

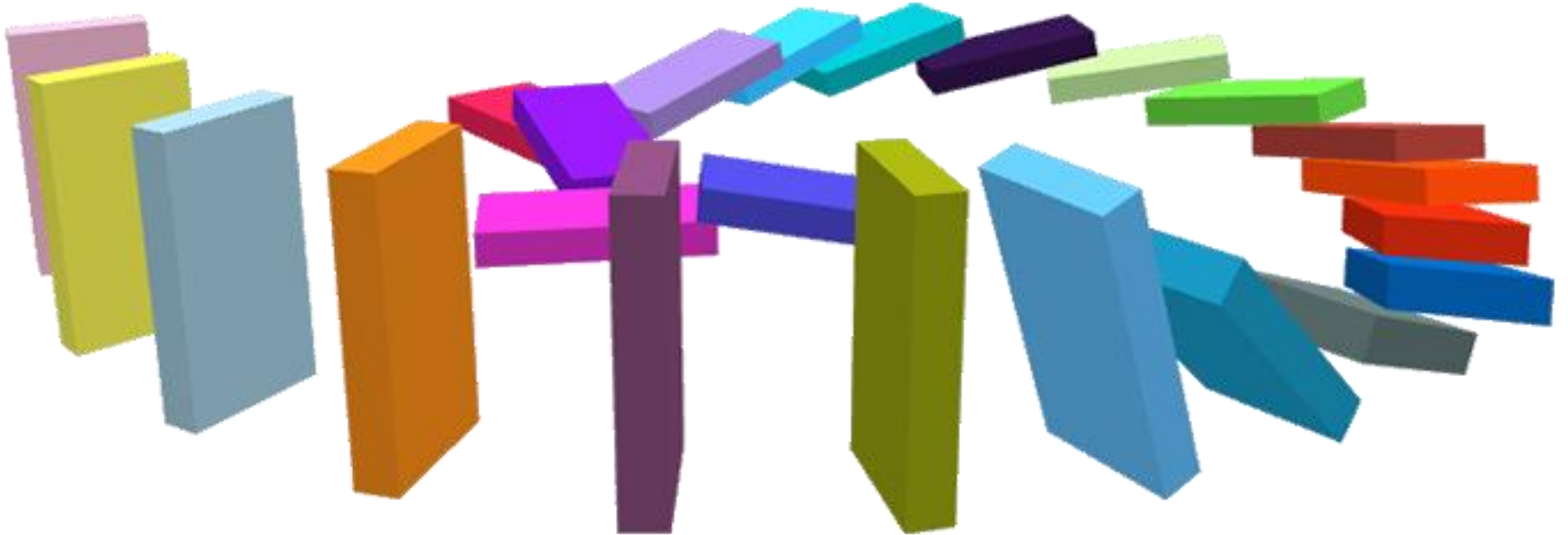
An aerial view of a city, likely King's Landing, in a state of complete destruction. A large, ornate building with a prominent dome is the central focus, surrounded by a vast field of rubble and debris. In the background, a body of water and distant mountains are visible under a cloudy sky. The scene is chaotic, with many small pieces of debris floating in the air, suggesting a recent explosion or battle.

CSC417 Physics-Based Animation ... starting at 11:10 am

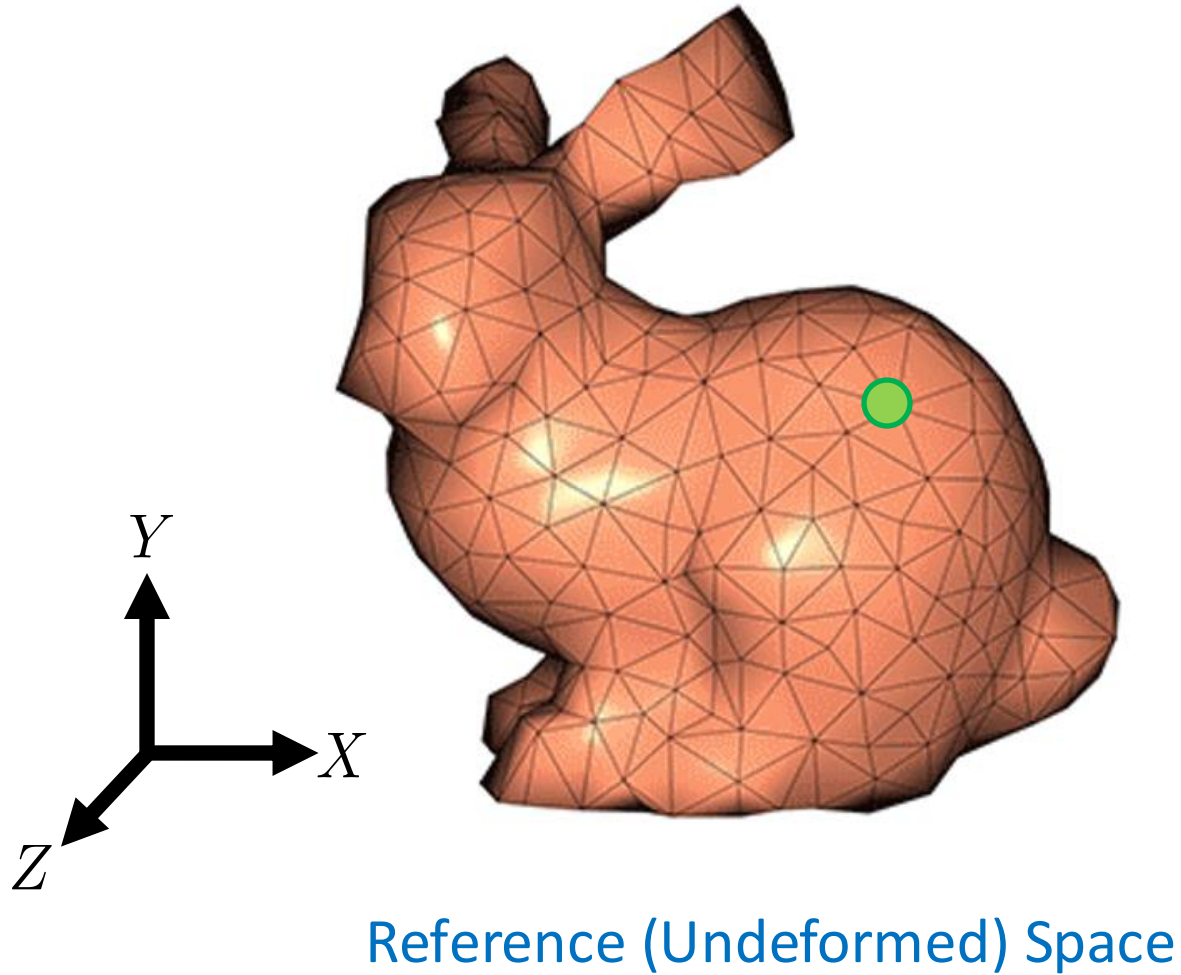
This Video: Rigid Body Simulation with Contact



What Makes an Object Rigid ?



Affine Body Dynamice



$$\mathbf{x}(\mathbf{X}, t) = \mathbf{J}(\mathbf{X})\mathbf{q}(t)$$



Solve using Optimization via Newton's Method

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M(\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

Questions from Previous Lecture ?

Optimization Problem for a single object

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

An aerial view of a city in ruins. A large, ornate building with a dark, domed roof is the central focus, surrounded by a vast field of rubble and debris. In the background, a body of water and distant mountains are visible under a cloudy sky. The scene depicts a scene of total destruction.

What about lots of objects ?

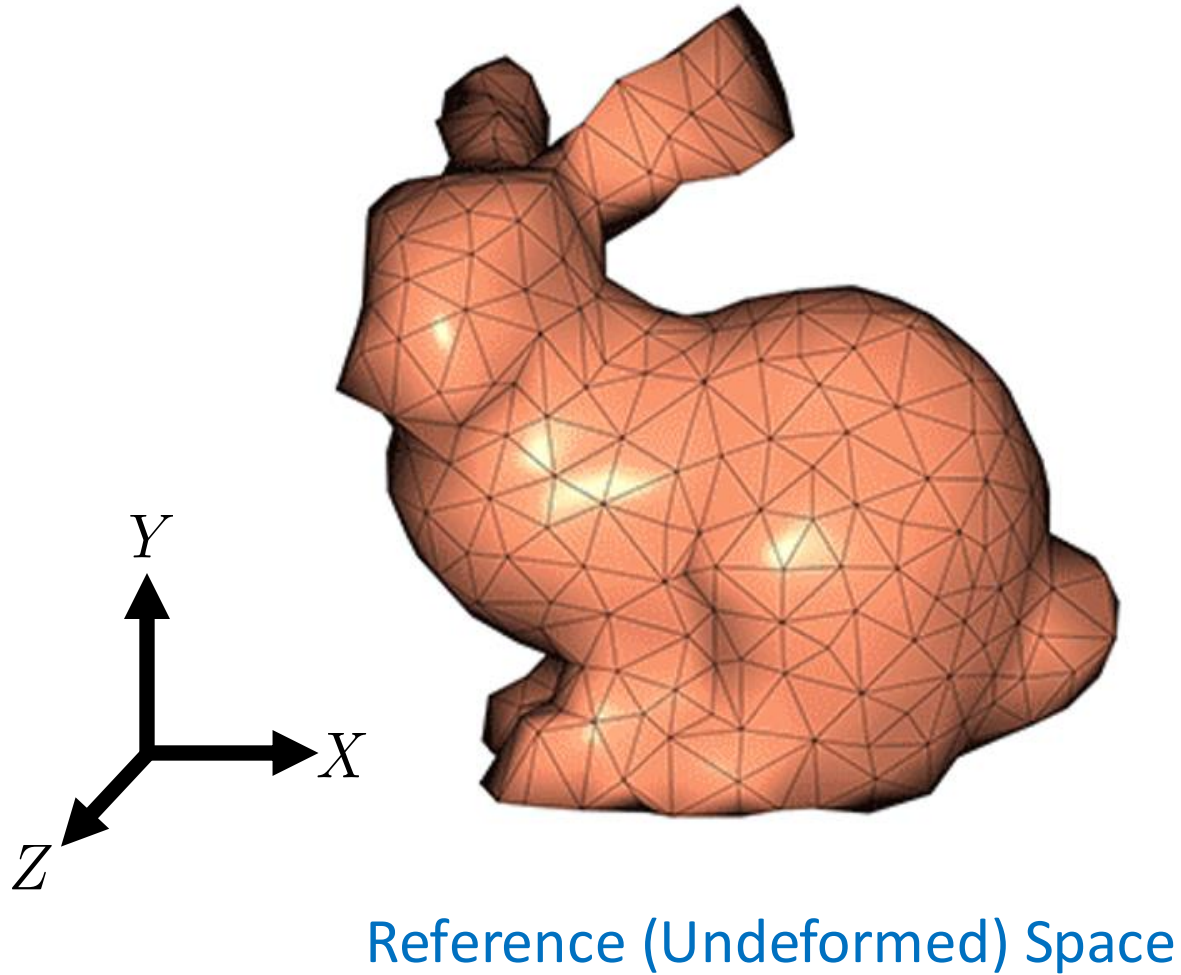
Two Problems with Our Current Approach

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

Problem 1: Solving this optimization problem only moves one object !!!

Problem 2: There's no term in this optimization that tells it how to handle collisions

Kinetic Energy of an Affine Body



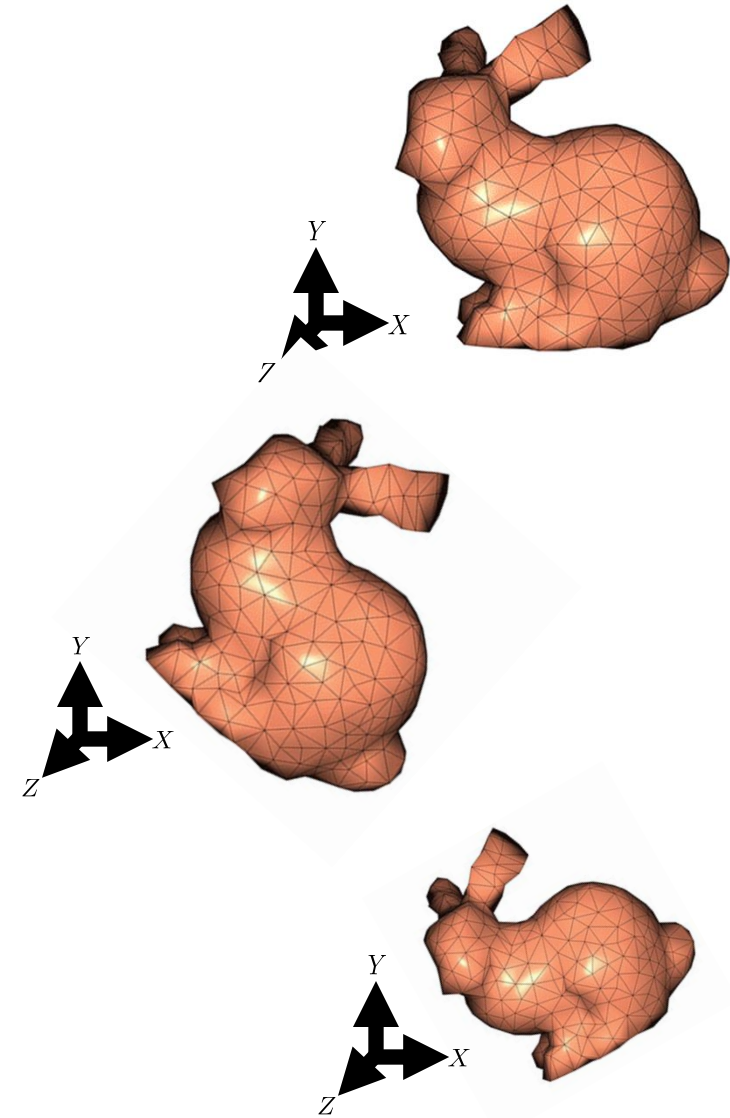
$$\frac{1}{2} \int_{\Omega} \rho \| \mathbf{v}(\mathbf{X}) \|^2 d\Omega$$

entire rigid body

infinitesimal volume

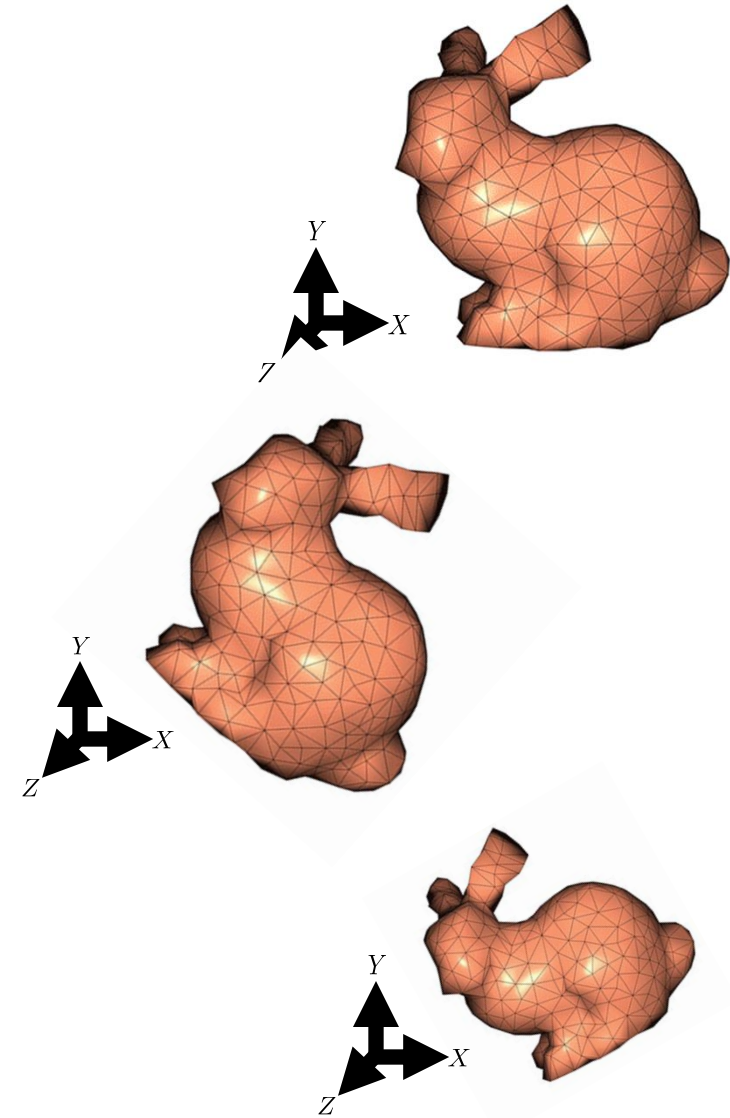


Kinetic Energy of many Affine Bodies



Reference (Undeformed) Spaces

Kinetic Energy of many Affine Bodies

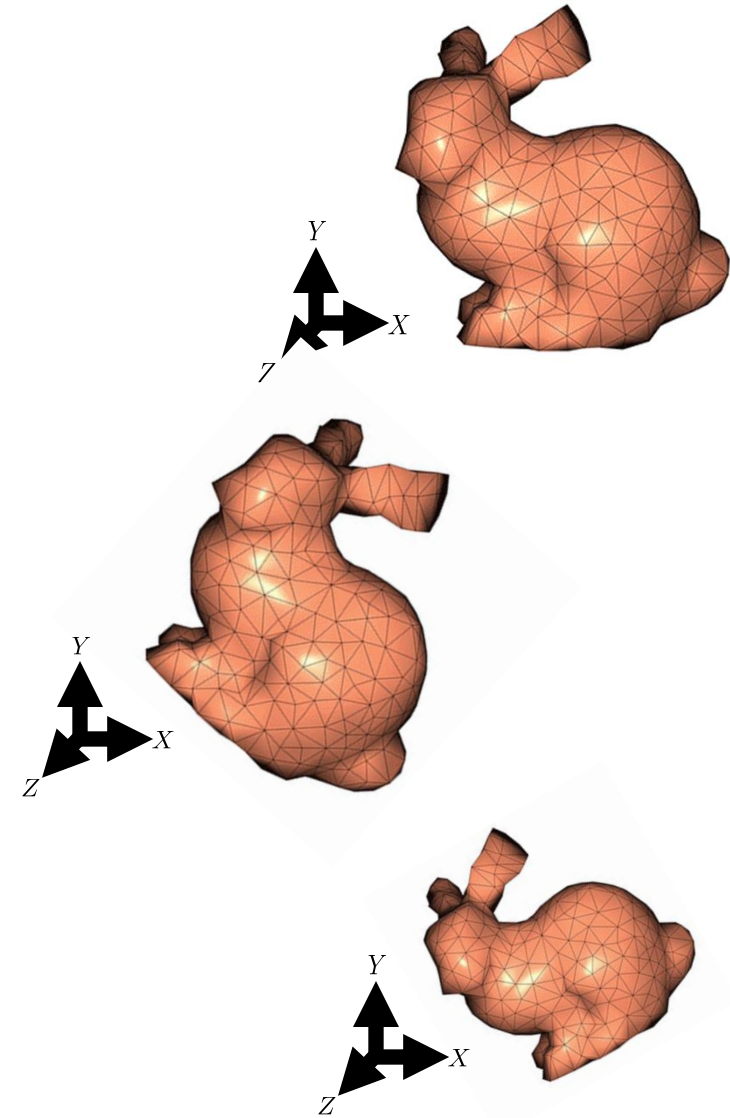


Number of Objects

$$\sum_{i=0}^N \frac{1}{2} \int_{\Omega_i} \rho_i ||\mathbf{v}_i(\mathbf{X})|| d\Omega_i$$

Reference (Undeformed) Spaces

Kinetic Energy of many Affine Bodies

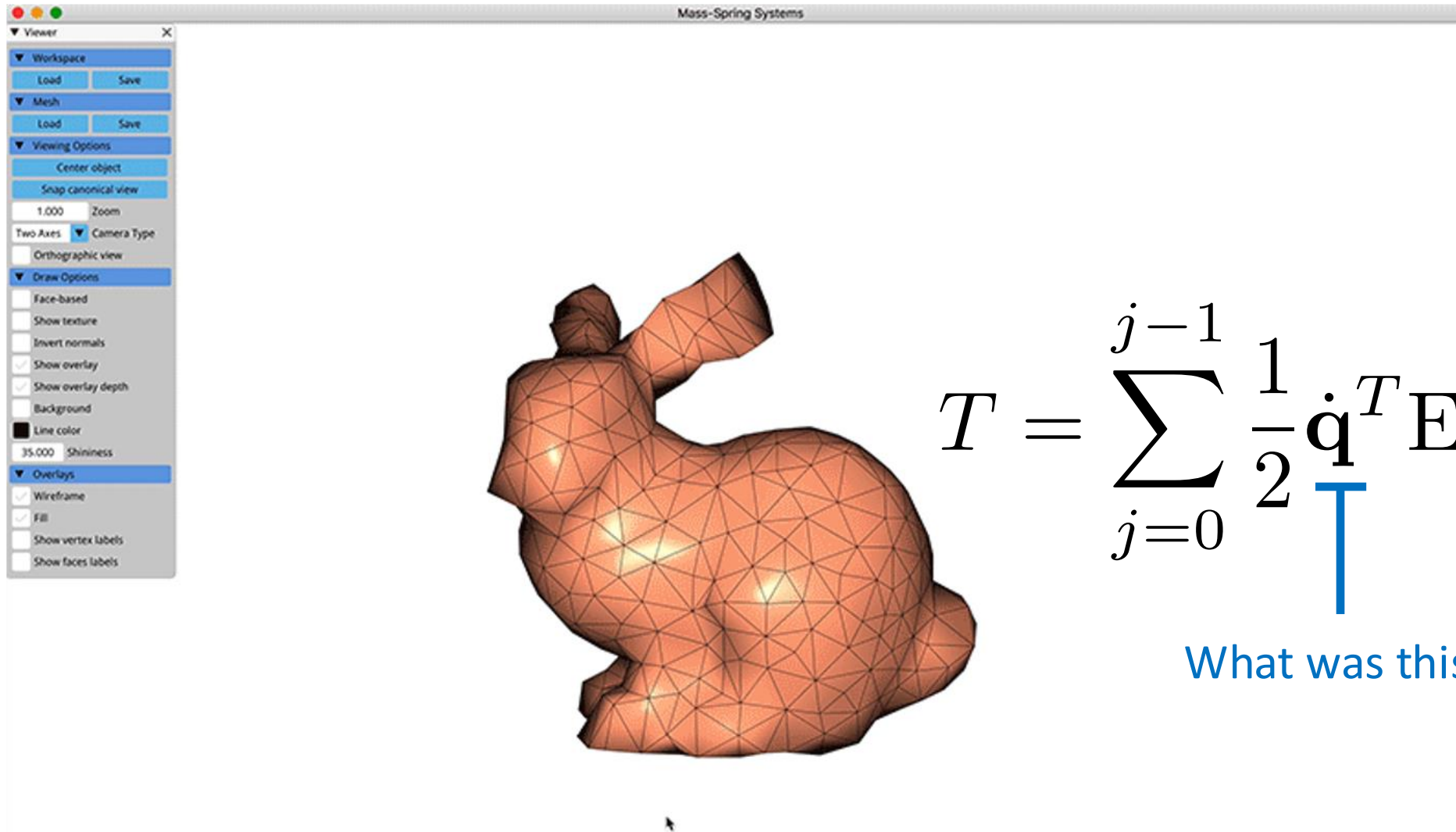


Number of Objects

$$\sum_{i=0}^N \frac{1}{2} \dot{\mathbf{q}}_i^T \mathbf{M}_i \dot{\mathbf{q}}_i$$

Reference (Undeformed) Spaces

Kinetic Energy for a Bunny using FEM

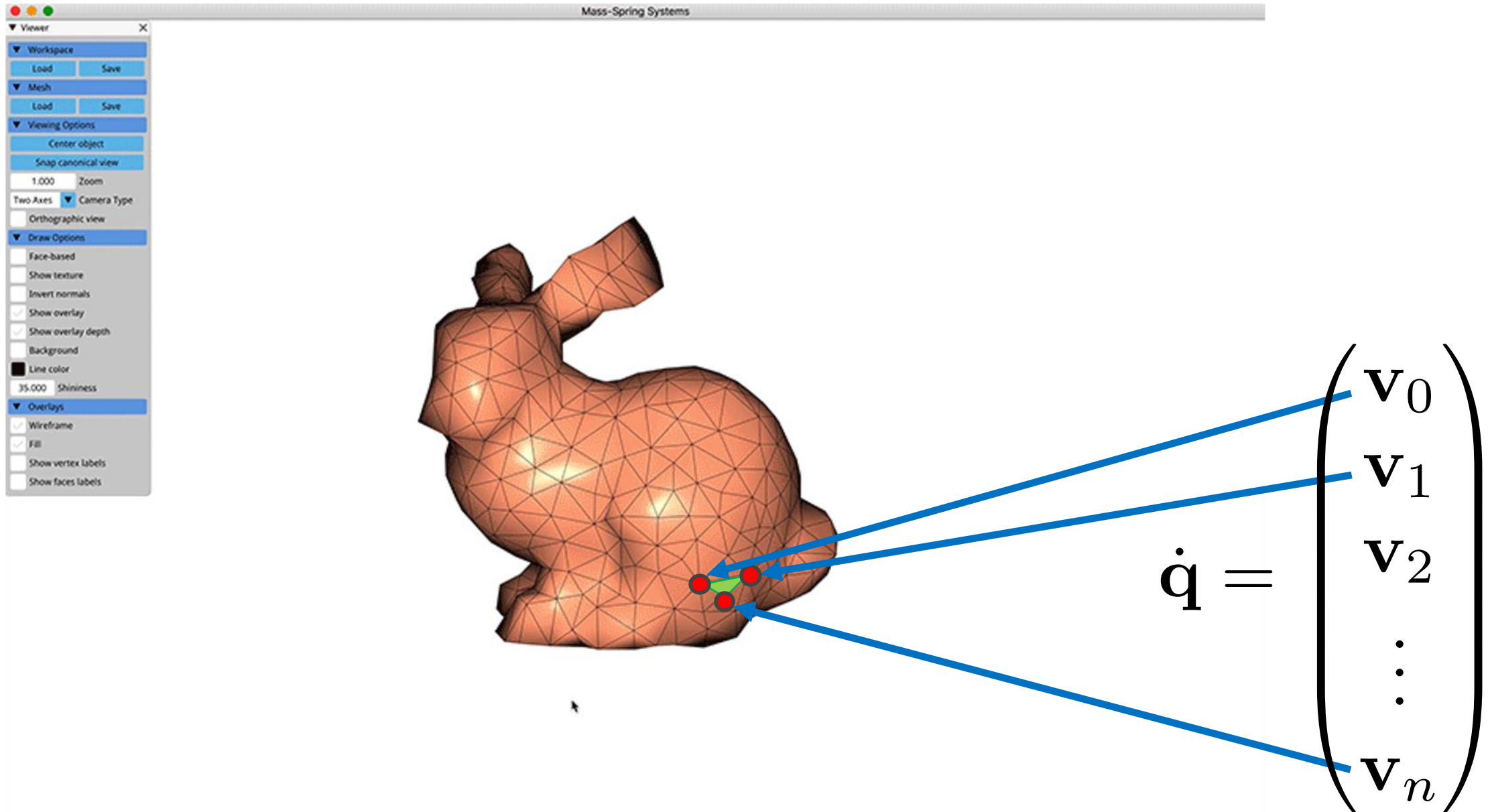


$$T = \sum_{j=0}^{j-1} \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{E}_j^T \mathbf{M}_j \mathbf{E}_j \dot{\mathbf{q}}$$

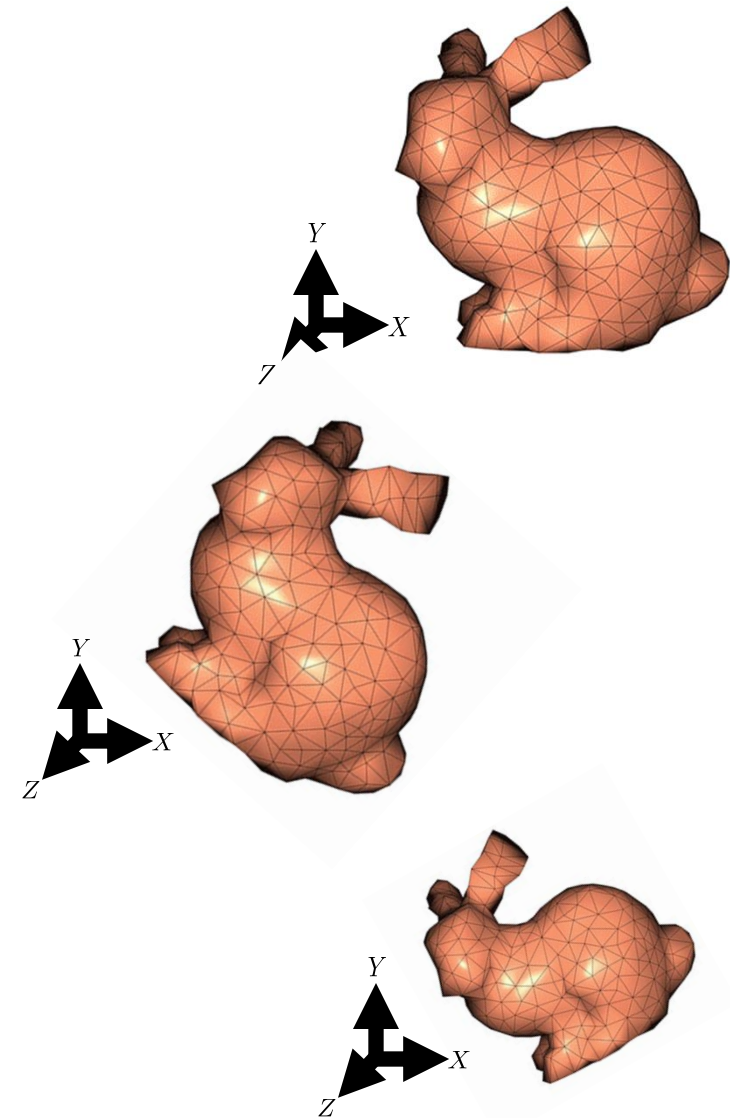
↑

What was this ?

Generalized Coordinates for Bunny FEM



Let's do the same thing



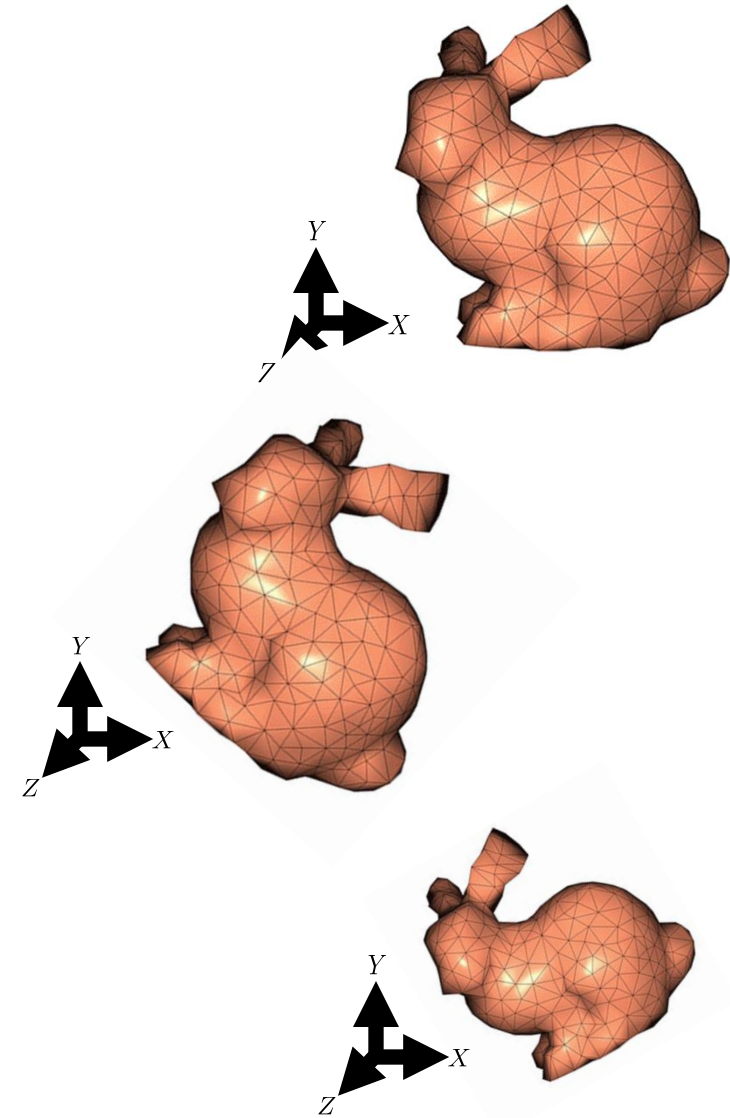
Number of Objects



$$\sum_{i=0}^N \frac{1}{2} \dot{\mathbf{q}}_i^T \mathbf{M}_i \dot{\mathbf{q}}_i$$

Reference (Undeformed) Spaces

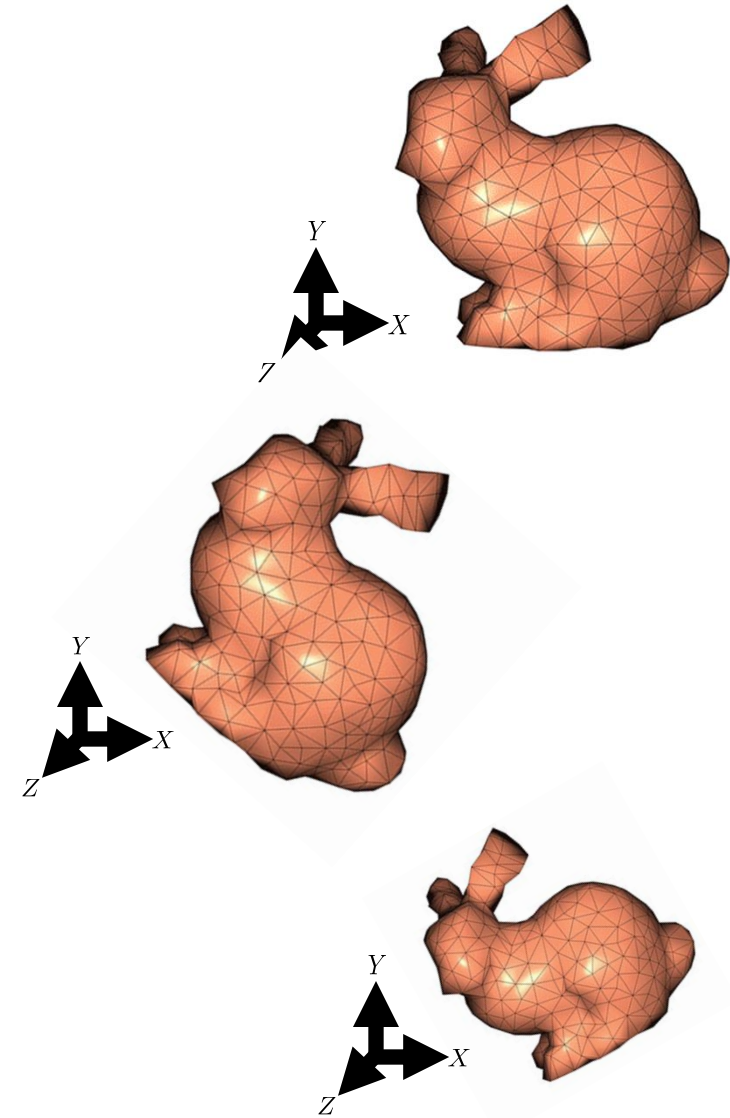
Generalized Velocity for MANY Affine Bodies



$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\mathbf{q}}_0 \\ \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix}$$

Reference (Undeformed) Spaces

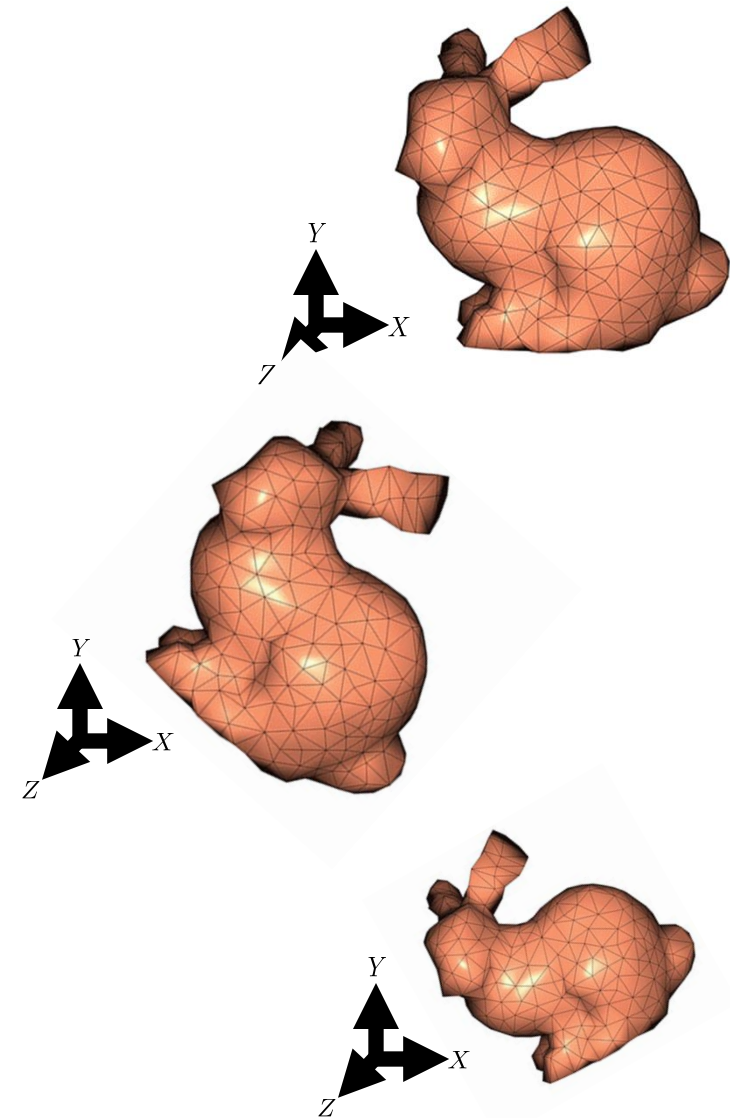
Generalized Coordinates for MANY Affine Bodies



$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}$$

Reference (Undeformed) Spaces

Let's do the same thing

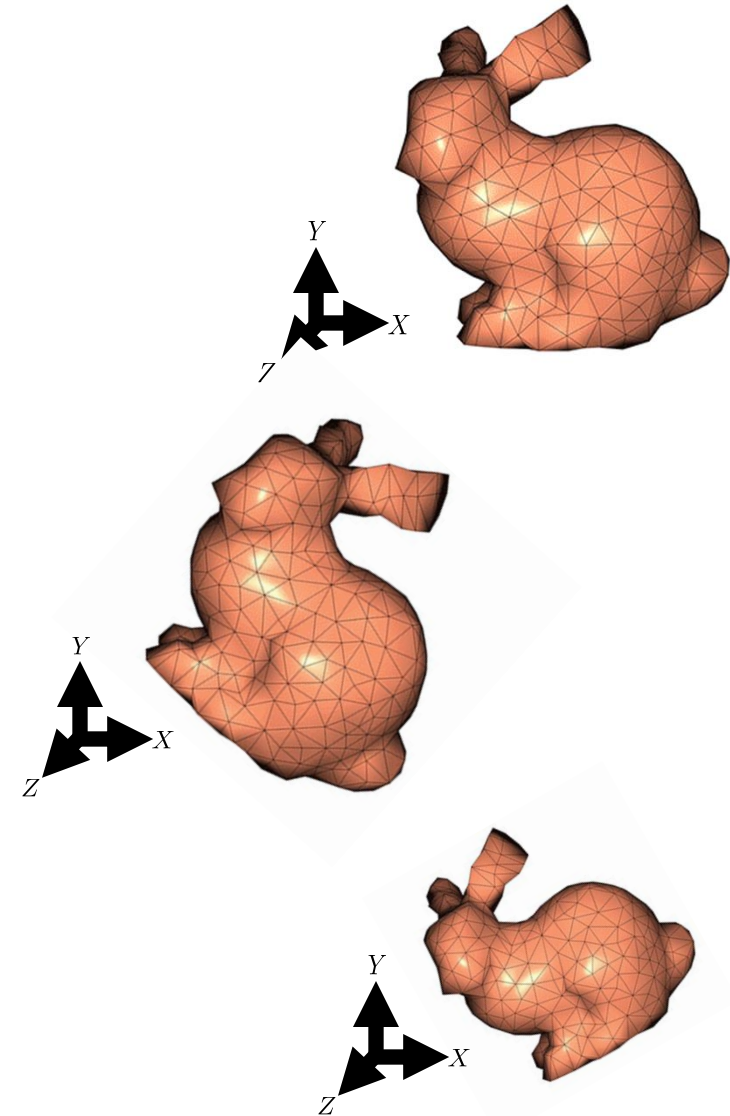


Number of Objects

$$\sum_{i=0}^N \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{E}_i^T \mathbf{M}_i \mathbf{E}_i \dot{\mathbf{q}}$$

Reference (Undeformed) Spaces

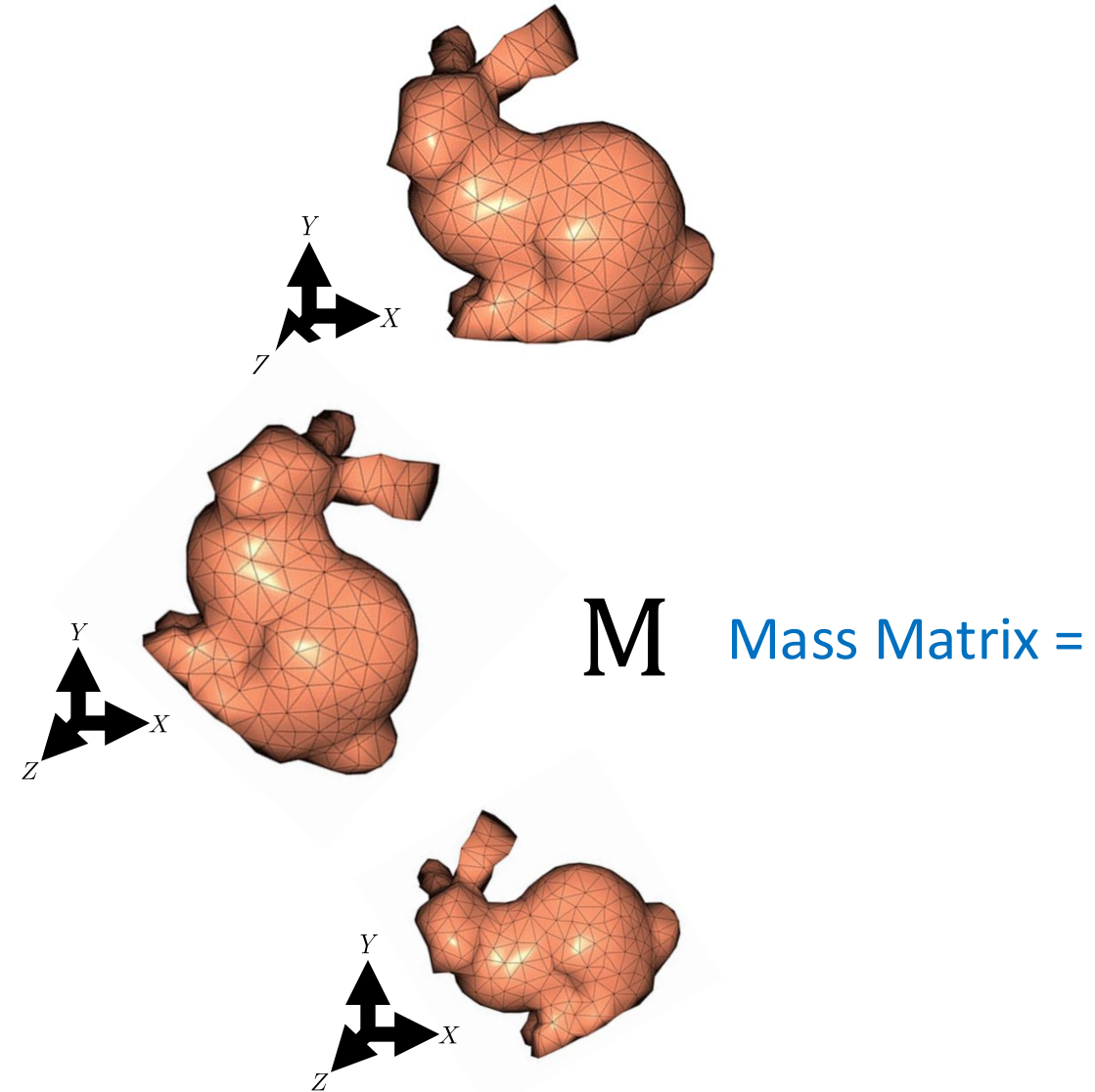
Mass Matrix for Affine Body System



Reference (Undeformed) Spaces

$$\frac{1}{2} \dot{\mathbf{q}}^T \left(\underbrace{\sum_{i=0}^N \mathbf{E}_i^T \mathbf{M}_i \mathbf{E}_i}_{\mathbf{M} \text{ Mass Matrix}} \right) \dot{\mathbf{q}}$$

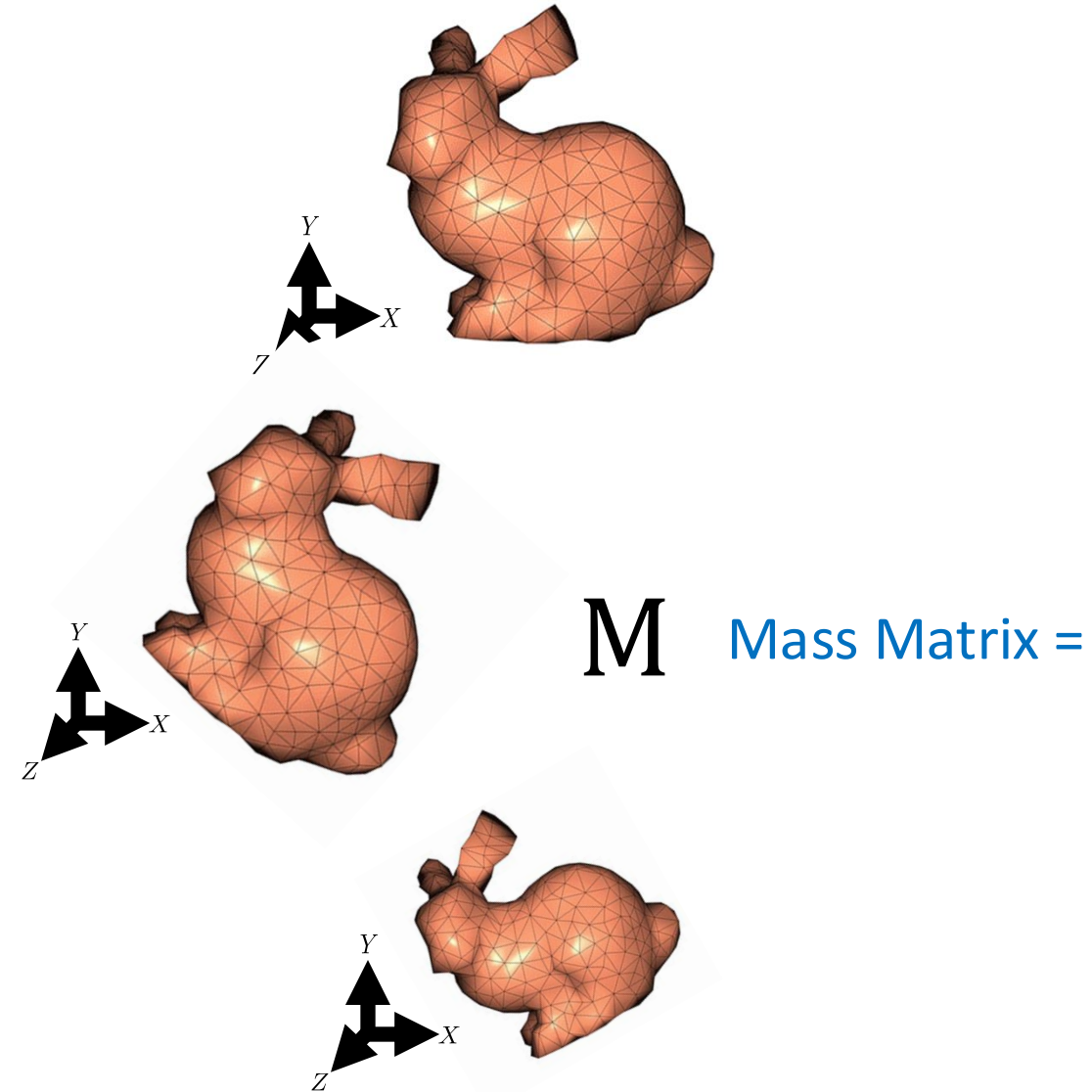
Block Structure of M ?



Reference (Undeformed) Spaces

	Object 1	Object 2	Object 3
Object 3	?	?	?
Object 2	?	?	?
Object 1	?	?	?

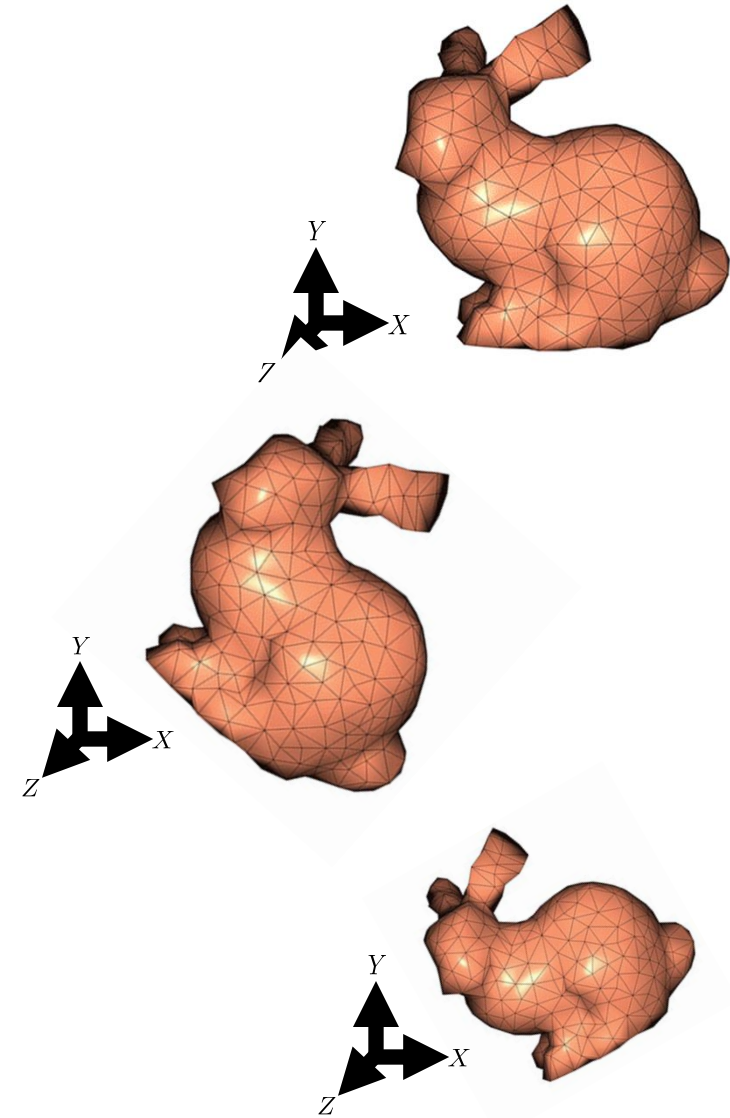
Block Structure of M ?



Reference (Undeformed) Spaces

	Object 1	Object 2	Object 3
Object 3	M_0	0	0
Object 2	0	M_1	0
Object 1	0	0	M_2

Potential Energy of Affine Body System



?

Reference (Undeformed) Spaces

Optimization Problem for a multi-object system

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

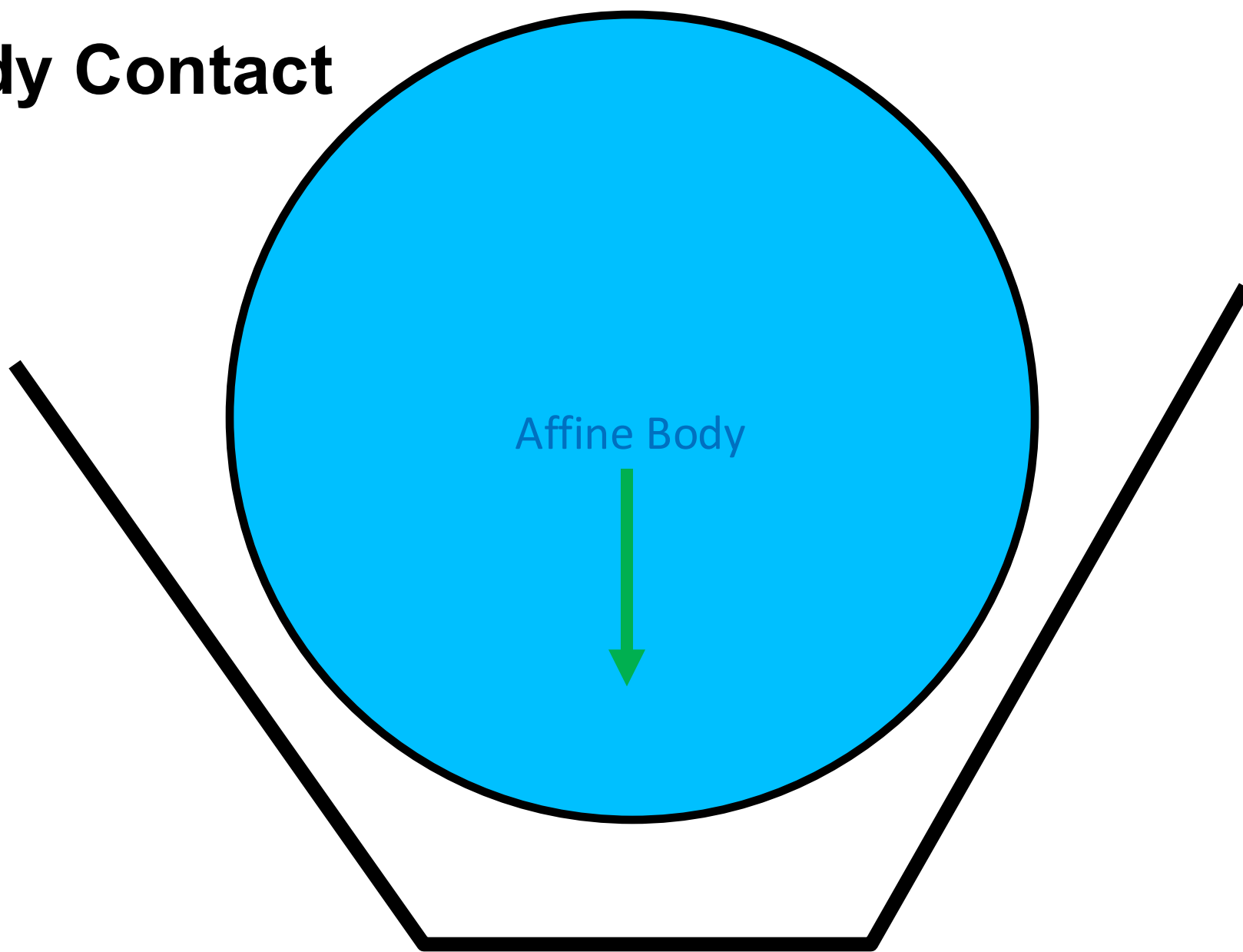
Two Problems with Our Current Approach

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

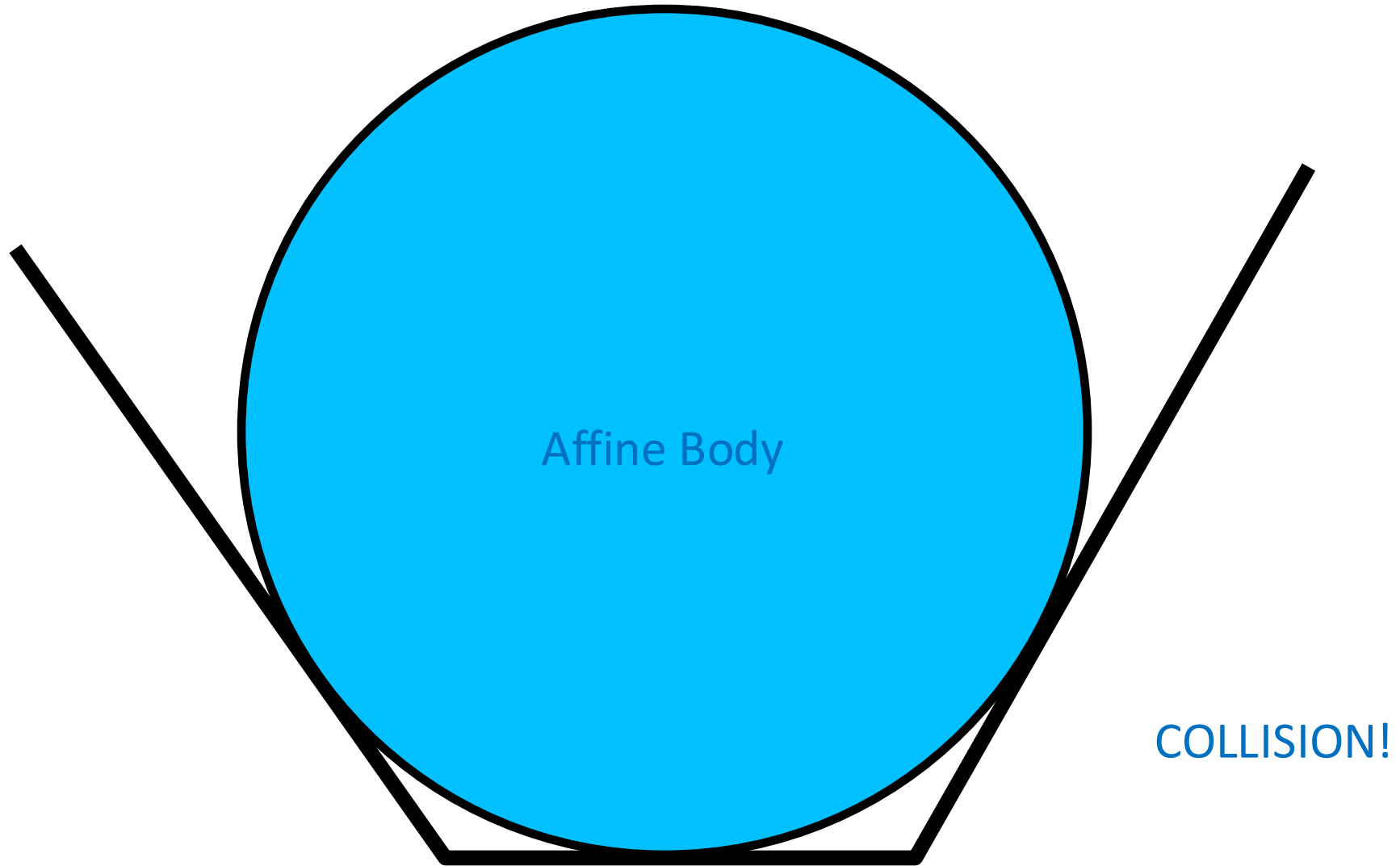
~~Problem 1: Solving this optimization problem only moves one object !!!~~

Problem 2: There's no term in this optimization that tells it how to handle collisions

Affine Body Contact



Affine Body Contact



Collisions in Simulation

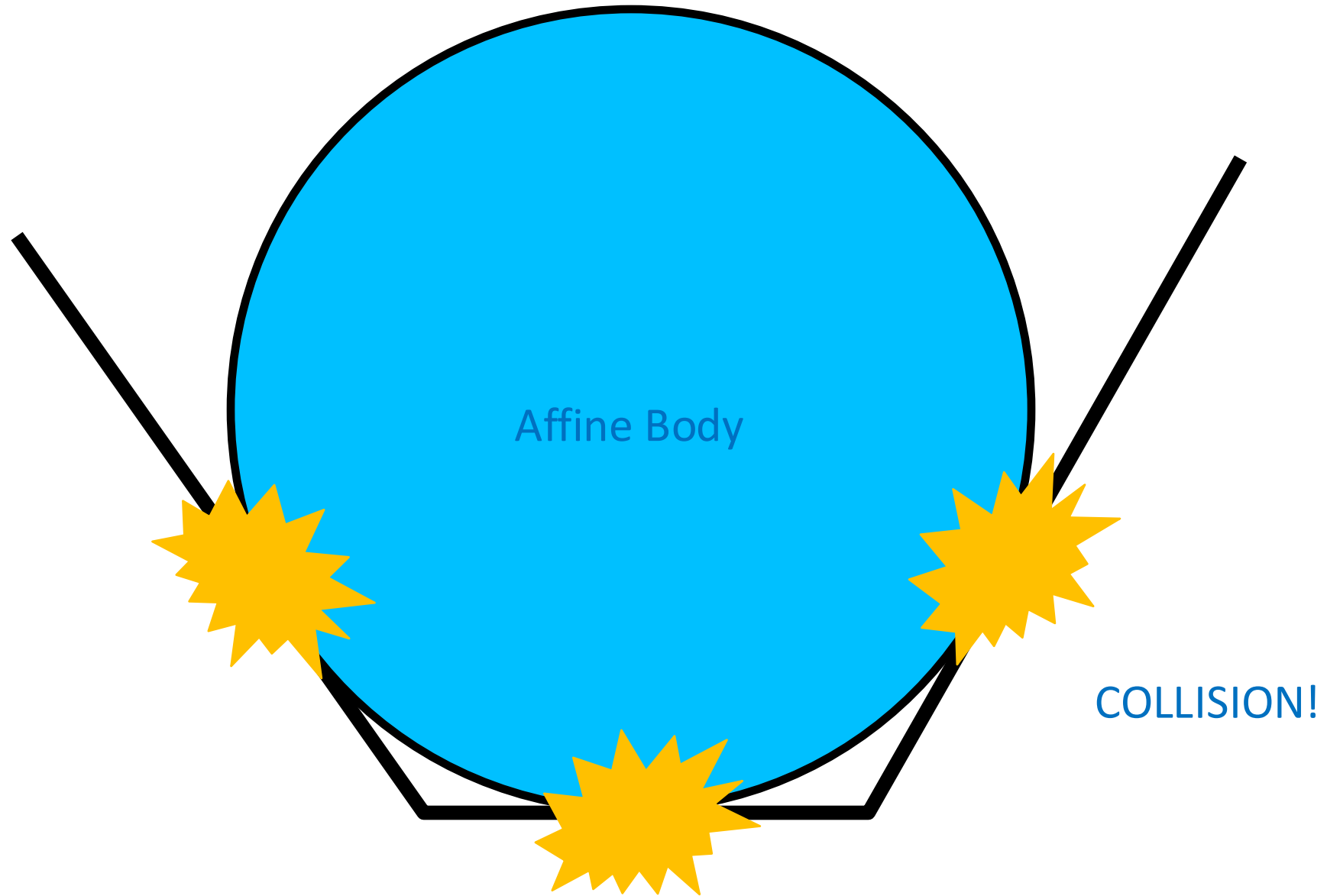
Two phases detection and response

Detection: Did I hit anything ?

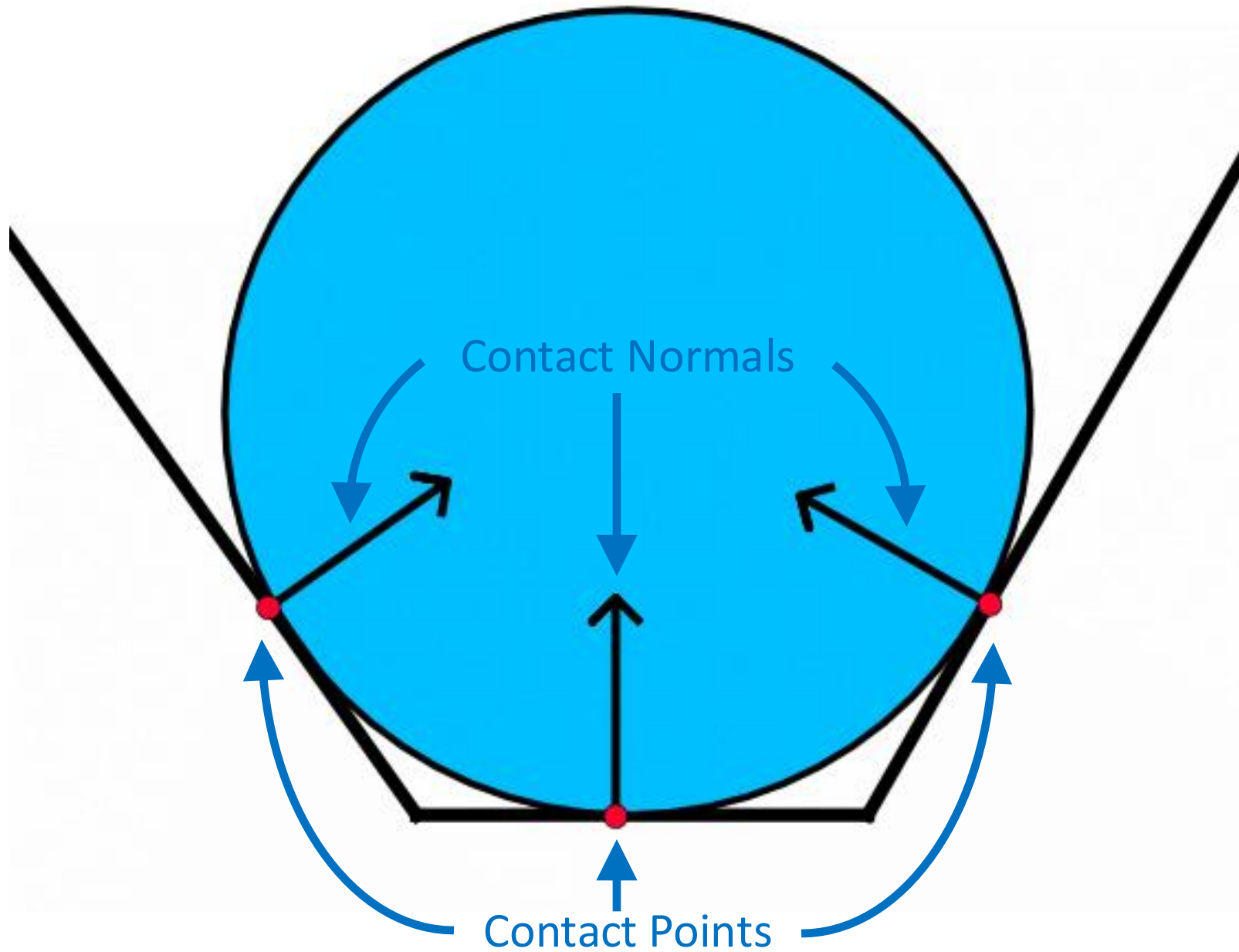
Response: I hit something ! What do I do ?



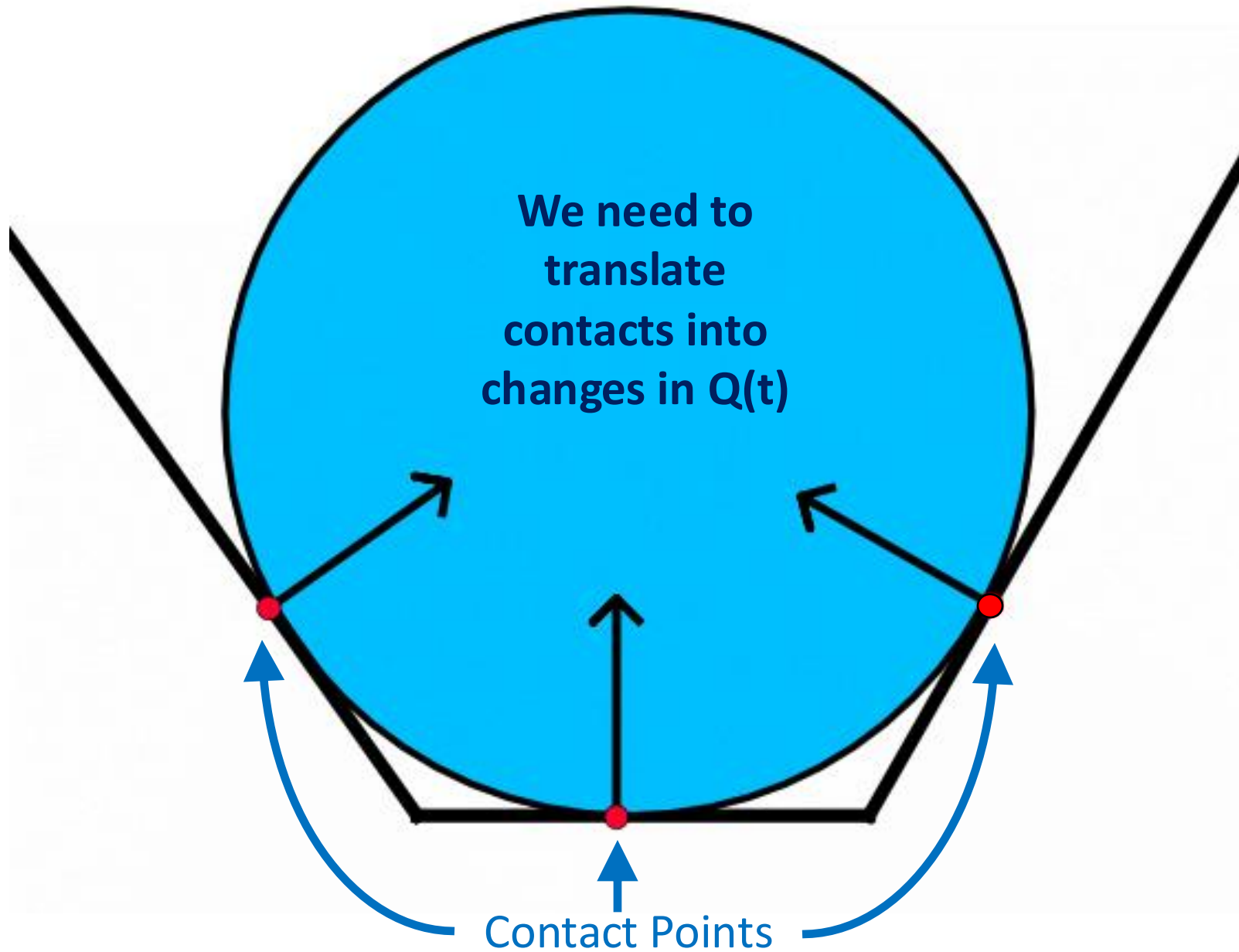
Affine Body Contact



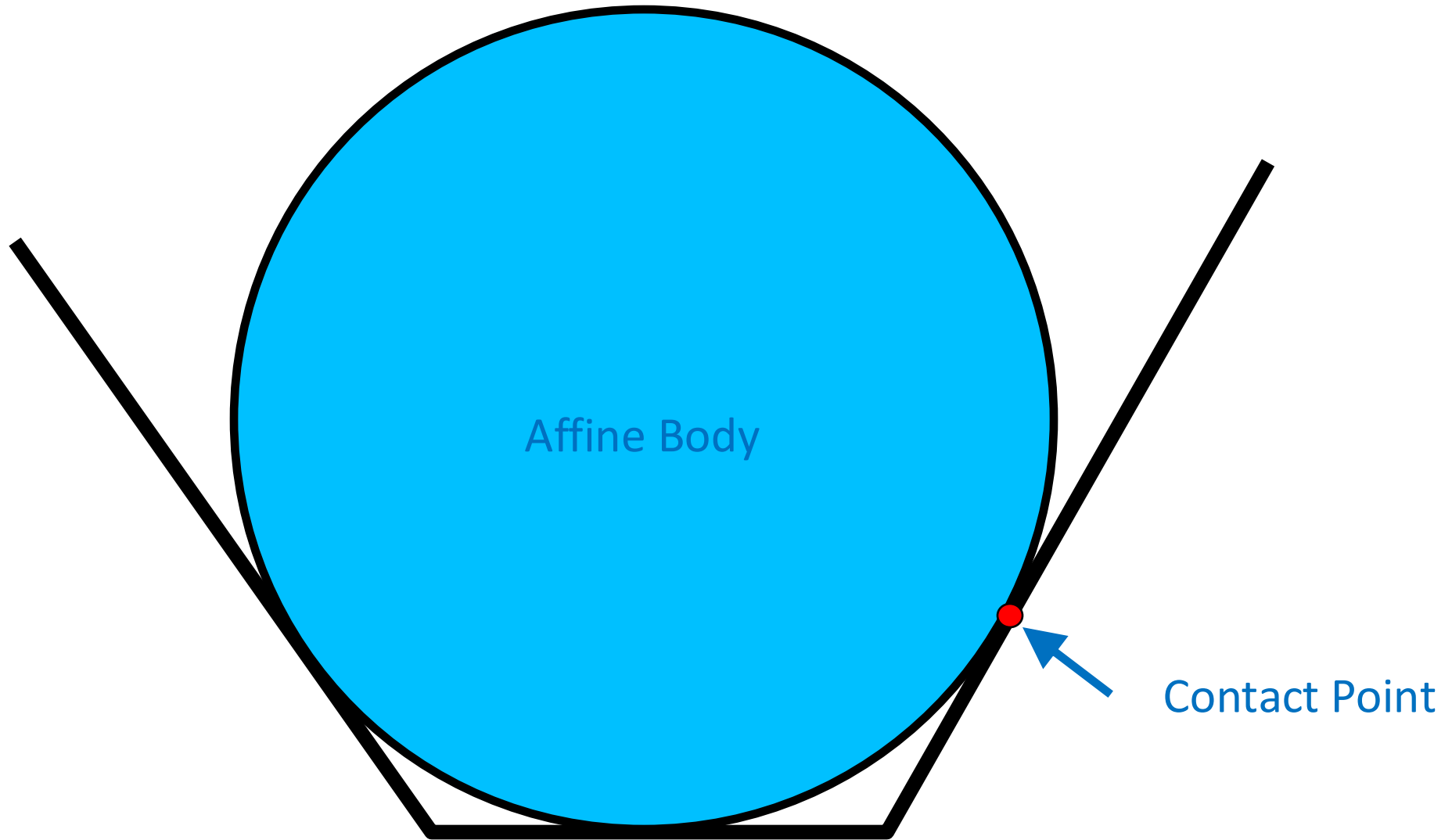
Affine Body Contact



Affine Body Contact

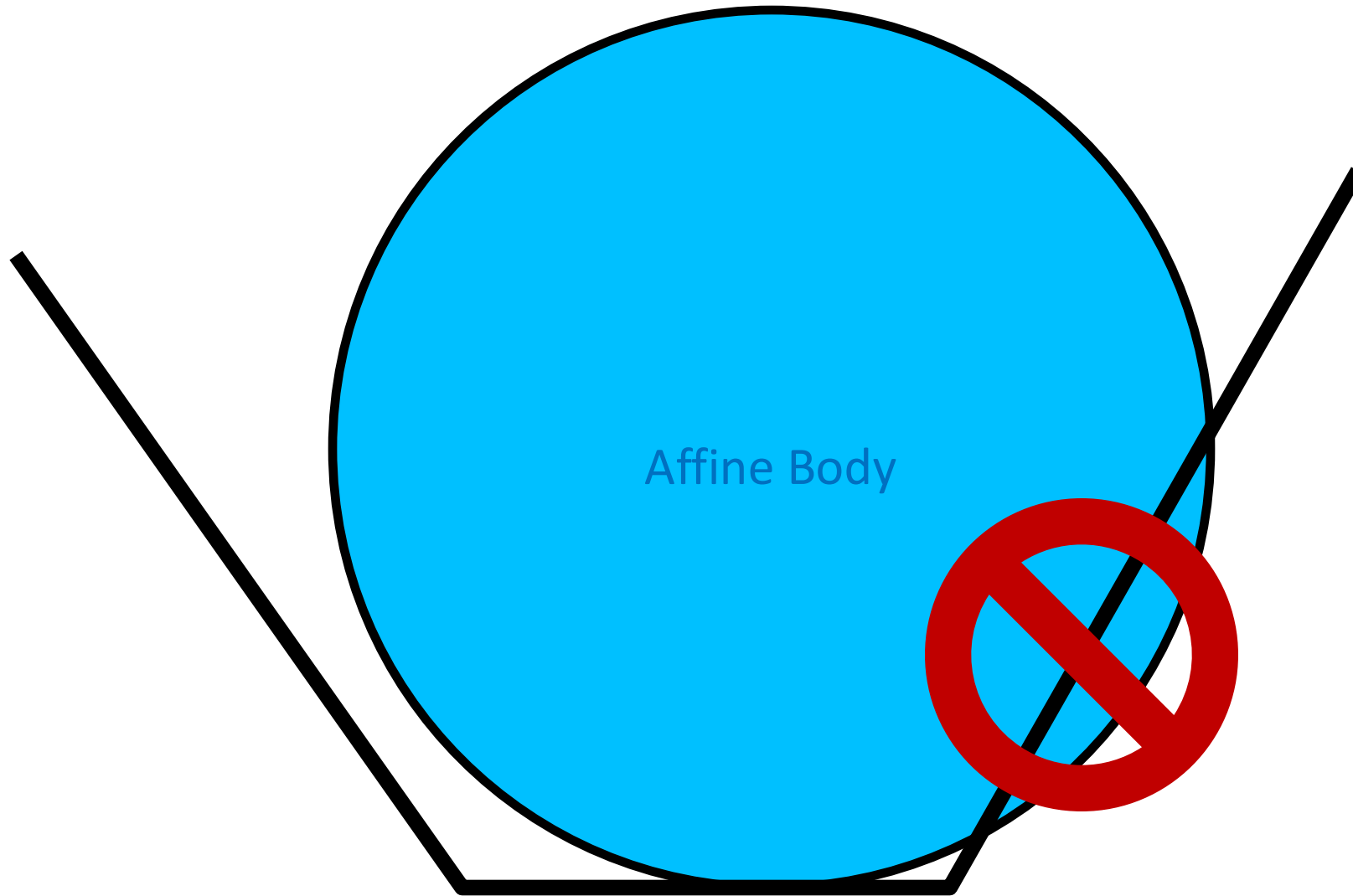


Three Rules of Contact Mechanics



Try to prevent interpenetration at contact point

Three Rules of Contact Mechanics



Try to prevent interpenetration at contact point



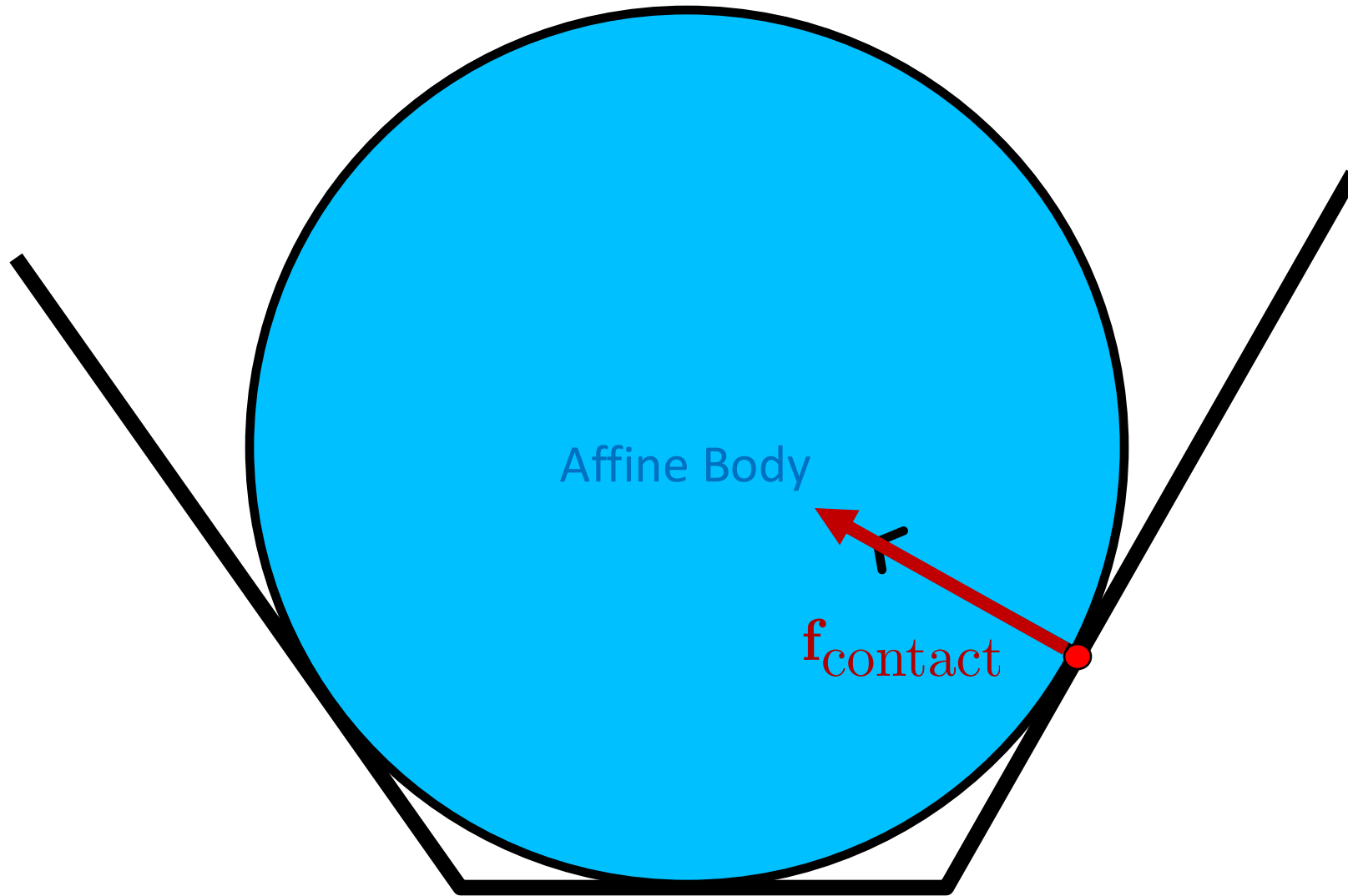
Three Rules of Contact Mechanics



Contact forces can only be applied when objects are in contact



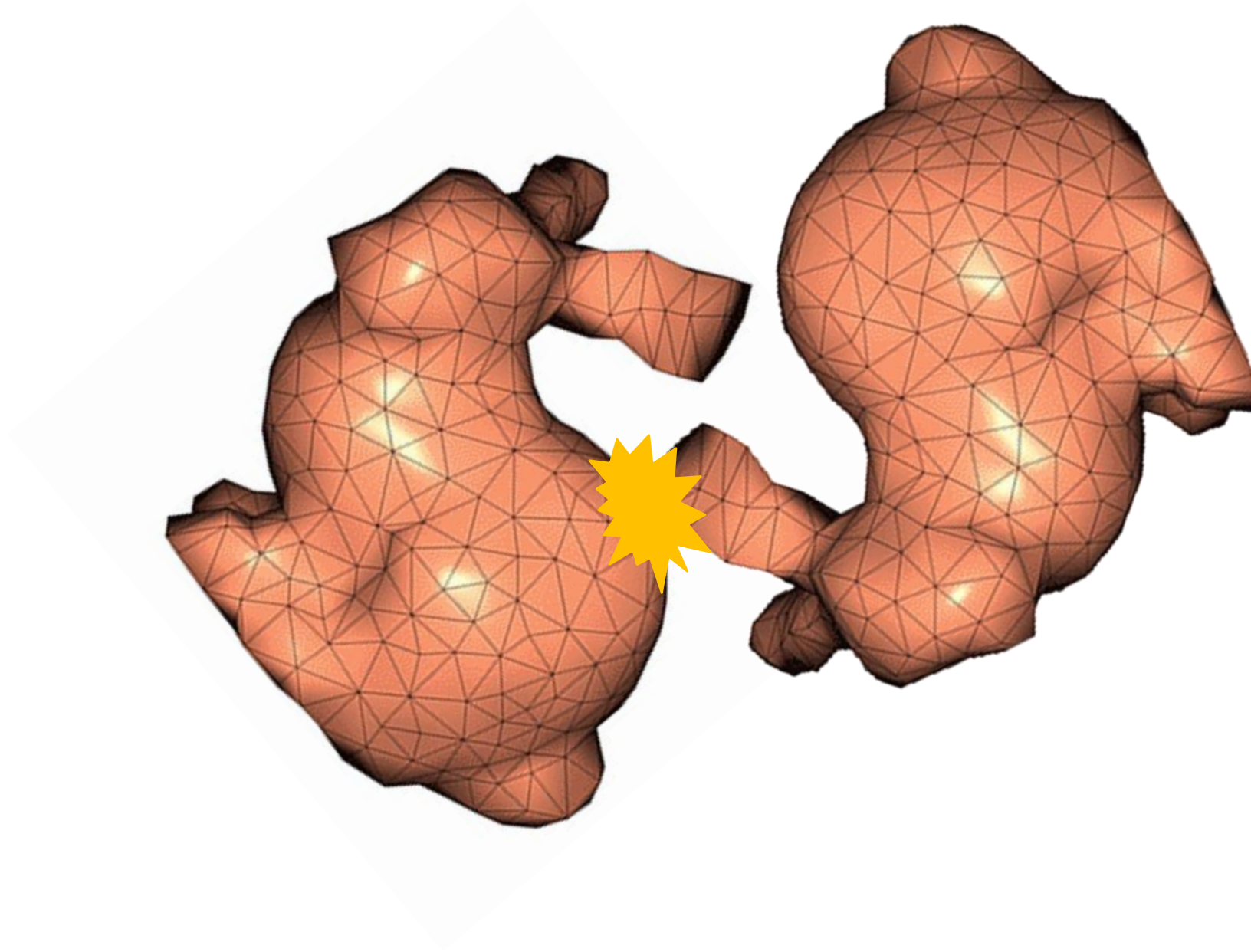
Three Rules of Contact Mechanics

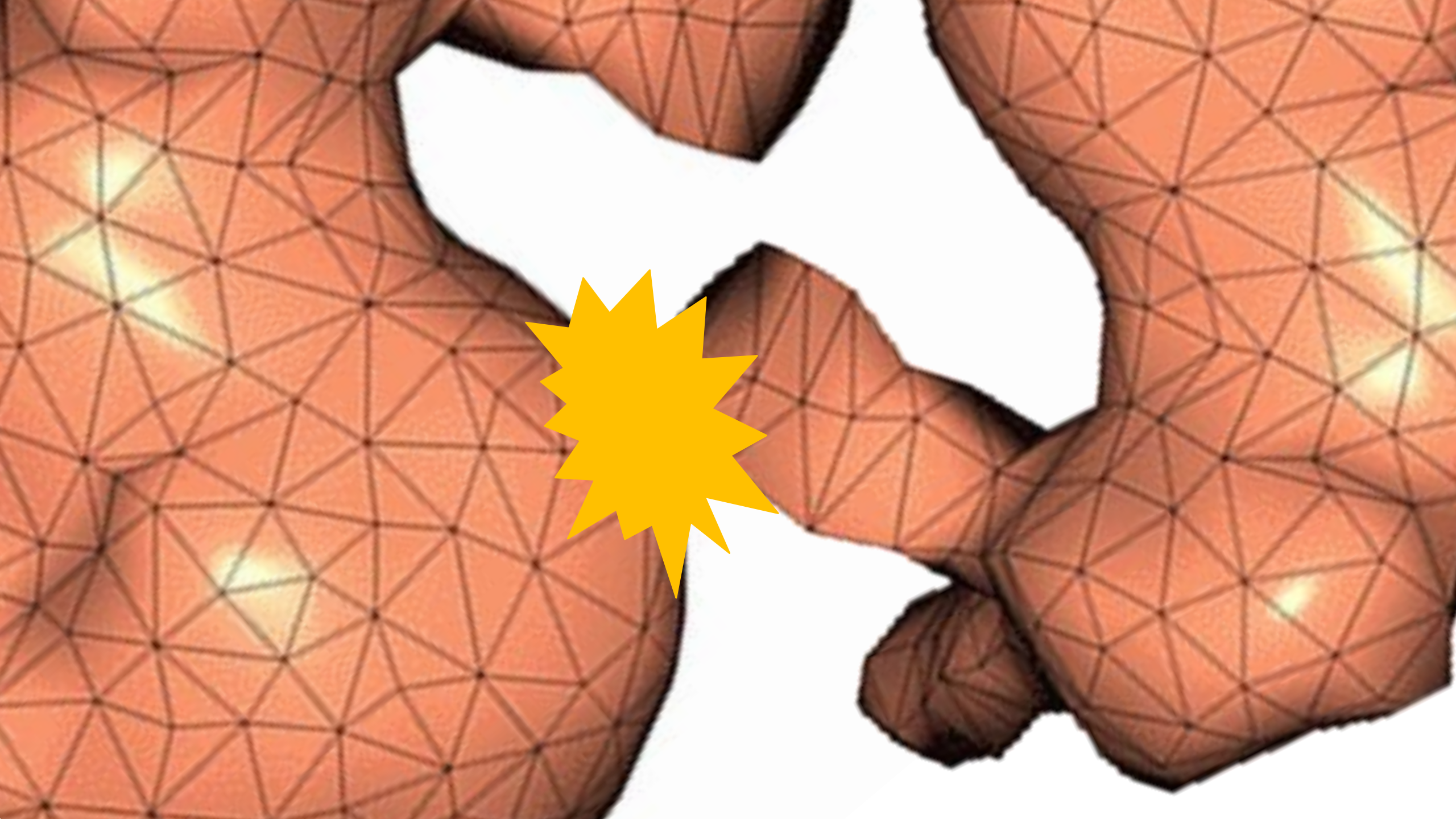


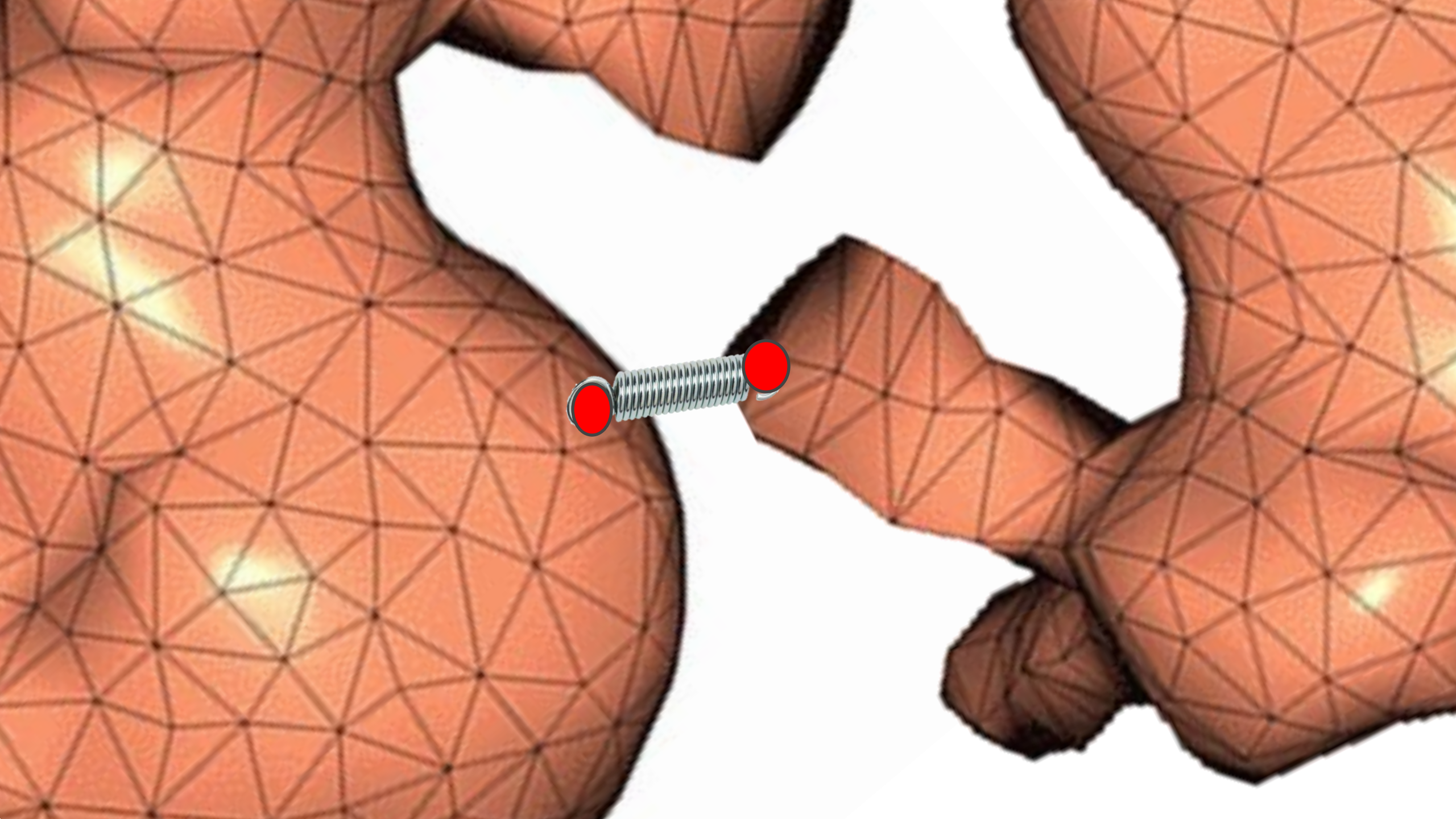
Contact forces “push” objects apart



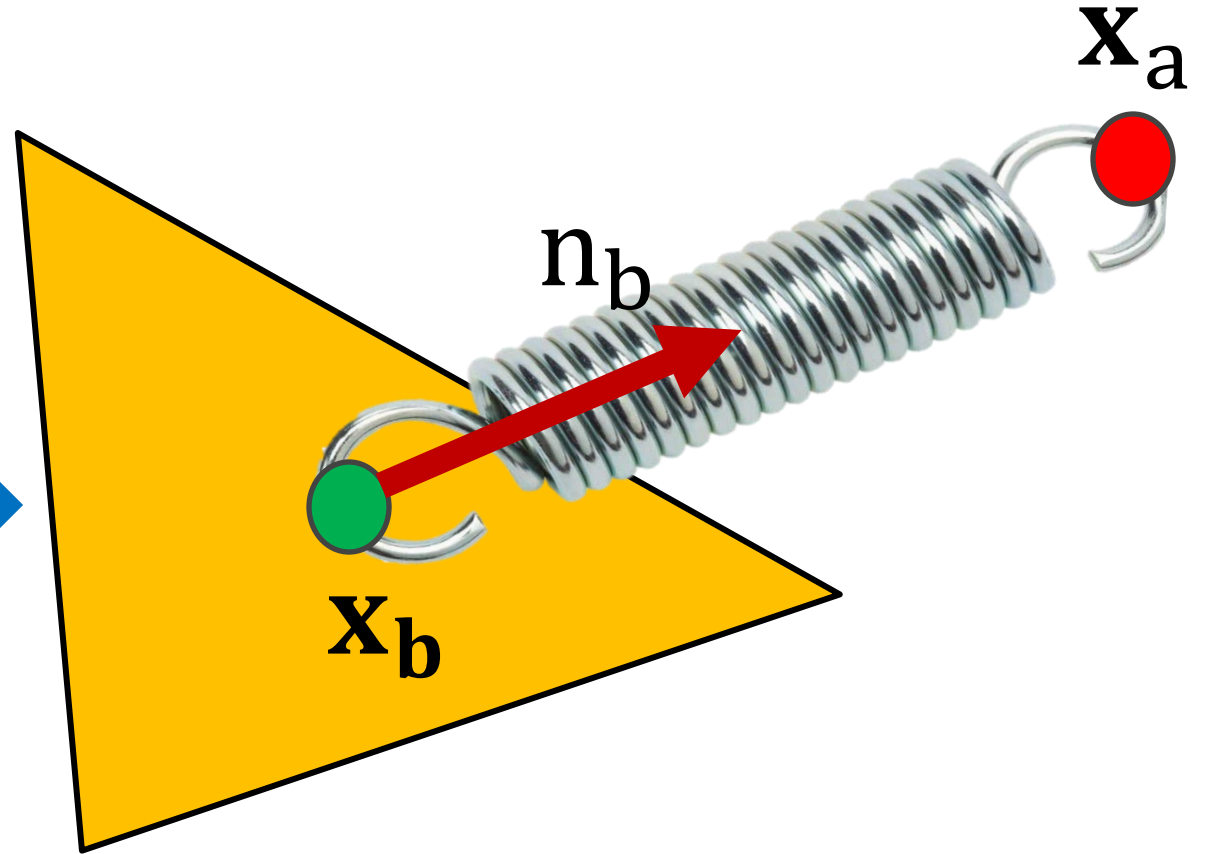
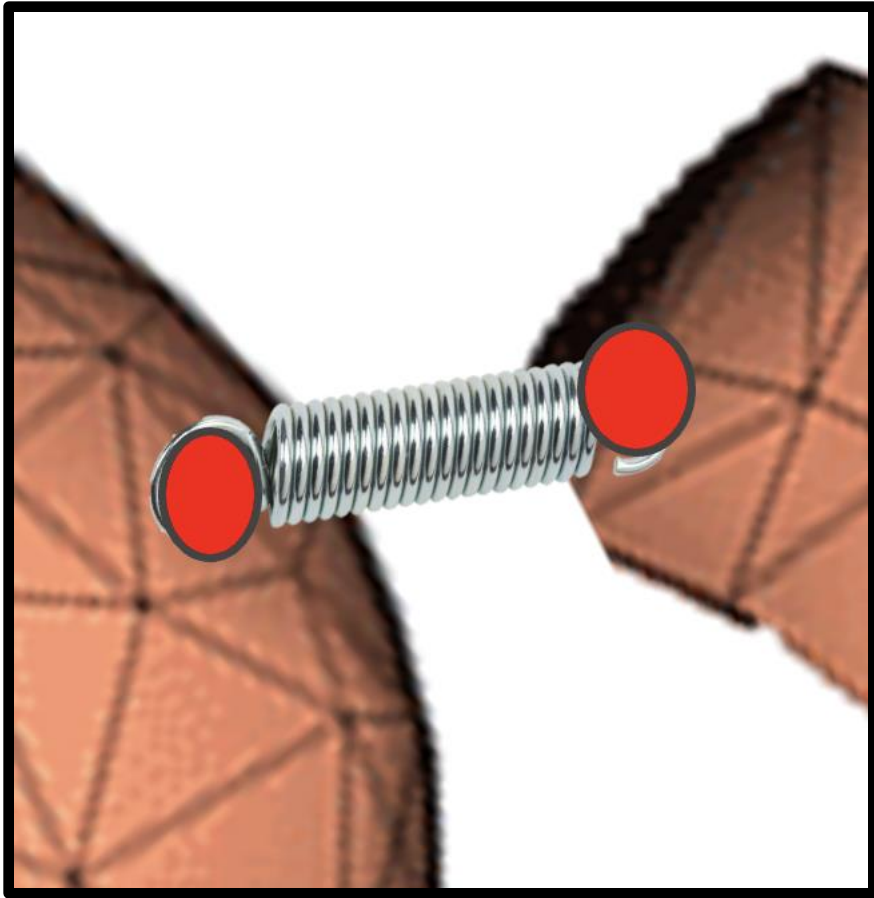
One Approach of Many – Penalty “Springs”



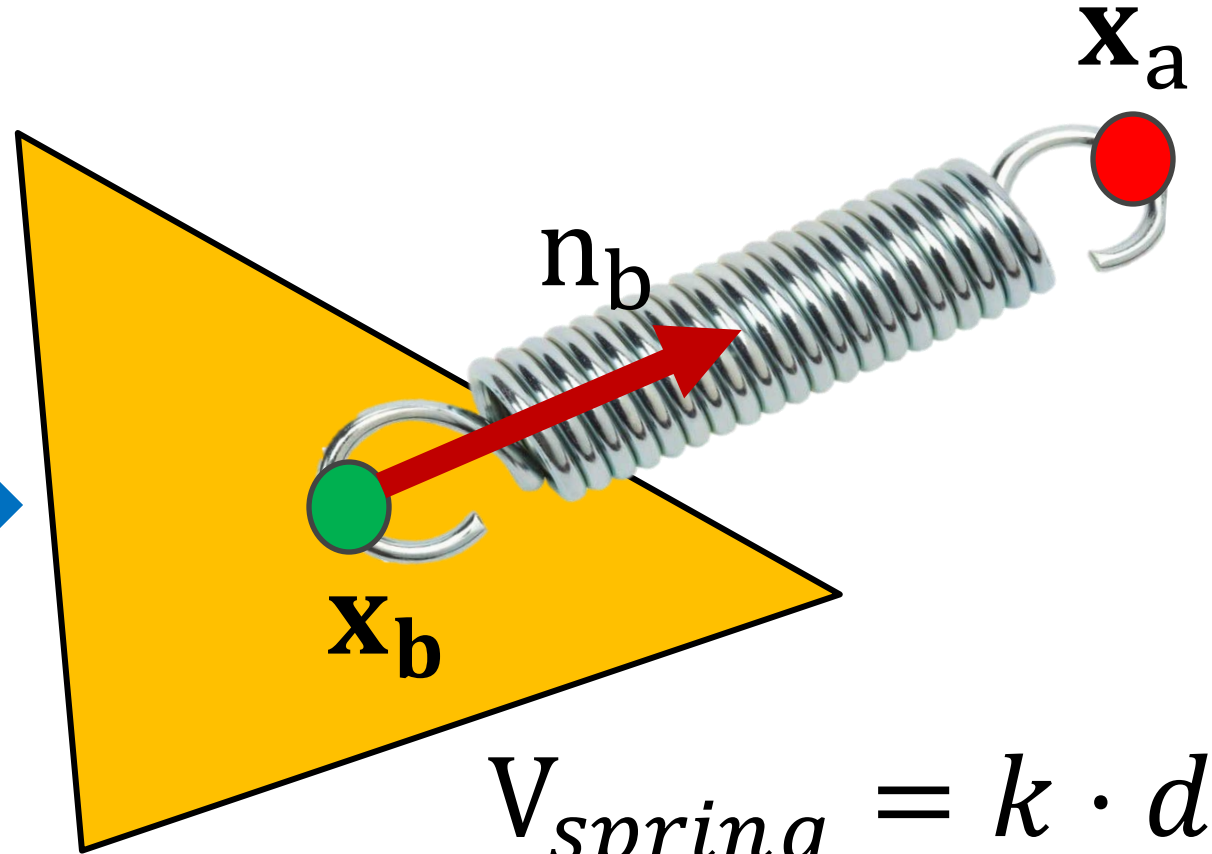
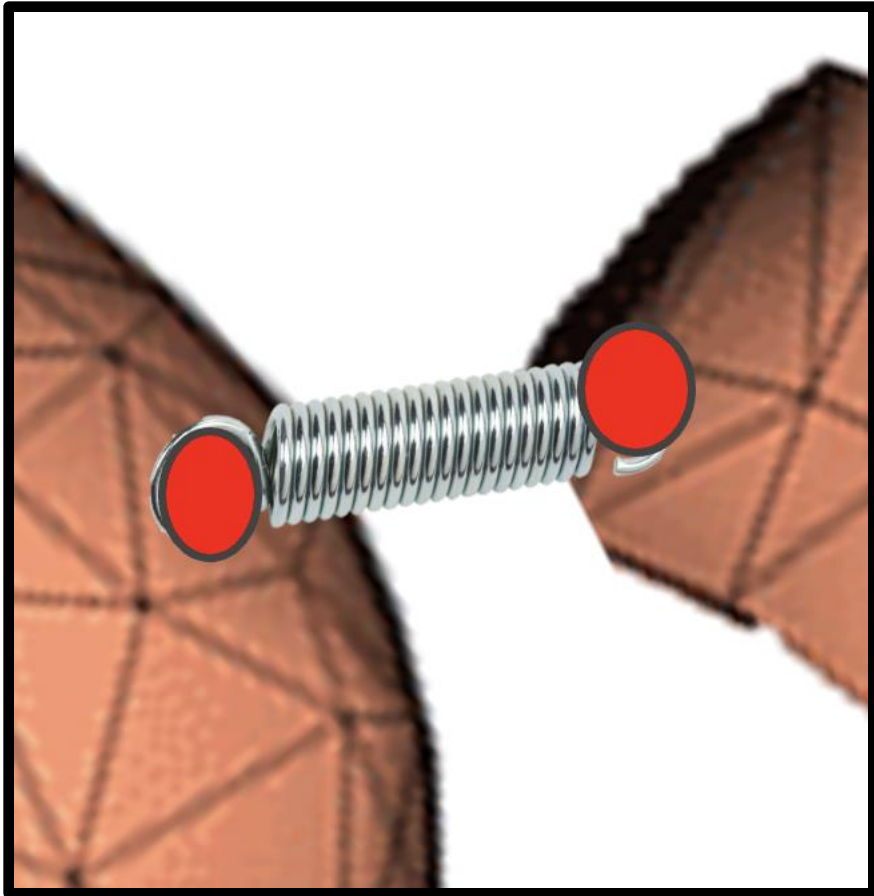




Triangle – Vertex Contacts



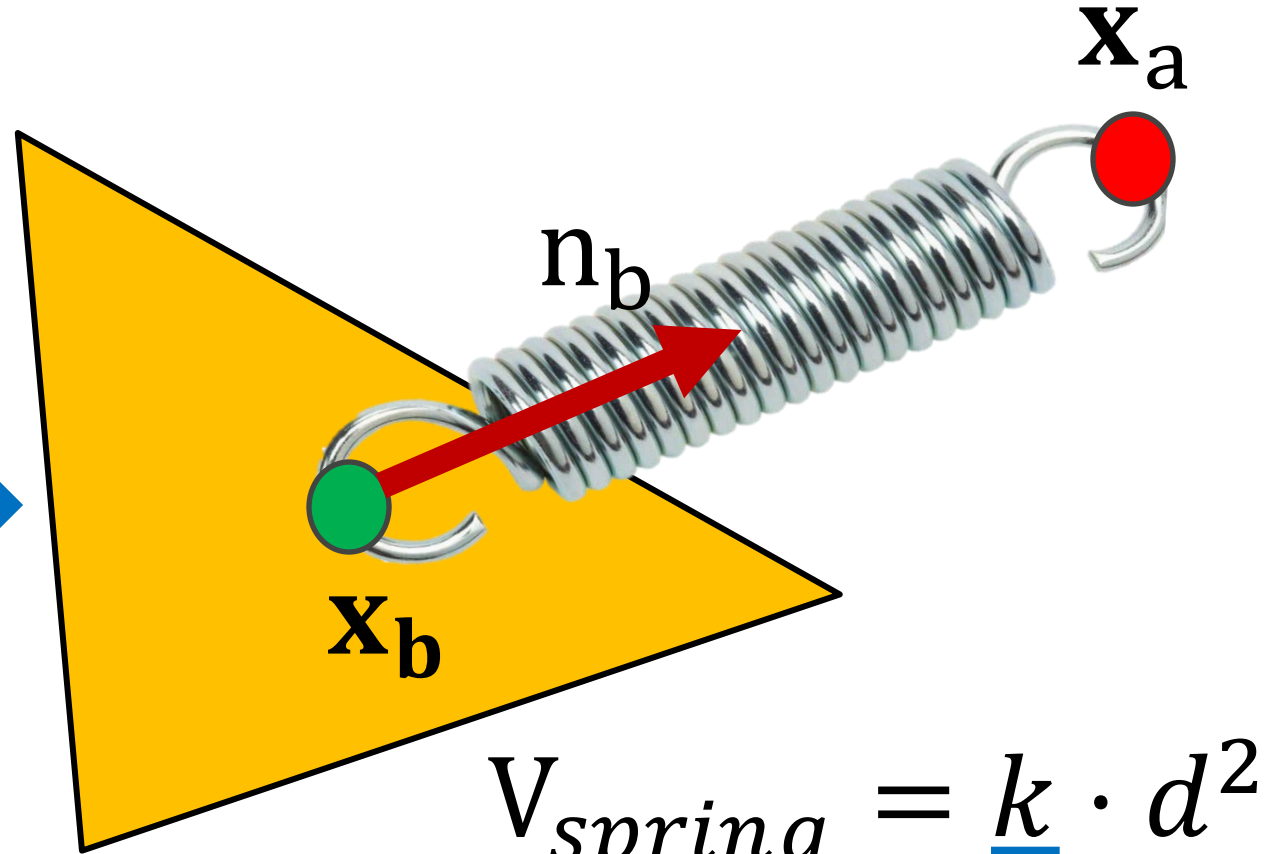
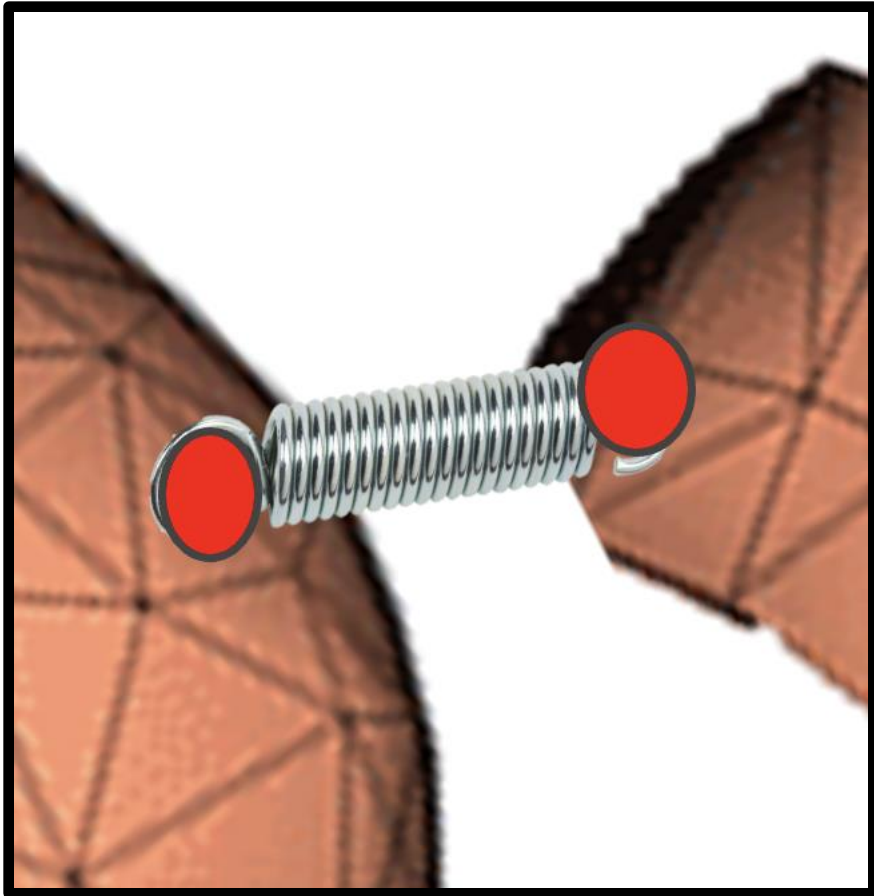
Triangle – Vertex Contacts



$$\underline{V_{spring} = k \cdot d^2}$$

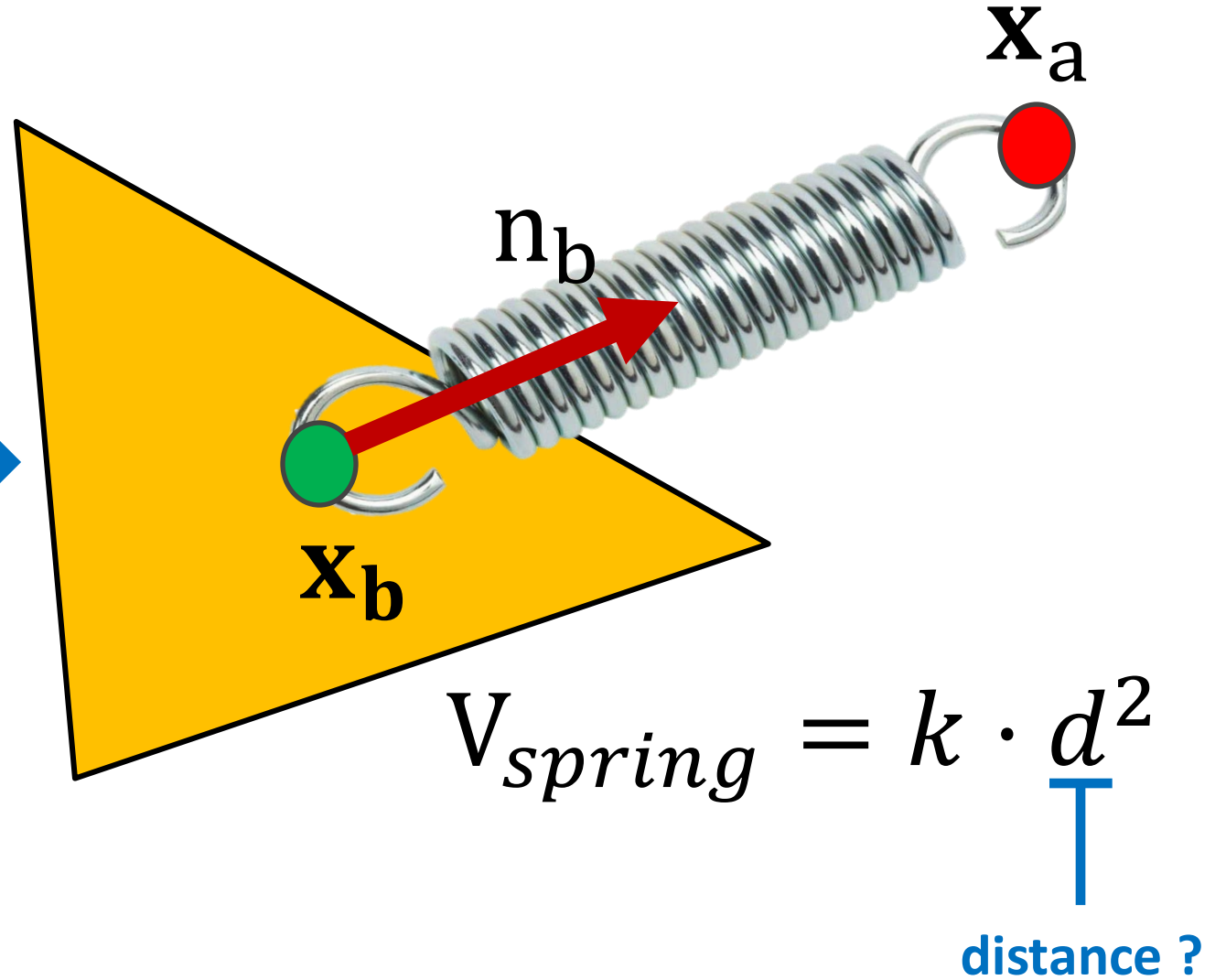
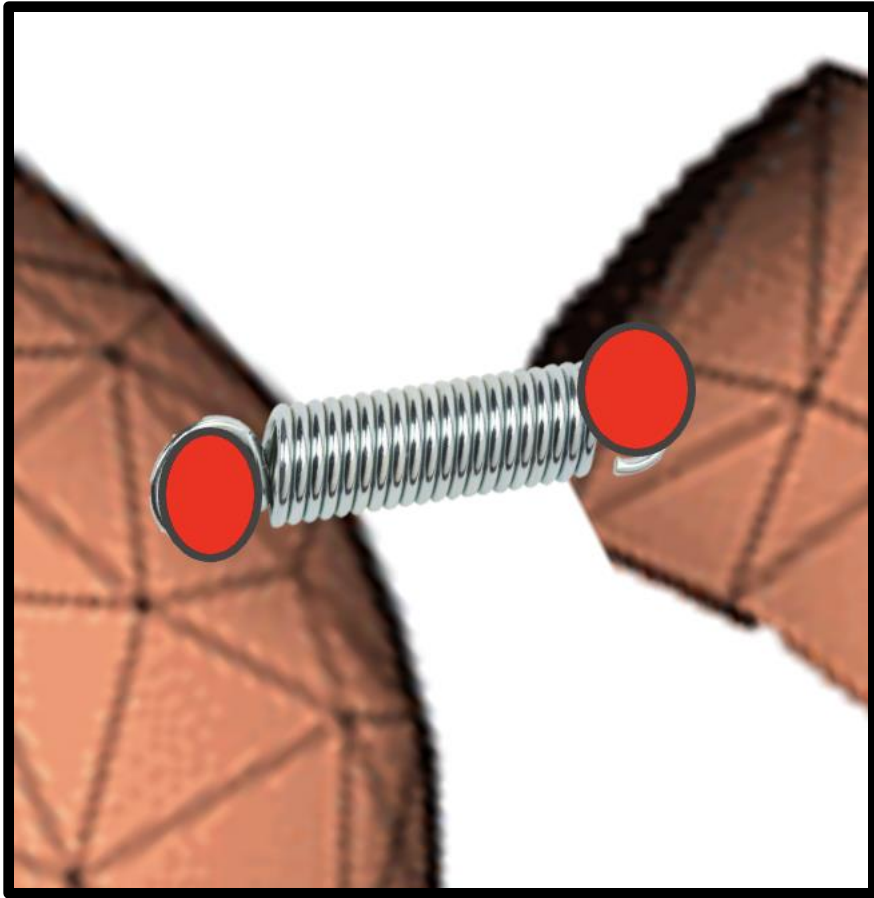
Standard energy form of a zero-rest length spring

Triangle – Vertex Contacts



$$V_{spring} = \underset{\substack{\uparrow \\ \text{Stiffness (user parameter)}}}{k} \cdot d^2$$

Triangle – Vertex Contacts



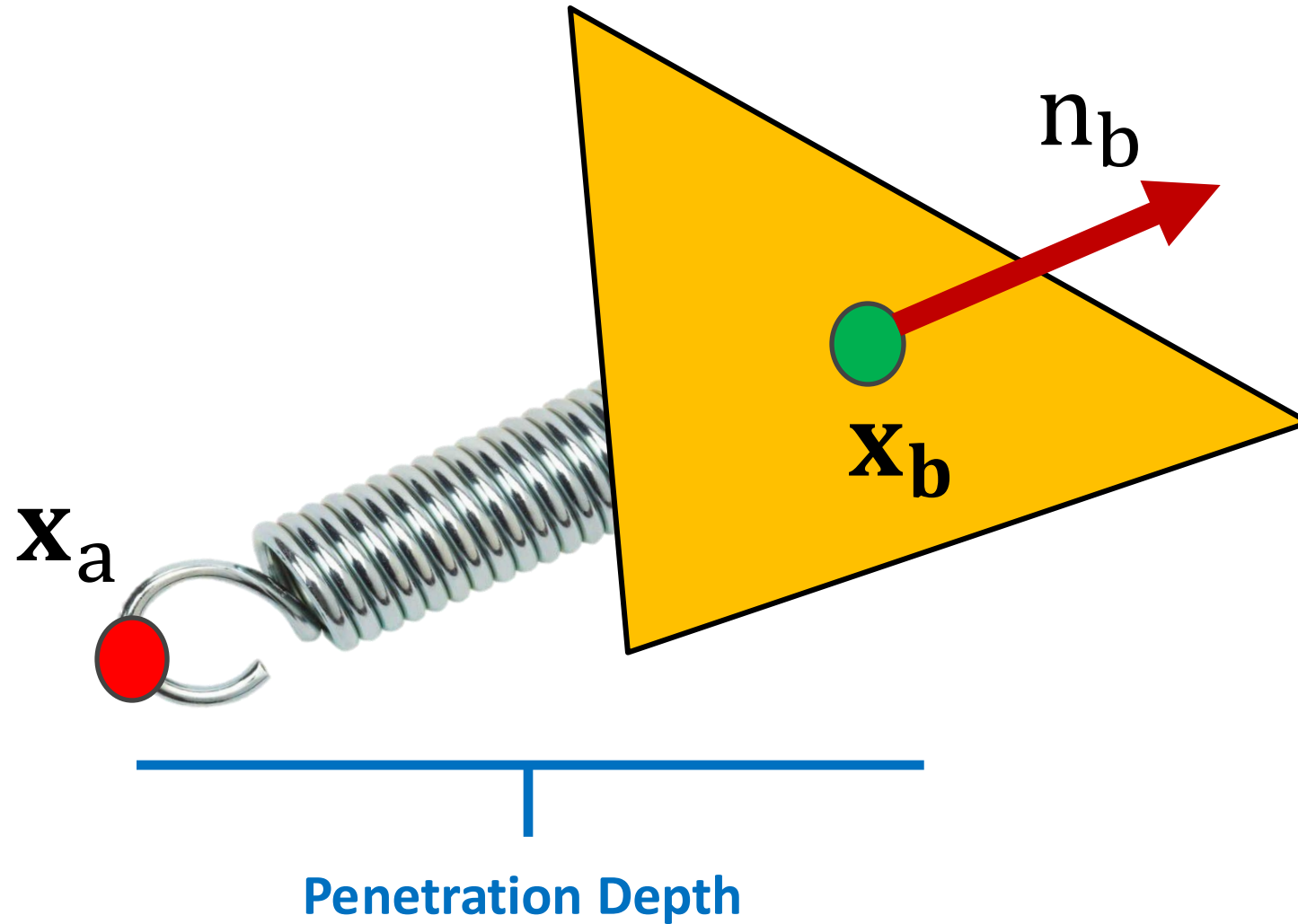
Remember the Rules

1. Contact Forces Prevent Penetration
2. Contact Force Only Push Objects Apart
3. Contact Forces Only Apply when Objects are in Contact

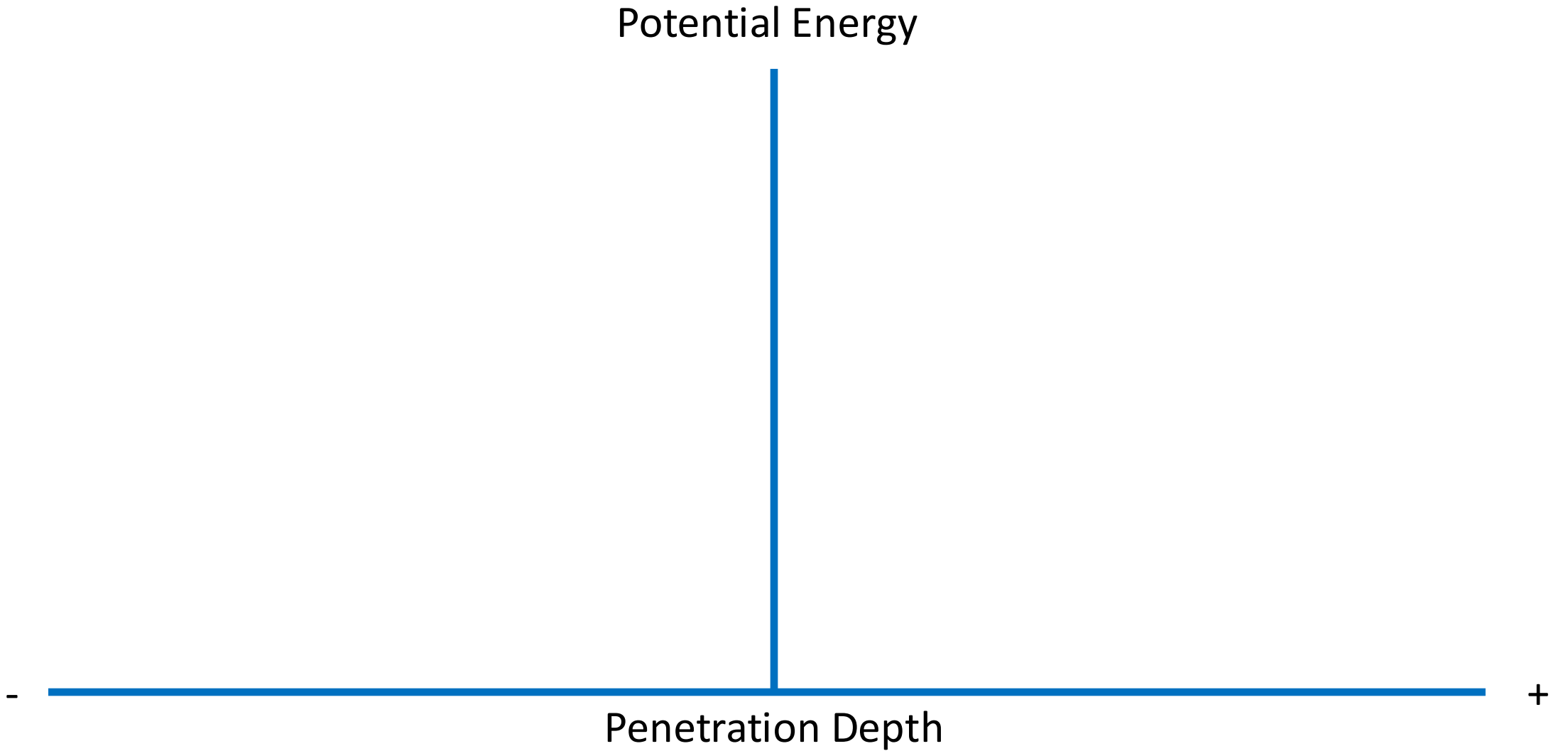
Remember the Rules

1. Contact Forces **UNDO** Penetration
2. Contact Force Only Push Objects Apart
3. Contact Forces Only Apply when Objects **Have Penetrated**

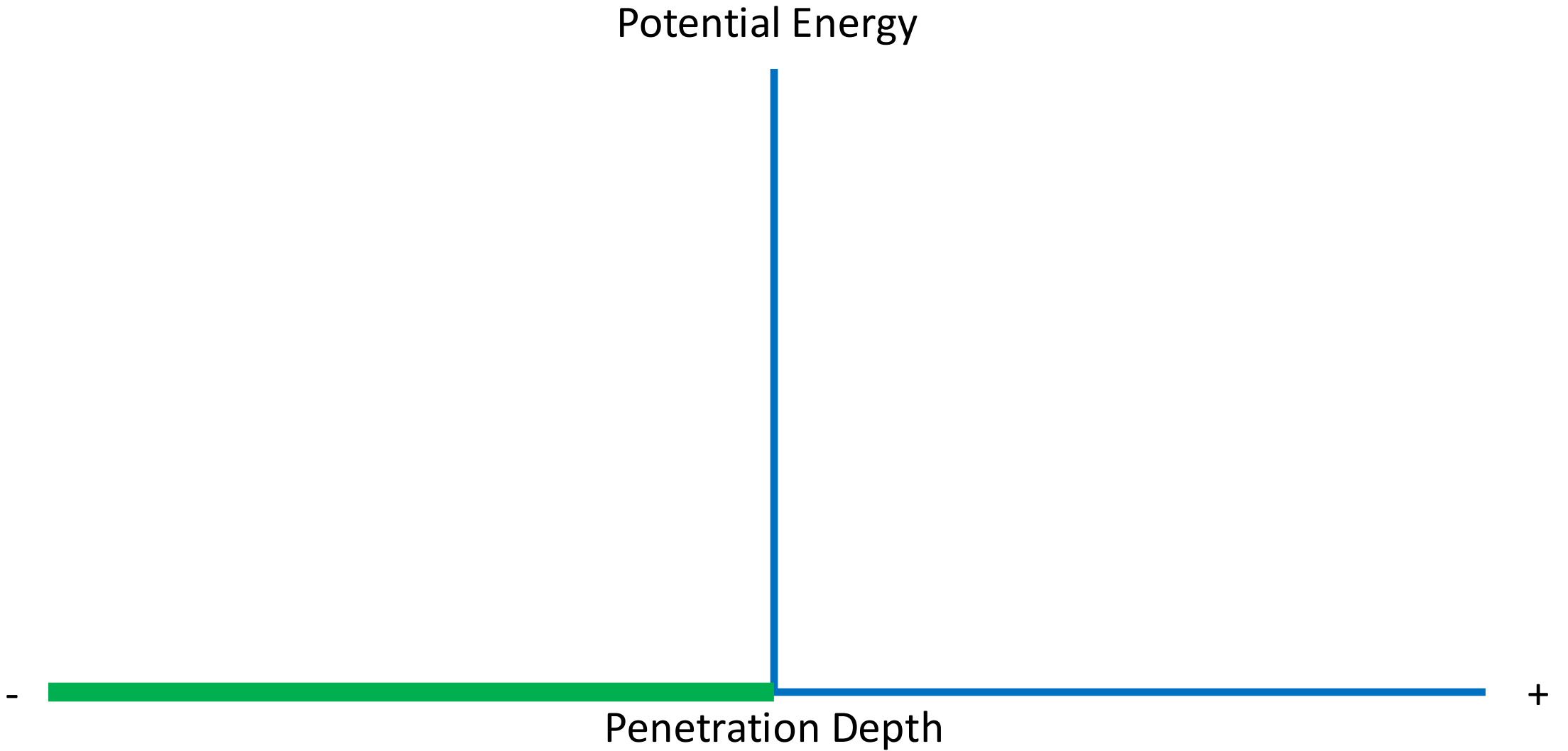
Triangle – Vertex Contacts



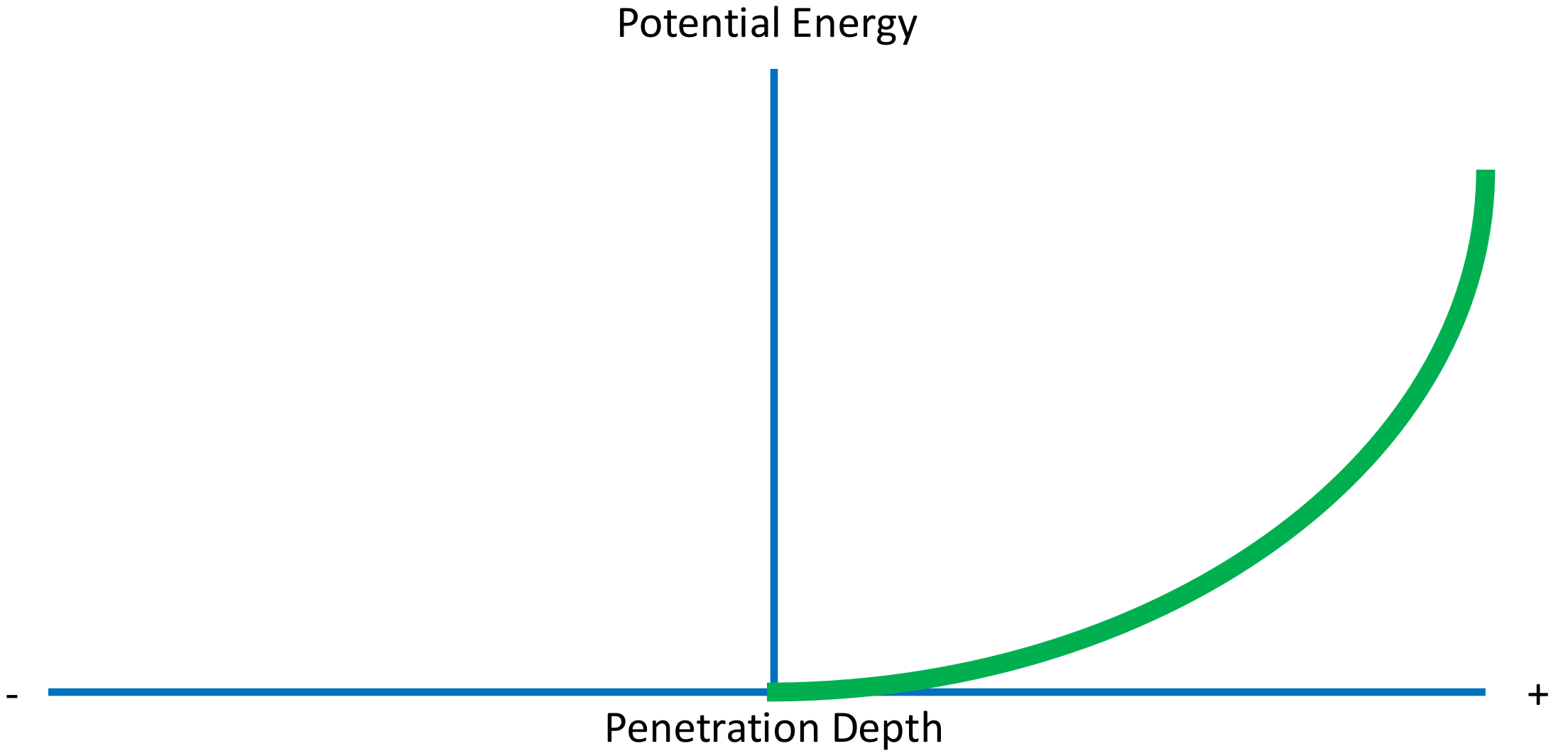
What should this energy look like ?



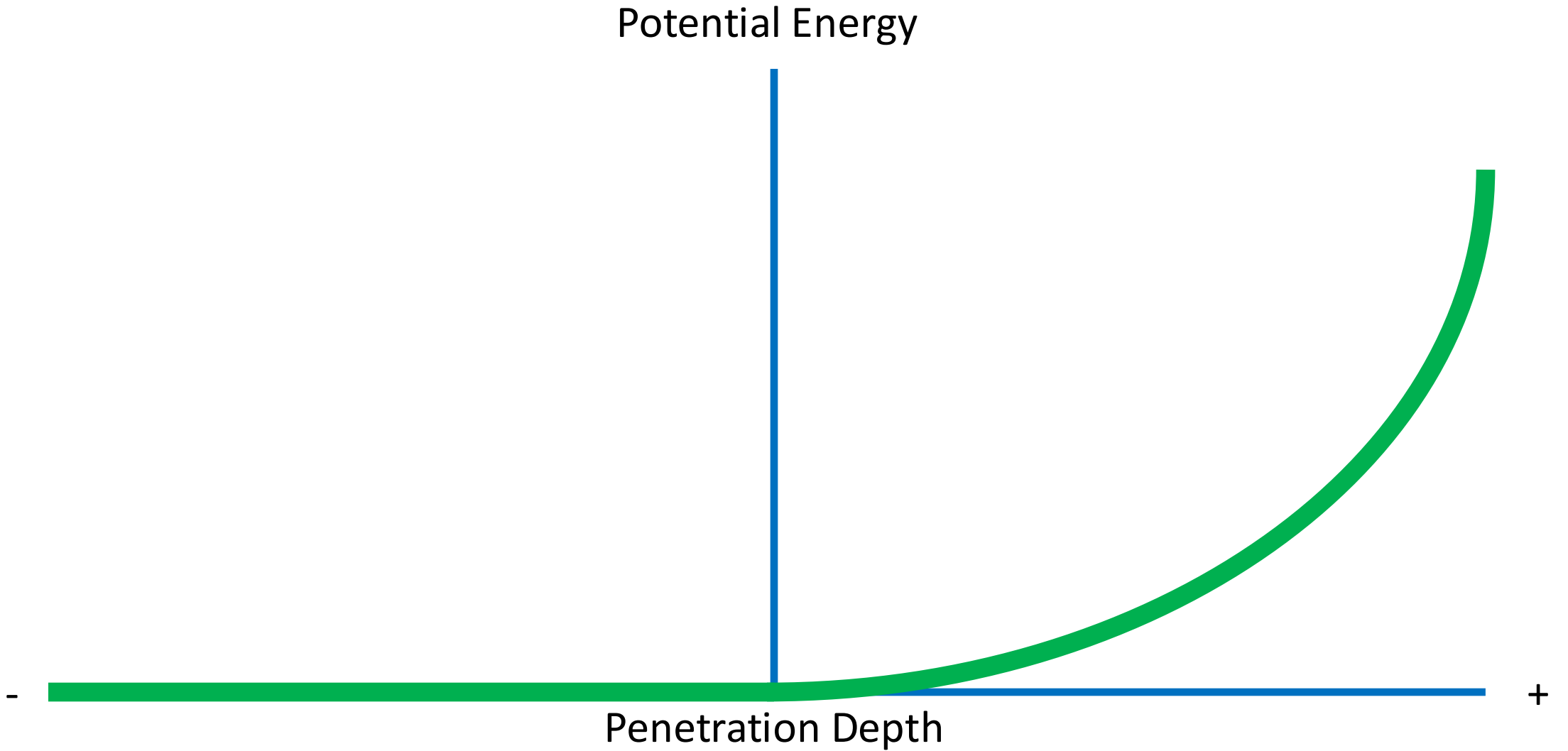
What should this energy look like ?



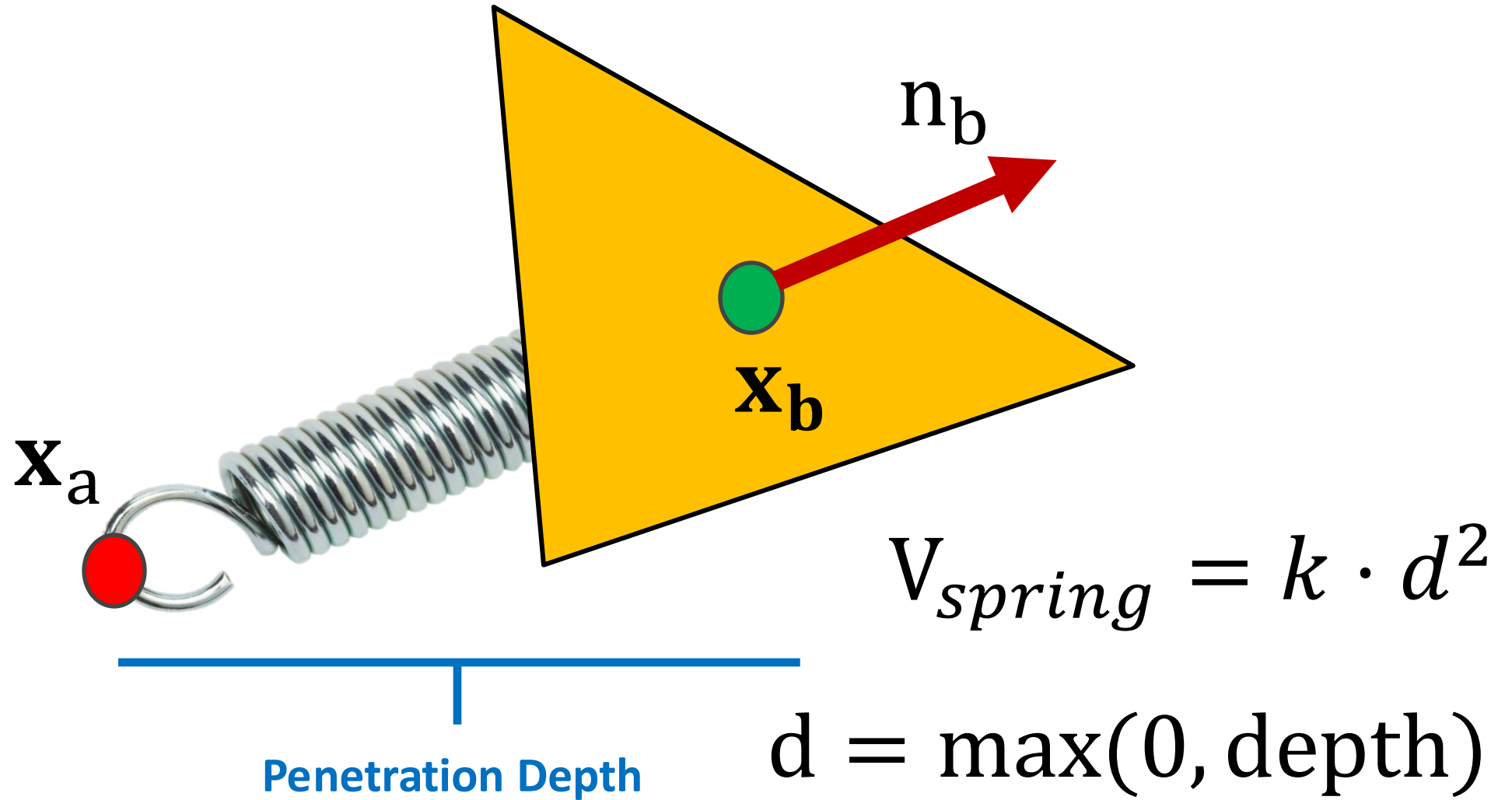
What should this energy look like ?



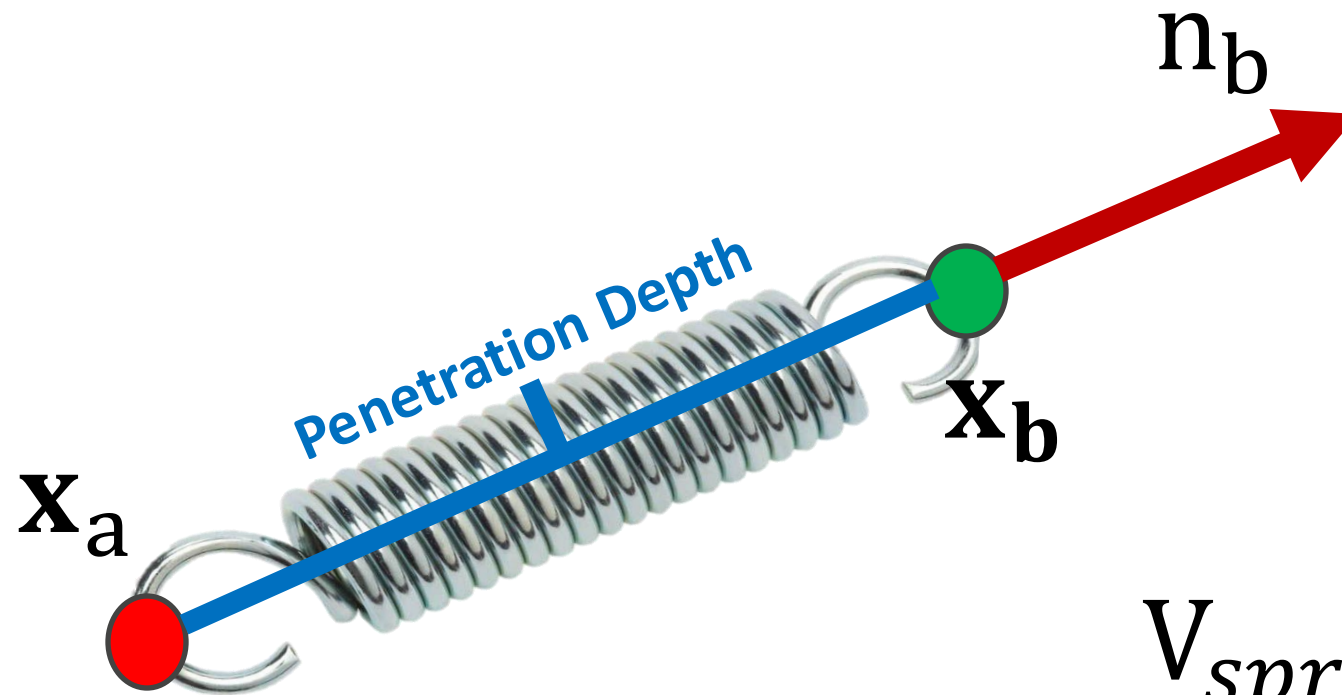
What should this energy look like ?



Triangle – Vertex Contacts



Triangle – Vertex Contacts

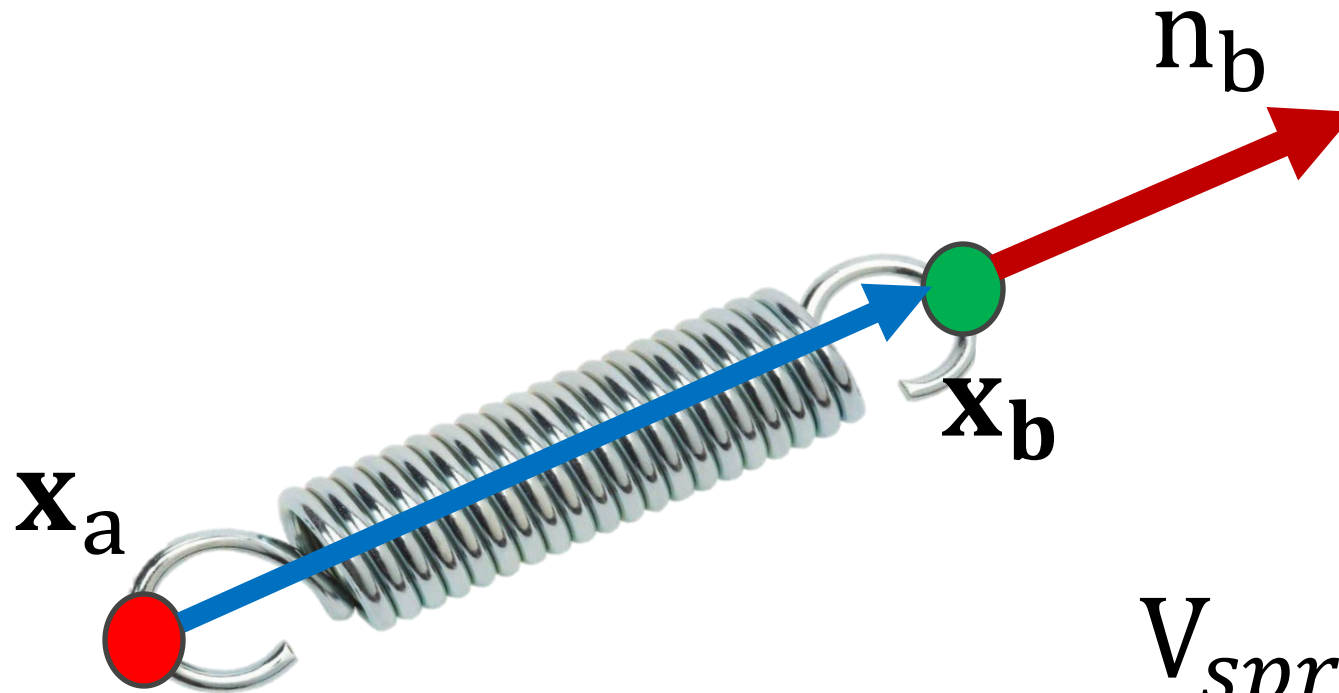


$$V_{spring} = k \cdot d^2$$

~~$$d = \max(0, \|\mathbf{x}_b - \mathbf{x}_a\|_2)$$~~

Triangle – Vertex Contacts

What does the normal tell us
about the sign of d ?

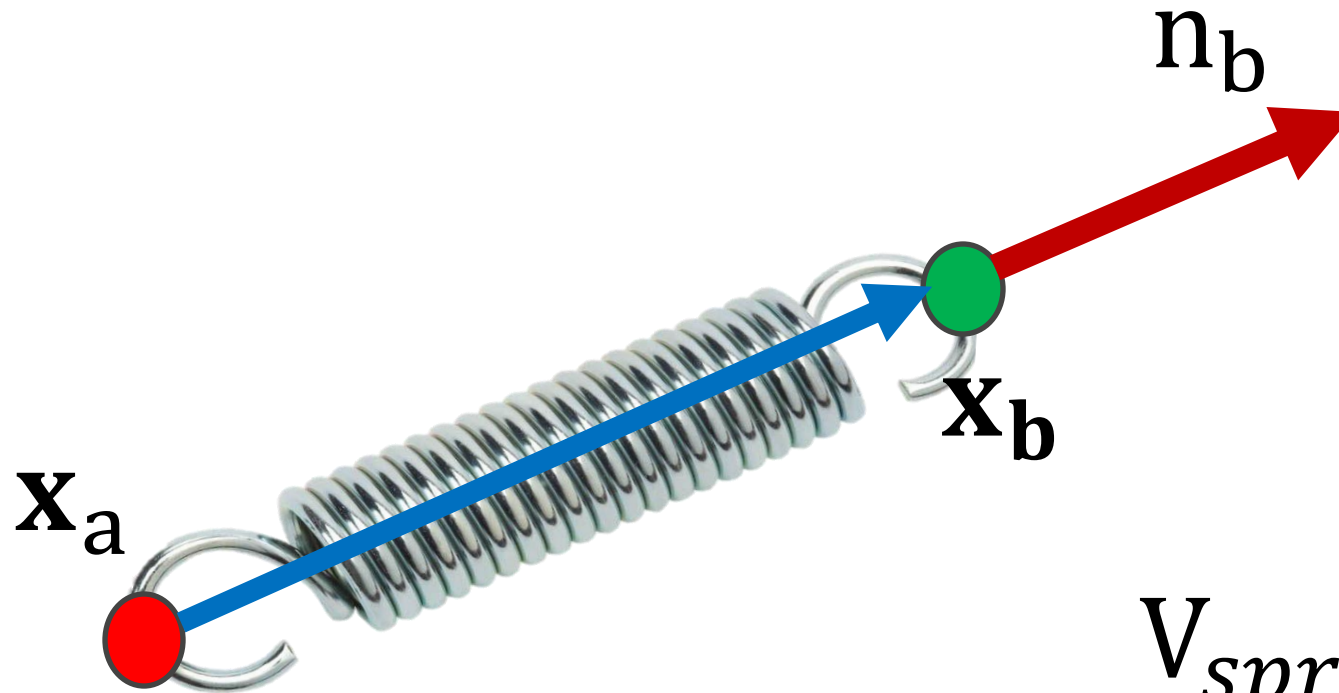


$$V_{spring} = k \cdot d^2$$

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Triangle – Vertex Contacts

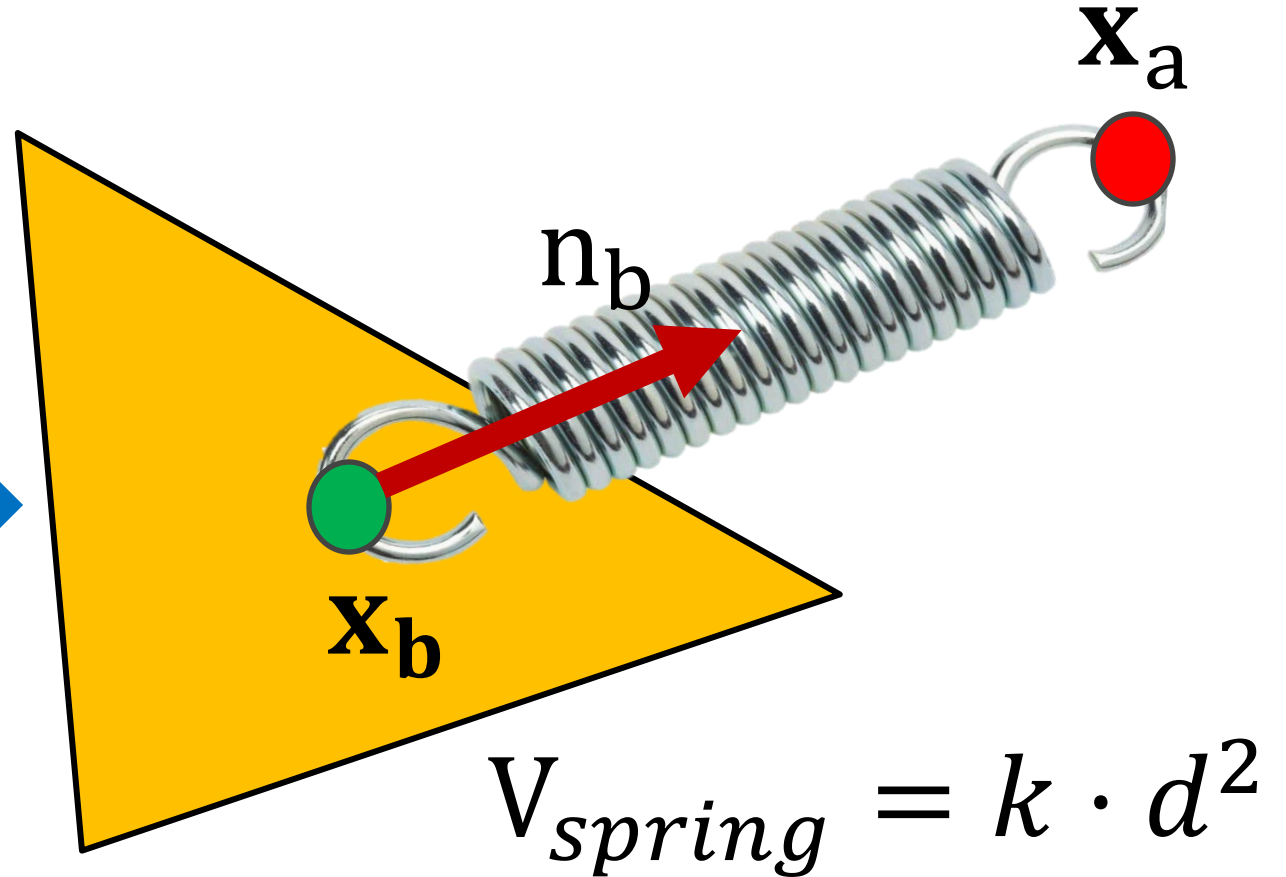
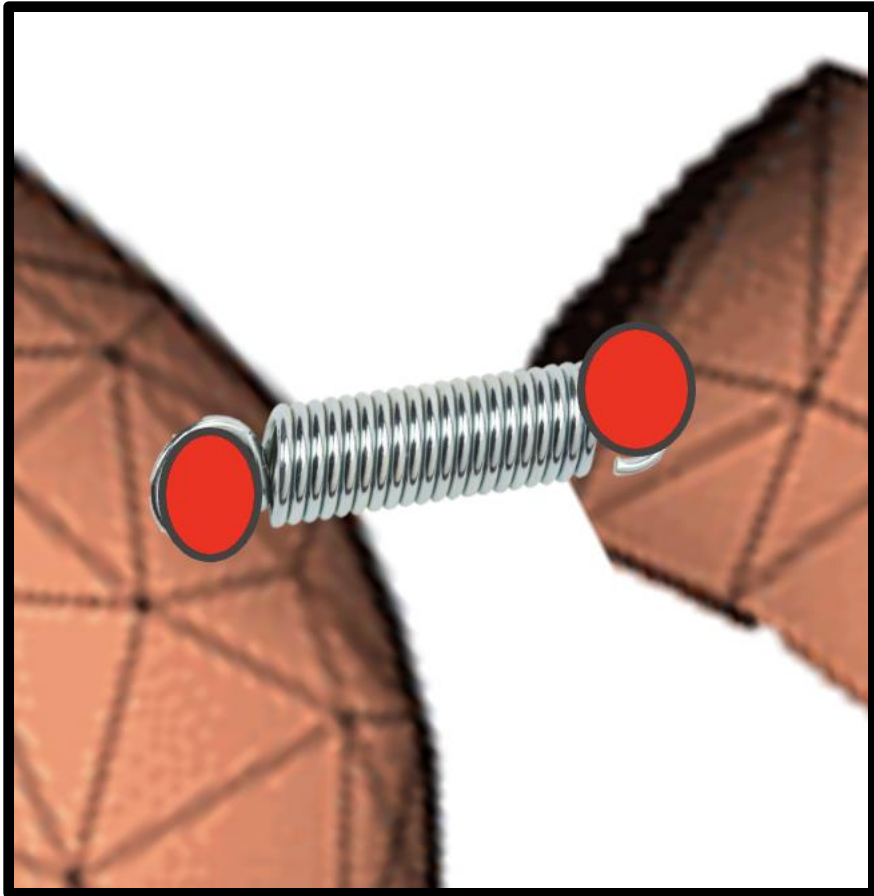
What does the normal tell us
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$$V_{spring} = k \cdot d^2$$

$$d = \max(0, (\mathbf{x}_b - \mathbf{x}_a)^T \mathbf{n}_b)$$

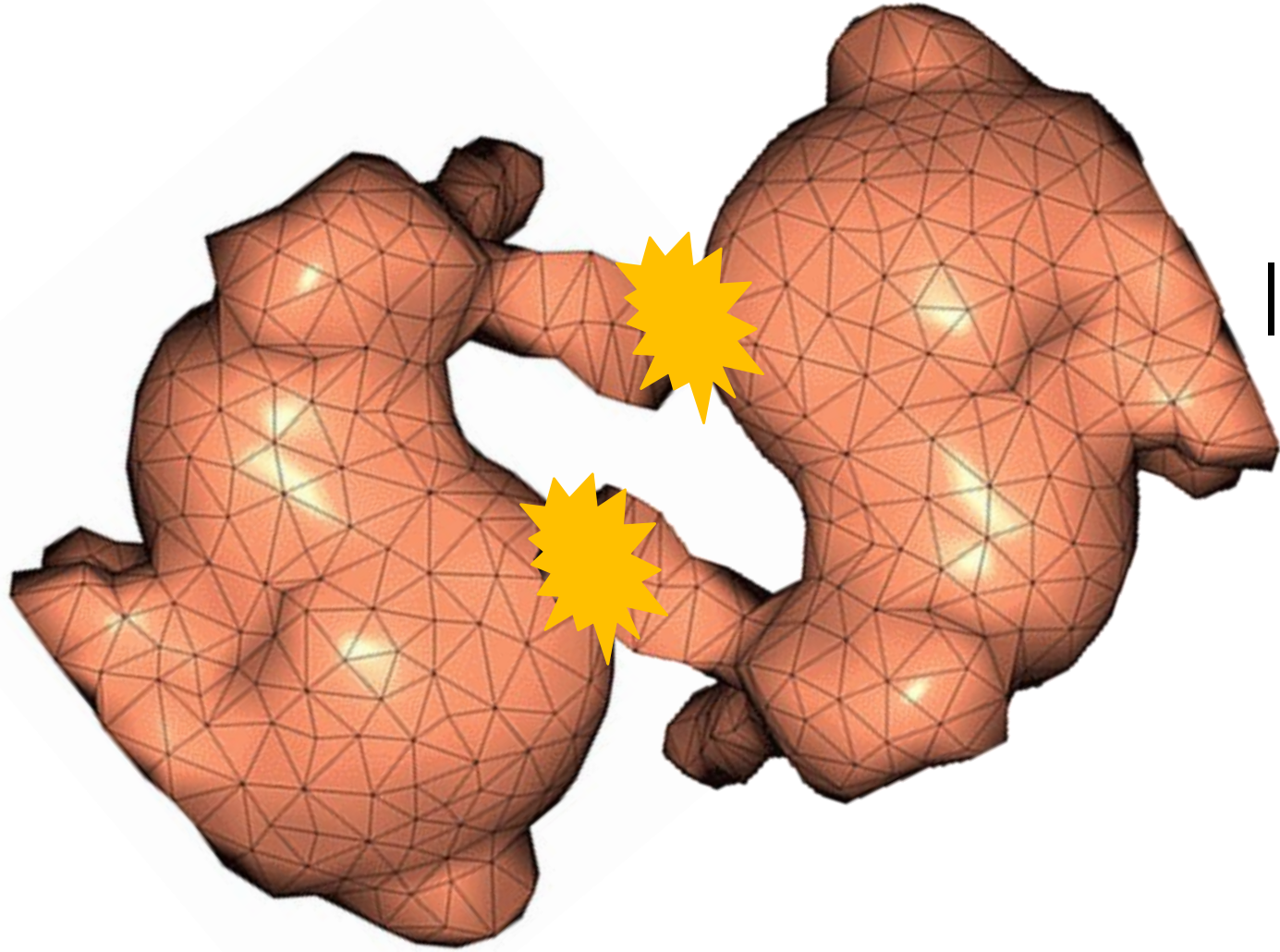
Triangle – Vertex Contacts



$$V_{spring} = k \cdot d^2$$

$$d = \max(0, (\mathbf{x}_b - \mathbf{x}_a)^T \mathbf{n}_b)$$

Contact Potential Energy



|Springs|

$$\sum_{j=0}$$

$$V_{spring}(\mathbf{x}(\mathbf{q}))$$

Two Problems with Our Current Approach

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 \underbrace{V(\mathbf{q}^{i+1})}_{V_{springs} + V_{affine}}$$

~~Problem 1: Solving this optimization problem only moves one object !!!~~

~~Problem 2: There's no term in this optimization that tells it how to handle collisions~~

Finding Contacts ?

list = [] # Empty list of penalty springs

For *A* in each Object

 For *B* in each Object

 if *A* == *B*

 continue

 else

 For each vertex, *v*, in *A*

 Find triangle, *t*, with least positive penetration in *B*

 Add spring between *v* and *t* to *list*

Finding Contacts ?

list = [] # Empty list of penalty springs

For *A* in each Object

 For *B* in each Object

 if *A* == *B*

 continue

 else

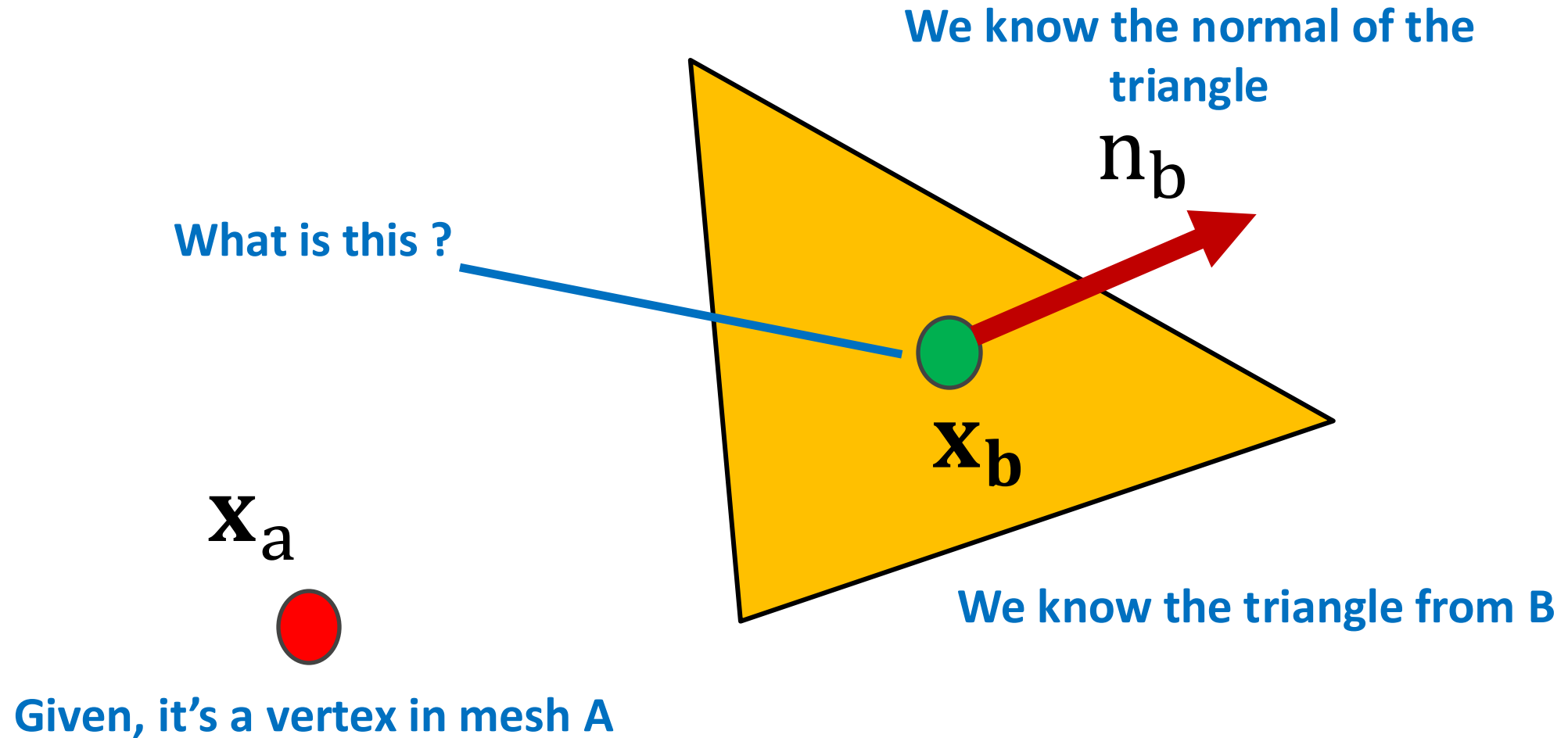
 For each vertex, *v*, in *A*

 Find triangle, *t*, with least positive penetration in *B*

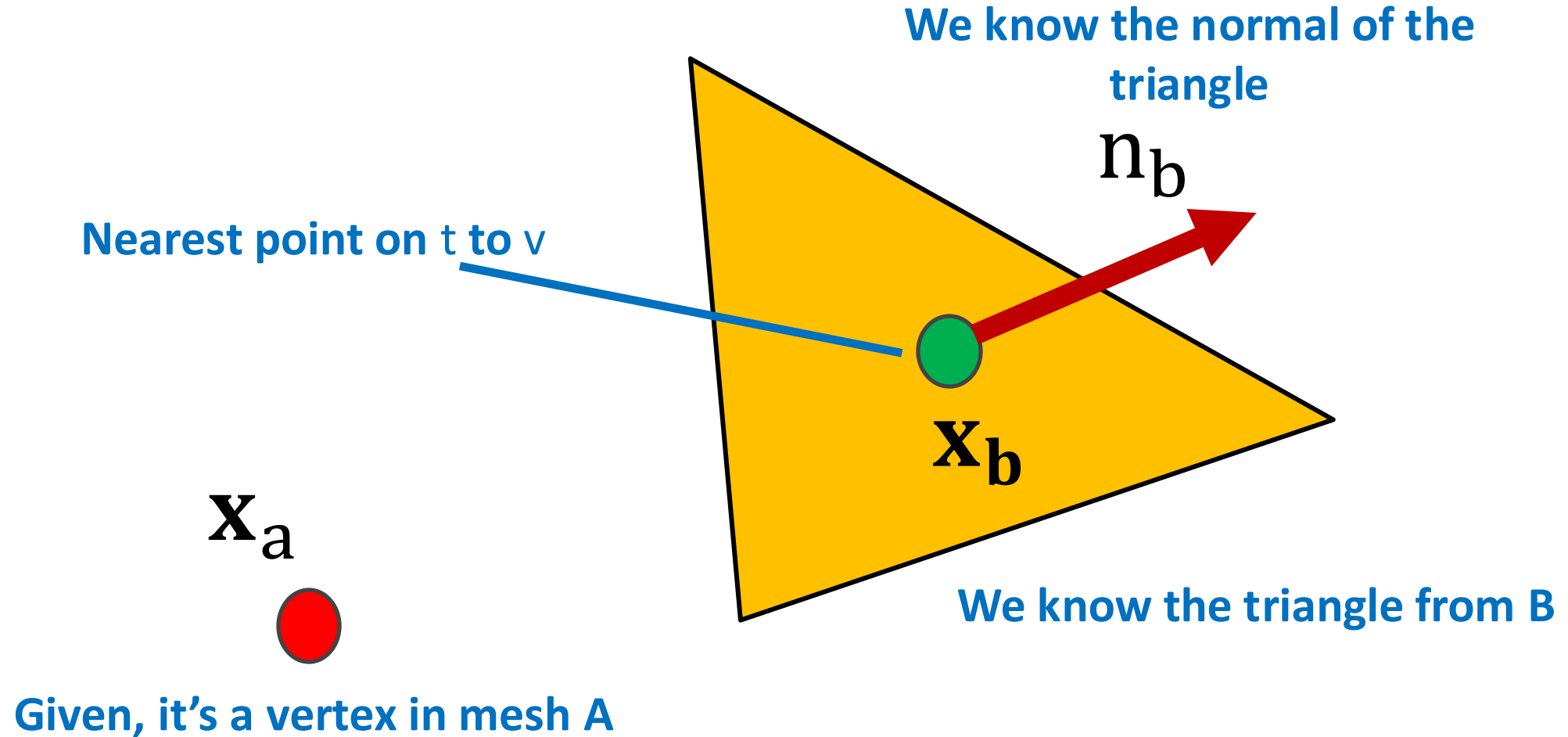
 Add spring between *v* and *t* to *list*

How exactly do we
compute this ?

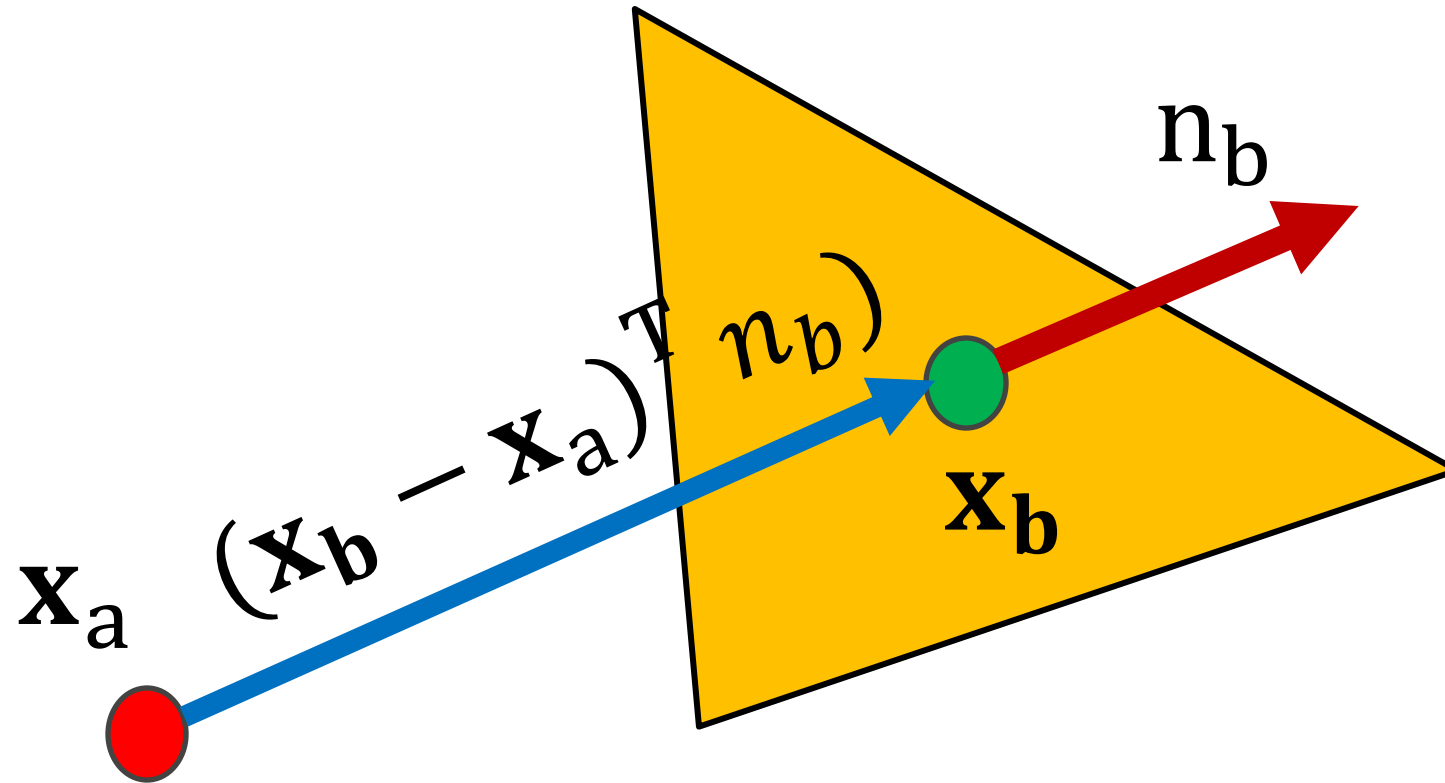
Calculating Penetration Depth For a Single Triangle



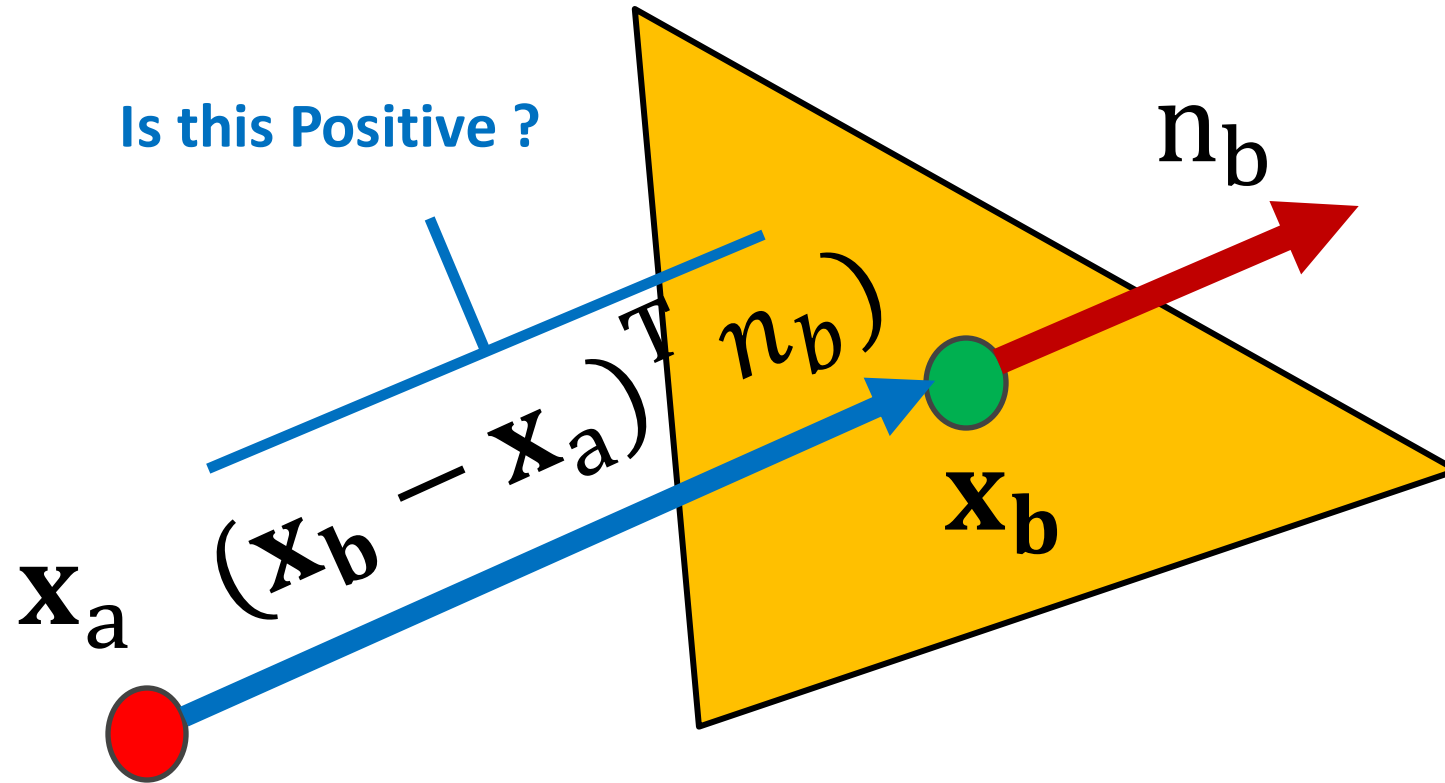
Calculating Penetration Depth For a Single Triangle



Calculating Penetration Depth For a Single Triangle

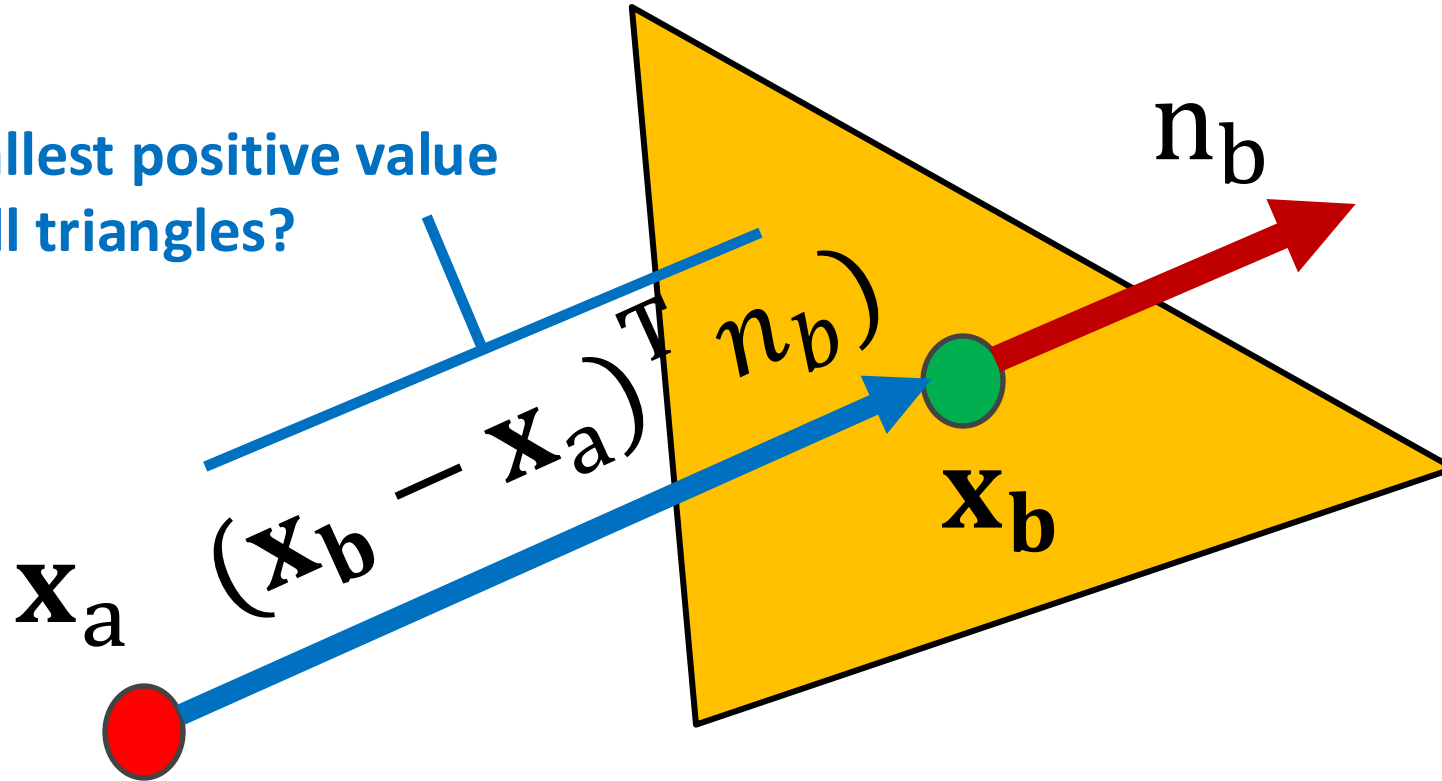


Calculating Penetration Depth For a Mesh

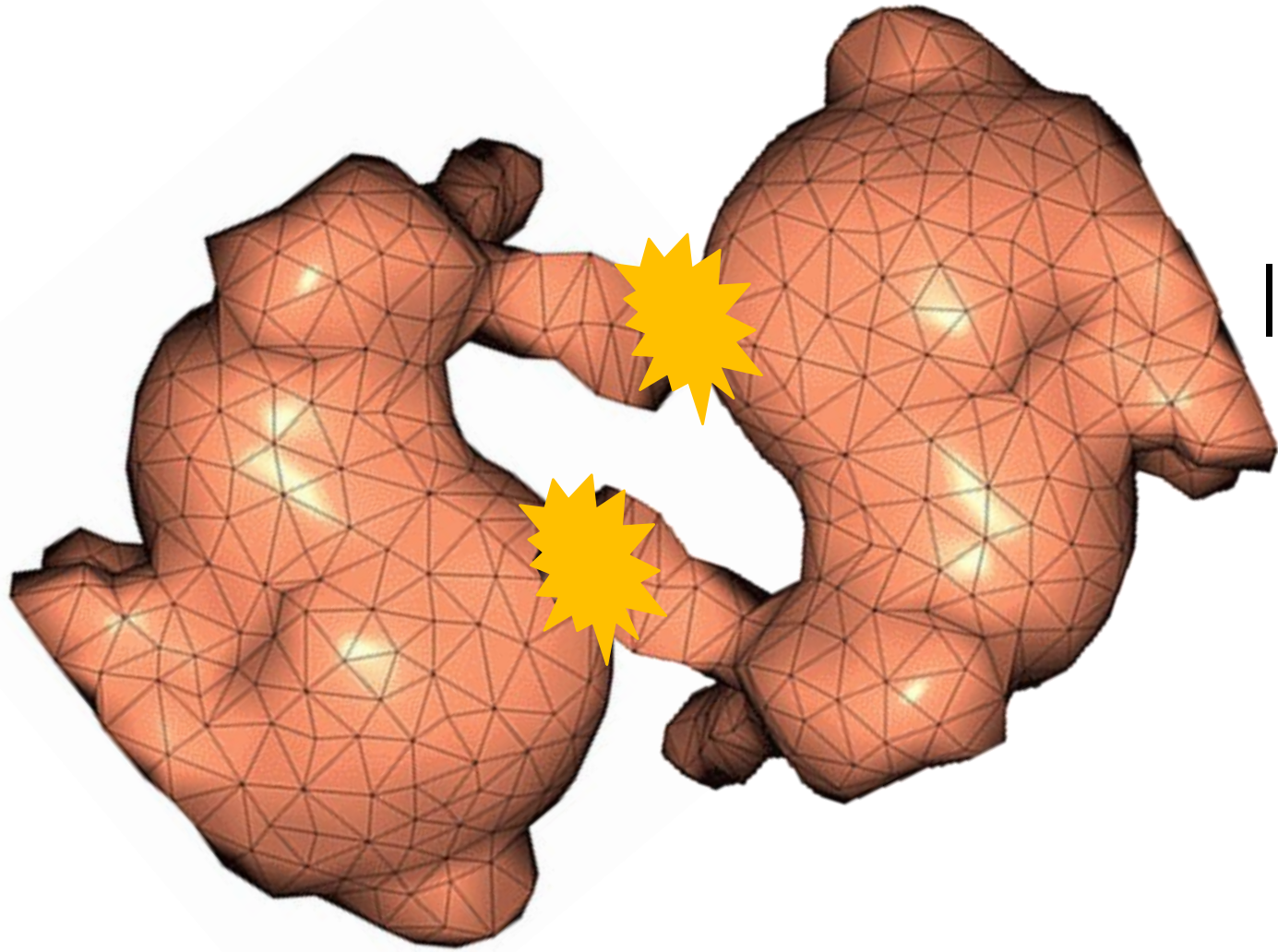


Calculating Penetration Depth For a Mesh

Is this the smallest positive value
over all triangles?



One last thing ...



$|\text{Springs}|$

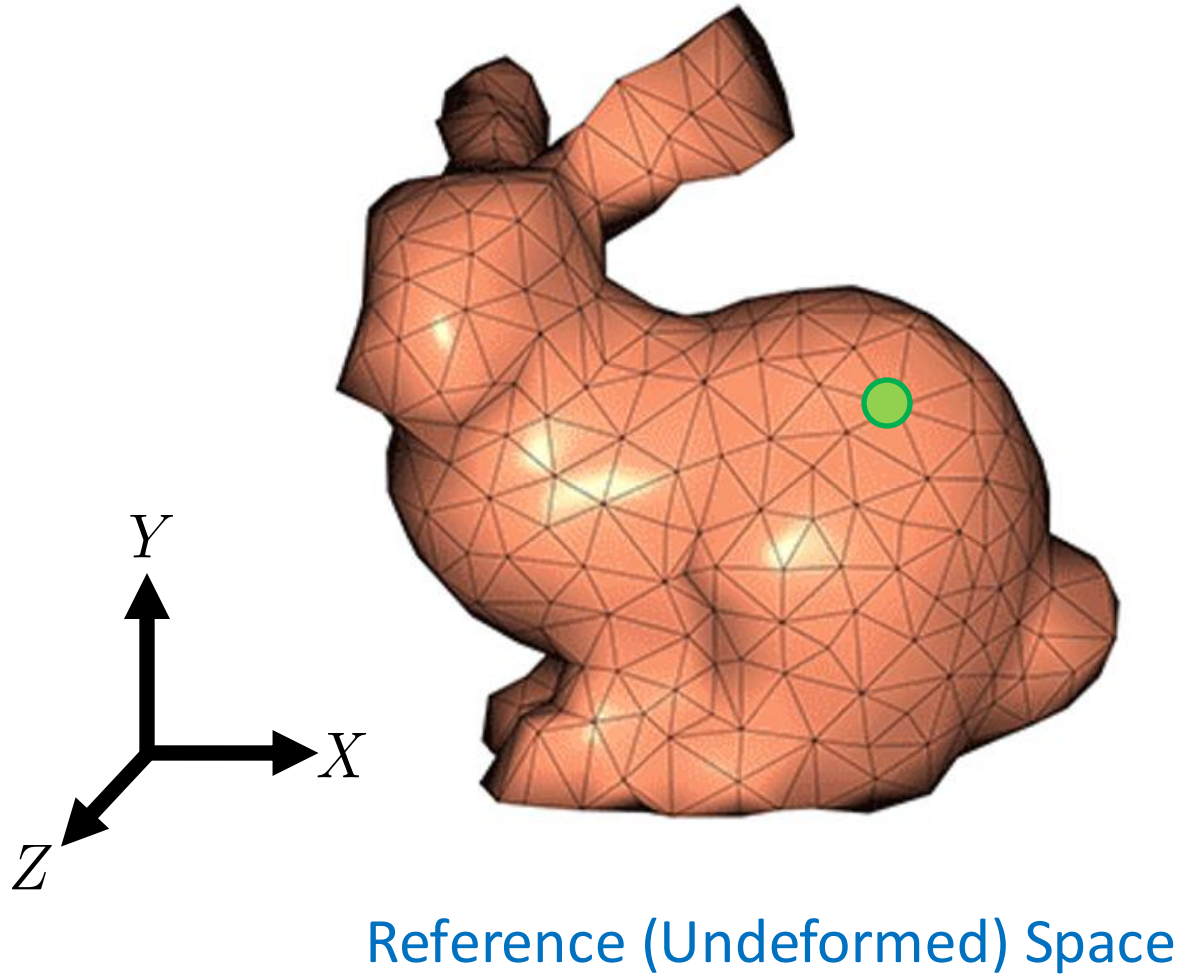
$$\sum_{j=0}$$

$$V_{spring}(\mathbf{x}(\mathbf{q}))$$



How do I compute this ?

Vectorized Generalized Coordinates

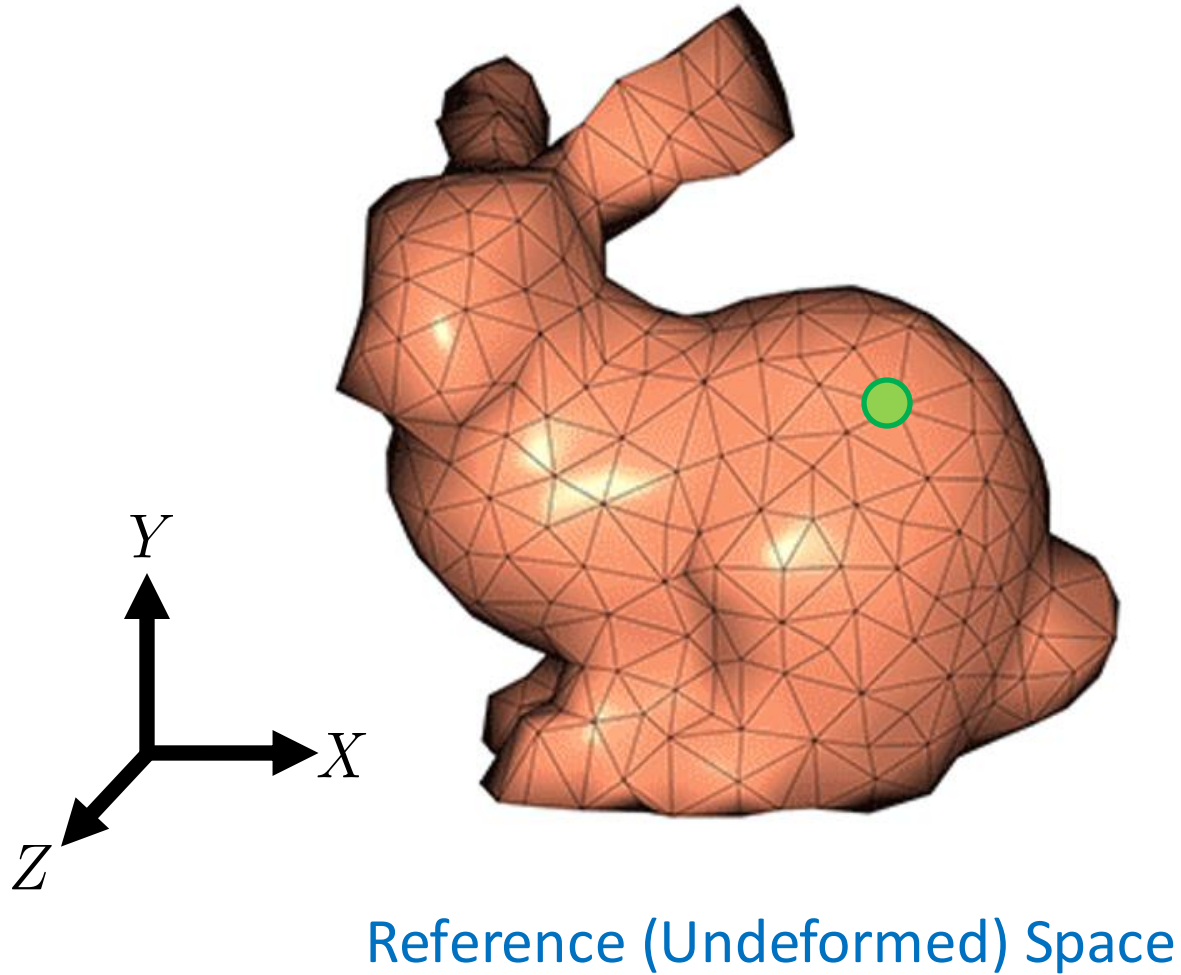


$$\mathbf{x}(\mathbf{X}, t) = \mathbf{J}(\mathbf{X})\mathbf{q}(t)$$

What's the problem?



Vectorized Generalized Coordinates

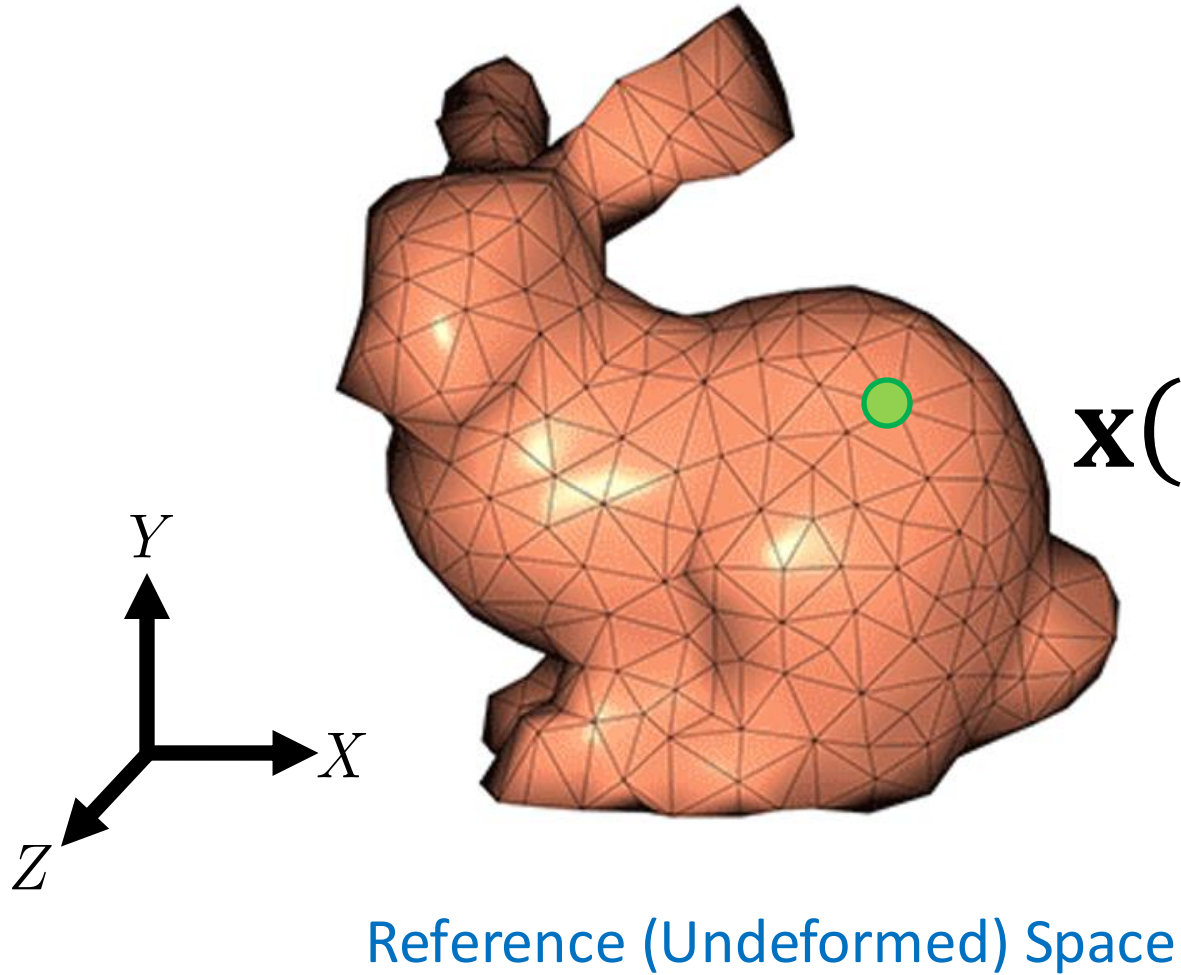


$$\mathbf{x}(\mathbf{X}, t) = \mathbf{J}(\mathbf{X})\mathbf{q}(t)$$

Given \mathbf{x} , need to FIND \mathbf{X} ... grrrrr

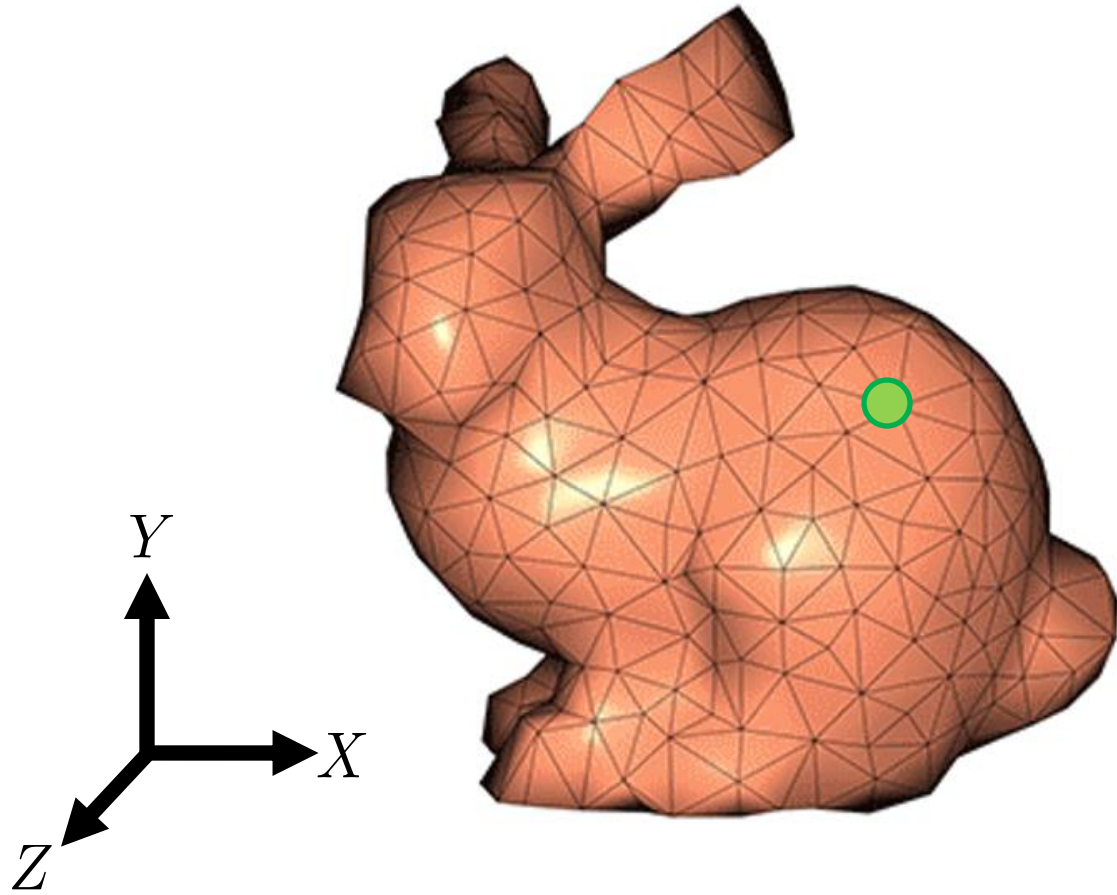


But what is the Deformation Gradient ?



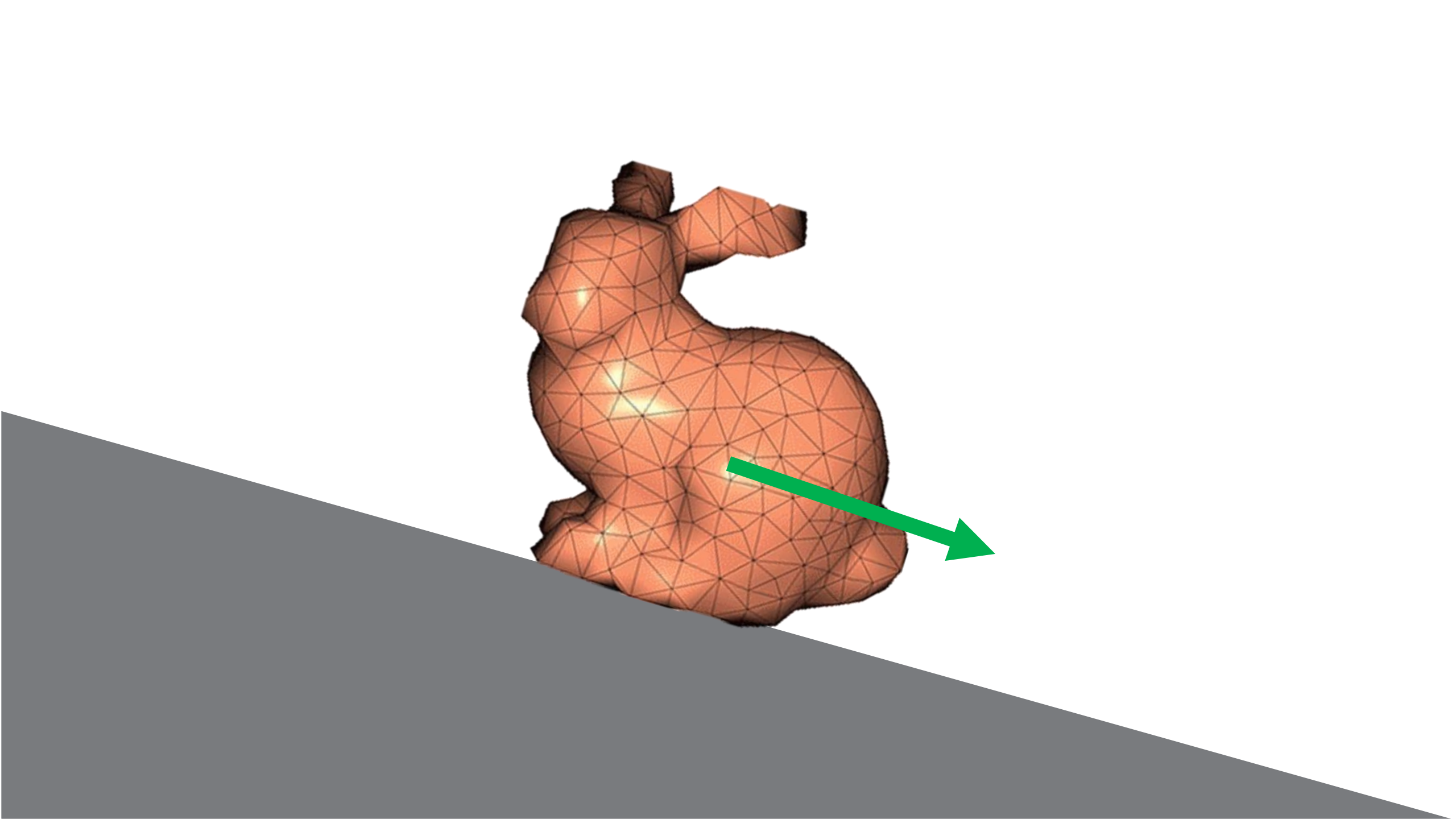
$$\mathbf{x}(\mathbf{X}, t) = \begin{bmatrix} q_0 & q_1 & q_2 \\ q_4 & q_5 & q_6 \\ q_8 & q_9 & q_{10} \end{bmatrix} \mathbf{X} + \begin{bmatrix} q_3 \\ q_7 \\ q_{11} \end{bmatrix}$$

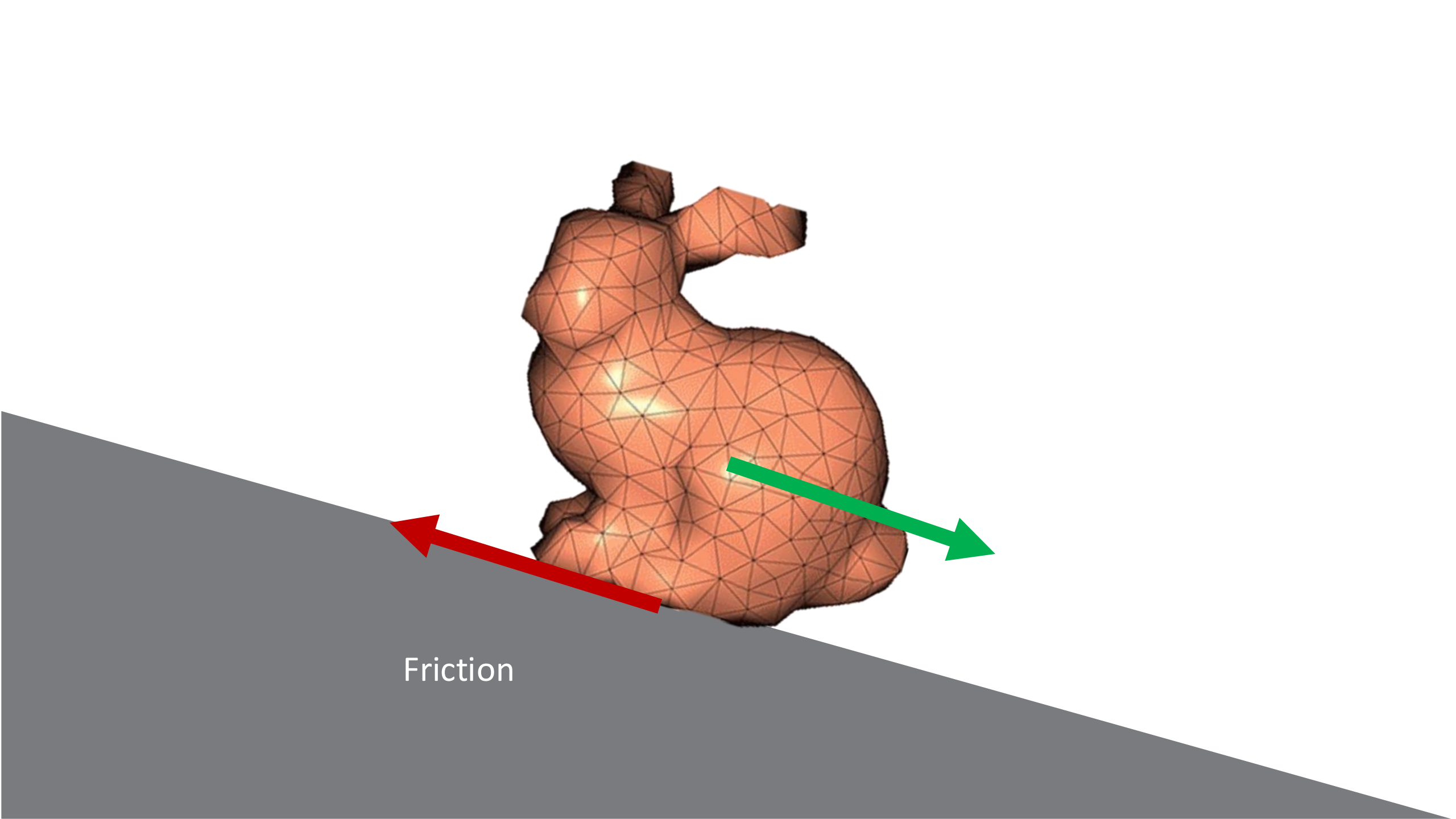
But what is the Deformation Gradient ?



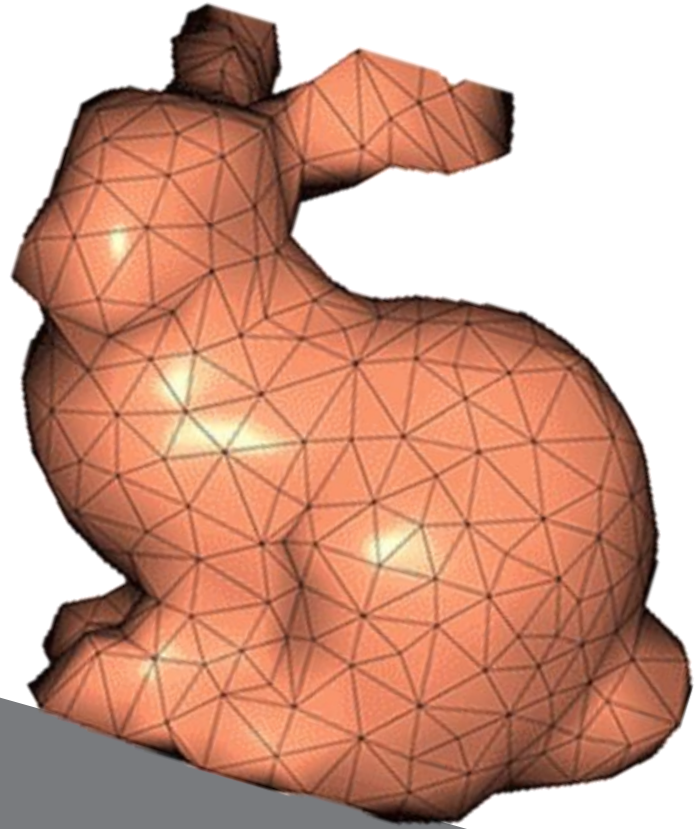
Reference (Undeformed) Space

$$\begin{bmatrix} q_0 & q_1 & q_2 \\ q_4 & q_5 & q_6 \\ q_8 & q_9 & q_{10} \end{bmatrix}^{-1} \left(\mathbf{x} - \begin{bmatrix} q_3 \\ q_7 \\ q_{11} \end{bmatrix} \right) = \mathbf{X}$$

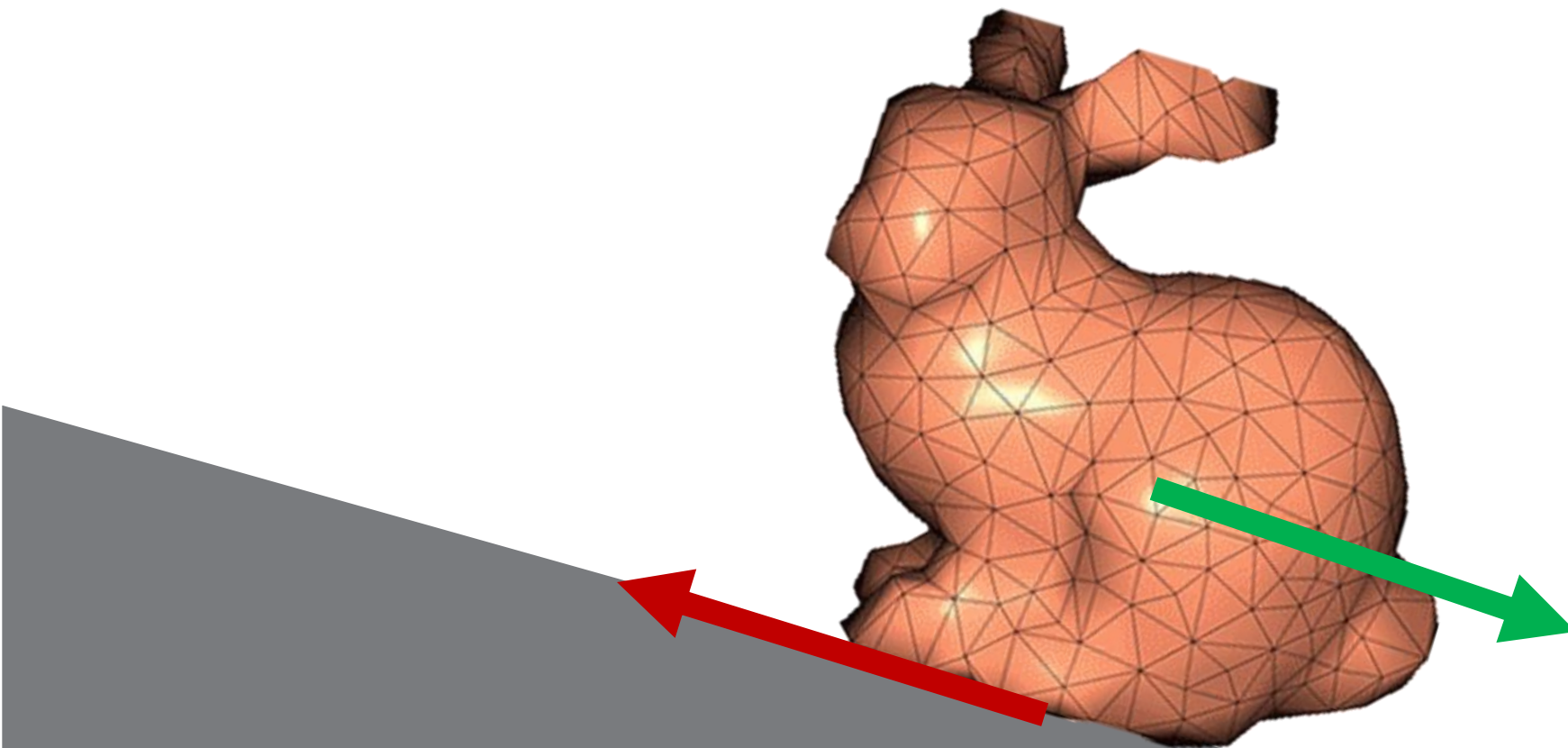




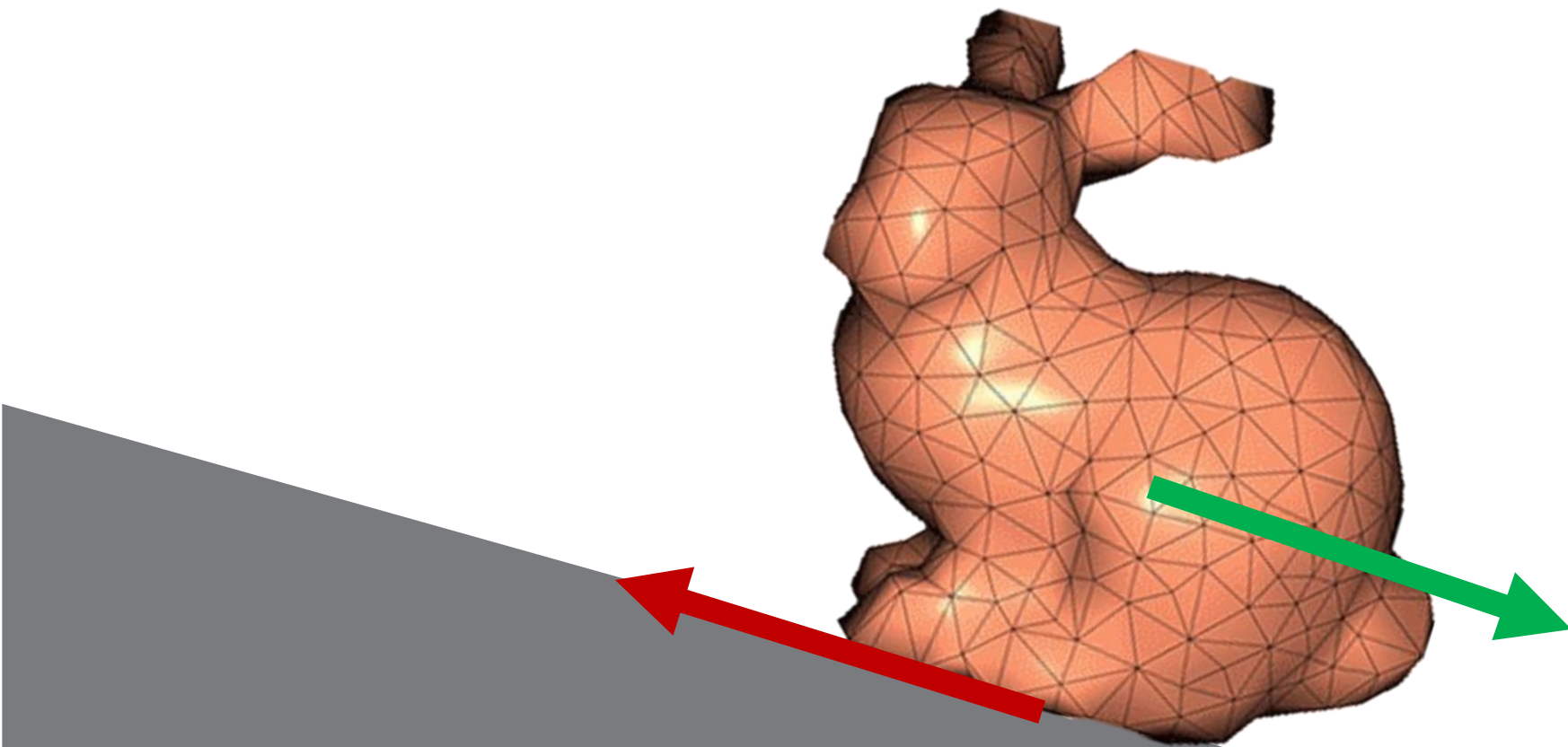
Friction



Static Friction: Holds things still

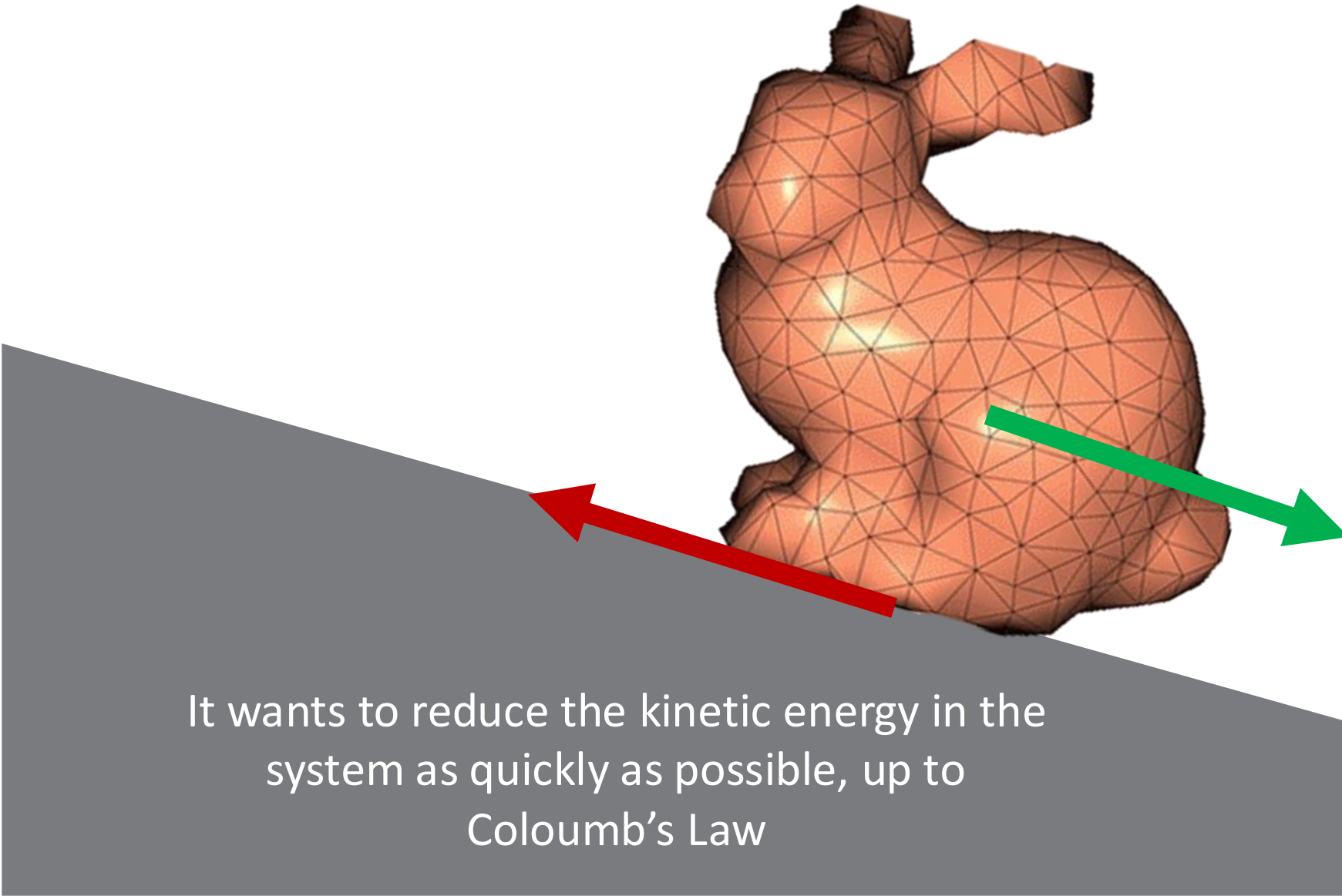


Dynamic Friction: Friction force
resists sliding when in motion

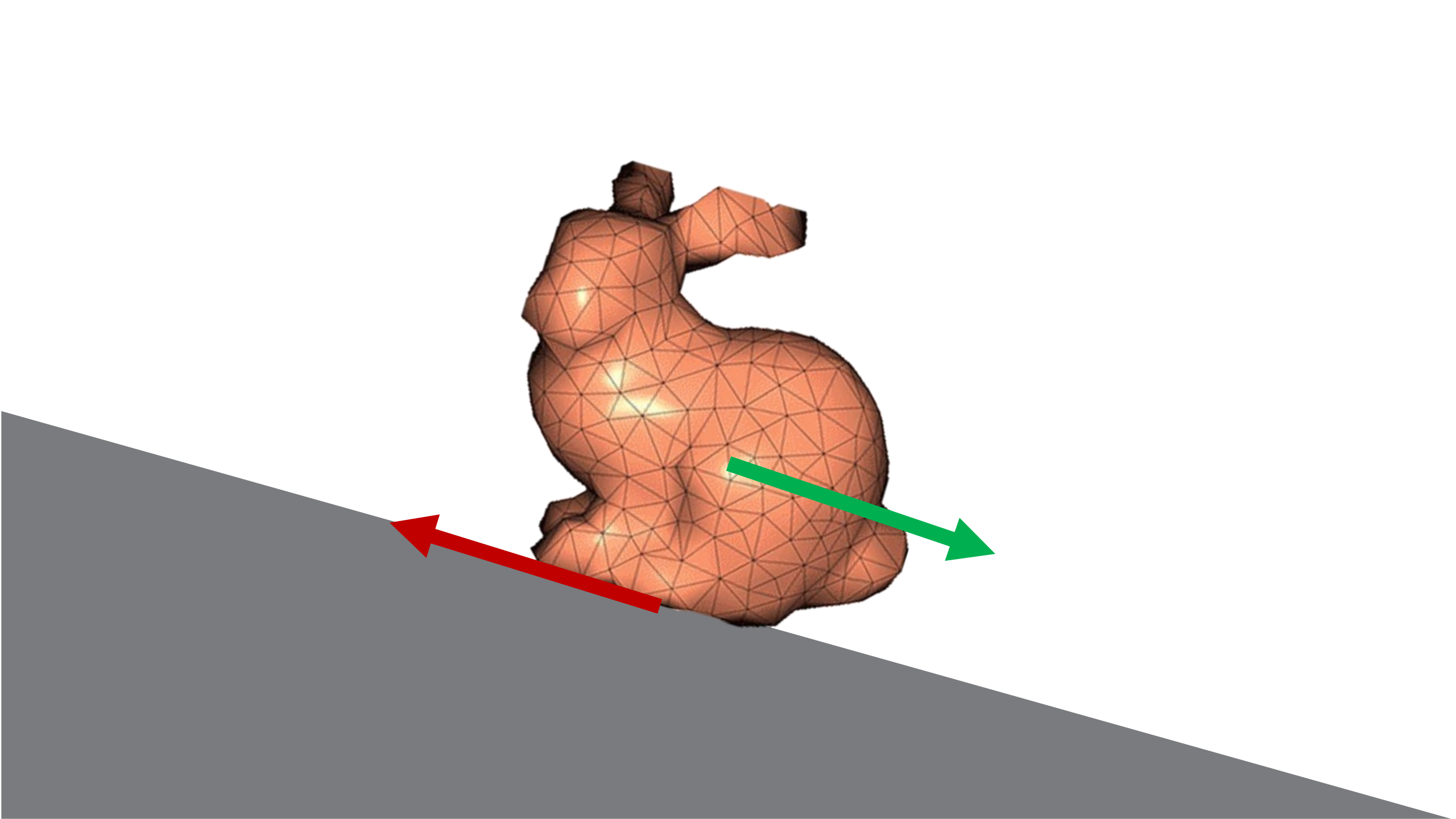


Coulomb's Law: $||\mathbf{f}|| \leq \mu ||\mathbf{c}||$

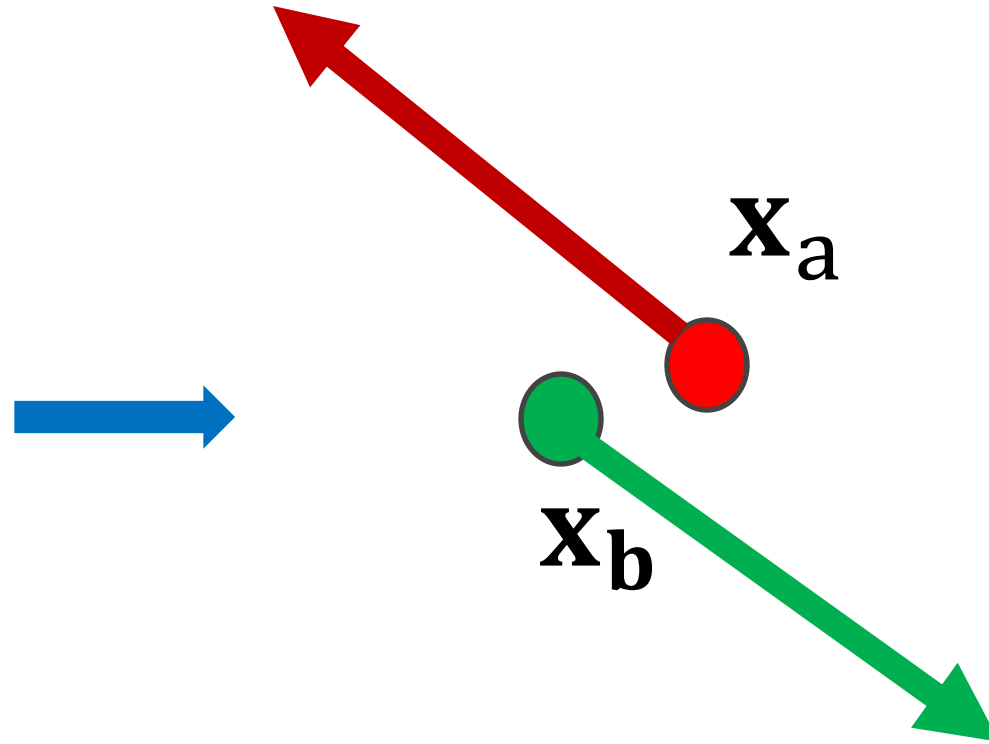
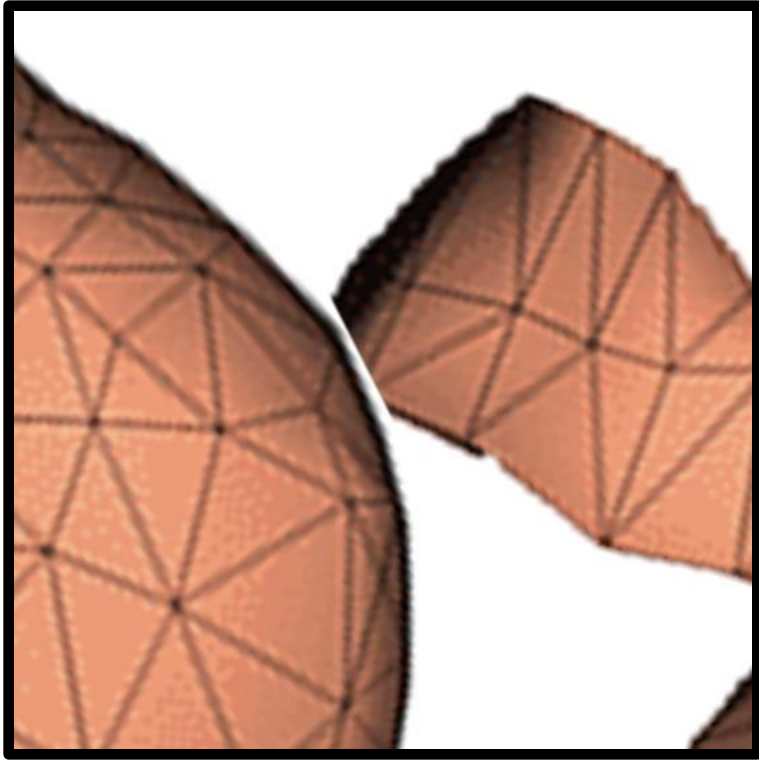
Friction is maximally dissipative



It wants to reduce the kinetic energy in the system as quickly as possible, up to Coloumb's Law

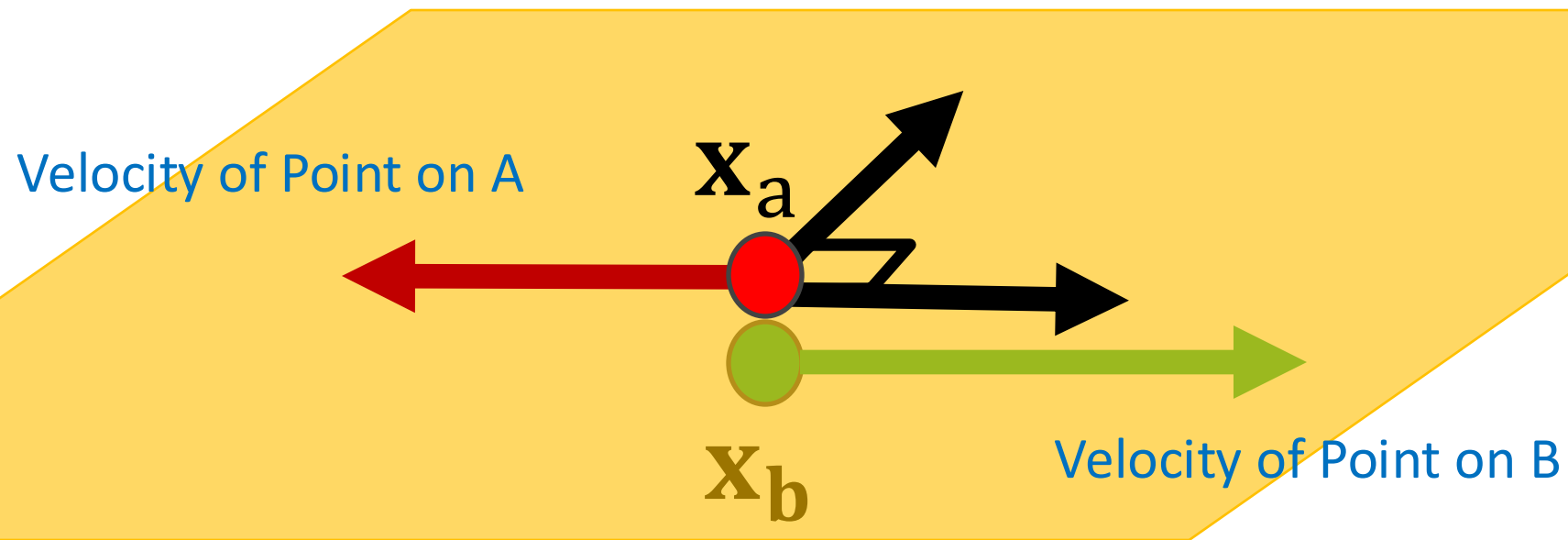


Friction Between Two Objects

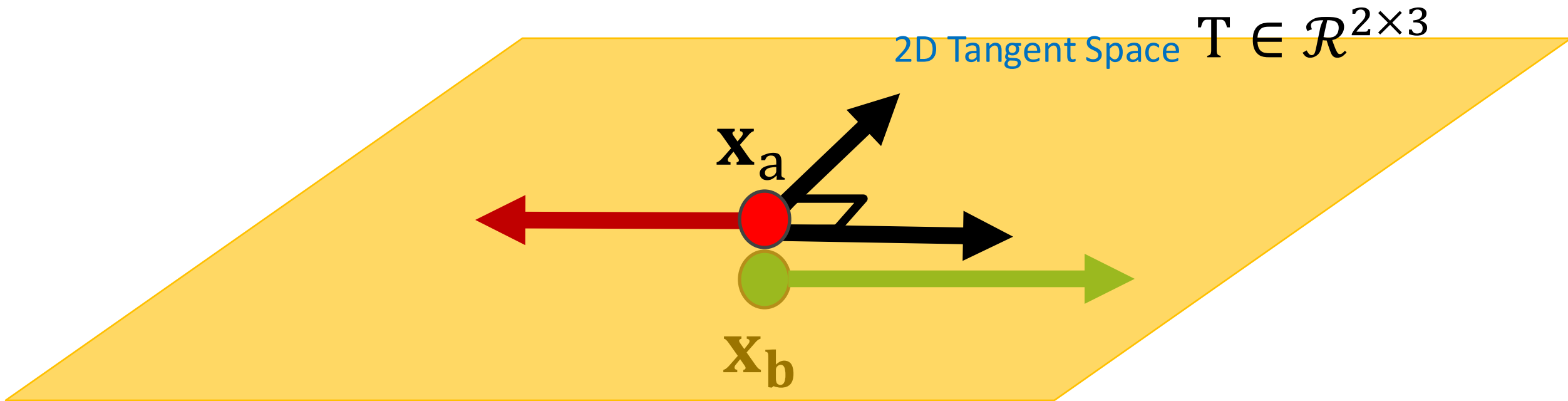


We apply friction between contact points where it opposes relative tangential velocity

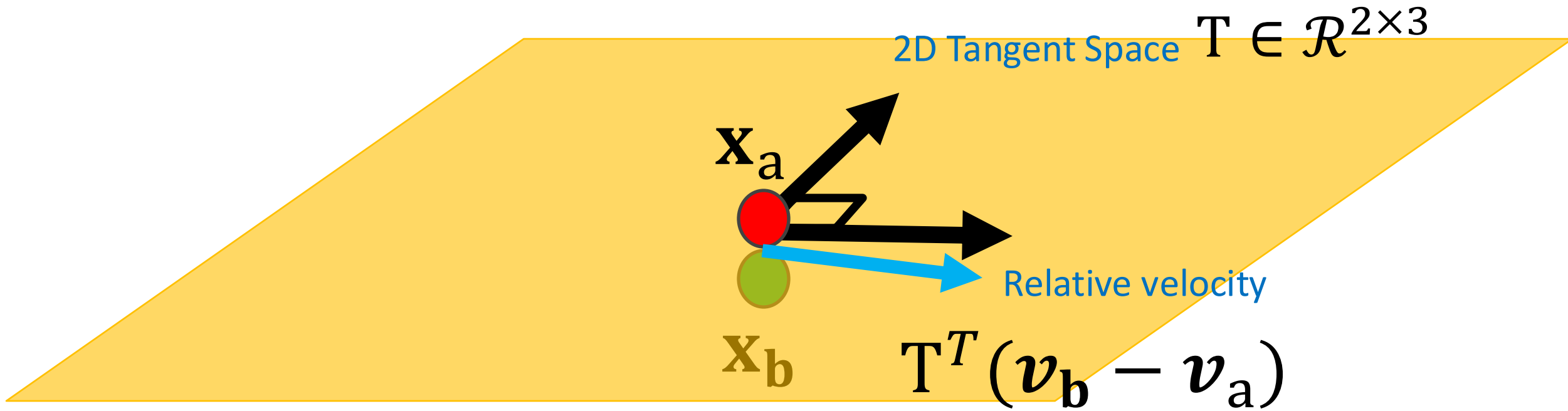
Friction Between Two Objects – the Tangent Space



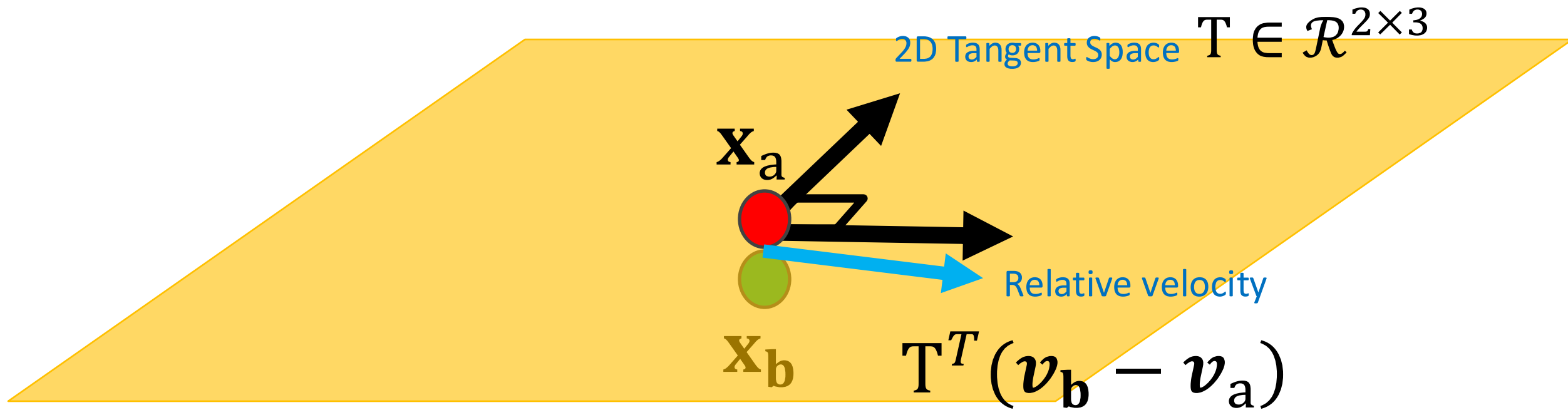
Friction Between Two Objects – the Tangent Space



Friction Between Two Objects – the Tangent Space



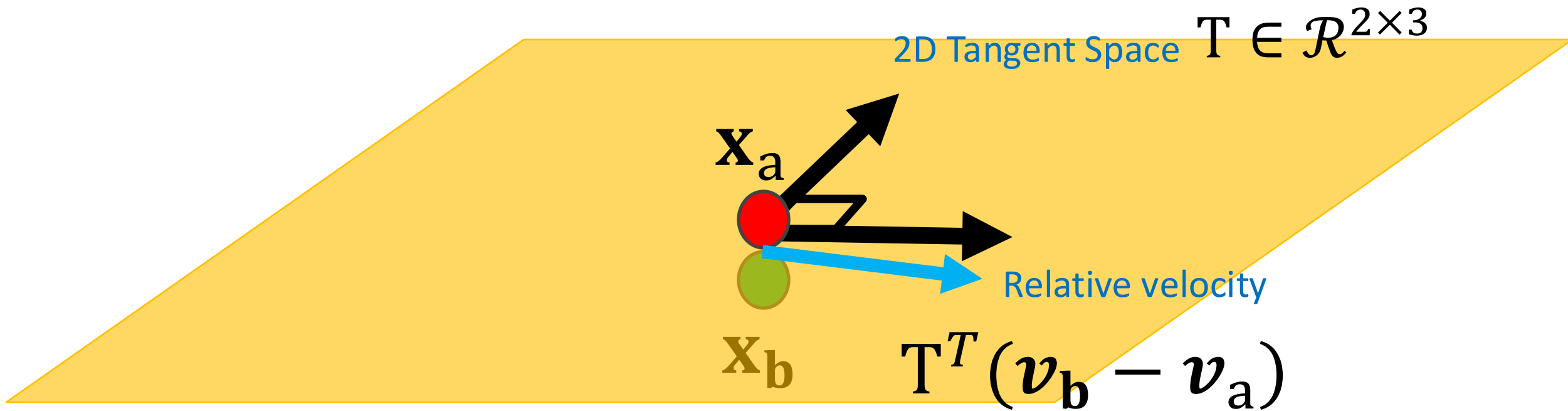
Friction Between Two Objects – the Tangent Space



An approximation:

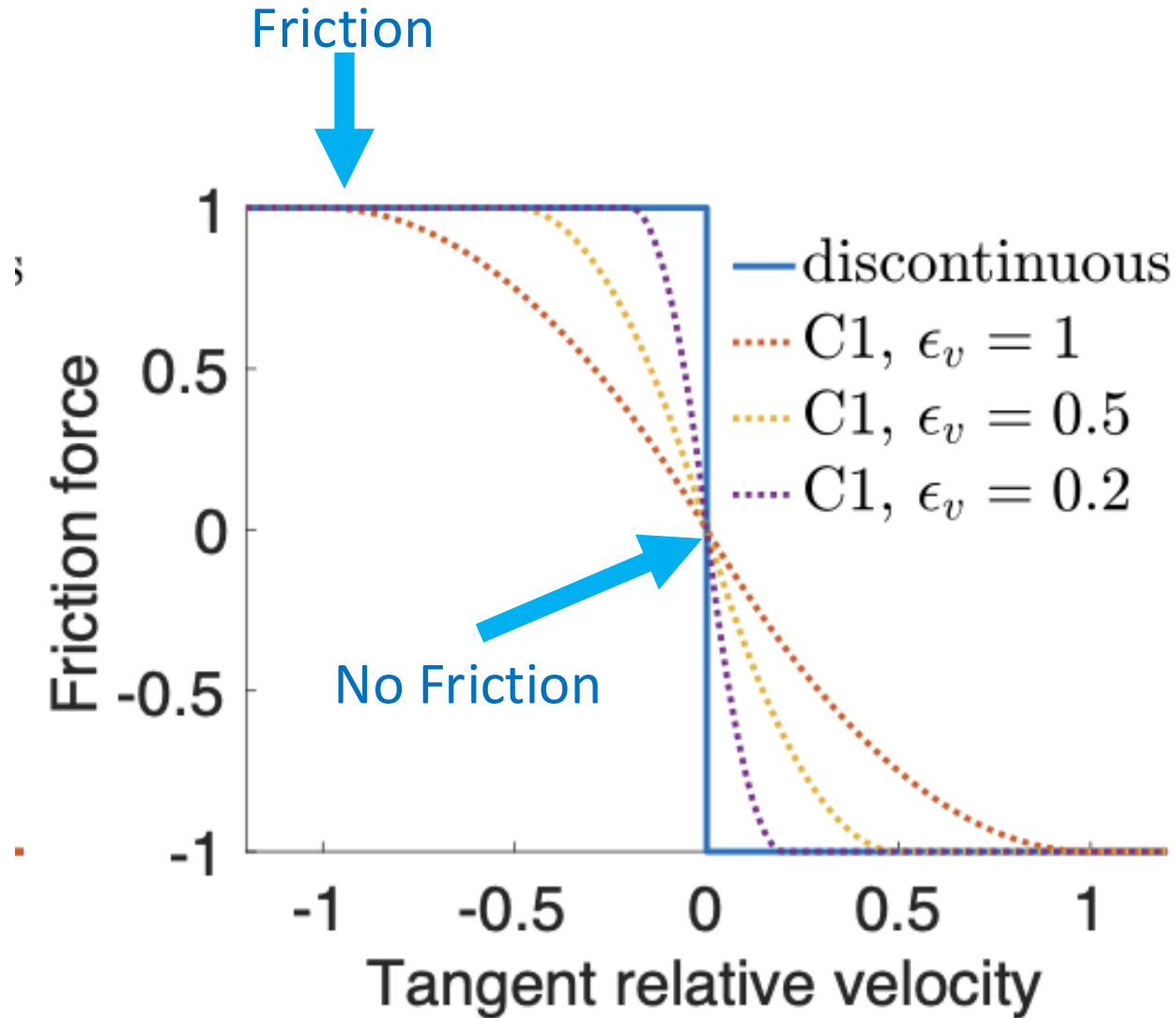
1. if relative velocity is zero, force of friction is zero
2. Otherwise friction opposes relative velocity with coloumb law magnitude.

Friction Between Two Objects – the Tangent Space



Ideally, we could write this out as an Energy and add it to our implicit integrator !

Introduce a “Threshold” Function



$$f_1(y) = \begin{cases} -\frac{y^2}{\epsilon_v^2 h^2} + \frac{2y}{\epsilon_v h}, & y \in (0, h\epsilon_v) \\ 1, & y \geq h\epsilon_v, \end{cases}$$

A Simple Friction Spring Energy

$$V_{friction}(\mathbf{q}) = \mu\lambda f_0(||\mathbf{v}_r^{t+1}(\mathbf{q})||)$$

T only computed at time t

$$\mathbf{v}_r^{t+1} = \mathbf{T}^T(\mathbf{v}_b - \mathbf{v}_a)$$

$$\lambda^t = ||\mathbf{c}||$$

A Simple Friction Spring Energy

Integral of f_1 wrt magnitude of tangential velocity



$$V_{friction}(\mathbf{q}) = \mu\lambda f_0(||\mathbf{v}_r^{t+1}(\mathbf{q})||)$$

\mathbf{T} only computed at time t

$$\mathbf{v}_r^{t+1} = \mathbf{T}^T(\mathbf{v}_b - \mathbf{v}_a)$$

$$\lambda^t = ||\mathbf{c}||$$

Multibody AND Contact AND Friction in One Solver

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M(\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 \underbrace{V(\mathbf{q}^{i+1})}_{V_{springs} + V_{affine} + V_{\{friction\}}}$$

This Video: Rigid Body Simulation with Contact

