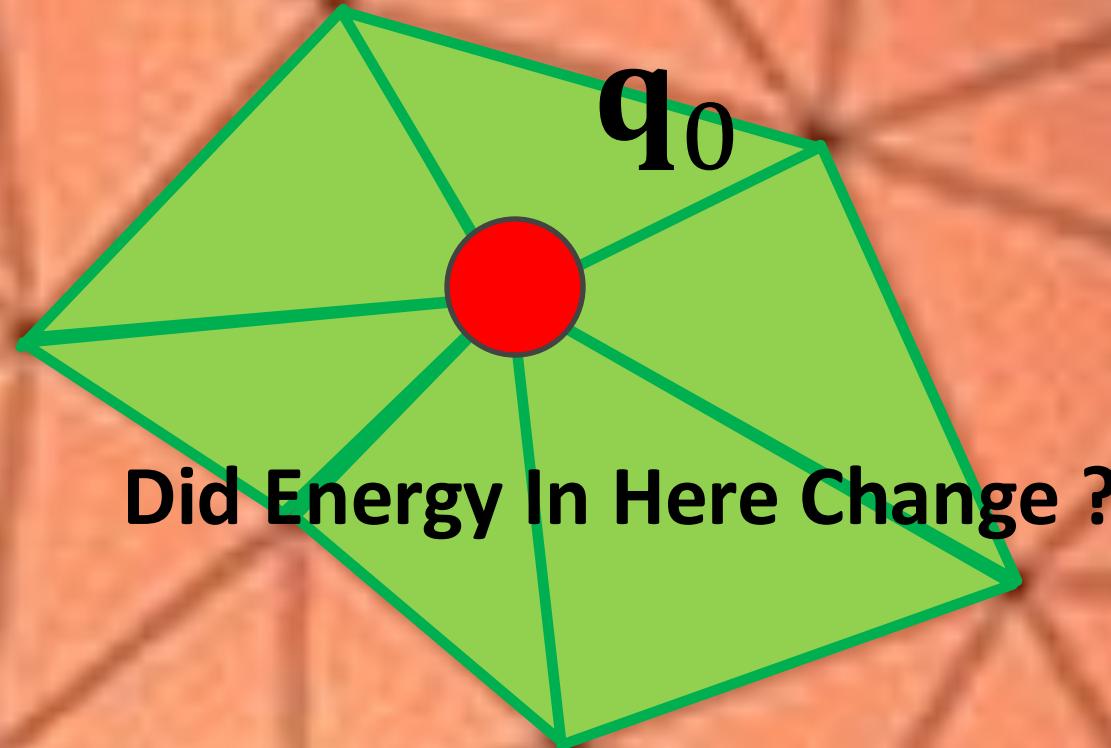
$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

T

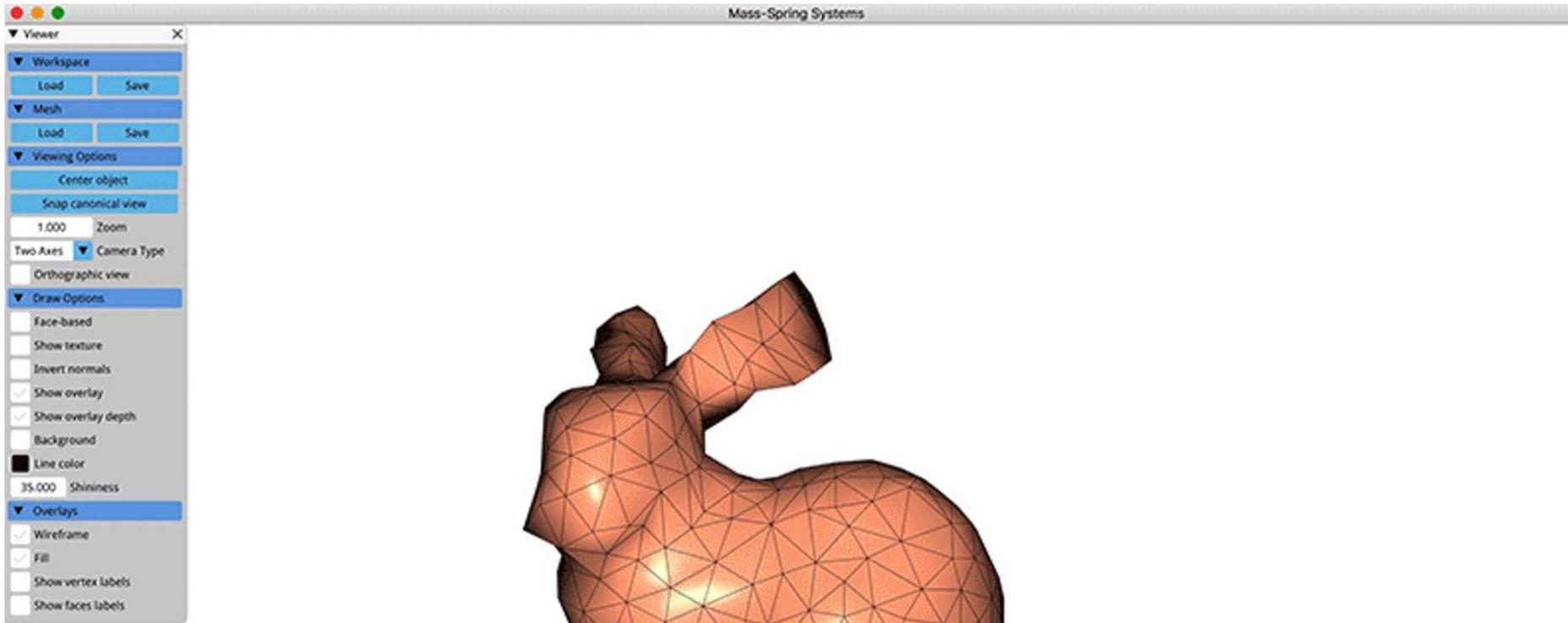
Momentum from other degrees-of-freedom



Did Energy Out Here Change ?



What happened to total energy then ?



So this suggests a new minimization strategy

For each vertex, j

Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

Repeat for awhile

Why is this fast ?

For each vertex, j

Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

Why is this fast ?

Why is this fast ?

Repeat for awhile

For each vertex, j

Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

Fast to solve using Newton's Method

Newton's Method

Choose an initial guess

$$i = 0$$

$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{H}^i \mathbf{d} = -\mathbf{g}^i$$

Here's our main cost

Choose α using line search

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

$$i = i + 1$$

Repeat until converged

So this suggests a new minimization strategy

For each vertex, j

Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

$$\mathbf{q}_j^{i+1} \in \mathcal{R}^3$$

$$\mathbf{g} = \frac{\partial E_j}{\partial \mathbf{q}_j^{i+1}} \in \mathcal{R}^3$$

$$\mathbf{H} = \frac{\partial^2 E_j}{\partial \mathbf{q}_j^{i+1} \partial \mathbf{q}_j^{i+1}} \in \mathcal{R}^{3 \times 3}$$

Newton's Method

Choose an initial guess

$$i = 0$$

$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{H}^i \mathbf{d} = -\mathbf{g}^i$$

Small, dense 3x3 linear system

Choose α using line search

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

$$i = i + 1$$

Repeat until converged

Why is this fast ?

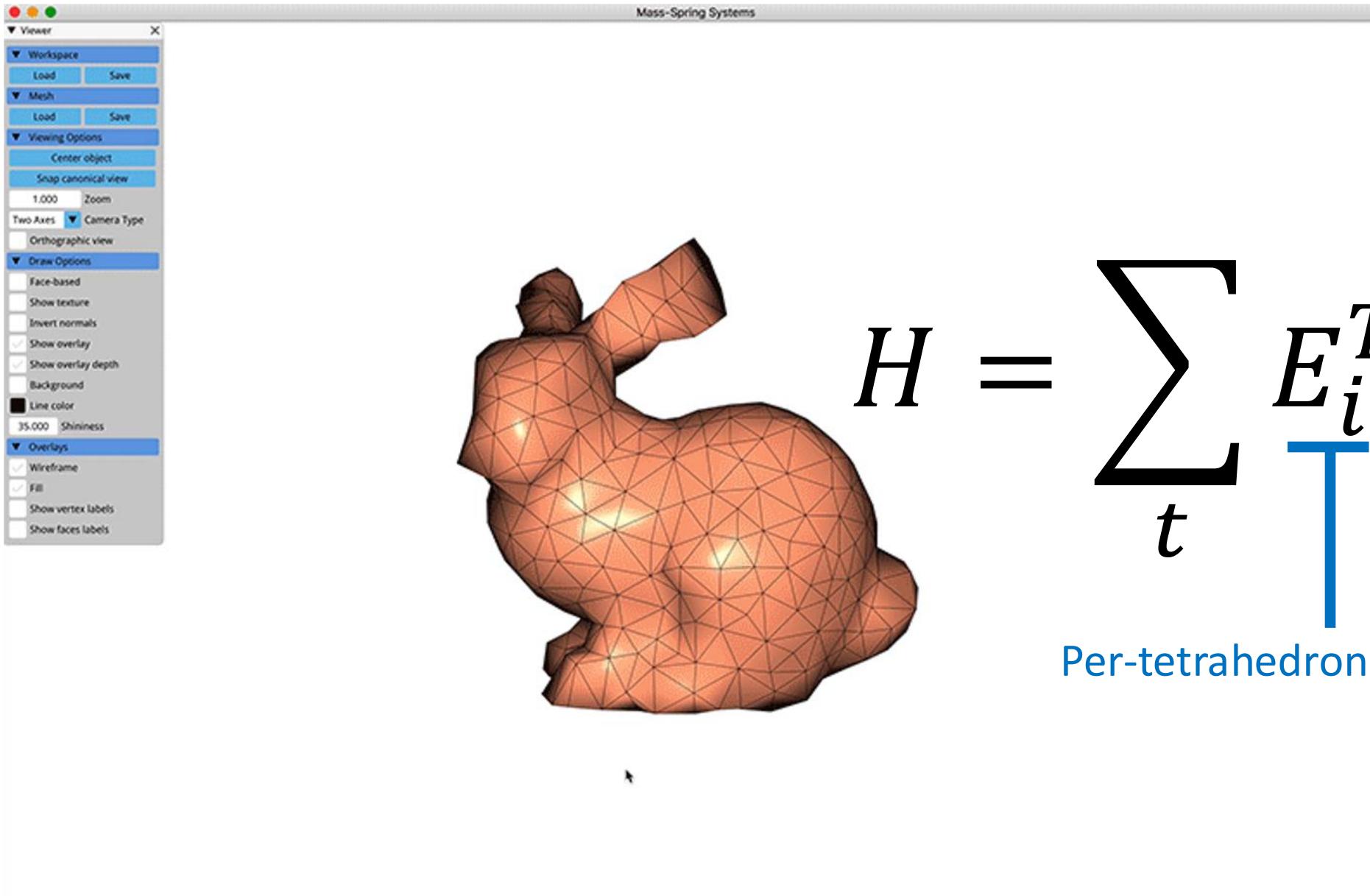
For each vertex, j

Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

Why is this fast ?

Assembly still visits every element 😞



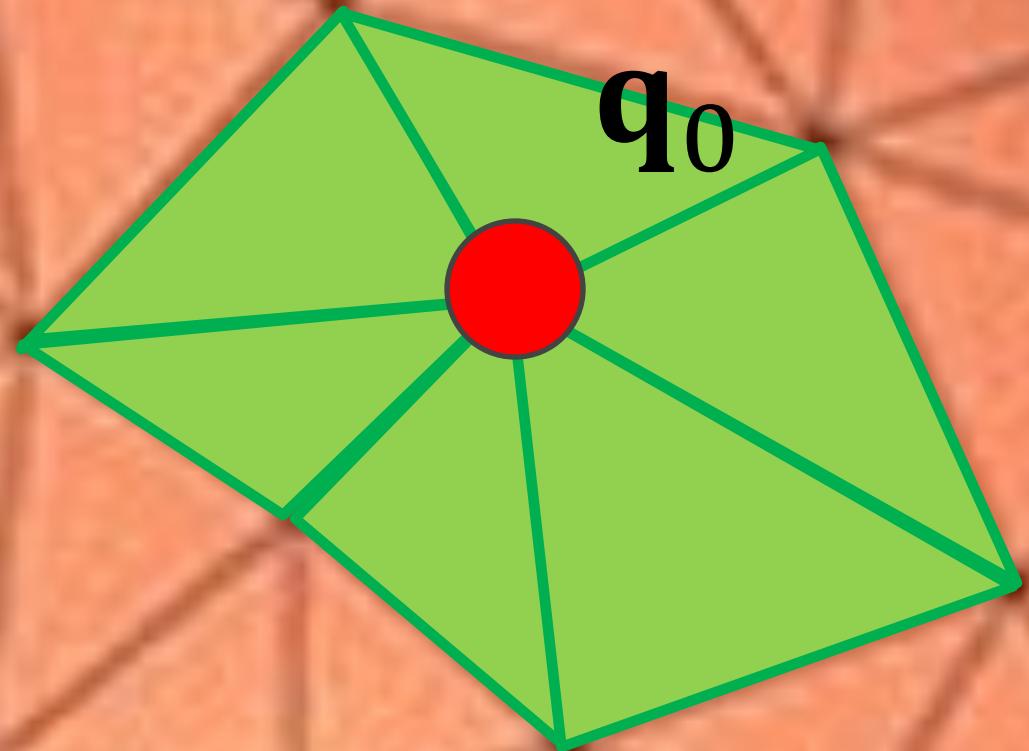
Why is this fast ?

For each vertex, j

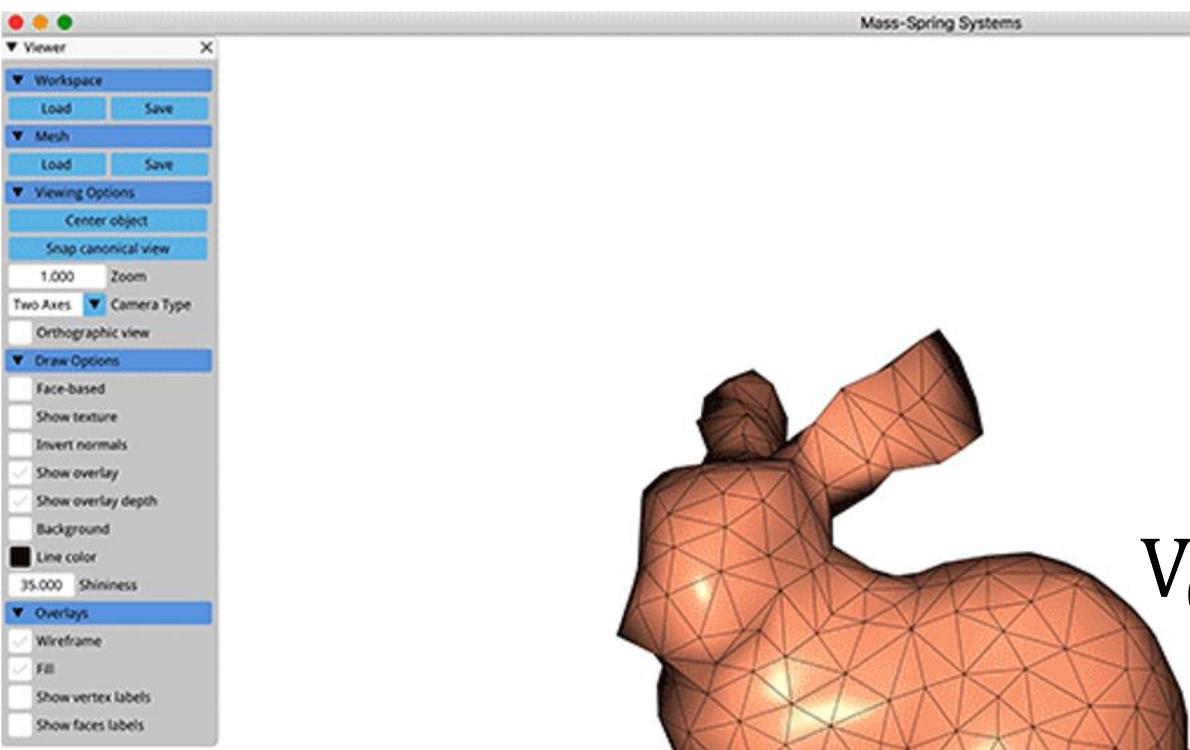
Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

Only depends on neighborhood of j



Potential Energy Structure (Way back in Lectures 2 and 3)



$$V_0(\mathbf{q}_0^{i+1}) = \sum_{\substack{i \in \mathcal{N} \\ \text{T}}} vol_i \psi(F_i(\mathbf{q}_0^{i+1}, \dots))$$

Tetrahedra containing \mathbf{q}_0



Other vertices of the tetrahedron

Why is this fast ?

For each vertex, j

Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

Why is this fast ?

Parallelization

For each vertex, j

Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

It's not obvious but we can do this in parallel

Parallelization

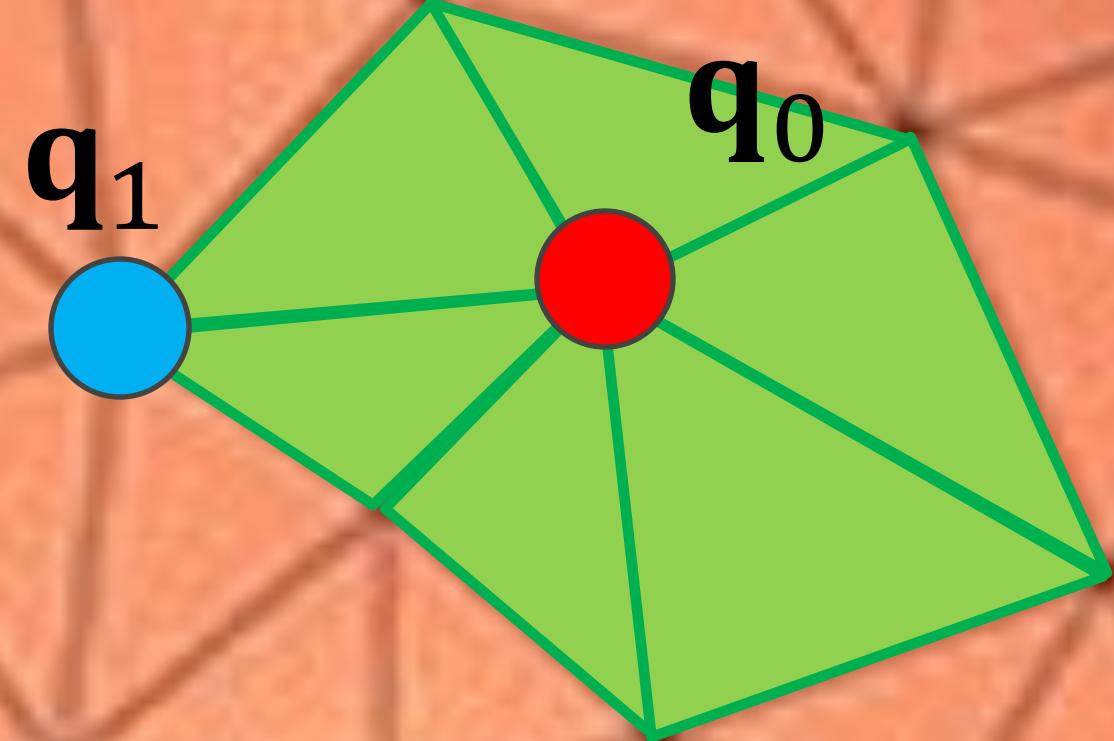
Repeat for awhile

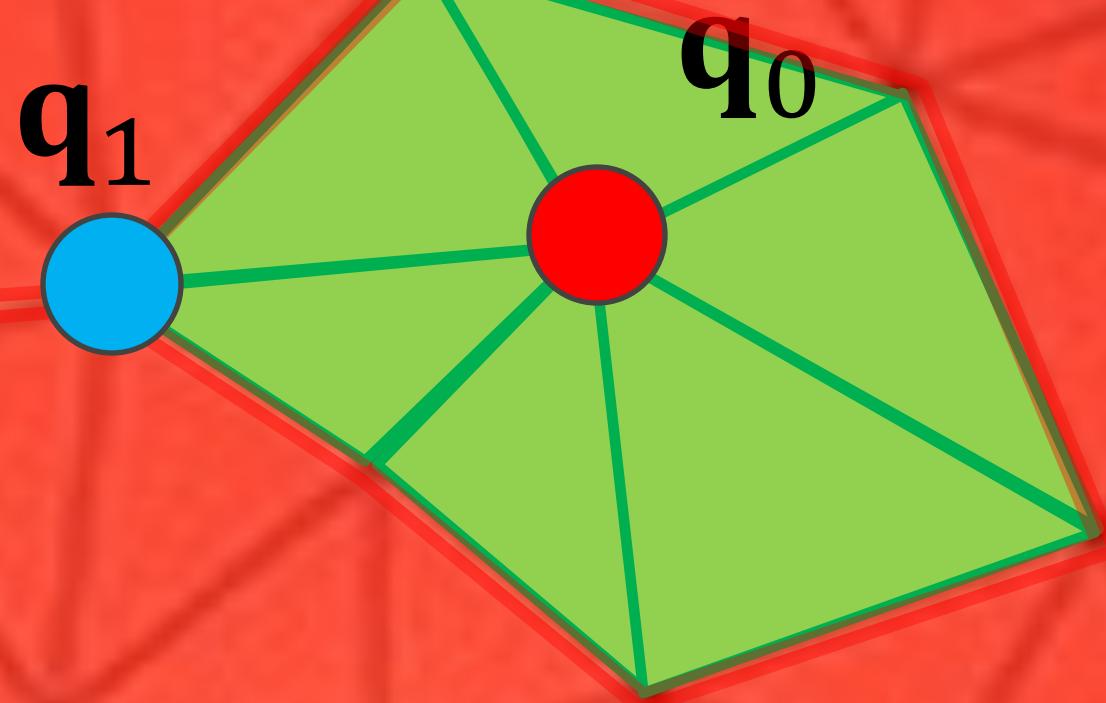
For each vertex, j **Parallelize this Loop !!!**

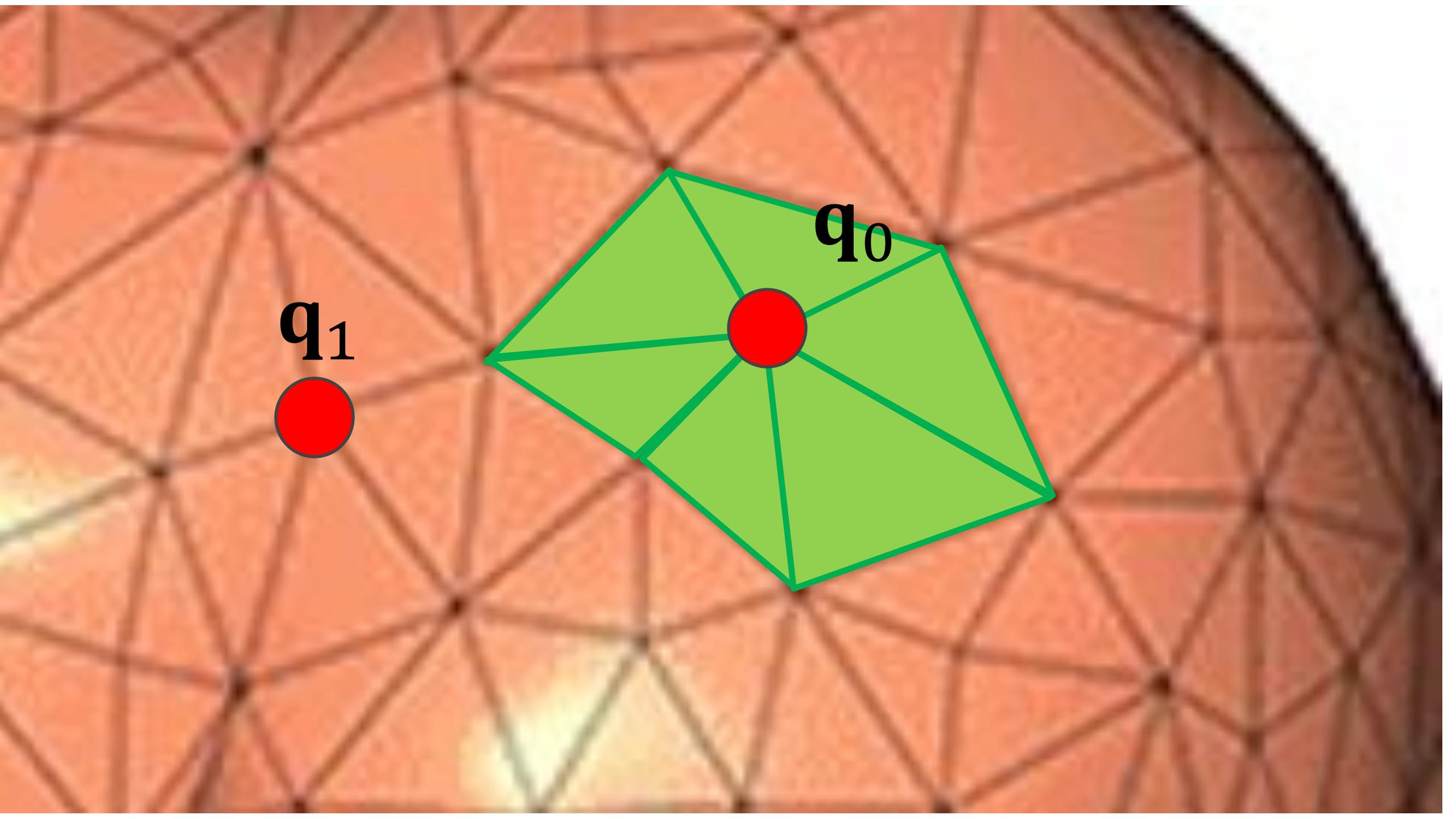
Minimize

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

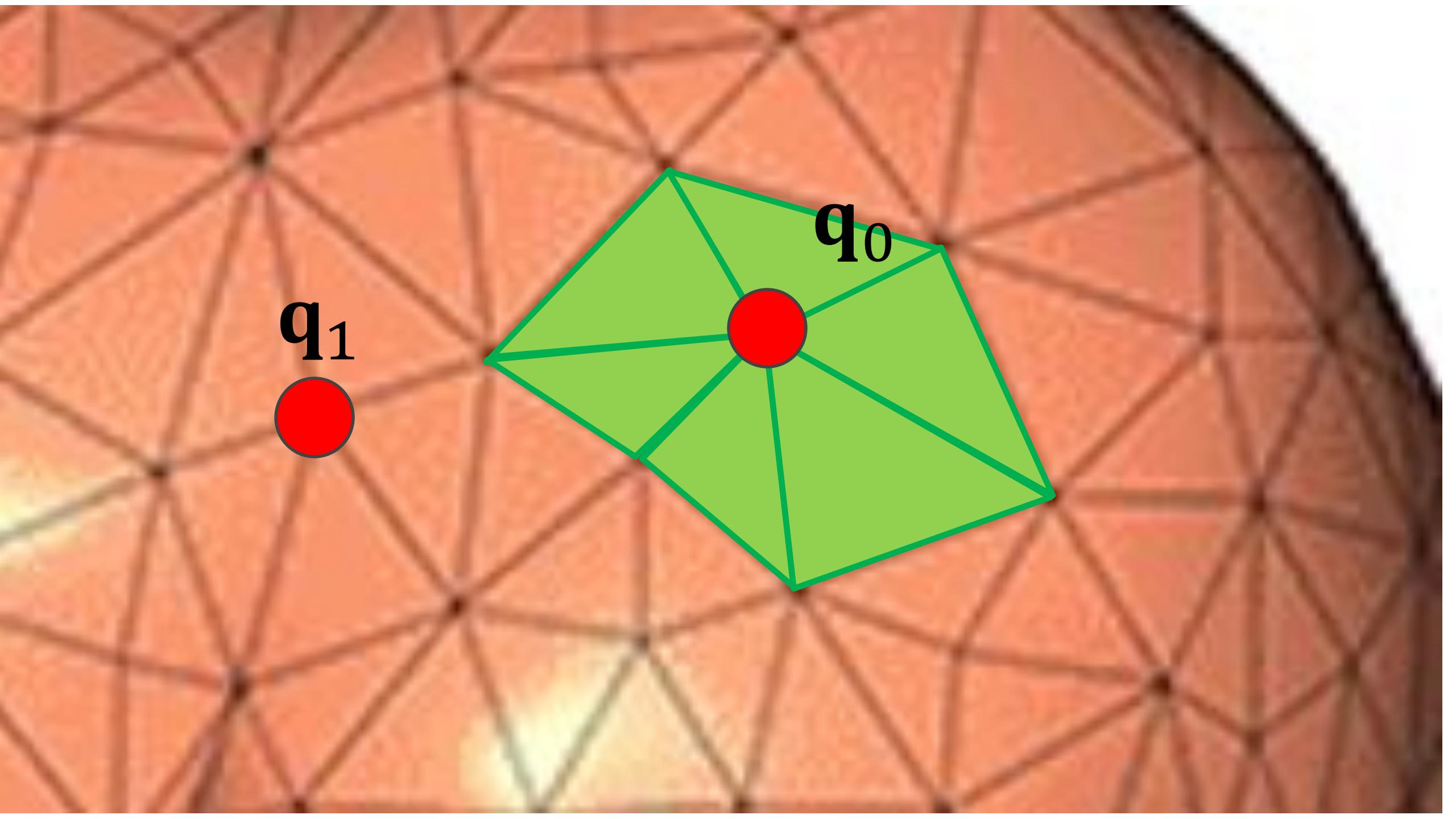
It's not obvious but we can do this in parallel



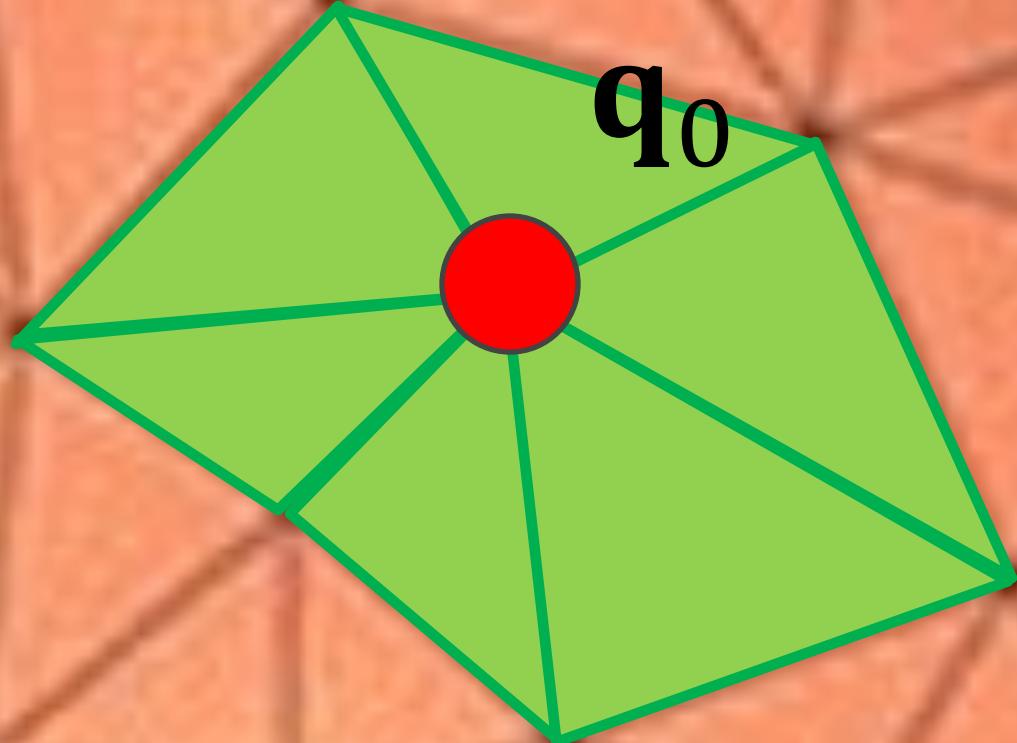


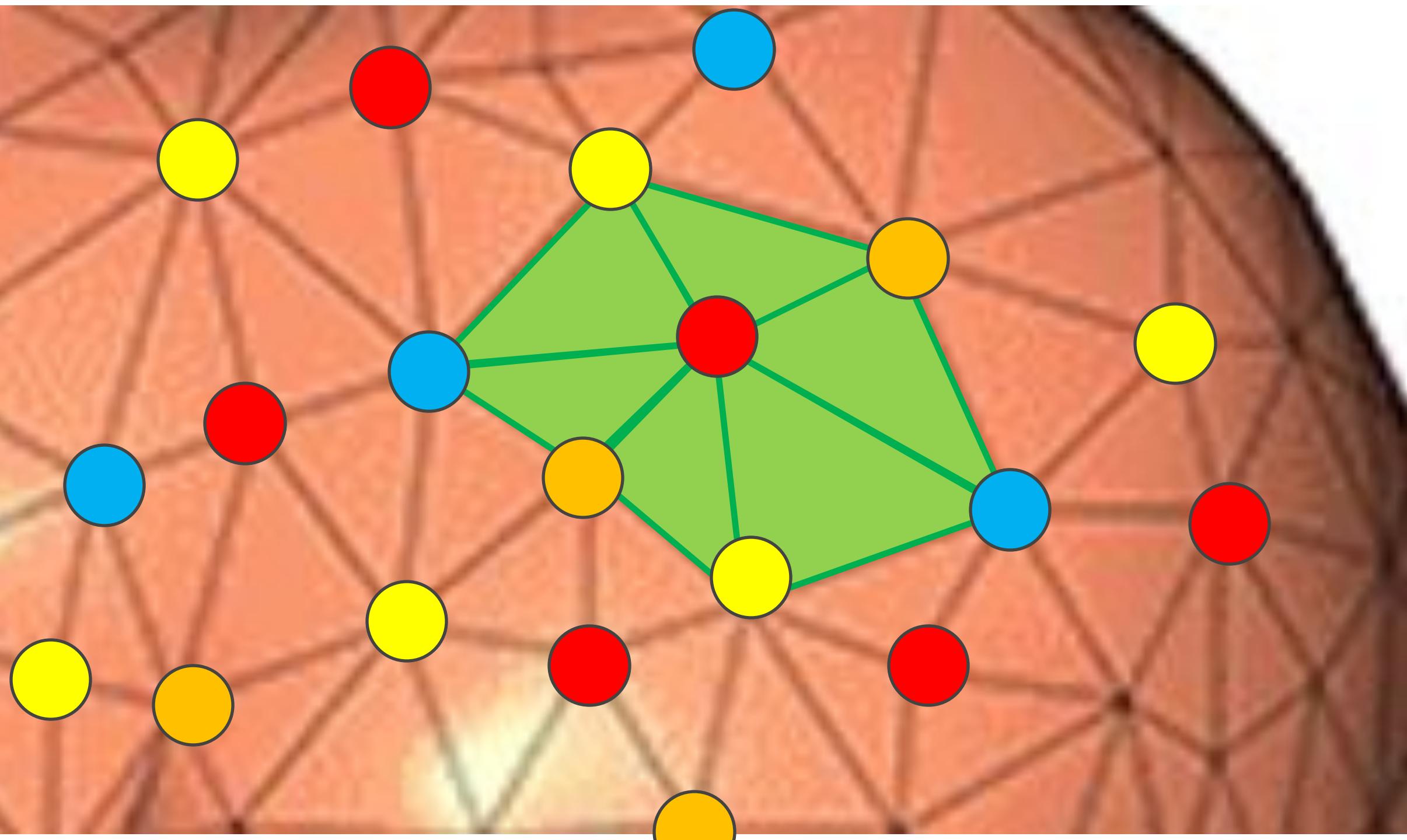


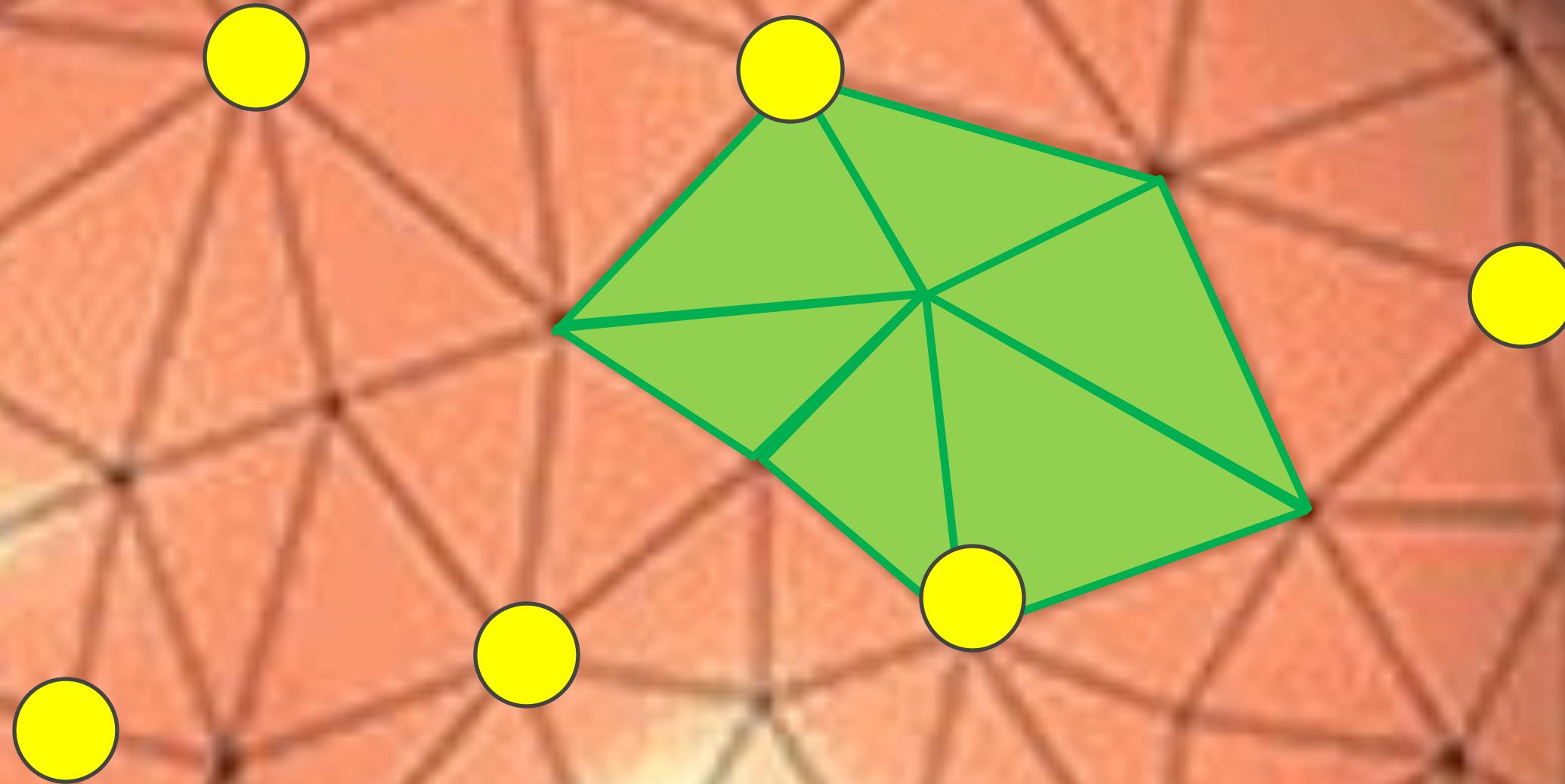
q_1



q_0

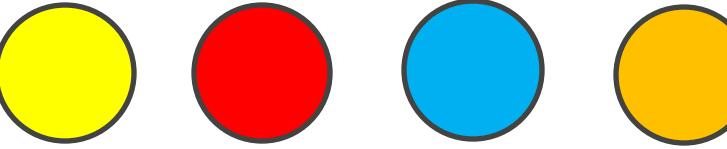






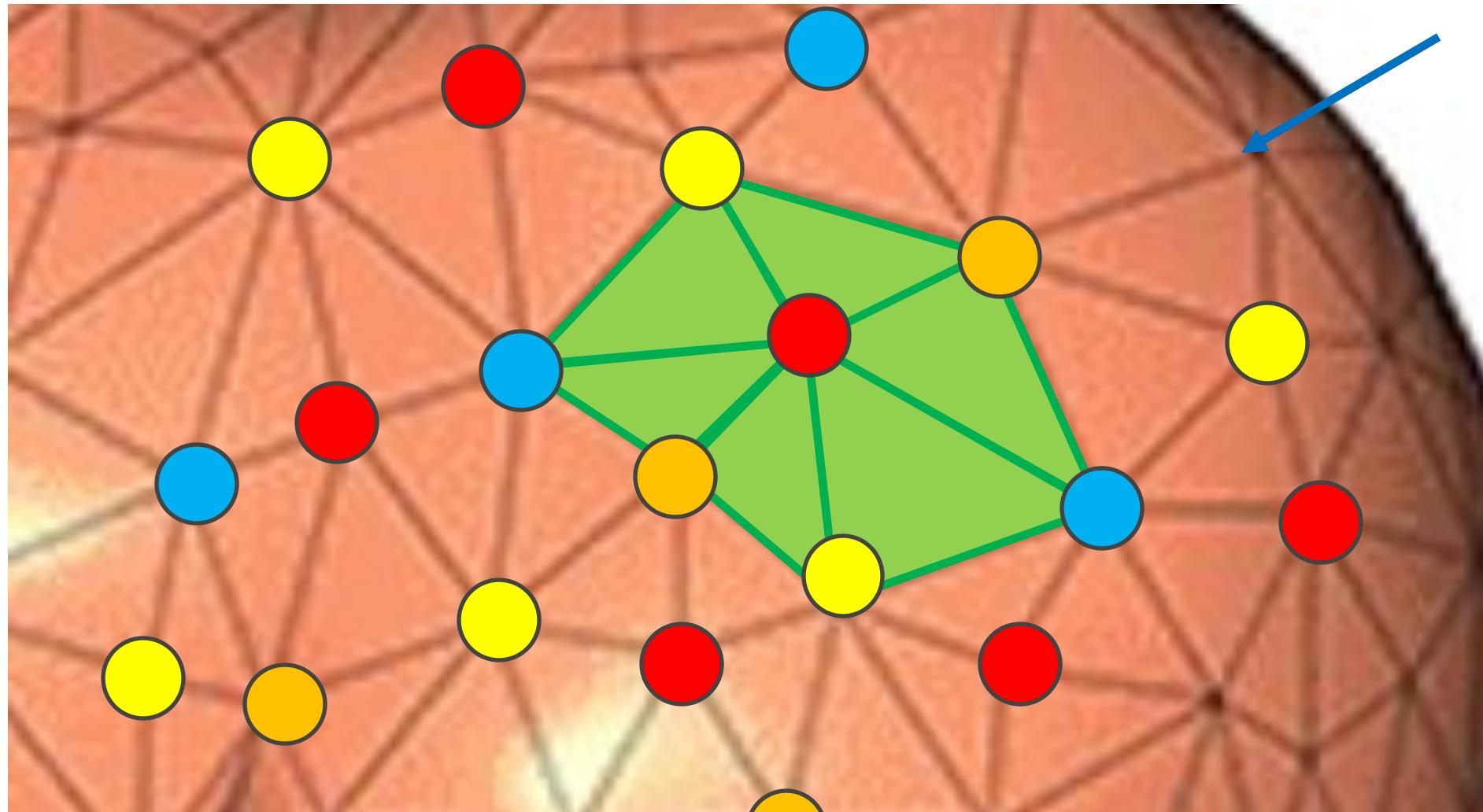
Greedy Graph Coloring

Pick a set of colors



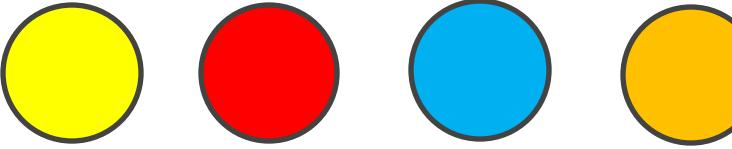
Order all vertices in the mesh

Assign vertex first feasible color



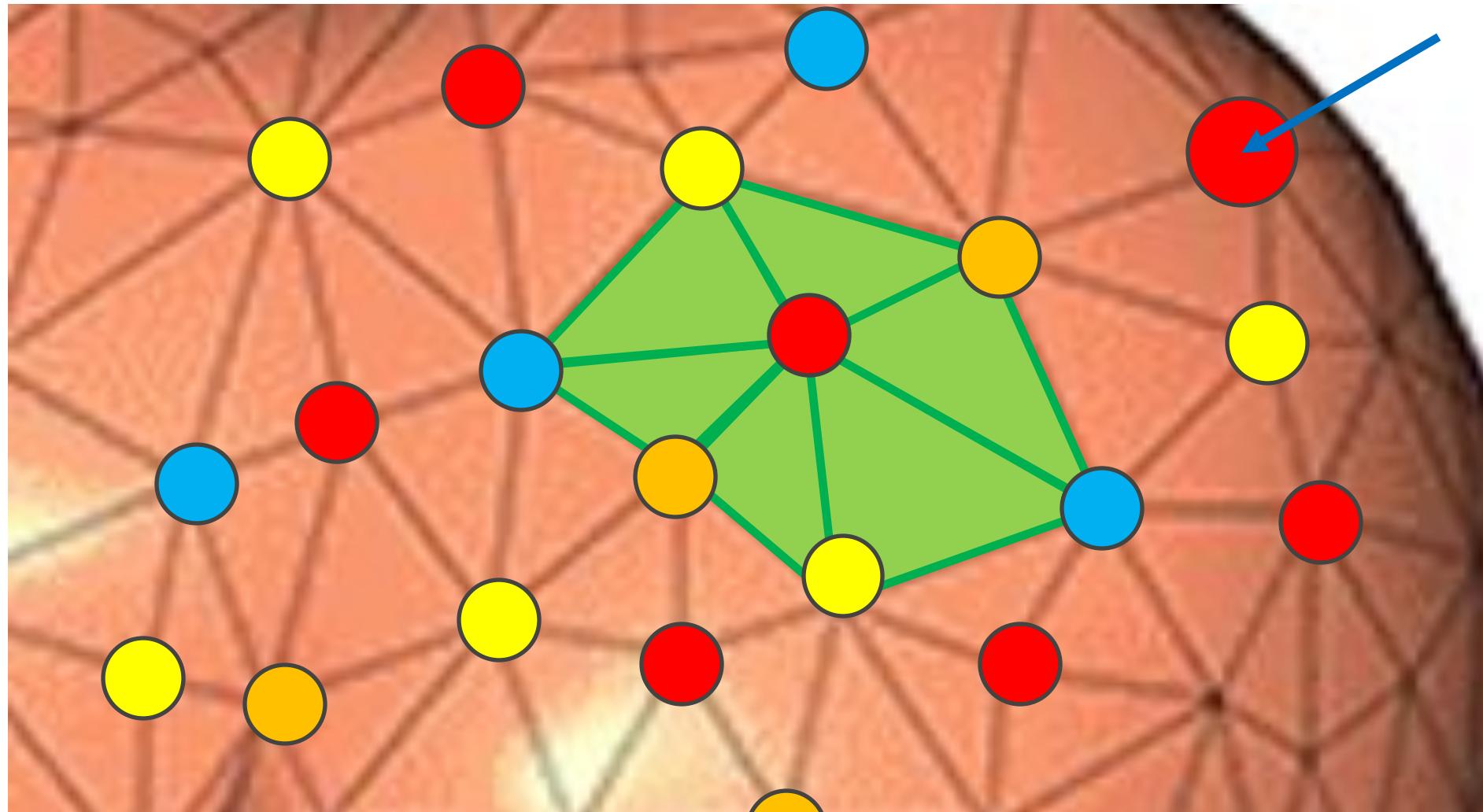
Greedy Graph Coloring

Pick a set of colors



Order all vertices in the mesh

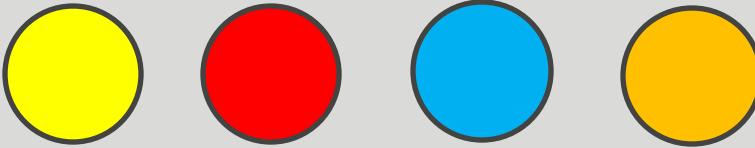
Assign vertex first feasible color



Repeat for awhile

Parallelization

For each color,



In parallel, minimize for all $j \in \text{color}$

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$





36M verts, 124M tets
avg frame time: 7.2s
max: 7.8s

Vertex Block Descent|Chen et al.

Pros and Cons

Pros

Very, very fast due to parallelized small solves

Cons

None ?

Pros and Cons

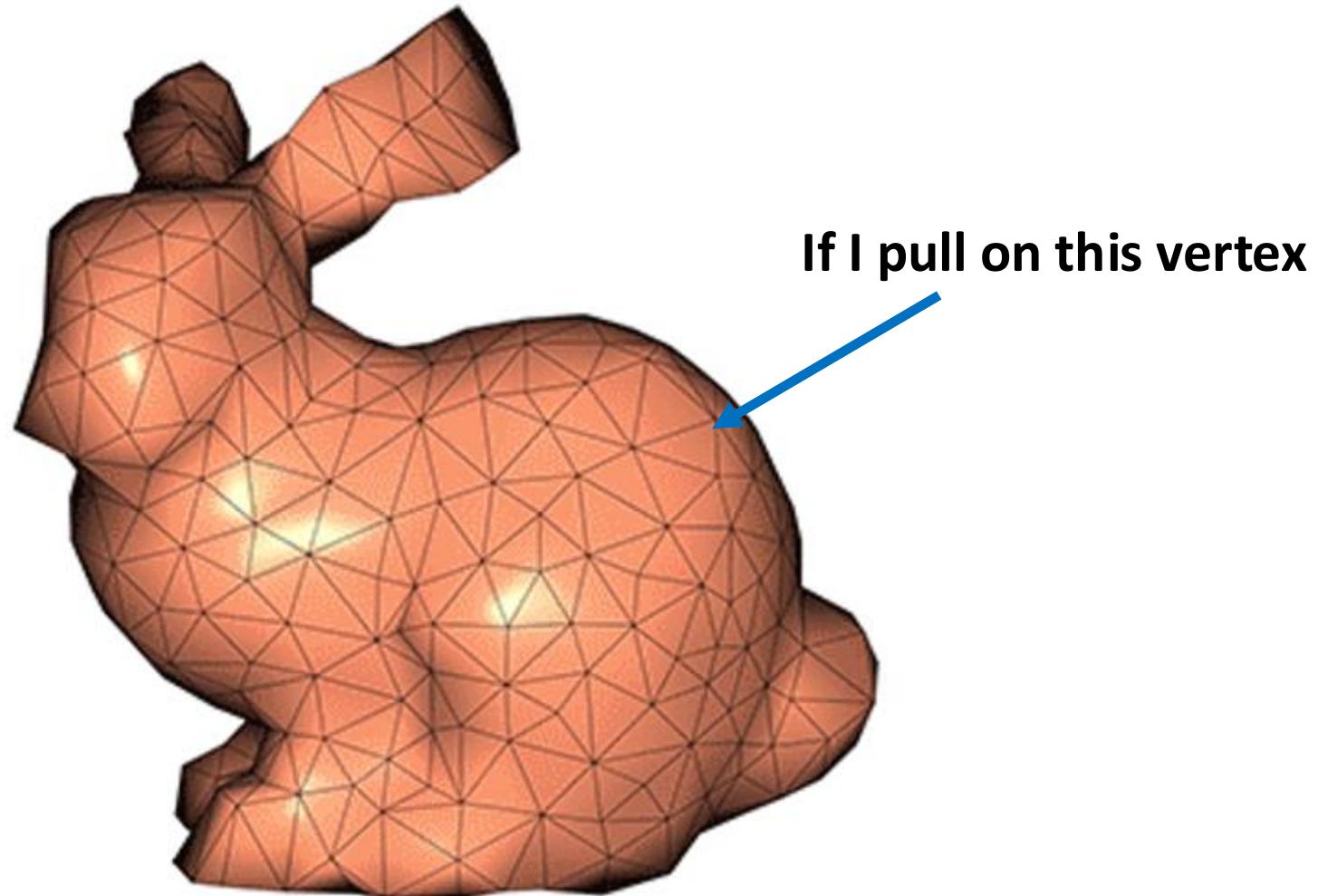
Pros

Very, very fast due to parallelized small solves

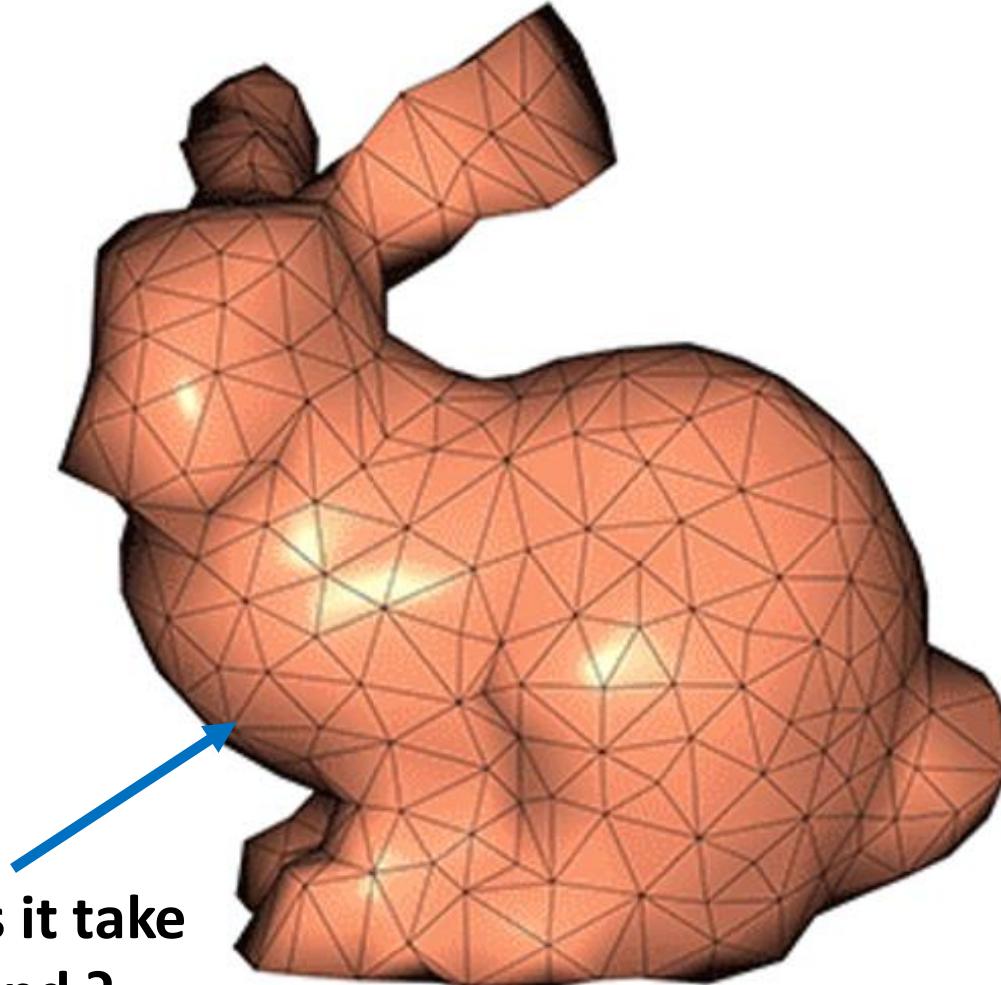
Cons

None ? (Sadly know)

Information Propagation

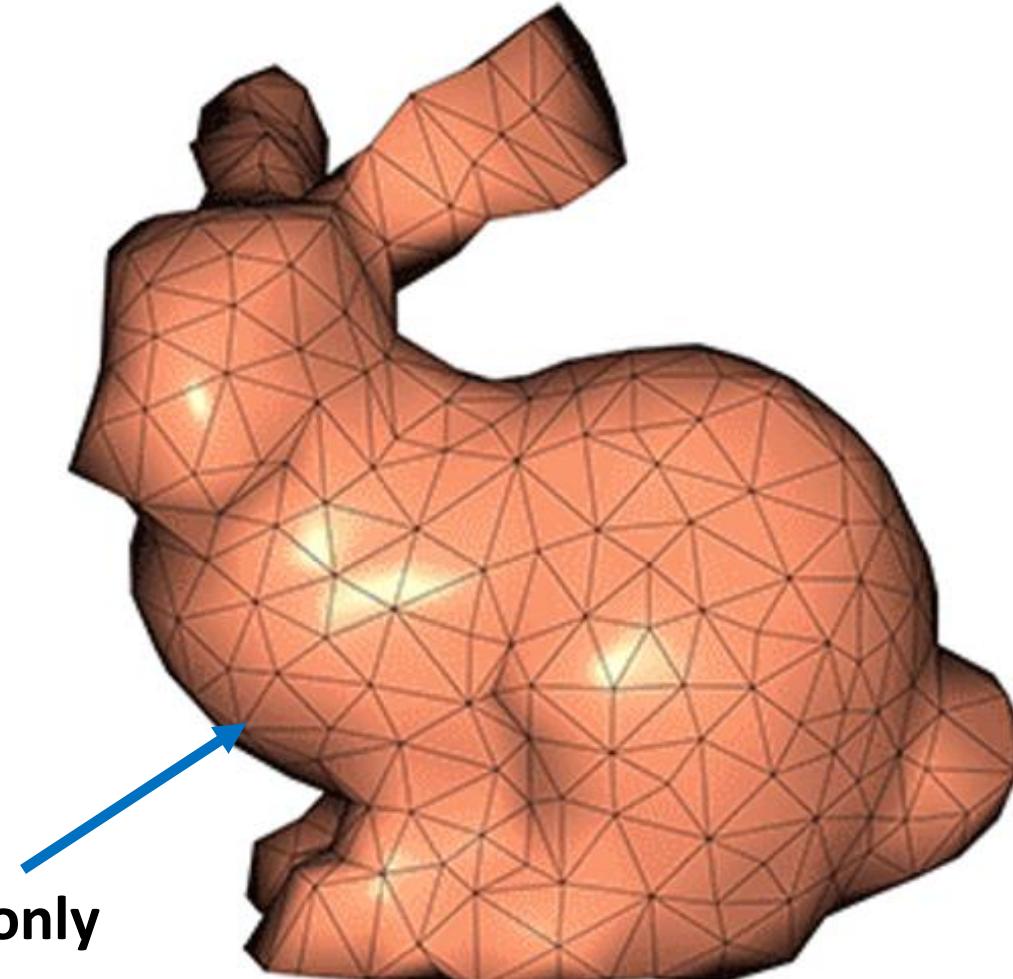


Information Propagation



How many iterations does it take
for this vertex to respond ?

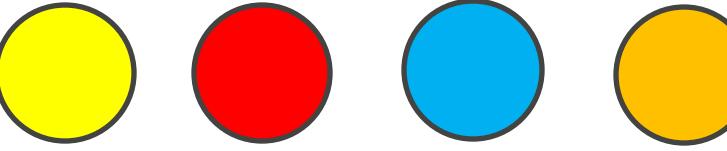
Information Propagation



A lot ! The deformation only
travels along one edge per
iteration

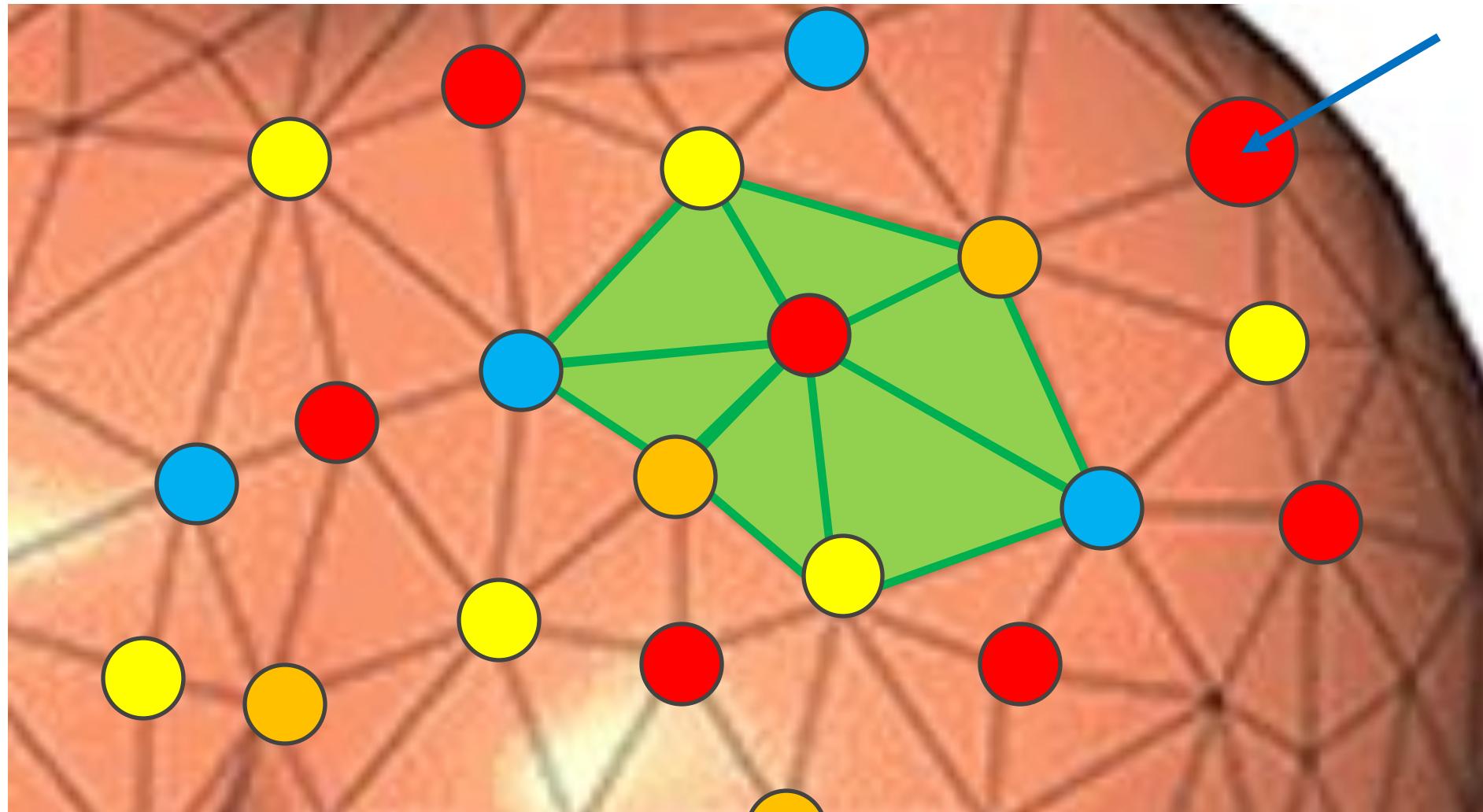
Greedy Graph Coloring

Pick a set of colors



Order all vertices in the mesh

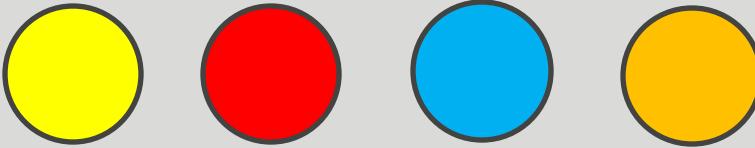
Assign vertex first feasible color



Repeat for awhile

Parallelization

For each color,



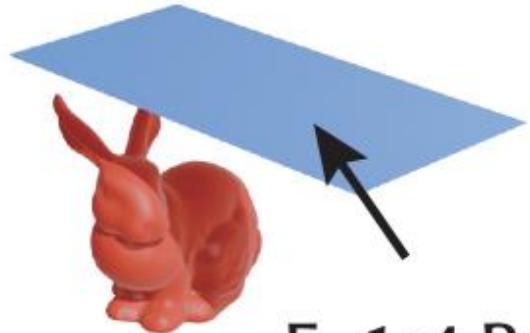
In parallel, minimize for all $j \in \text{color}$

$$E(\mathbf{q}_j^{i+1}) = \frac{1}{2} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i)^T M_{jj} (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + \mathbf{p}_j^T (\mathbf{q}_j^{i+1} - \tilde{\mathbf{q}}_j^i) + h^2 V_j(\mathbf{q}_j^{i+1})$$

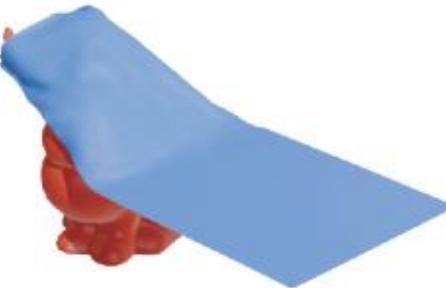


Information Propagation

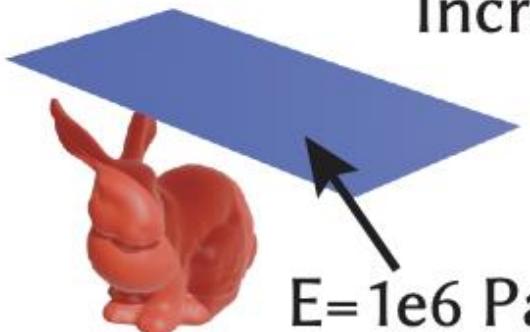
Varying stiffness in VBD [Chen et al. 2024]



$E=1e4$ Pa



— Increase Stiffness → **Fails with same settings**



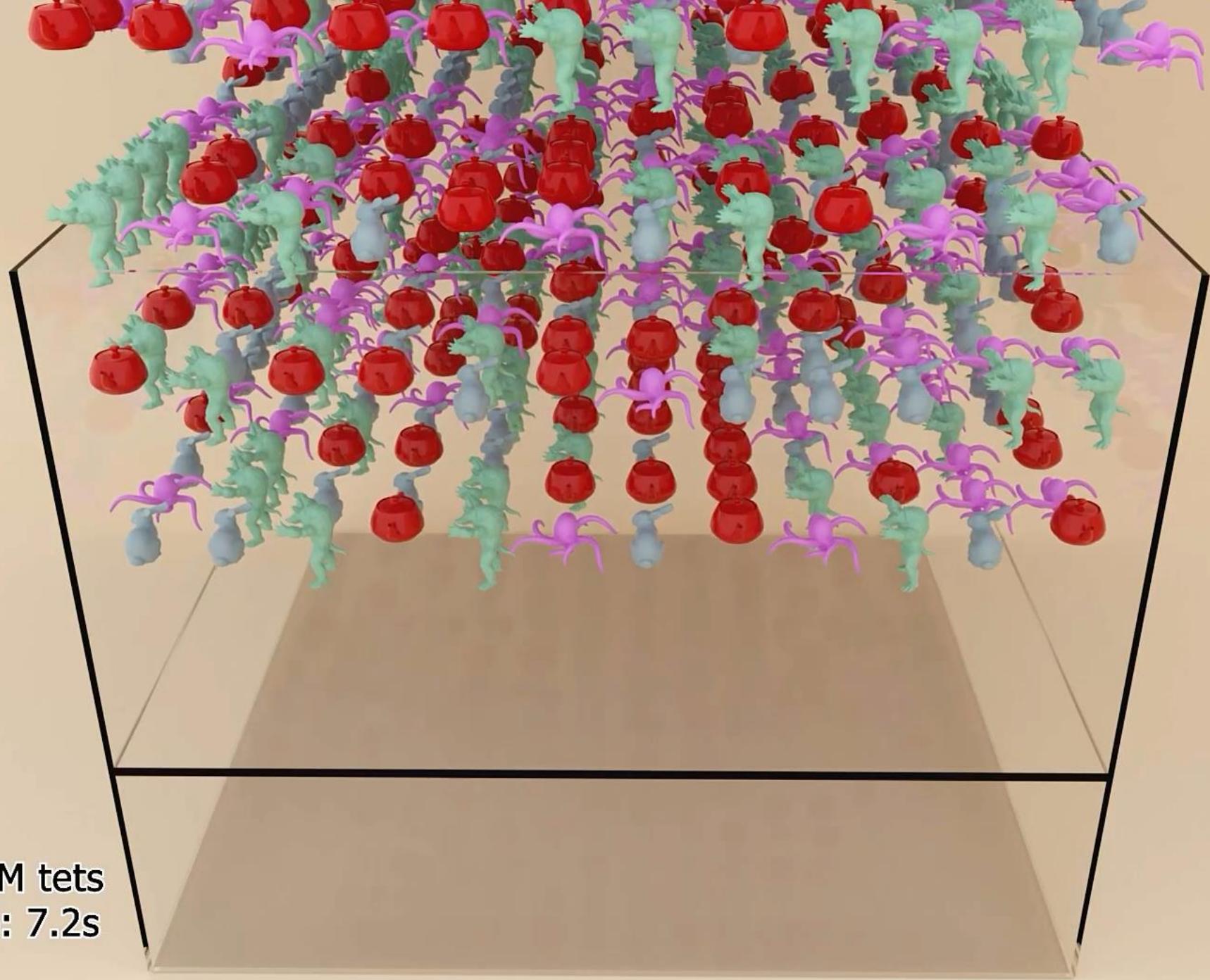
$E=1e6$ Pa



Information Propagation

High resolution geometry and high stiffness exacerbates the issue

Not a deal breaker because you can tune your parameters and geometry to get reasonable output, but not a black box an algorithm as pure Newton's method.



36M verts, 124M tets
avg frame time: 7.2s
max: 7.8s

Next Lecture: NO MORE STRUCTURED LECTURES

