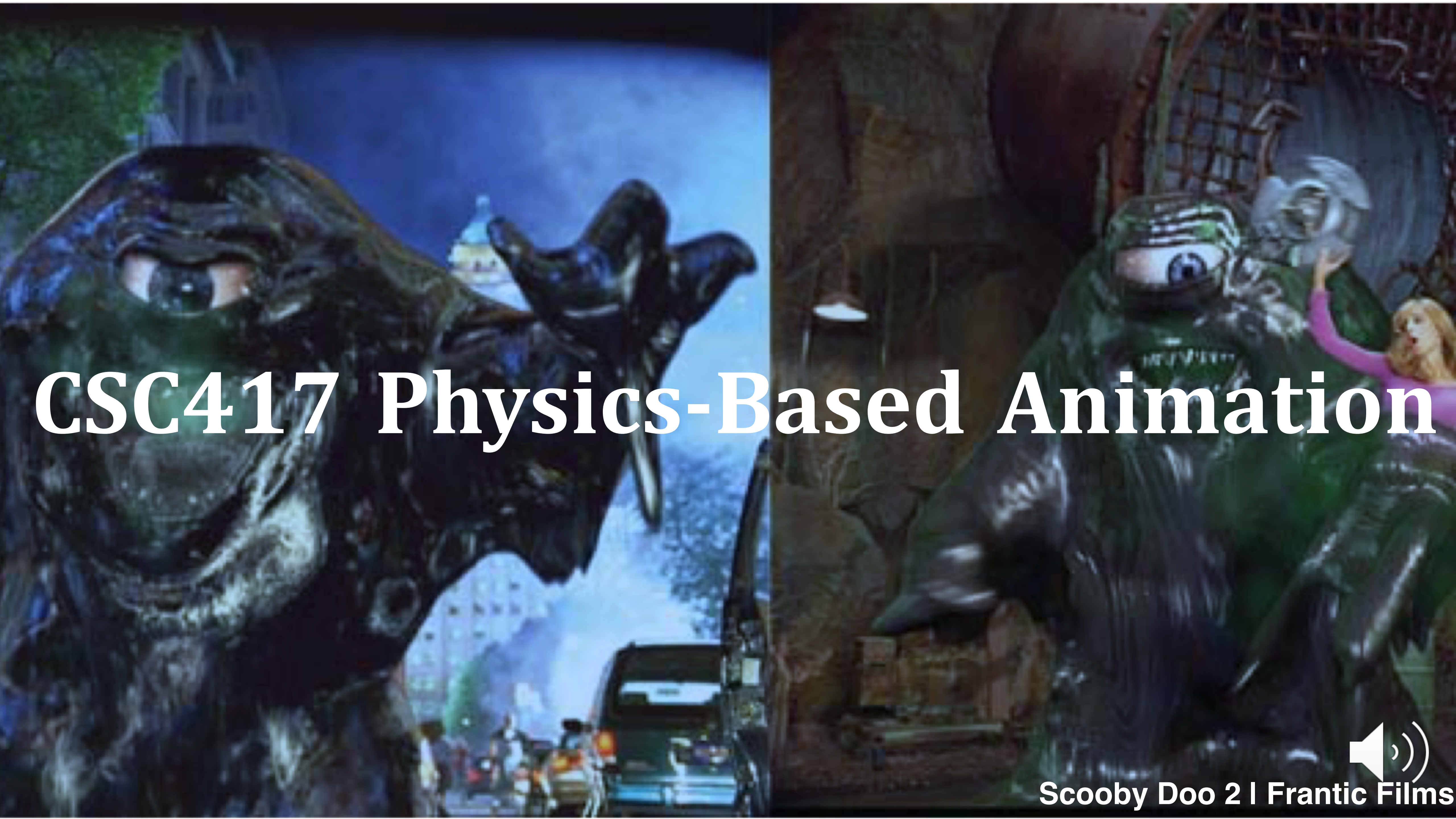
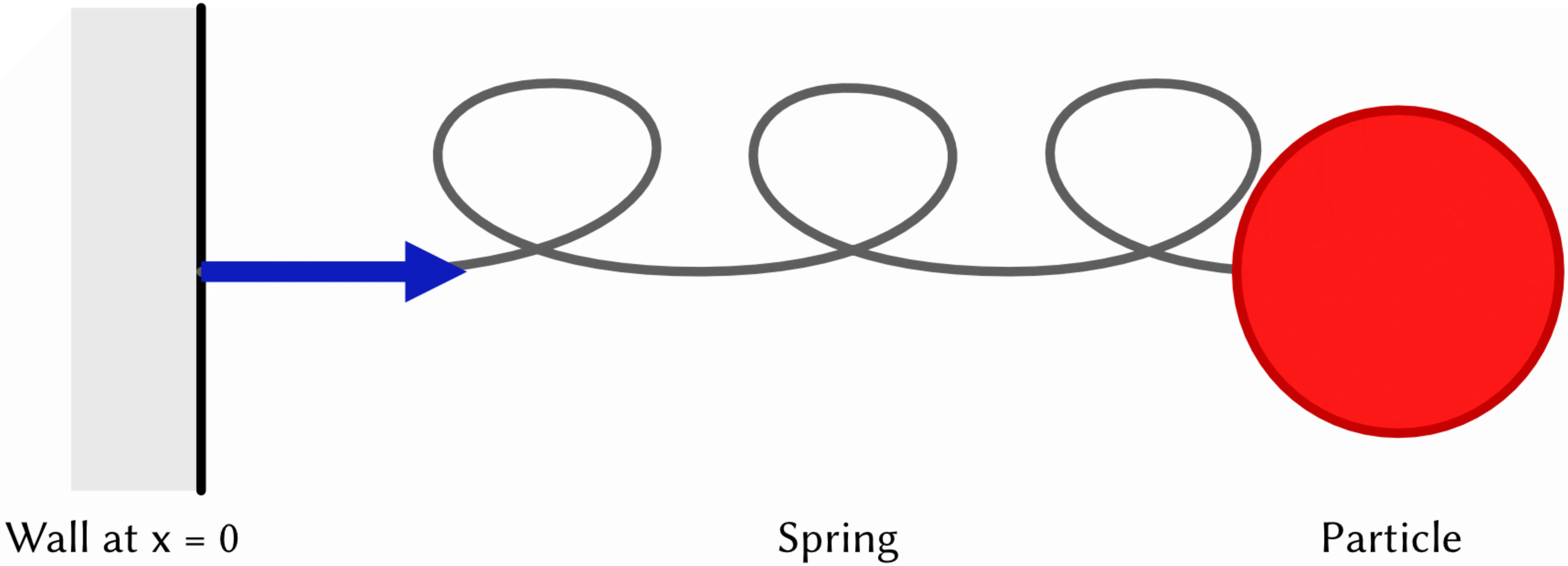
CSC417 Physics-Based Animation



Scooby Doo 2 | Frantic Films

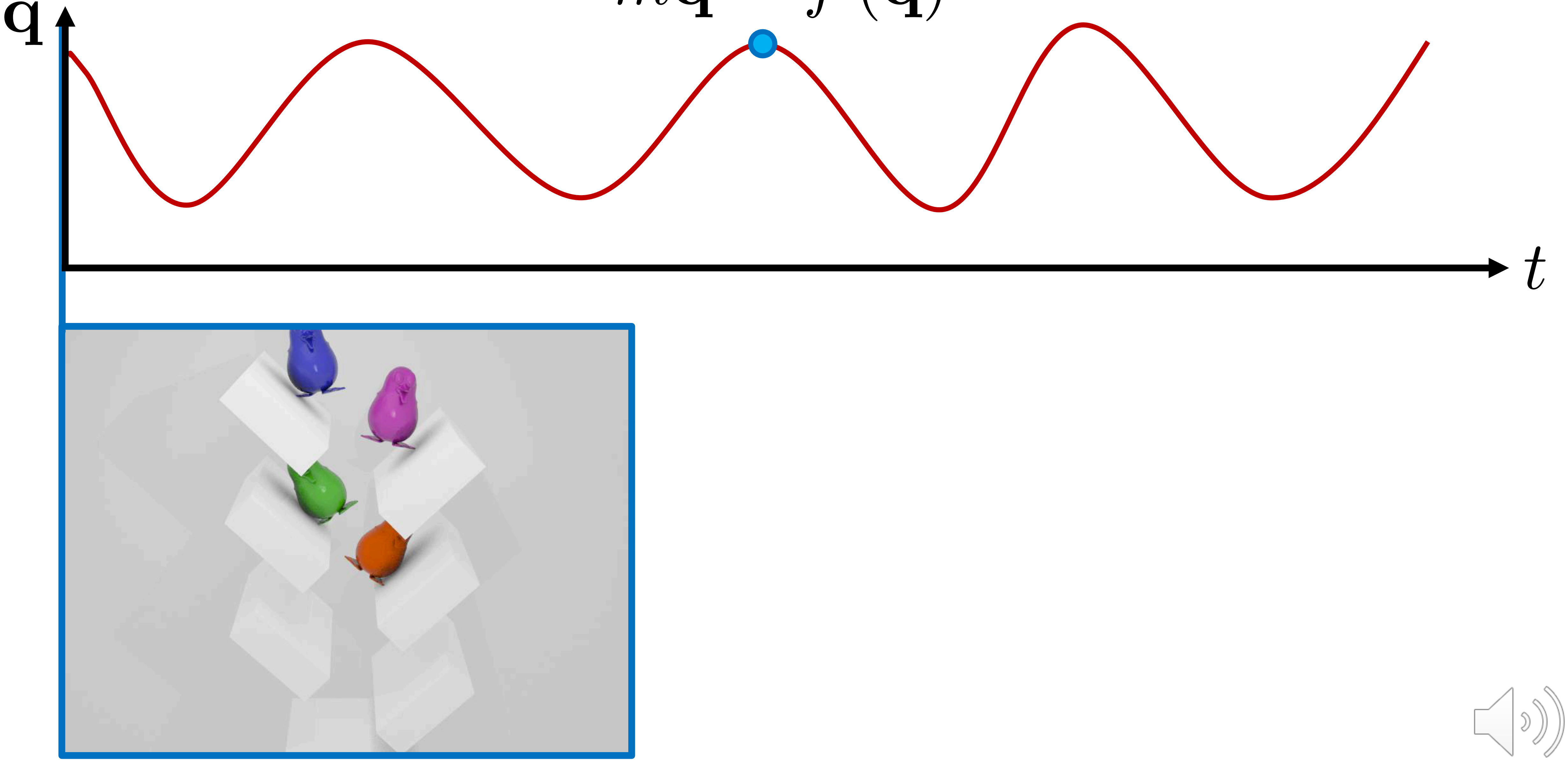
Last Video: Time Integration


Wall at x = 0
Spring
Particle

This Video: Enter the Third Dimension !

Time Integration

$$m\ddot{\mathbf{q}} = f\left(\mathbf{q}\right)$$



Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}$$

A Single Spring in 3D

Particle 0

Particle 1

The Setup

rest length

$\perp$

l_0

Particle 0

$\mathbf{x}_0\left(t\right) \in \mathbb{R}^3$

Particle 1

$\mathbf{x}_1\left(t\right) \in \mathbb{R}^3$

Generalized Coordinates for Mass Spring System

Particle 0
$\mathbf{x}_0\left(t\right)$

Particle 1
$\mathbf{x}_1\left(t\right)$

Generalized Coordinates for Mass Spring System

Particle 0

$\mathbf{x}_0\left(t\right)$

Particle 1

$\mathbf{x}_1\left(t\right)$

$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}$$

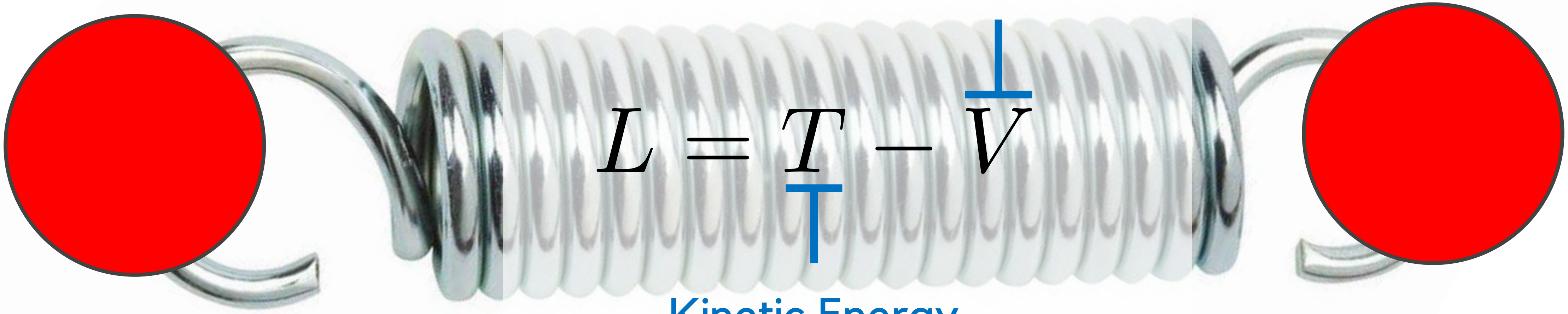
Generalized Coordinates for Mass Spring System

$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}$$

Generalized Coordinates for Mass Spring System

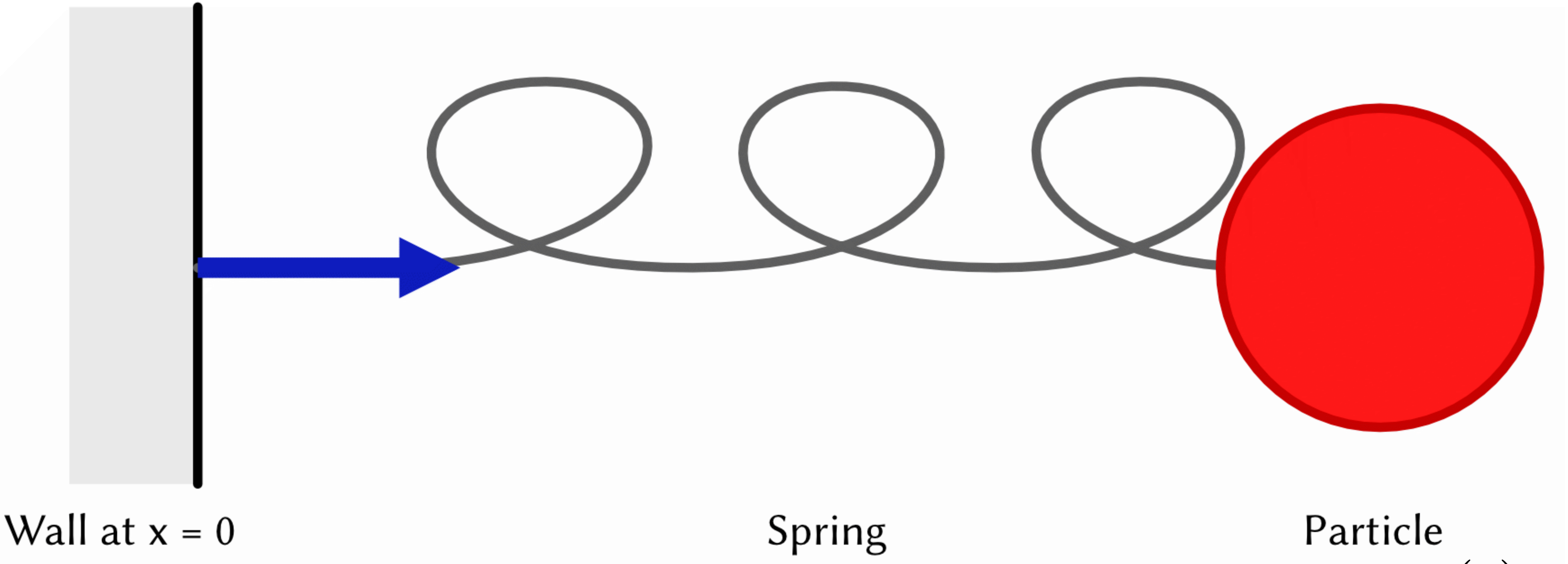
$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{\mathbf{x}}_0 \\ \dot{\mathbf{x}}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \end{pmatrix}$$

The Lagrangian



$$L = T - V$$

Kinetic Energy for a 1D Mass-Spring System



$$q = x(t)$$
$$\dot{q} = v(t)$$

Kinetic Energy for a 1D Mass-Spring System

Kinetic Energy in 1D $$\frac{1}{2}m\underbrace{v^2}$$

Reimagine as $\underbrace{||v||_2^2}$

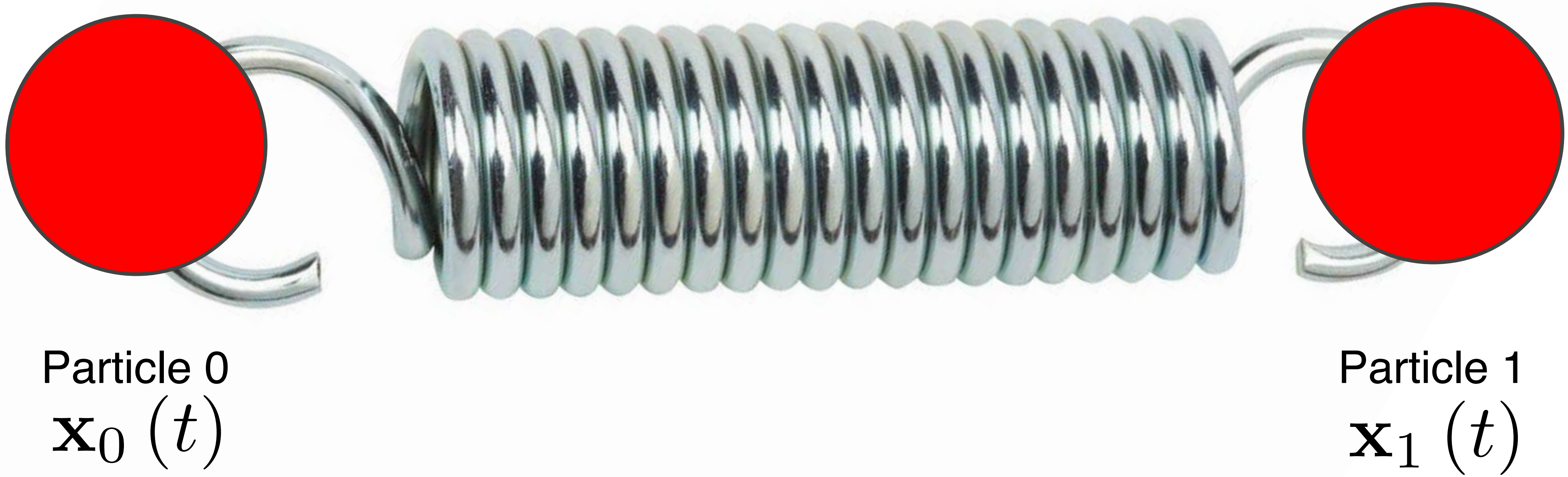
Euclidean 2-norm squared

OR

squared magnitude of velocity

KINDA SILLY IN 1D ☺

Kinetic Energy for 3D Mass-Spring System



Kinetic Energy for 3D Mass-Spring System

Particle 0

$\underbrace{\mathbf{v}_0}_{\text{velocity}}$

Particle 1

$\underbrace{\mathbf{v}_1}_{\text{velocity}}$

Kinetic Energy for 3D Mass-Spring System

Particle 0

$$\frac{1}{2}m\|\mathbf{v}_0\|_2^2$$

Kinetic Energy

Particle 1

$$\frac{1}{2}m\|\mathbf{v}_1\|_2^2$$

Kinetic Energy

Kinetic Energy for 3D Mass-Spring System

$$\underbrace{\overset{\text{Particle 0}}{\frac{1}{2}m\|\mathbf{v}_0\|_2^2}}_{\text{Kinetic Energy}} + \underbrace{\overset{\text{Particle 1}}{\frac{1}{2}m\|\mathbf{v}_1\|_2^2}}_{\text{Kinetic Energy}}$$

Kinetic Energy for a 3D Mass-Spring System

Total Kinetic Energy $\sum_{i=0}^{1}\frac{1}{2}m\|\mathbf{v}_i\|_2^2 = \sum_{i=0}^{1}\frac{1}{2}m\mathbf{v}_i^T\mathbf{v}_i$

$$\sum_{i=0}^{1}\frac{1}{2}\mathbf{v}_i^T\underbrace{\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}}_{\mathrm{M}_i}\mathbf{v}_i$$

Generalized Coordinates for Mass Spring System

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{\mathbf{x}}_0 \\ \dot{\mathbf{x}}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \end{pmatrix}$$

Kinetic Energy for a 3D Mass-Spring System

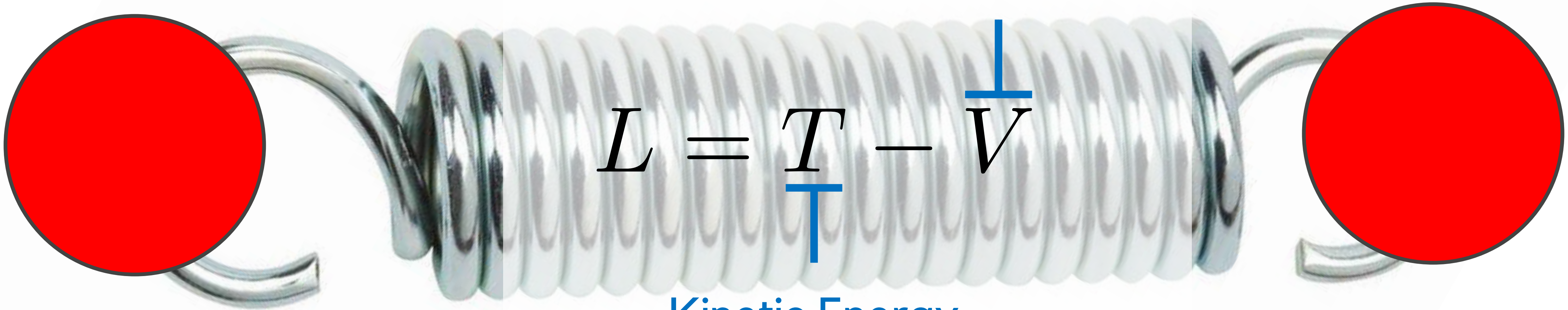
Total Kinetic Energy

$$\sum_{i=0}^{1} \frac{1}{2}\mathbf{v}_i^T \underbrace{\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}}_{\mathrm{M}_i} \mathbf{v}_i$$

$$\frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\mathrm{M}}_{\begin{pmatrix} \mathrm{M}_0 & \mathbf{0} \\ \mathbf{0} & \mathrm{M}_1 \end{pmatrix}} \dot{\mathbf{q}}$$

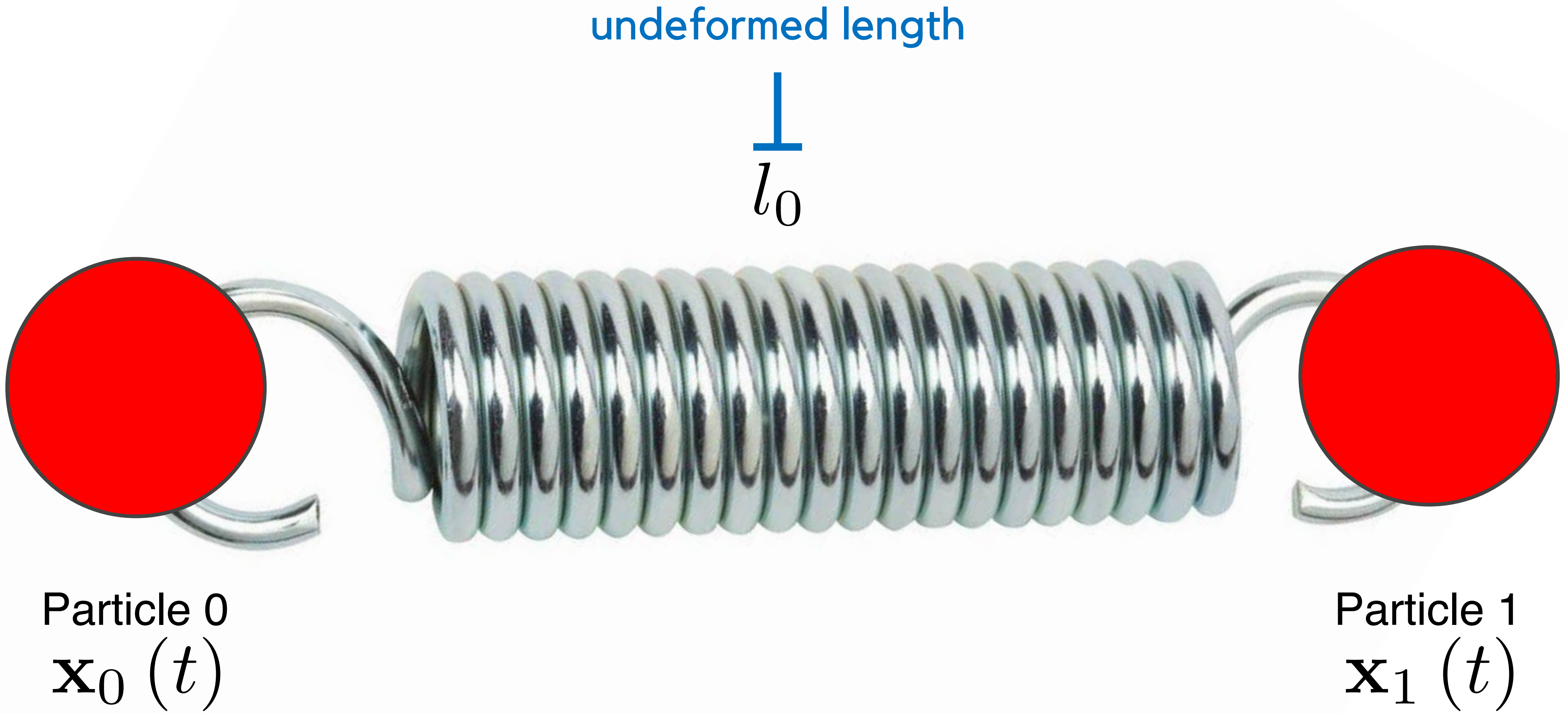
block diagonal mass matrix

The Lagrangian



$$L = T - V$$

Potential Energy for a 3D Spring



Potential Energy for a 3D Spring

1. Spring should go back to original length when all external forces are removed i.e rest length should be a minimum

2. Rigid motion (translation and/or rotation) should not change the energy

3. Energy should depend only on particle positions

Potential Energy for a 3D Spring

undeformed length

l_0

Particle 0

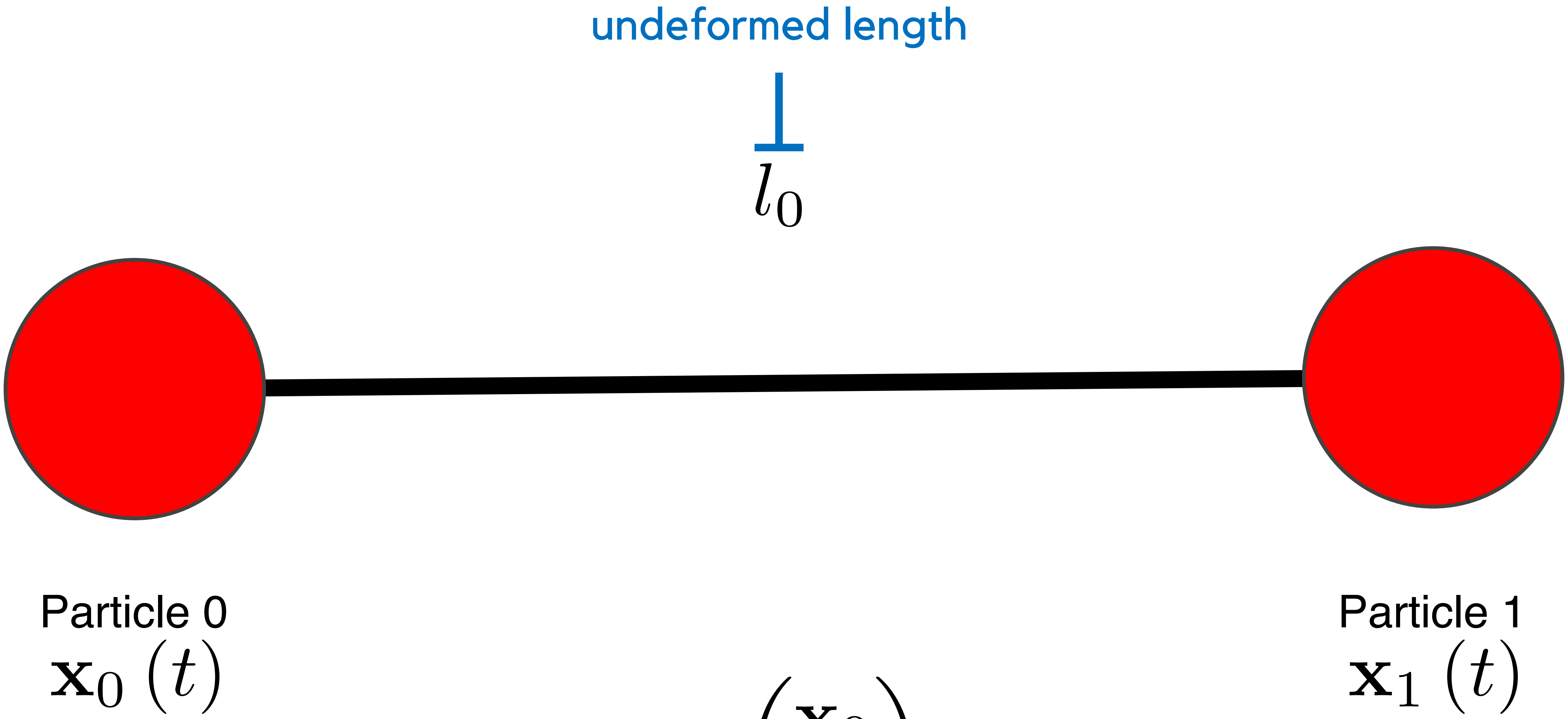
$\mathbf{x}_0\left(t\right)$

Particle 1

$\mathbf{x}_1\left(t\right)$

$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}$$

Potential Energy for a 3D Spring



$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}$$

Potential Energy for a 3D Spring

Strain $l - l_0$

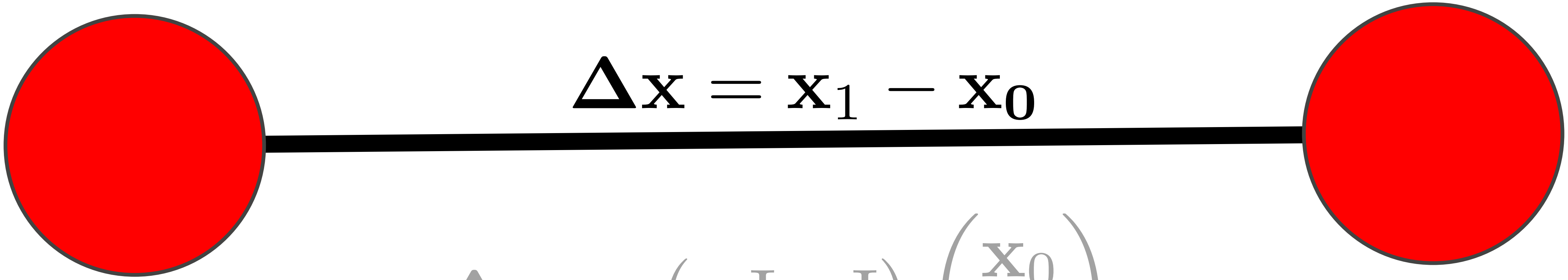
deformed length

Potential Energy $\frac{1}{2}k\left(l - l_0\right)^2$

stiffness parameter

Potential Energy for a 3D Spring

undeformed length



$$\Delta \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_0$$

$$\Delta \mathbf{x} = \underbrace{\begin{pmatrix} -I & I \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}}_{\mathbf{q}}$$

Particle 0
$\mathbf{x}_0(t)$

Particle 1
$\mathbf{x}_1(t)$

Potential Energy for a 3D Spring

Strain $l - l_0$

$$l = \sqrt{\boldsymbol{\Delta}\mathbf{x}^T \boldsymbol{\Delta}\mathbf{x}} = \sqrt{\mathbf{q}^T \mathrm{B}^T \mathrm{B}\mathbf{q}}$$

Potential Energy $\frac{1}{2} \underbrace{k}_{\text{stiffness parameter}} \left(l - l_0\right)^2$

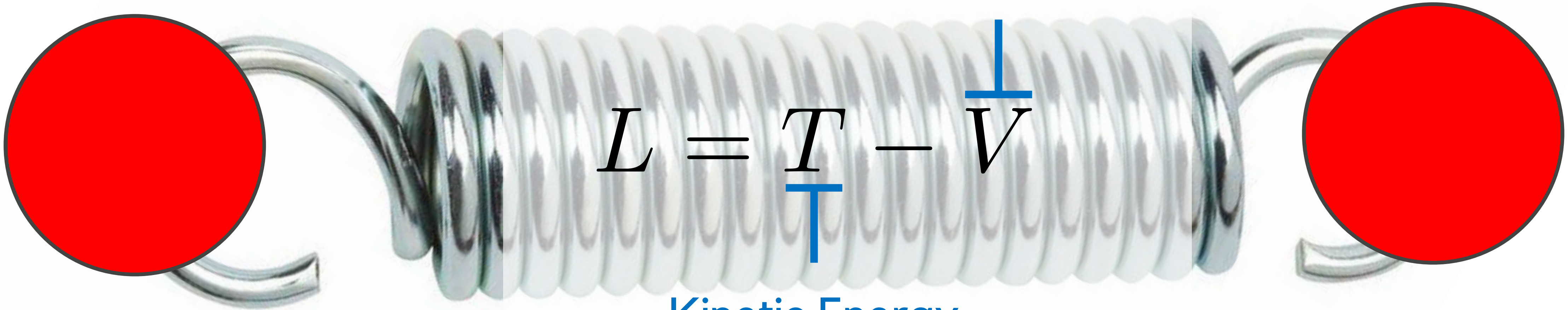
Potential Energy for a 3D Spring

Strain $l - l_0$

$$l = \sqrt{\mathbf{\Delta x}^T \mathbf{\Delta x}} = \sqrt{\mathbf{q}^T \mathrm{B}^T \mathrm{B} \mathbf{q}}$$

Potential Energy $V\left(\mathbf{q}\right) = \frac{1}{2} k \left(\sqrt{\mathbf{q}^T \mathrm{B}^T \mathrm{B} \mathbf{q}} - l_0\right)^2$

The Lagrangian



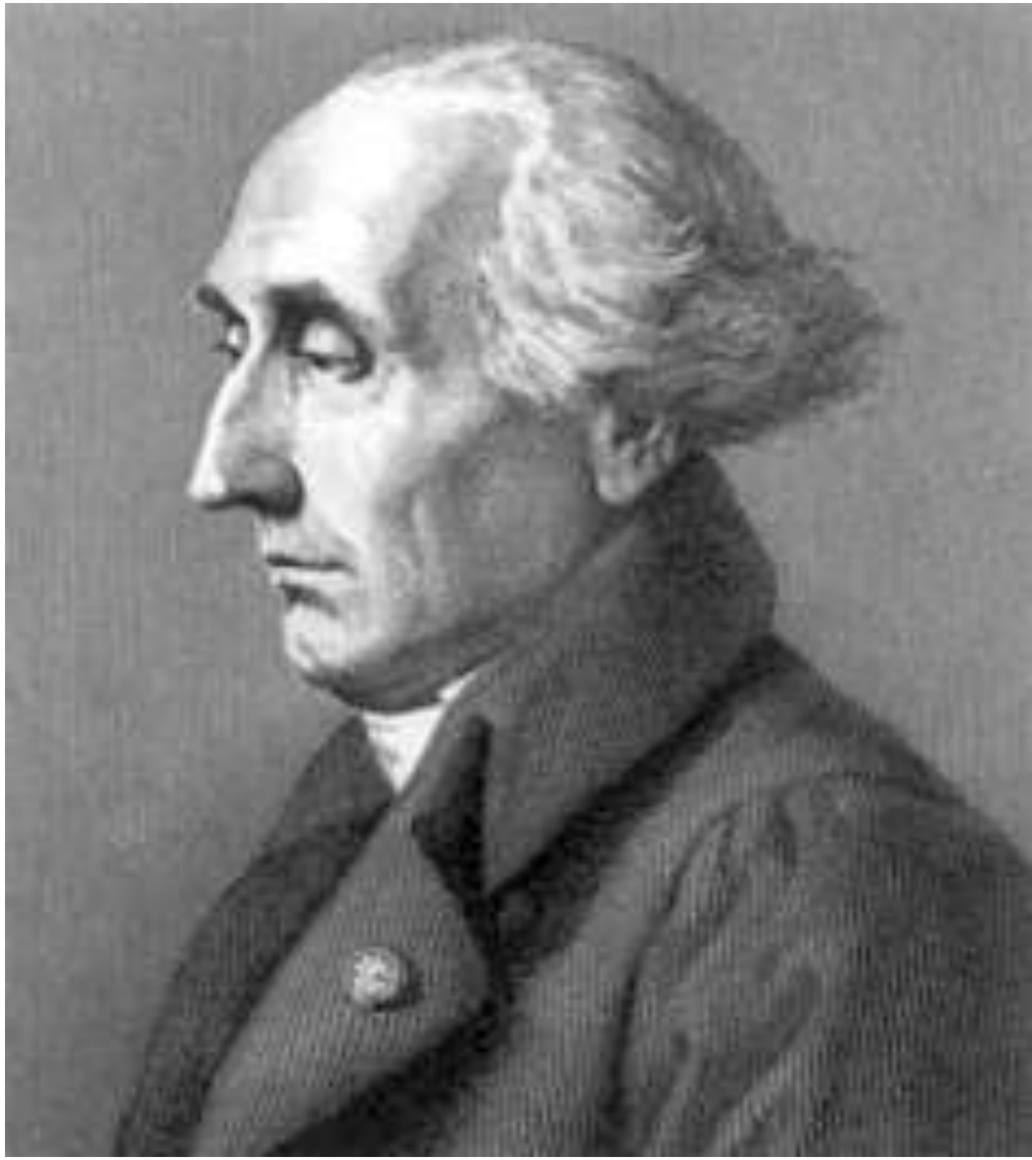
$$L = T - V$$

The Lagrangian

$$\frac{1}{2}k\left(\sqrt{\mathbf{q}^T \mathrm{B}^T \mathrm{B}\mathbf{q}} - l_0\right)^2$$

$$L = T - V$$

$$\frac{1}{2}\dot{\mathbf{q}}^T \mathrm{M}\dot{\mathbf{q}}$$



Euler-Lagrange Equation

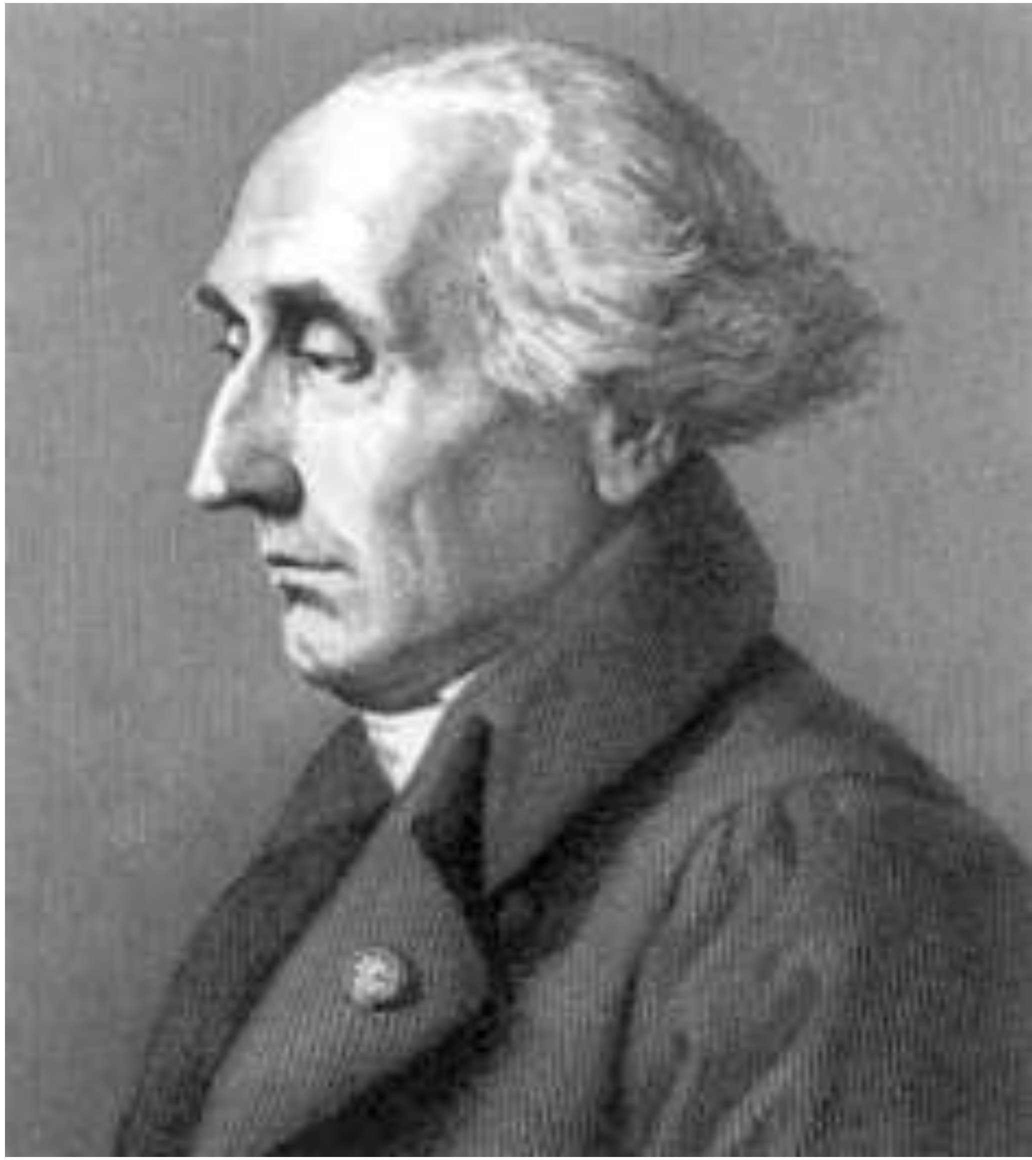
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}$$

The Lagrangian

$$\frac{1}{2}k\left(\sqrt{\mathbf{q}^T\mathrm{B}^T\mathrm{B}\mathbf{q}} - l_0\right)^2$$

$$L = T - V$$

$$\frac{1}{2}\dot{\mathbf{q}}^T\mathrm{M}\dot{\mathbf{q}}$$



Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \underbrace{-\frac{\partial V}{\partial \mathbf{q}}}_{\text{Generalized Forces } \mathbf{f}}$$

Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = -\frac{\partial V}{\partial \mathbf{q}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{1}{2}\dot{\mathbf{q}}^T \mathrm{M} \dot{\mathbf{q}}\right)$$

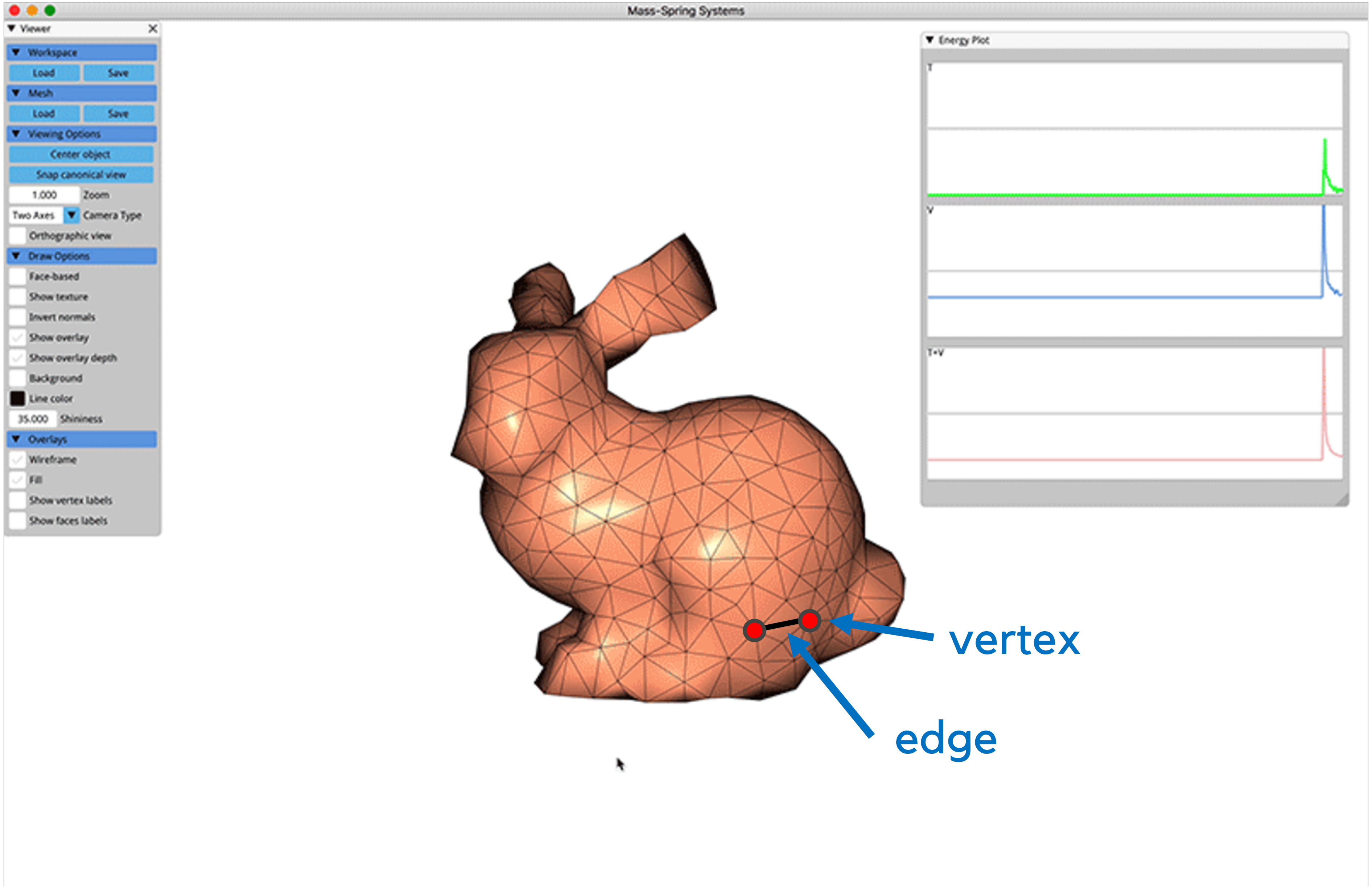
$$\frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{1}{2}\dot{\mathbf{q}}^T \mathrm{M} \dot{\mathbf{q}}\right) = \frac{d}{dt} M \dot{\mathbf{q}}$$

$$\frac{d}{dt} M \dot{\mathbf{q}} = M \ddot{\mathbf{q}}$$

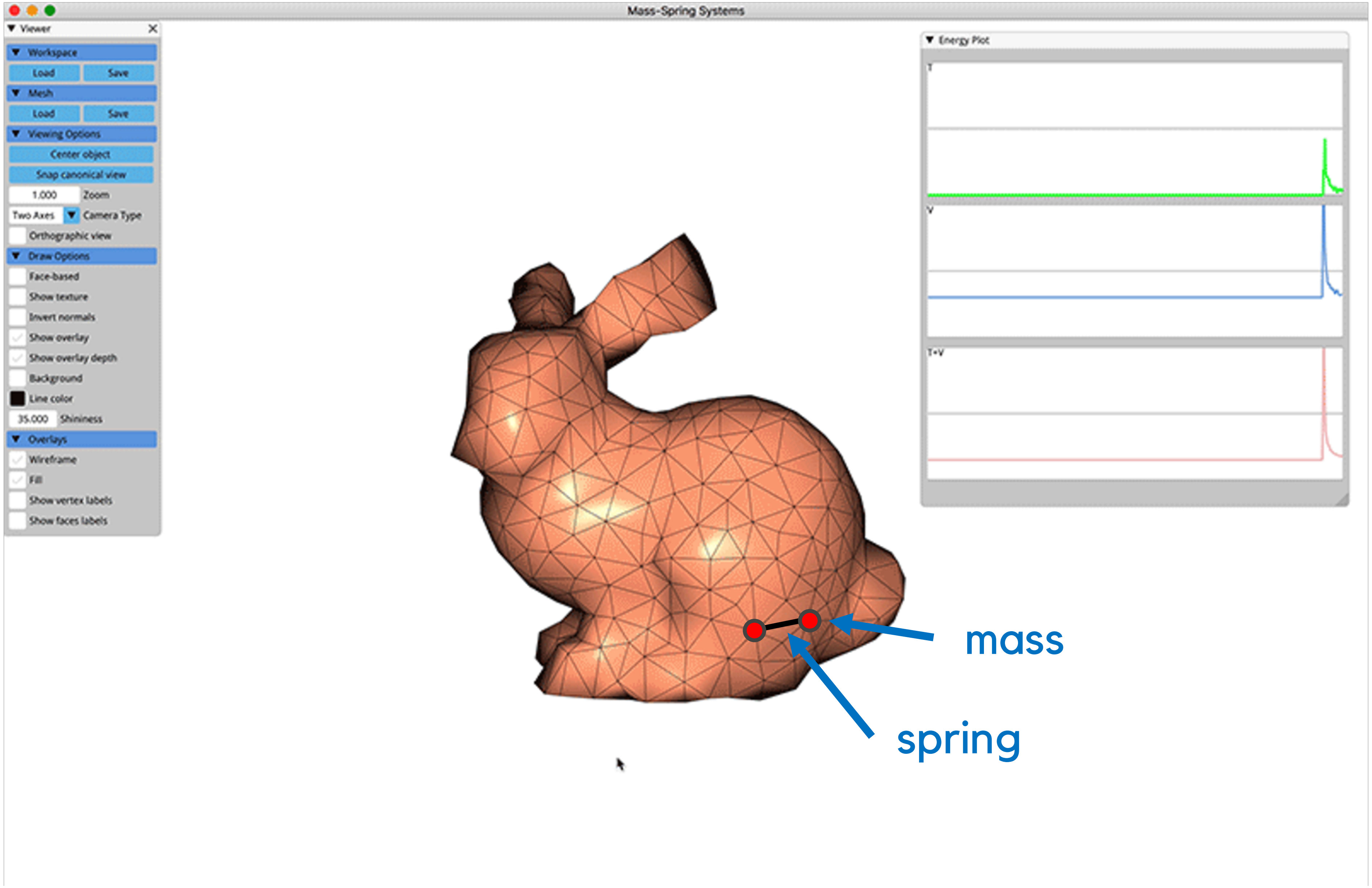
Equations of Motion

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

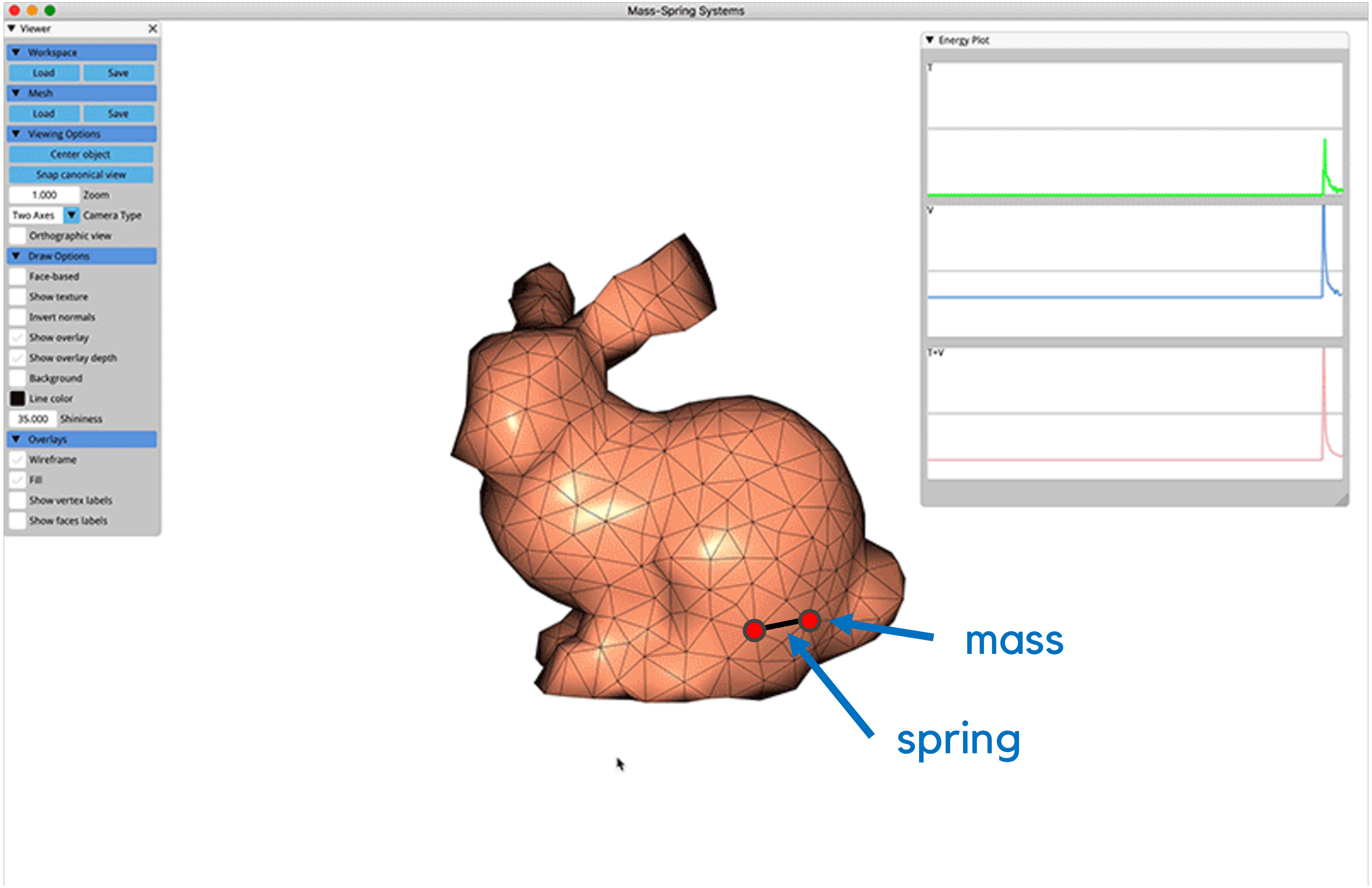
Large Mass-Spring Systems in 3D



Large Mass-Spring Systems in 3D



Generalized Coordinates for Bunny Spring System



Generalized Coordinates for Bunny Spring System

undeformed length

l_0

Particle 0

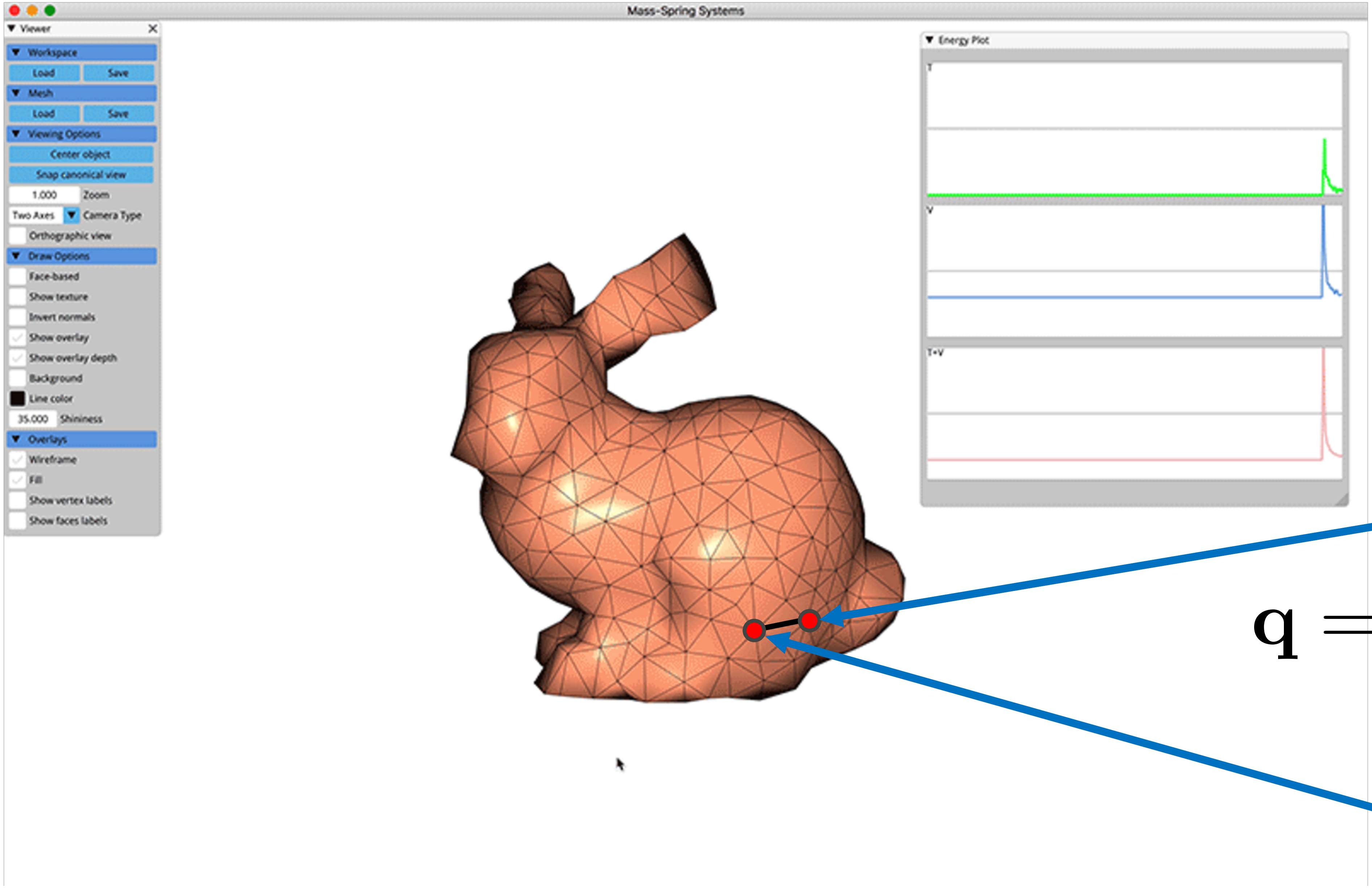
$\mathbf{x}_0\,(t)$

Particle 1

$\mathbf{x}_1\,(t)$

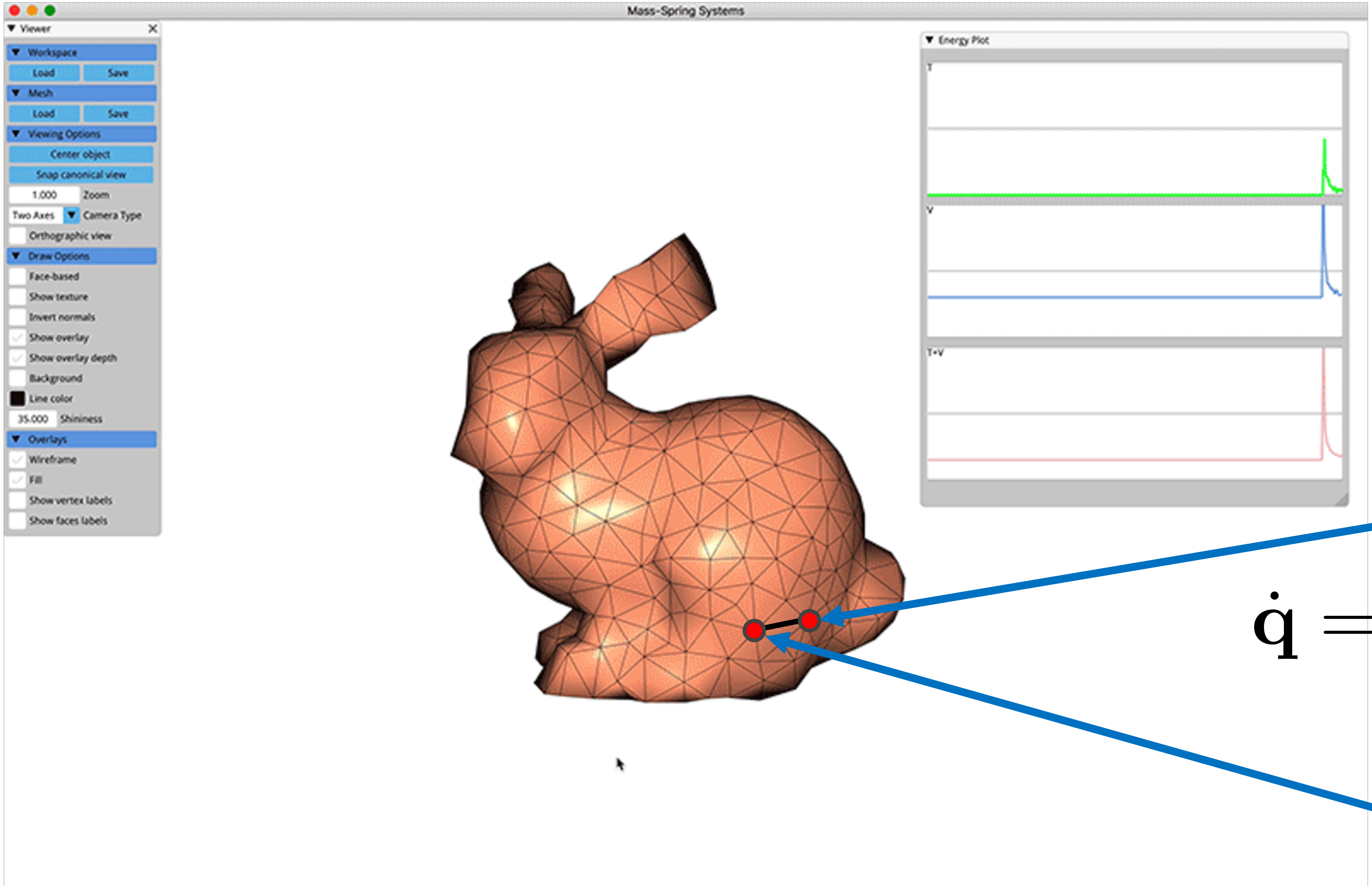
$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}$$

Generalized Coordinates for Bunny Spring System



$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Generalized Coordinates for Bunny Spring System



$$\dot{\mathbf{q}} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix}$$

Kinetic Energy for Bunny Spring System

Total Kinetic Energy $$\sum_{i=0}^{n-1} \frac{1}{2}\mathbf{v}_i^T \underbrace{\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}}_{\mathrm{M}_i} \mathbf{v}_i$$

$$\frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\mathrm{M}}_{\begin{pmatrix} \mathrm{M}_0 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathrm{M}_n \end{pmatrix}} \dot{\mathbf{q}}$$

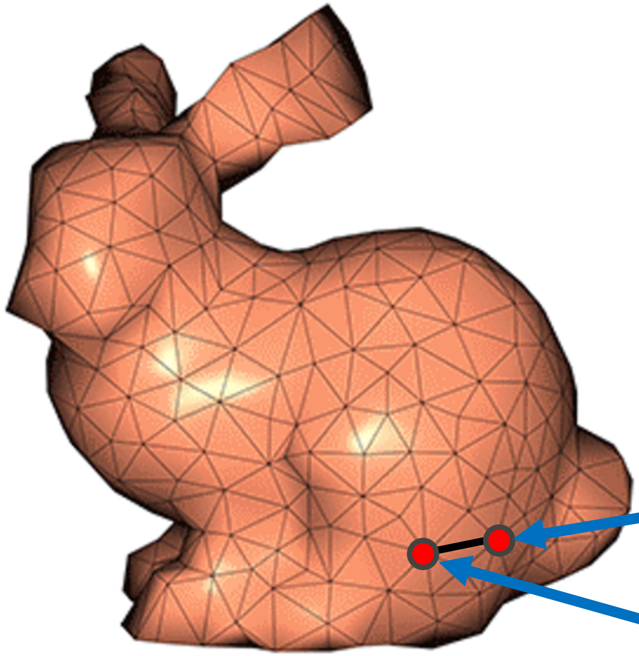
$$\begin{pmatrix} \mathrm{M}_0 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathrm{M}_n \end{pmatrix} \in \mathbb{R}^{3n \times 3n}$$

block diagonal mass matrix

Potential Energy for Bunny Spring System

Total Potential Energy $$V = \sum_{j=0}^{m-1} V_j\left(\mathbf{x}_A, \mathbf{x}_B\right)$$

potential energy for each spring



$\mathbf{x}_A$

$\mathbf{x}_B$

need this mapping

$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Aside: Selection Matrices

$$\begin{pmatrix} \mathrm{A}_0 & \underbrace{\mathrm{A}_1}_{\mathbb{R}^{3\times 3}} & \cdots & \mathrm{A}_n \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} = \sum_{i=1}^{n} \mathrm{A}_i \mathbf{x}_i$$

Select i^{th} particle position:

$$\mathrm{A}_i = I$$
$$\mathrm{A}_{j\neq i} = \mathbf{0}$$

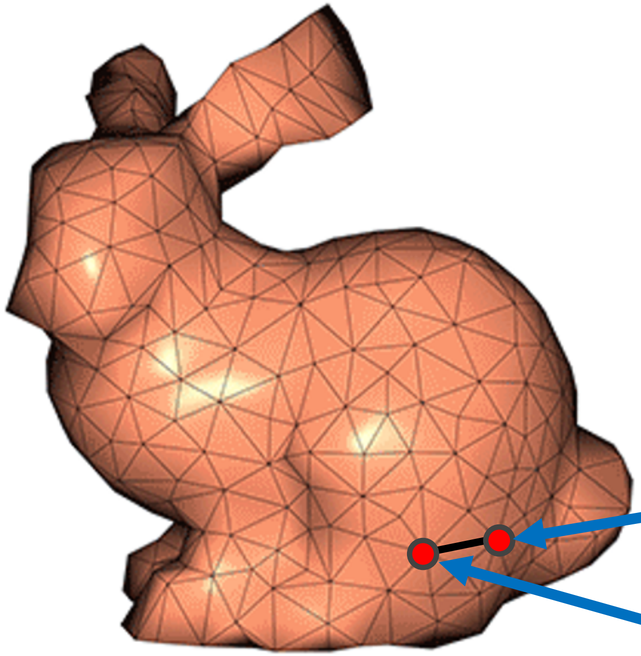
Aside: Selection Matrices

$$\underbrace{\begin{pmatrix} \mathbf{0} & \mathrm{I} & \cdots & \mathbf{0} \end{pmatrix}}_{\mathrm{S}_1} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} = \mathbf{x}_1$$

Potential Energy for Bunny Spring System

Total Potential Energy $$V = \sum_{j=0}^{m-1} V_j\left(\mathbf{x}_A, \mathbf{x}_B\right)$$

potential energy for each spring



$\mathbf{x}_A$

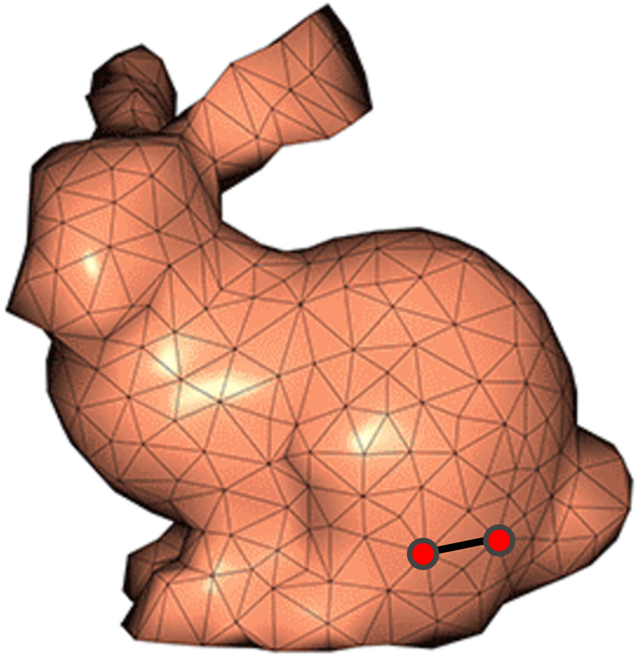
$\mathbf{x}_B$

need this mapping

$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Potential Energy for Bunny Spring System

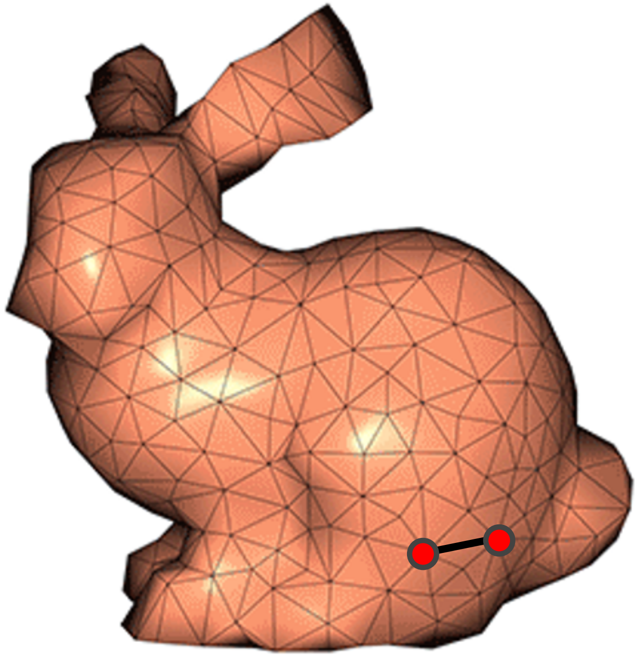
Total Potential Energy $V = \sum_{j=0}^{m-1} \underbrace{V_j}_{\text{potential energy for each spring}} (\mathbf{x}_A, \mathbf{x}_B)$



$$\mathbf{q}_j = \begin{pmatrix} \mathbf{x}_A \\ \mathbf{x}_B \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{S}_1 \\ \mathrm{S}_n \end{pmatrix}}_{\mathrm{E}_j} \mathbf{q}$$

Potential Energy for Bunny Spring System

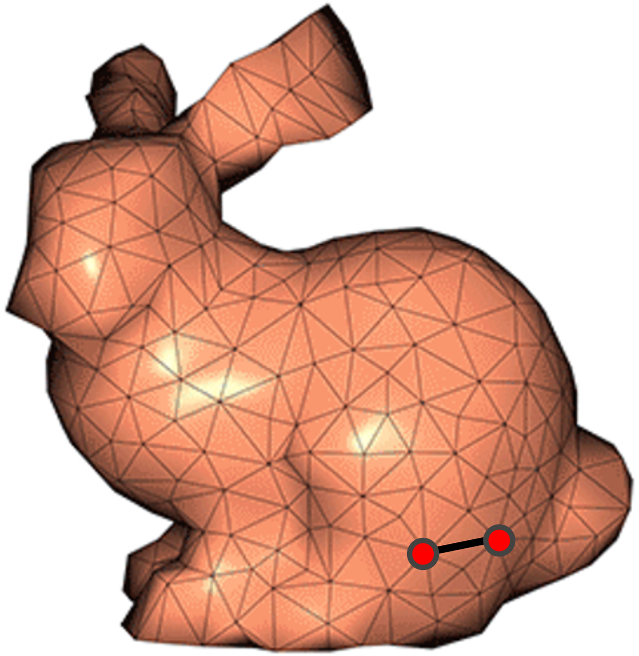
Total Potential Energy $V = \sum_{j=0}^{m-1} \underbrace{V_j}_{\text{potential energy for each spring}} (\mathbf{q}_j)$



$$\mathbf{q}_j = \begin{pmatrix} \mathbf{x}_A \\ \mathbf{x}_B \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{S}_1 \\ \mathrm{S}_n \end{pmatrix}}_{\mathrm{E}_j} \mathbf{q}$$

Potential Energy for Bunny Spring System

Total Potential Energy $V = \sum_{j=0}^{m-1} \underbrace{V_j}_{\text{potential energy for each spring}} \left(\mathrm{E}_j \mathbf{q}\right)$



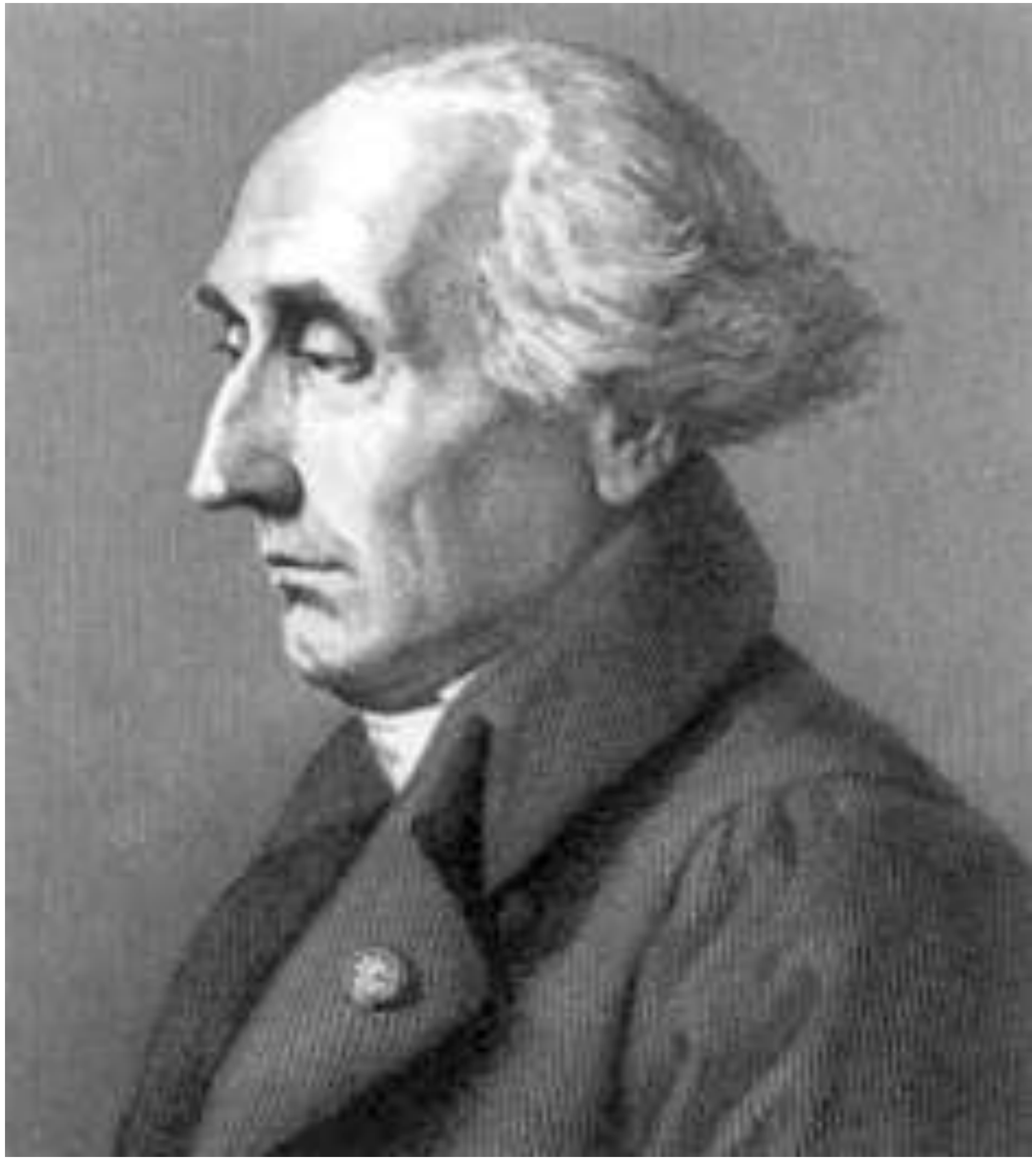
$$\mathbf{q}_j = \begin{pmatrix} \mathbf{x}_A \\ \mathbf{x}_B \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{S}_1 \\ \mathrm{S}_n \end{pmatrix}}_{\mathrm{E}_j} \mathbf{q}$$

The Lagrangian

Potential Energy

$$L = T - V$$

Kinetic Energy

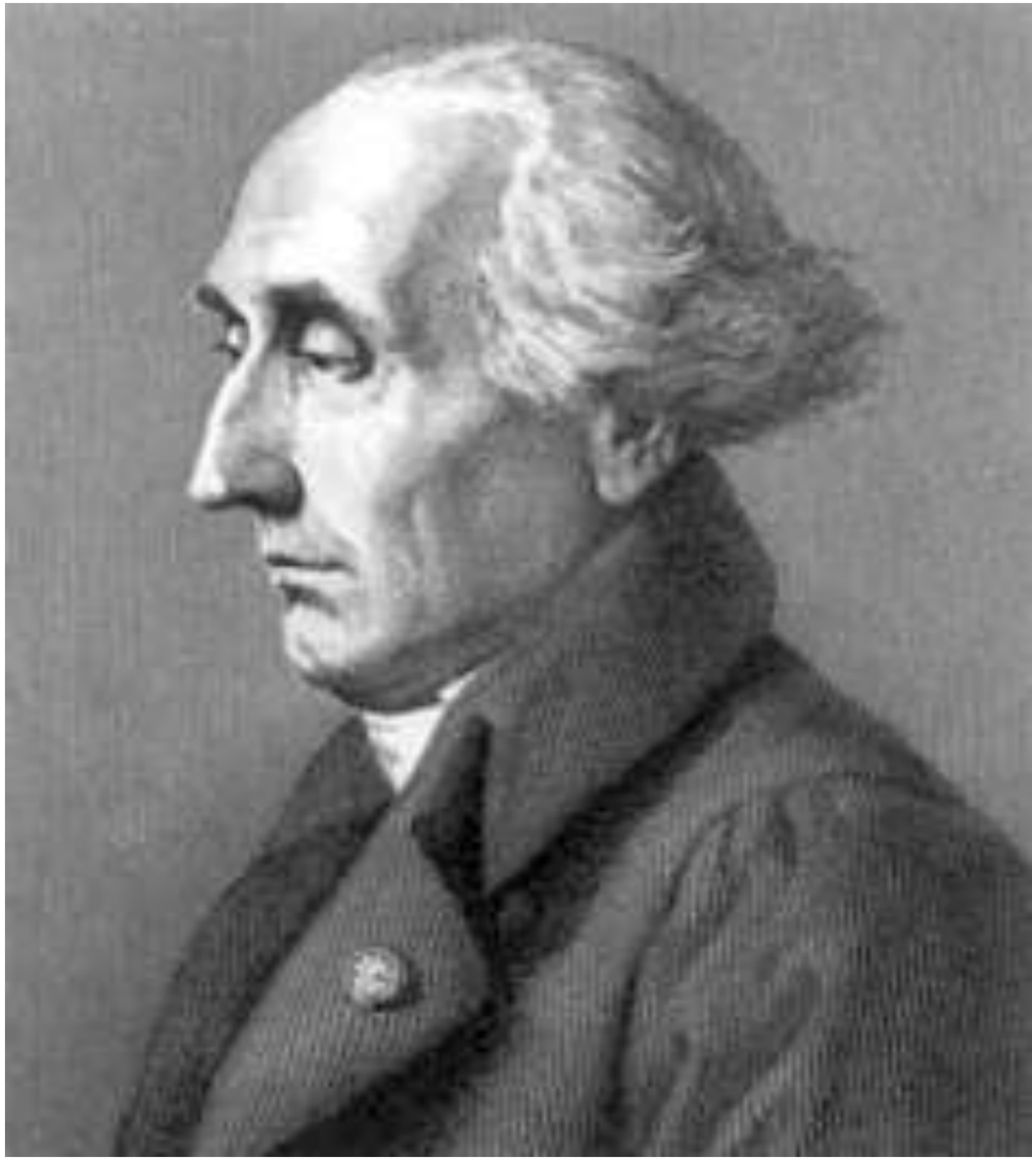


The Lagrangian

$$\sum_{j=0}^{m-1} V_j \left(\mathrm{E}_j \mathbf{q} \right)$$

$$L = T - V$$

$$\frac{1}{2} \dot{\mathbf{q}}^T \mathrm{M} \dot{\mathbf{q}}$$



Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \underbrace{-\frac{\partial V}{\partial \mathbf{q}}}_{\text{Generalized Forces } \mathbf{f}}$$

Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = -\frac{\partial V}{\partial \mathbf{q}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{1}{2}\dot{\mathbf{q}}^T \mathrm{M} \dot{\mathbf{q}}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{1}{2}\dot{\mathbf{q}}^T \mathrm{M} \dot{\mathbf{q}}\right) = \frac{d}{dt} M \dot{\mathbf{q}}$$

$$\frac{d}{dt} M \dot{\mathbf{q}} = M \ddot{\mathbf{q}}$$

Equations of Motion

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Generalized Forces

$$-\frac{\partial V}{\partial \mathbf{q}} = -\frac{\partial}{\partial \mathbf{q}} \sum_{j=0}^{m-1} V_j \left(\mathrm{E}_j \mathbf{q}\right)$$

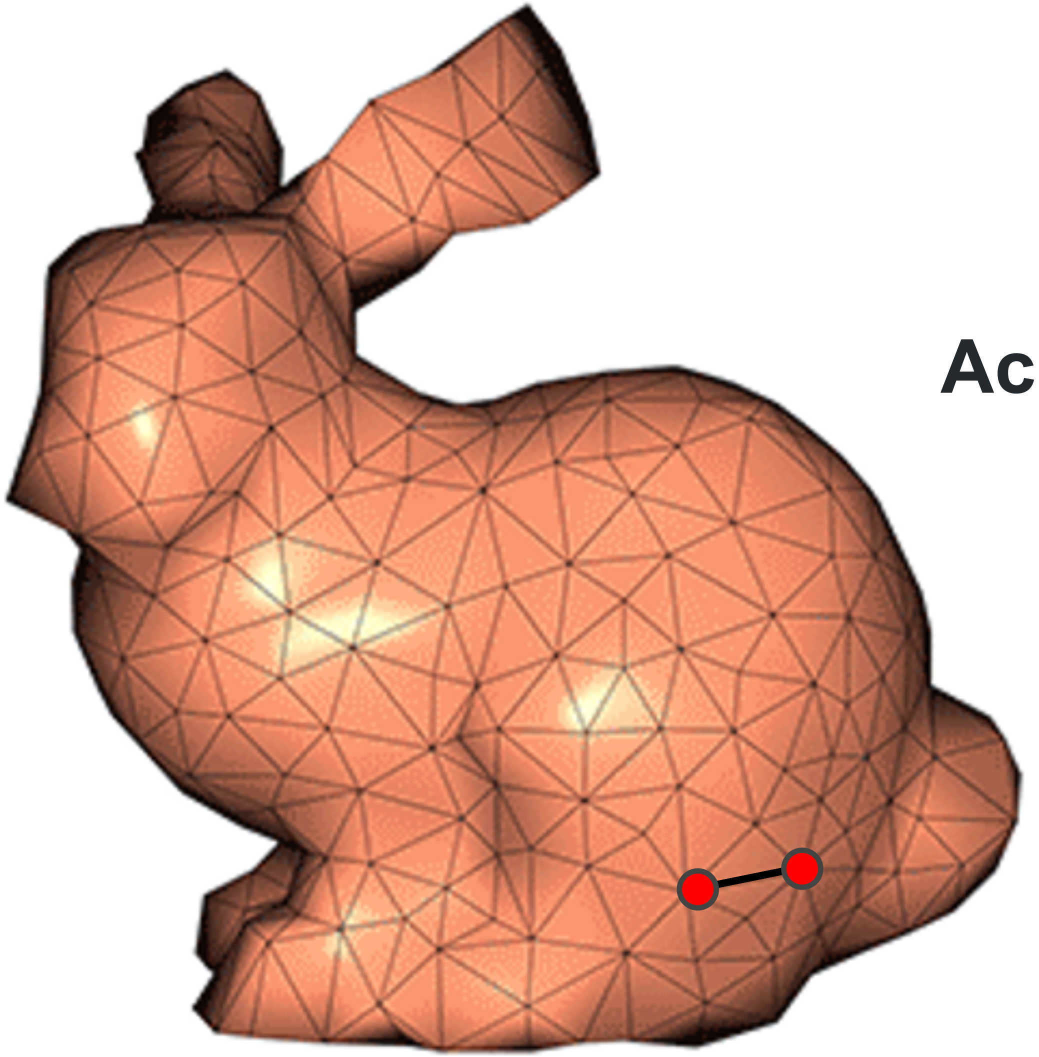
$$-\frac{\partial}{\partial \mathbf{q}} \sum_{j=0}^{m} V_j \left(\mathrm{E}_j \mathbf{q}\right) = -\sum_{j=0}^{m-1} \frac{\partial}{\partial \mathbf{q}} V_j \left(\mathrm{E}_j \mathbf{q}\right)$$

$$-\sum_{j=0}^{m} \frac{\partial}{\partial \mathbf{q}} V_j \left(\mathrm{E}_j \mathbf{q}\right) = -\sum_{j=0}^{m-1} \mathrm{E}_j^T \underbrace{\frac{\partial V_j}{\partial \mathbf{q_j}} \left(\mathbf{q}_j\right)}_{\text{per-spring gradient}}$$

Generalized Forces

$$-\frac{\partial V}{\partial \mathbf{q}} = -\sum_{j=0}^{m-1} \mathrm{E}_j^T \underbrace{\frac{\partial V_j}{\partial \mathbf{q_j}}(\mathbf{q}_j)}_{\text{per-spring gradient}} = \sum_{j=0}^{m-1} \mathrm{E}_j^T \underbrace{\mathbf{f}_j(\mathbf{q}_j)}_{\text{per-spring generalized force}}$$

Assembly of Forces



Accumulate forces on particles

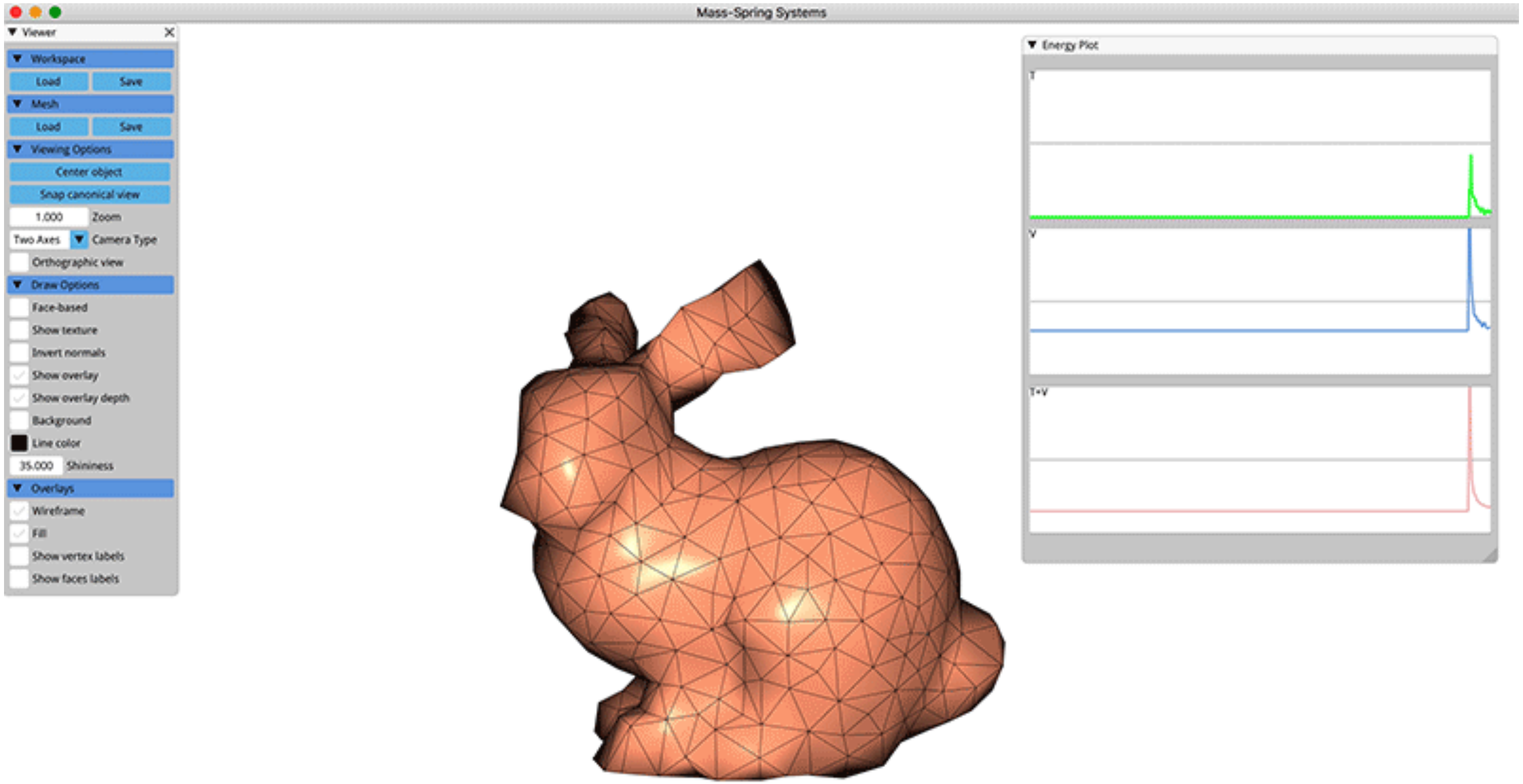
force on particle A

$$\mathbf{f}_j = \begin{pmatrix} \mathbf{f}_A \\ \mathbf{f}_B \end{pmatrix} \qquad \mathrm{E}_j^T \qquad \mathbf{f} = \begin{pmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_n \end{pmatrix}$$

force on particle B

Time Integration

Linearly-Implicit Time Integration



Equations of Motion

$$M\ddot{\mathbf{q}} = \mathbf{f}$$

Linearly-Implicit Time Integration

$$\mathbf{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}\left(\mathbf{q}^{t+1}\right)$$

$$\mathbf{q}^{t+1} = \mathbf{q}^{t} + \Delta t \dot{\mathbf{q}}^{t+1}$$

Backward Euler

$$\mathbf{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}\left(\mathbf{q}^{t} + \Delta t \dot{\mathbf{q}}^{t+1}\right)$$

Substitute

First Order Approximation

$$\mathbf{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}\left(\mathbf{q}^{t}\right) + \Delta t^2 \underbrace{\frac{\partial \mathbf{f}}{\partial \mathbf{q}}}_{\text{stiffness matrix } \mathrm{K}} \dot{\mathbf{q}}^{t+1}$$

Linearly-Implicit Time Integration

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}\left(\mathbf{q}^{t}\right) + \Delta t^{2} \underbrace{\frac{\partial \mathbf{f}}{\partial \mathbf{q}}}_{\text{stiffness matrix } \mathrm{K}} \dot{\mathbf{q}}^{t+1}$$

Rearrange

$$\left(\mathrm{M} - \Delta t^{2}\mathrm{K}\right)\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}\left(\mathbf{q}^{t}\right)$$

Solve

$$\mathbf{q}^{t+1} = \mathbf{q}^{t} + \Delta t \dot{\mathbf{q}}^{t+1}$$

Update Position

The Stiffness Matrix

By Definition

$$\mathrm{K} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}$$

$$\mathbf{f} = -\frac{\partial}{\partial \mathbf{q}} \sum_{j=0}^{m-1} V_j\left(\mathrm{E}_j \mathbf{q}\right)$$

$$\mathrm{K} = -\frac{\partial^2}{\partial \mathbf{q}^2} \sum_{j=0}^{m-1} V_j\left(\mathrm{E}_j \mathbf{q}\right)$$

$$\mathrm{K} = -\sum_{j=0}^{m-1} \frac{\partial^2}{\partial \mathbf{q}^2} V_j\left(\mathrm{E}_j \mathbf{q}\right)$$

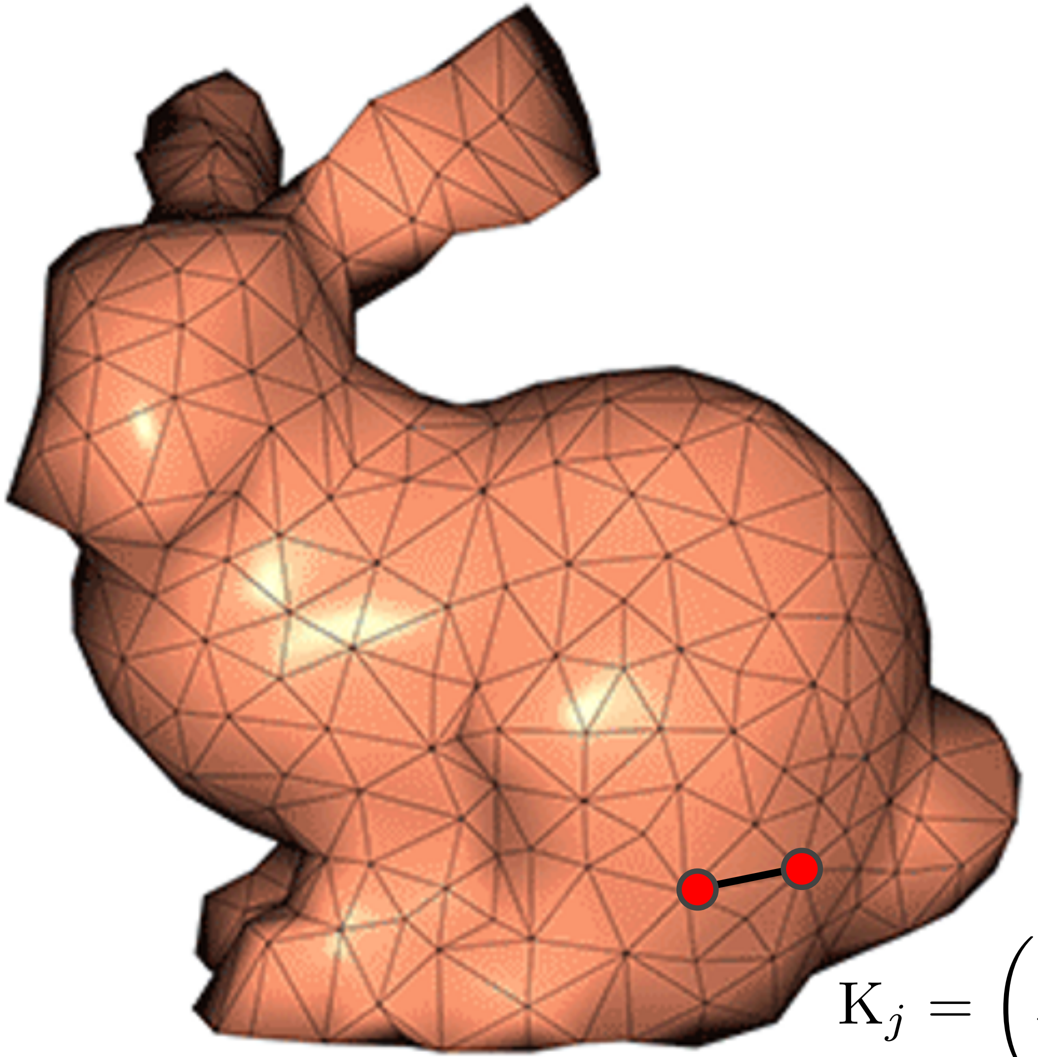
The Stiffness Matrix

$$\mathrm{K} = -\sum_{j=0}^{m-1} \frac{\partial^2}{\partial \mathbf{q}^2} V_j \left(\mathrm{E}_j \mathbf{q}\right)$$

$$\mathrm{K} = \sum_{j=0}^{m-1} \left(-\mathrm{E}_j^T \frac{\partial^2 V_j}{\partial \mathbf{q}_j^2} \mathrm{E}_j \right)$$

$$\mathrm{K} = \sum_{j=0}^{m} \left(\mathrm{E}_j^T \mathrm{K}_j \mathrm{E}_j\right) \longmapsfrom \mathrm{K}_j = -\frac{\partial^2 V_j}{\partial \mathbf{q}_j^2}$$

Assembly of Stiffness Matrix

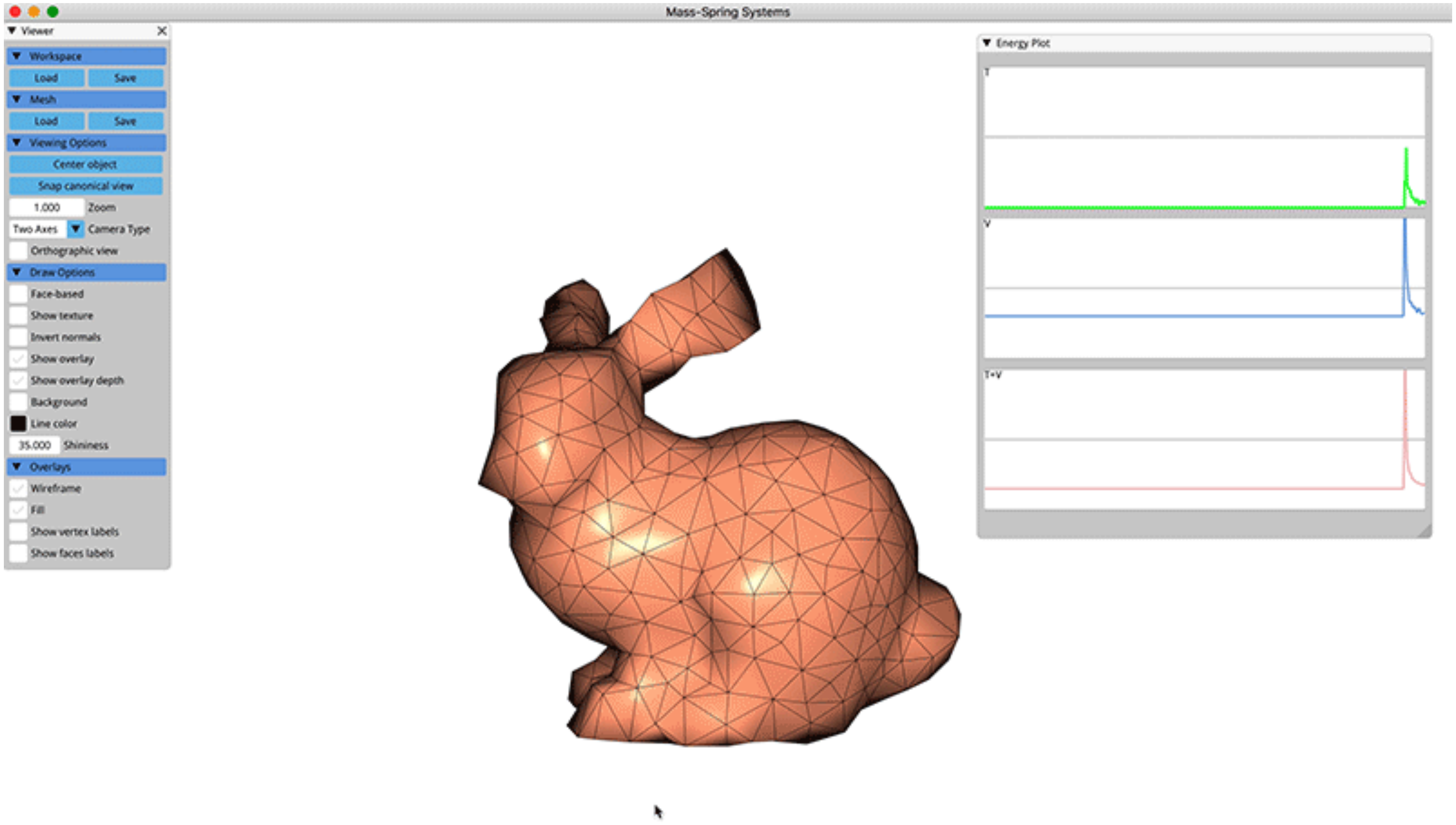


Accumulate matrices

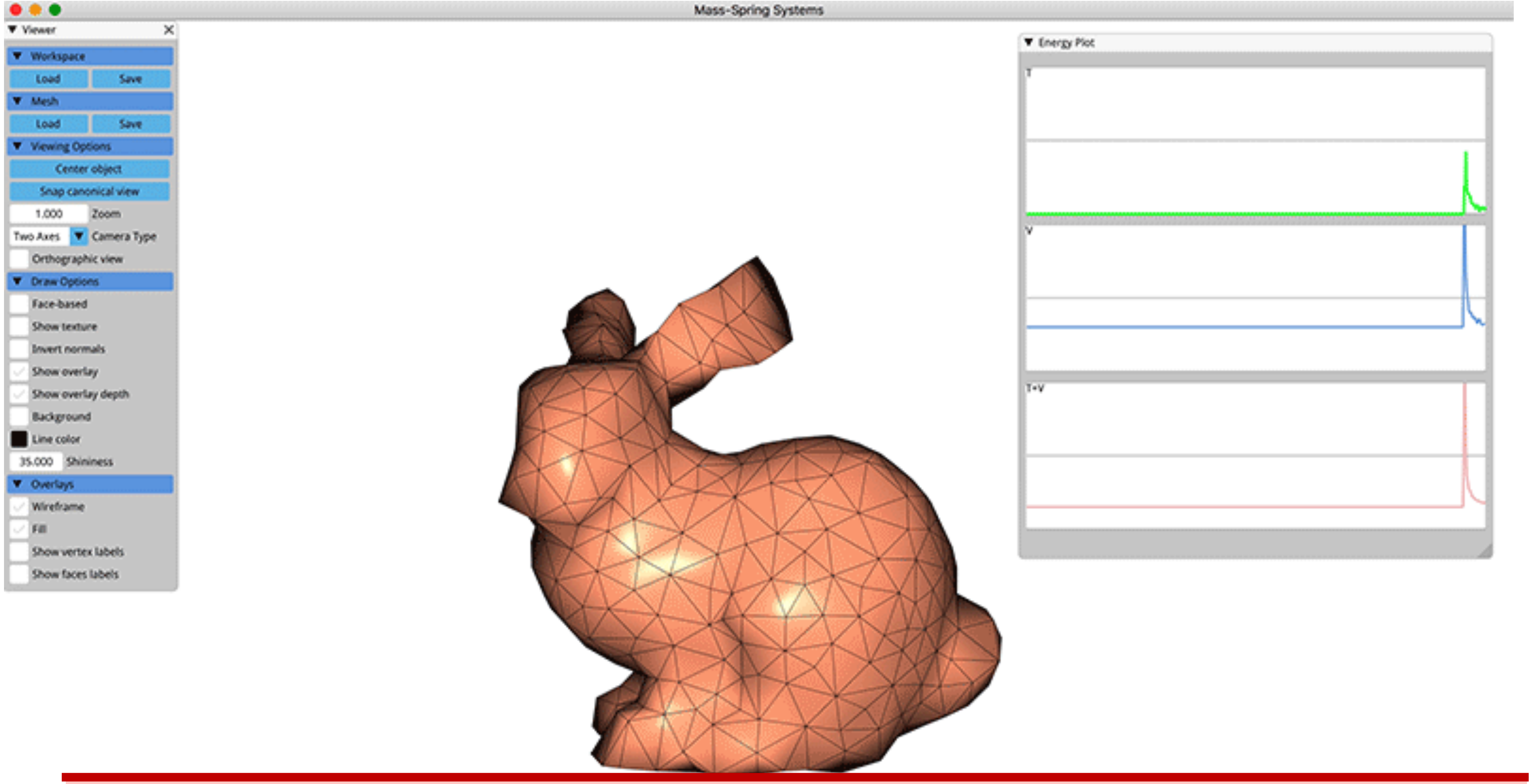
$$\begin{pmatrix} \mathrm{K}_{00} & \cdots & \mathrm{K}_{0n} \\ \vdots & \ddots & \vdots \\ \mathrm{K}_{n0} & \cdots & \mathrm{K}_{nn} \end{pmatrix}$$

$$\mathrm{K}_j = \begin{pmatrix} \mathrm{K}_{AA} & \mathrm{K}_{AB} \\ \mathrm{K}_{AB}^T & \mathrm{K}_{BB} \end{pmatrix}$$

Linearly-Implicit Time Integration

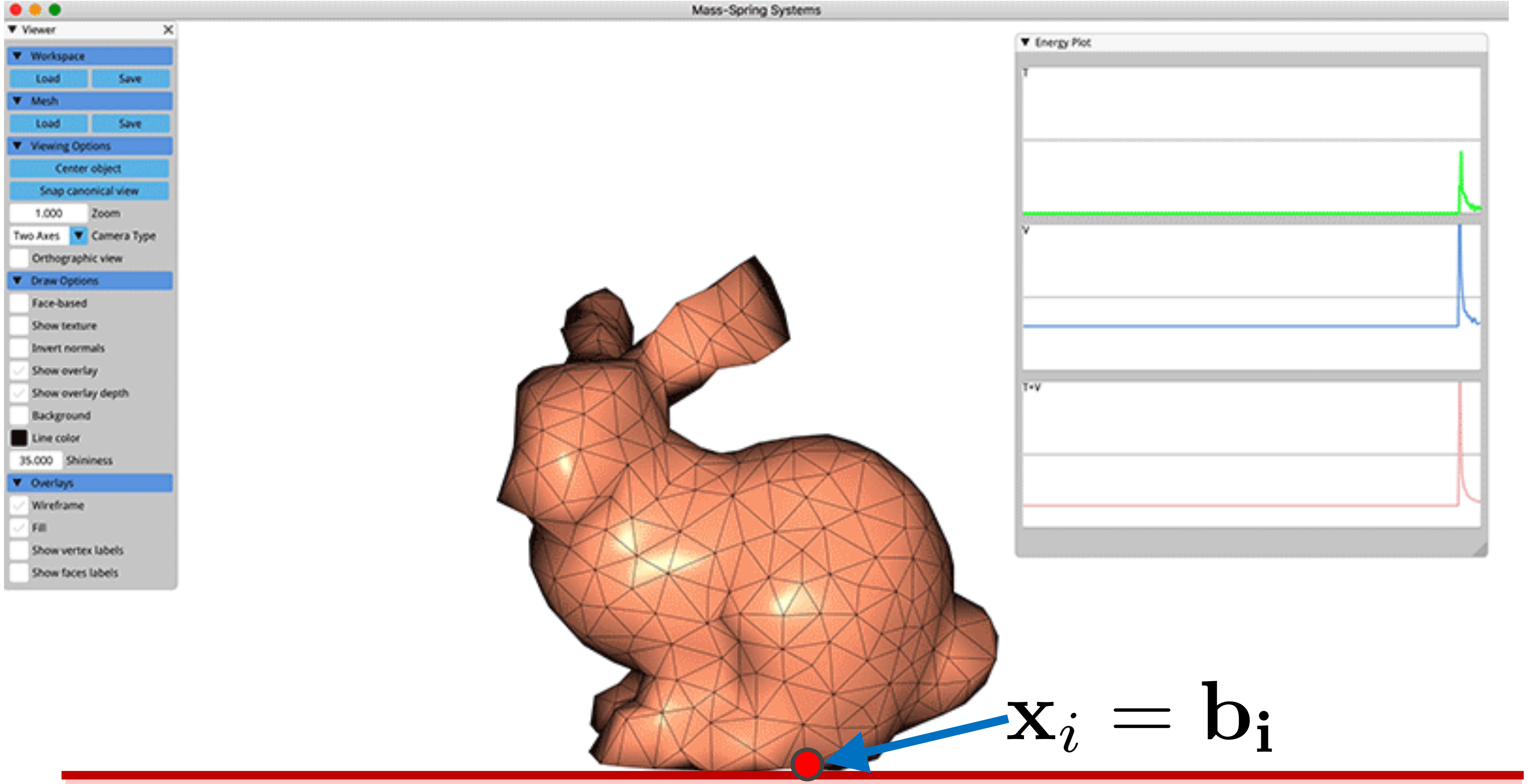


Fixed Boundary Conditions



Bunny is fixed to the floor

Fixed Boundary Conditions



$$\mathbf{x}_i = \mathbf{b_i}$$

Bunny is fixed to the floor

Fixed Boundary Conditions

$$\mathbf{q}_i = \mathbf{b_i}$$

degrees-of-freedom (DOF)

$$\hat{\mathbf{q}} = \mathrm{P}\mathbf{q}$$

selection matrix
that selects
NON-FIXED points

$$\mathrm{P}^T\hat{\mathbf{q}} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{0} \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Zero in fixed positions

Fixed Boundary Conditions

$$\mathbf{q}_i = \mathbf{b_i}$$

degrees-of-freedom (DOF)

$$\hat{\mathbf{q}} = \mathrm{P}\mathbf{q}$$

selection matrix
that selects
NON-FIXED points

$$\mathrm{P}^T\hat{\mathbf{q}} = \underbrace{\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{0} \\ \vdots \\ \mathbf{x}_n \end{pmatrix} + \underbrace{\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{b}_i \\ \vdots \\ \mathbf{0} \end{pmatrix}}_{\mathbf{b}}}_{\mathbf{q}}$$

Fixed Boundary Conditions

$$\mathbf{q} = \mathrm{P}^T \hat{\mathbf{q}} + \mathbf{b}$$

$$\dot{\mathbf{q}} = \mathrm{P}^T \dot{\hat{\mathbf{q}}} + \cancel{\dot{\mathbf{b}}}$$

Substitute into discrete update

$$\left(\mathrm{M} - \Delta t^2 \mathrm{K}\right) \dot{\mathbf{q}}^{t+1} = M \dot{\mathbf{q}}^t + \Delta t \mathbf{f}\left(\mathbf{q}^t\right)$$

$$\mathbf{q}^{t+1} = \mathbf{q}^t + \Delta t \dot{\mathbf{q}}^{t+1}$$

Fixed Boundary Conditions

$$\mathbf{q} = \mathrm{P}^T \hat{\mathbf{q}} + \mathbf{b}$$

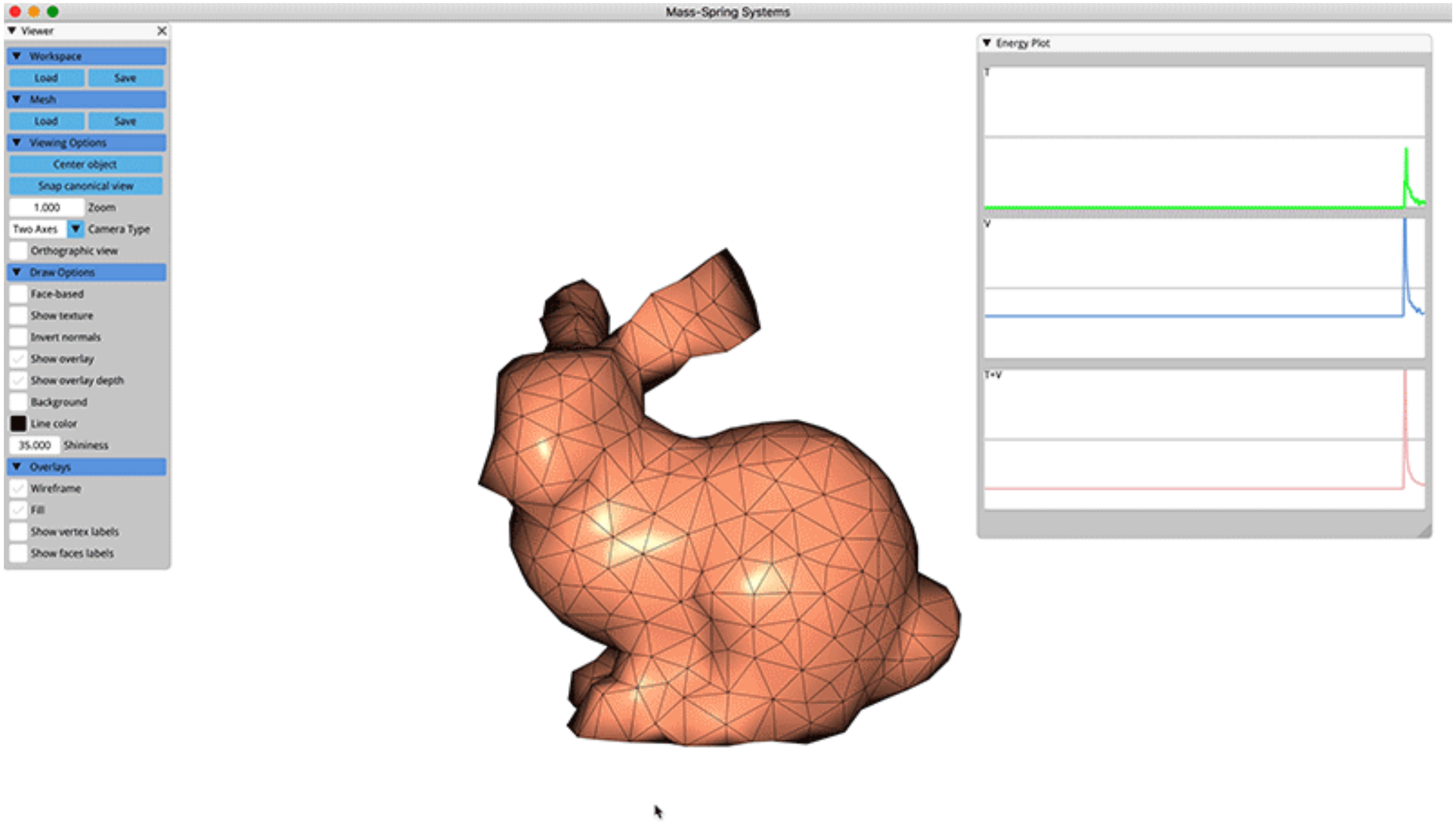
$$\dot{\mathbf{q}} = \mathrm{P}^T \dot{\hat{\mathbf{q}}} \cancel{+ \dot{\mathbf{b}}}$$

Substitute into discrete update

$$\mathrm{P}\left(\mathrm{M} - \Delta t^2 \mathrm{K}\right)\mathrm{P}^T \dot{\hat{\mathbf{q}}}^{t+1} = \mathrm{P} M \dot{\mathbf{q}}^t + \Delta t \mathrm{P} \mathbf{f}\left(\mathbf{q}^t\right)$$

$$\mathbf{q}^{t+1} = \mathbf{q}^t + \Delta t \mathrm{P}^T \dot{\hat{\mathbf{q}}}^{t+1}$$

Linearly-Implicit Time Integration



Next Week: