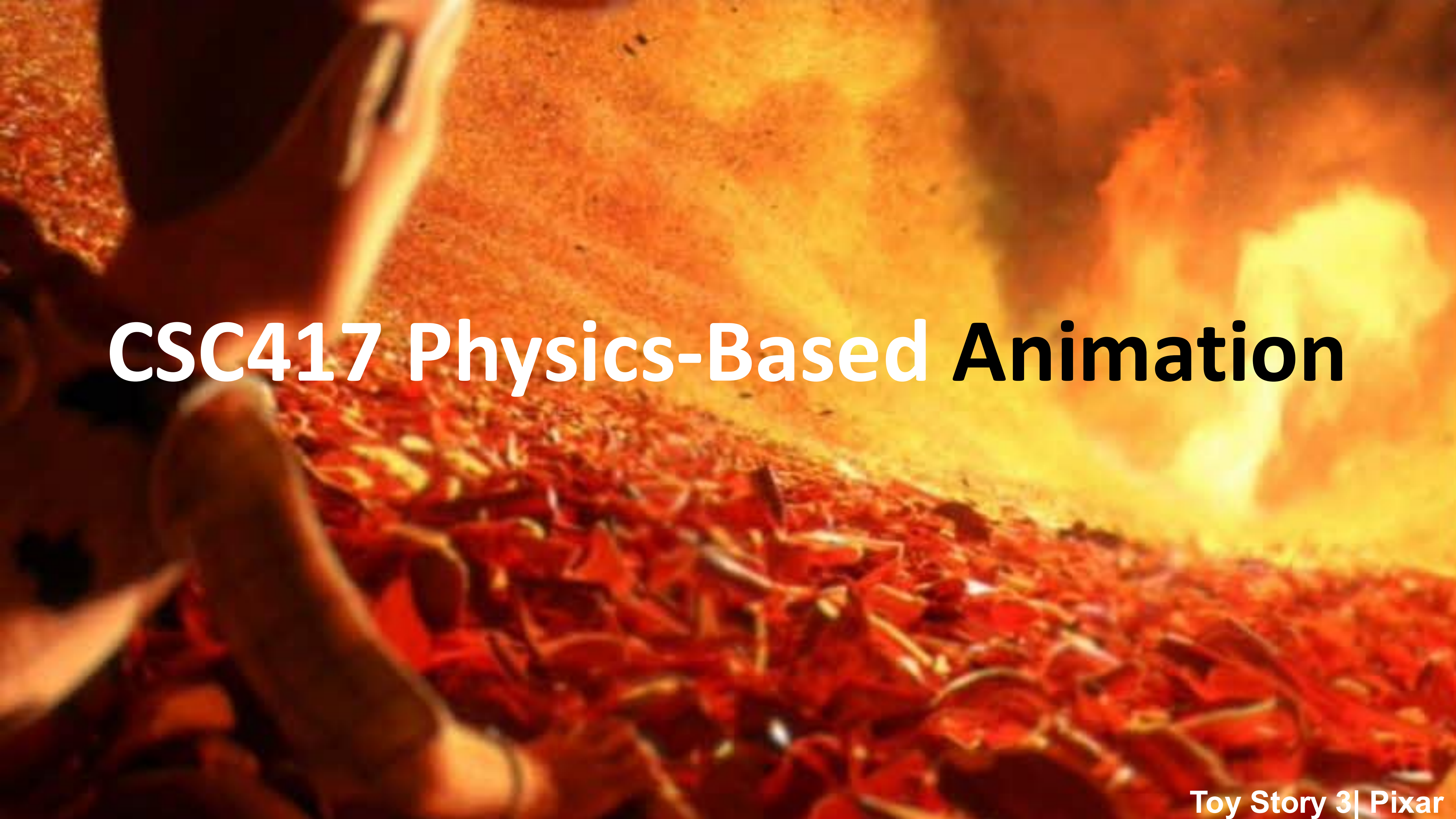
- Modal Warping
- Skinning Subspaces (do an introduction to skinning here)

- **Coordinate-Based Methods (fugure work slide)**
  - Handle-based deformation with harmonic coordinates
  - Variational form of harmonic Coordinates
  - Connection to modal subspaces: similar variational form, different constraints.
- **Nonlinear Modes + Others (future work slide)**
  - Linear vs nonlinear neural modes
  - Latent space dynamics
    - Fluid latent space dynamics (nils thuerey stuff)
    - Dave's elastic latent space physics one
    - Data-free learning of kinematics (the Sharpe one)
  - (optional) Linear modes with a nonlinear energy (simplicits)

CSC417 Physics-Based Animation



Toy Story 3| Pixar

Last Week: Affine Body Simulation with Contact

Questions from Previous Lecture ?

We Solve This Every Time Step

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 V\left(\mathbf{q}^{\mathrm{i+1}}\right)$$

Gradient of what equals this ? Let's guess, then check

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \left. \frac{\partial E}{\partial \mathbf{v}} \right|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

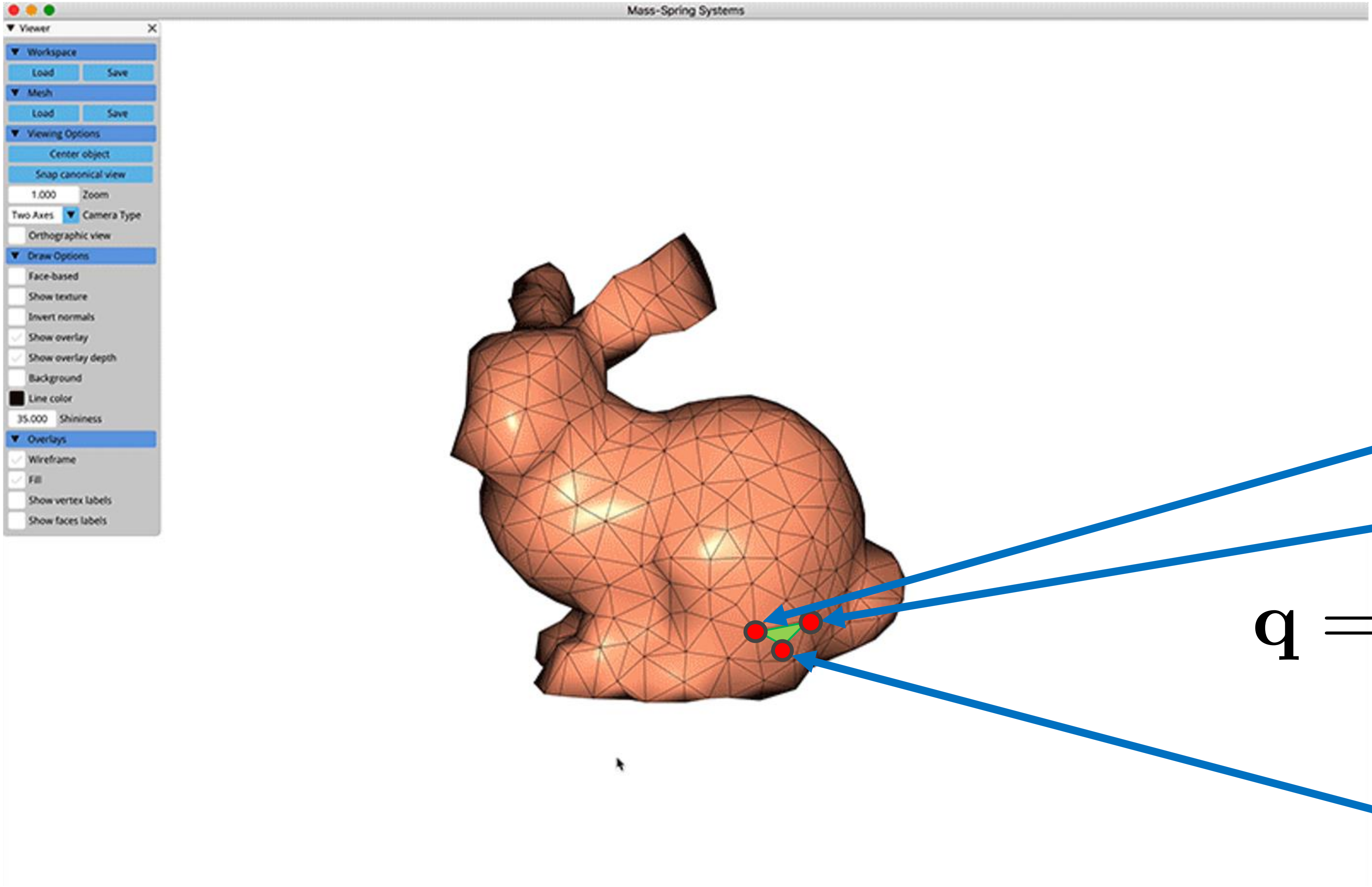
Choose α using line search

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

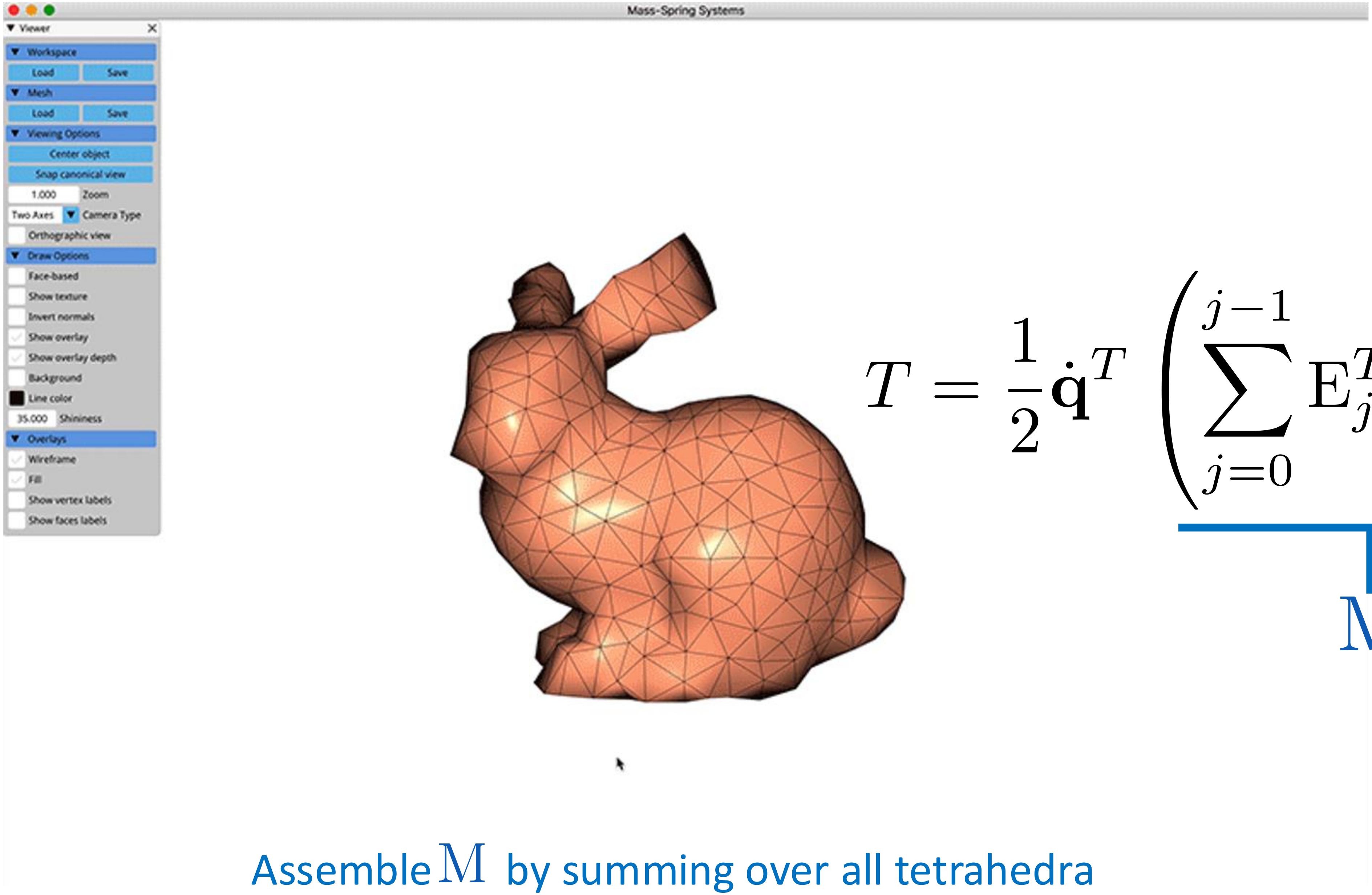
Repeat until converged

Spatial Discretization -- Finite Elements



$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

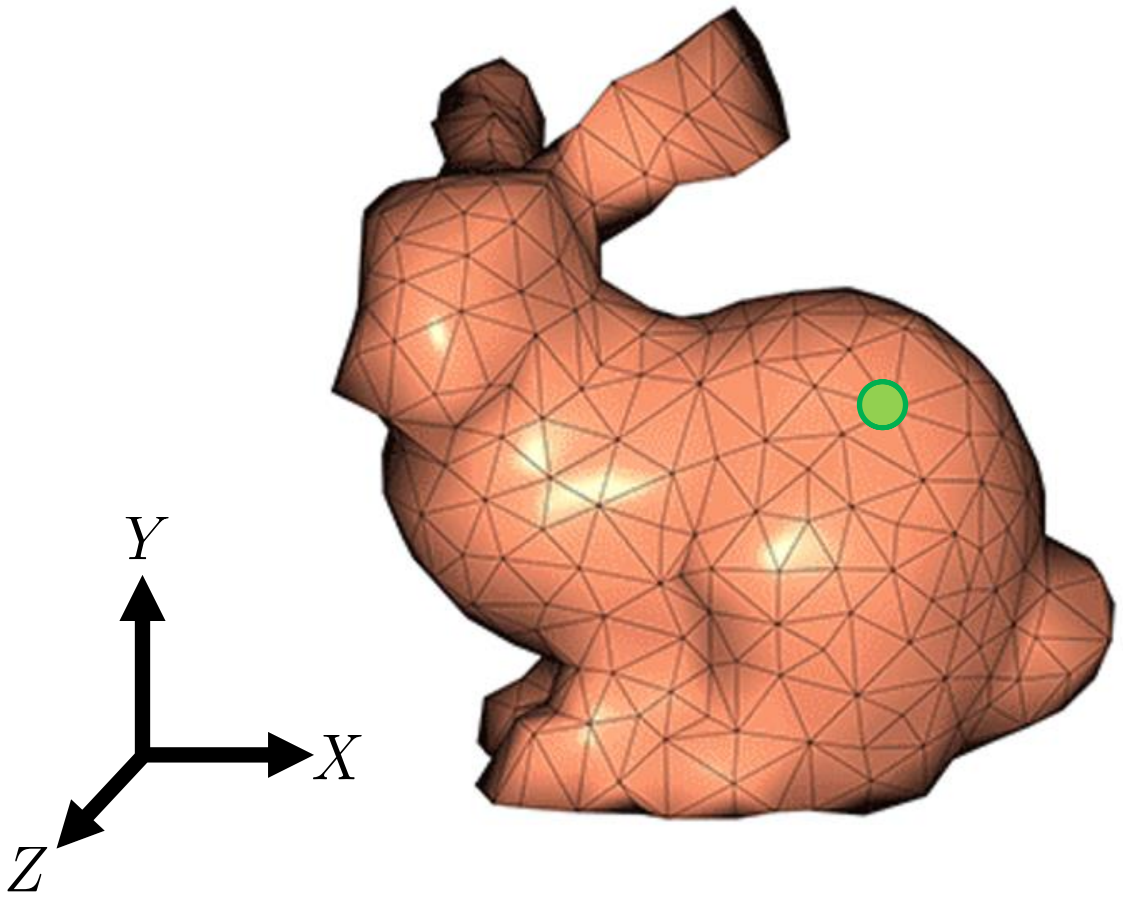
Kinetic Energy for a Bunny



$$T = \frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\left(\sum_{j=0}^{j-1} \mathrm{E}_j^T \mathrm{M}_j \mathrm{E}_j\right)}_{\mathrm{M}} \dot{\mathbf{q}}$$

Assemble M by summing over all tetrahedra

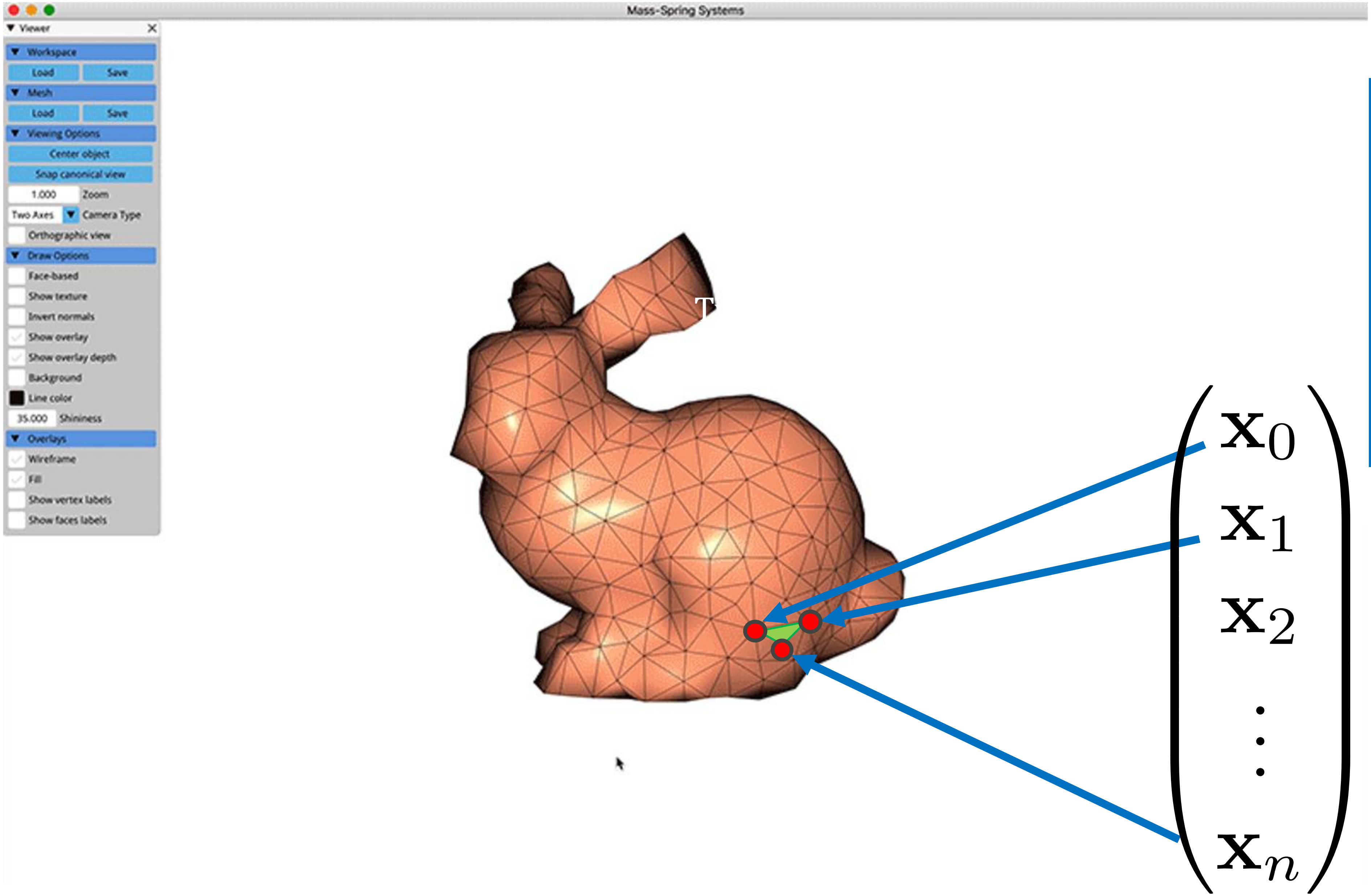
Affine Body Dynamics – Reduced-Order Elasticity



Reference (Undeformed) Space

$$\mathbf{x}(\mathbf{X}, t) = \mathrm{J}(\boldsymbol{X})\mathrm{q}(\mathrm{t})$$

Spatial Discretization -- Finite Elements



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} = \mathrm{U}\boldsymbol{r}$$

Reduced Space

|r| << |q|

Pros and Cons of Model Reduction

Pros

- Runtime depends on smaller number of degrees of freedom
- Many calculations become independent of mesh resolution

Cons

- Loss of expressivity (reduced model can’t represent all the motions the full space model can)

How Do We Build a Reduced-Order Model That’s Compact and Expressive ?

How Do We Build a Reduced-Order Model That's Compact and Expressive ?

1. Modal Analysis

2. Data-Driven Deformation Modes

Modal Analysis Starts Here

$$M\ddot{\boldsymbol{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Modal Analysis Starts Here

$$M\ddot{\boldsymbol{q}} + \frac{\partial V}{\partial \mathbf{q}} = 0$$

Linearize Around Rest Shape

$$M\left(\ddot{\boldsymbol{q}(0) + \boldsymbol{u}(\mathrm{t})}\right) + \frac{\partial V\left(\boldsymbol{q}(0) + \boldsymbol{u}(\mathrm{t})\right)}{\partial \mathbf{q}} = 0$$

Linearize Around Rest Shape

$$M\ddot{\boldsymbol{u}}(\mathrm{t}) + \underbrace{\frac{\partial V^2}{\partial \mathbf{q}^2}}_{\mathrm{H}}\boldsymbol{u}(\mathrm{t}) = 0$$

Linearize Around Rest Shape

$$M\ddot{\boldsymbol{u}}(\mathrm{t}) + \mathrm{H}\boldsymbol{u}(\mathrm{t}) = 0$$

A linear, homogenous ODE!
How can we solve it ?

Assume we know the solution !

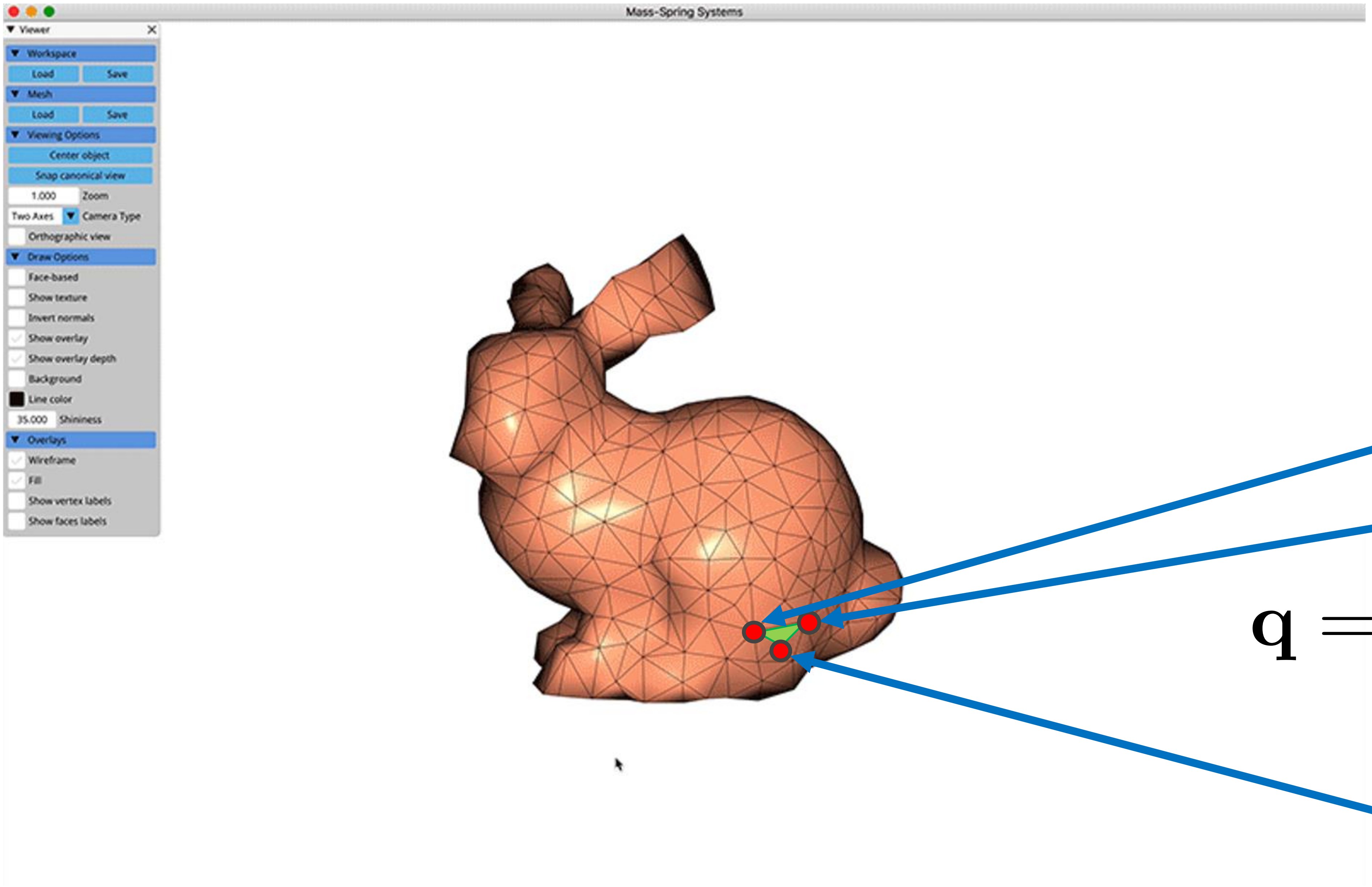
$\mathcal{R}^{3n}$ vector

$$\mathbf{u} = \boldsymbol{v} e^{\lambda t}$$

Complex constant

$$\ddot{\mathbf{u}} = \lambda^2 \boldsymbol{v} e^{\lambda t}$$

Spatial Discretization -- Finite Elements



$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Assume we know the solution !

$\mathcal{R}^{3n}$ vector

$$\mathbf{u} = \boldsymbol{v} e^{\lambda t}$$

Complex constant

$$\ddot{\mathbf{u}} = \lambda^2 \boldsymbol{v} e^{\lambda t}$$

Substitute our “guess” into the equation

$$\lambda^2 M\mathbf{u}e^{\lambda t} - K\mathbf{u}e^{\lambda t} = \mathbf{0}$$

Substitute our "guess" into the equation

$$\lambda^2 M\mathbf{u}\cancel{e^{\lambda t}} - K\mathbf{u}\cancel{e^{\lambda t}} = \mathbf{0}$$

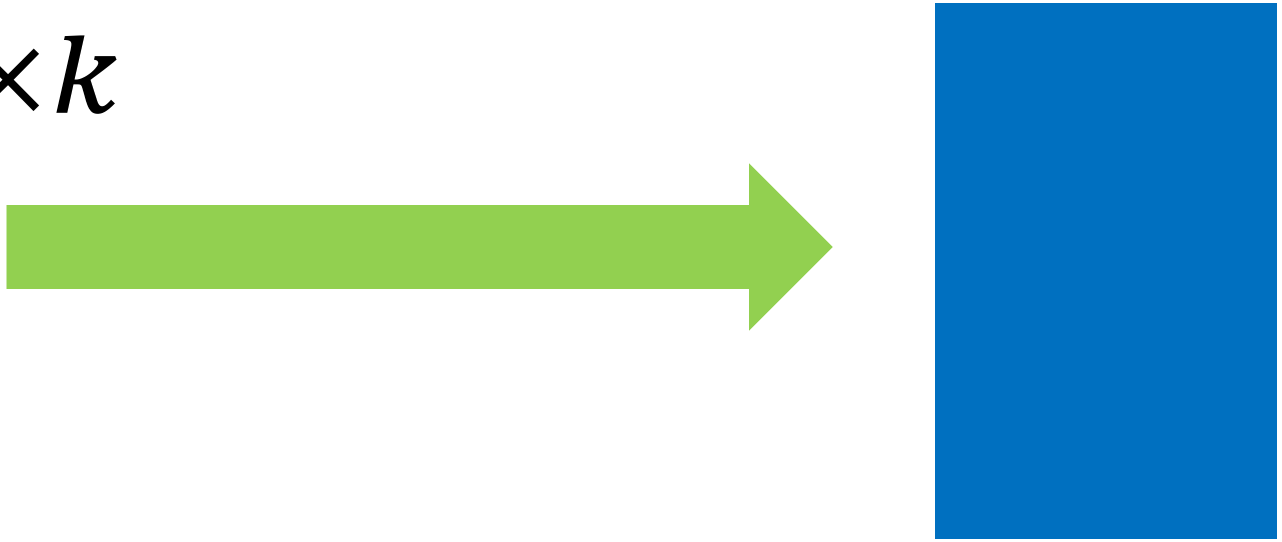
Generalized Eigenvector Problem

$$\underbrace{\lambda}_{\text{Eigenvalue}}^2 M\mathbf{u} - K\underbrace{\mathbf{u}}_{\text{Eigenvector}} = \mathbf{0}$$

Modal Analysis

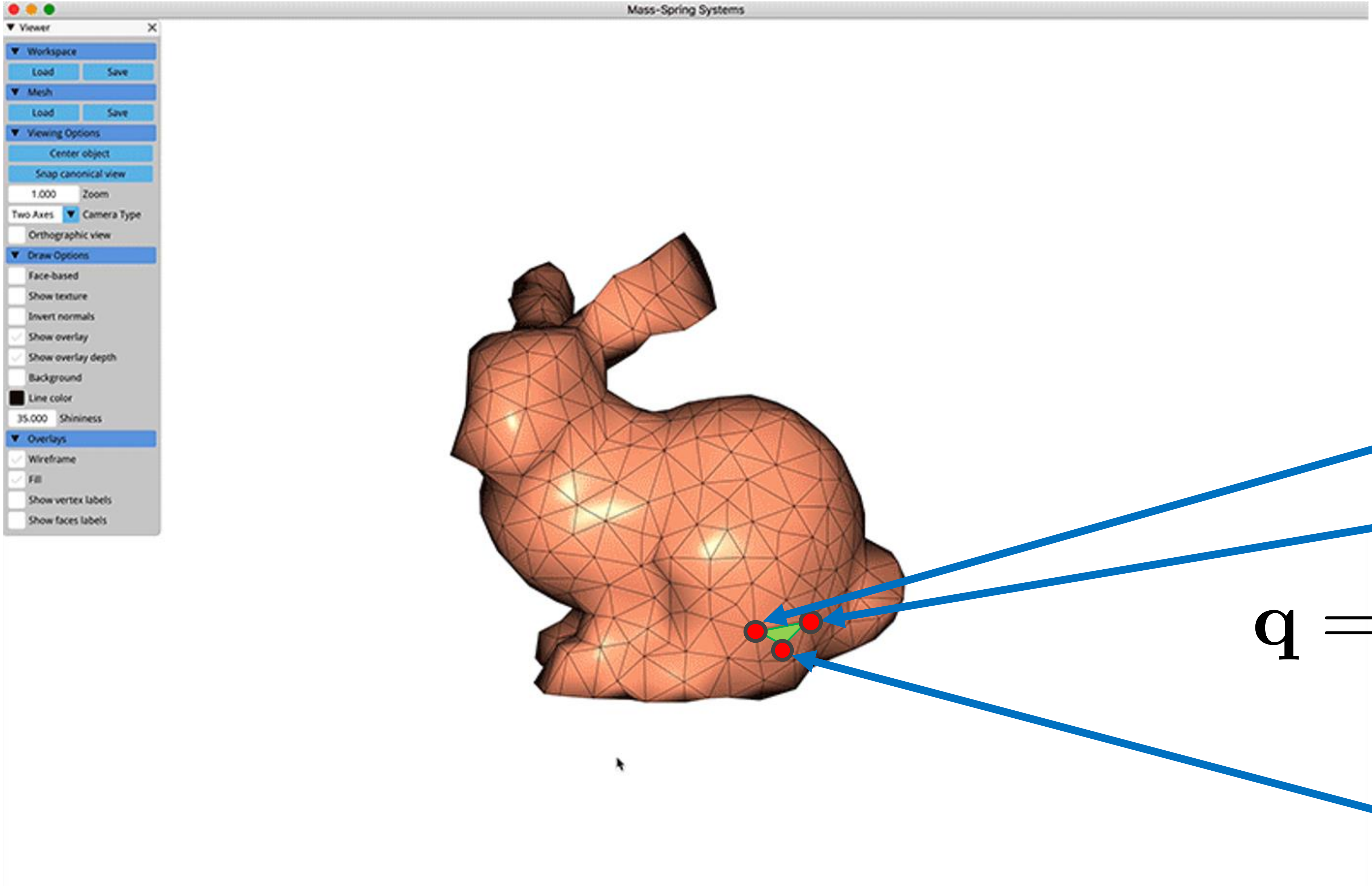
So a good reduced space is a vector of the *k* lowest order Eigenvectors

$$U \in \mathcal{R}^{n \times k}$$



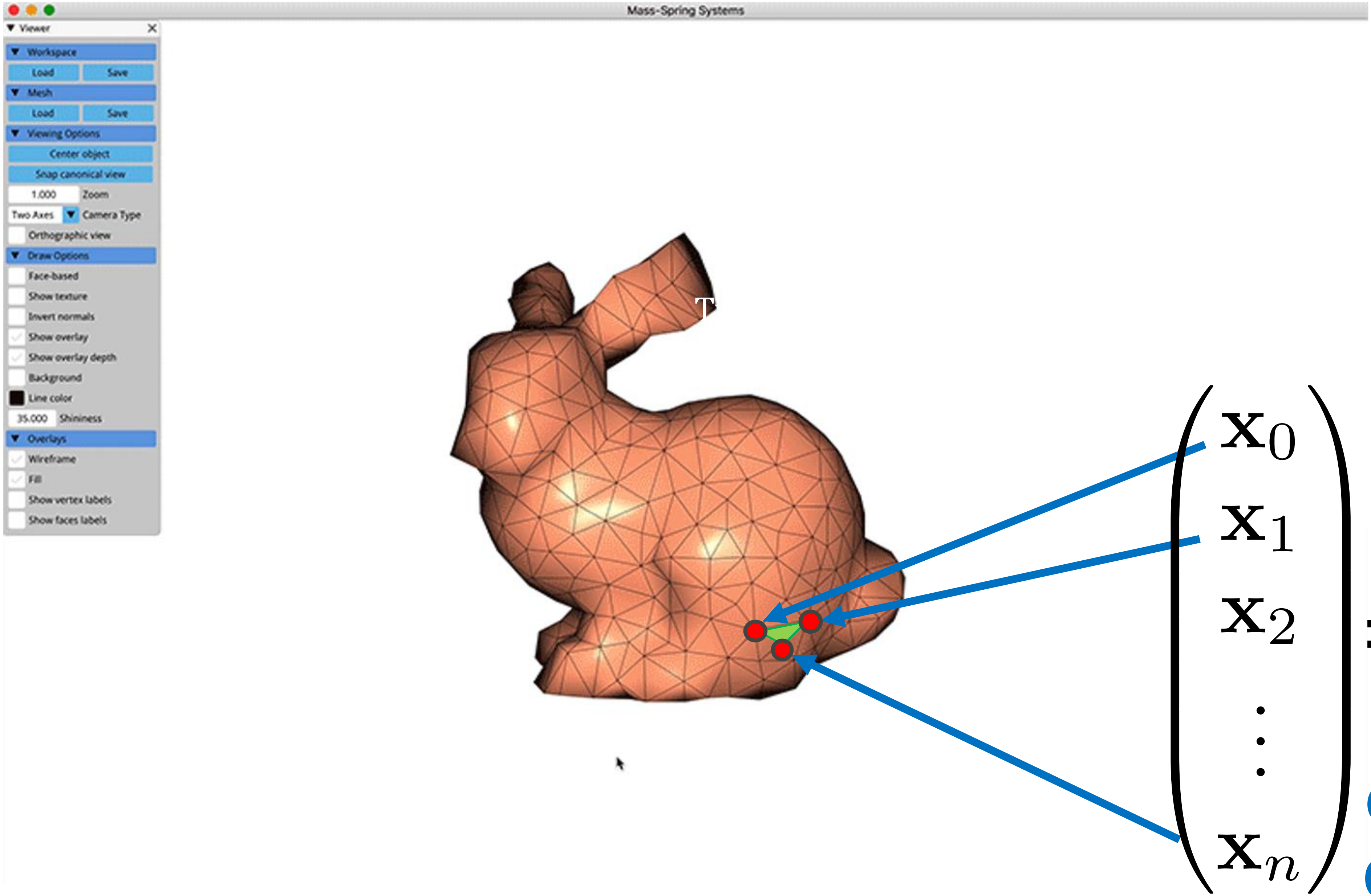
Reduced Space

Spatial Discretization -- Finite Elements



$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Spatial Discretization -- Finite Elements



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} : q(0) + U\boldsymbol{r}$$

Generalized Coordinates

An Aside: Variational Modal Analysis

$$U^* = \arg\min_U tr(U^T H U)$$

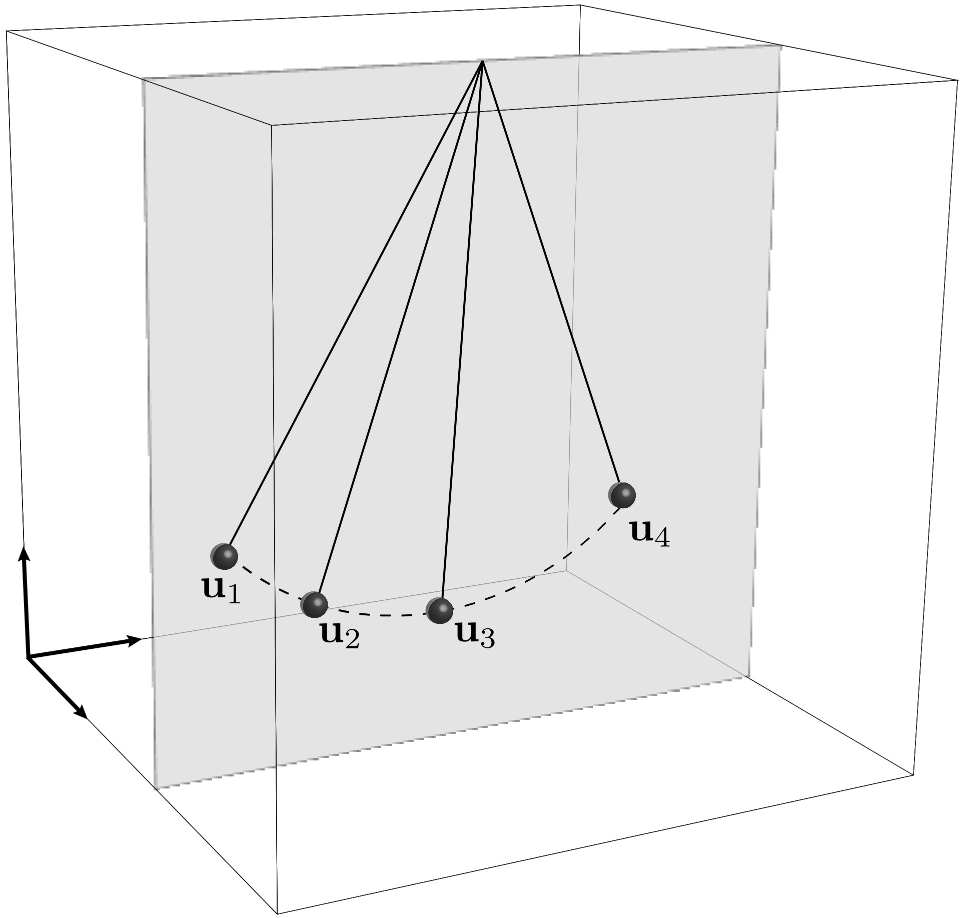
$$s.t\ U^T M U = I$$

How Do We Build a Reduced-Order Model That's Compact and Expressive ?

1. ~~Modal Analysis~~

2. Data-Driven Deformation Modes

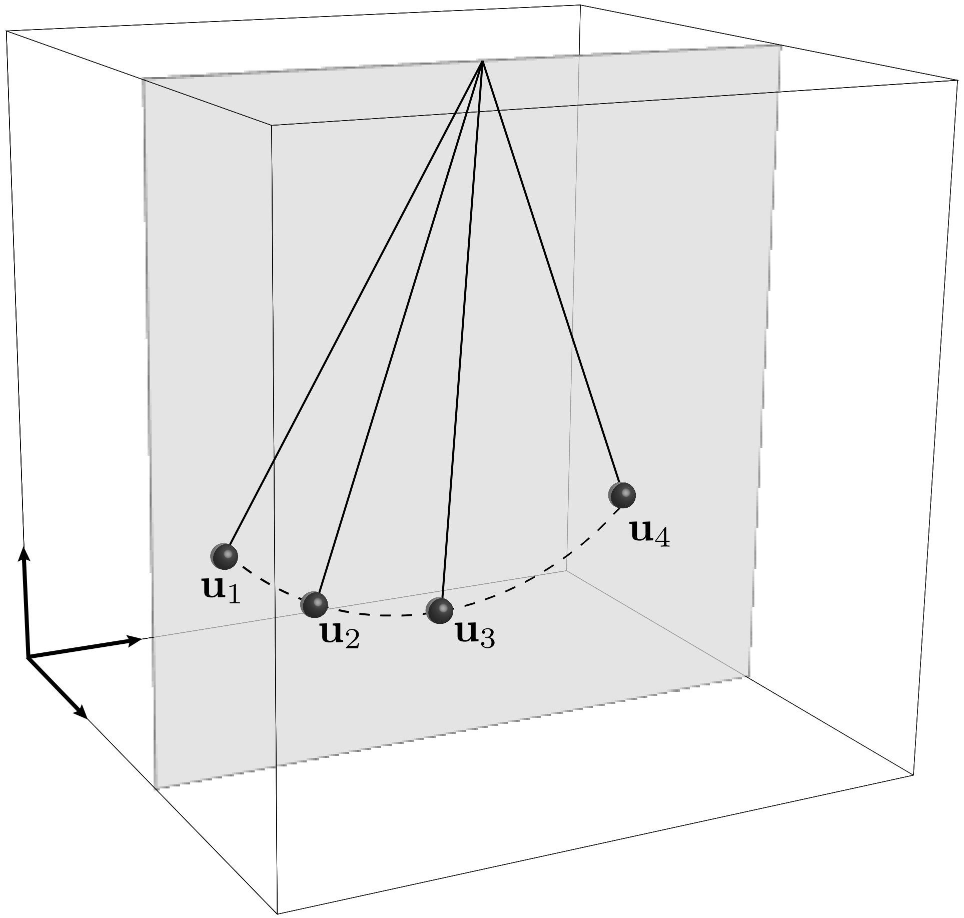
Data-Driven Deformation Modes



Collect Snapshots

$$Q = [q_0, q_1, q_2 \ldots]$$

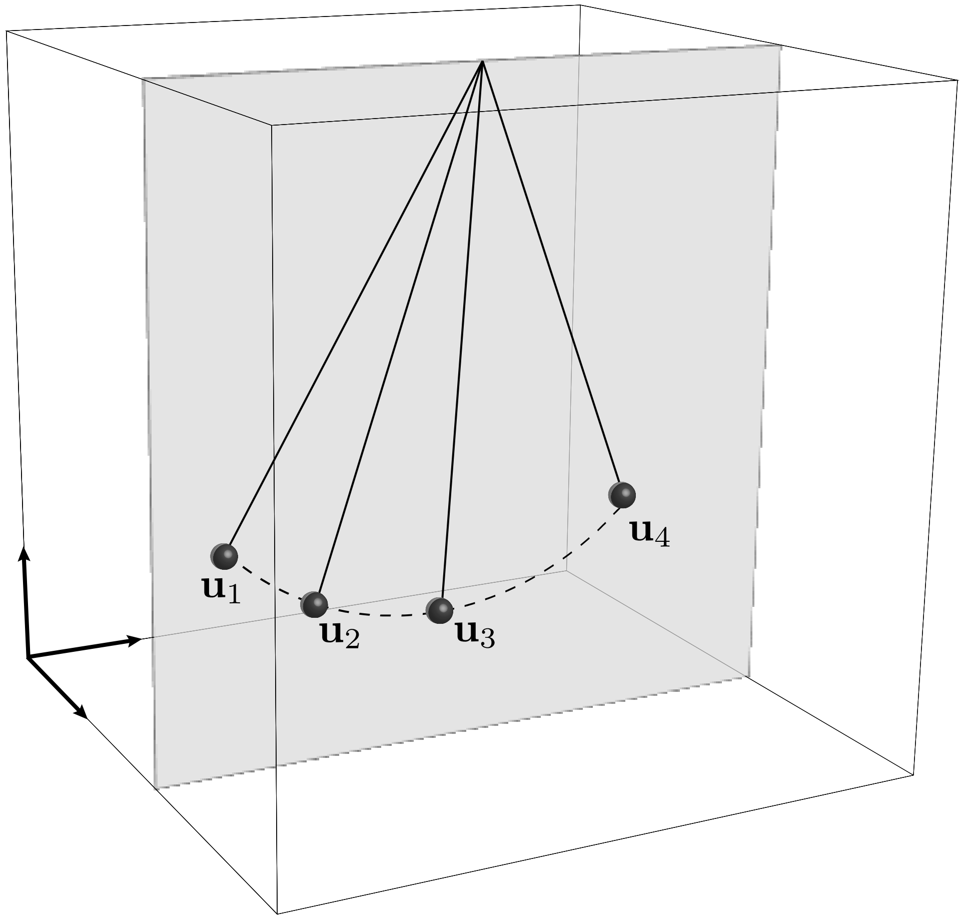
Data-Driven Deformation Modes



Collect Snapshots

$$Q = [q_0, q_1, q_2 \dots]$$

Probabilistic Reconstruction

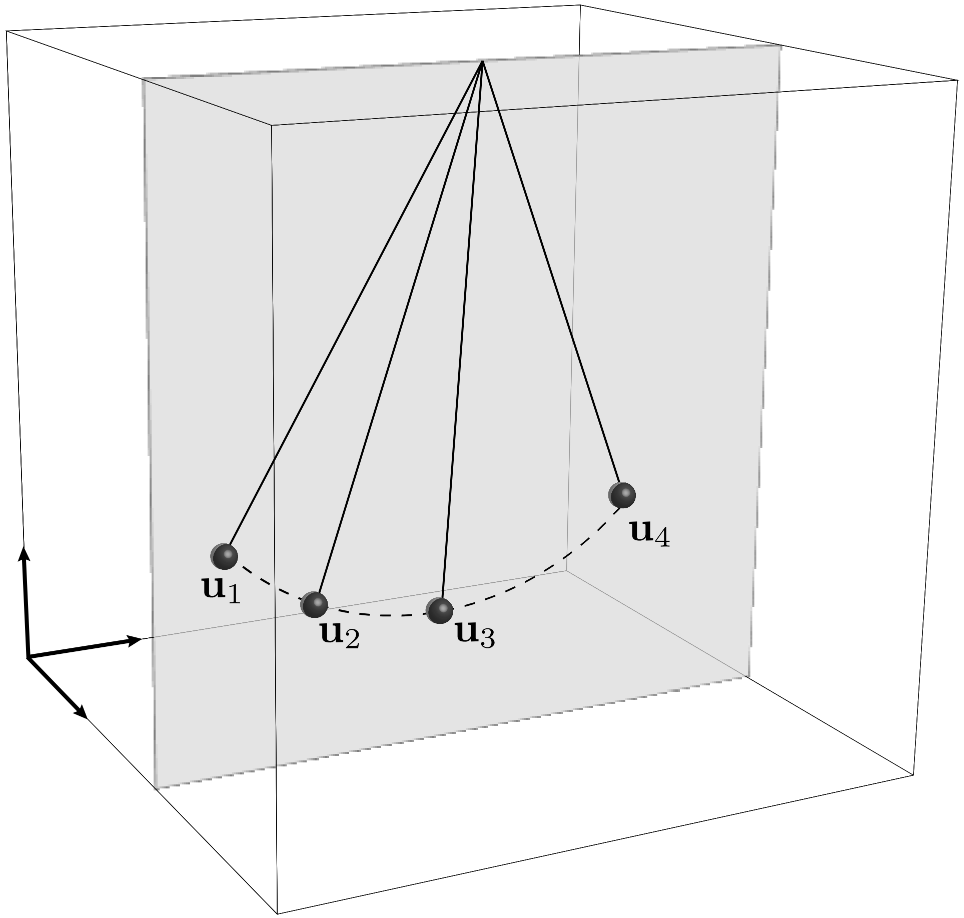


Given a **bunch of states**, assume their **displacements** from the **mean state** can be **modelled linearly**

$$s.t\ \mathbf{q_i} = \boldsymbol{\mu}(\mathbf{0}) + U\mathbf{r_i}$$

Mean pose from data

Probabilistic Reconstruction



Given a **bunch of states**, assume their **displacements** from the **mean state** can be **modelled linearly**

$$s.t\ \underbrace{(\mathbf{q_i} - \boldsymbol{\mu}(\mathbf{0}))}_{\text{Displacement } \boldsymbol{u}_\mathrm{i}} = U\mathbf{r_i}$$

Probabilistic Reconstruction

Find U that minimizes expected reconstruction error

$$\boldsymbol{U}^* = \arg\min \sum_{\boldsymbol{i}} ||\mathbf{u_i} - \boldsymbol{U}\mathbf{r_i}||_2^2 \underbrace{\boldsymbol{p_{data}}(\mathbf{u}_{\boldsymbol{i}})}_{\text{Data distribution (unknown)}}$$

Probabilistic Reconstruction

Find U that minimizes expected reconstruction error

$$\boldsymbol{U}^* = \arg\min \sum_{\boldsymbol{i}} ||\mathbf{u_i} - \boldsymbol{U}\mathbf{r_i}||_2^2 \underbrace{\boldsymbol{p_{data}}(\mathbf{u}_{\boldsymbol{i}})}_{\text{Data distribution (unknown)}}$$

$$s.t\ U^T U = I$$

Prevent collapse of reduced space

Some useful identities

Projection onto an orthogonal subspace

$$\mathbf{r}_{\mathrm{i}} = \boldsymbol{U}^{\boldsymbol{T}}\,\mathbf{u}_{\boldsymbol{i}}$$

Vector norms as traces

$$||\boldsymbol{v} - \mathrm{A}\boldsymbol{v}||_2^2 \quad = \mathrm{trace}(A\boldsymbol{v}\boldsymbol{v}^{\boldsymbol{T}})$$

Linearity of Expectation

$$\sum\nolimits_{\mathrm{i}} \mathrm{trace}(AB_{\boldsymbol{i}})\boldsymbol{p}_{\boldsymbol{data}} = \mathrm{trace}(A \sum\nolimits_{\boldsymbol{i}} B_{\boldsymbol{i}}\boldsymbol{p}_{\boldsymbol{data}})$$

Probabilistic Reconstruction

Ok let's apply some identities

$$U^* = \arg\min \sum_i ||\mathbf{u_i} - U\mathbf{r_i}||_2^2 \underbrace{p_{data}(\mathbf{u}_i)}_{\text{Data distribution (unknown)}}$$

$$s.t\ U^T U = I$$

Prevent collapse of reduced space

Probabilistic Reconstruction

Step 1: Get rid of r

$$\boldsymbol{U}^* = \arg\min \sum_{\boldsymbol{i}} \left|\left|\mathbf{u_i} - \boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\mathbf{u_i}\right|\right|_2^2 \boldsymbol{p_{data}}(\mathbf{u}_{\boldsymbol{i}})$$

$$s.t\ U^T U = I$$

Prevent collapse of reduced space

Probabilistic Reconstruction

Step 2: Get rid of norm

$$\boldsymbol{U}^* = \arg\min \sum_{\boldsymbol{i}} \operatorname{trace}((\mathrm{I} - \boldsymbol{U}\boldsymbol{U}^T)\mathbf{u_i}\mathbf{u_i}^T)\boldsymbol{p}_{data}(\mathbf{u}_{\boldsymbol{i}})$$

$$s.t\ U^T U = I$$

Prevent collapse of reduced space

Probabilistic Reconstruction

Step 3: Apply linearity of expectation

$$U^* = \arg\min \operatorname{trace}\Big((\mathrm{I} - \boldsymbol{U}\boldsymbol{U}^T) \sum_{\boldsymbol{i}} \underbrace{\mathbf{u_i}\mathbf{u_i}^T \boldsymbol{p_{data}}(\mathbf{u}_{\boldsymbol{i}})}\Big)$$

$$s.t\ U^T U = I$$

Assume data is zero mean, unit variance so this is just the covariance $\boldsymbol{\Sigma}$

Probabilistic Reconstruction

Step 4: Get rid of terms independent of U

$$\boldsymbol{U}^* = \arg\min -\mathrm{trace}(\boldsymbol{U}\boldsymbol{U}^T\Sigma)$$
$$s.t\ U^T U = I$$

Probabilistic Reconstruction

Step 5: Rearrange and get rid of minus sign

$$\boldsymbol{U}^* = \arg\max \operatorname{trace}(\boldsymbol{U}^T \Sigma \boldsymbol{U})$$
$$s.t\ U^T U = I$$

An Aside: Variational Modal Analysis

$$U^* = \arg\min_U tr(U^T H U)$$

$$s.t\ U^T M U = I$$

Probabilistic Reconstruction

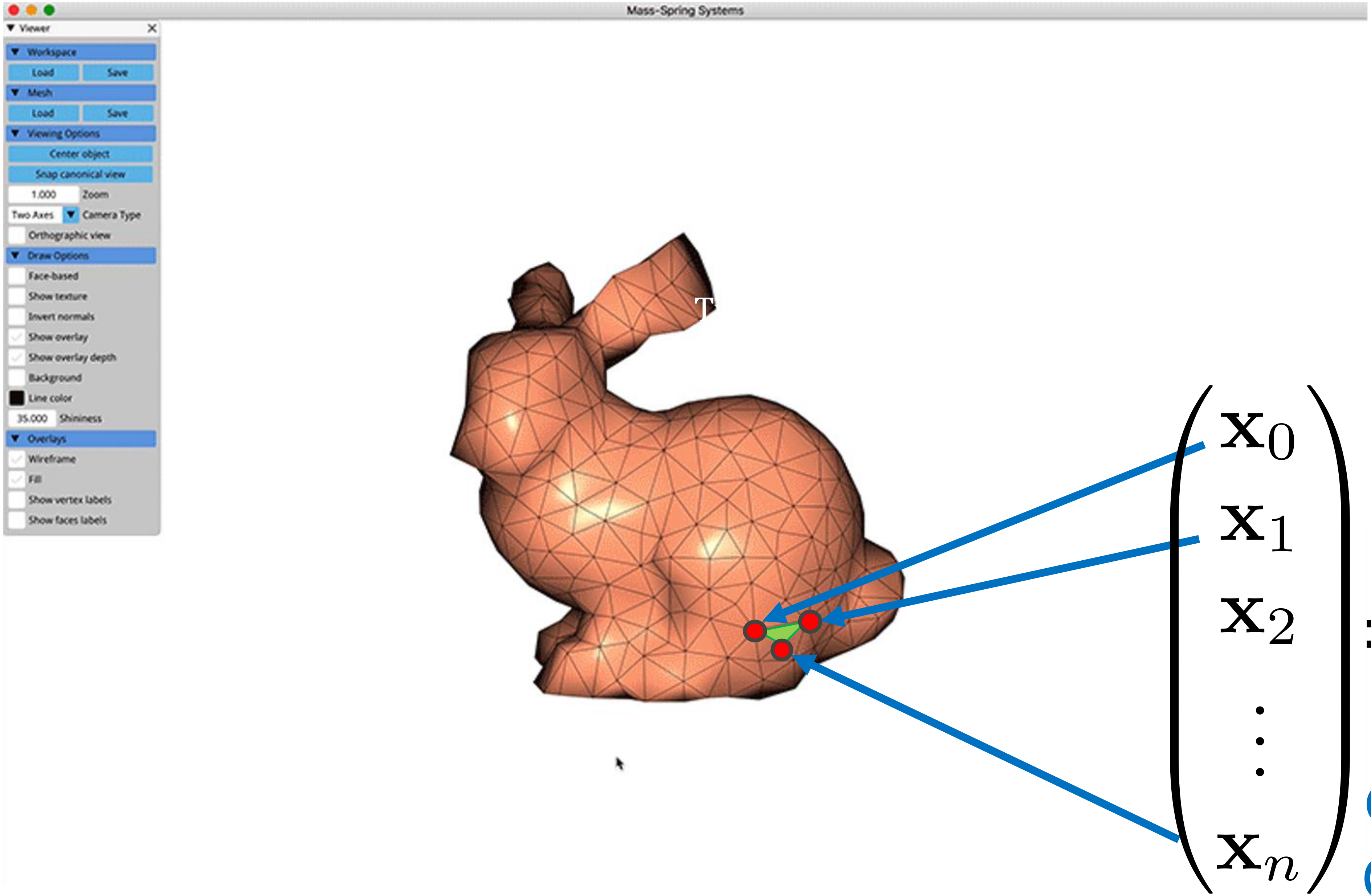
$$\boldsymbol{U}^* = \arg\max \operatorname{trace}(\boldsymbol{U}^T \Sigma \boldsymbol{U})$$

$$s.t\ U^T U = I$$

Principal Component Analysis

$$\boldsymbol{U}^* = \arg\max \operatorname{trace}(\boldsymbol{U}^T \Sigma \boldsymbol{U})$$
$$s.t\ U^T U = I$$

Spatial Discretization -- Finite Elements



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} : q(0) + U\boldsymbol{r}$$

Generalized Coordinates

How do we use this reduced space ?

Answer, direct substitution

$$E\left(\mathbf{q}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right)^{\mathrm{T}} M\left(\mathbf{q}^{\mathbf{i+1}} - \widetilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 V\left(\mathbf{q}^{\mathrm{i+1}}\right)$$

Gradient of what equals this ? Let's guess, then check

How do we use this reduced space ?

Answer, direct substitution

$$E\left(\mathrm{r}^{\mathbf{i+1}}\right) = \frac{1}{2}\left(\mathrm{r}^{\mathbf{i+1}} - \widetilde{\mathrm{r}^{\mathbf{i}}}\right)^{\mathrm{T}} \underbrace{U^{T} M \mathrm{U}}_{\textbf{Reduced Mass Matrix}} \left(\mathrm{r}^{\mathbf{i+1}} - \widetilde{\mathrm{r}^{\mathbf{i}}}\right) + h^{\mathbf{2}} V\left(q(0) + U^{t} r^{i+1}\right)$$

What are the reduced gradient and Hessian ?

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left|\left|\frac{\partial E}{\partial \mathbf{v}}\bigg|_{\mathbf{v}^i}\right|\right| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i\mathbf{d} = -\mathbf{g}^i$$

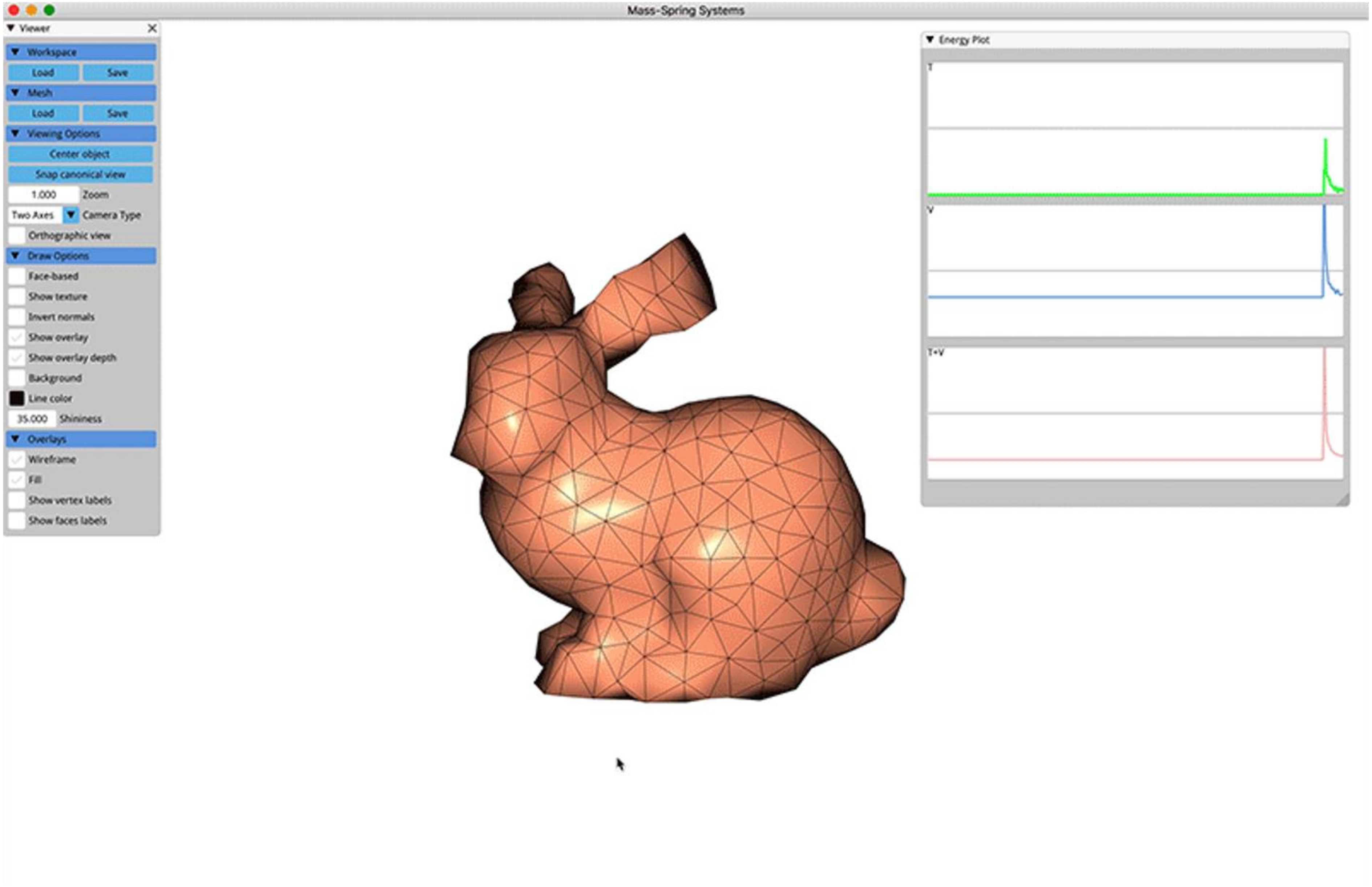
H and **g** are now reduced so smaller and faster to solve

Choose α using line search

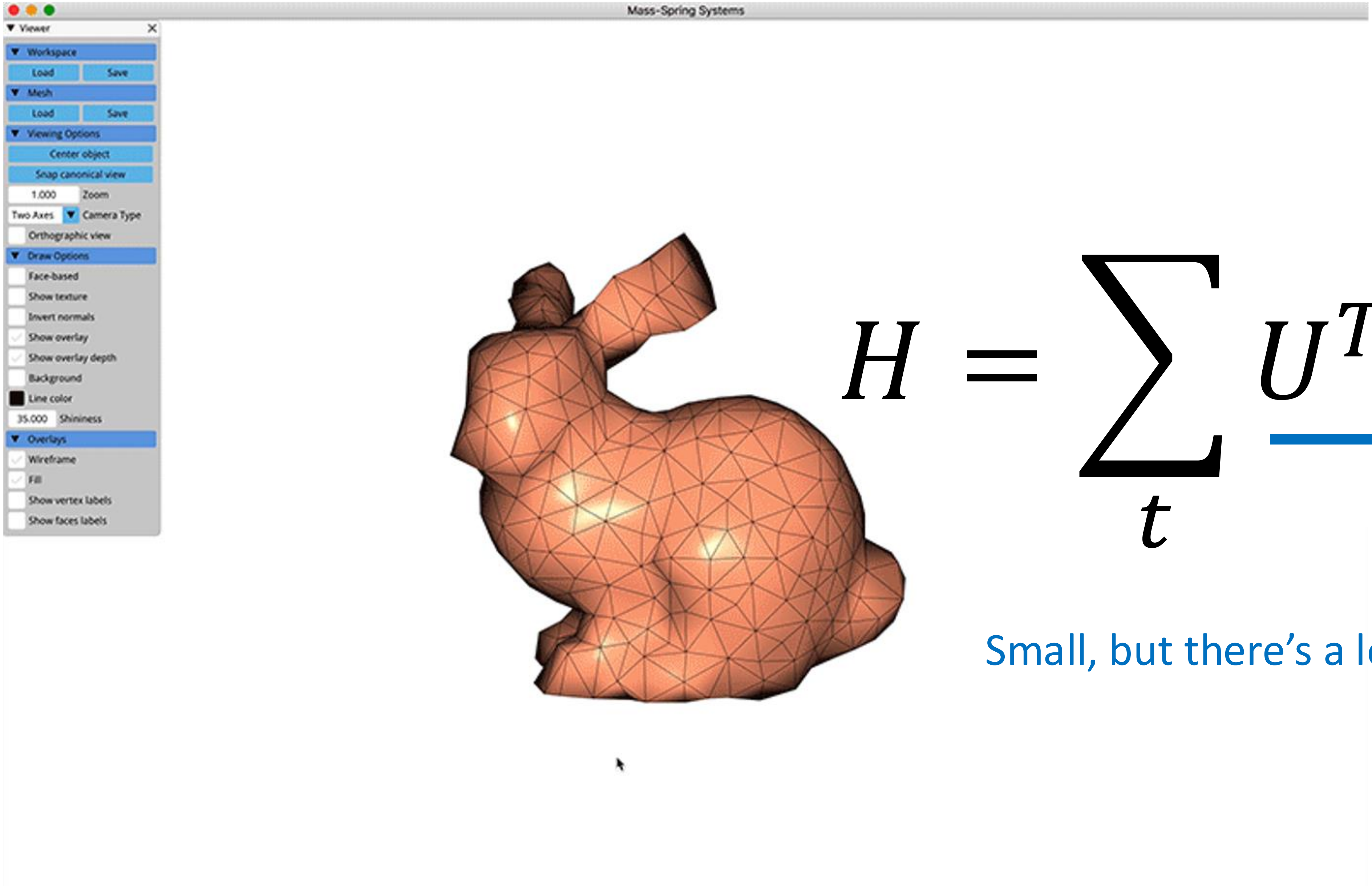
Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha\mathbf{d}$$
$$i = i + 1$$

Problem, it’s still slow! Why ?

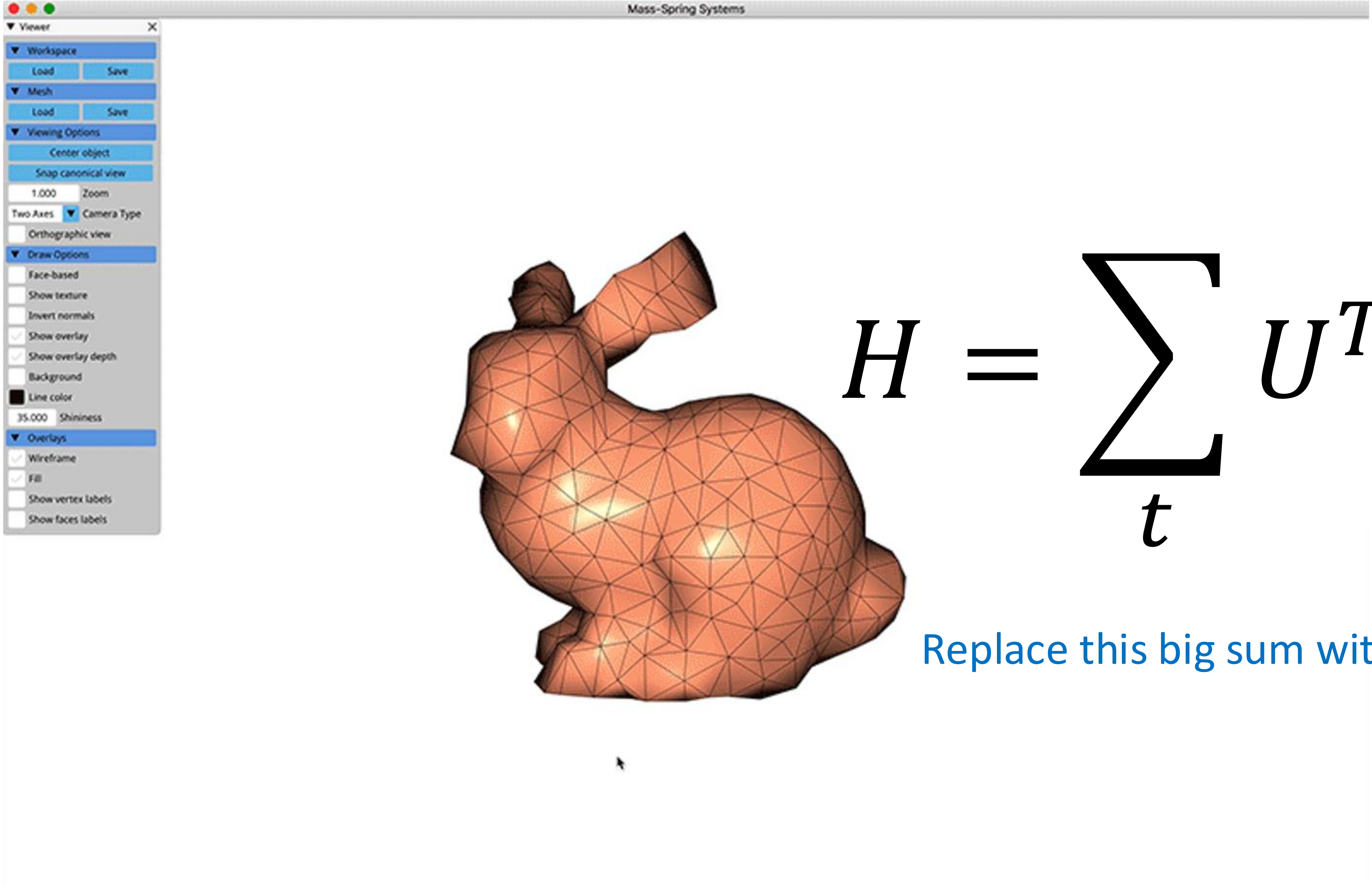


Assembly still visits every element ☹



$$H = \sum_t \underbrace{U^T H_i U}_{\text{Small, but there's a lot of them}}$$

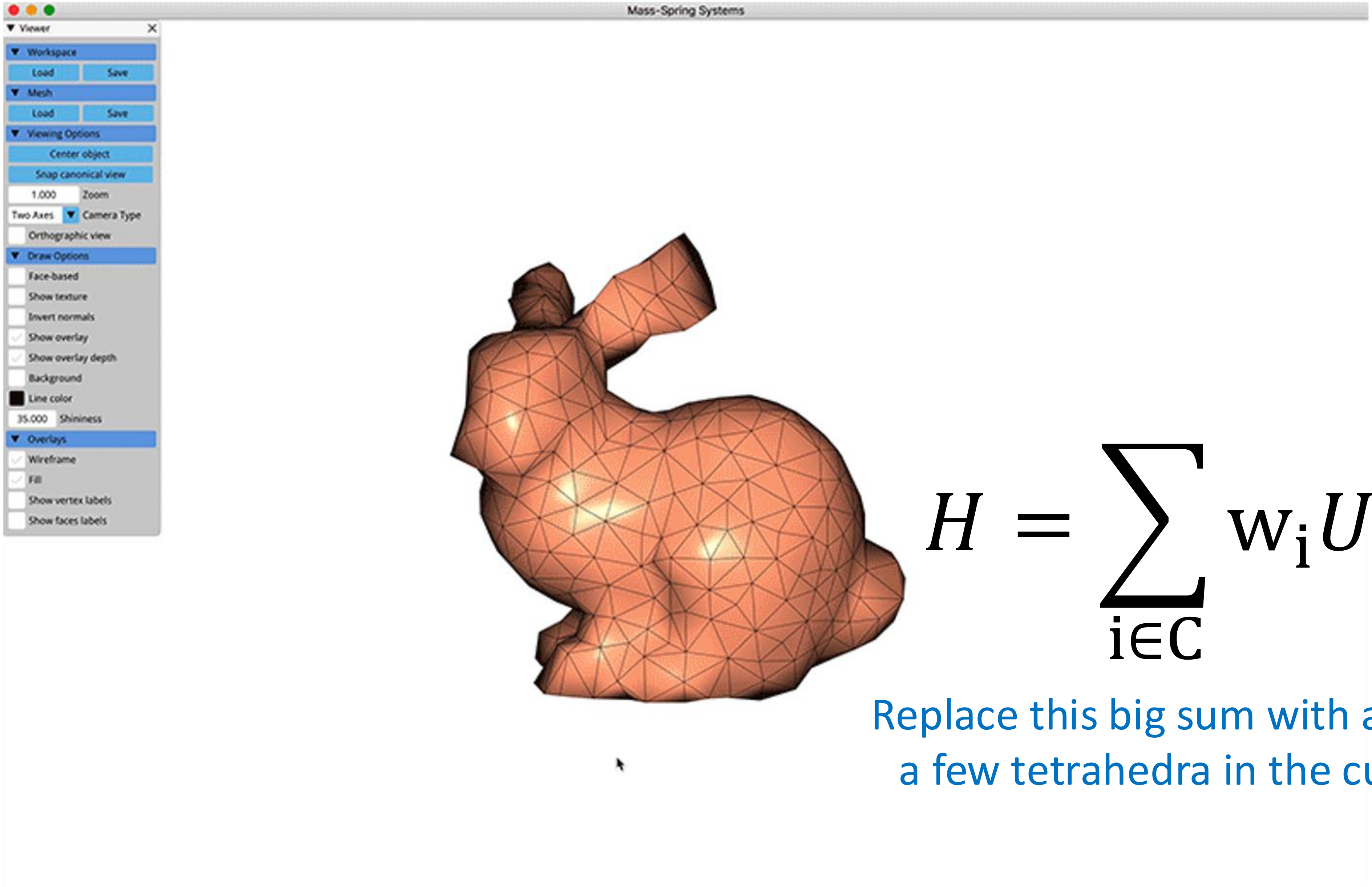
Optimal Quadrature



$$H = \sum_t U^T H_i U$$

Replace this big sum with a small sum

Optimal Quadrature

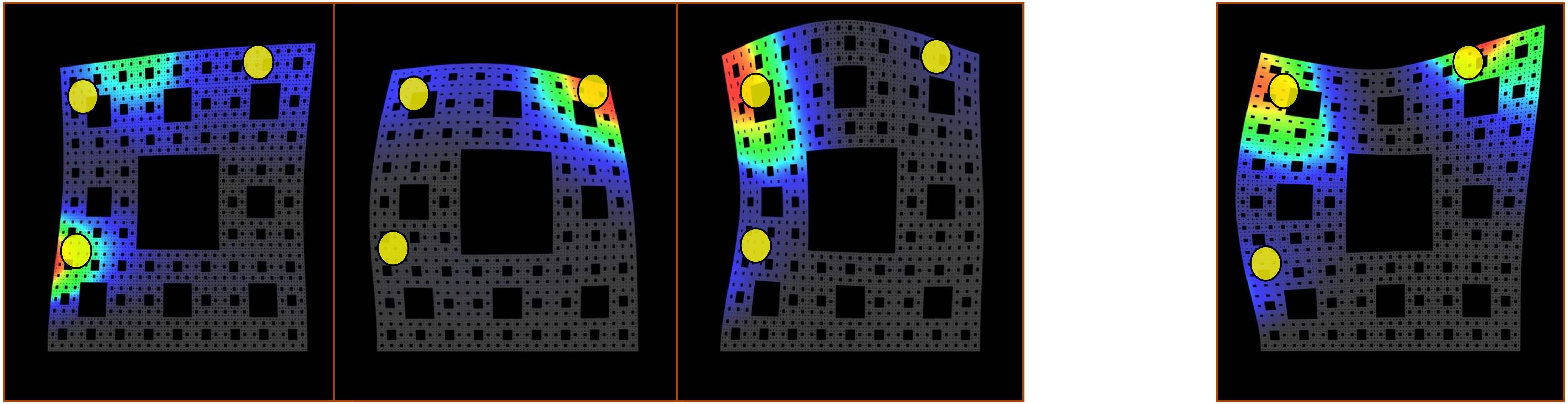


$$H = \sum_{\mathrm{i} \in \mathrm{C}} \mathrm{w_i} U^T H_i U$$

Replace this big sum with a small sum over a few tetrahedra in the cubuature set C.

One Method: Data Driven

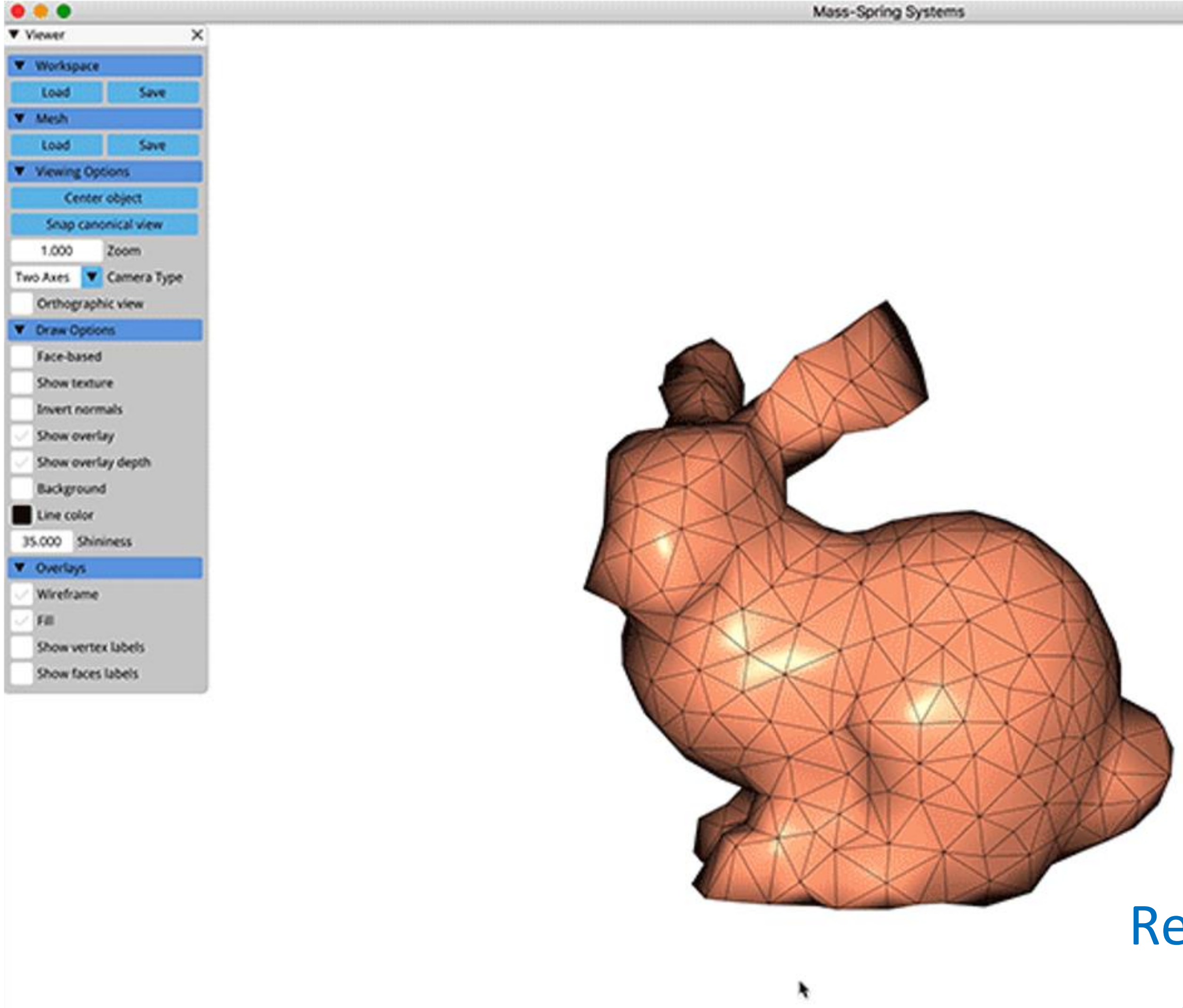
$$\{q_i, g_i\}$$



$\mathbf{q}^{(1)}$ $\mathbf{q}^{(2)}$ $\mathbf{q}^{(3)}$ $\mathbf{q}^{(T)}$

Optimized Cubature | An et al.

Optimal Quadrature



$$\boldsymbol{g} = \sum_{\mathrm{i} \in \mathrm{C}} \mathrm{w_i} \boldsymbol{g}_{\mathrm{i}}$$

Replace this big sum with a small sum over a few tetrahedra in the cubature set C.

Cubature for all poses

$$\underbrace{\begin{pmatrix} \boldsymbol{g}_{\mathbf{0}}^{0} & \cdots & \boldsymbol{g}_{\boldsymbol{n}}^{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{g}_{\mathbf{0}}^{\mathrm{T}} & \cdots & \boldsymbol{g}_{\mathrm{n}}^{\mathrm{T}} \end{pmatrix}}_{\text{Per-tetrahedron gradient values}} \overbrace{\begin{pmatrix} \boldsymbol{w}_{\mathbf{0}} \\ \vdots \\ \boldsymbol{w}_{\boldsymbol{n}} \end{pmatrix}}^{\text{Weights}} = \underbrace{\begin{pmatrix} \mathrm{g}_{\mathbf{0}} \\ \vdots \\ \mathrm{g}_{\boldsymbol{n}} \end{pmatrix}}_{\text{Per-pose value}}$$

Cubature for all poses

$$\underbrace{\begin{pmatrix} \boldsymbol{g}_0^0 & \cdots & \boldsymbol{g}_n^0 \\ \vdots & \ddots & \vdots \\ \boldsymbol{g}_0^{\mathrm{T}} & \cdots & \boldsymbol{g}_{\mathrm{n}}^{\mathrm{T}} \end{pmatrix}}_{G} \overbrace{\begin{pmatrix} \boldsymbol{w}_0 \\ \vdots \\ \boldsymbol{w}_n \end{pmatrix}}^{\mathrm{W}} = \underbrace{\begin{pmatrix} \mathrm{g}_0 \\ \vdots \\ \mathrm{g}_n \end{pmatrix}}_{\mathrm{g}}$$

Solve using Non-linear least squares

Non-Linear Least Squares

$$\boldsymbol{U}^* = \arg\min \left|\left|\boldsymbol{G}\boldsymbol{w} - \boldsymbol{g}\right|\right|_2^2$$
$$s.t\ \mathbf{w} \geq 0$$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left|\left|\frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i}\right|\right| < \text{tol}$$

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H and **g** are now reduced so smaller and faster to solve

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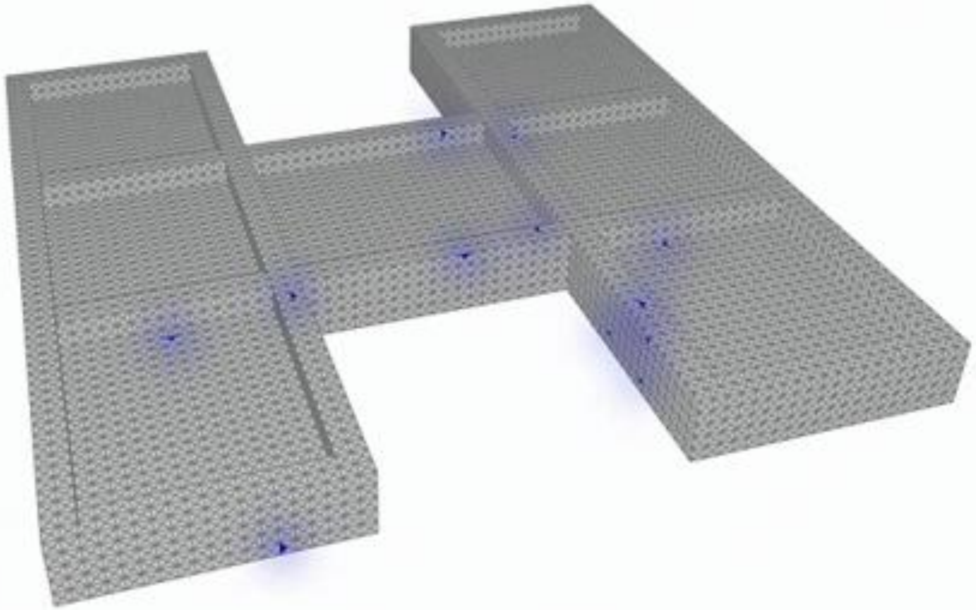
Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha\mathbf{d}$$
$$i = i + 1$$

Repeat until converged

PCA Reanalysis of a Balloon



Next Video: Fast Solvers for Elastodynamics !