CSC417 Physics-Based Animation

… starting at 3:10 pm

Doctor Strange | Frame Store

Capture and Modeling of Non-Linear Heterogeneous Soft Tissue | Bickel et al

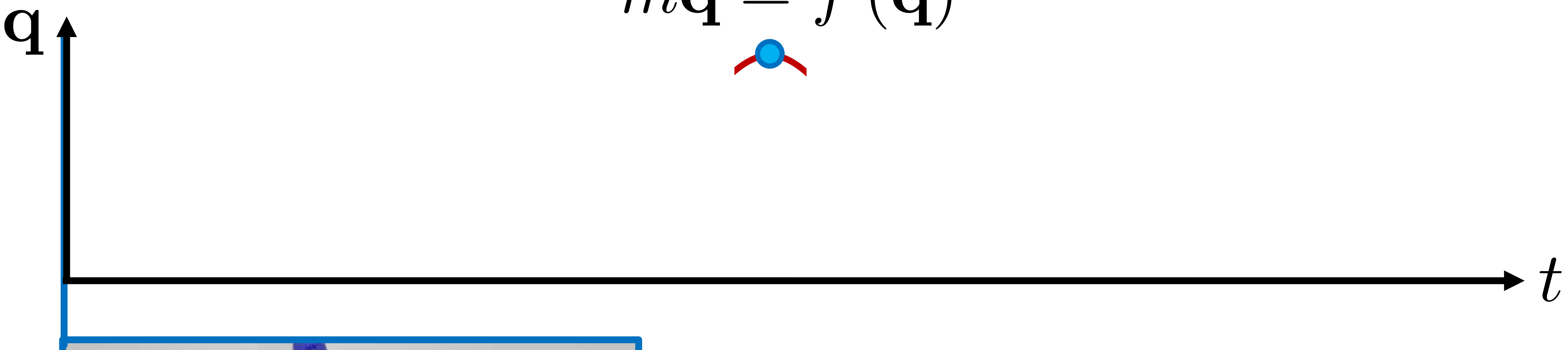
Questions from Previous Lecture ?

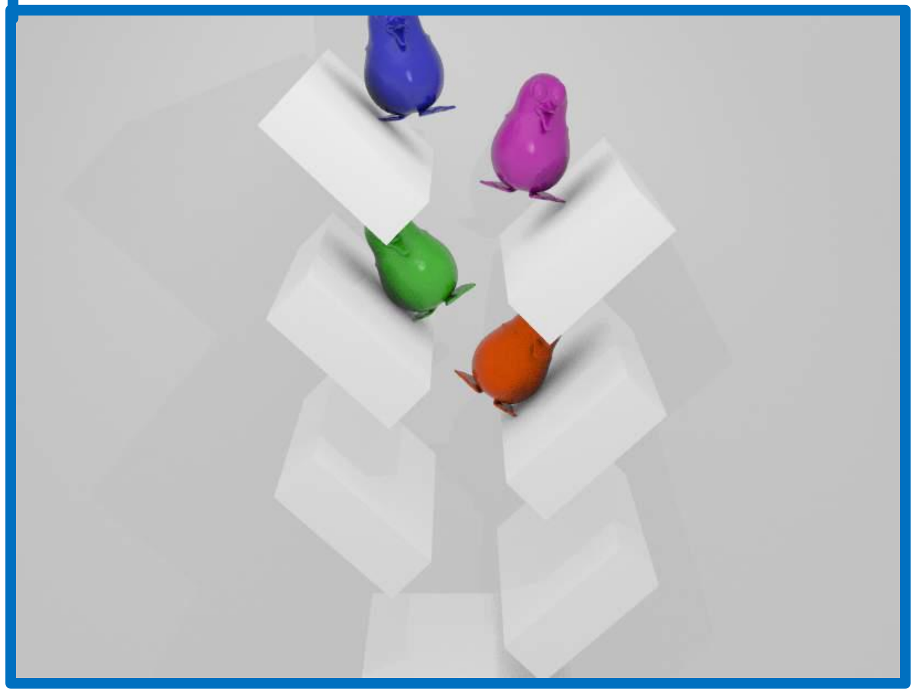
Equations of Motion

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Time Integration

$$m\ddot{\mathbf{q}} = f\left(\mathbf{q}\right)$$

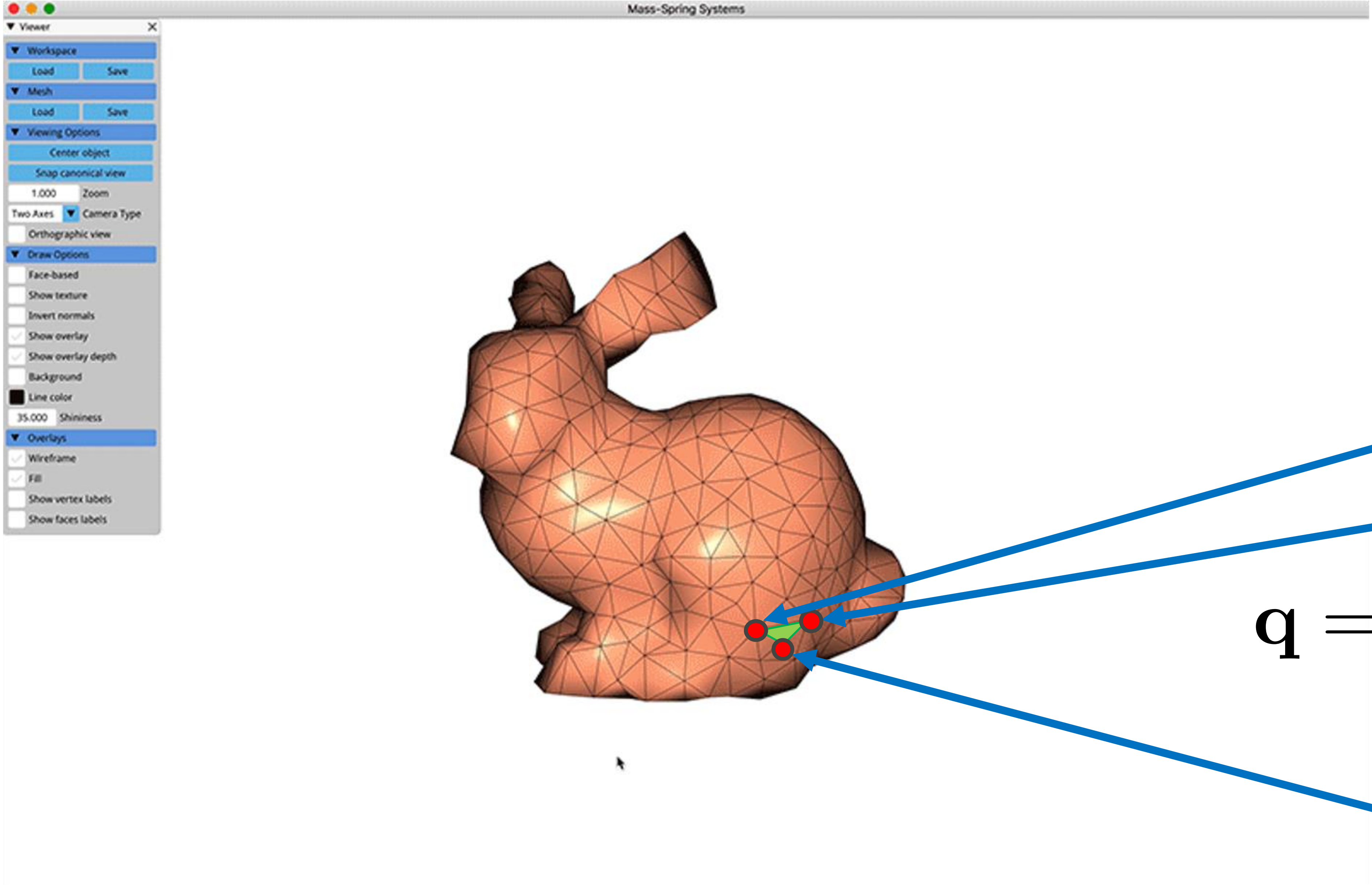




How to Solve This ?

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Spatial Discretization -- Finite Elements

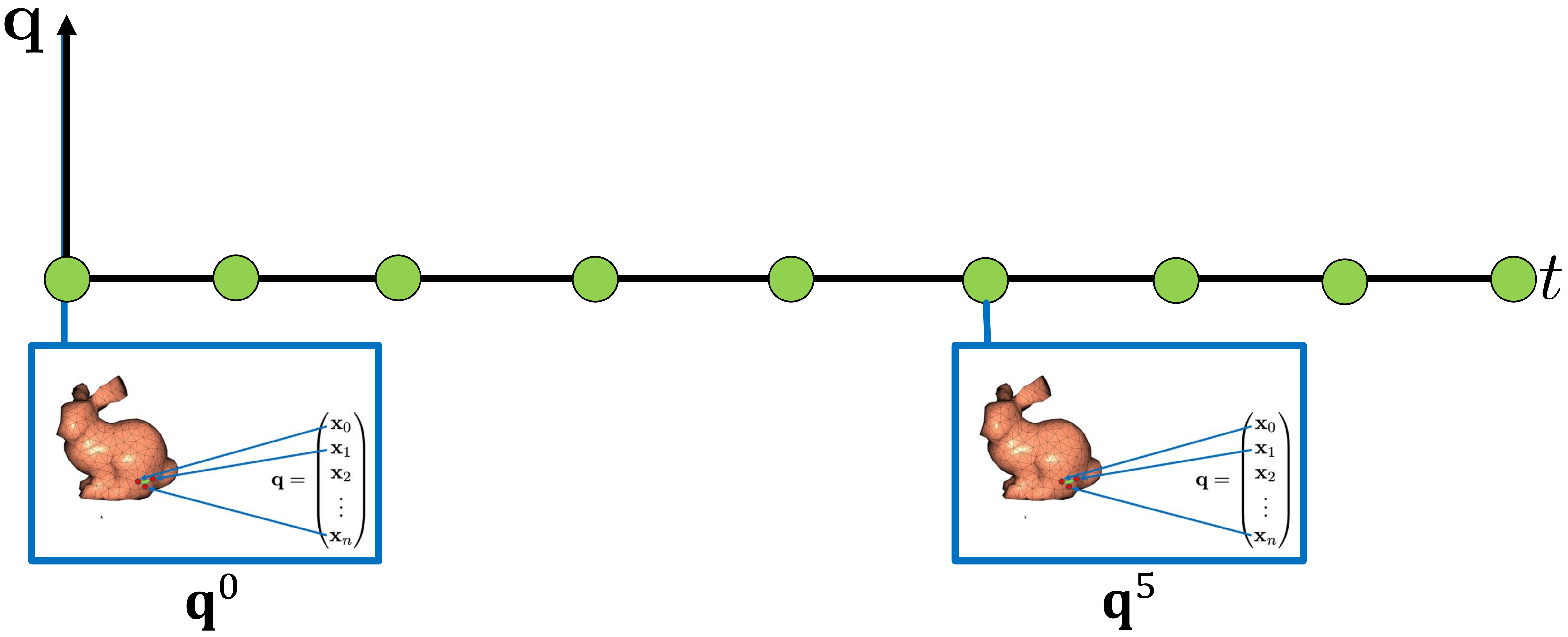


$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

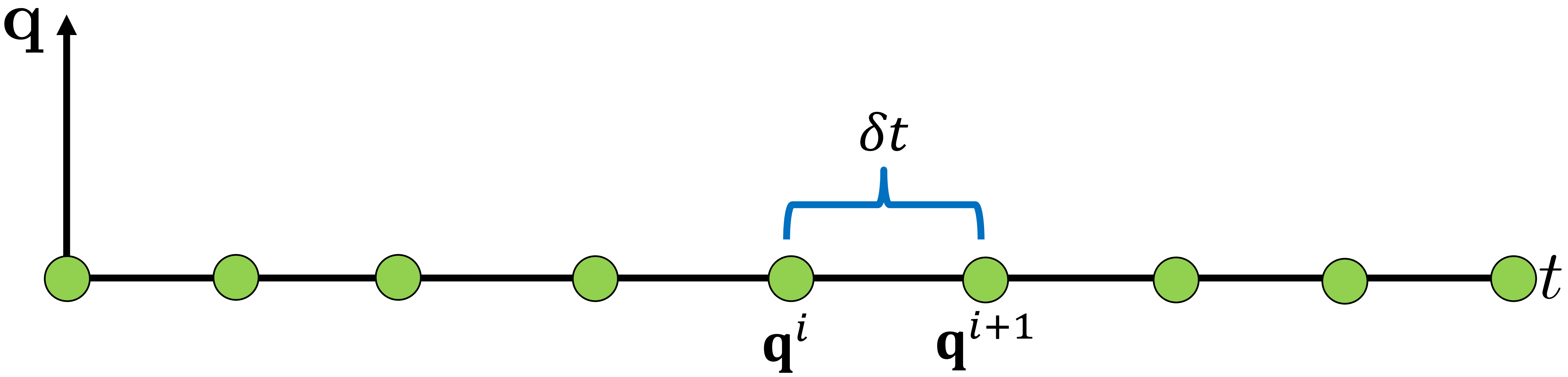
How to Solve This ?

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Time Discretization



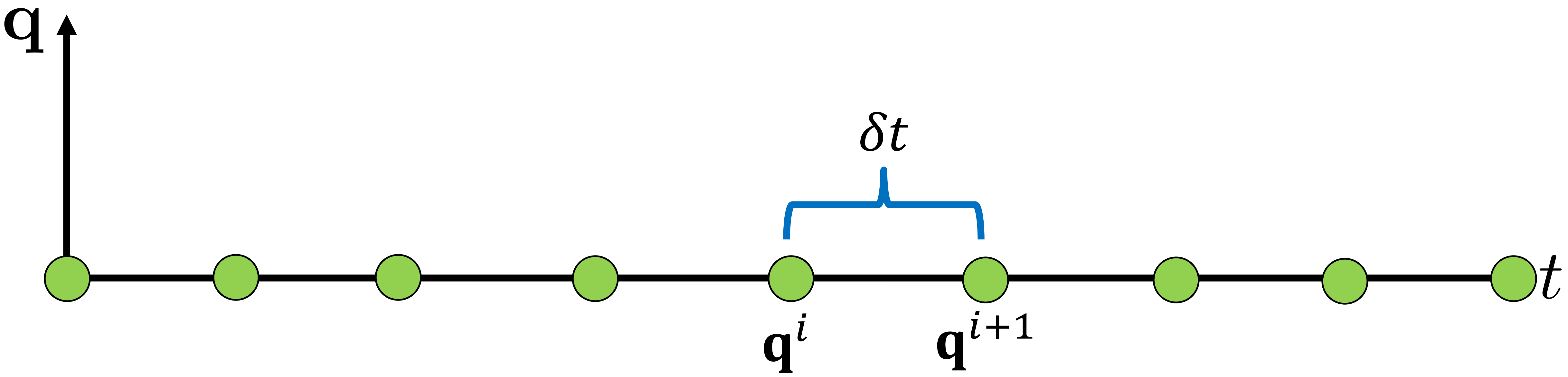
Time Discretization



Continuous Derivative

$$\dot{\boldsymbol{q}} = \lim_{t \to 0} \frac{\mathbf{q}(\mathbf{t} + \delta \mathbf{t}) - \mathbf{q}(\mathbf{t})}{\delta t}$$

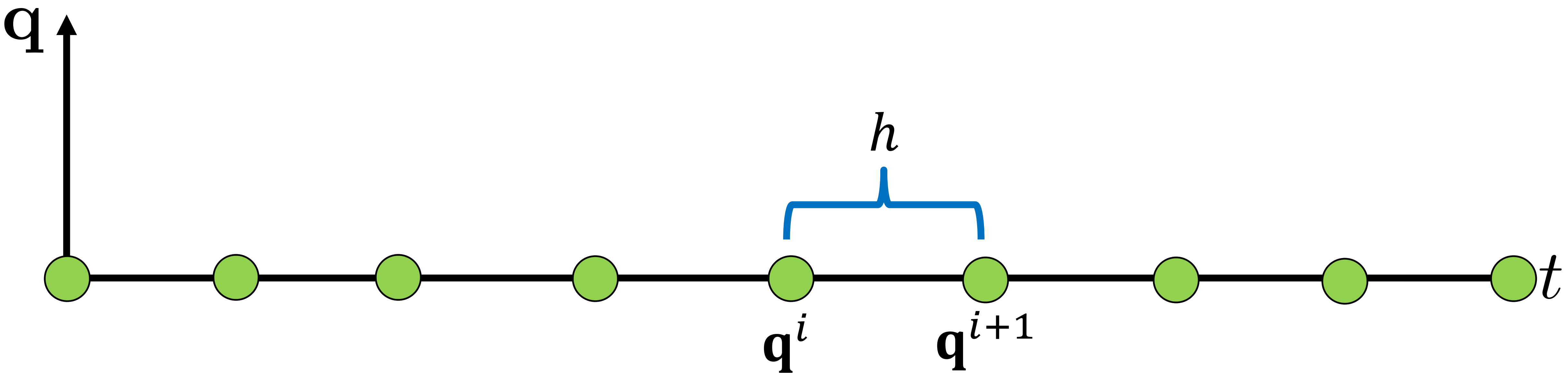
Time Discretization



Finite Difference Derivative

$$\dot{\mathbf{q}} = \frac{\boldsymbol{q}(\mathbf{t} + \delta\mathbf{t}) - \boldsymbol{q}(\mathbf{t})}{\delta t}$$

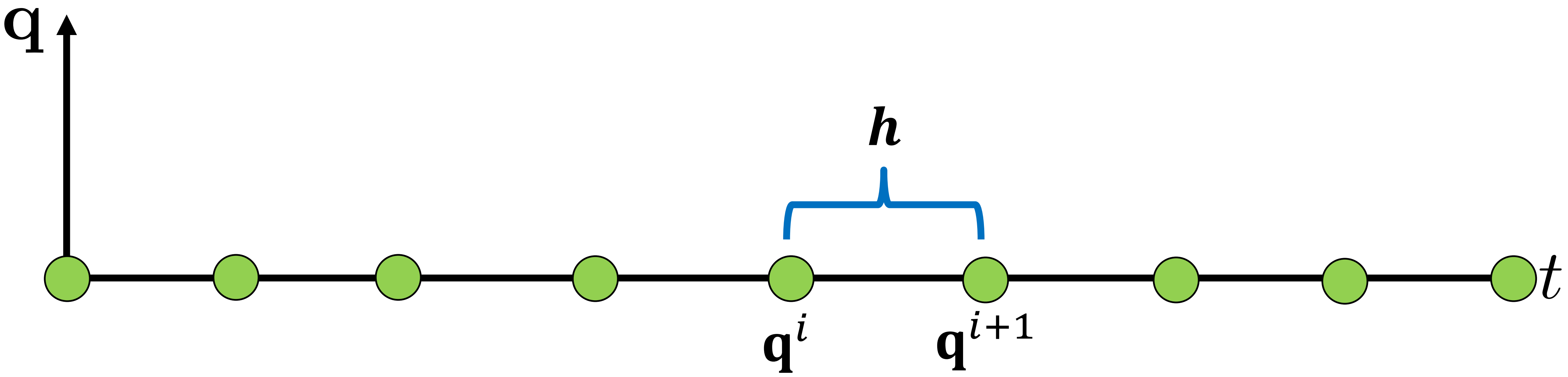
Time Discretization



Finite Difference Derivative

$$\dot{\mathbf{q}}^{i+1} = \frac{\mathbf{q}^{i+1} - \mathbf{q}^{i}}{h}$$

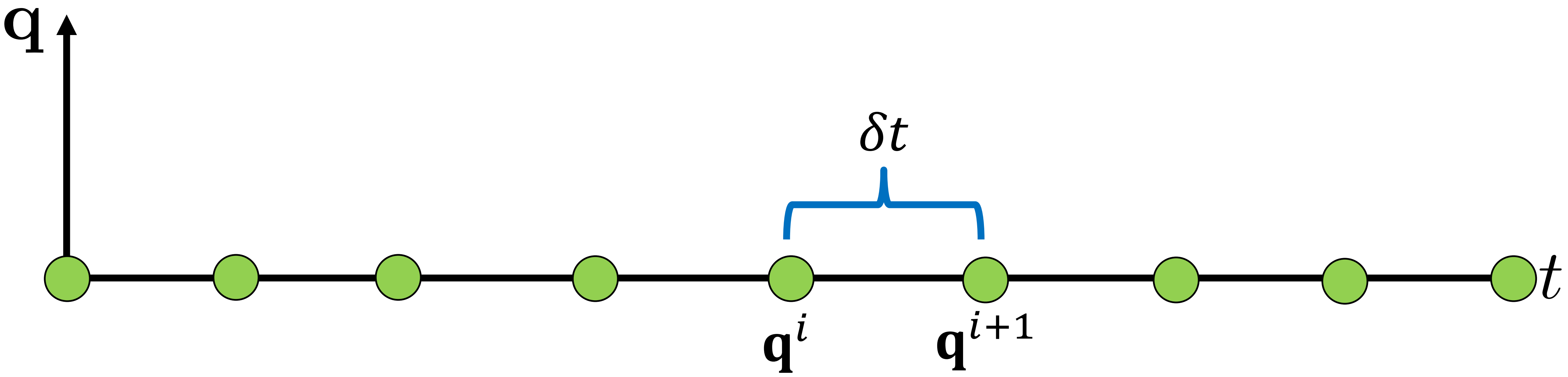
Time Discretization



Finite Difference

$$\ddot{\mathbf{q}} = ?????$$

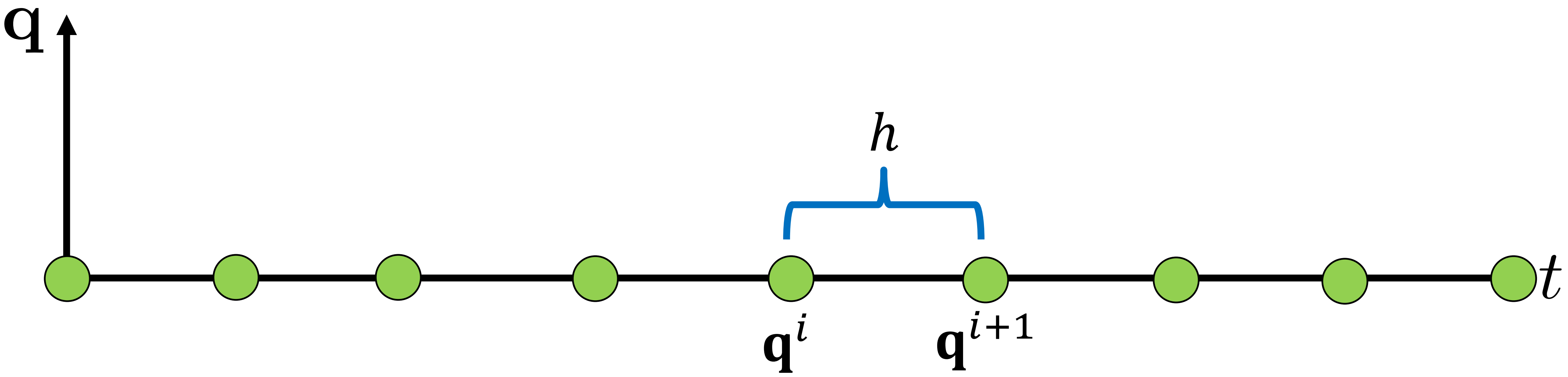
Time Discretization



Continuous Derivative

$$\ddot{\mathbf{q}} = \lim_{t\to 0} \frac{\dot{\mathbf{q}}(\mathbf{t}+\delta\mathbf{t}) - \dot{\mathbf{q}}(\mathbf{t})}{\delta t}$$

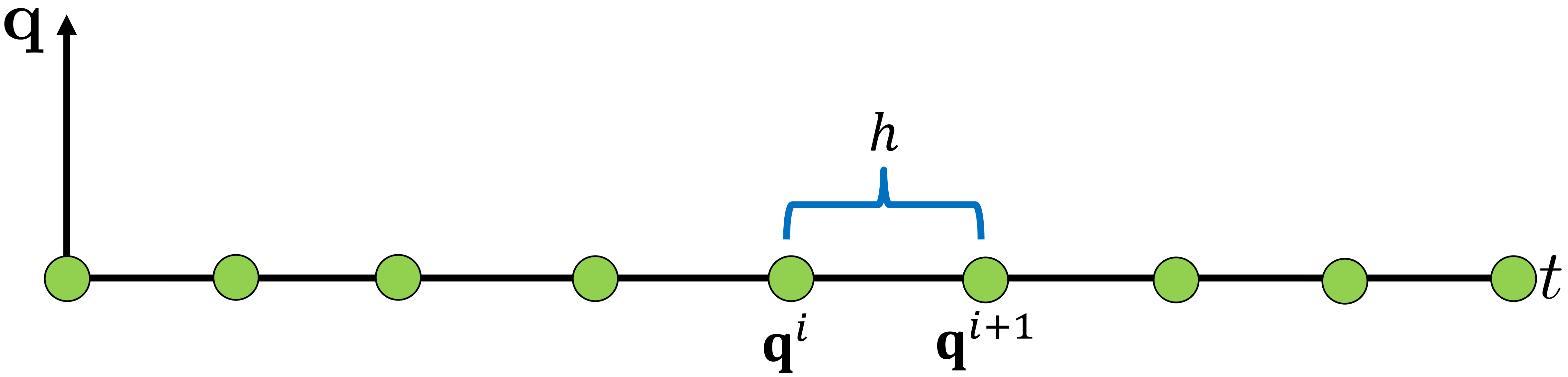
Time Discretization



Finite Difference Derivative

$$\ddot{\mathbf{q}}^i = \frac{\dot{\mathbf{q}}^{i+1} - \dot{\mathbf{q}}^i}{h}$$

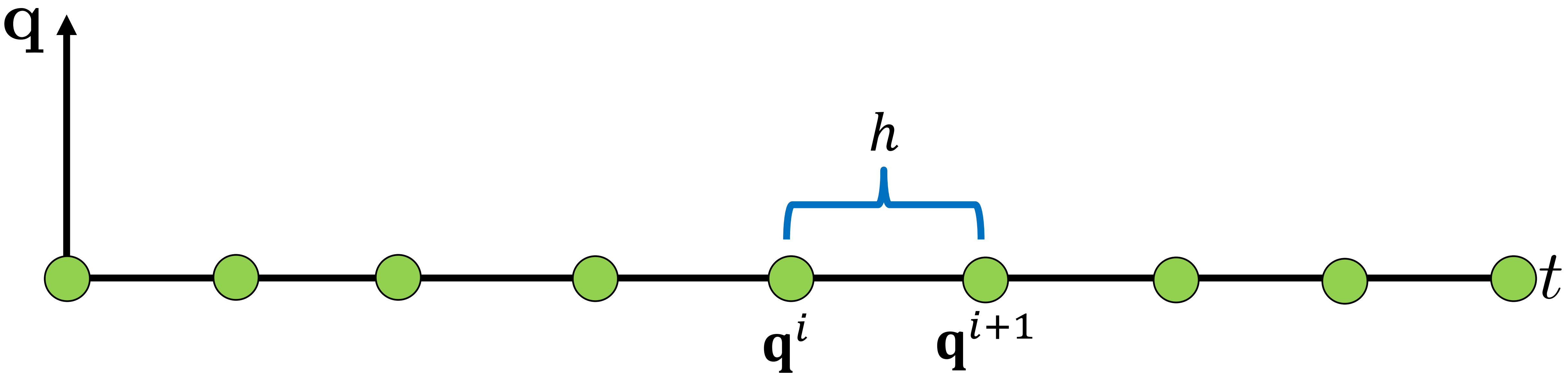
Time Discretization



Finite Difference Derivative

$$\ddot{\mathbf{q}}^i = \frac{\mathbf{1}}{h}\left(\frac{\mathbf{q}^{i+1} - \mathbf{q}^i}{h} - \frac{\mathbf{q}^i - \mathbf{q}^{i-1}}{h}\right)$$

Time Discretization



Finite Difference Derivative

$$\ddot{\mathbf{q}}^i = \frac{\mathbf{1}}{h^2}\left(\mathbf{q}^{i+1} - \underbrace{\mathbf{2q}^i + \mathbf{q}^{i-1}}_{\tilde{\mathbf{q}}^i}\right)$$

Discretize in Time

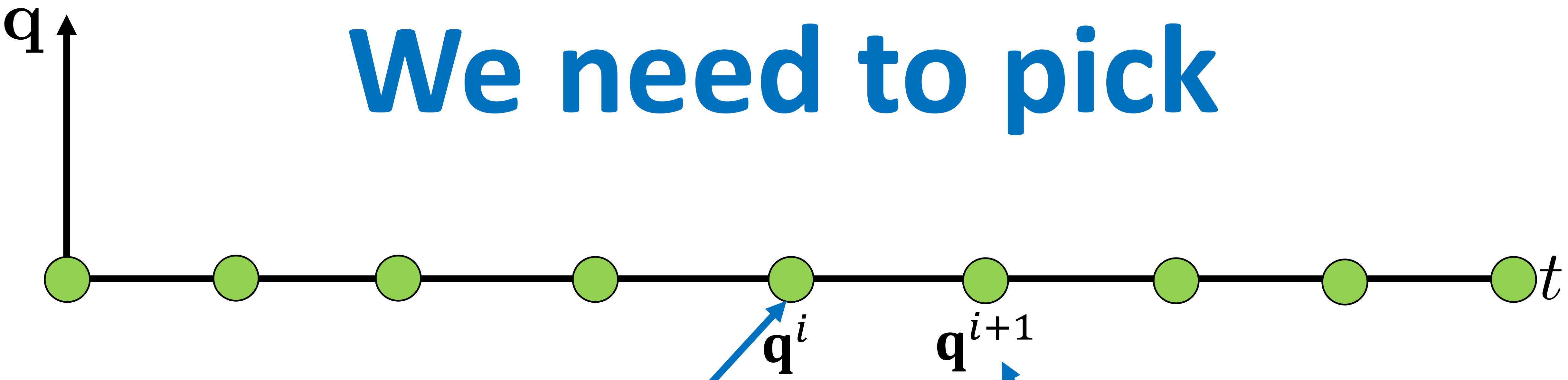
$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Discretize in Time

$$M \frac{1}{h^2}\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) = -\frac{\partial V}{\partial \mathbf{q}}$$

What about this ?

We need to pick

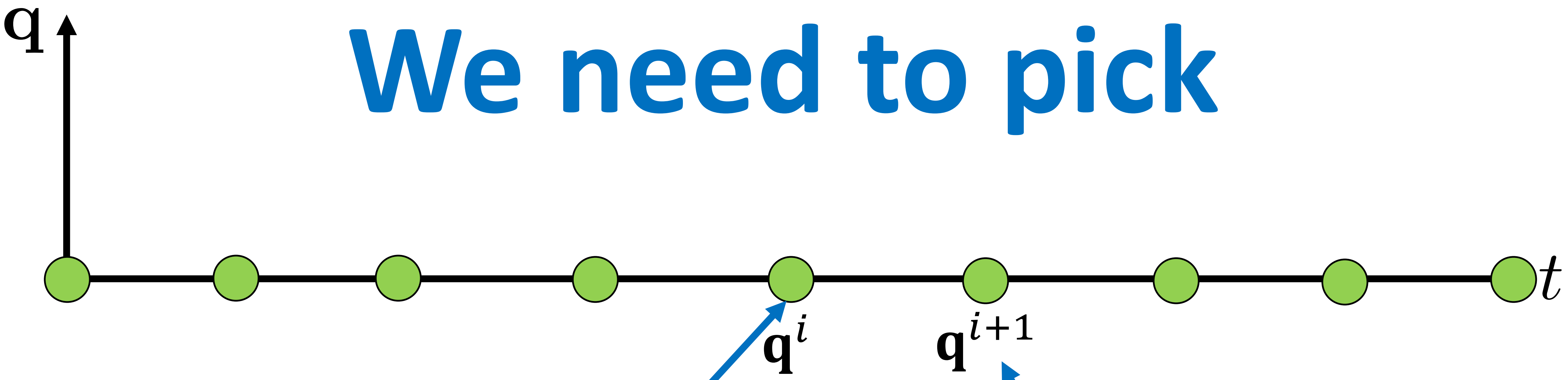


$$\frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i}}\right)$$

OR

$$\frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right)$$

We need to pick



$$\frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}^{\mathbf{i}})$$

Explicit

OR

$$\frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}^{\mathbf{i}+1})$$

Implicit

Explicit Integrator

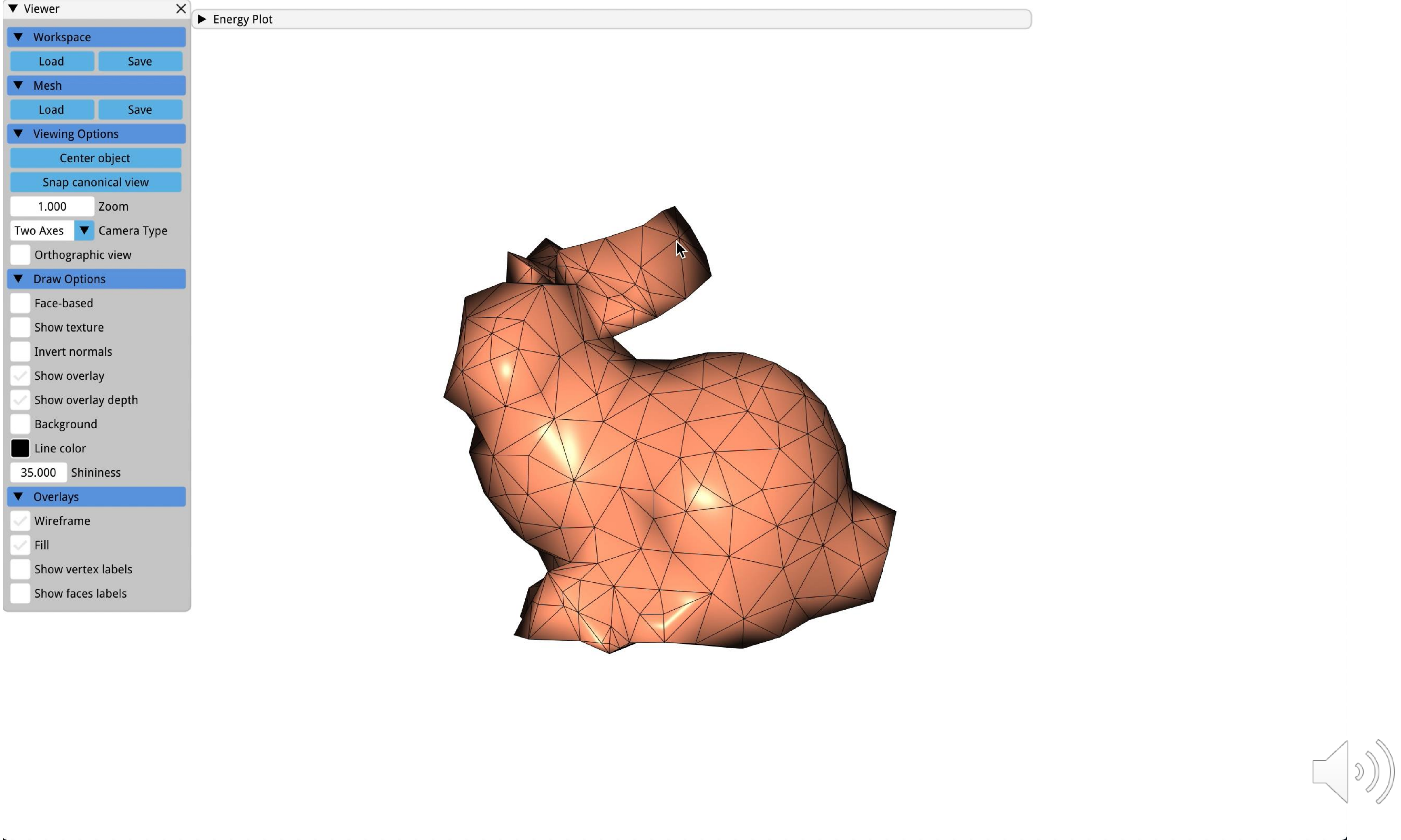
$$M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}^{\mathbf{i}}}\right) = -h^{2} \frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}^{\mathbf{i}})$$

Forward Euler

Implicit Integrator

$$M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}^{\mathbf{i}}}\right) = -h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right)$$

Backward Euler: More stable, harder to solve



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How do we Solve This ?

$$M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}^{\mathbf{i}}}\right) = -h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right.$$

Root Finding ?

$$M\left(\mathbf{q}^{\mathbf{i+1}}-\widetilde{\mathbf{q}^{\mathbf{i}}}\right)+h^{2}\frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right)=0$$

Nonlinear Root Finding is Hard

How do we Solve This ?

$$M\left(\mathbf{q}^{\mathbf{i}+1}-\widetilde{\mathbf{q}^{\mathbf{i}}}\right)+h^{2}\frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i}+1}\right)=0$$

What other type of problem involves finding the zero of a function?

Optimization

$$\nabla \mathbf{g}\left(\mathbf{q}^{\mathbf{i+1}}\right) = 0$$

Find where gradient is zero

How do we Solve This ?

$$M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}^{\mathbf{i}}}\right) + h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right) = 0$$

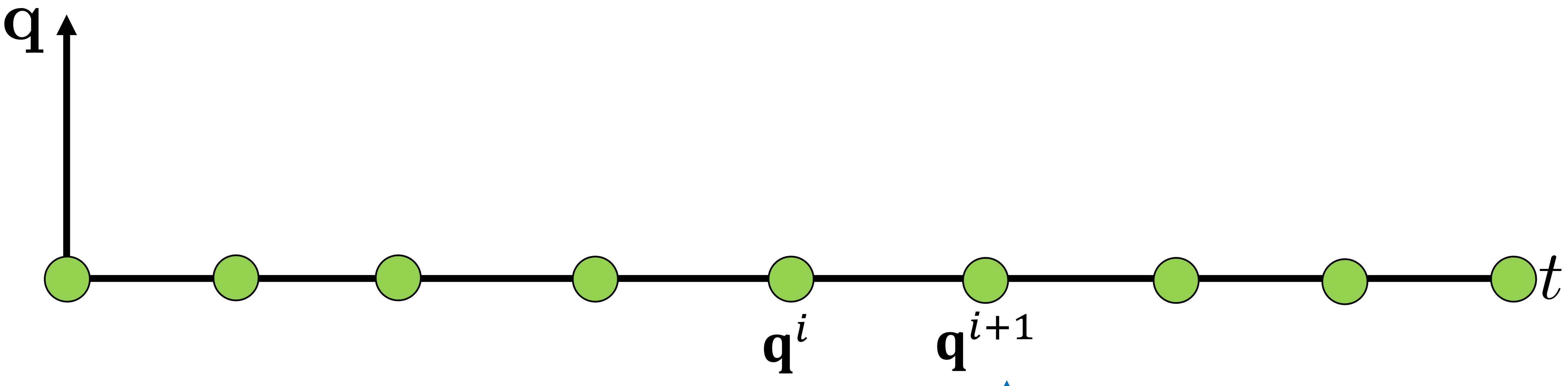
Gradient of what equals this ? Let's guess, then check

How do we Solve This ?

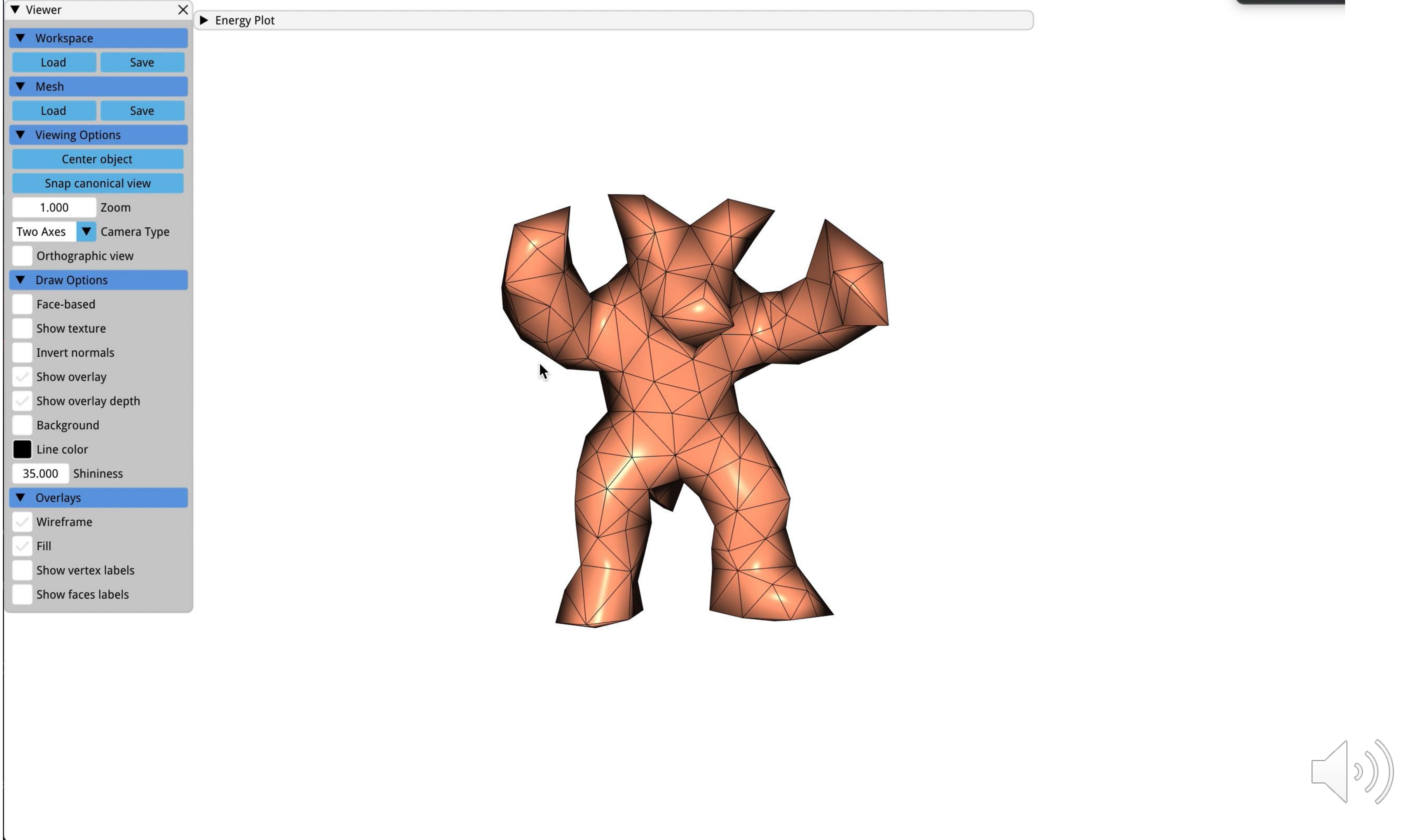
$$E(\mathbf{q}^{\mathbf{i+1}}) = \frac{1}{2}(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}})^{\mathrm{T}} M(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}) + h^2 V(\mathbf{q}^{\mathrm{i+1}})$$

Gradient of what equals this ? Let’s guess, then check

For each Time Step, Use Optimization !



$$\mathbf{q}^{\mathbf{i+1}} = \arg\min_{\mathbf{q}} E(\mathbf{q})$$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How do we Solve This ?

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 V\left(\mathbf{q}^{\mathrm{i+1}}\right)$$

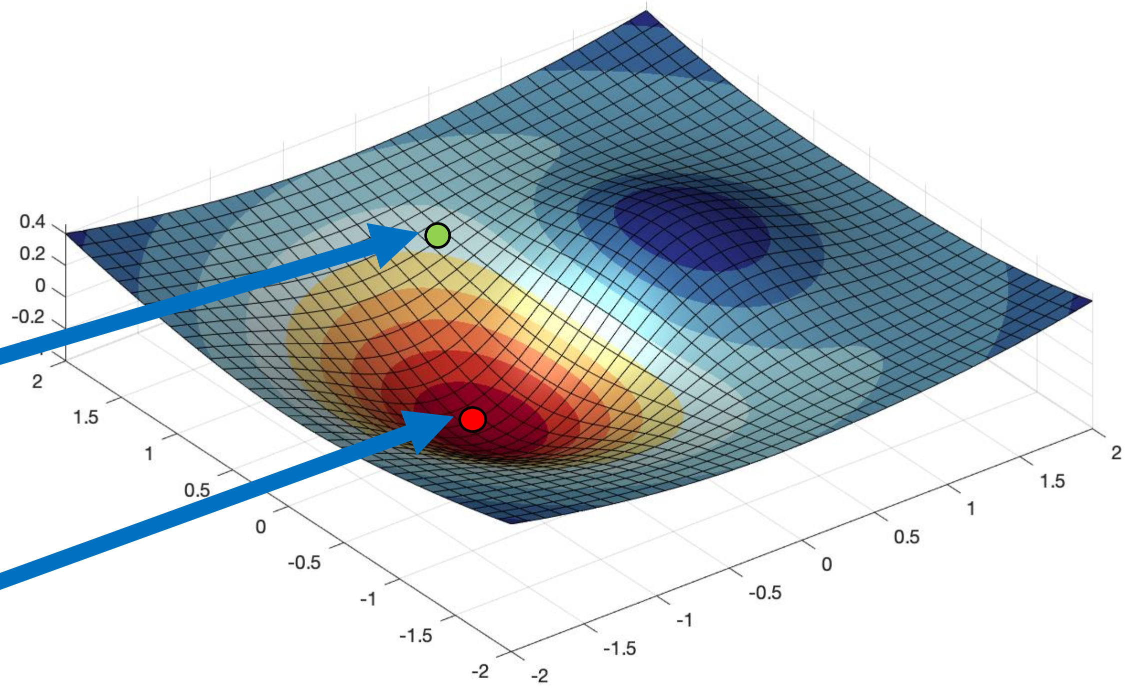
Gradient of what equals this ? Let's guess, then check

WARNING I DID NOT HAVE TIME TO RE-TYPESET THIS MATH SO THE OTPIMIZATION PROBLEM IS SLIGHTLY DIFFERENT FROM THE ONE ABOVE

BUT DON'T PANIC THE ALGORITHM IS STILL THE SAME !!!!!

Backward Euler using Optimization

$$\mathbf{v}^* = \arg\min_{\mathbf{v}} \underbrace{\frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t \mathbf{v}\right)}_{E(\mathbf{v})}$$



start here

end here

Gradient-Based Optimization

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left|\left|\frac{\partial E}{\partial \mathbf{v}}\bigg|_{\mathbf{v}^i}\right|\right| < \text{tol}$$

Choose search direction

$$\mathbf{d} = \text{??????}$$

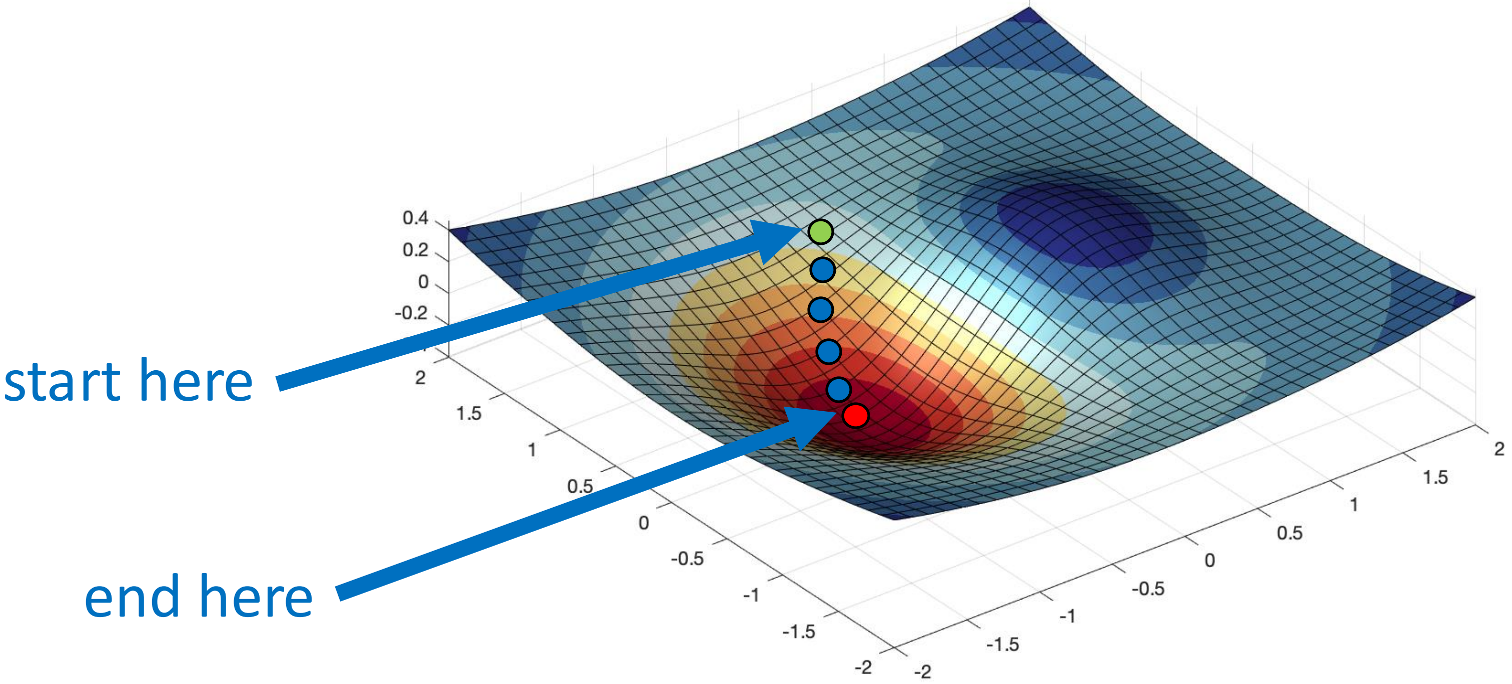
Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha\mathbf{d}$$
$$i = i + 1$$

Repeat until converged

Backward Euler using Optimization

$$\mathbf{v}^* = \arg\min_{\mathbf{v}} \underbrace{\frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t \mathbf{v}\right)}_{E(\mathbf{v})}$$



Gradient Descent

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \left. \frac{\partial E}{\partial \mathbf{v}} \right|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

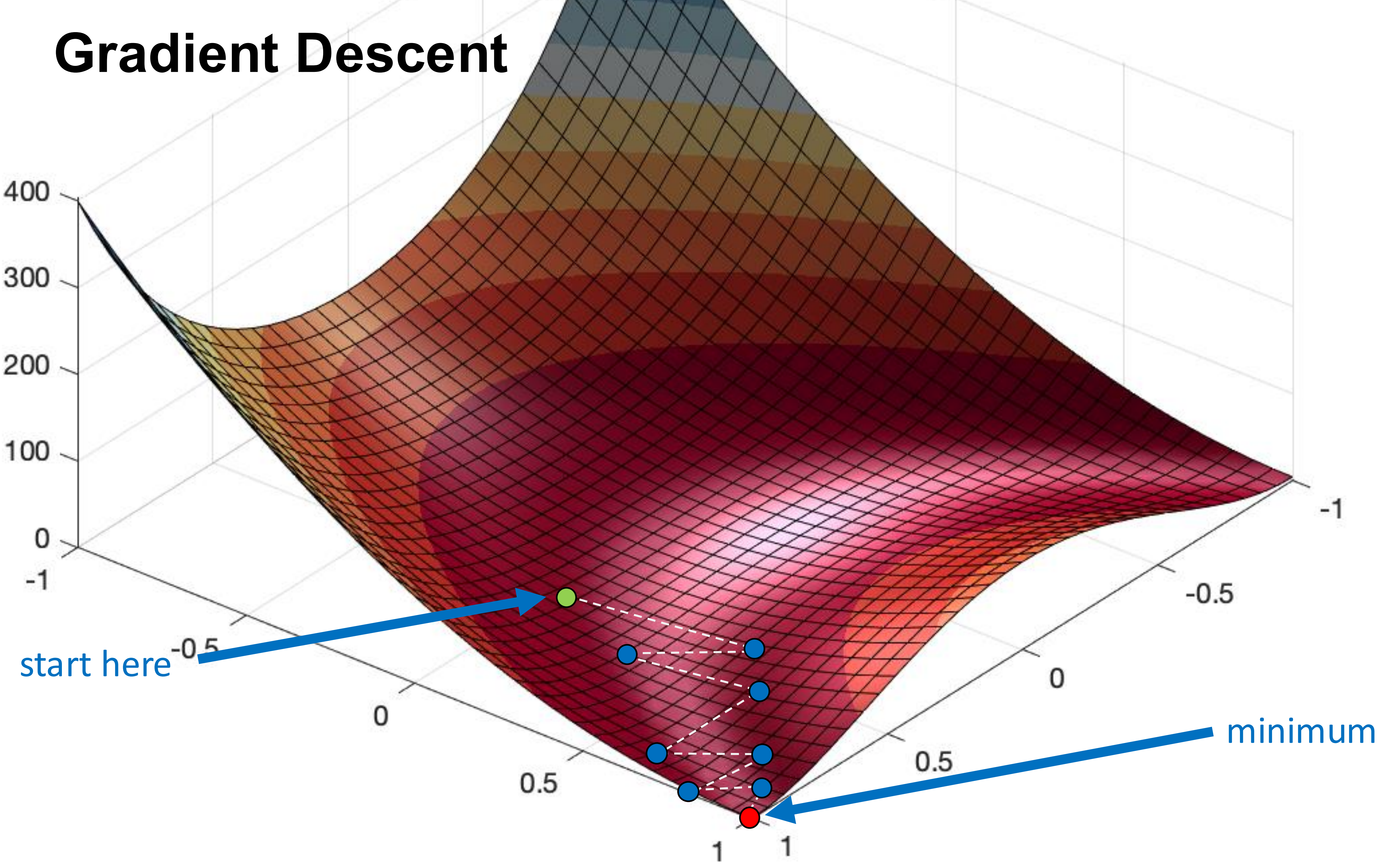
$$\mathbf{d} = -\left. \frac{\partial E}{\partial \mathbf{v}} \right|_{\mathbf{v}^i}$$

Use search direction to update current guess

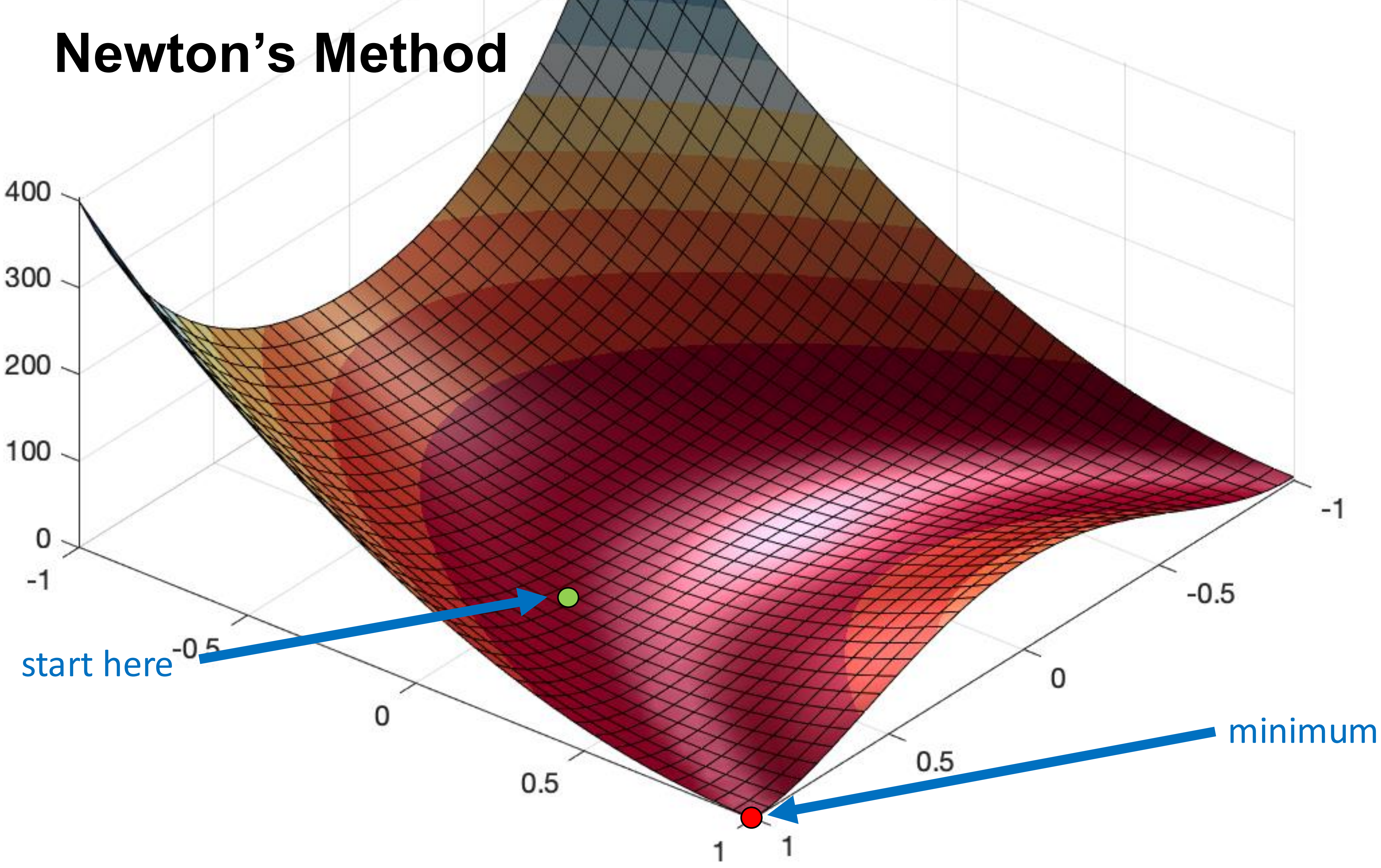
$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Repeat until converged

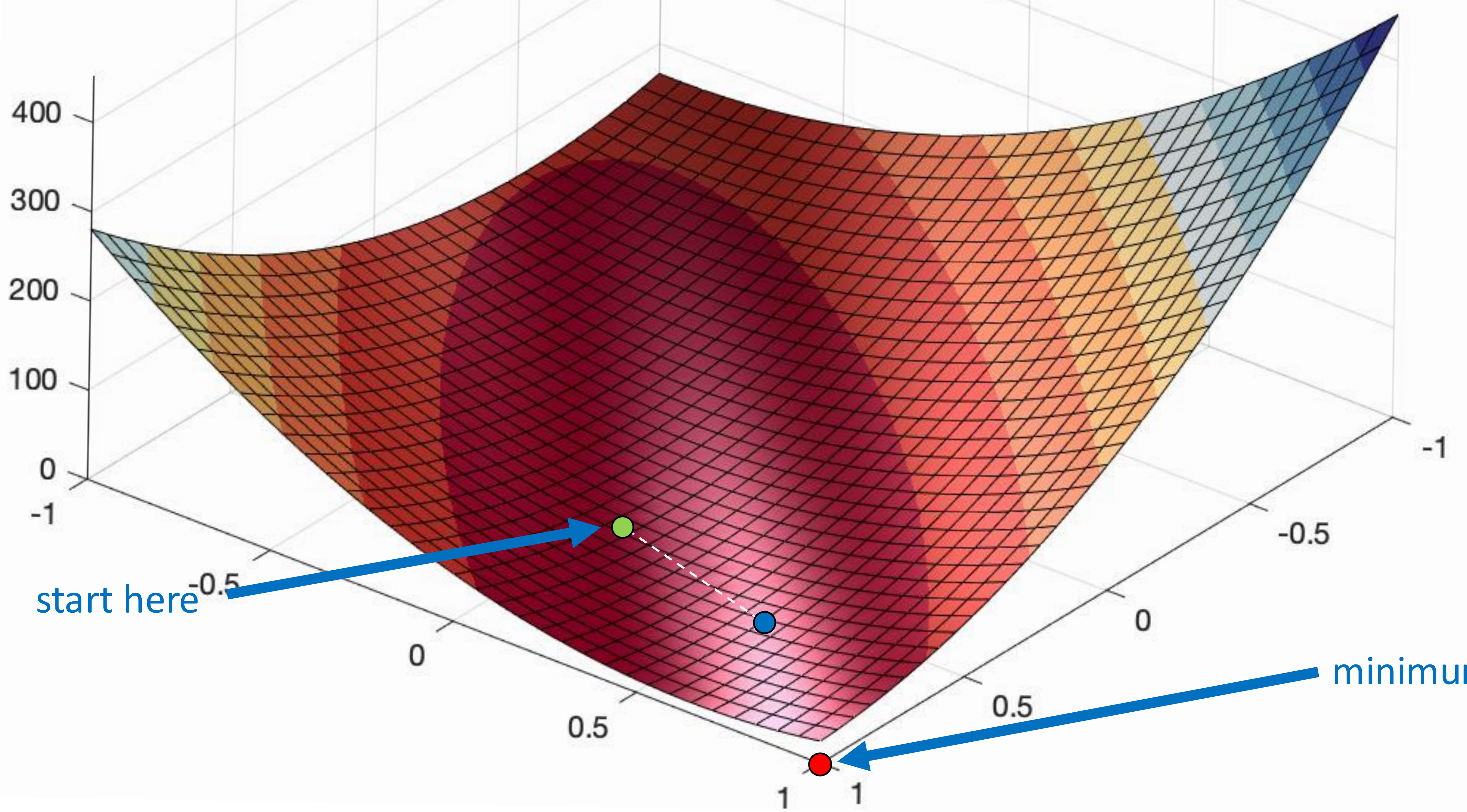
Gradient Descent



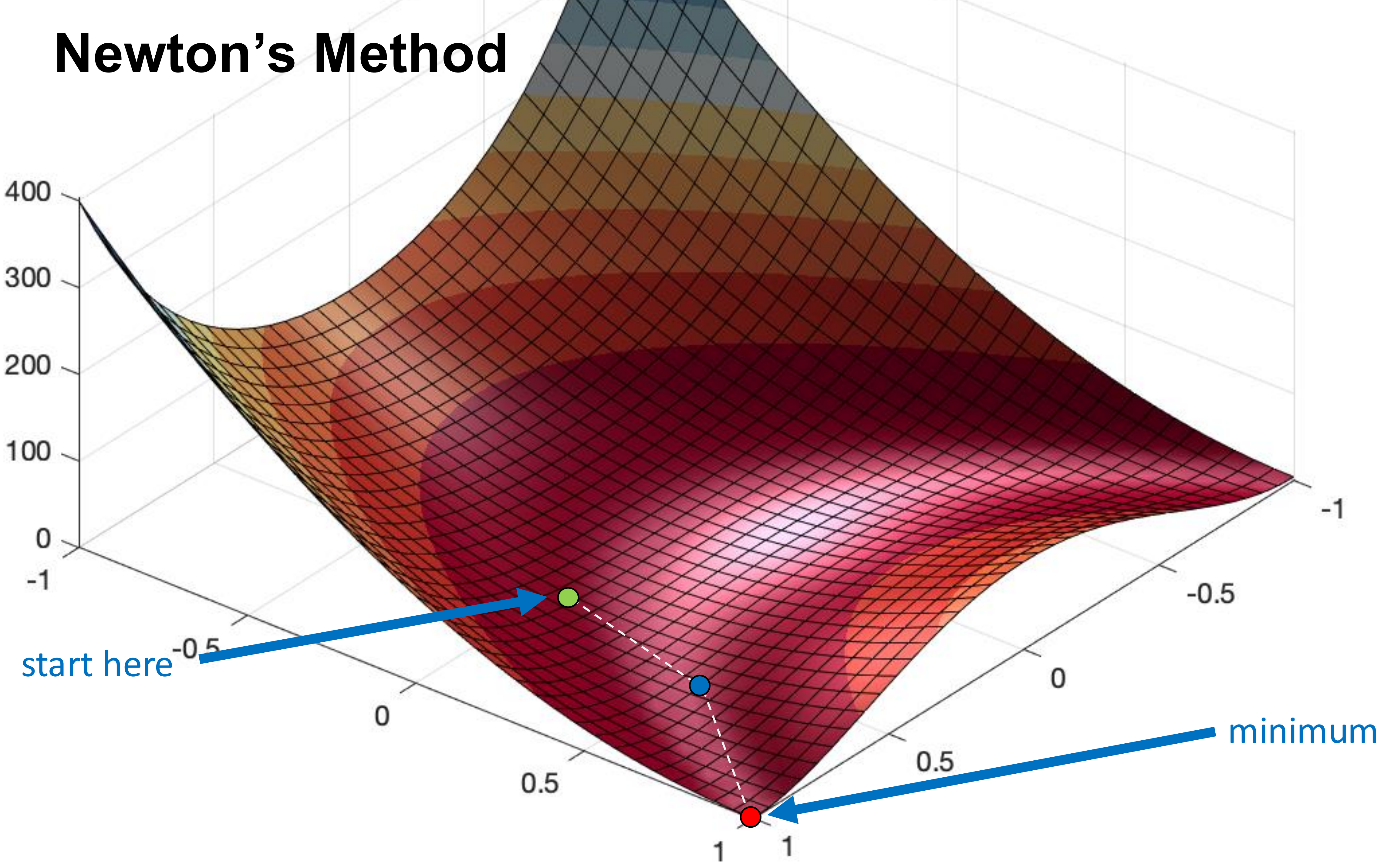
Newton's Method



Newton's Method



Newton's Method



Newton's Method

Repeat until converged

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left|\left|\frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i}\right|\right| < \text{tol}$$

Choose search direction

$$\mathbf{d} =$$ Minimize quadratic approximation

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha\mathbf{d}$$
$$i = i + 1$$

Aside: Quadratic Approximations

$$\underbrace{\mathbf{y}^*}_{\text{optimal solution}} = \arg\min_{\mathbf{y}} \underbrace{E}_{\text{energy/cost function}}(\mathbf{y})$$

generic optimization problem

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} E\left(\mathbf{y}^i + \tilde{\mathbf{d}}\right)$$

Second Order Taylor Approximation

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \frac{1}{2}\tilde{\mathbf{d}}^T \underbrace{\mathrm{H}^i}_{\left.\frac{\partial^2 E}{\partial \mathbf{y}^2}\right|_{\mathbf{y}^i}} \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \underbrace{\mathbf{g}^i}_{\left.\frac{\partial E}{\partial \mathbf{y}}\right|_{\mathbf{y}^i}} + \underbrace{\cancel{E^i}}_{E\left(\mathbf{y}^i\right)}$$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\bigg|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \frac{1}{2}\tilde{\mathbf{d}}^T \mathrm{H}^i \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \mathbf{g}^i$$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Repeat until converged

Aside: Minimizing a Quadratic Function

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \underbrace{\frac{1}{2}\tilde{\mathbf{d}}^T \mathrm{H}^i \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \mathbf{g}^i}$$

Minimum when gradient is equal to zero

$$\mathrm{H}^i \mathbf{d} + \mathbf{g}^i = \mathbf{0}$$

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left|\left|\frac{\partial E}{\partial \mathbf{v}}\bigg|_{\mathbf{v}^i}\right|\right| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Gradient and Hessian

$$E\left(\mathbf{v}\right) = \frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t \mathbf{v}\right)$$

$$\frac{\partial E}{\partial \mathbf{v}} = \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + \Delta t \underbrace{\left.\frac{\partial V}{\partial \mathbf{q}}\right|_{\mathbf{q}^t + \Delta t \mathbf{v}}}_{\text{negative generalized forces}}$$

$$\frac{\partial^2 E}{\partial \mathbf{v}^2} = \mathrm{M} + \Delta t^2 \underbrace{\left.\frac{\partial^2 V}{\partial \mathbf{q}^2}\right|_{\mathbf{q}^t + \Delta t \mathbf{v}}}_{\text{negative stiffness matrix}}$$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left|\left|\frac{\partial E}{\partial \mathbf{v}}\bigg|_{\mathbf{v}^i}\right|\right| < \text{tol}$$

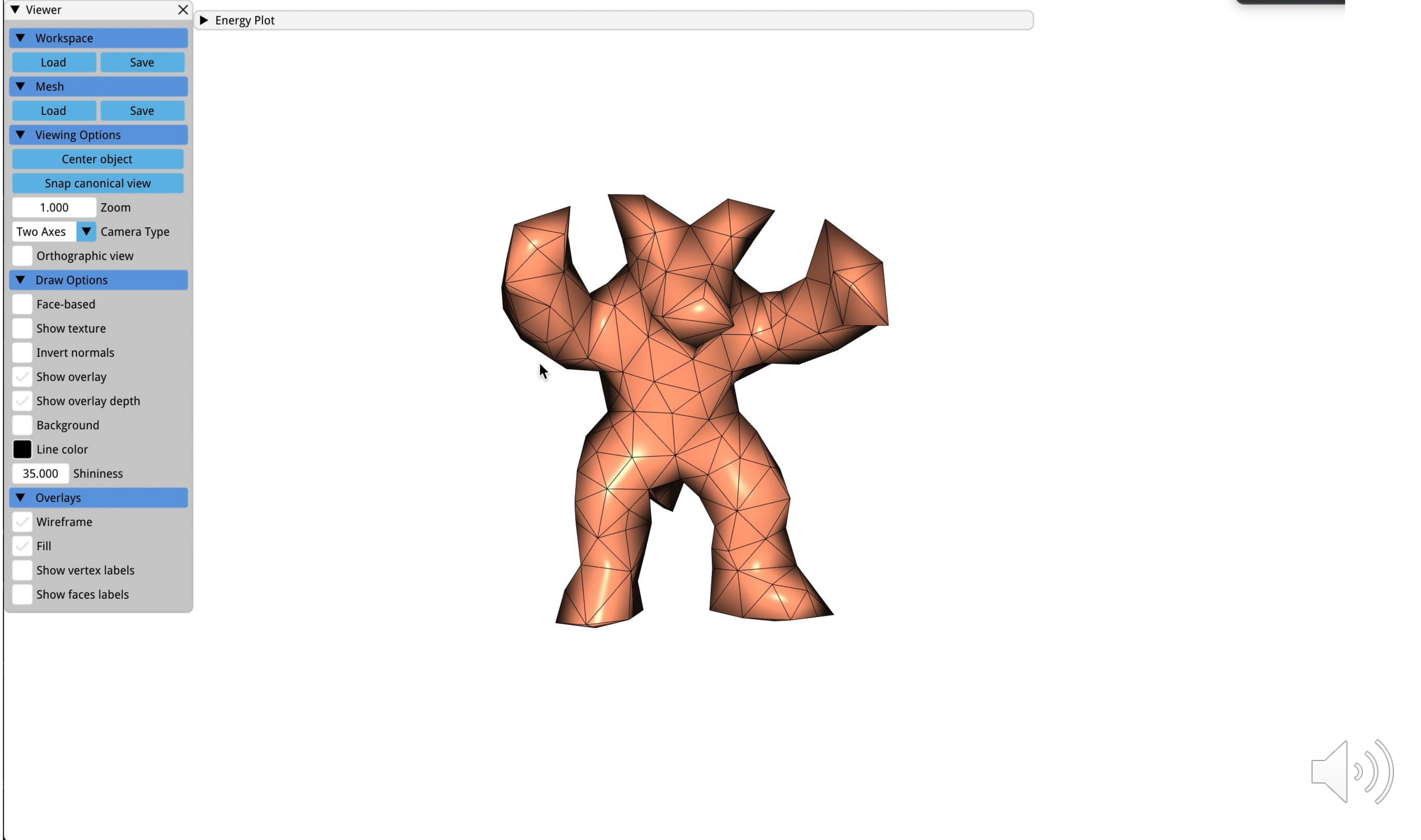
Choose search direction

$$\mathrm{H}^i\mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha\mathbf{d}$$
$$i = i + 1$$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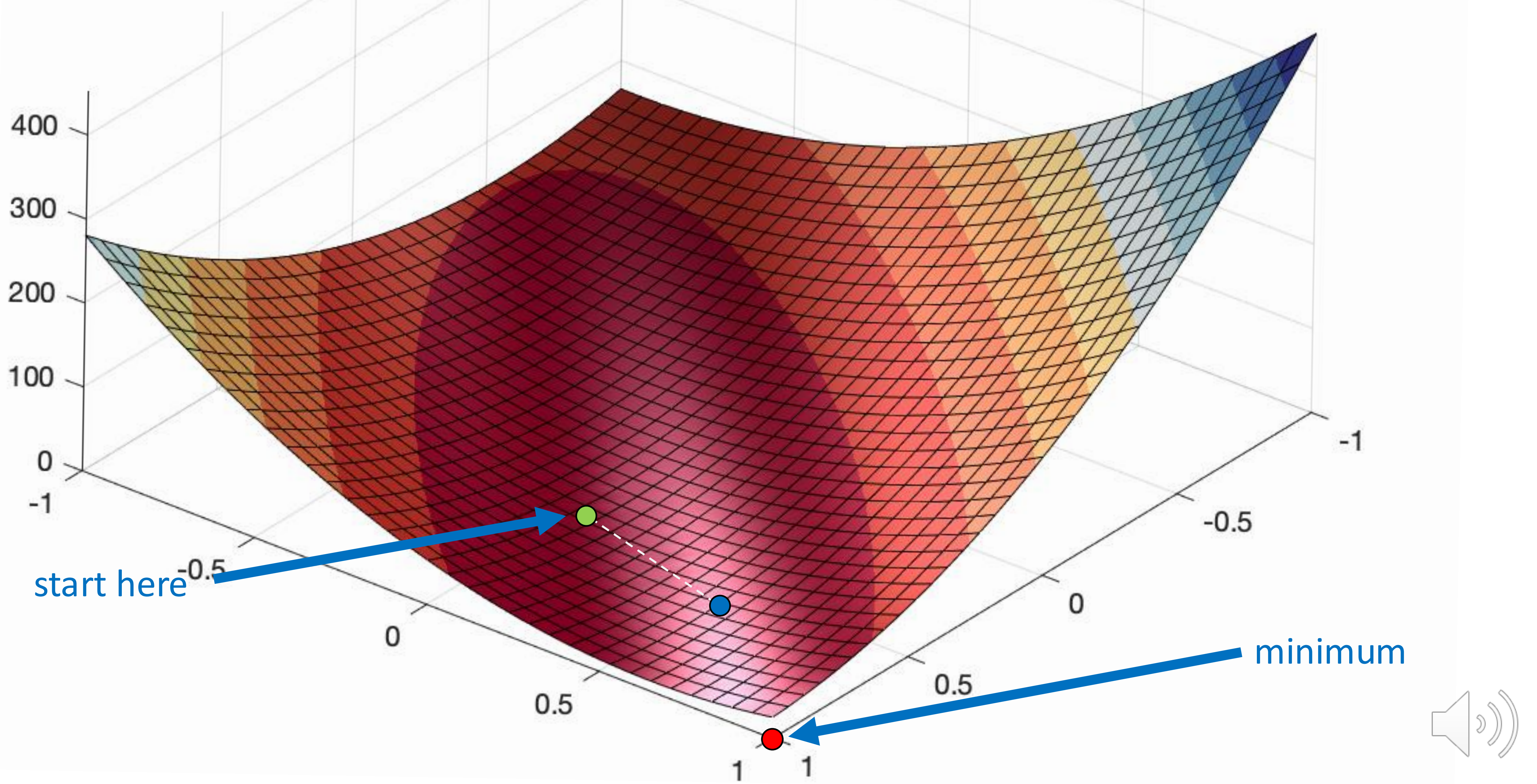
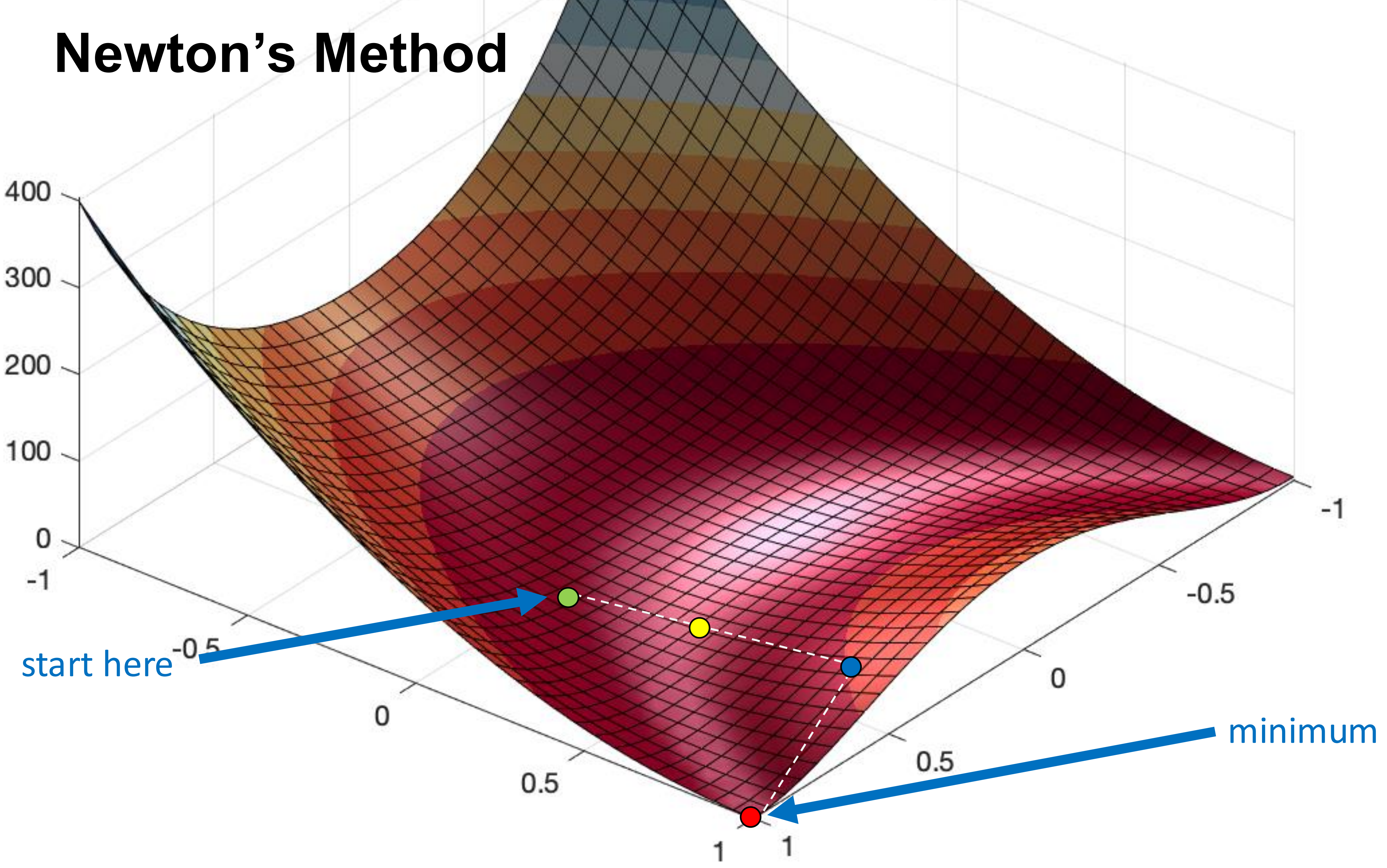
Newton's Method



Newton’s Method



Newton's Method

Repeat until converged

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \left. \frac{\partial E}{\partial \mathbf{v}} \right|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \left. \frac{\partial E}{\partial \mathbf{v}} \right|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Choose α using line search

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Line Search

$$\mathbf{y}^* = \arg\min_{\mathbf{y}} E(\mathbf{y})$$

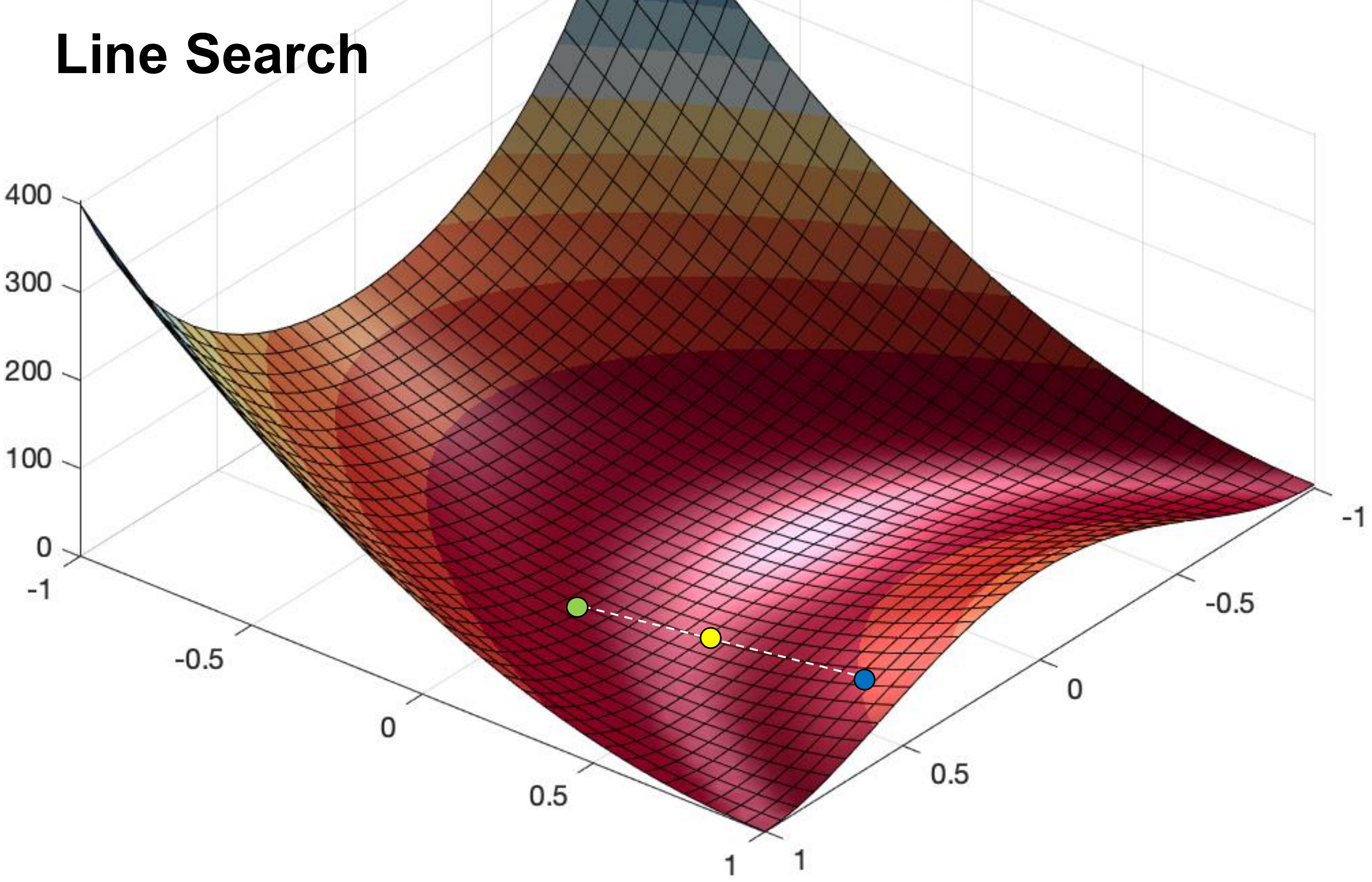
Once we compute $\mathbf{d}$...

One-dimensional optimization

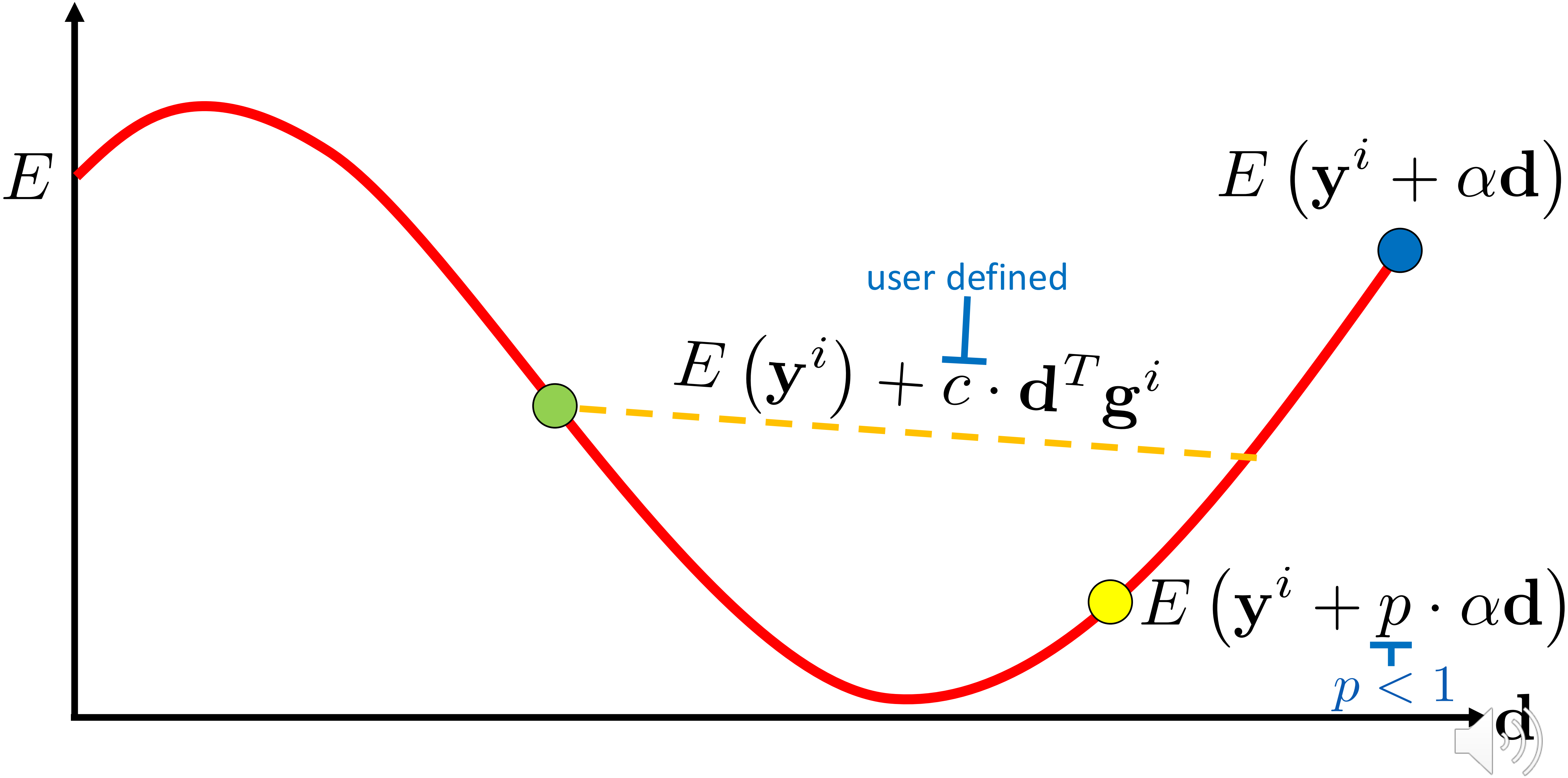
$$\alpha^* = \arg\min_{\alpha} E\left(\mathbf{y}^i + \alpha \mathbf{d}\right)$$

Solve by 1D Search

Line Search



Line Search


E
$E\left(\mathbf{y}^i + \alpha\mathbf{d}\right)$
user defined
$E\left(\mathbf{y}^i\right) + c \cdot \mathbf{d}^T\mathbf{g}^i$
$E\left(\mathbf{y}^i + p \cdot \alpha\mathbf{d}\right)$
$p < 1$
$\mathbf{d}$

Backtracking Line Search

Choose an initial guess

$$\alpha = \alpha_{max}$$

Repeat

Stop if sufficient decrease achieved or you get stuck

$$E\left(\mathbf{v}^i + \alpha \mathbf{d}\right) <= E\left(\mathbf{v}^i\right) + c \cdot \mathbf{d}^T \mathbf{g}^i \text{ or } \alpha < \text{tol}$$

Reduce α

$$\alpha = p \cdot \alpha$$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