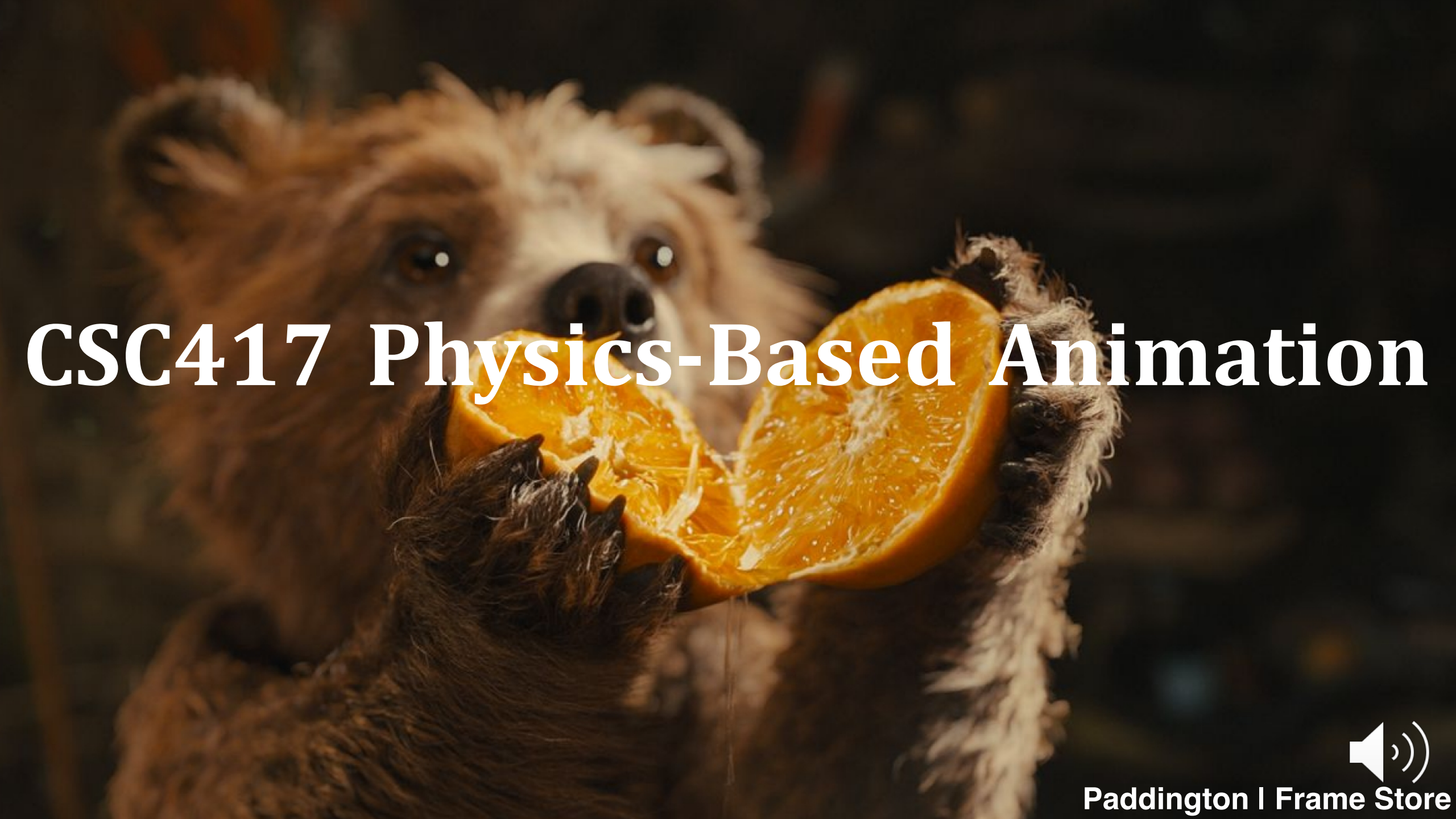
CSC417 Physics-Based Animation



Last Video: The Finite Element Method

This Video: More Finite Element Method !

Equations of Motion

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Linearly-Implicit Time Integration

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}\left(\mathbf{q}^{t+1}\right)$$

$$\mathbf{q}^{t+1} = \mathbf{q}^{t} + \Delta t \dot{\mathbf{q}}^{t+1}$$

Backward Euler

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}\left(\mathbf{q}^{t} + \Delta t \dot{\mathbf{q}}^{t+1}\right)$$

Substitute

First Order Approximation

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}\left(\mathbf{q}^{t}\right) + \Delta t^2 \underbrace{\frac{\partial \mathbf{f}}{\partial \mathbf{q}}}_{\text{stiffness matrix } \mathrm{K}} \dot{\mathbf{q}}^{t+1}$$

Linearly-Implicit Time Integration

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t\mathbf{f}\left(\mathbf{q}^{t}\right) + \Delta t^{2}\underbrace{\frac{\partial \mathbf{f}}{\partial \mathbf{q}}}_{\text{stiffness matrix } \mathrm{K}}\dot{\mathbf{q}}^{t+1}$$

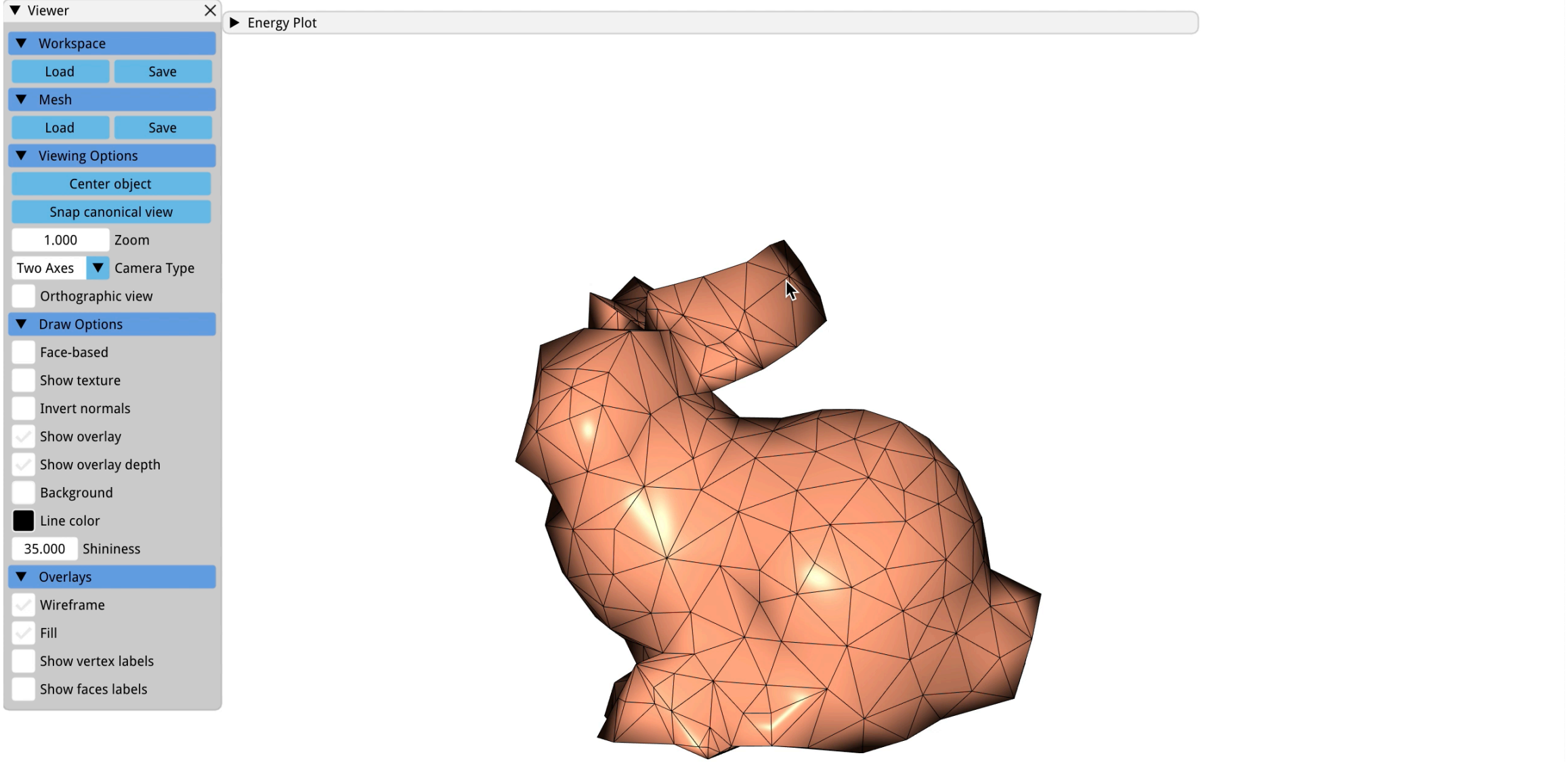
Rearrange

$$\left(\mathrm{M} - \Delta t^{2}\mathrm{K}\right)\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t\mathbf{f}\left(\mathbf{q}^{t}\right)$$

Solve

$$\mathbf{q}^{t+1} = \mathbf{q}^{t} + \Delta t\dot{\mathbf{q}}^{t+1}$$

Update Position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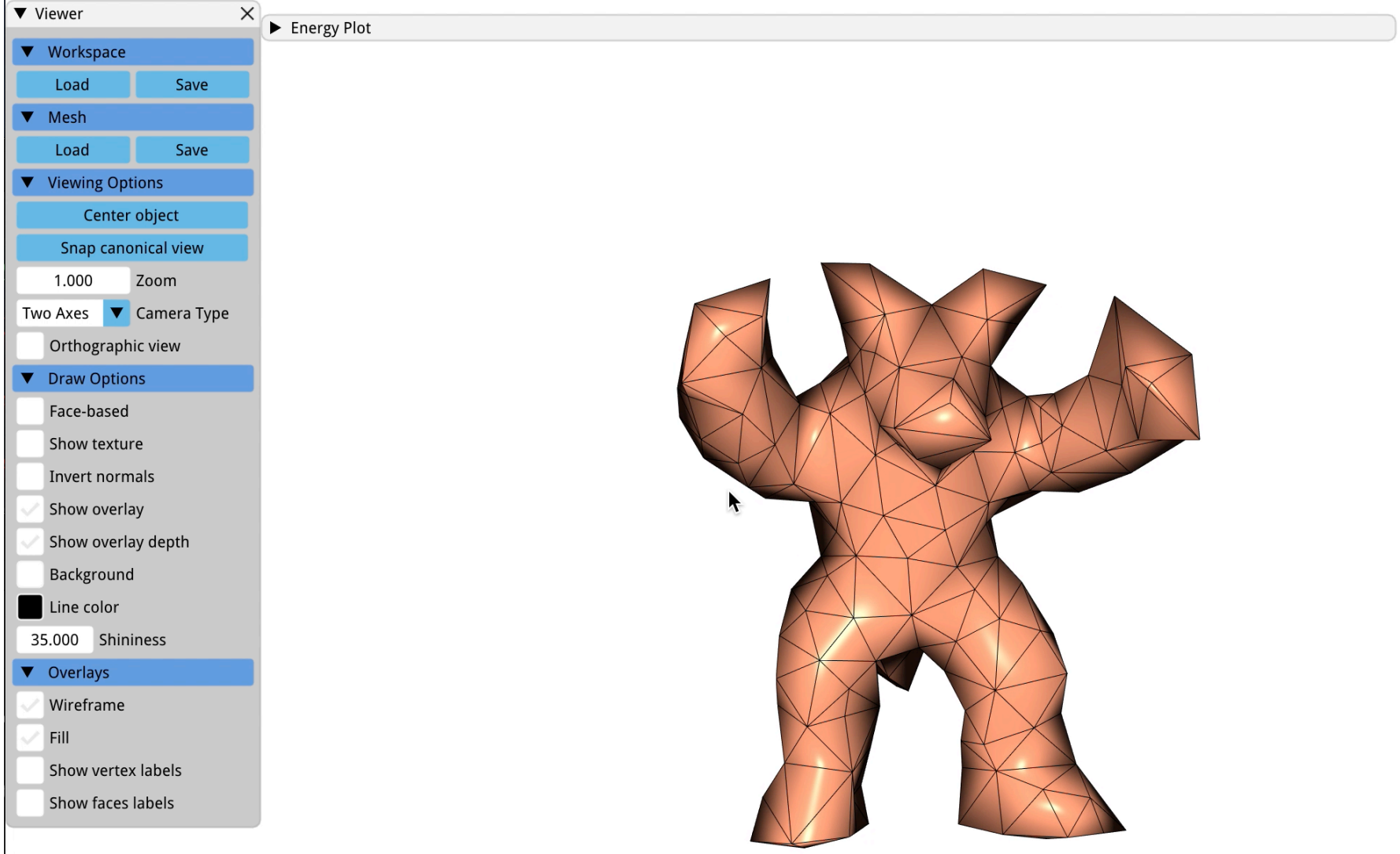
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Fully-Implicit Time Integration

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t\mathbf{f}\left(\mathbf{q}^{t+1}\right)$$

$$\mathbf{q}^{t+1} = \mathbf{q}^{t} + \Delta t\dot{\mathbf{q}}^{t+1}$$

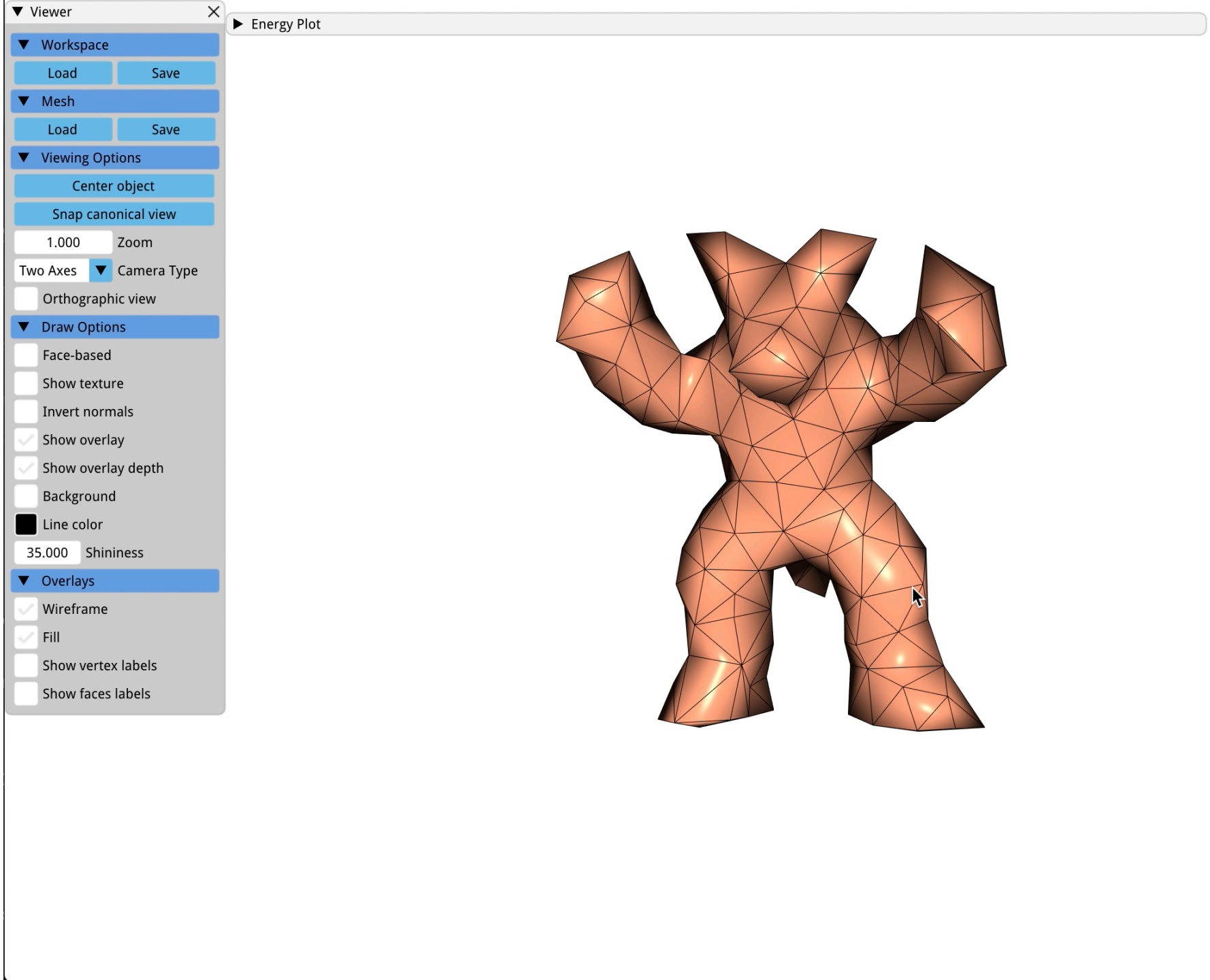
Backward Euler

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t\mathbf{f}\left(\mathbf{q}^{t} + \Delta t\dot{\mathbf{q}}^{t+1}\right)$$

Substitute

First Order Approximation

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t\mathbf{f}\left(\mathbf{q}^{t}\right) + \Delta t^{2}\underbrace{\frac{\partial \mathbf{f}}{\partial \mathbf{q}}}_{\text{stiffness matrix } \mathrm{K}}\dot{\mathbf{q}}^{t+1}$$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Backward Euler using Optimization

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t\mathbf{f}\left(\mathbf{q}^{t+1}\right)$$

$$\mathbf{q}^{t+1} = \mathbf{q}^{t} + \Delta t\dot{\mathbf{q}}^{t+1}$$

Backward Euler

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} - \mathrm{M}\dot{\mathbf{q}}^{t} - \Delta t\mathbf{f}\left(\mathbf{q}^{t} + \Delta t\dot{\mathbf{q}}^{t+1}\right) = \mathbf{0}$$

Nonlinear root finding is hard ☹, maybe we can do something else …

Optimization

$$\mathbf{y}^* = \arg\min_{\mathbf{y}} E(\mathbf{y})$$

generic optimization problem

optimal solution

energy/cost function

$$\left.\frac{\partial E}{\partial \mathbf{y}}\right|_{\mathbf{y}^*} = 0$$

first-order optimality

$$\mathbf{g}(\mathbf{y}^*) = 0$$

root finding

Backward Euler using Optimization

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = M\dot{\mathbf{q}}^{t} + \Delta t\mathbf{f}\left(\mathbf{q}^{t+1}\right)$$

$$\mathbf{q}^{t+1} = \mathbf{q}^{t} + \Delta t\dot{\mathbf{q}}^{t+1}$$

Backward Euler

$$\underline{\mathrm{M}\dot{\mathbf{q}}^{t+1} - \mathrm{M}\dot{\mathbf{q}}^{t} - \Delta t\mathbf{f}\left(\mathbf{q}^{t} + \Delta t\dot{\mathbf{q}}^{t+1}\right)} = \mathbf{0}$$

$$\left.\frac{\partial E}{\partial \dot{\mathbf{q}}}\right|_{\dot{\mathbf{q}}^{t+1}} = 0$$

Backward Euler using Optimization

$$\mathrm{M}\mathbf{v} - \mathrm{M}\dot{\mathbf{q}}^t - \Delta t \mathbf{f}\left(\mathbf{q}^t + \Delta t \mathbf{v}\right) = \mathbf{0}$$

linear — quadratic

$$\Delta t \frac{\partial V}{\partial \mathbf{q}}$$

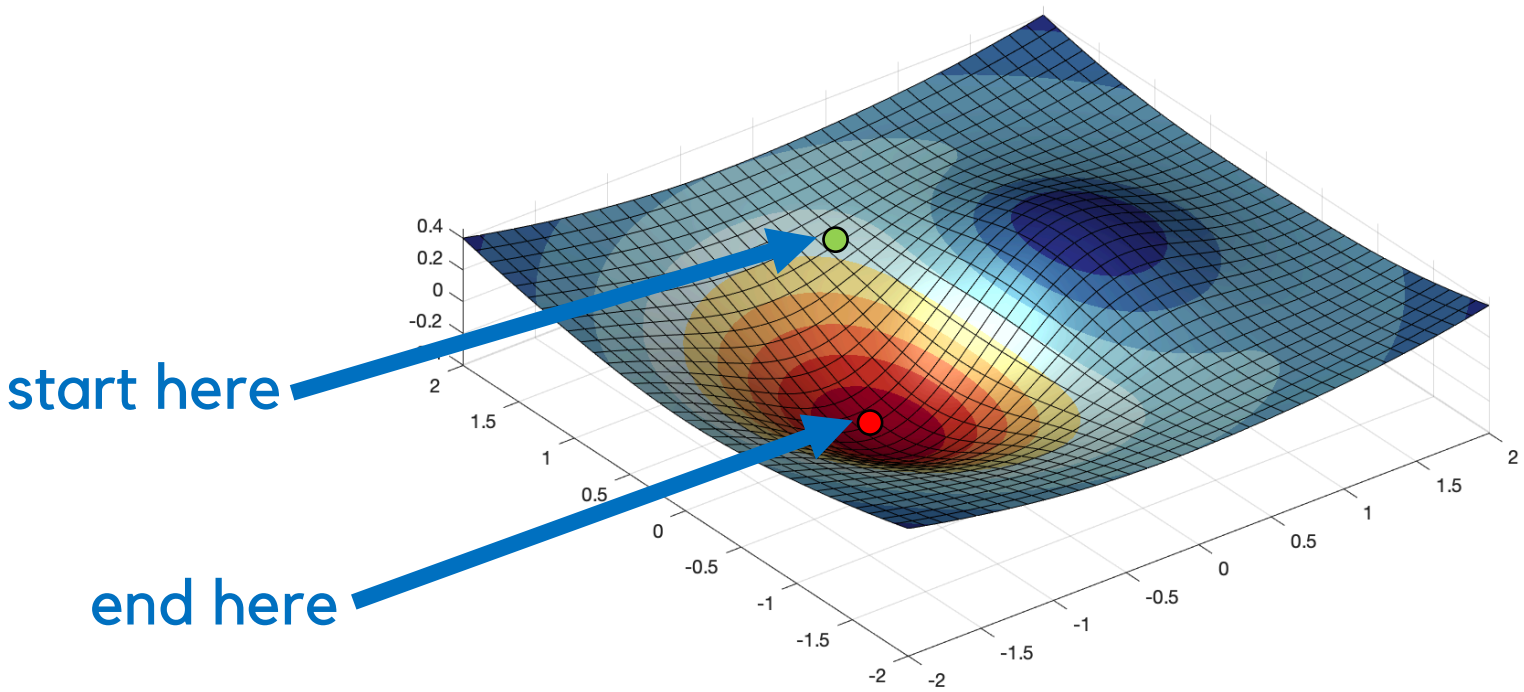
chain rule

$$\frac{\partial \mathbf{q}}{\partial \mathbf{v}}^T \frac{\partial V}{\partial \mathbf{q}}$$

$$\frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t \mathbf{v}\right)$$

Backward Euler using Optimization

$$\mathbf{v}^* = \arg\min_{\mathbf{v}} \underbrace{\frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t \mathbf{v}\right)}_{E(\mathbf{v})}$$



Gradient-Based Optimization

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

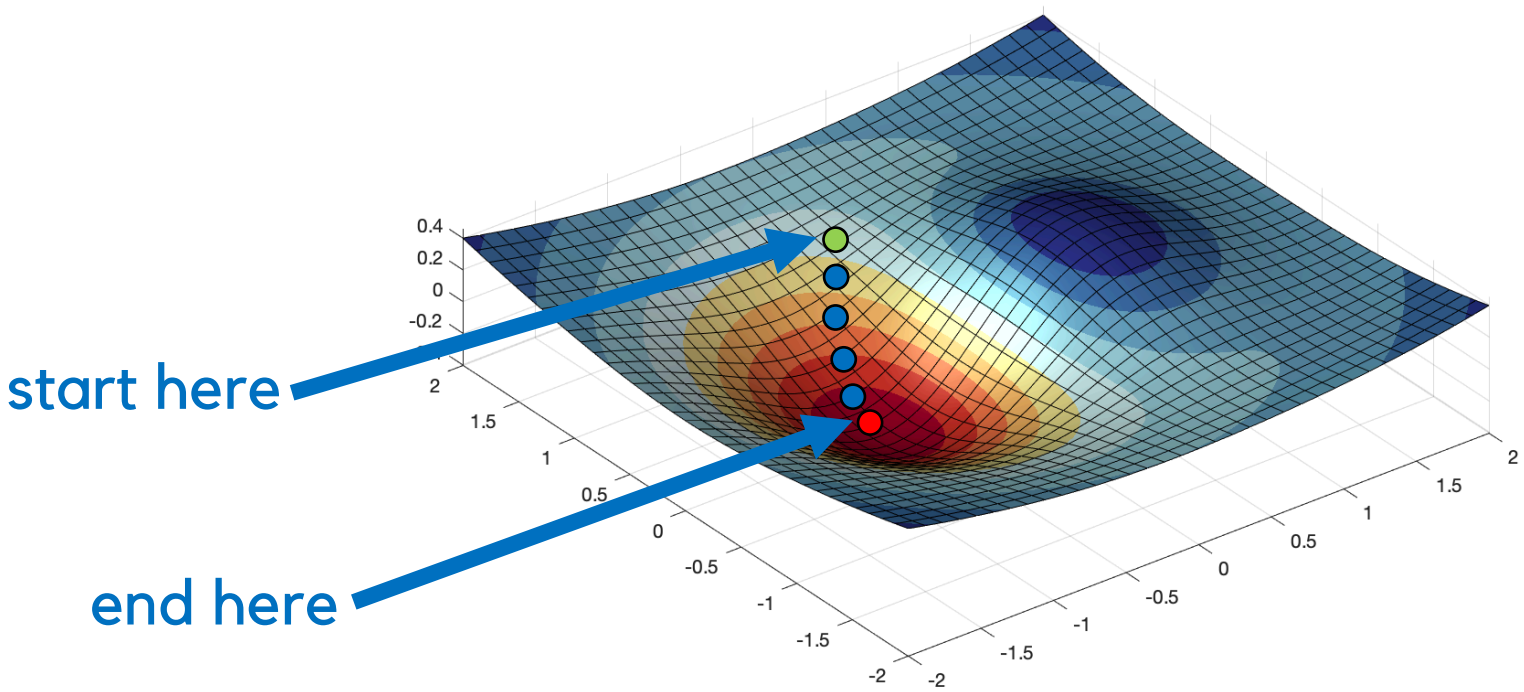
$$\mathbf{d} = \quad ??????$$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Backward Euler using Optimization

$$\mathbf{v}^* = \arg\min_{\mathbf{v}} \underbrace{\frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t \mathbf{v}\right)}_{E(\mathbf{v})}$$



Gradient Descent

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\bigg|_{\mathbf{v}^i} \right\| < \text{tol}$$

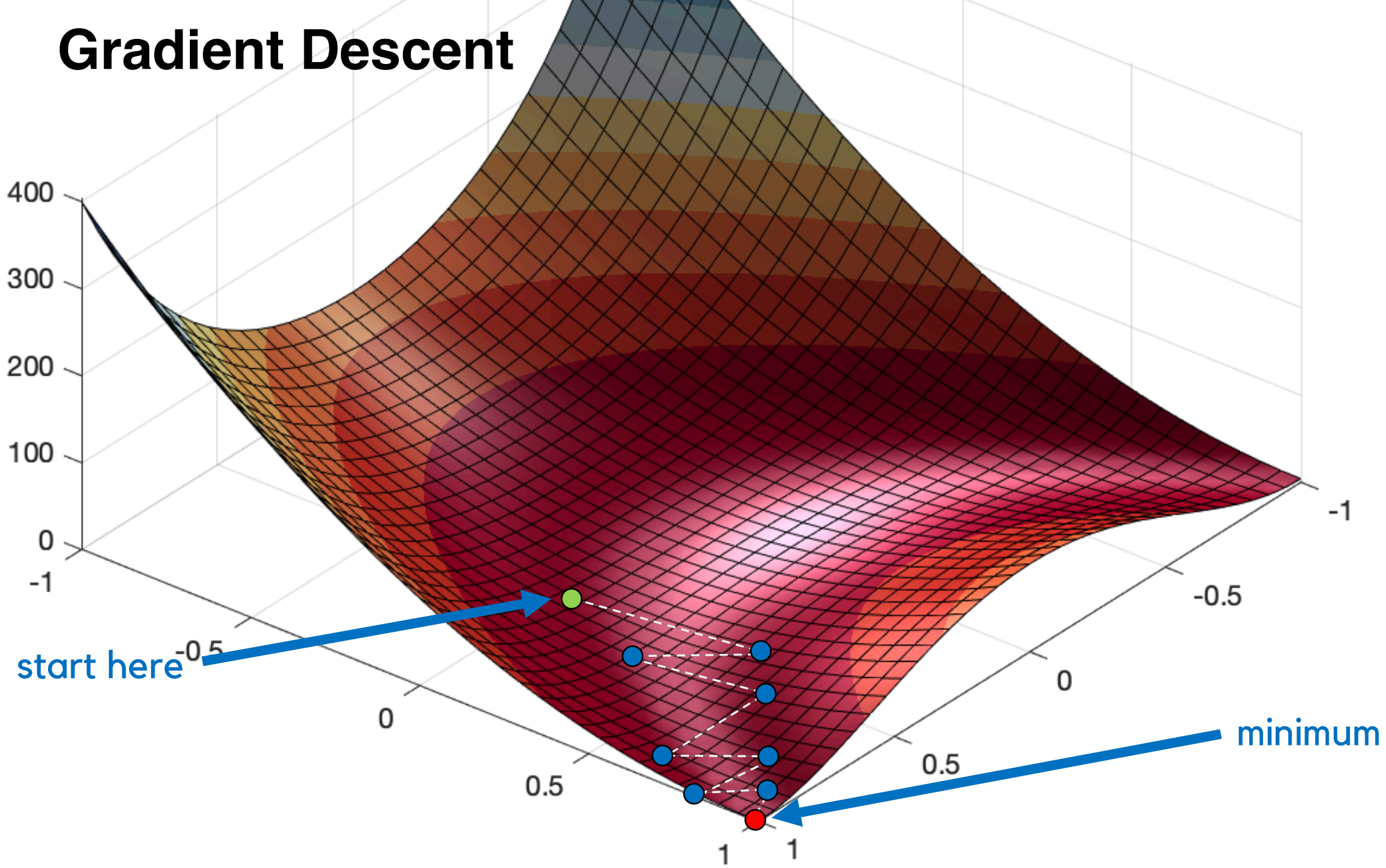
Choose search direction

$$\mathbf{d} = -\frac{\partial E}{\partial \mathbf{v}}\bigg|_{\mathbf{v}^i}$$

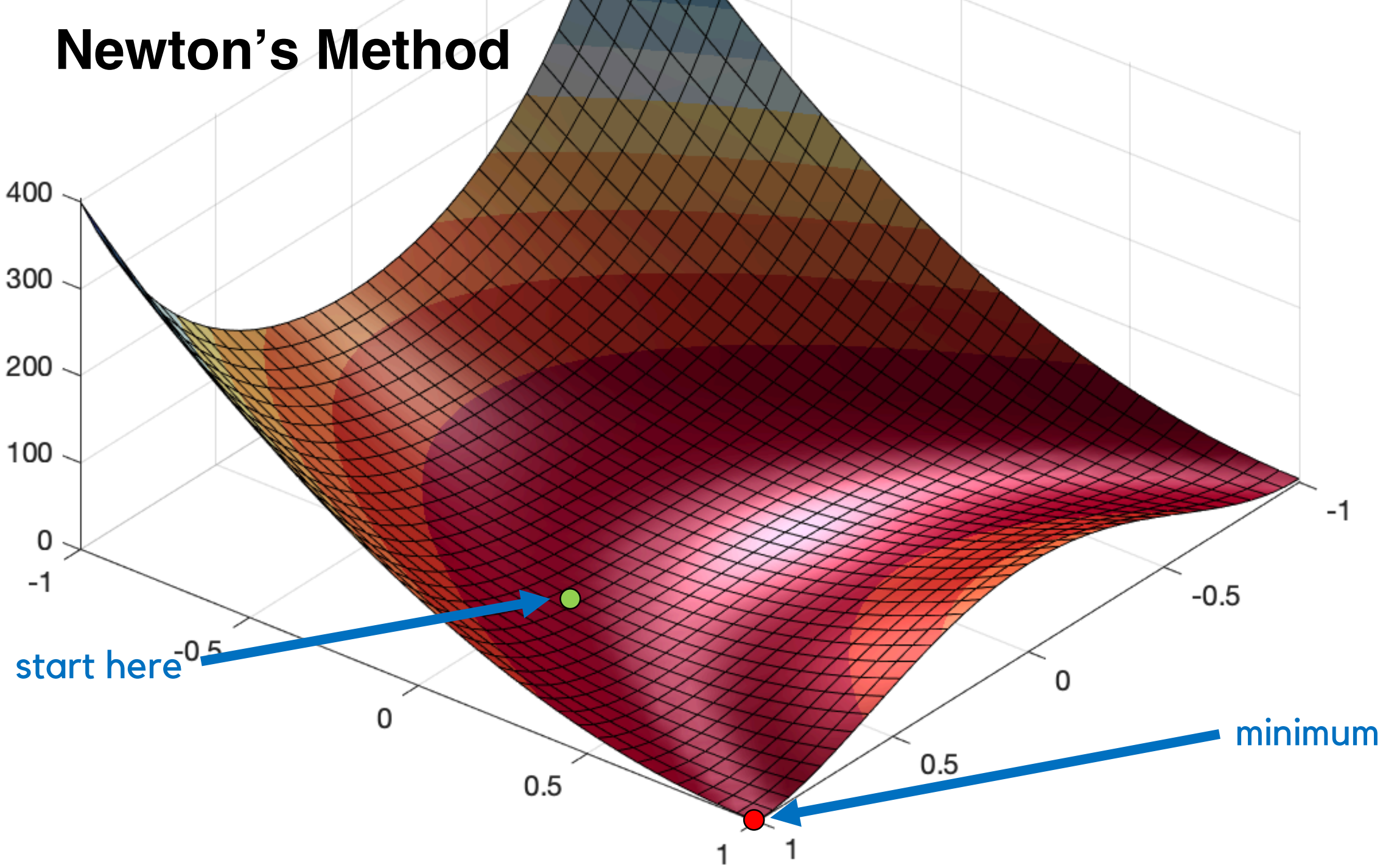
Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

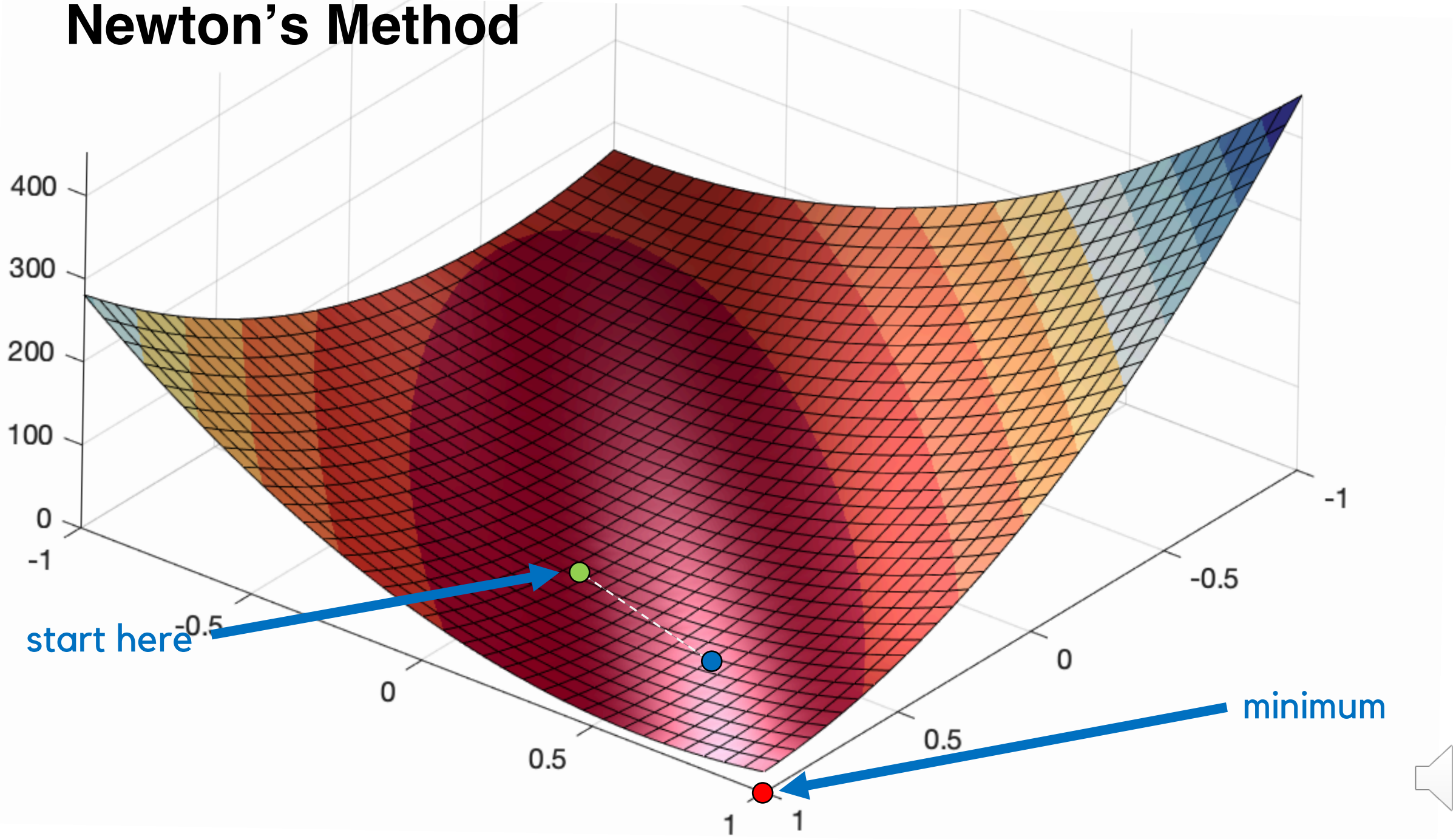
Gradient Descent



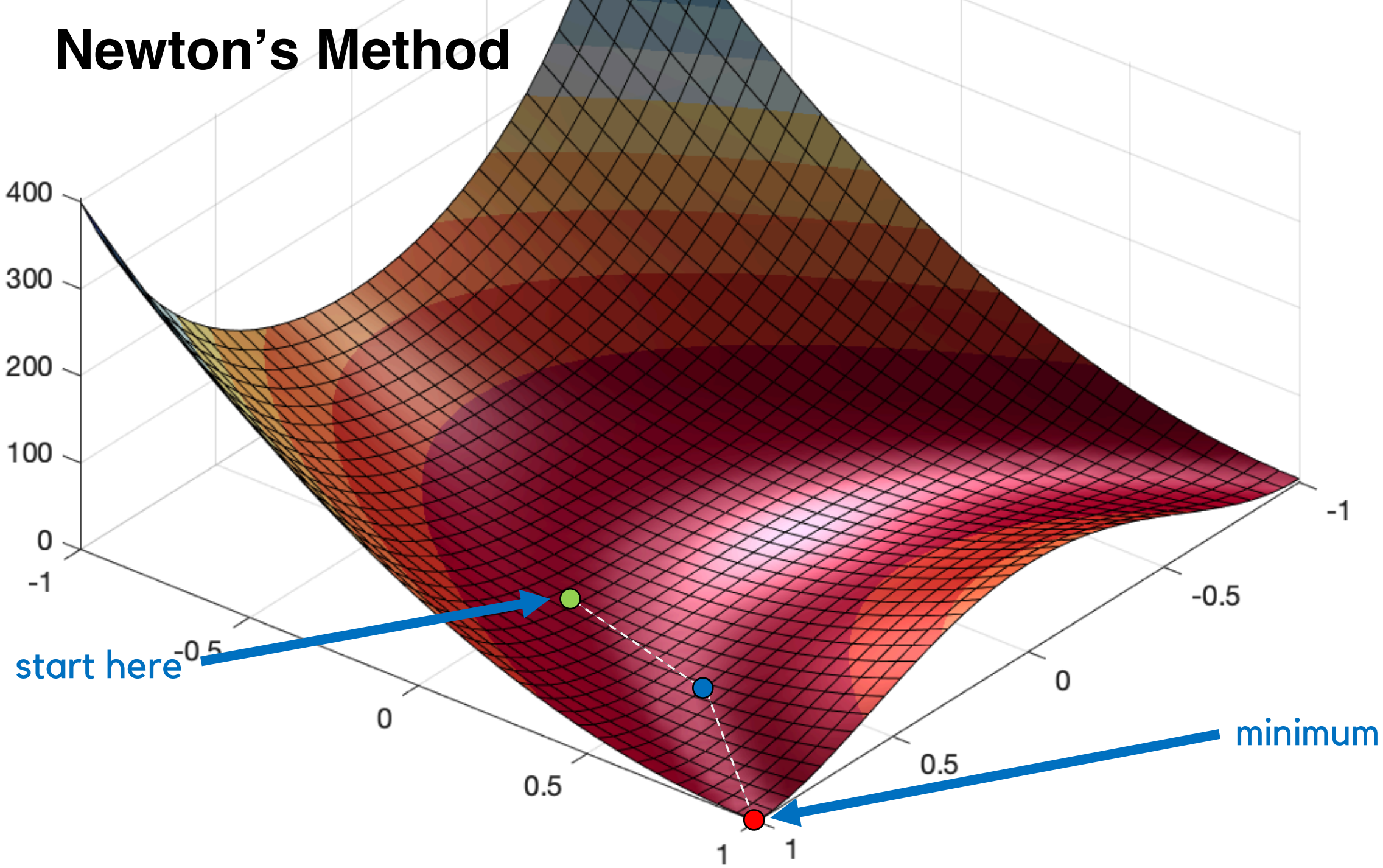
Newton's Method



Newton's Method



Newton's Method



Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{d} = \text{Minimize quadratic approximation}$$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Aside: Quadratic Approximations

$$\mathbf{y}^* = \arg\min_{\mathbf{y}} E(\mathbf{y})$$

generic optimization problem

optimal solution

energy/cost function

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} E\left(\mathbf{y}^i + \tilde{\mathbf{d}}\right)$$

Second Order Taylor Approximation

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \frac{1}{2}\tilde{\mathbf{d}}^T \mathrm{H}^i \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \mathbf{g}^i + \cancel{E^i}$$

$$\left.\frac{\partial^2 E}{\partial \mathbf{y}^2}\right|_{\mathbf{y}^i} \qquad \left.\frac{\partial E}{\partial \mathbf{y}}\right|_{\mathbf{y}^i} \qquad E\left(\mathbf{y}^i\right)$$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \frac{1}{2}\tilde{\mathbf{d}}^T \mathrm{H}^i \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \mathbf{g}^i \,.$$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Aside: Minimizing a Quadratic Function

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \underbrace{\frac{1}{2}\tilde{\mathbf{d}}^T \mathrm{H}^i \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \mathbf{g}^i}_{\text{Minimum when gradient is equal to zero}}$$

$$\mathrm{H}^i \mathbf{d} + \mathbf{g}^i = \mathbf{0}$$

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Gradient and Hessian

$$E(\mathbf{v}) = \frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t\mathbf{v}\right)$$

$$\frac{\partial E}{\partial \mathbf{v}} = \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + \Delta t \underbrace{\frac{\partial V}{\partial \mathbf{q}}\bigg|_{\mathbf{q}^t + \Delta t\mathbf{v}}}_{\text{negative generalized forces}}$$

$$\frac{\partial^2 E}{\partial \mathbf{v}^2} = \mathrm{M} + \Delta t^2 \underbrace{\frac{\partial^2 V}{\partial \mathbf{q}^2}\bigg|_{\mathbf{q}^t + \Delta t\mathbf{v}}}_{\text{negative stiffness matrix}}$$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

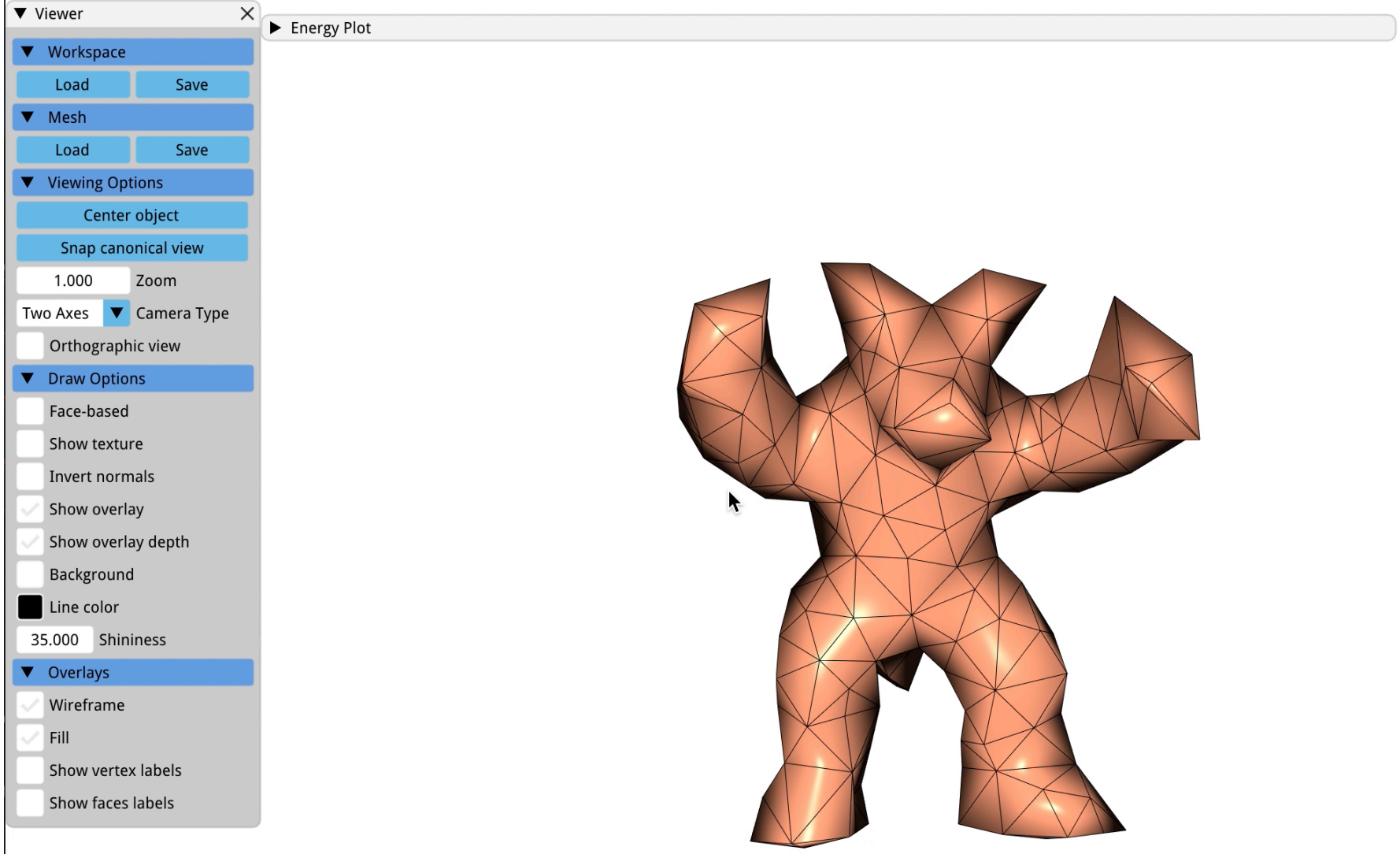
Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$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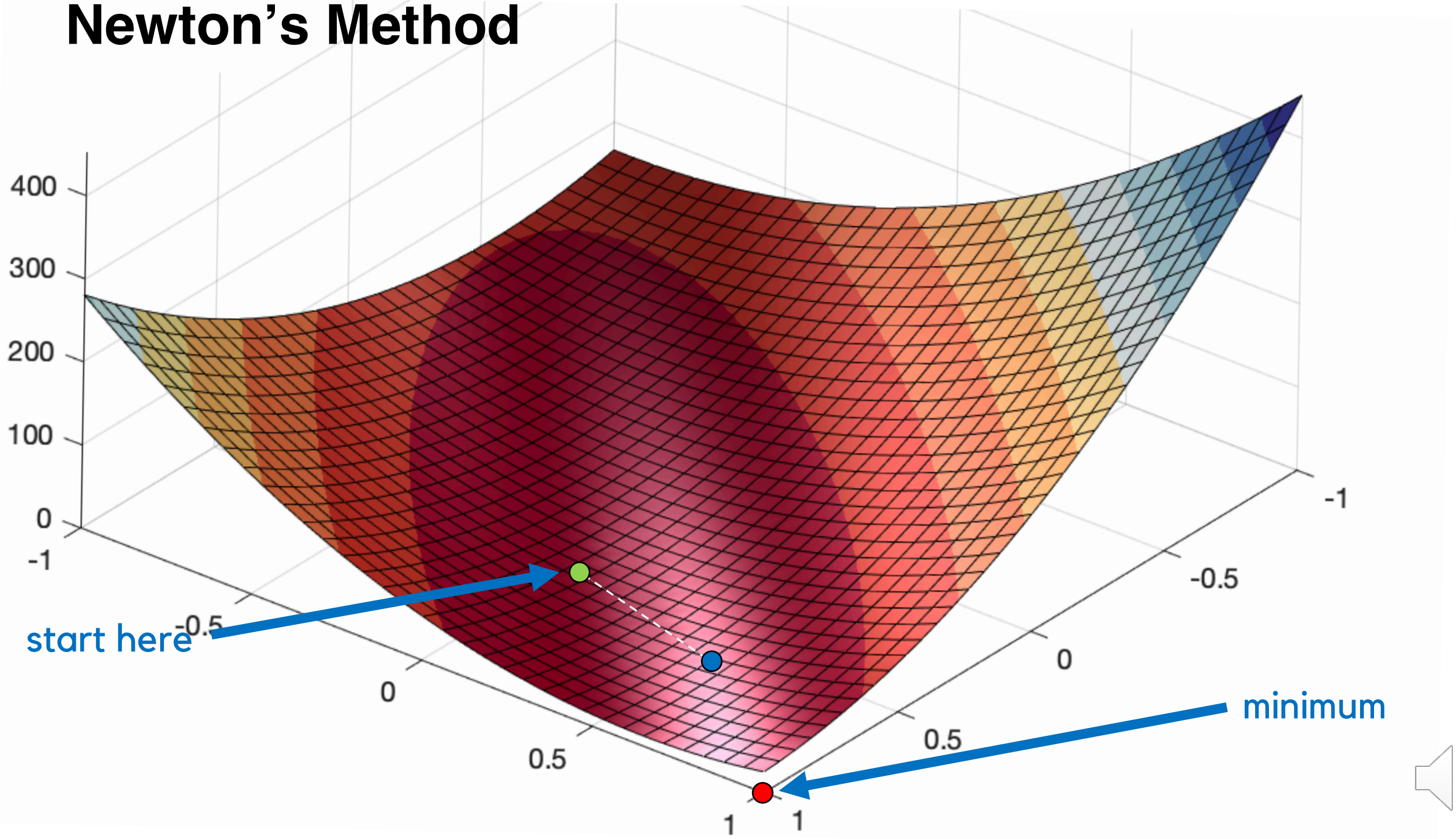
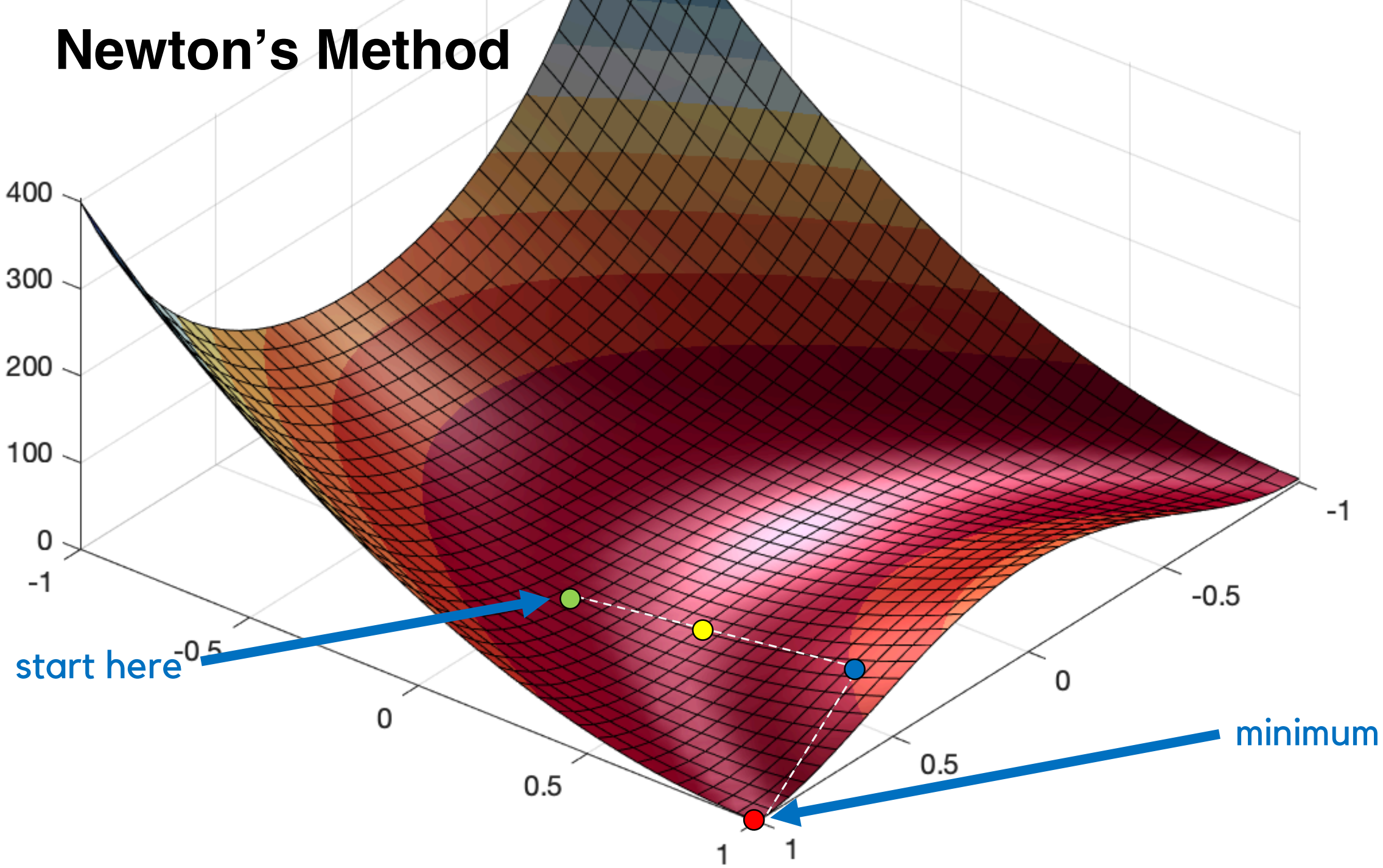
Newton's Method



Newton's Method



Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Choose α using line search

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Line Search

$$\mathbf{y}^* = \arg\min_{\mathbf{y}} E(\mathbf{y})$$

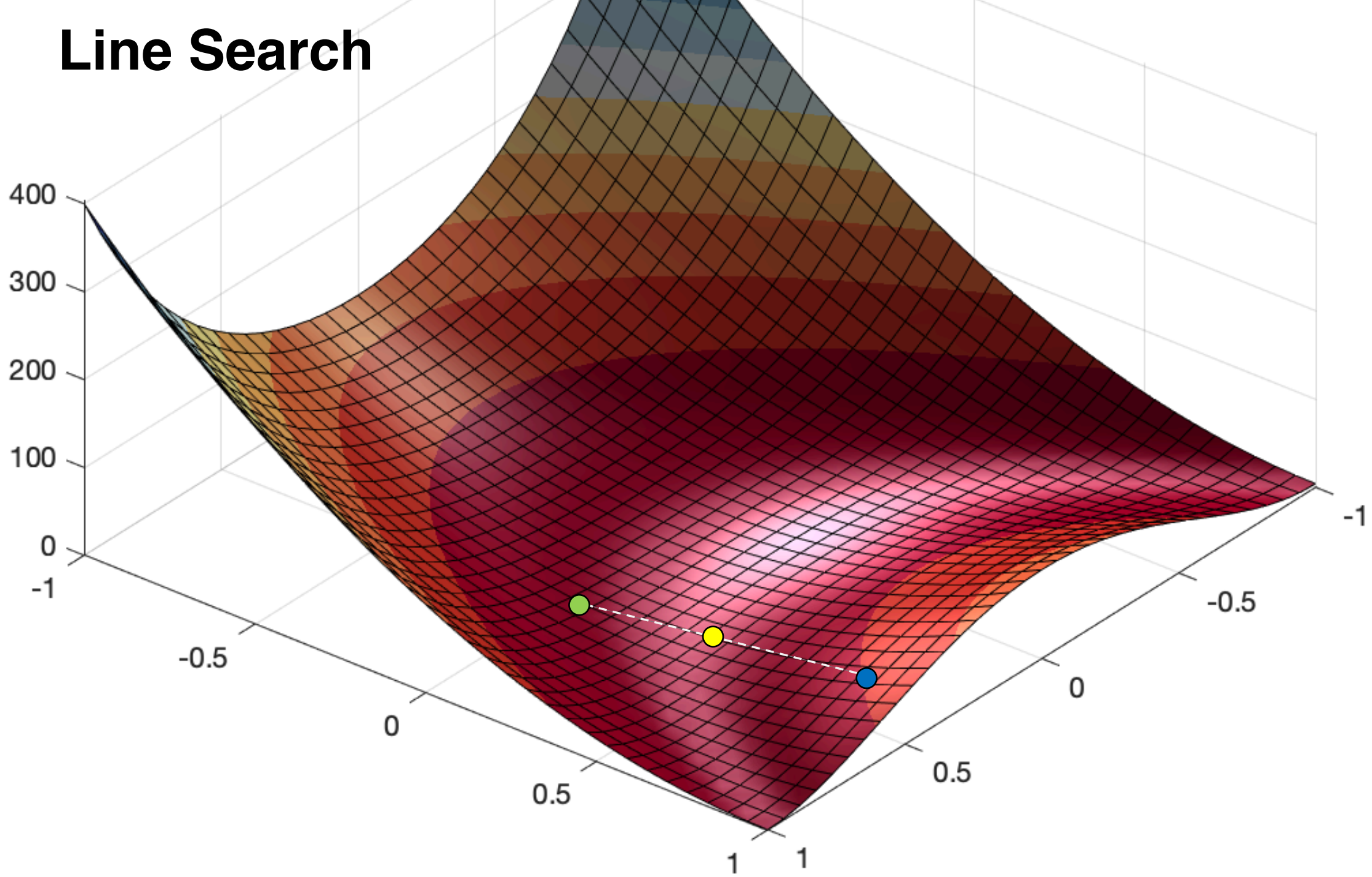
Once we compute $\mathbf{d}$ ...

One-dimensional optimization

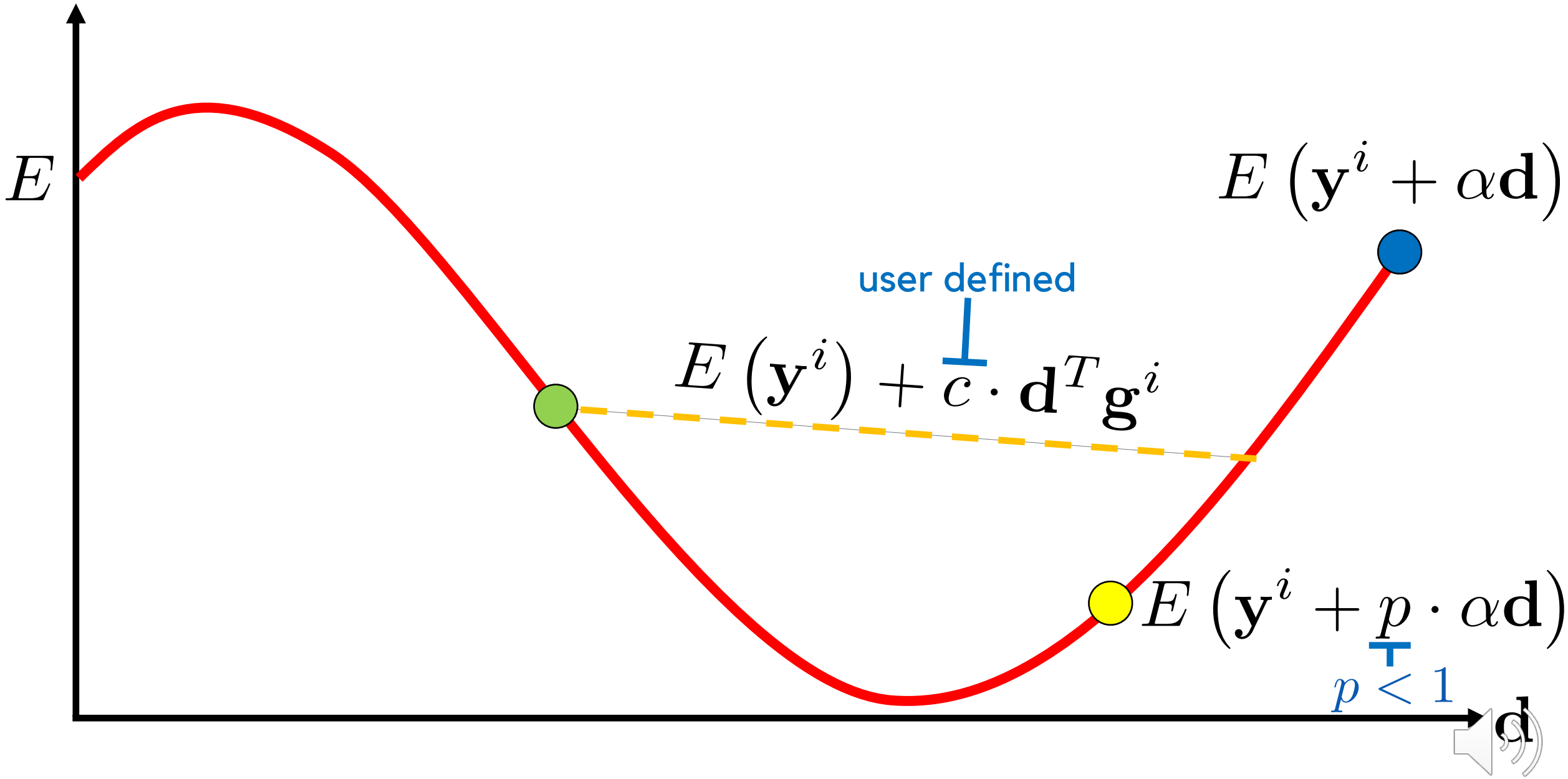
$$\alpha^* = \arg\min_{\alpha} E\left(\mathbf{y}^i + \alpha \mathbf{d}\right)$$

Solve by 1D Search

Line Search



Line Search


E
$E\left(\mathbf{y}^i + \alpha\mathbf{d}\right)$
user defined
$E\left(\mathbf{y}^i\right) + c \cdot \mathbf{d}^T\mathbf{g}^i$
$E\left(\mathbf{y}^i + p \cdot \alpha\mathbf{d}\right)$
$p < 1$
$\mathbf{d}$

Backtracking Line Search

Choose an initial guess

$$\alpha = \alpha_{max}$$

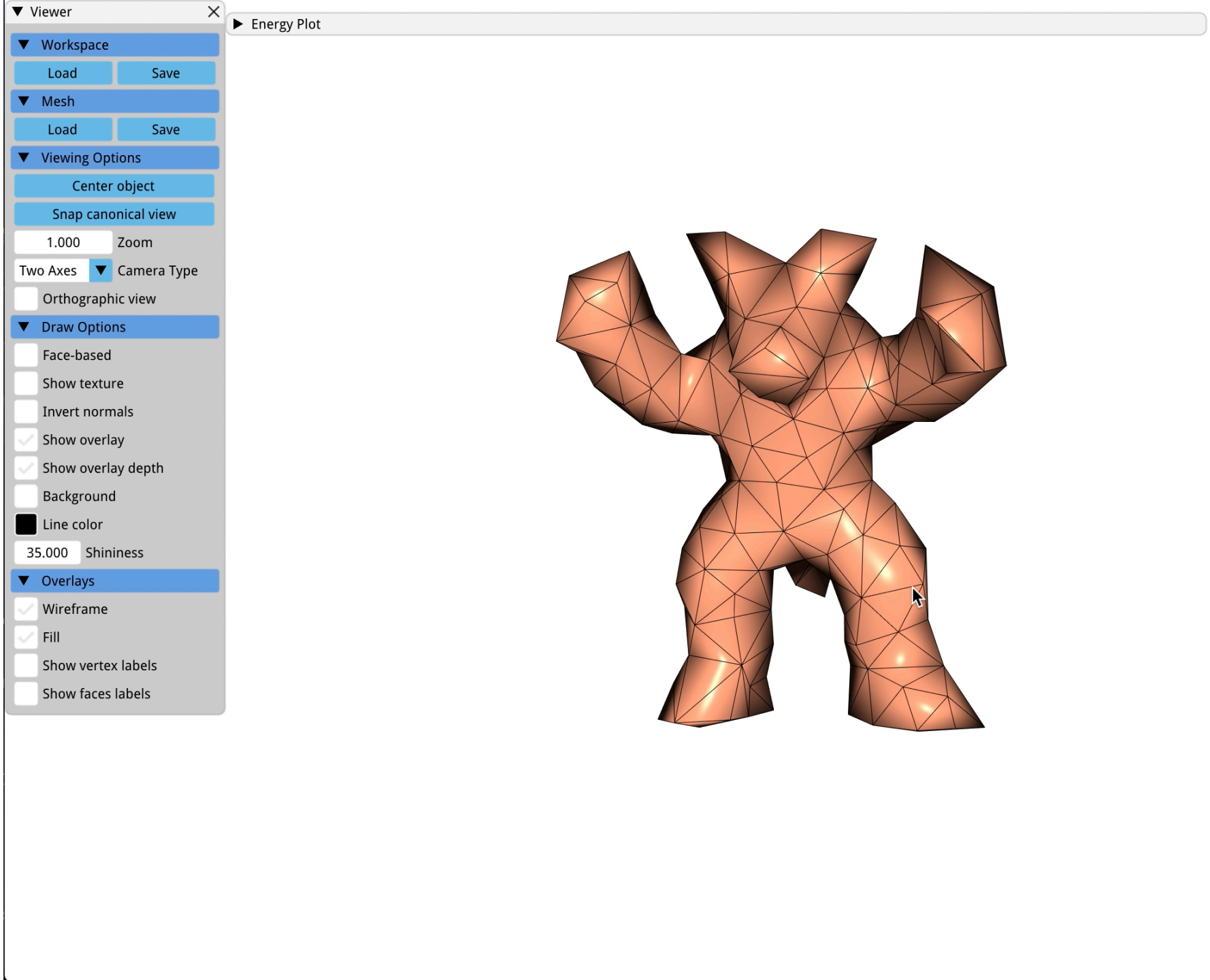
Repeat

Stop if sufficient decrease achieved or you get stuck

$$E\left(\mathbf{v}^i + \alpha \mathbf{d}\right) <= E\left(\mathbf{v}^i\right) + c \cdot \mathbf{d}^T \mathbf{g}^i \text{ or } \alpha < \text{tol}$$

Reduce α

$$\alpha = p \cdot \alpha$$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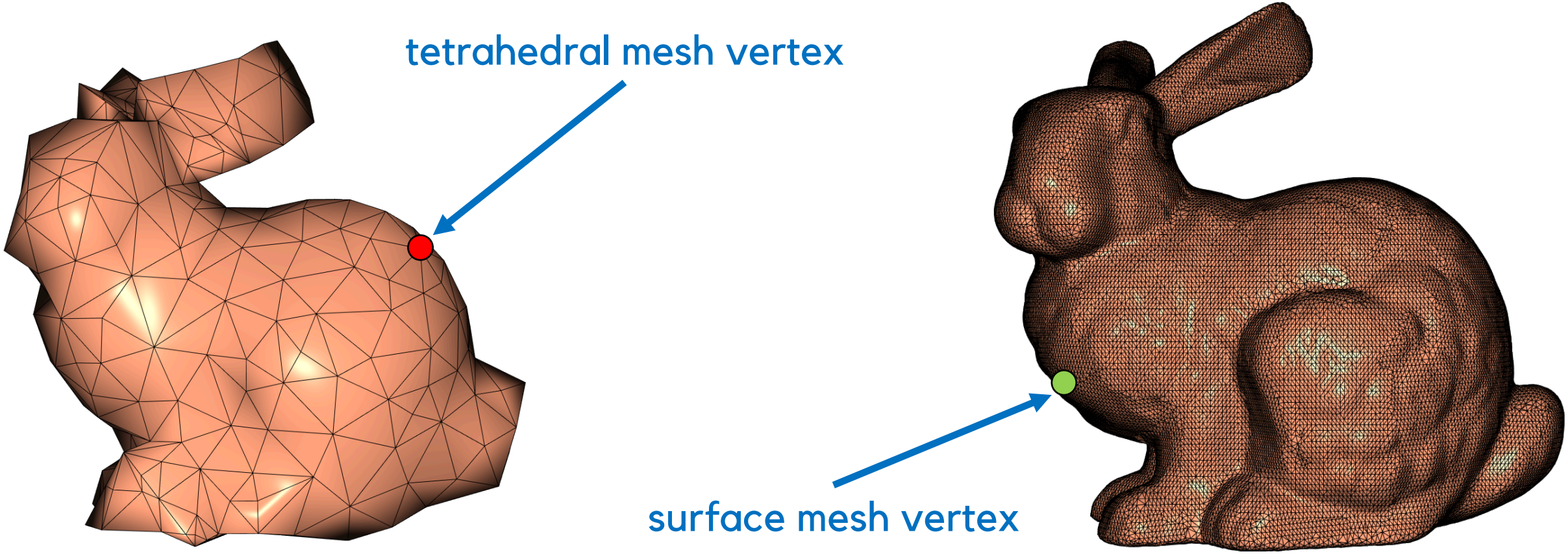
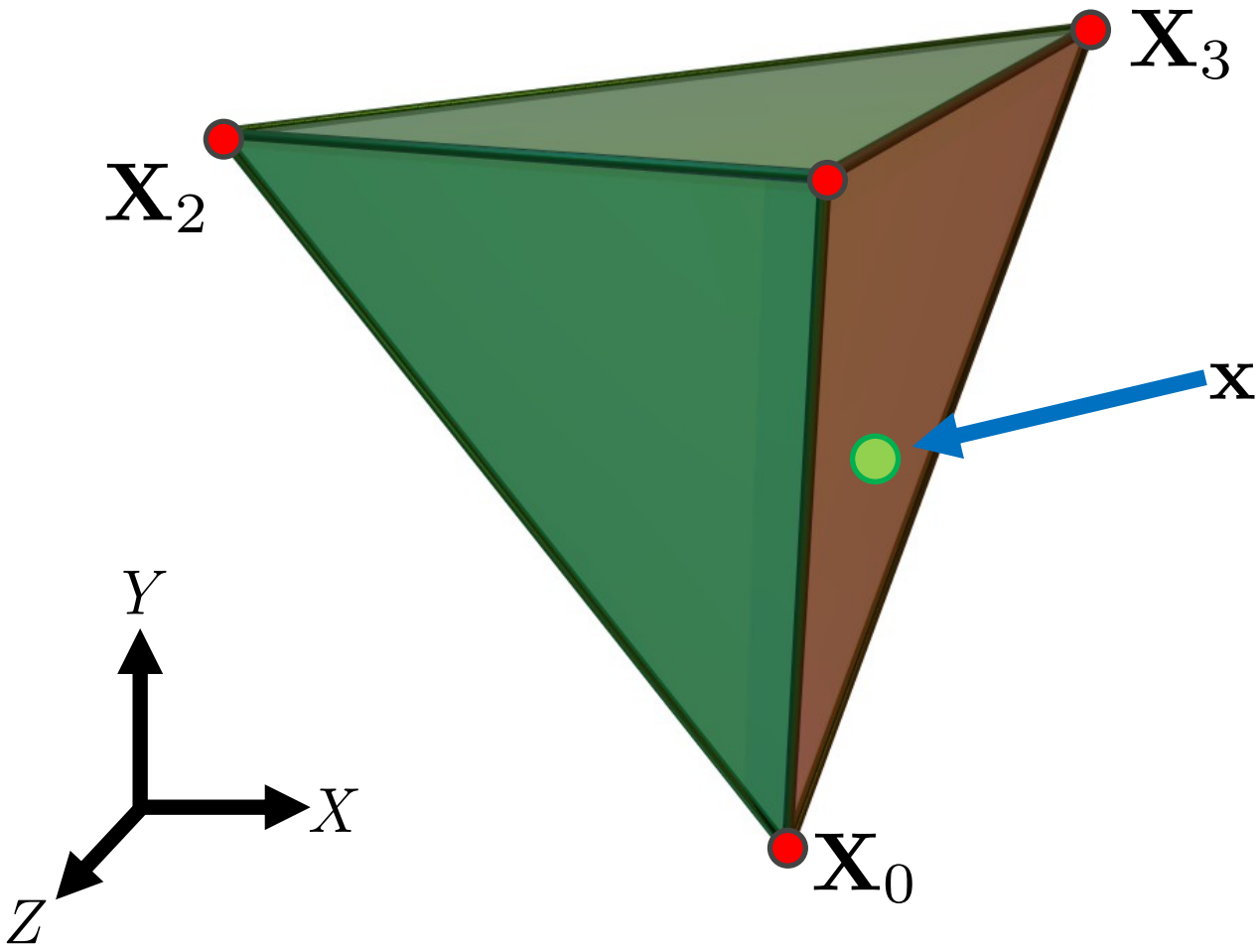


Simulation Tetrahedral Mesh

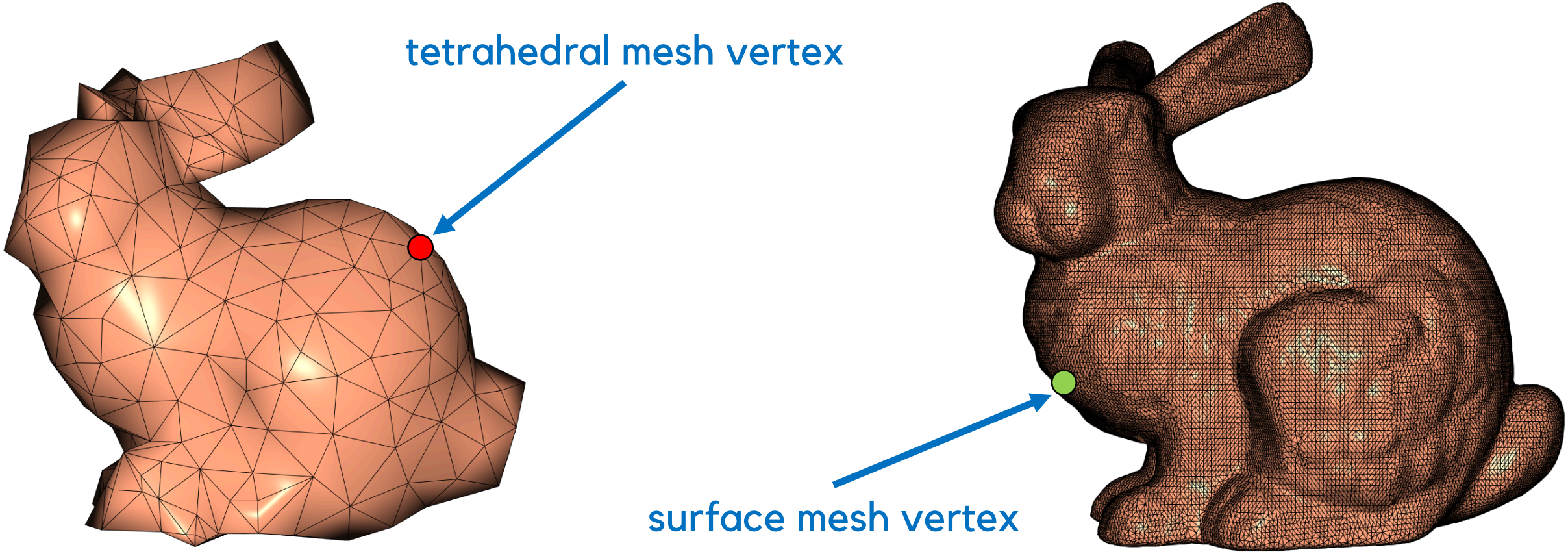
Visualization Triangle Mesh

Finite Elements for Deformation



Reference (Undeformed) Space

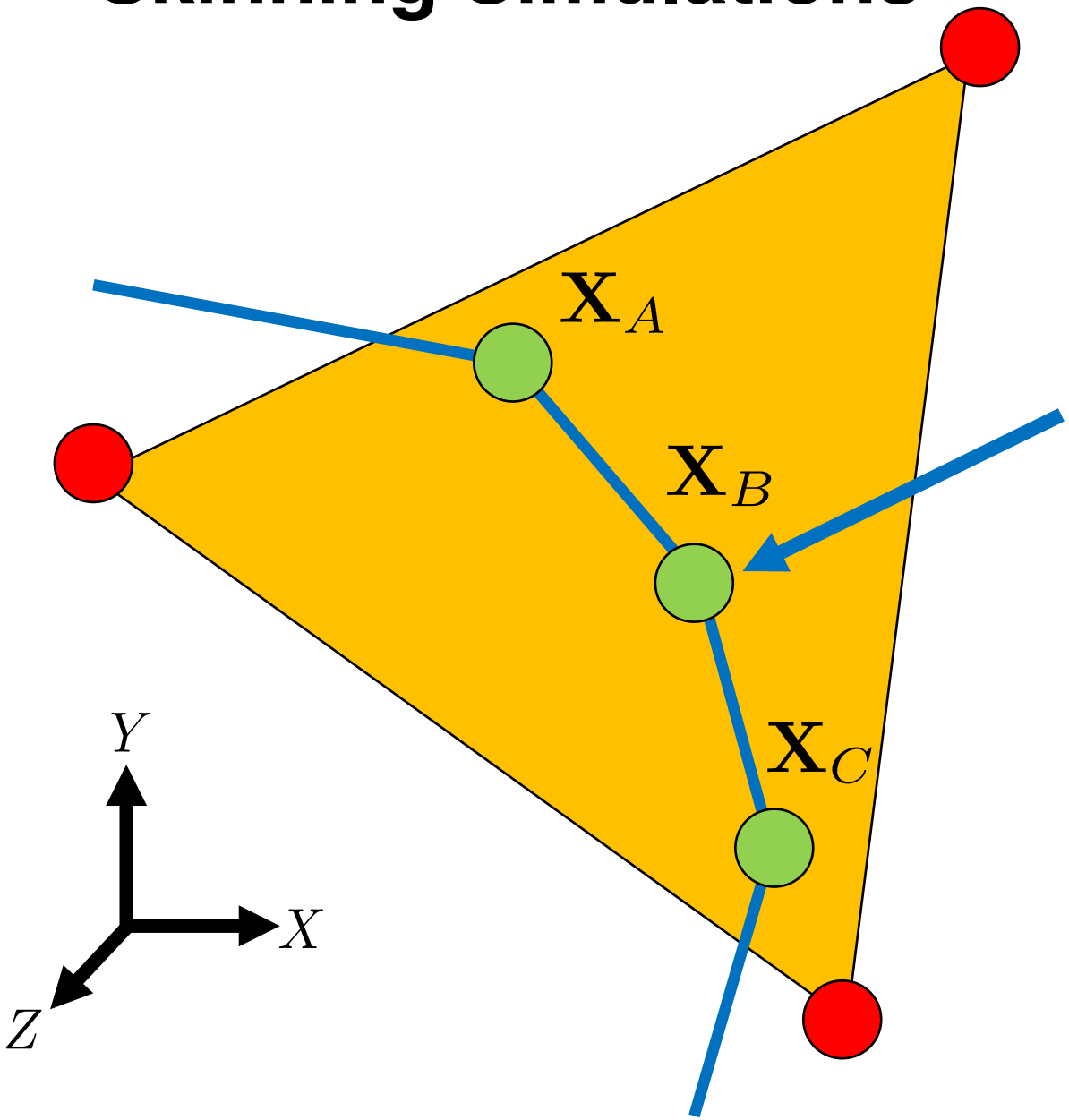
$$\mathbf{x}(\mathbf{X}) = \underbrace{\begin{pmatrix} \phi_0 \mathrm{I} & \phi_1 \mathrm{I} & \phi_2 \mathrm{I} & \phi_3 \mathrm{I} \end{pmatrix}}_{\mathrm{N}(\mathbf{X})} \underbrace{\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}}_{\mathbf{q}(t)}$$



Simulation Tetrahedral Mesh

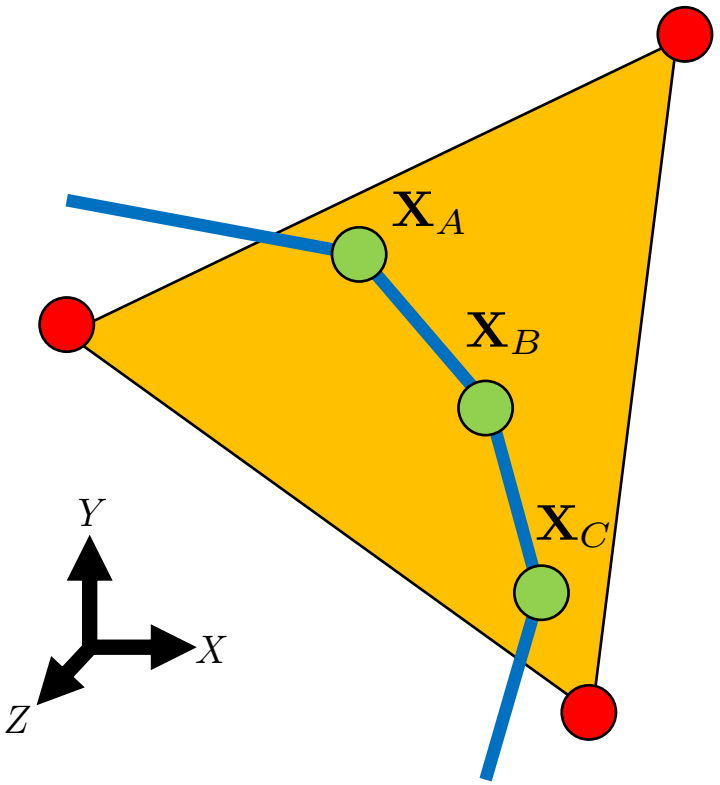
Visualization Triangle Mesh

Skinning Simulations



$$\mathbf{x}(\mathbf{X}) = \underbrace{\begin{pmatrix} \phi_0 \mathrm{I} & \phi_1 \mathrm{I} & \phi_2 \mathrm{I} & \phi_3 \mathrm{I} \end{pmatrix}}_{\mathrm{N}(\mathbf{X})} \underbrace{\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}}_{\mathbf{q}(t)}$$

Skinning Simulations

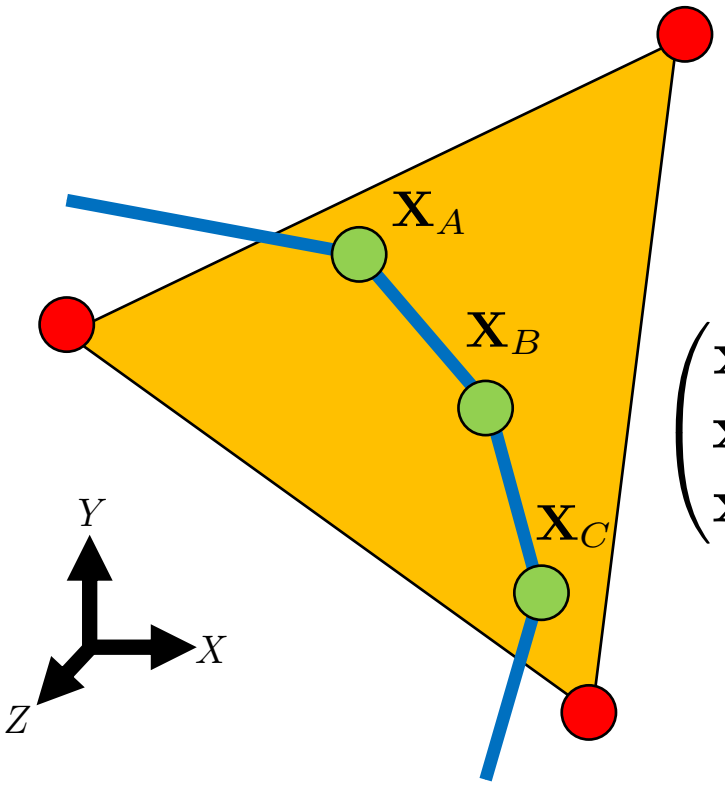


$$\mathbf{x}(\mathbf{X}_A) = \begin{pmatrix} \phi_0(\mathbf{X}_A)\mathrm{I} & \phi_1(\mathbf{X}_A)\mathrm{I} & \phi_2(\mathbf{X}_A)\mathrm{I} & \phi_3(\mathbf{X}_A)\mathrm{I} \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$$

$$\mathbf{x}(\mathbf{X}_B) = \begin{pmatrix} \phi_0(\mathbf{X}_B)\mathrm{I} & \phi_1(\mathbf{X}_B)\mathrm{I} & \phi_2(\mathbf{X}_B)\mathrm{I} & \phi_3(\mathbf{X}_B)\mathrm{I} \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$$

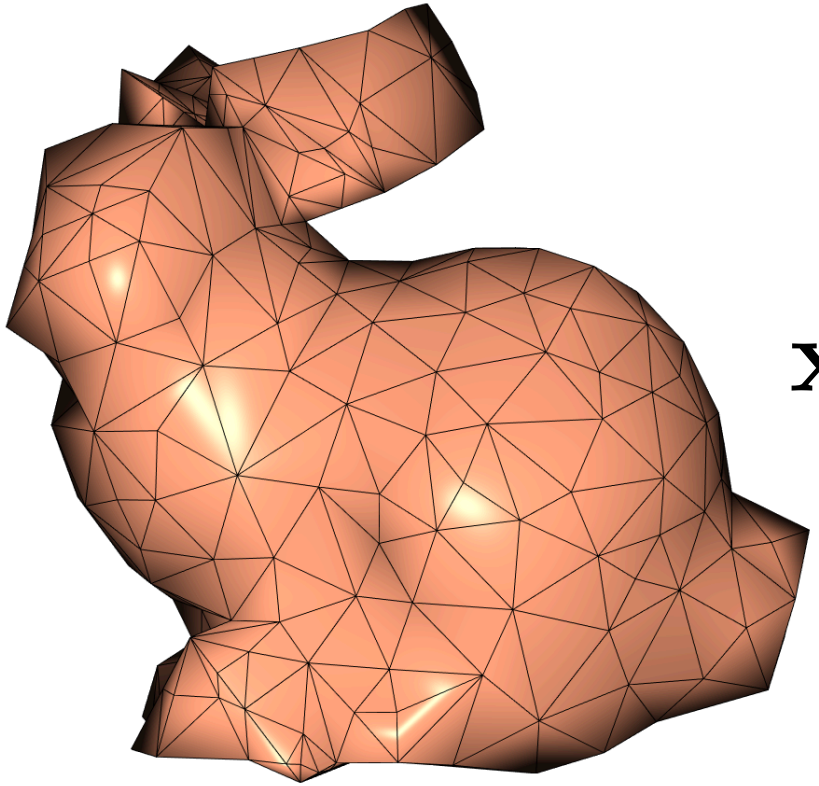
$$\mathbf{x}(\mathbf{X}_C) = \begin{pmatrix} \phi_0(\mathbf{X}_C)\mathrm{I} & \phi_1(\mathbf{X}_C)\mathrm{I} & \phi_2(\mathbf{X}_C)\mathrm{I} & \phi_3(\mathbf{X}_C)\mathrm{I} \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$$

Skinning Simulations

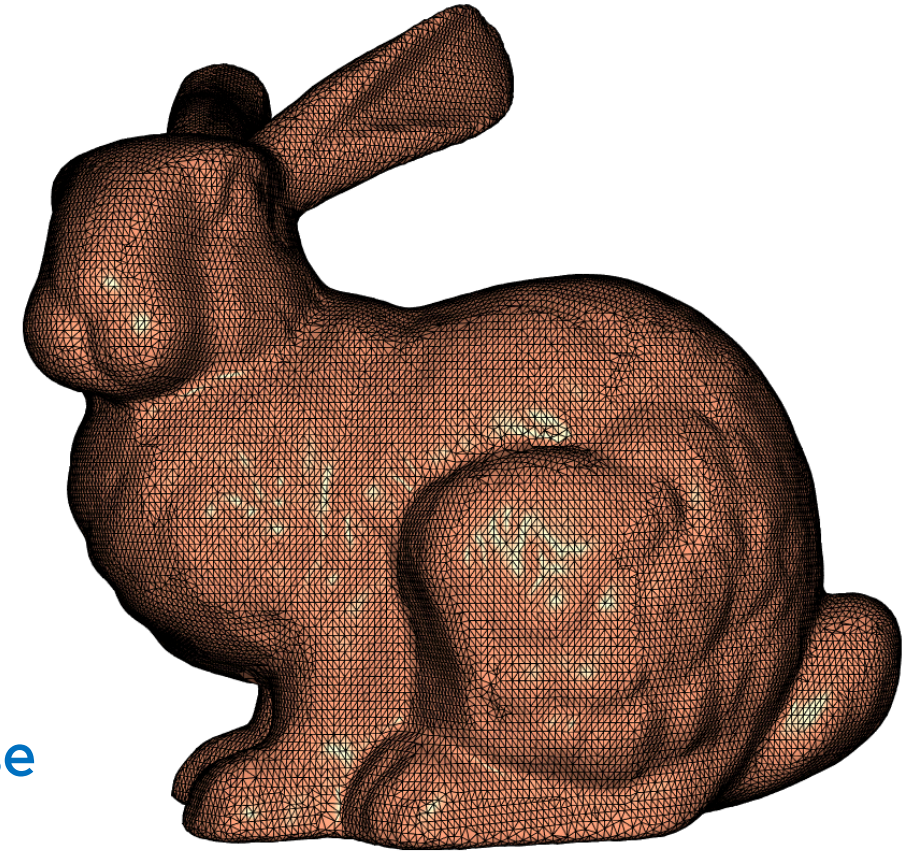


$$\begin{pmatrix} \mathbf{x}(\mathbf{X}_A) \\ \mathbf{x}(\mathbf{X}_B) \\ \mathbf{x}(\mathbf{X}_C) \end{pmatrix} = \underbrace{\begin{pmatrix} \phi_0(\mathbf{X}_A)\mathrm{I} & \phi_1(\mathbf{X}_A)\mathrm{I} & \phi_2(\mathbf{X}_A)\mathrm{I} & \phi_3(\mathbf{X}_A)\mathrm{I} \\ \phi_0(\mathbf{X}_B)\mathrm{I} & \phi_1(\mathbf{X}_B)\mathrm{I} & \phi_2(\mathbf{X}_B)\mathrm{I} & \phi_3(\mathbf{X}_B)\mathrm{I} \\ \phi_0(\mathbf{X}_C)\mathrm{I} & \phi_1(\mathbf{X}_C)\mathrm{I} & \phi_2(\mathbf{X}_C)\mathrm{I} & \phi_3(\mathbf{X}_C)\mathrm{I} \end{pmatrix}}_{\text{skinning weights } \mathrm{W}} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$$

Skinning Simulations

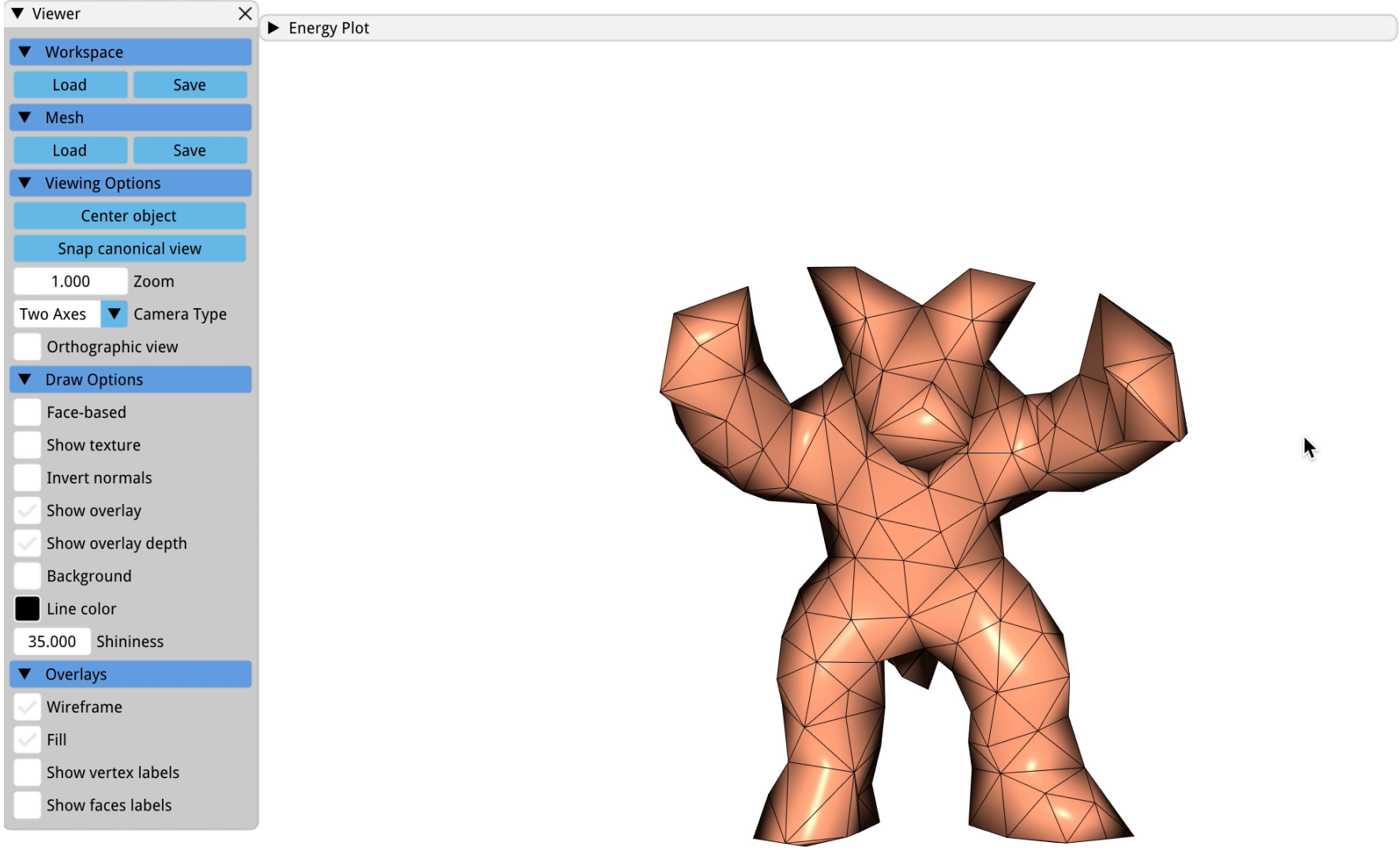


$$\mathbf{x}_{\text{surface}} = \underbrace{\mathbf{W}}_{\text{very sparse}}\mathbf{q}$$

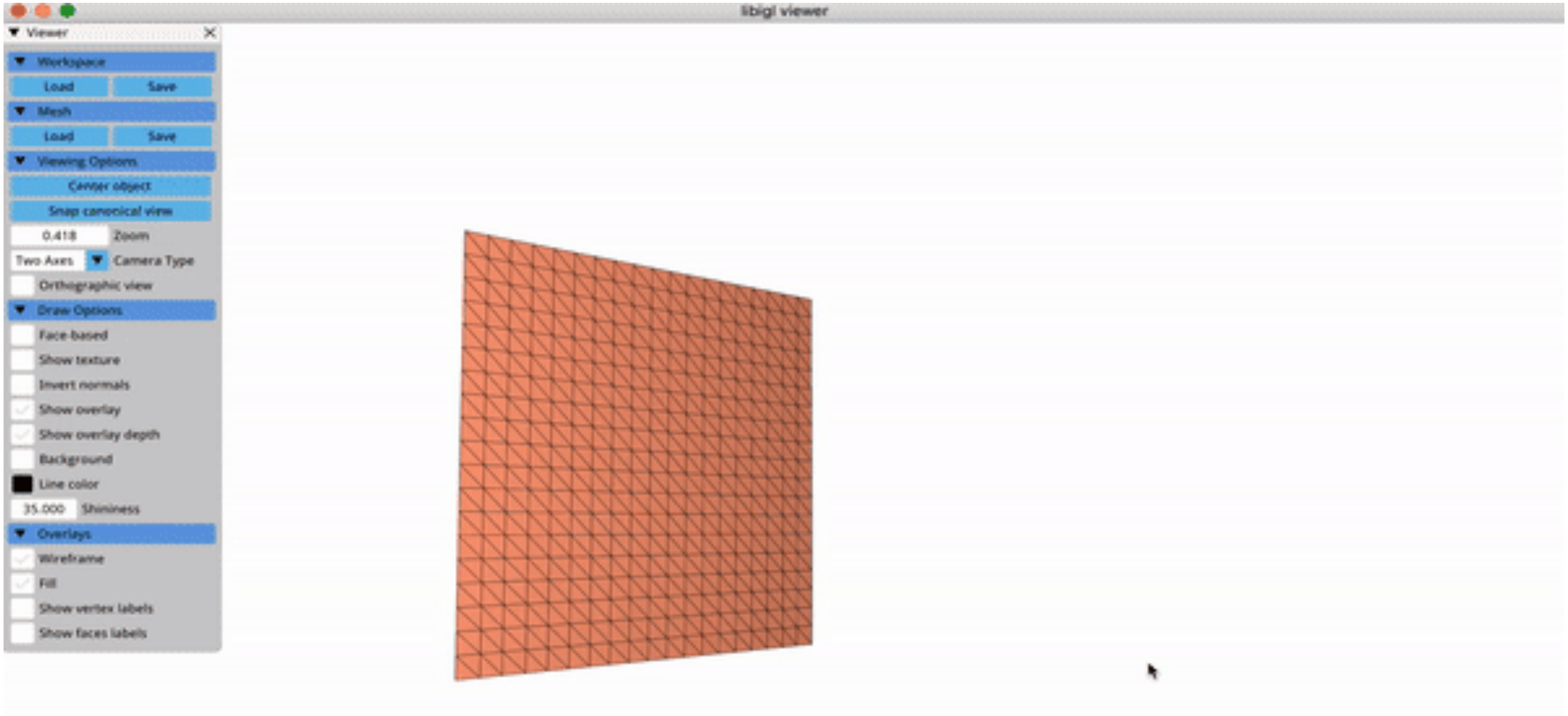


Simulation Tetrahedral Mesh

Visualization Triangle Mesh


Viewer
Energy Plot
Workspace
Load
Save
Mesh
Load
Save
Viewing Options
Center object
Snap canonical view
1.000
Zoom
Two Axes
Camera Type
Orthographic view
Draw Options
Face-based
Show texture
Invert normals
Show overlay
Show overlay depth
Background
Line color
35.000
Shininess
Overlays
Wireframe
Fill
Show vertex labels
Show faces labels

Next Week:


libigl viewer
Viewer
Workspace
Load
Save
Mesh
Load
Save
Viewing Options
Center object
Snap canonical view
0.418
Zoom
Two Axes
Camera Type
Orthographic view
Draw Options
Face-based
Show texture
Invert normals
Show overlay
Show overlay depth
Background
Line color
35.000
Shininess
Overlays
Wireframe
Fill
Show vertex labels
Show faces labels