

# CSC417 Physics-Based Animation

# Last Week: Affine Body Simulation with Contact



**Questions from Previous Lecture ?**



# We Solve This Every Time Step

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

Gradient of what equals this ? Let's guess, then check

# Newton's Method

Choose an initial guess

$$i = 0$$

$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{H}^i \mathbf{d} = -\mathbf{g}^i \quad \text{Solve linear system to get } \mathbf{d}$$

Choose  $\alpha$  using line search

Use search direction to update current guess

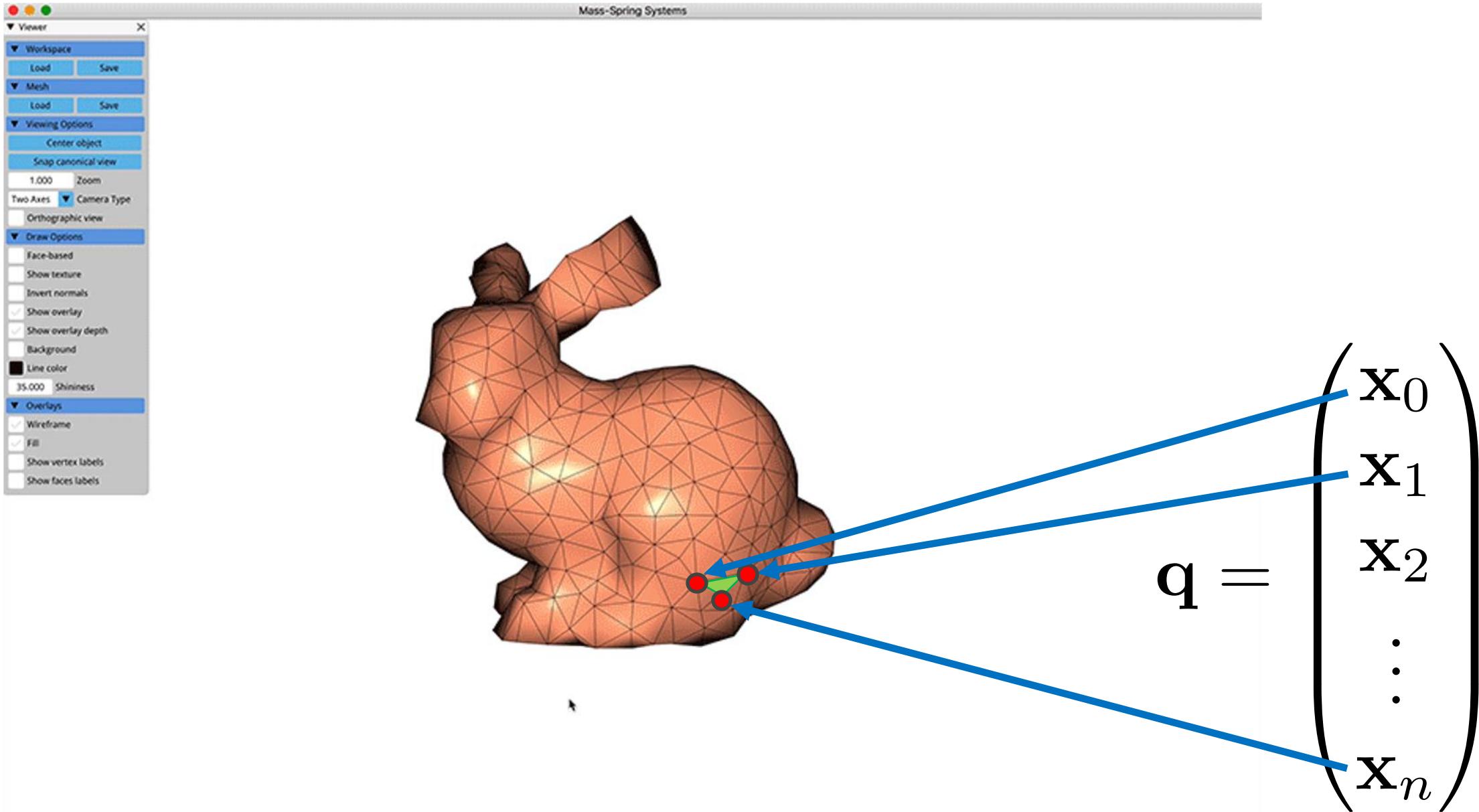
$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

$$i = i + 1$$

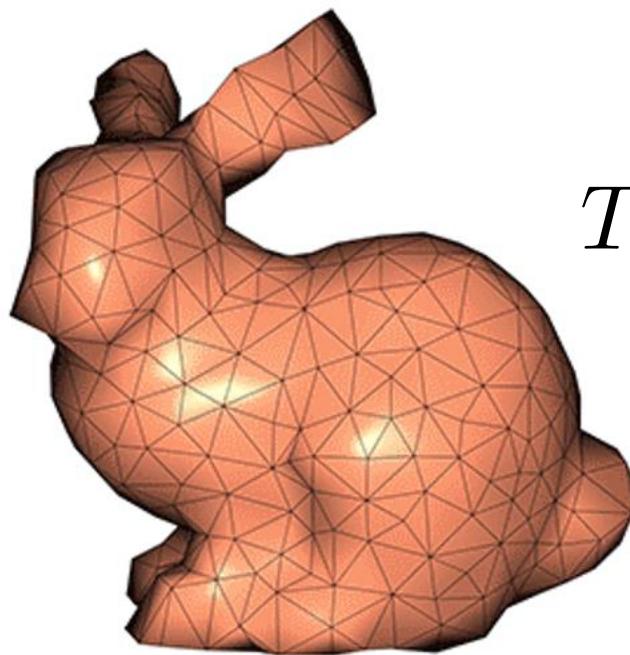
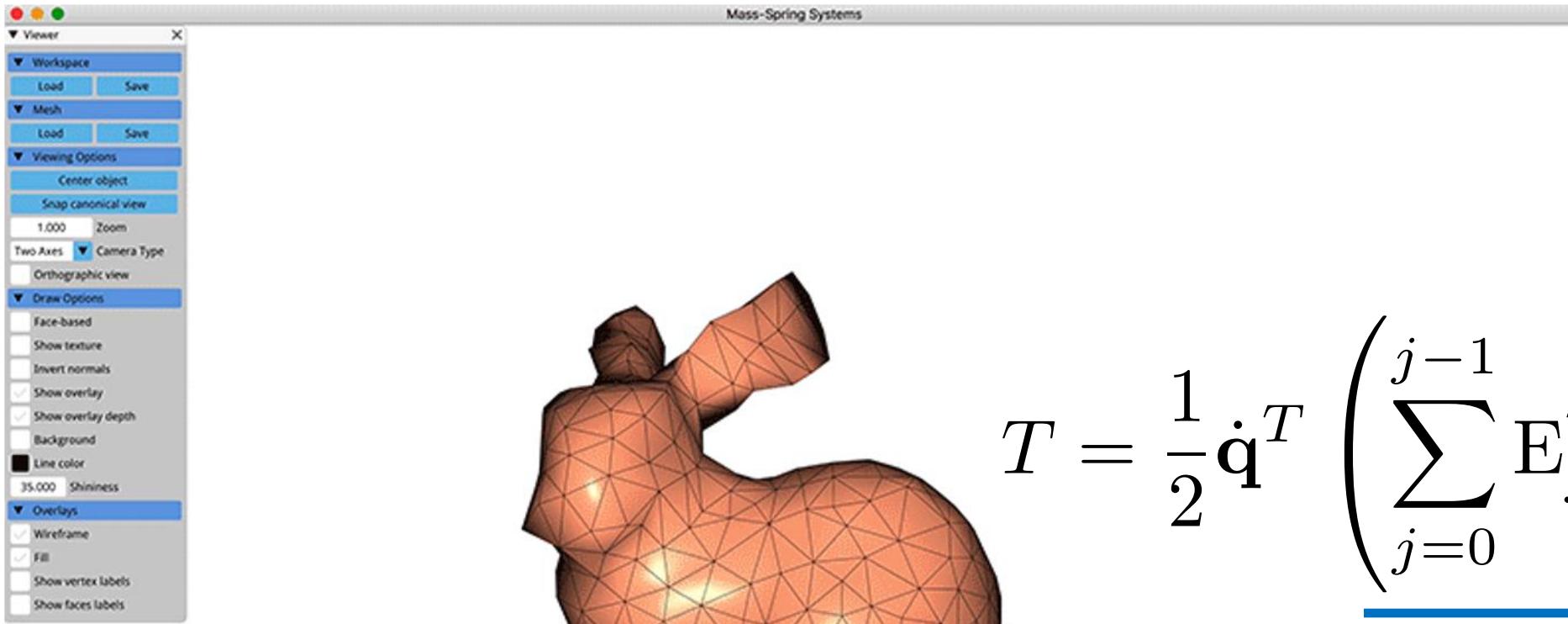
Repeat until converged



# Spatial Discretization -- Finite Elements



# Kinetic Energy for a Bunny

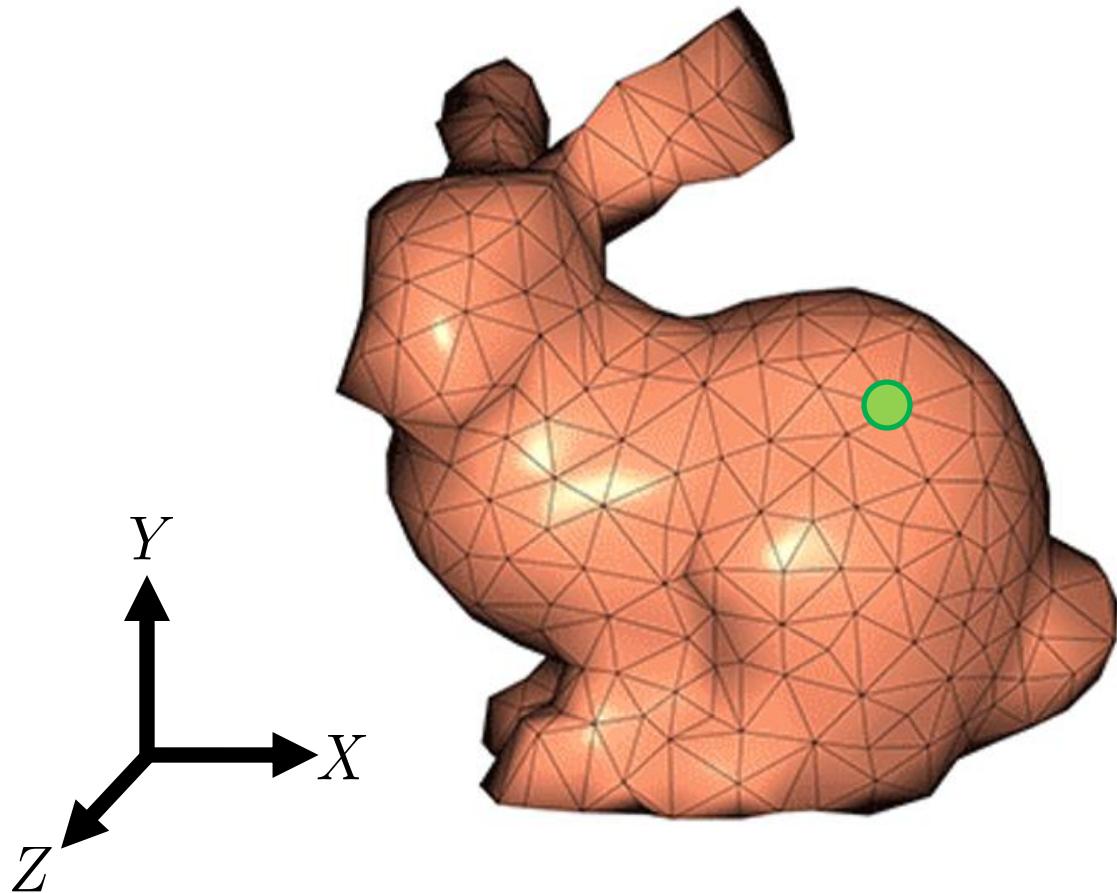


$$T = \frac{1}{2} \dot{\mathbf{q}}^T \left( \sum_{j=0}^{j-1} \mathbf{E}_j^T \mathbf{M}_j \mathbf{E}_j \right) \dot{\mathbf{q}}$$

$\overline{\mathbf{M}}$

Assemble  $\mathbf{M}$  by summing over all tetrahedra

# Affine Body Dynamics – Reduced-Order Elasticity

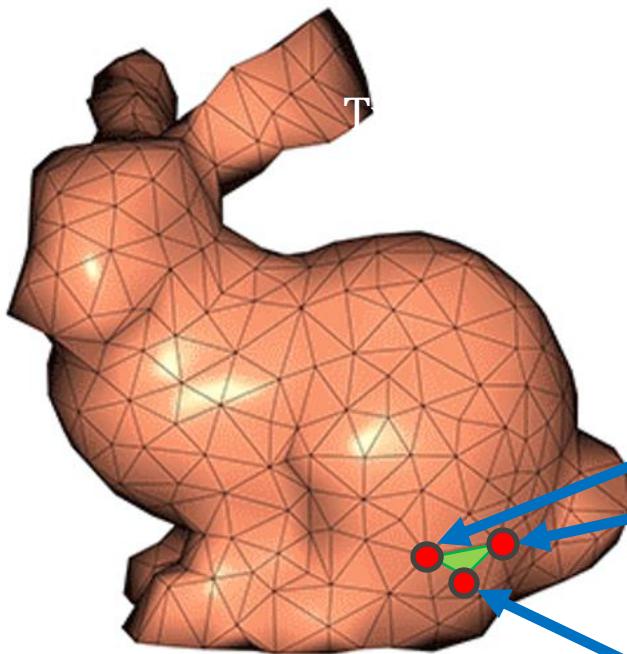
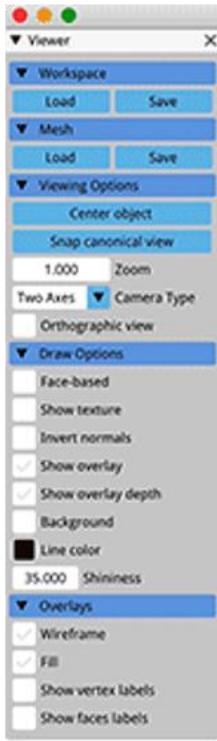


Reference (Undeformed) Space

$$\mathbf{x}(\mathbf{X}, t) = \mathbf{J}(\mathbf{X})\mathbf{q}(t)$$



# Spatial Discretization -- Finite Elements



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} = \mathbf{U}\mathbf{r}$$

Reduced  
Space

$$|\mathbf{r}| \ll |\mathbf{q}|$$

# Pros and Cons of Model Reduction

## Pros

- Runtime depends on smaller number of degrees of freedom
- Many calculations become independent of mesh resolution

## Cons

- Loss of expressivity (reduced model can't represent all the motions the full space model can)

# How Do We Build a Reduced-Order Model That's Compact and Expressive ?

# How Do We Build a Reduced-Order Model That's Compact and Expressive ?

1. Modal Analysis
2. Data-Driven Deformation Modes

# Modal Analysis Starts Here

$$M \ddot{q} = - \frac{\partial V}{\partial q}$$

# Modal Analysis Starts Here

$$M\ddot{q} + \frac{\partial V}{\partial q} = 0$$

# Linearize Around Rest Shape

$$M(\ddot{q}(0) + \ddot{u}(t)) + \frac{\partial V(q(0) + u(t))}{\partial q} = 0$$

# Linearize Around Rest Shape

$$M \ddot{u}(t) + \frac{\partial V^2}{\partial q^2} u(t) = 0$$

$\top$   
 $H$

# Linearize Around Rest Shape

$$M\ddot{u}(t) + Hu(t) = 0$$

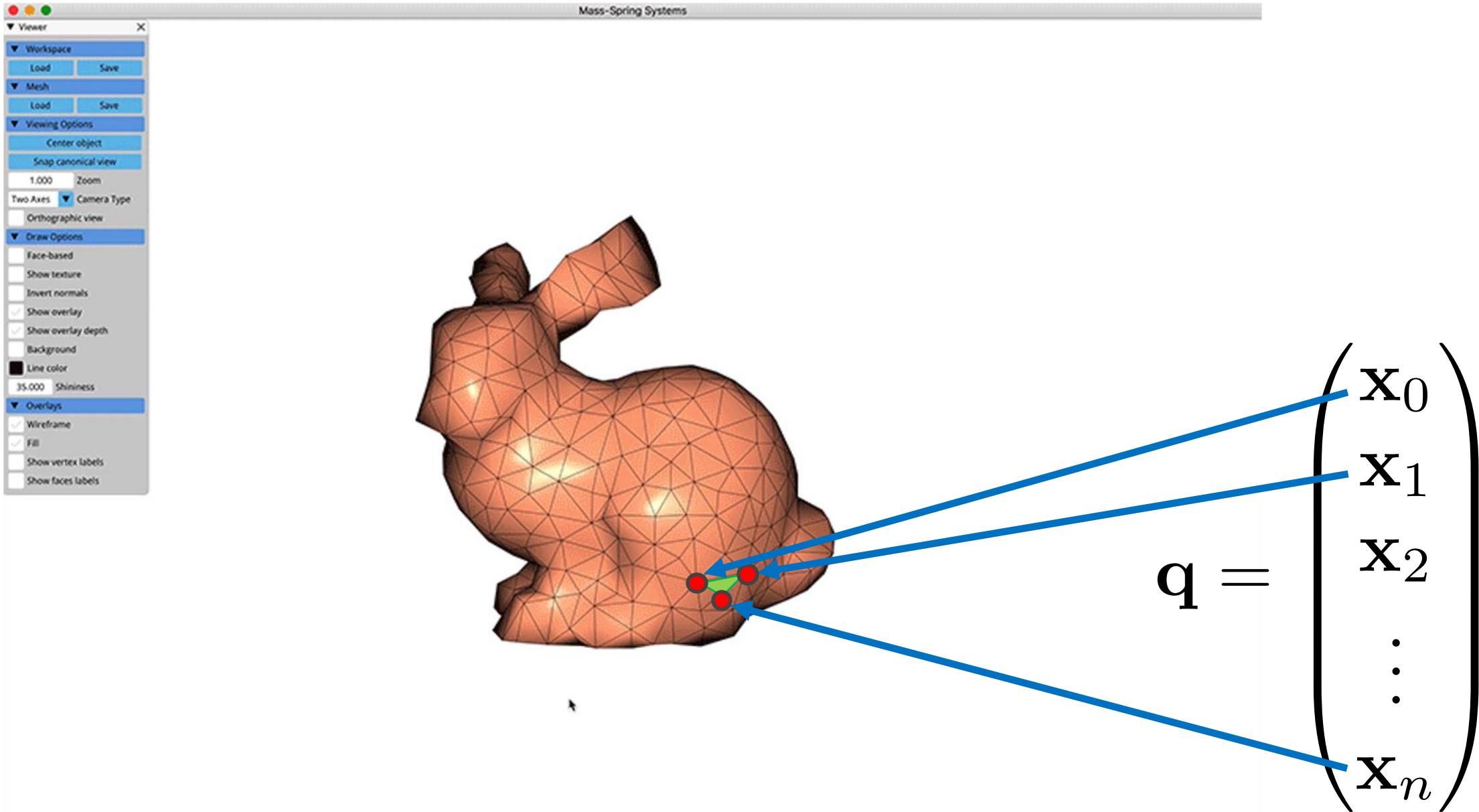
**A linear, homogenous ODE!**

**How can we solve it ?**

# Assume we know the solution !

$$\mathbf{u} = \underset{\substack{\mathcal{R}^{3n} \text{ vector} \\ \text{Complex constant}}}{\underset{\top}{v} e^{\lambda t}}$$
$$\ddot{\mathbf{u}} = \lambda^2 v e^{\lambda t}$$

# Spatial Discretization -- Finite Elements



# Assume we know the solution !

$$\mathbf{u} = \underset{\substack{\mathcal{R}^{3n} \text{ vector} \\ \text{Complex constant}}}{\underset{\top}{v} e^{\lambda t}}$$
$$\ddot{\mathbf{u}} = \lambda^2 v e^{\lambda t}$$

Substitute our “guess” into the equation

$$\lambda^2 M u e^{\lambda t} - K u e^{\lambda t} = 0$$

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$$\lambda^2 M u e^{\cancel{\lambda} t} - K u e^{\cancel{\lambda} t} = 0$$

# Generalized Eigenvector Problem

$$\underline{\lambda^2 M \mathbf{u}} - \underline{K \mathbf{u}} = 0$$

# Modal Analysis

So a good reduced space is a vector of the  $k$  lowest order  
Eigenvectors

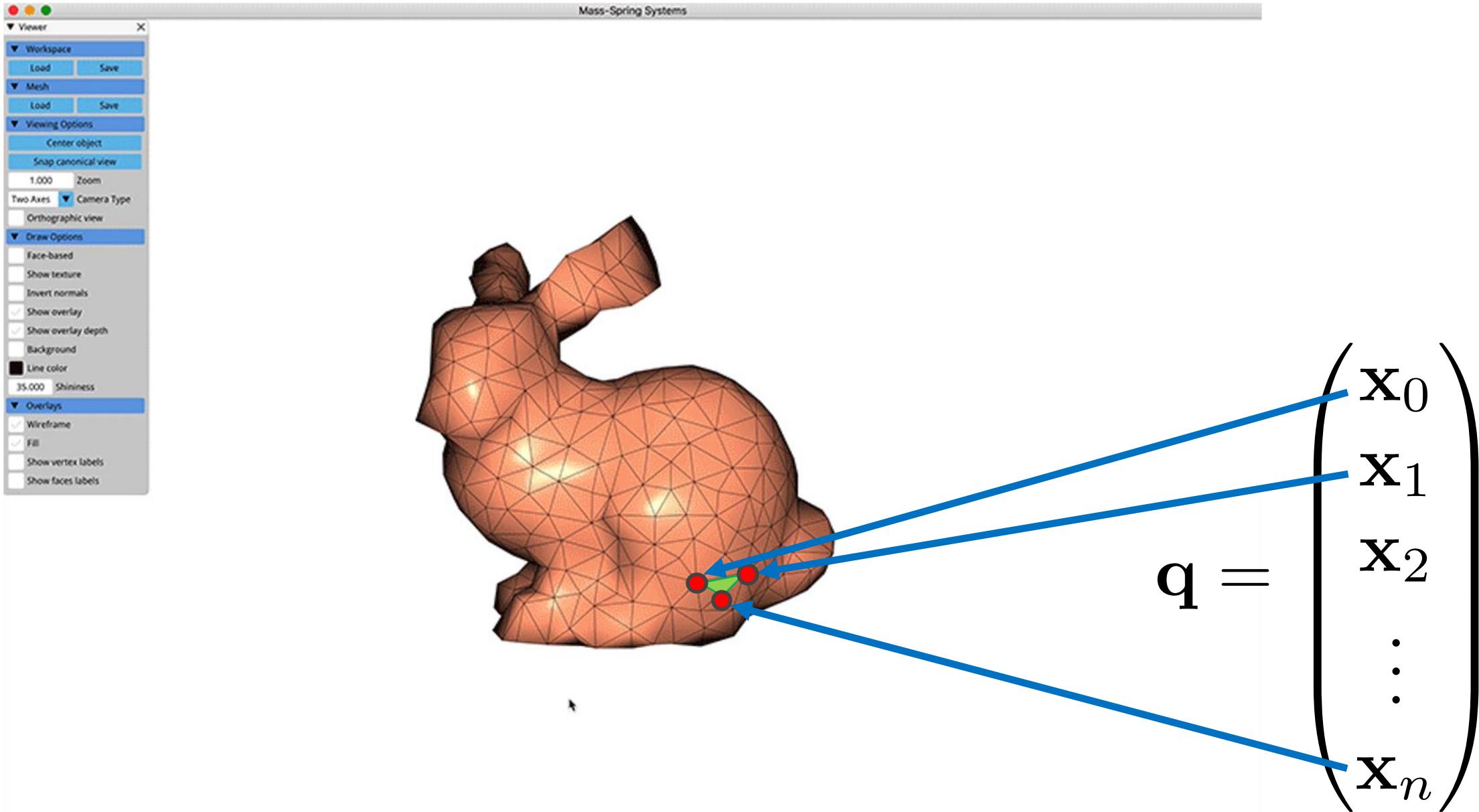
$$U \in \mathcal{R}^{n \times k}$$



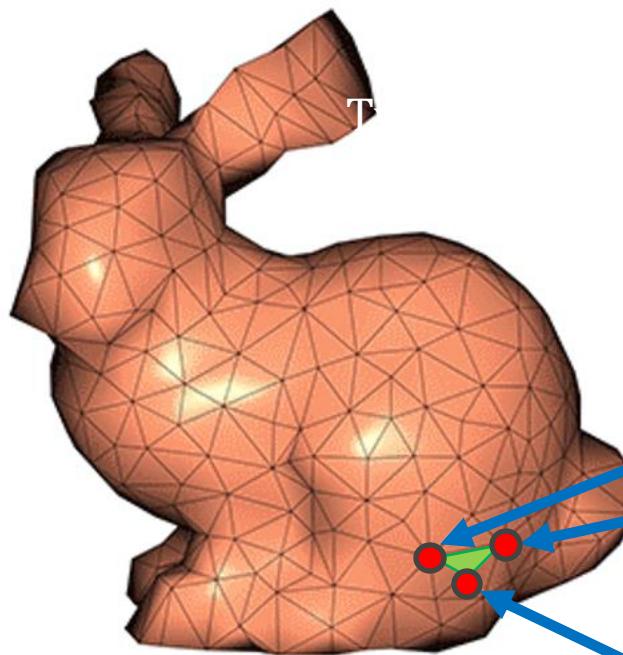
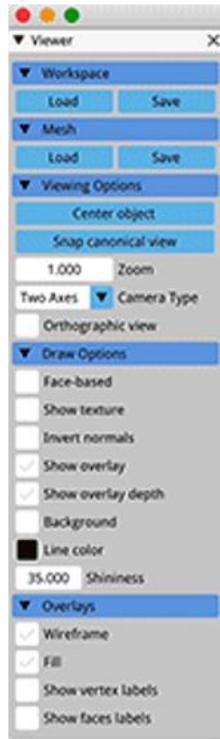
Reduced Space



# Spatial Discretization -- Finite Elements



# Spatial Discretization -- Finite Elements



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} : q(0) + U_r$$

Generalized  
Coordinates

## An Aside: Variational Modal Analysis

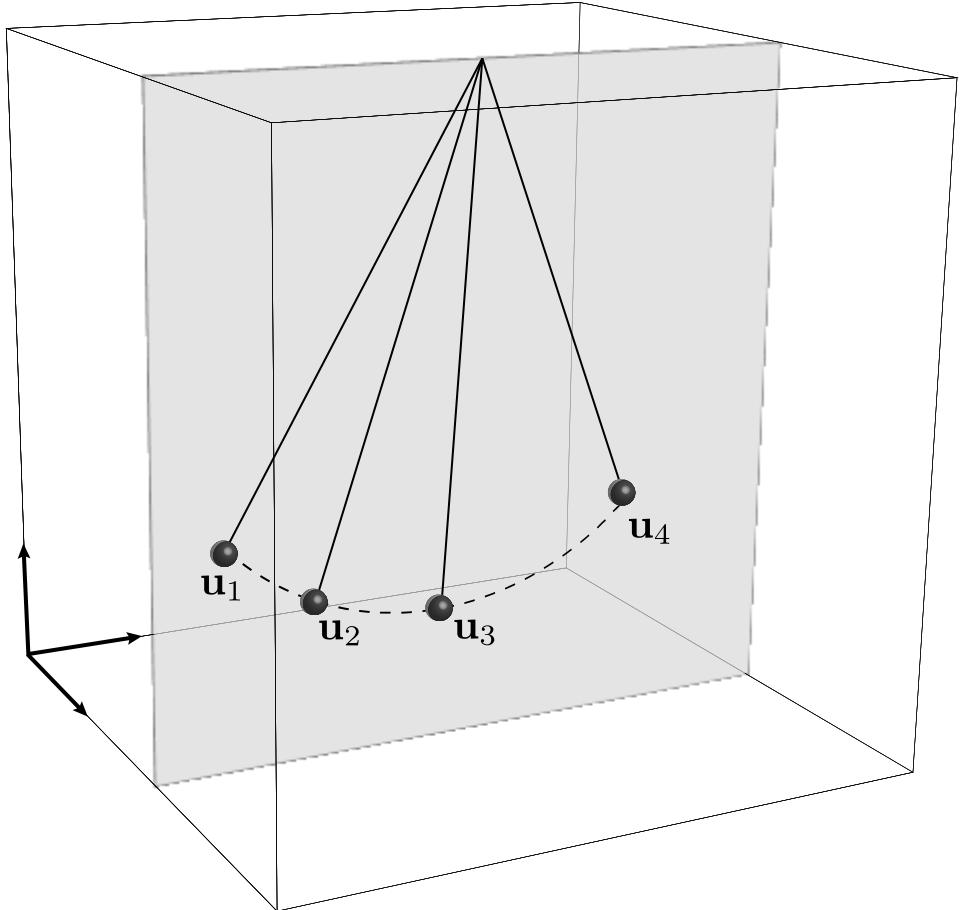
$$U^* = \arg \min_U \text{tr}(U^T H U)$$

$$\text{s.t. } U^T M U = I$$

# How Do We Build a Reduced-Order Model That's Compact and Expressive ?

1. ~~Modal Analysis~~
2. Data-Driven Deformation Modes

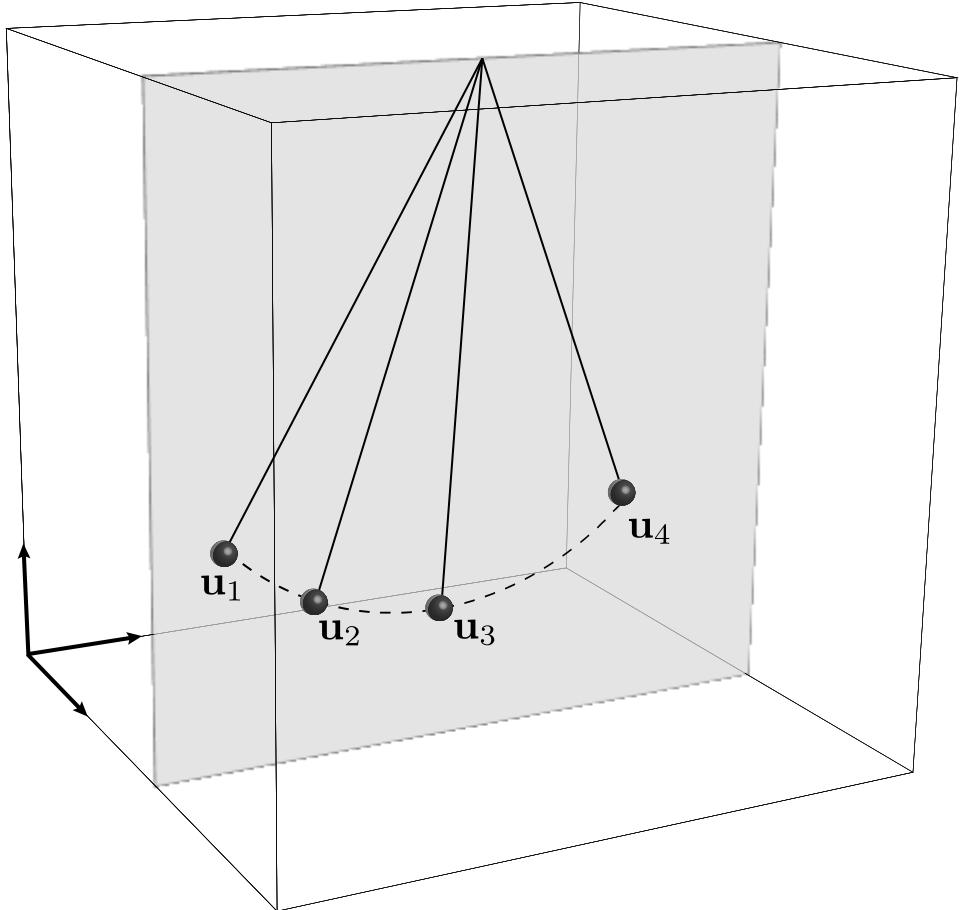
# Data-Driven Deformation Modes



Collect Snapshots

$$Q = [q_0, q_1, q_2 \dots]$$

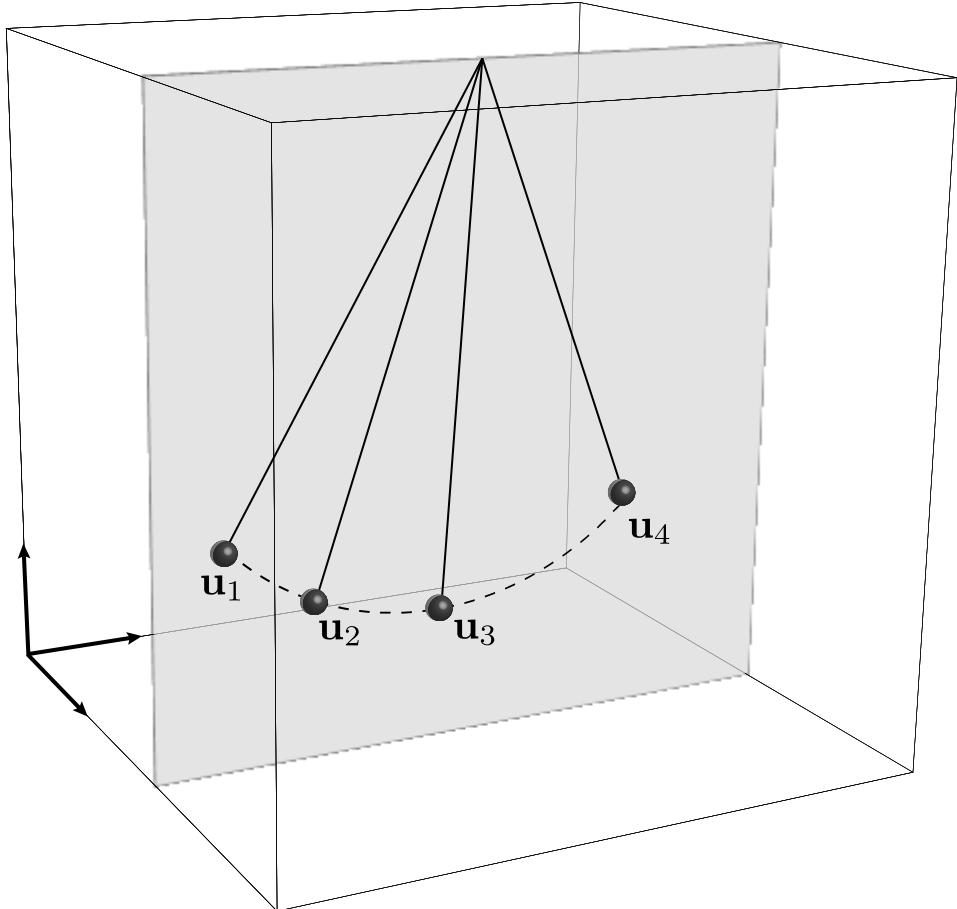
# Data-Driven Deformation Modes



Collect Snapshots

$$Q = [q_0, q_1, q_2 \dots]$$

# Probabilistic Reconstruction



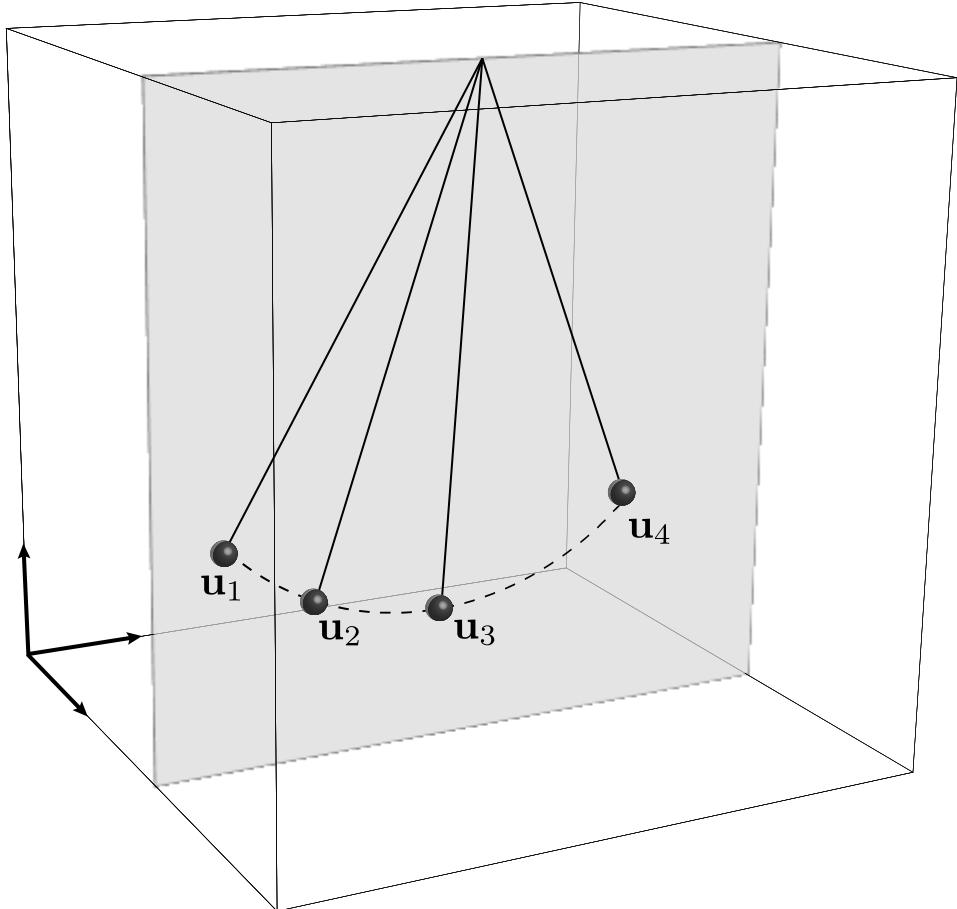
Given a **bunch of states**, assume their **displacements** from the **mean state** can be **modelled linearly**

$$s.t \quad \mathbf{q}_i = \boldsymbol{\mu}(0) + U\mathbf{r}_i$$

|

Mean pose from data

# Probabilistic Reconstruction



Given a **bunch of states**, assume their **displacements** from the **mean state** can be **modelled linearly**

$$= \frac{s \cdot t (\mathbf{q}_i - \boldsymbol{\mu}(0))}{U \mathbf{r}_i}$$

Displacement  $\mathbf{u}_i$

# Probabilistic Reconstruction

Find  $U$  that minimizes expected reconstruction error

$$U^* = \arg \min \sum_i \|u_i - Ur_i\|_2^2 \frac{p_{data}(u_i)}{\text{Data distribution (unknown)}}$$

# Probabilistic Reconstruction

Find  $U$  that minimizes expected reconstruction error

$$U^* = \arg \min \sum_i \|u_i - Ur_i\|_2^2 \frac{p_{data}(u_i)}{\text{Data distribution (unknown)}}$$

$$s.t \ U^T U = I$$

Prevent collapse of reduced space

# Some useful identities

Projection onto an orthogonal subspace

$$\mathbf{r}_i = U^T \mathbf{u}_i$$

Vector norms as traces

$$\|v - Av\|_2^2 = \text{trace}(Avv^T)$$

Linearity of Expectation

$$\sum_i \text{trace}(AB_i)p_{data} = \text{trace}(A \sum_i B_i p_{data})$$

# Probabilistic Reconstruction

Ok let's apply some identities

$$U^* = \arg \min \sum_i \| \mathbf{u}_i - U \mathbf{r}_i \|_2^2 \frac{p_{data}(\mathbf{u}_i)}{\text{Data distribution (unknown)}}$$

$$s.t \quad U^T U = I$$

Prevent collapse of reduced space

# Probabilistic Reconstruction

Step 1: Get rid of r

$$U^* = \arg \min \sum_i \left\| \mathbf{u}_i - UU^T \mathbf{u}_i \right\|_2^2 p_{data}(\mathbf{u}_i)$$

$$s.t \quad U^T U = I$$

Prevent collapse of reduced space

# Probabilistic Reconstruction

Step 2: Get rid of norm

$$U^*$$

$$= \arg \min \sum_i \text{trace}((I - UU^T)\mathbf{u}_i\mathbf{u}_i^T)p_{data}(\mathbf{u}_i)$$

$$s.t \quad U^T U = I$$

Prevent collapse of reduced space

# Probabilistic Reconstruction

Step 3: Apply linearity of expectation

$$U^* = \arg \min \text{trace}((I - UU^T) \sum_i \mathbf{u}_i \mathbf{u}_i^T p_{data}(\mathbf{u}_i))$$



$$s.t \ U^T U = I$$

This just the covariance  $\Sigma$

# Probabilistic Reconstruction

Step 4: Get rid of terms independent of U

$$\begin{aligned} U^* = \arg \min -\text{trace}(UU^T\Sigma) \\ s.t \; U^T U = I \end{aligned}$$

# Probabilistic Reconstruction

Step 5: Rearrange and get rid of minus sign

$$\begin{aligned} U^* &= \arg \max \text{trace}(U^T \Sigma U) \\ s.t \quad &U^T U = I \end{aligned}$$

## An Aside: Variational Modal Analysis

$$U^* = \arg \min_U \text{tr}(U^T H U)$$

$$\text{s.t. } U^T M U = I$$

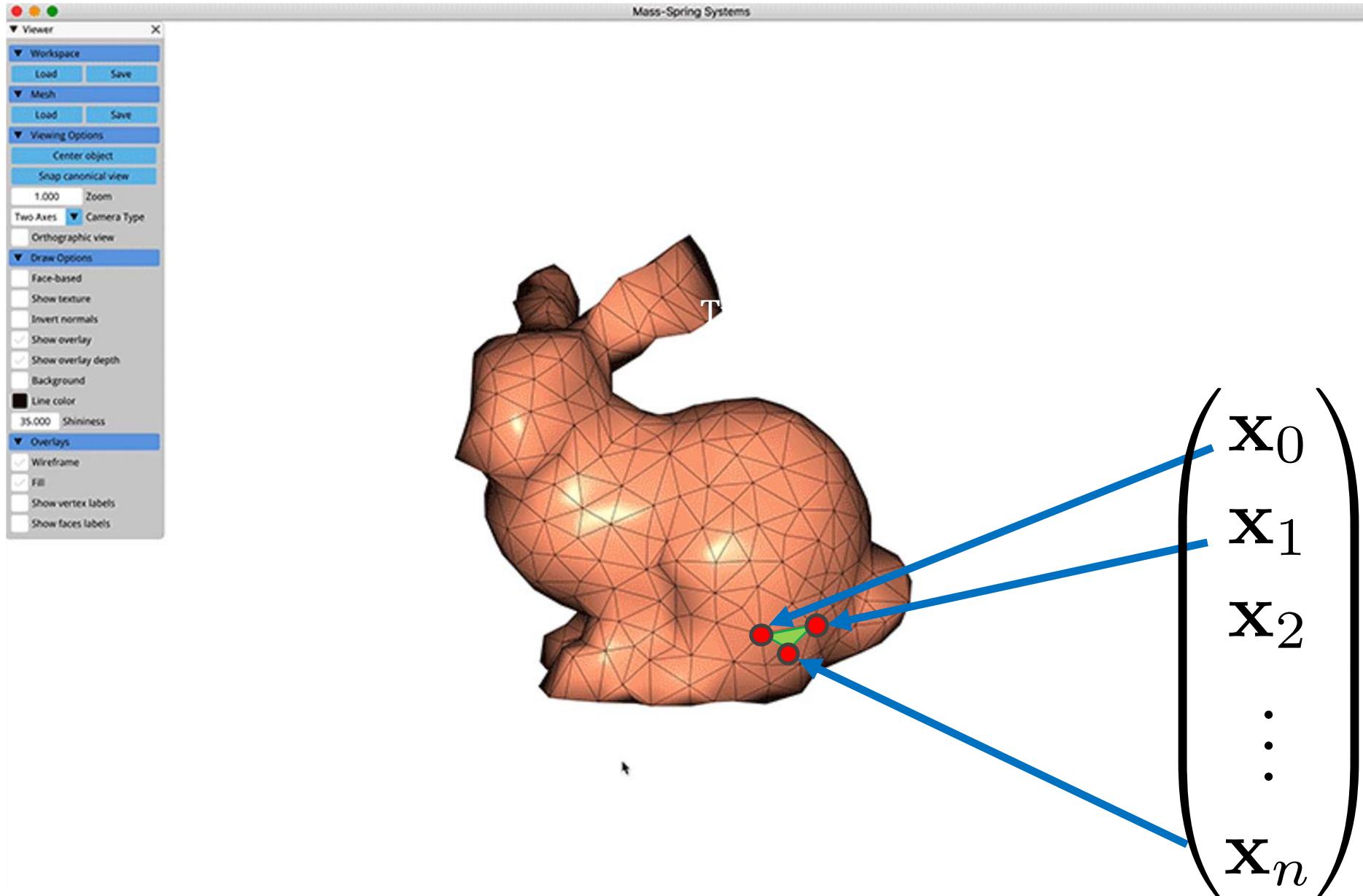
# Probabilistic Reconstruction

$$\begin{aligned} U^* &= \arg \max \text{trace}(U^T \Sigma U) \\ s.t \quad &U^T U = I \end{aligned}$$

# Principal Component Analysis

$$\begin{aligned} U^* &= \arg \max \text{trace}(U^T \Sigma U) \\ s.t \quad &U^T U = I \end{aligned}$$

# Spatial Discretization -- Finite Elements



# How do we use this reduced space ?

Answer, direct substitution

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

Gradient of what equals this ? Let's guess, then check

# How do we use this reduced space ?

Answer, direct substitution

$$E(r^{i+1}) = \frac{1}{2} (r^{i+1} - \tilde{r}^i)^T U^T M U (r^{i+1} - \tilde{r}^i) + h^2 V(q(0) + U^t r^{i+1})$$



**Reduced Mass Matrix**

**What are the reduced gradient and Hessian ?**

# Newton's Method

Choose an initial guess

$$i = 0$$

$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{H}^i \mathbf{d} = -\mathbf{g}^i$$

$\mathbf{H}$  and  $\mathbf{g}$  are now reduced so smaller and faster to solve

Choose  $\alpha$  using line search

Use search direction to update current guess

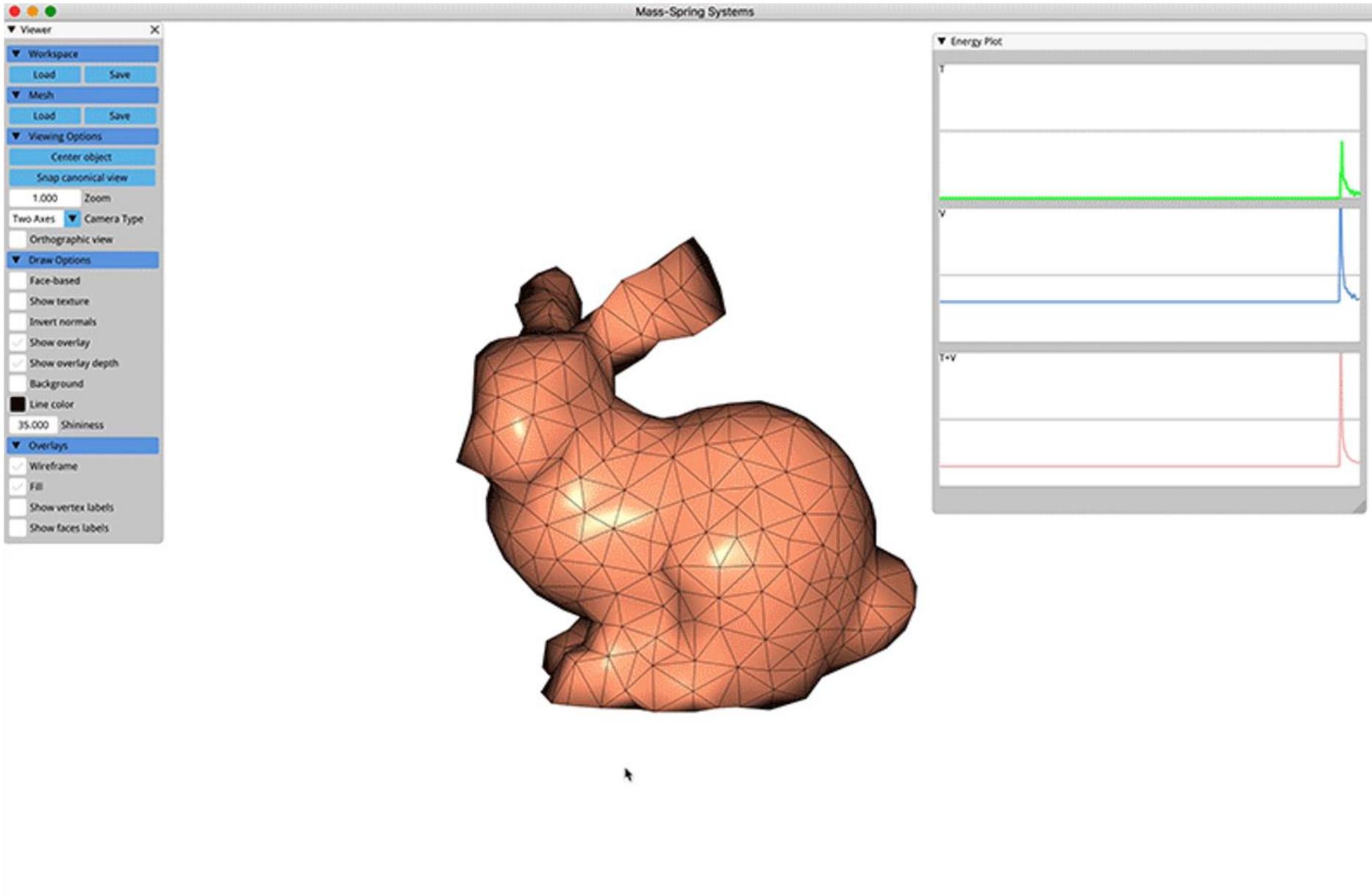
$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

$$i = i + 1$$

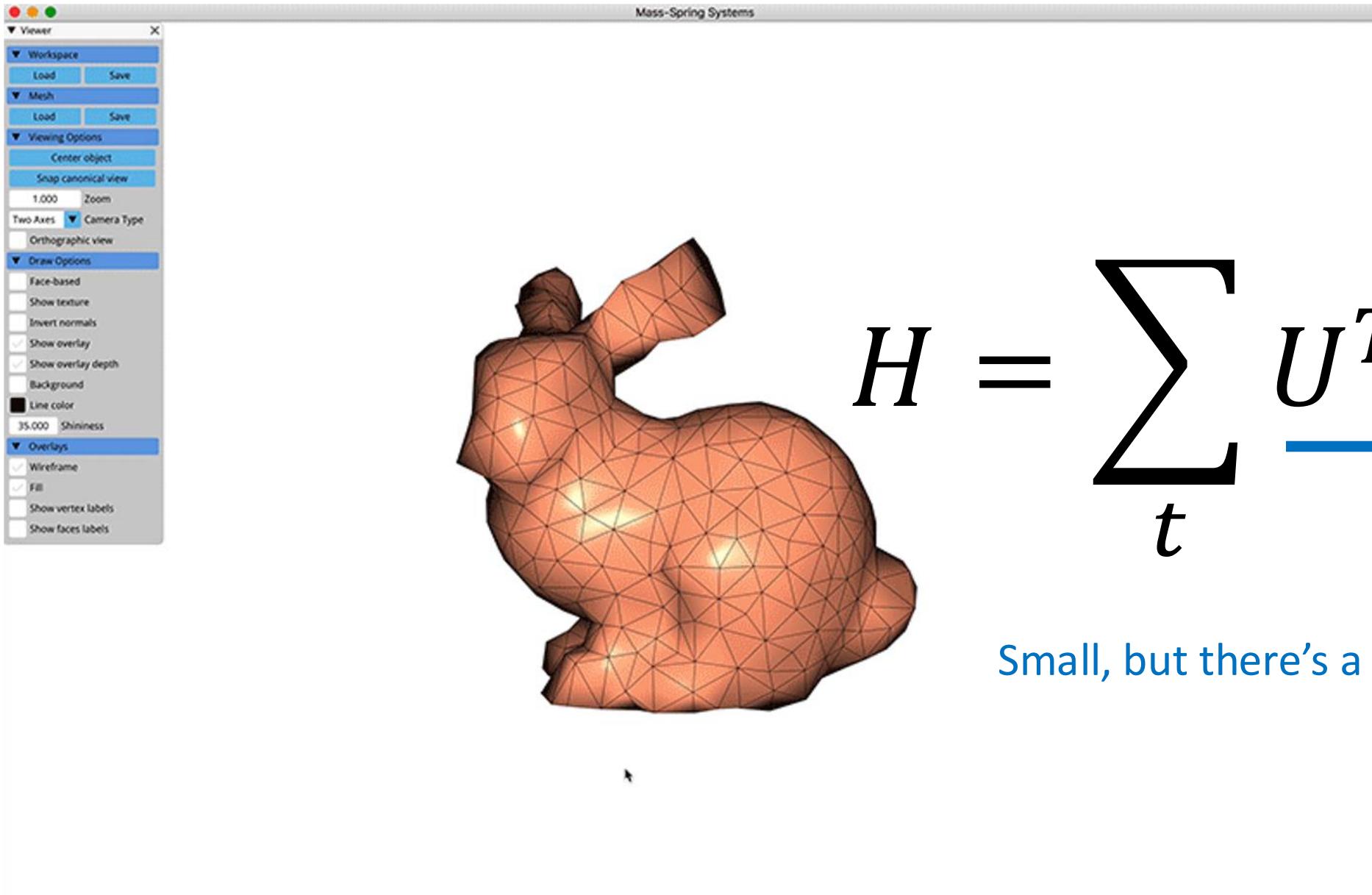
Repeat until converged



# Problem, it's still slow! Why ?



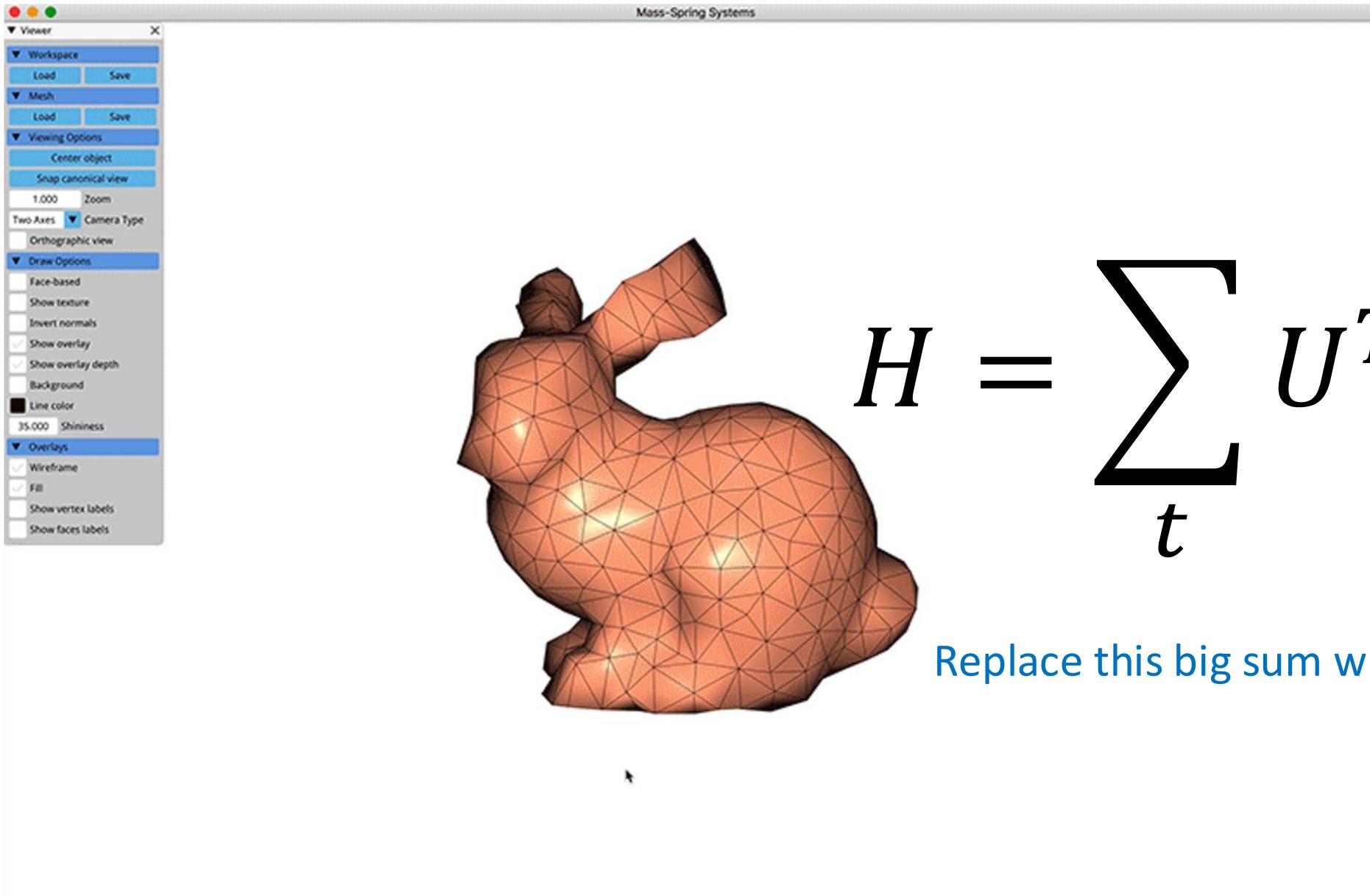
# Assembly still visits every element 😞



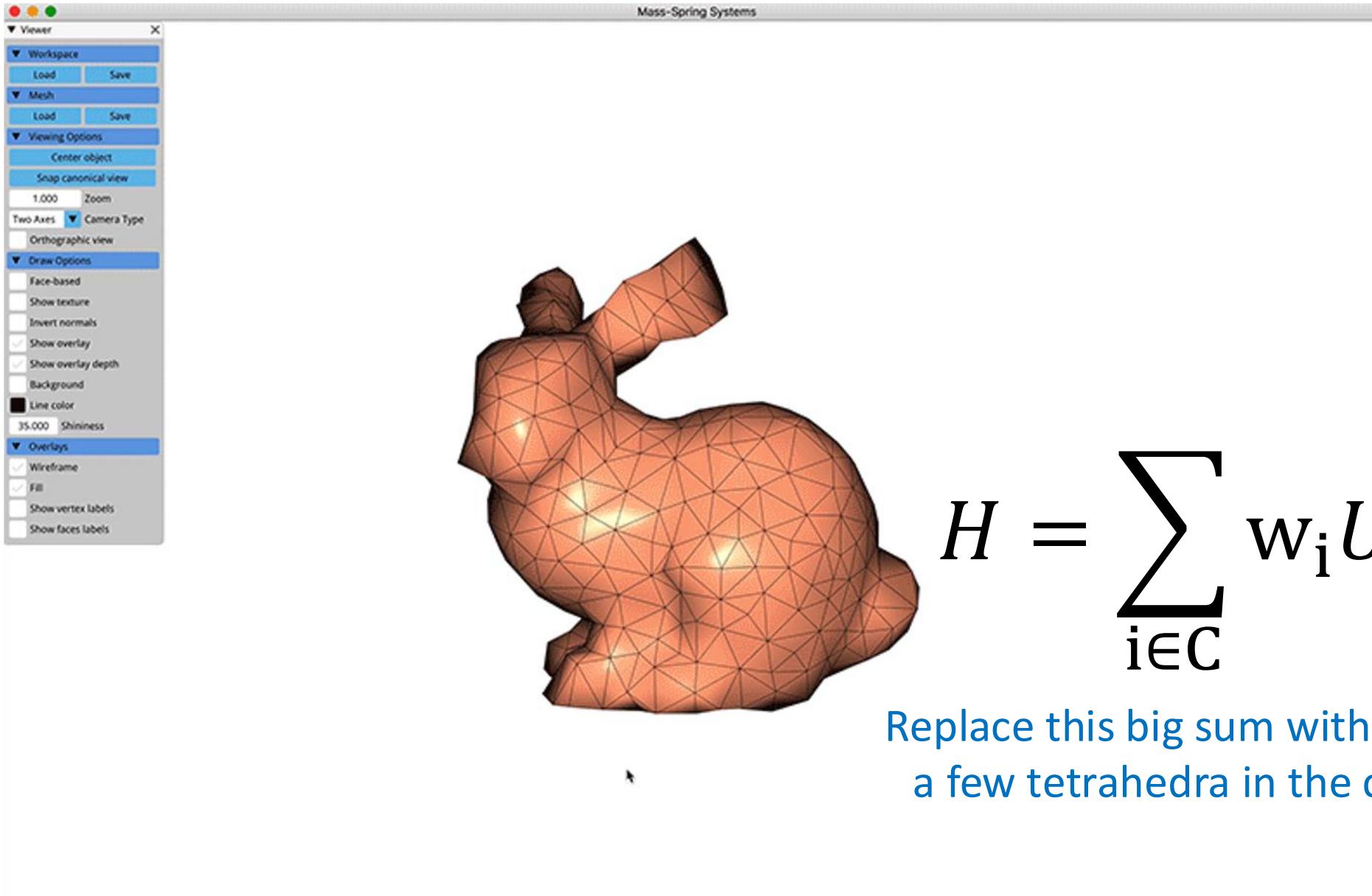
$$H = \sum_t U^T H_i U$$

Small, but there's a lot of them

# Optimal Quadrature



# Optimal Quadrature

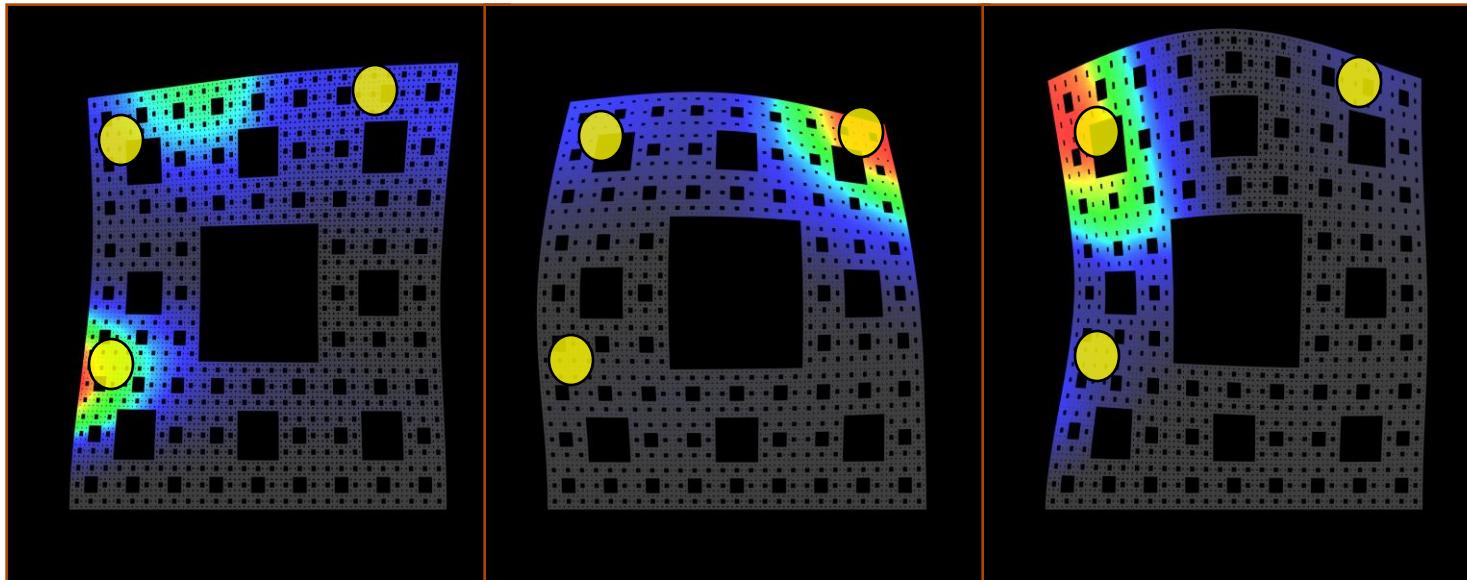


$$H = \sum_{i \in C} w_i U^T H_i U$$

Replace this big sum with a small sum over a few tetrahedra in the cubature set  $C$ .

# One Method: Data Driven

$$\{q_i, g_i\}$$



$q^{(1)}$

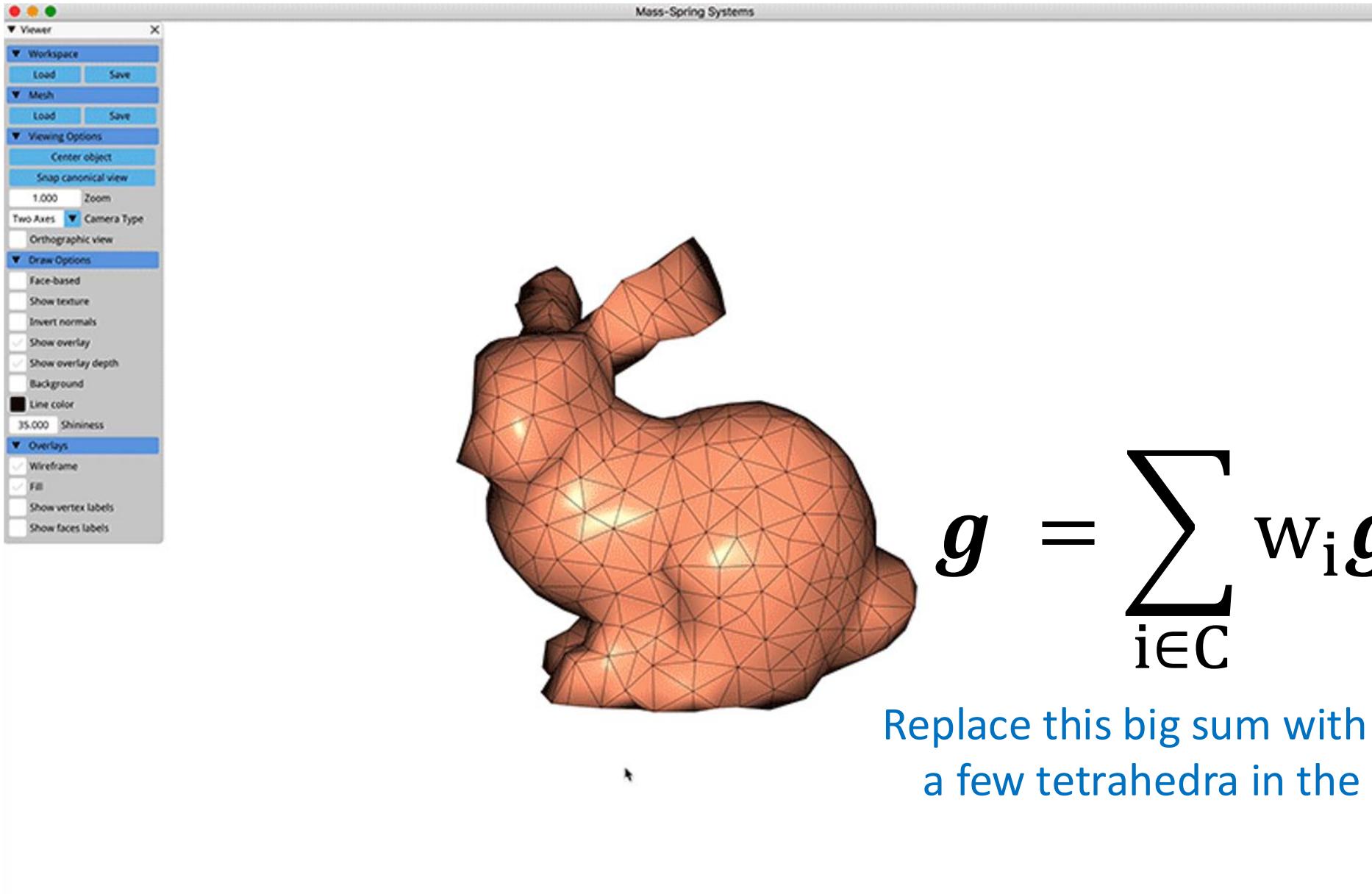
$q^{(2)}$

$q^{(3)}$

$q^{(T)}$

Optimized Cubature | An et al.

# Optimal Quadrature



# Cubature for all poses

$$\begin{pmatrix} g_0^0 & \cdots & g_n^0 \\ \vdots & \ddots & \vdots \\ g_0^T & \cdots & g_n^T \end{pmatrix} \begin{matrix} \text{Weights} \\ \downarrow \\ \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \end{matrix} = \begin{pmatrix} g_0 \\ \vdots \\ g_n \end{pmatrix}$$

Per-tetrahedron gradient values

Per-pose value

# Cubature for all poses

$$\begin{array}{c} \text{W} \\ \top \\ \left( \begin{array}{ccc} g_0^0 & \cdots & g_n^0 \\ \vdots & \ddots & \vdots \\ g_0^T & \cdots & g_n^T \end{array} \right) \left( \begin{array}{c} w_0 \\ \vdots \\ w_n \end{array} \right) = \left( \begin{array}{c} g_0 \\ \vdots \\ g_n \end{array} \right) \\ \hline G \\ \top \\ g \end{array}$$

Solve using Non-linear least squares

# Non-Linear Least Squares

$$U^* = \arg \min_{s.t. w \geq 0} \|Gw - g\|_2^2$$

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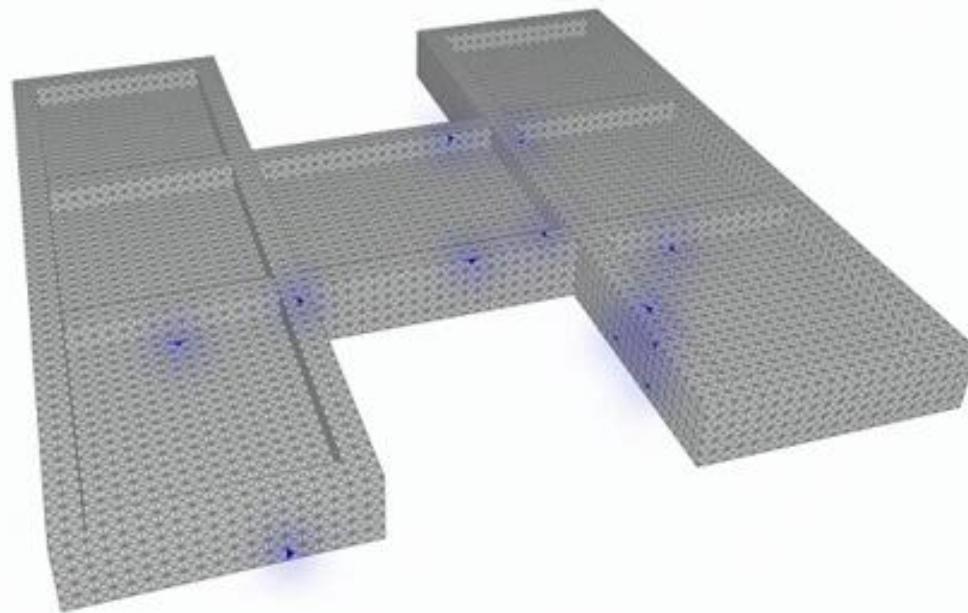
$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

$$i = i + 1$$

Repeat until converged



# PCA Reanalysis of a Balloon



# Next Video: Fast Solvers for Elastodynamics !

