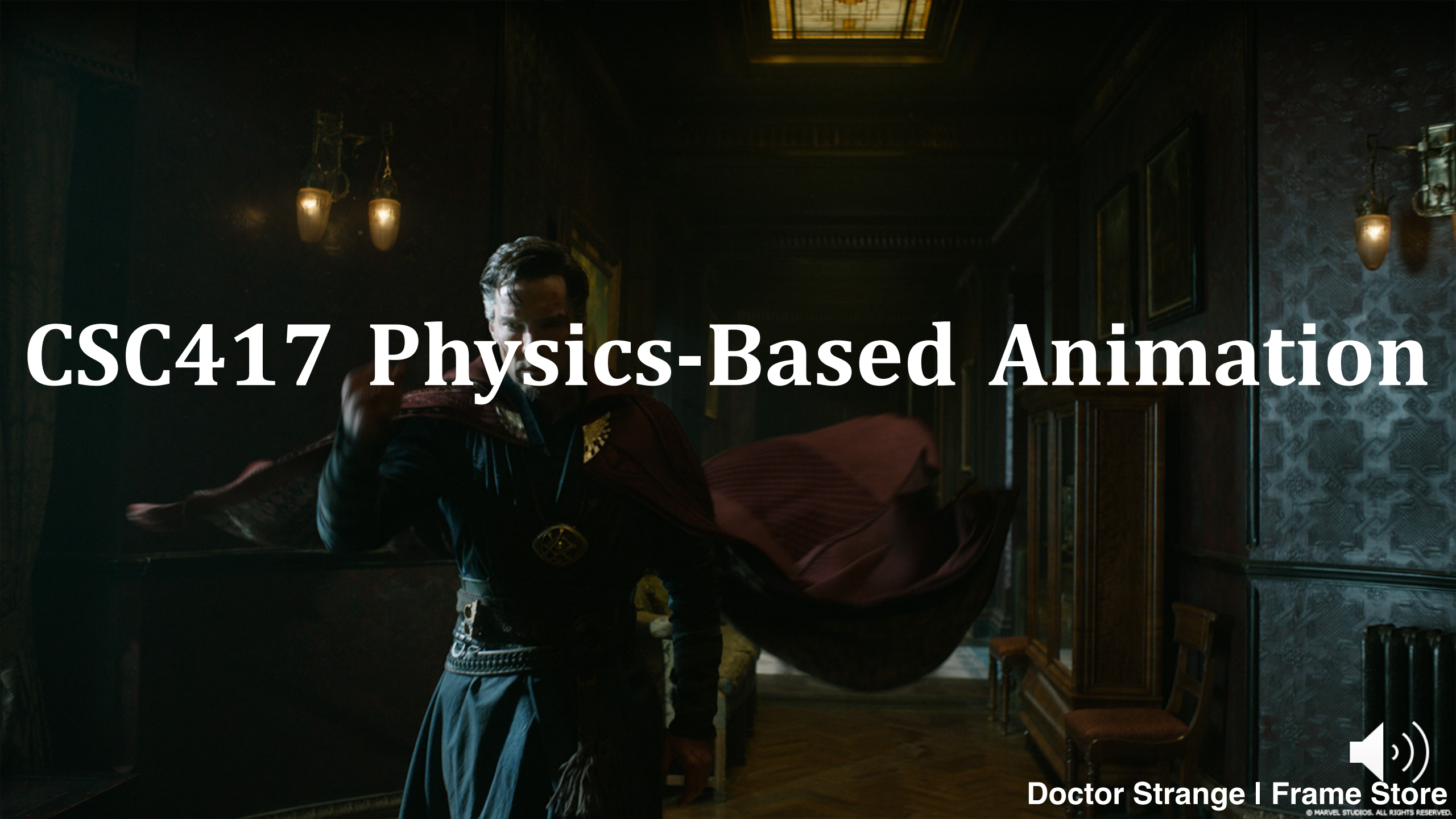
# CSC417 Physics-Based Animation



Doctor Strange | Frame Store

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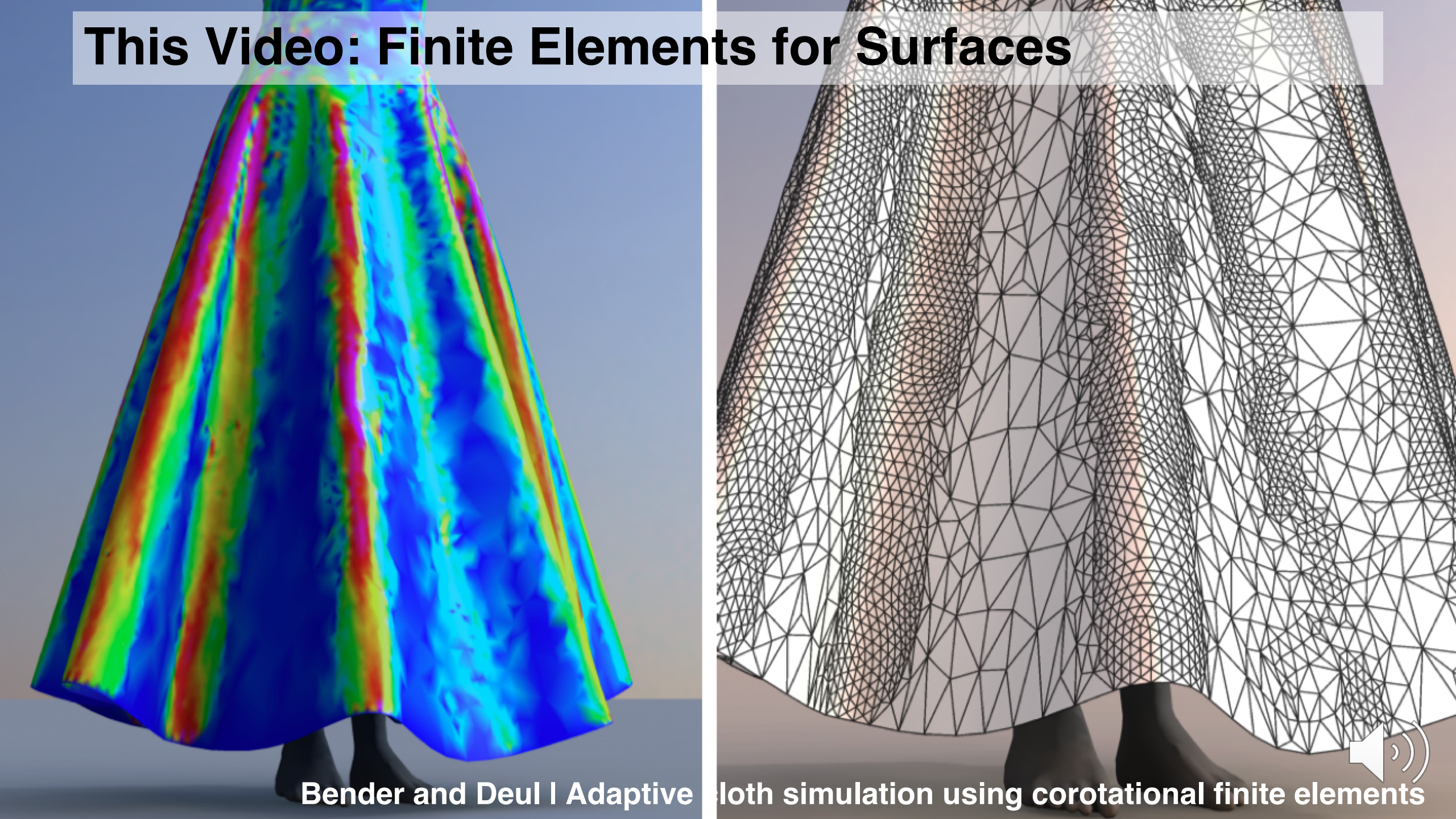
# Last Video: More Finite Element Method

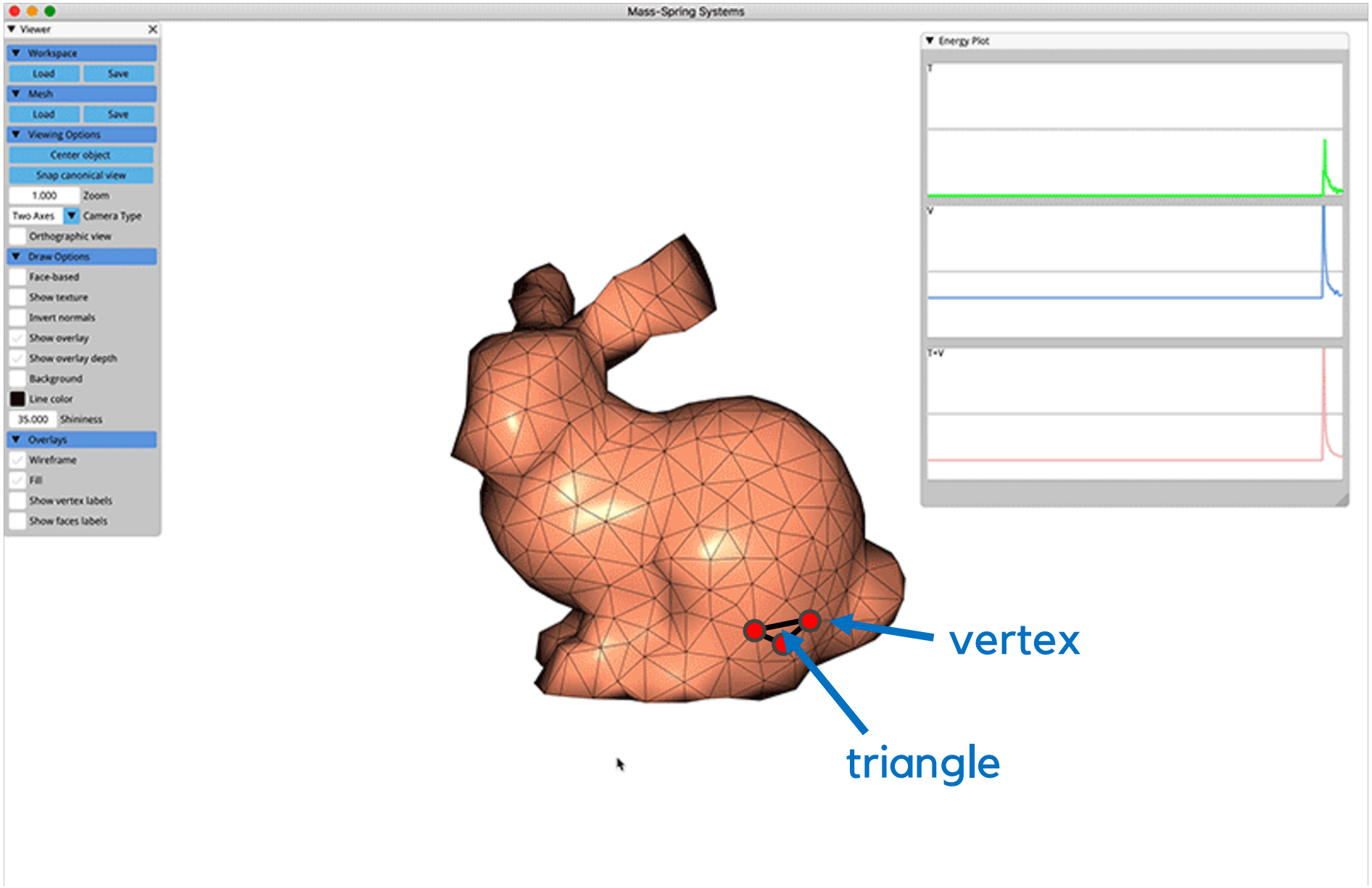
# This Video: Cloth Simulation

Data-Driven Elastic Models for Cloth: Modelling and Measurement | Wang et al.

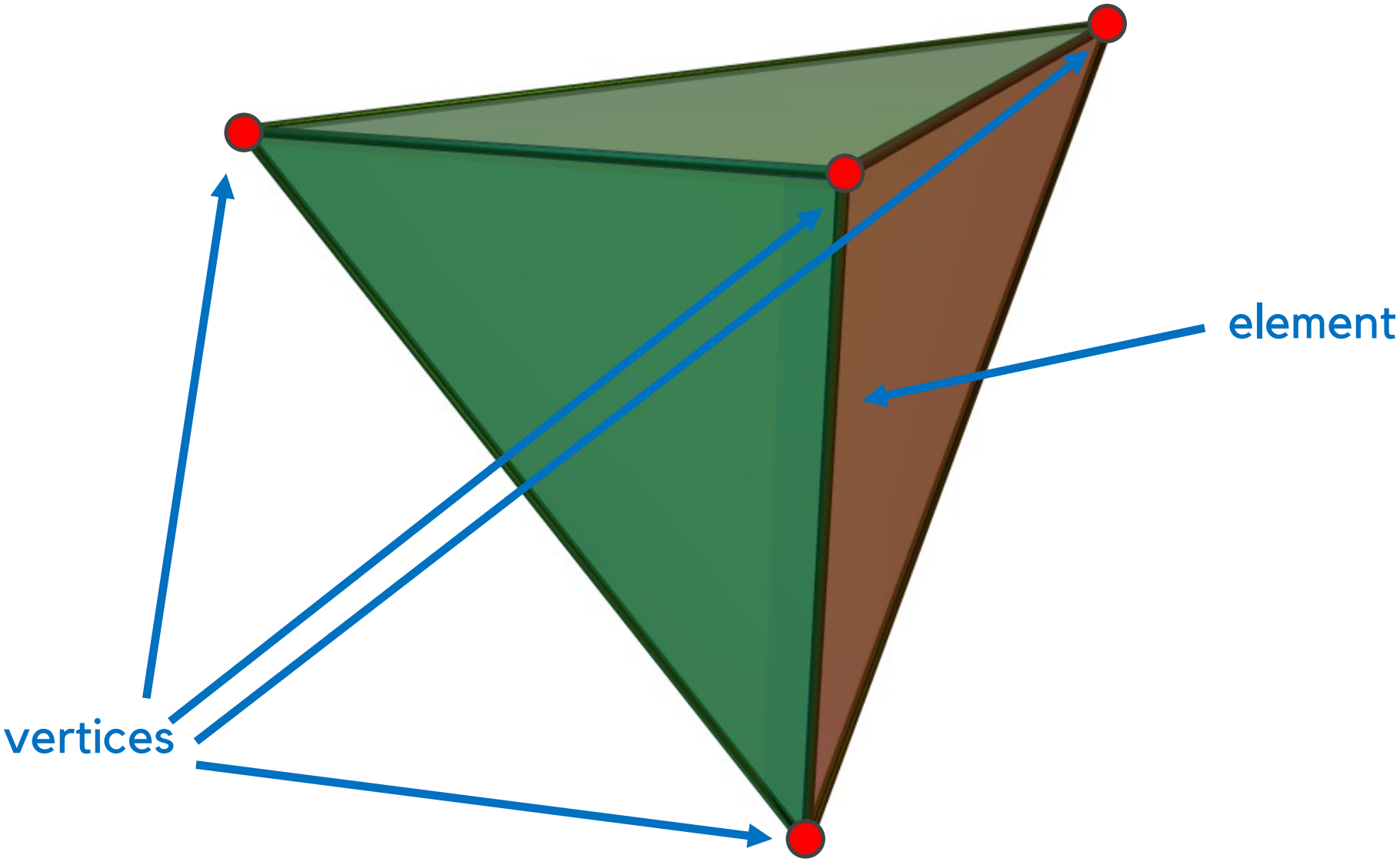
# This Video: Finite Elements for Surfaces



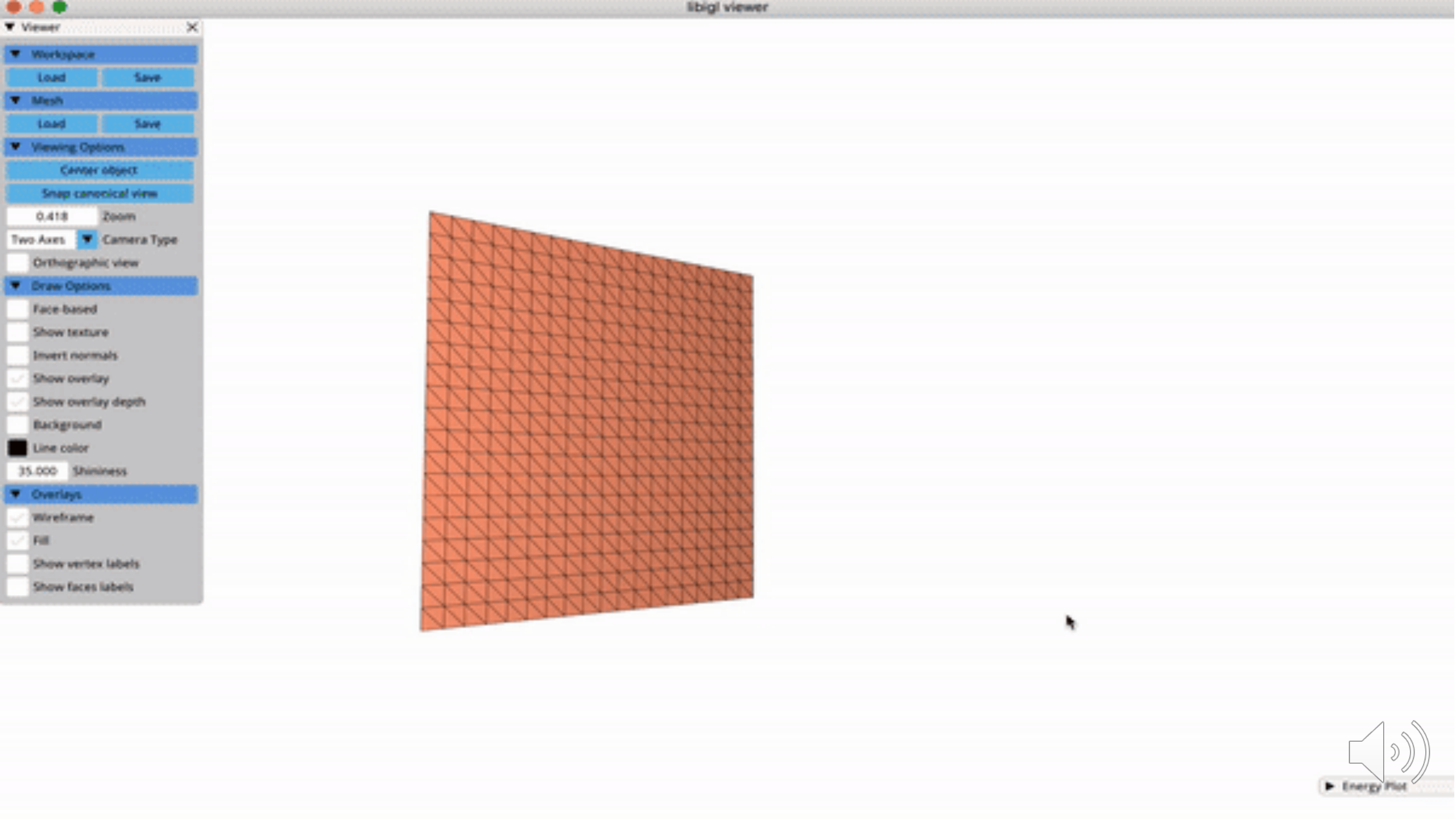
# Finite Elements for Volumes



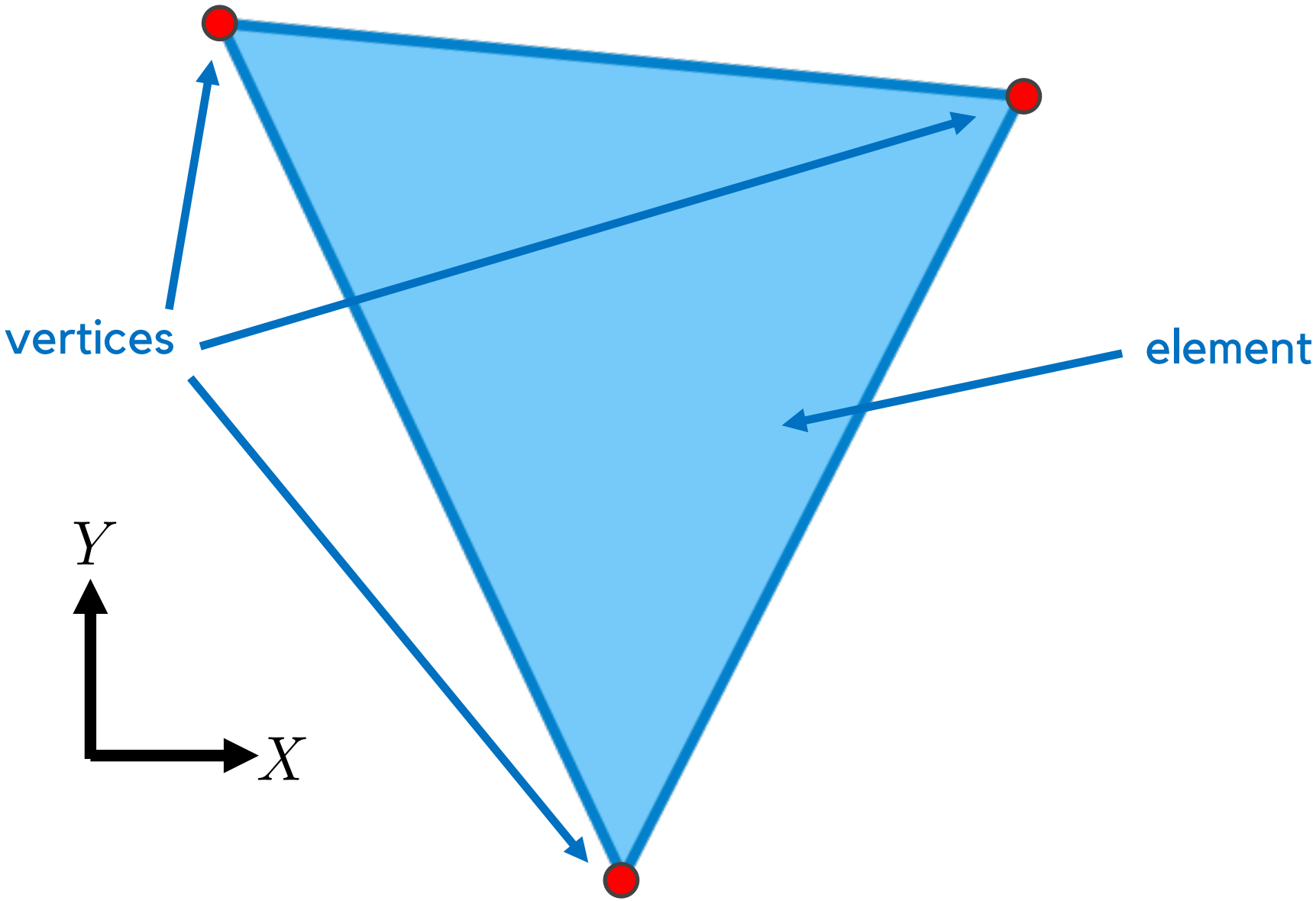
# Tetrahedral Finite Elements



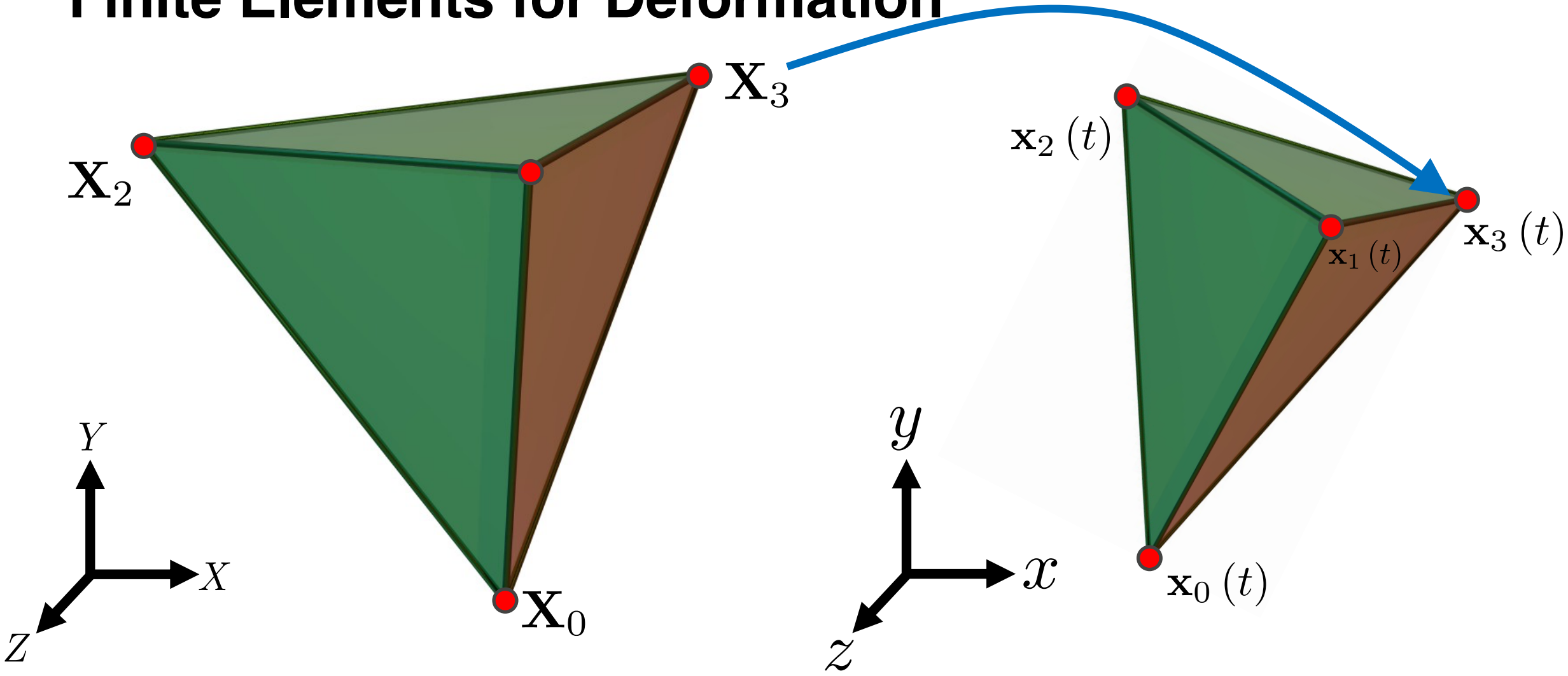

TetWild | https://github.com/Yixin-Hu/TetWild


libigl viewer
Viewer
Workspace
Load
Save
Mesh
Load
Save
Viewing Options
Center object
Snap canonical view
0.418
Zoom
Two Axes
Camera Type
Orthographic view
Draw Options
Face-based
Show texture
Invert normals
Show overlay
Show overlay depth
Background
Line color
35.000
Shininess
Overlays
Wireframe
Fill
Show vertex labels
Show faces labels
Energy Plot

# Triangular Finite Elements



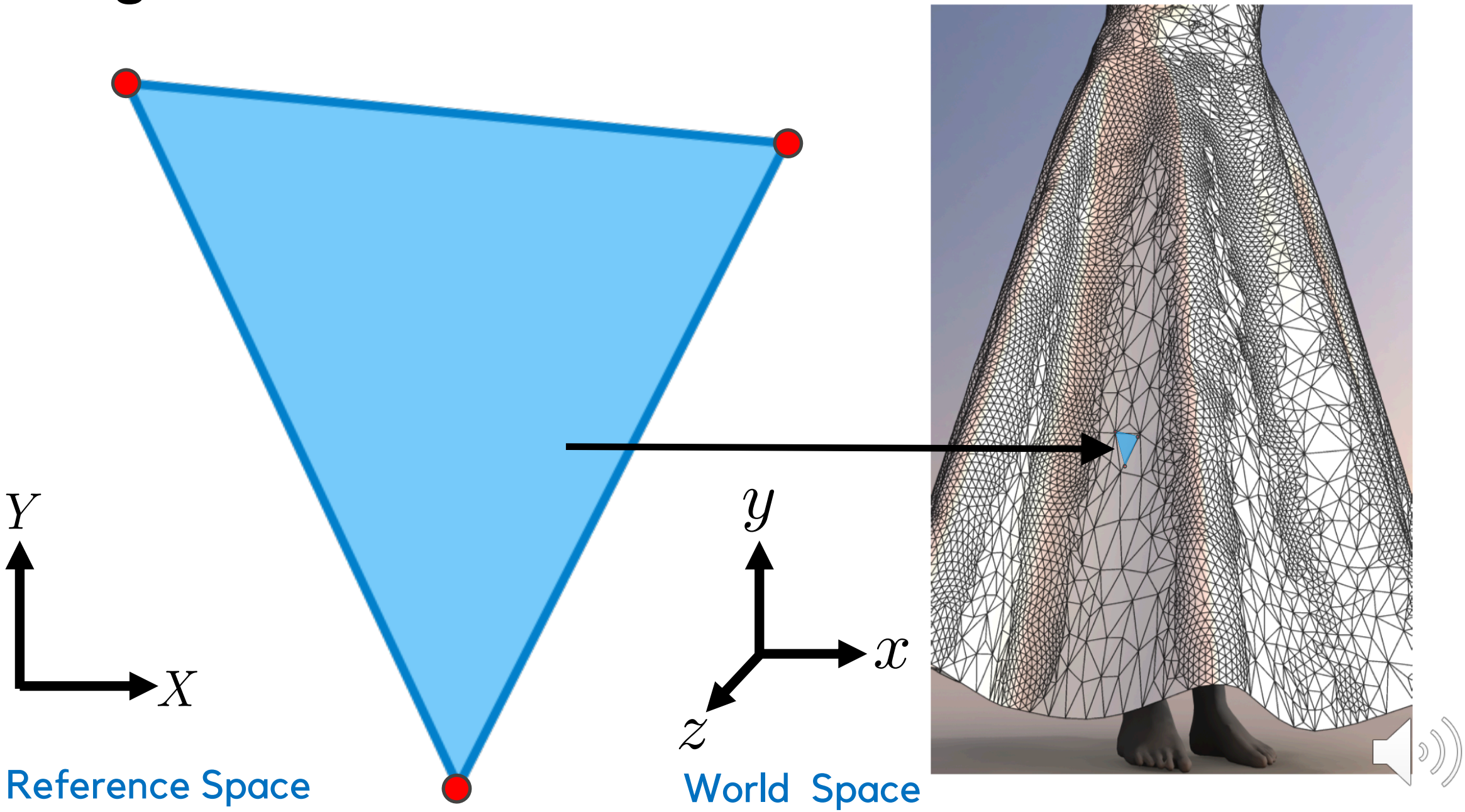
# Finite Elements for Deformation



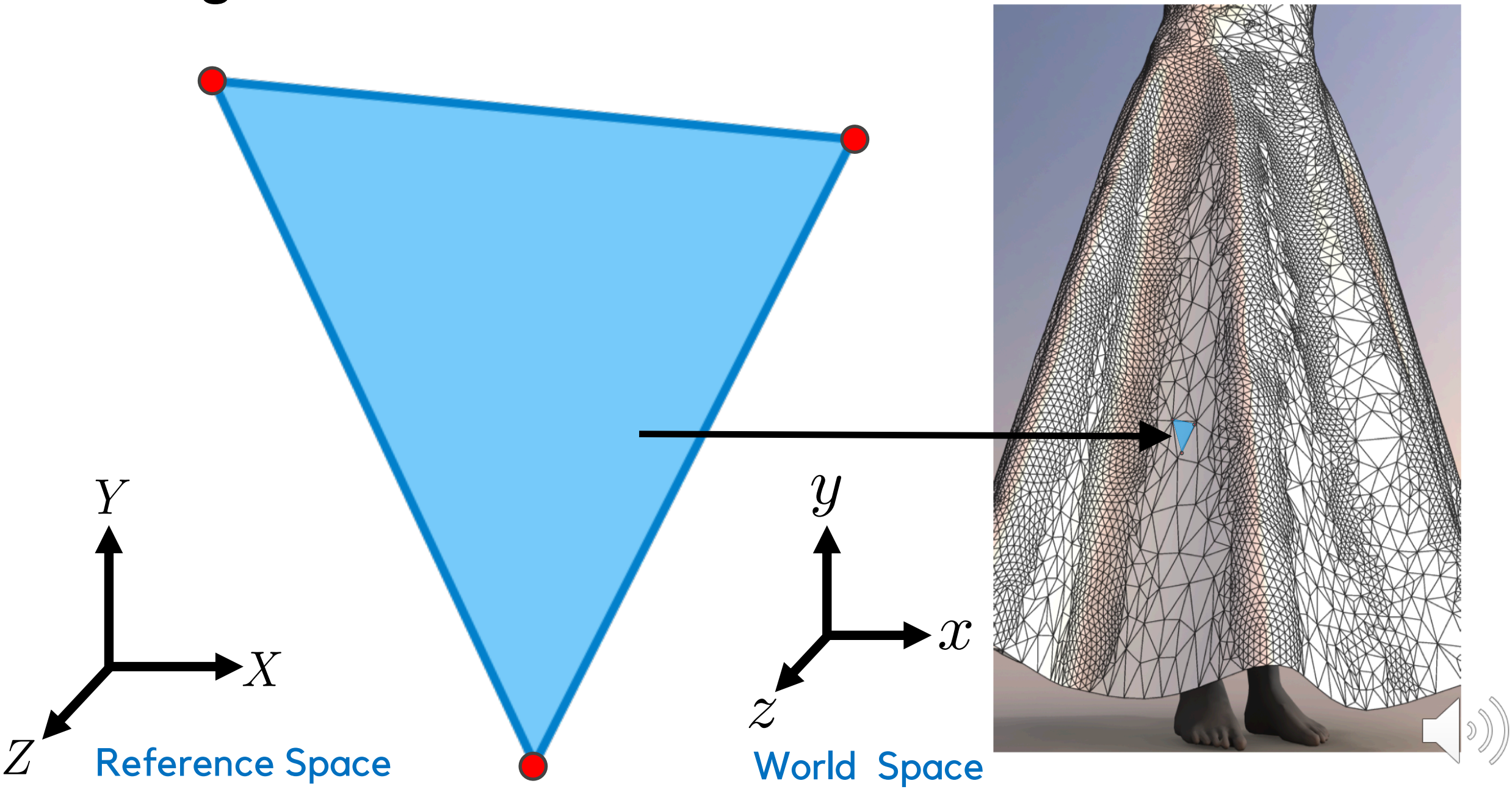

Reference (Undeformed) Space

World (Deformed) Space

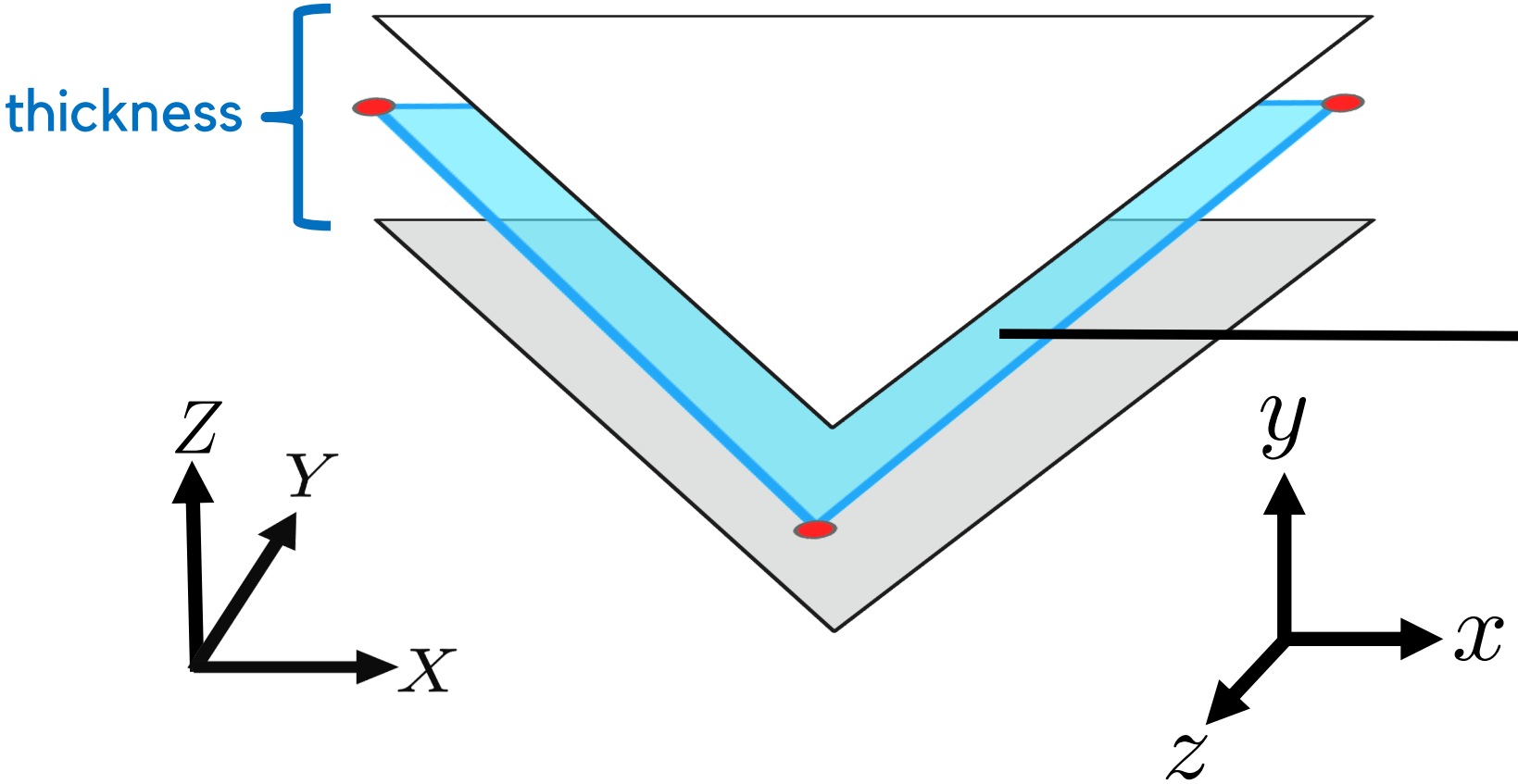
# Triangular Finite Elements

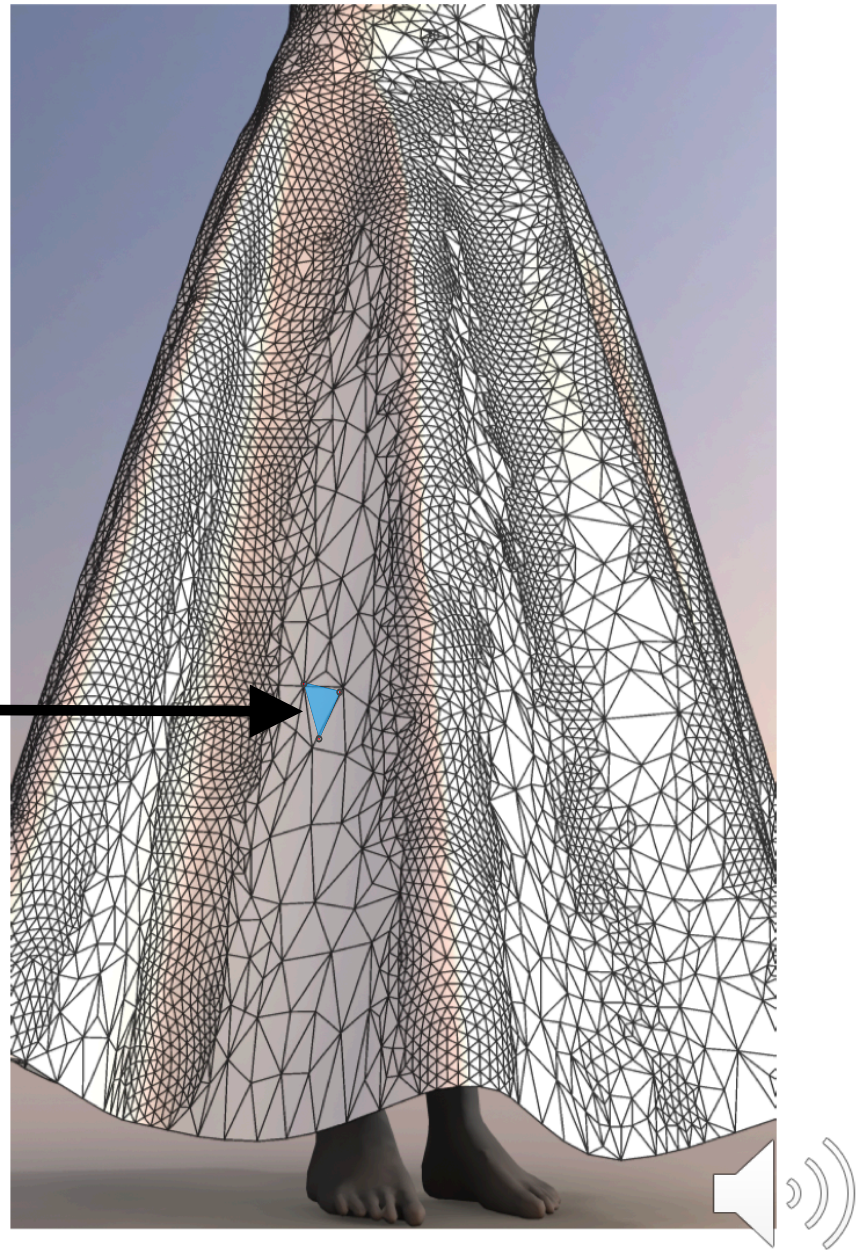


# Triangular Finite Elements



# Triangular Finite Elements

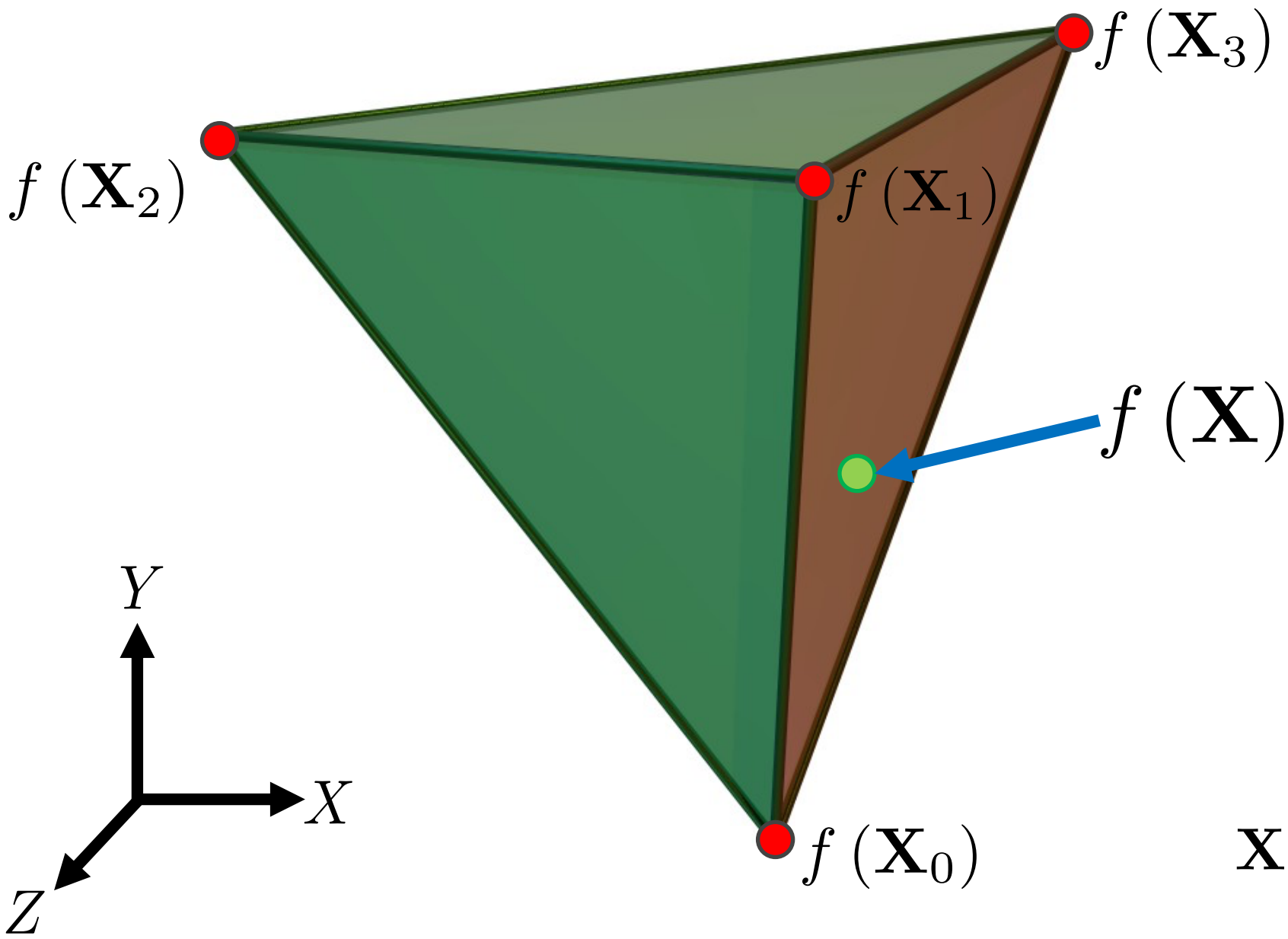


Reference Space

World Space

# Finite Elements



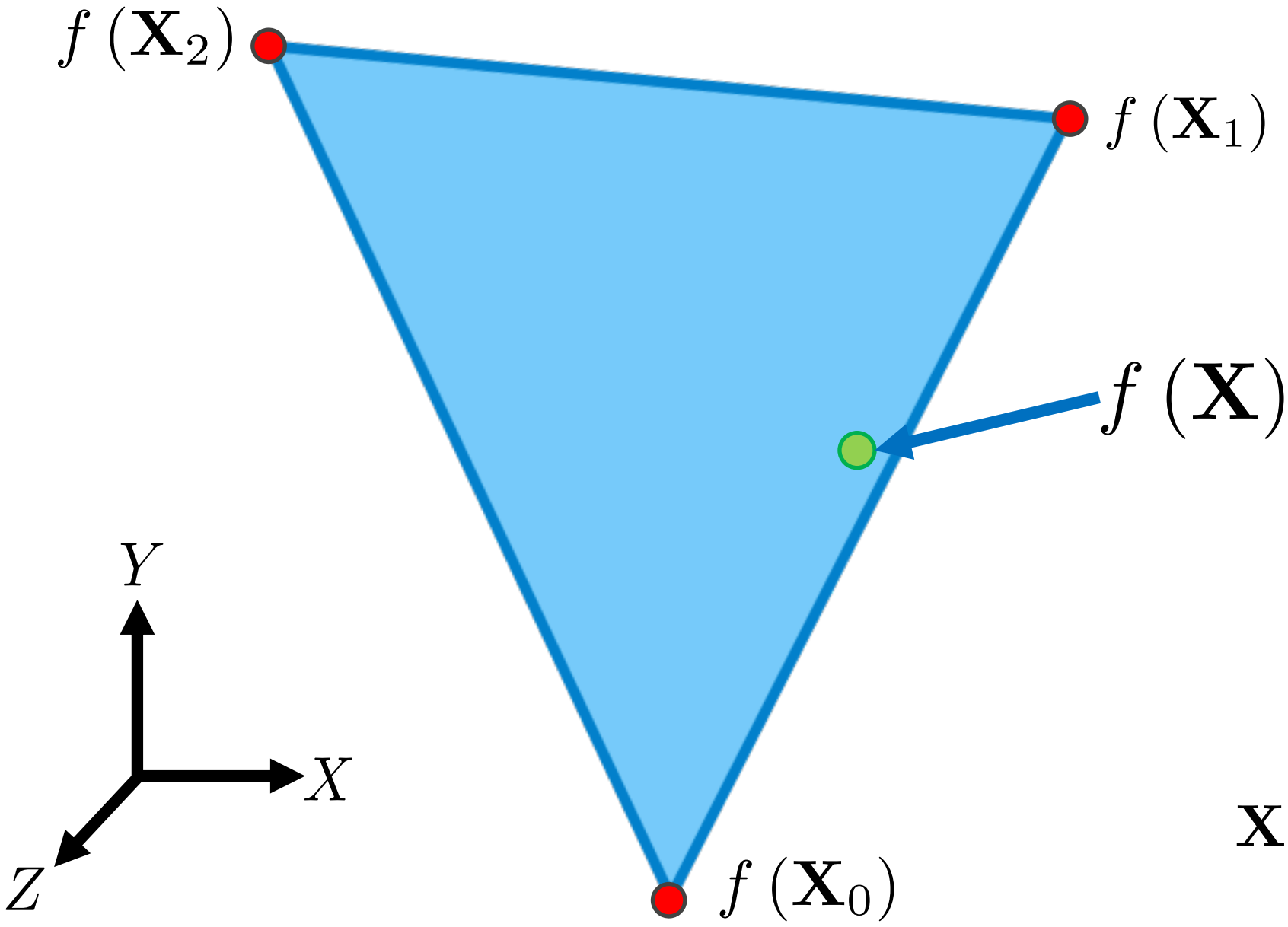

shape function

$$f(\mathbf{X}) = \sum_{i=0}^{i=3} f_i \phi_i(\mathbf{X})$$

$f(\mathbf{X}_i)$

$$\mathbf{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{R}^3$$

# Triangular Finite Elements



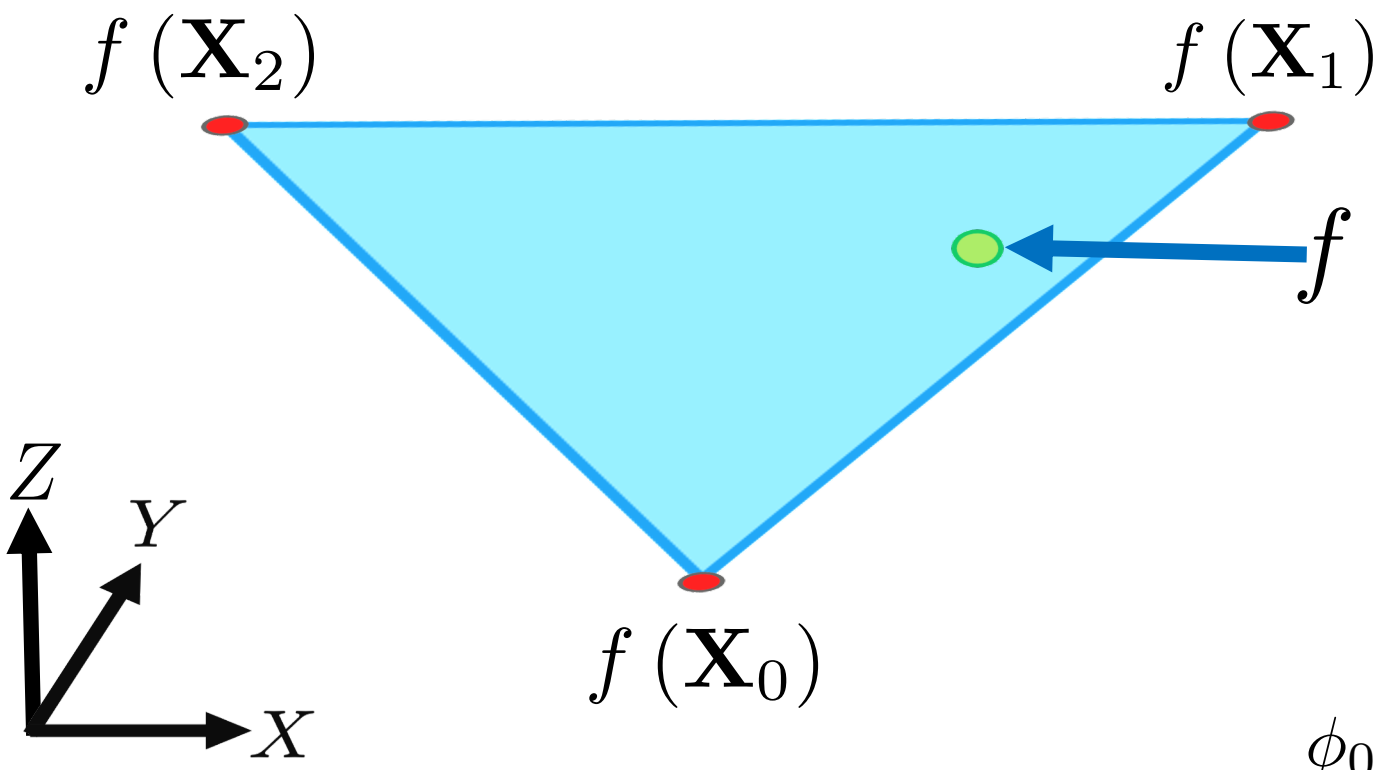

shape function

$$f(\mathbf{X}) = \sum_{i=0}^{i=2} f_i \phi_i(\mathbf{X})$$

$f(\mathbf{X}_i)$

$$\mathbf{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{R}^3$$

# Triangular Finite Elements



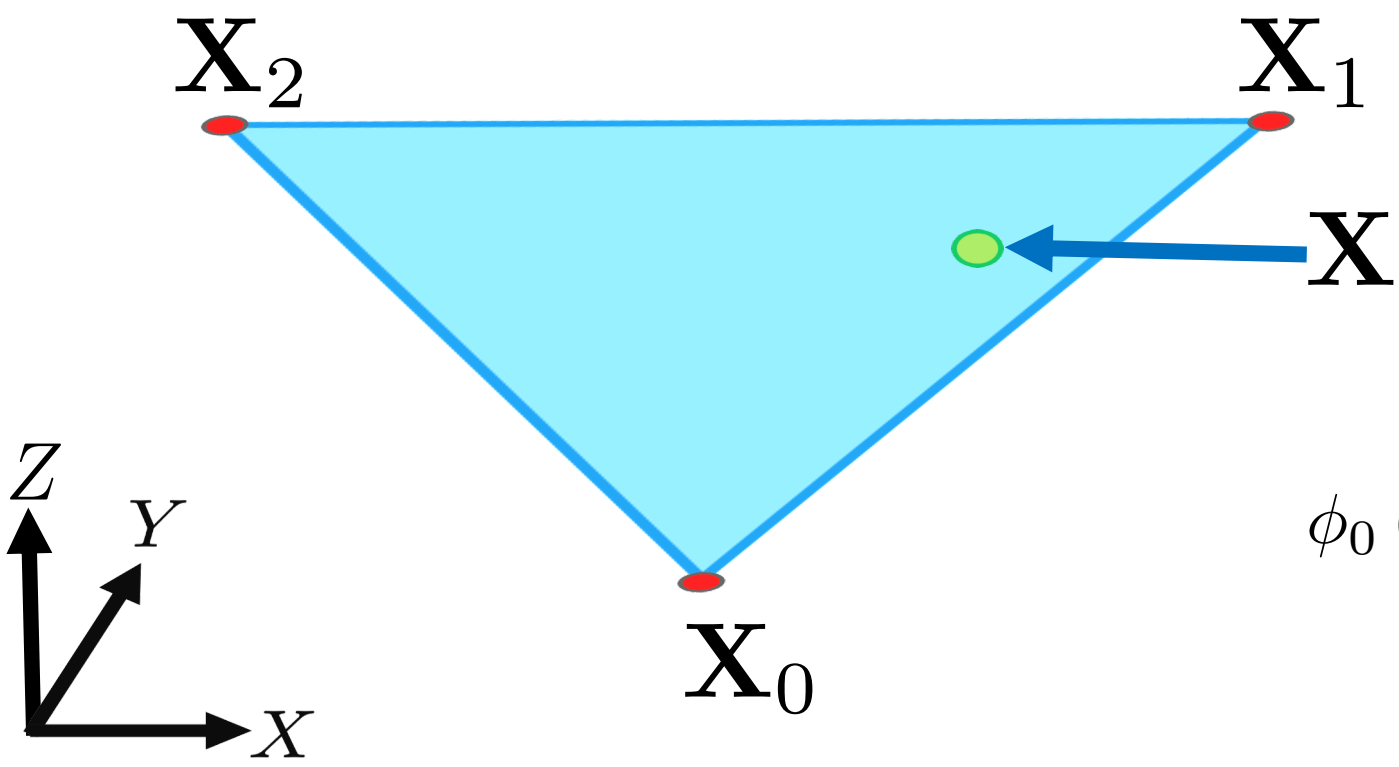

shape function

$$f(\mathbf{X}) = \sum_{i=0}^{i=2} f_i \phi_i(\mathbf{X})$$

$f(\mathbf{X}_i)$

$$\phi_0(\mathbf{X}) = 1 - \phi_1(\mathbf{X}) - \phi_2(\mathbf{X})$$

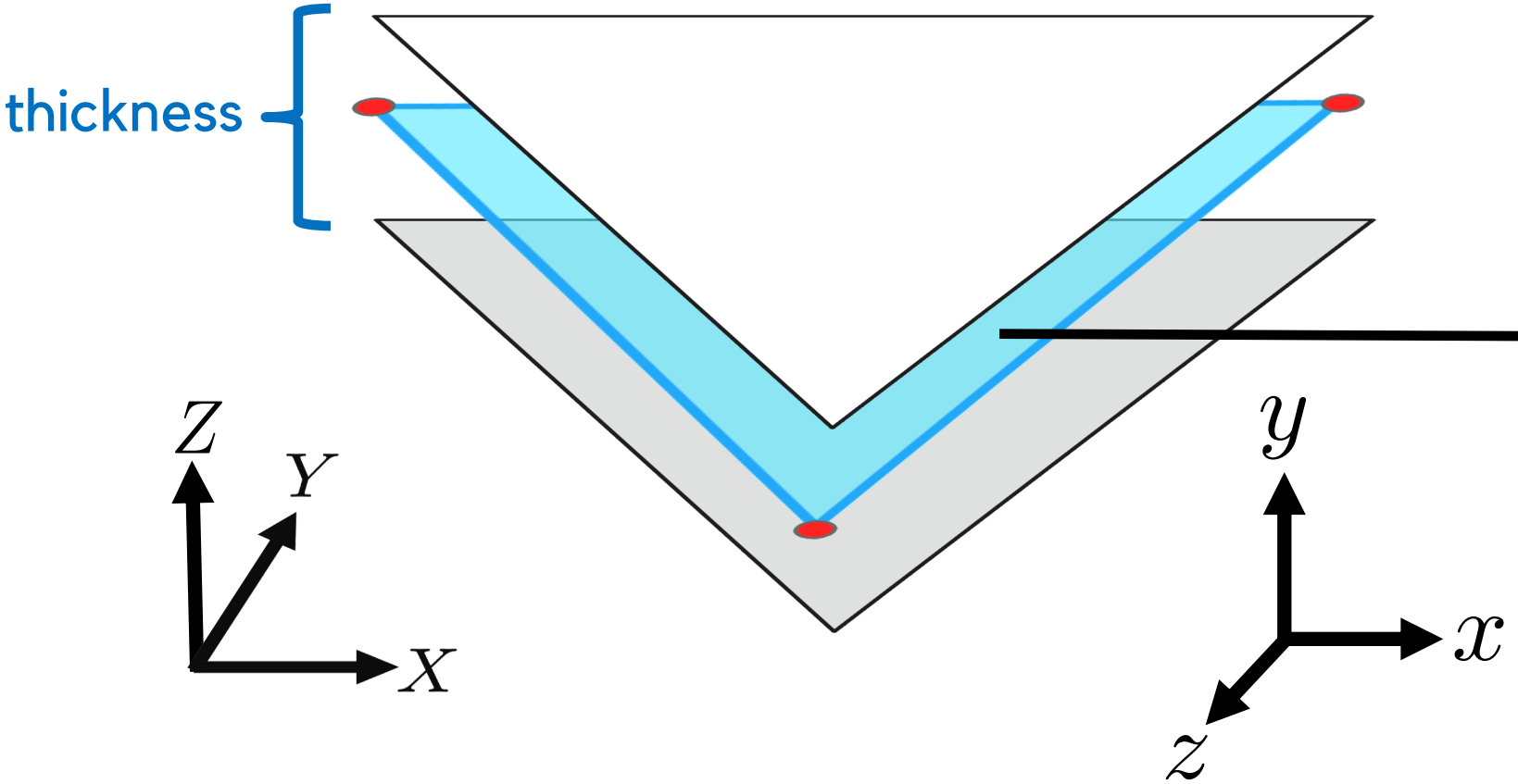
# Triangular Finite Elements

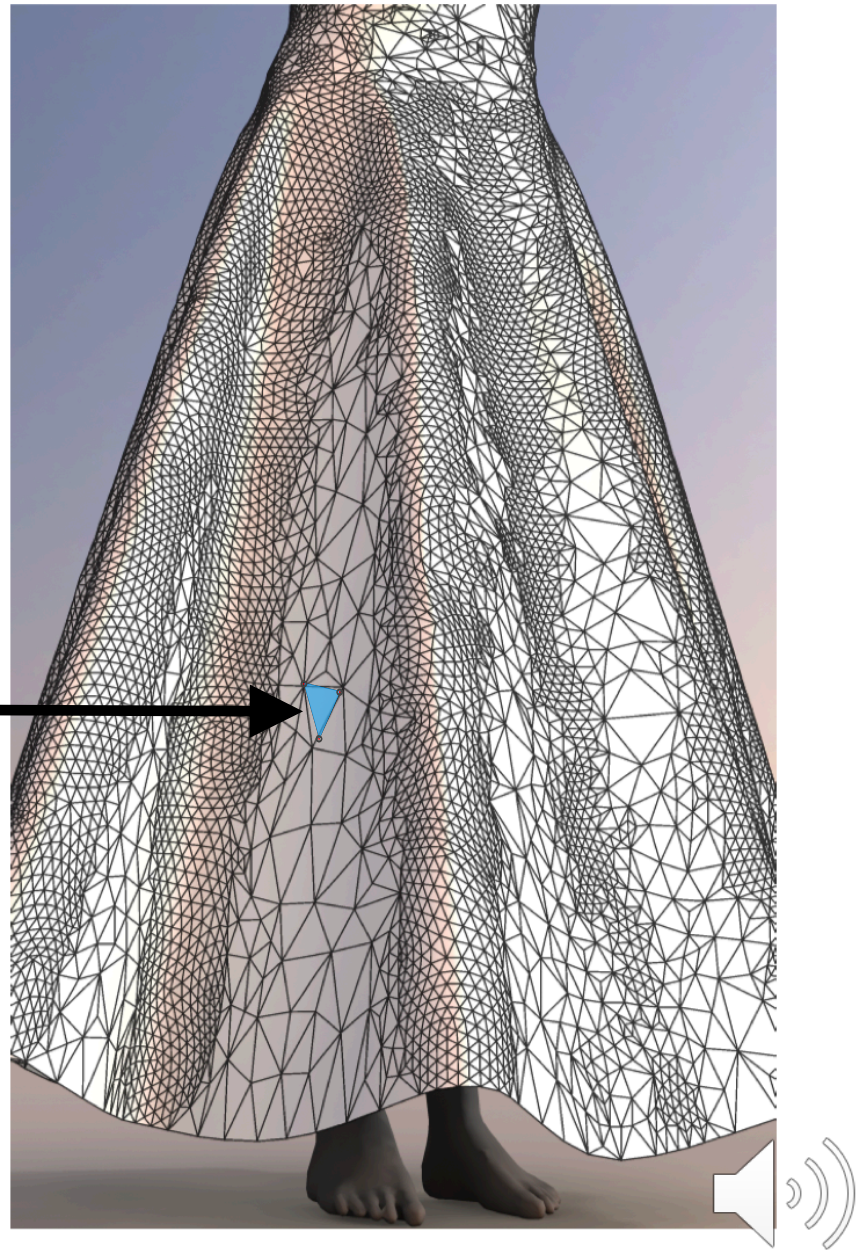



$$\mathbf{X} = \sum_{i=0}^{i=2} \mathbf{X}_i \phi_i \left(\mathbf{X}\right)$$

$$\phi_0 \left(\mathbf{X}\right) = 1 - \phi_1 \left(\mathbf{X}\right) - \phi_2 \left(\mathbf{X}\right)$$

# Triangular Finite Elements

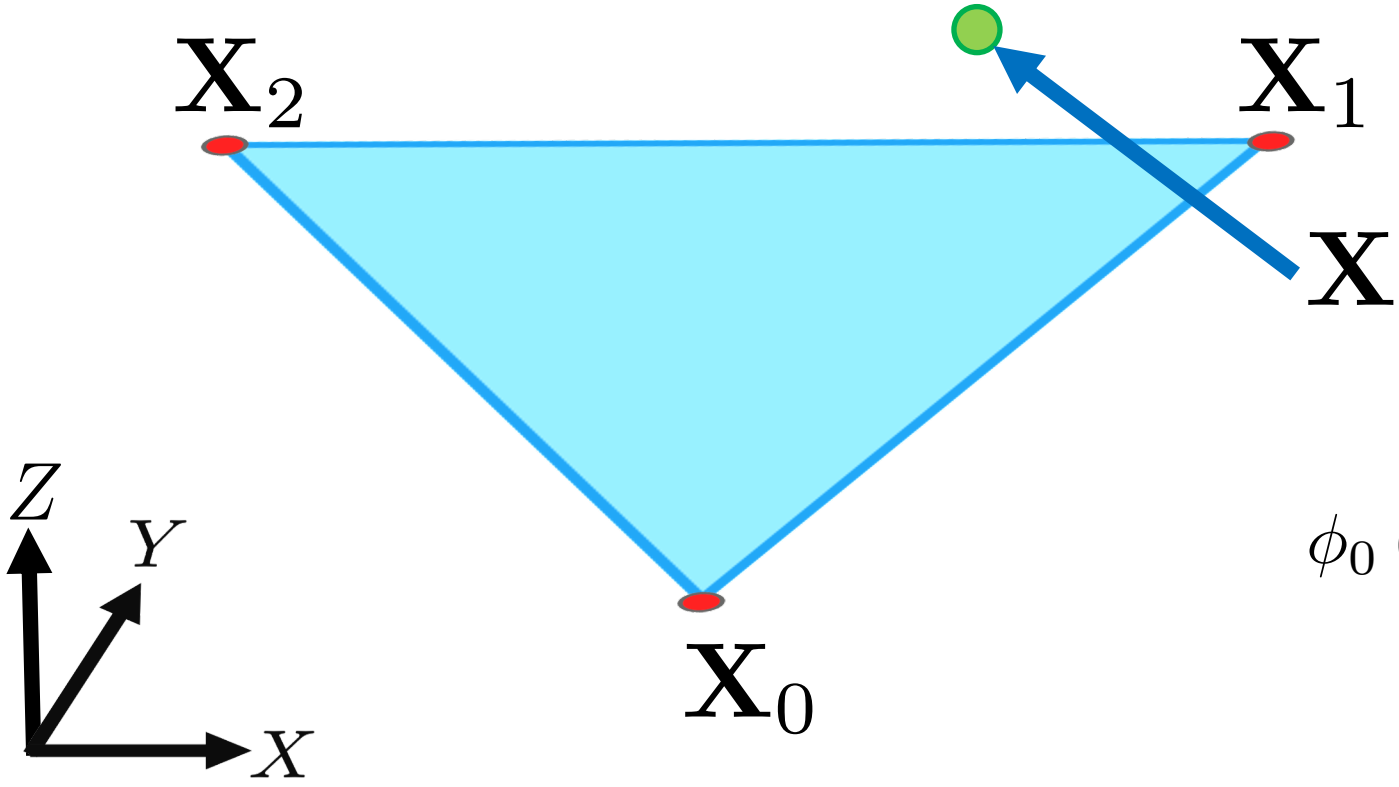


Reference Space

World Space

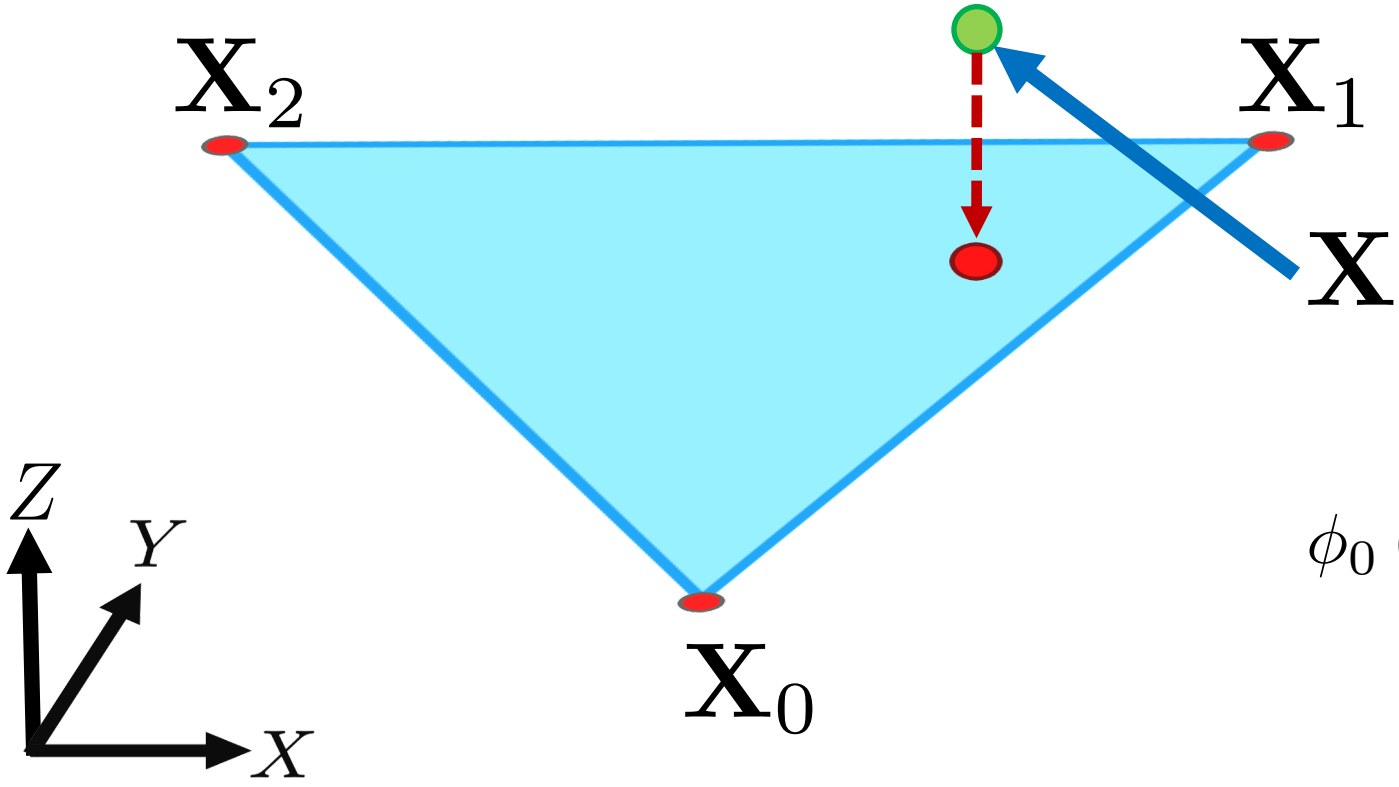
# Triangular Finite Elements




$$\mathbf{X} = \sum_{i=0}^{i=2} \mathbf{X}_i \phi_i \left(\mathbf{X}\right)$$

$$\phi_0 \left(\mathbf{X}\right) = 1 - \phi_1 \left(\mathbf{X}\right) - \phi_2 \left(\mathbf{X}\right)$$

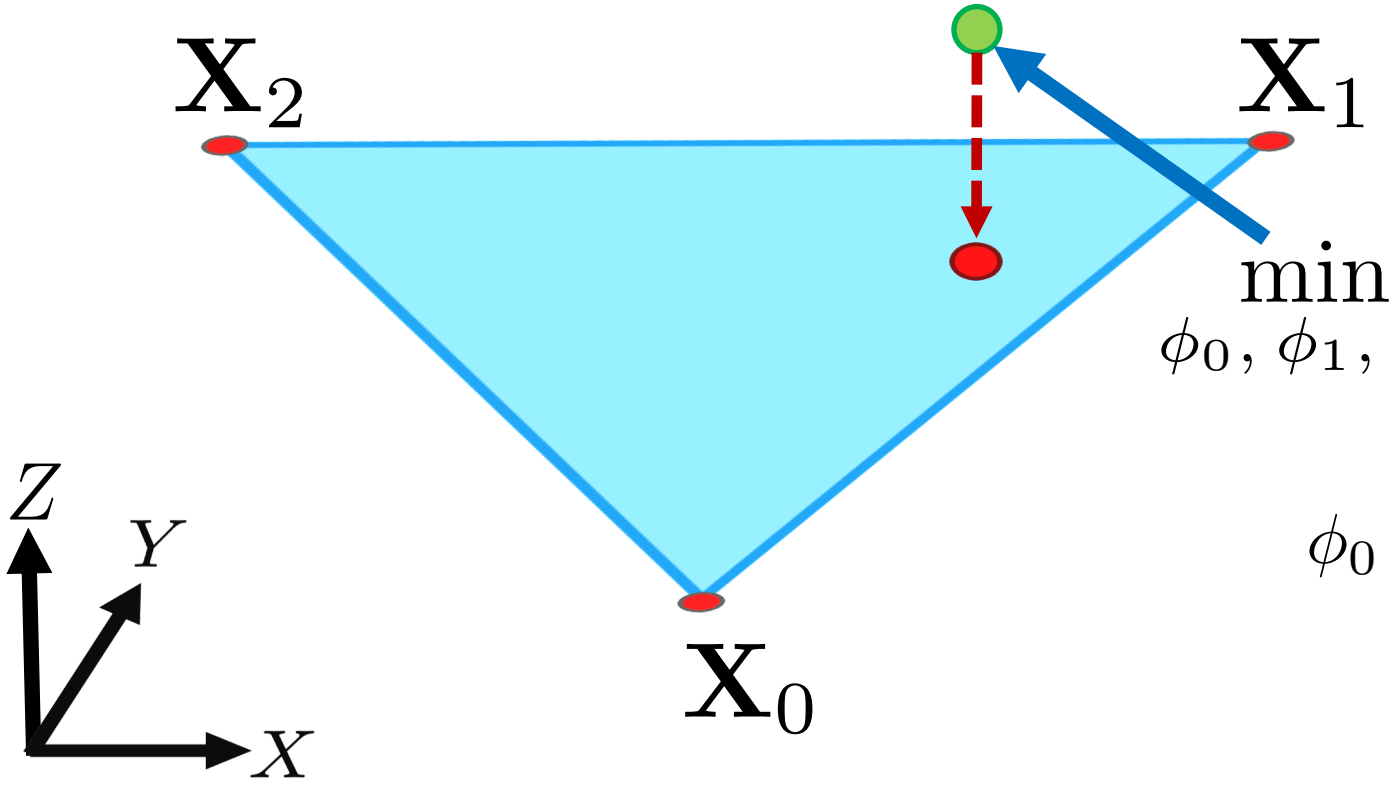
# Triangular Finite Elements




$$\mathbf{X} = \sum_{i=0}^{i=2} \mathbf{X}_i \phi_i \left(\mathbf{X}\right)$$

$$\phi_0 \left(\mathbf{X}\right) = 1 - \phi_1 \left(\mathbf{X}\right) - \phi_2 \left(\mathbf{X}\right)$$

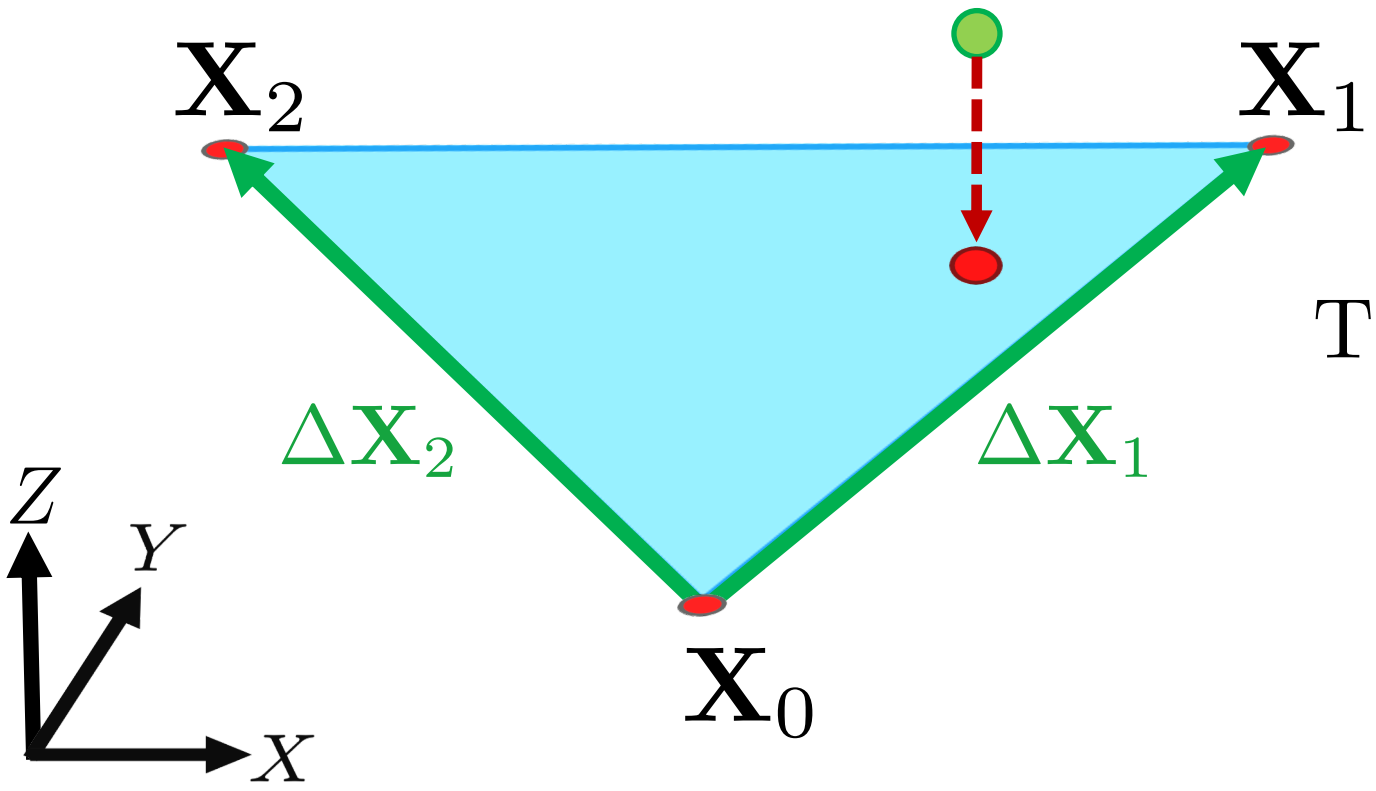
# Triangular Finite Elements




$$\min_{\phi_0,\,\phi_1,\,\phi_2} \left\| \sum_{i=0}^{i=2} \mathbf{X}_i \phi_i\left(\mathbf{X}\right) - \mathbf{X} \right\|_2^2$$

$$\phi_0\left(\mathbf{X}\right) = 1 - \phi_1\left(\mathbf{X}\right) - \phi_2\left(\mathbf{X}\right)$$

# Triangular Finite Elements

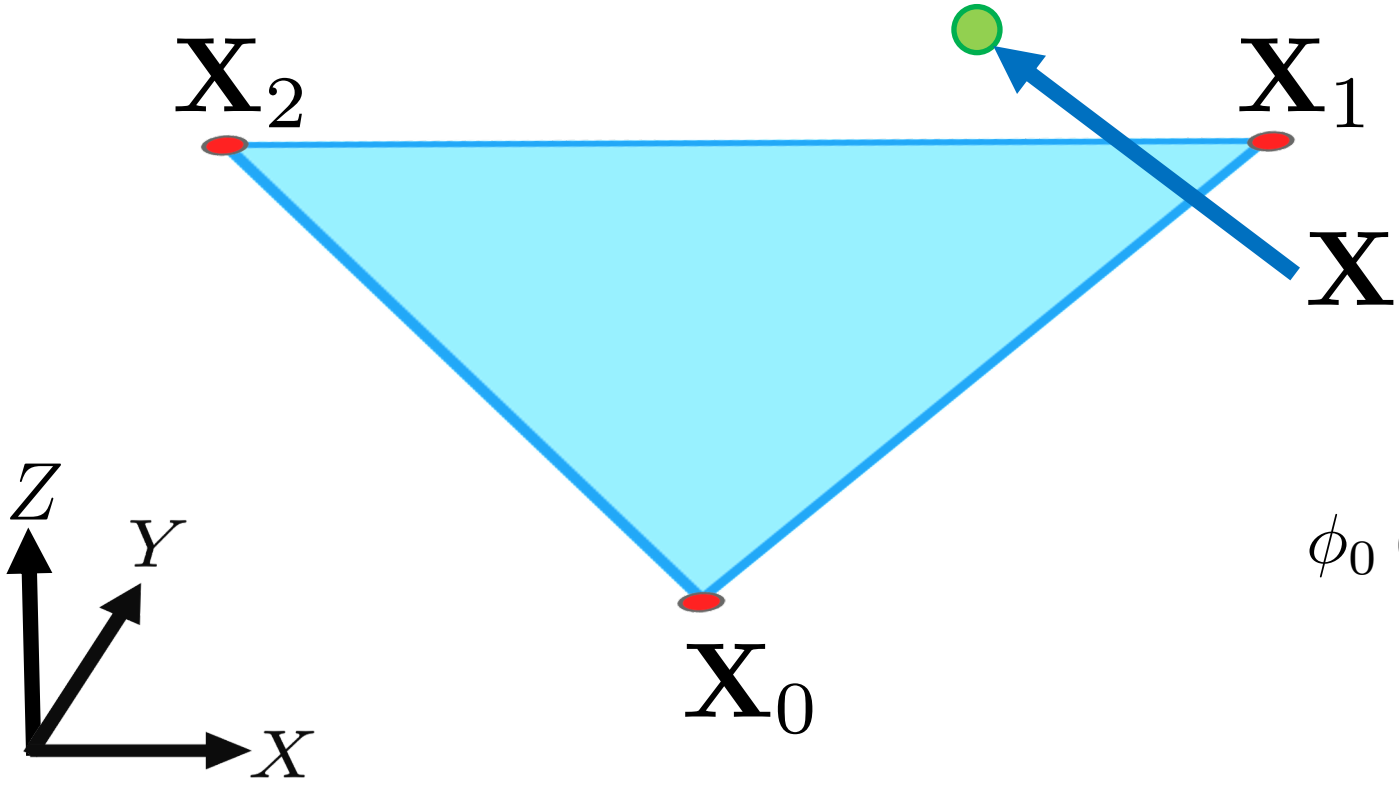



$$\mathrm{T} = \begin{pmatrix} \Delta\mathbf{X}_1 & \Delta\mathbf{X}_2 \end{pmatrix}$$

$$\mathrm{T}^T\mathrm{T}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \mathrm{T}^T\left(\mathbf{X} - \mathbf{X}_0\right)$$

$$\phi_0\left(\mathbf{X}\right) = 1 - \phi_1\left(\mathbf{X}\right) - \phi_2\left(\mathbf{X}\right)$$

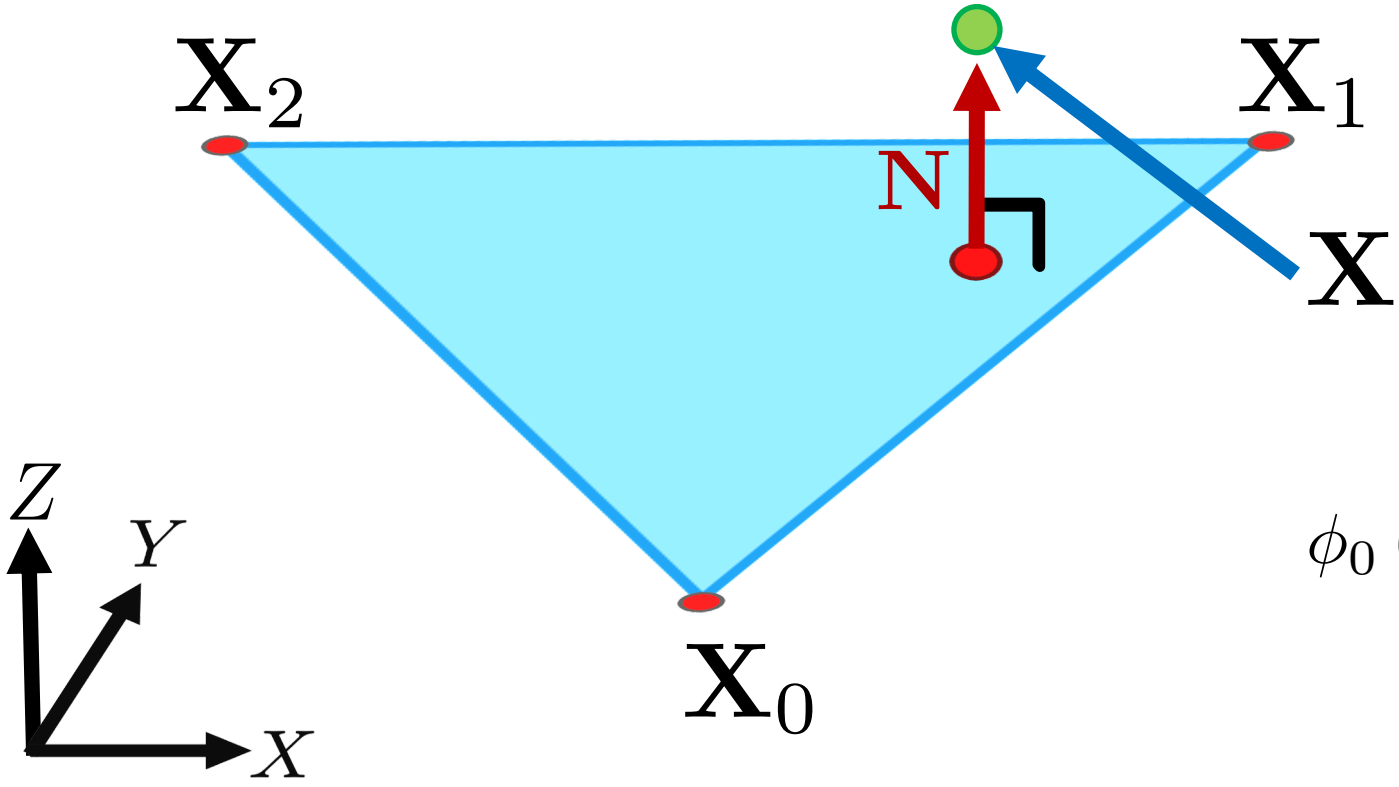
# Triangular Finite Elements




$$\mathbf{X} \neq \sum_{i=0}^{i=2} \mathbf{X}_i \phi_i \left(\mathbf{X}\right)$$

$$\phi_0 \left(\mathbf{X}\right) = 1 - \phi_1 \left(\mathbf{X}\right) - \phi_2 \left(\mathbf{X}\right)$$

# Triangular Finite Elements

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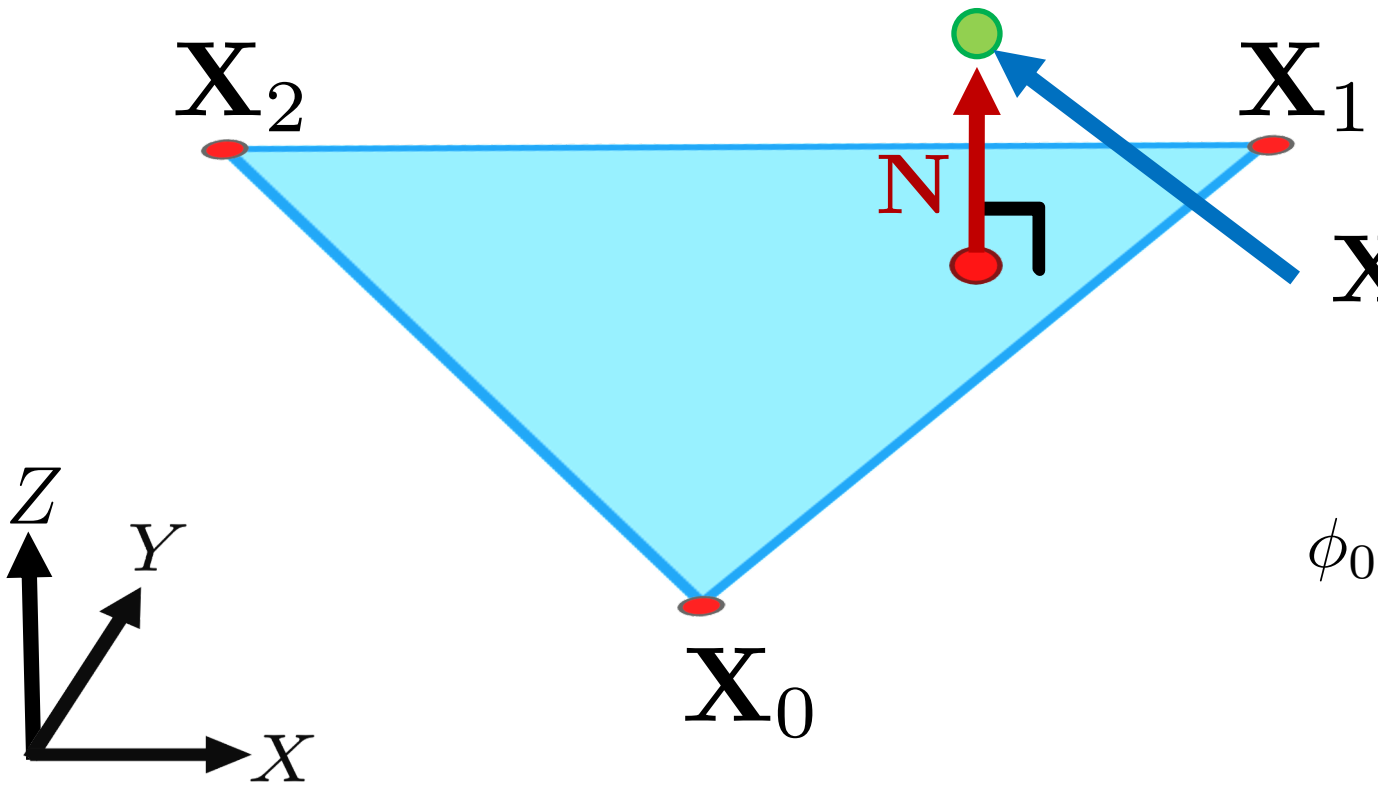
$$\phi_0 \left(\mathbf{X}\right) = 1 - \phi_1 \left(\mathbf{X}\right) - \phi_2 \left(\mathbf{X}\right)$$

# Triangular Finite Elements

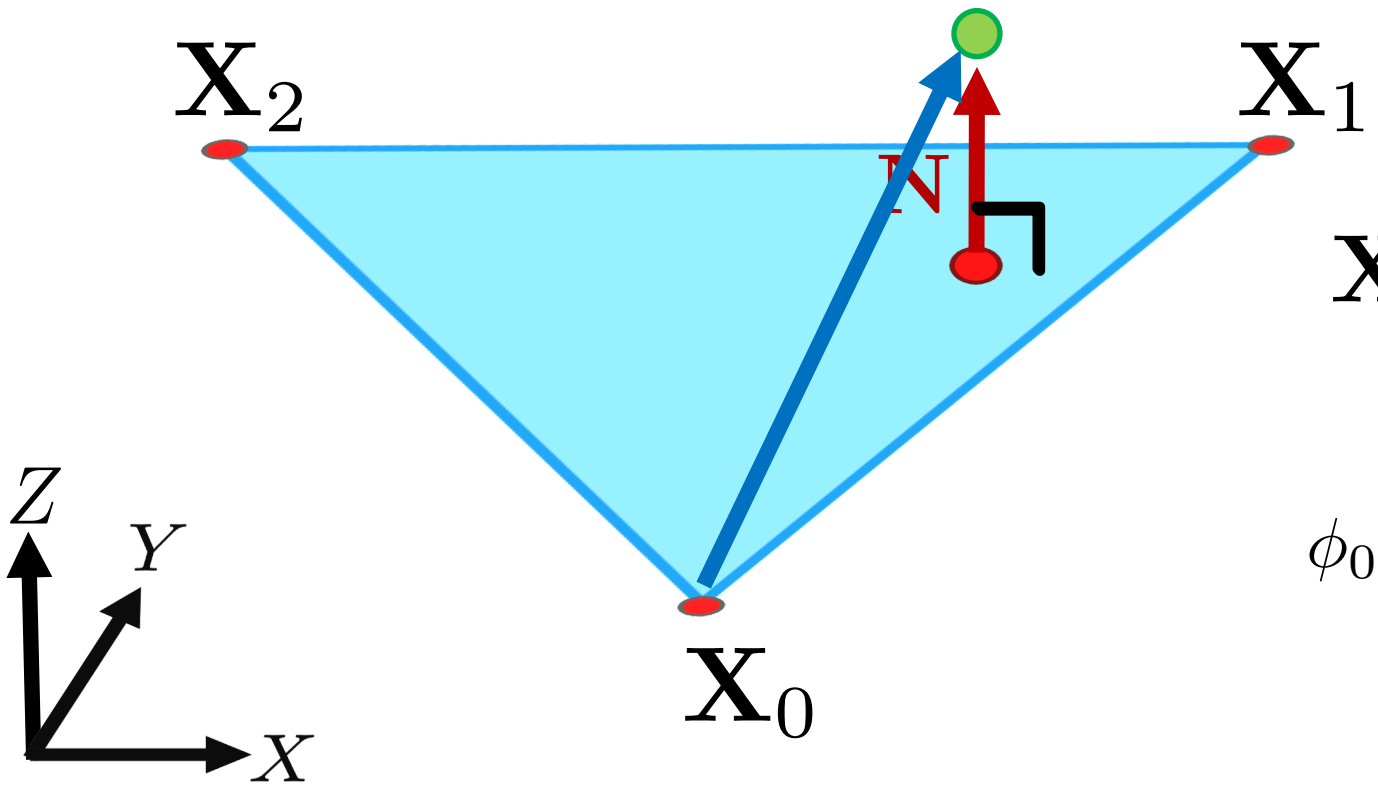



distance

$$\mathbf{X} = \sum_{i=0}^{i=2} \mathbf{X}_i \phi_i(\mathbf{X}) + \alpha \mathbf{N}$$

$$\phi_0(\mathbf{X}) = 1 - \phi_1(\mathbf{X}) - \phi_2(\mathbf{X})$$

# Triangular Finite Elements

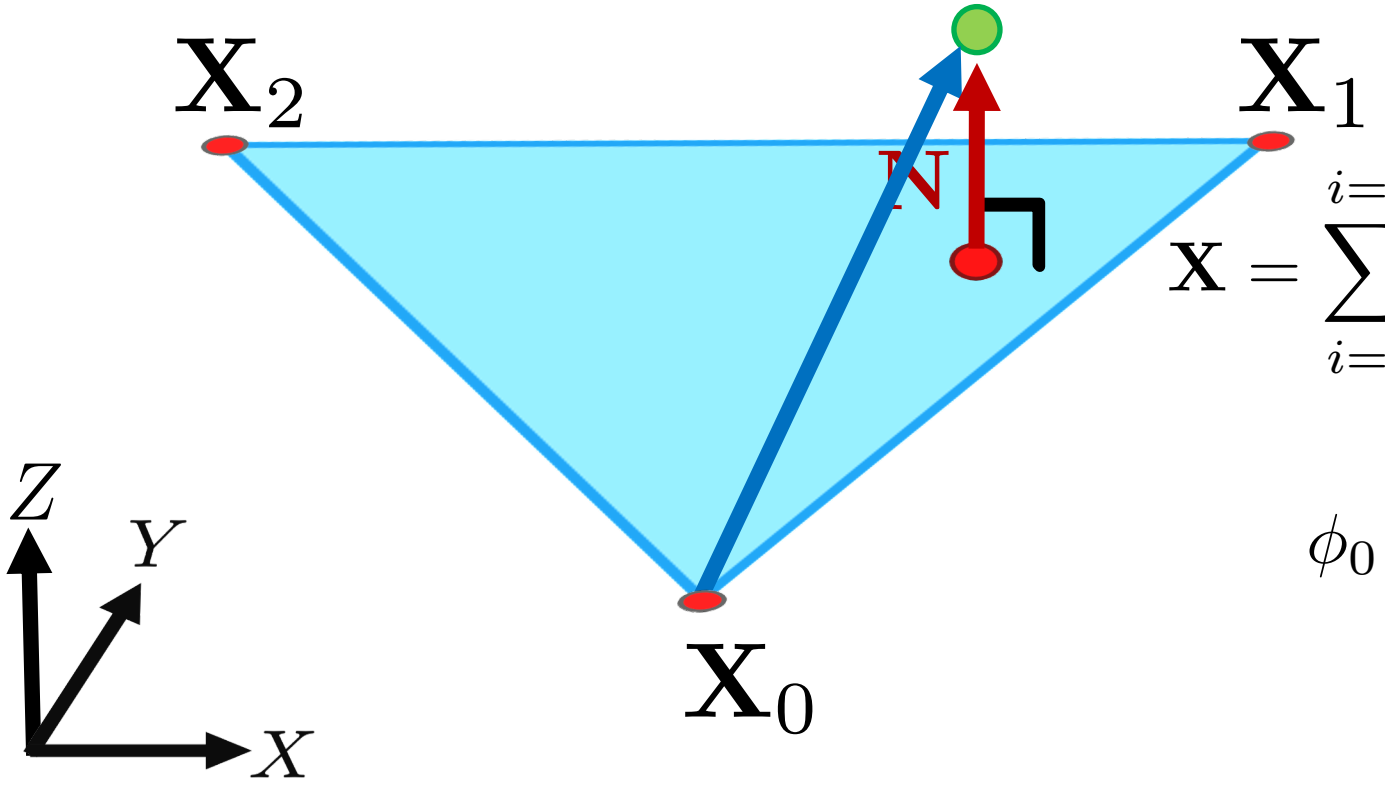



distance

$$\mathbf{X} = \sum_{i=0}^{i=2} \mathbf{X}_i \phi_i \left(\mathbf{X}\right) + \alpha \mathbf{N}$$

$$\phi_0 \left(\mathbf{X}\right) = 1 - \phi_1 \left(\mathbf{X}\right) - \phi_2 \left(\mathbf{X}\right)$$

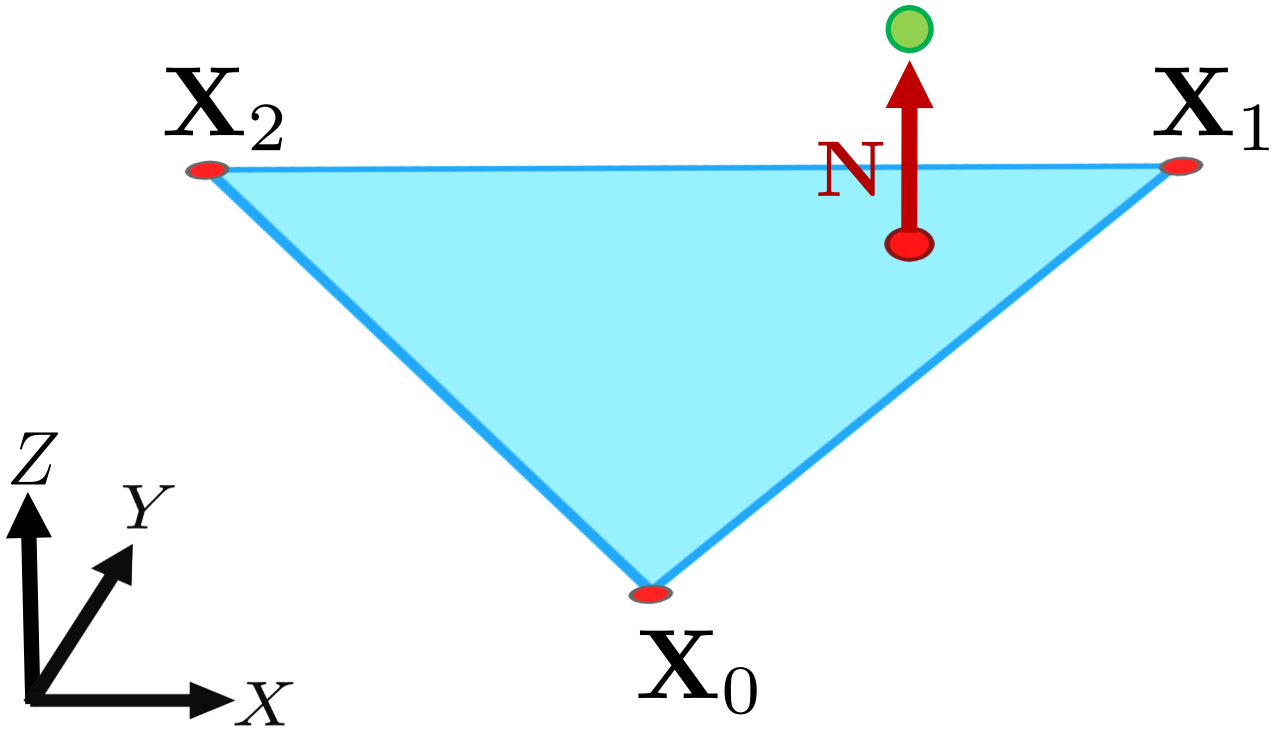
# Triangular Finite Elements



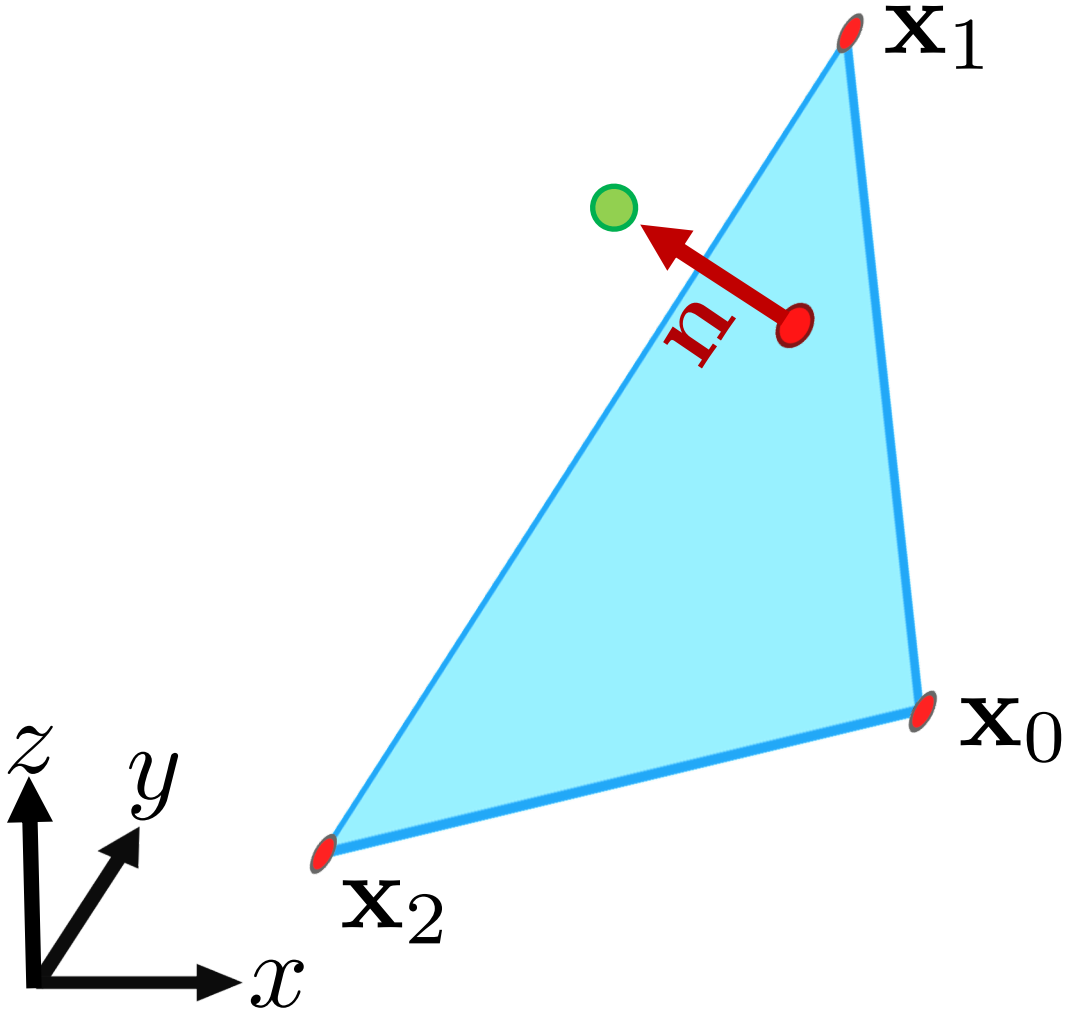

$$\mathbf{X} = \sum_{i=0}^{i=2} \mathbf{X}_i \phi_i(\mathbf{X}) + \overbrace{(\mathbf{X} - \mathbf{X}_0)^T \mathbf{N} \cdot \mathbf{N}}^{\alpha}$$

$$\phi_0(\mathbf{X}) = 1 - \phi_1(\mathbf{X}) - \phi_2(\mathbf{X})$$

# Triangular Finite Elements



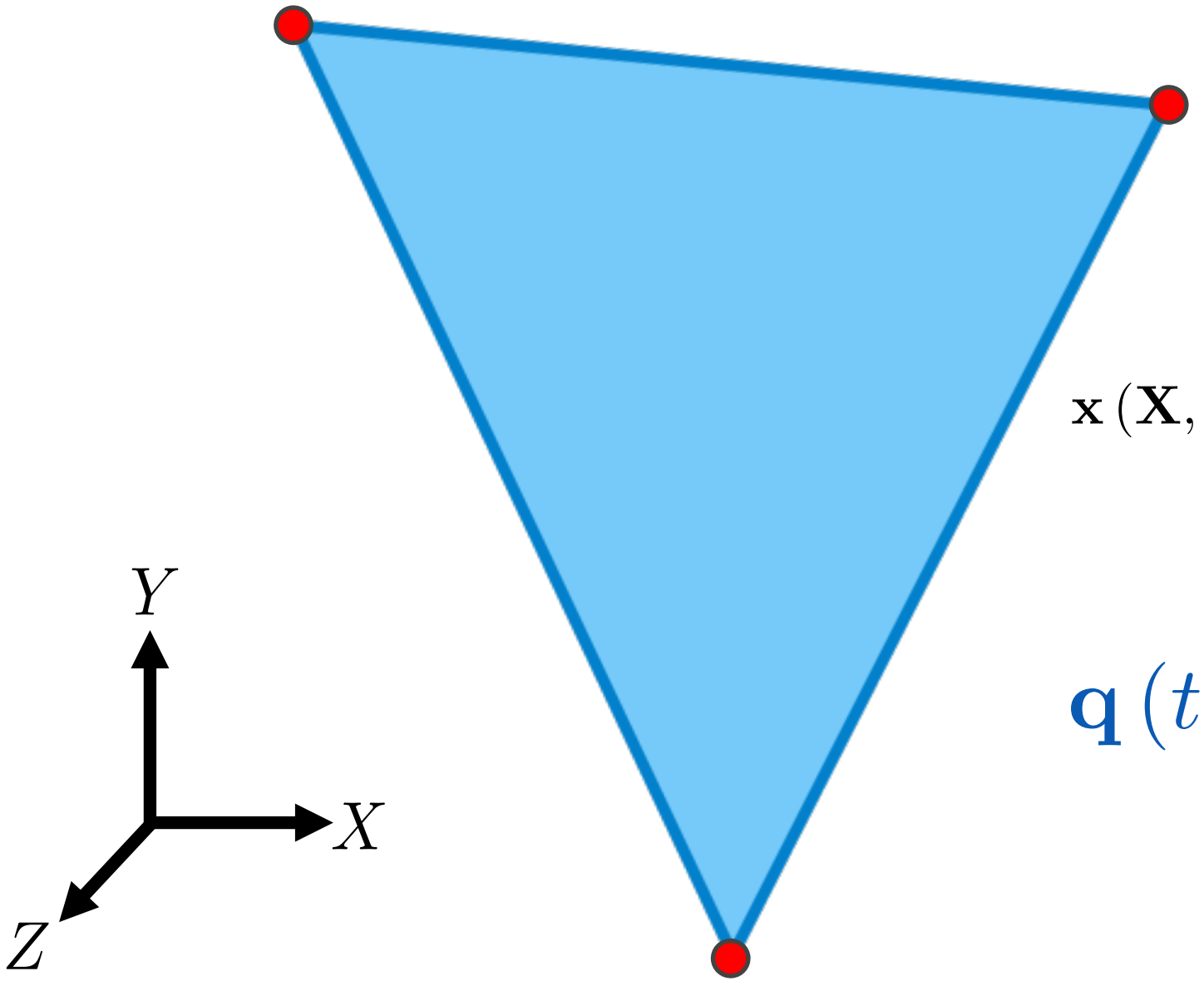

Reference Space




World Space

$$\mathbf{x}(\mathbf{X}, t) = \sum_{i=0}^{i=2} \mathbf{x}_i \phi_i(\mathbf{X}) + (\mathbf{X} - \mathbf{X}_0)^T \mathbf{N} \cdot \mathbf{n}$$

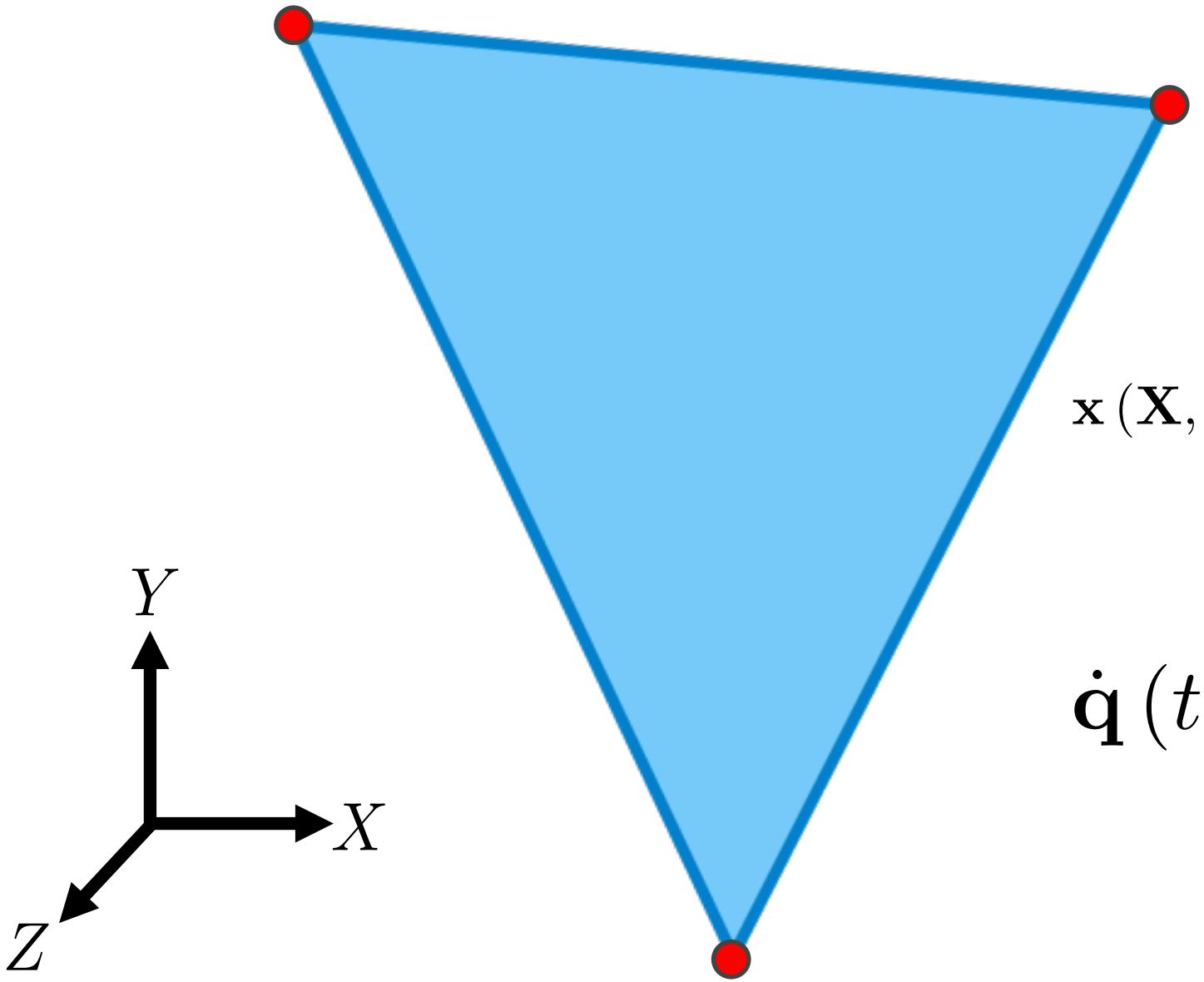
# Generalized Coordinates




$$\mathbf{x}(\mathbf{X},t) = \sum_{i=0}^{i=2} \mathbf{x}_i \phi_i(\mathbf{X}) + (\mathbf{X}-\mathbf{X}_0)^T \mathbf{N} \cdot \mathbf{n}(\mathbf{q}(t))$$

$$\mathbf{q}(t) = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$$

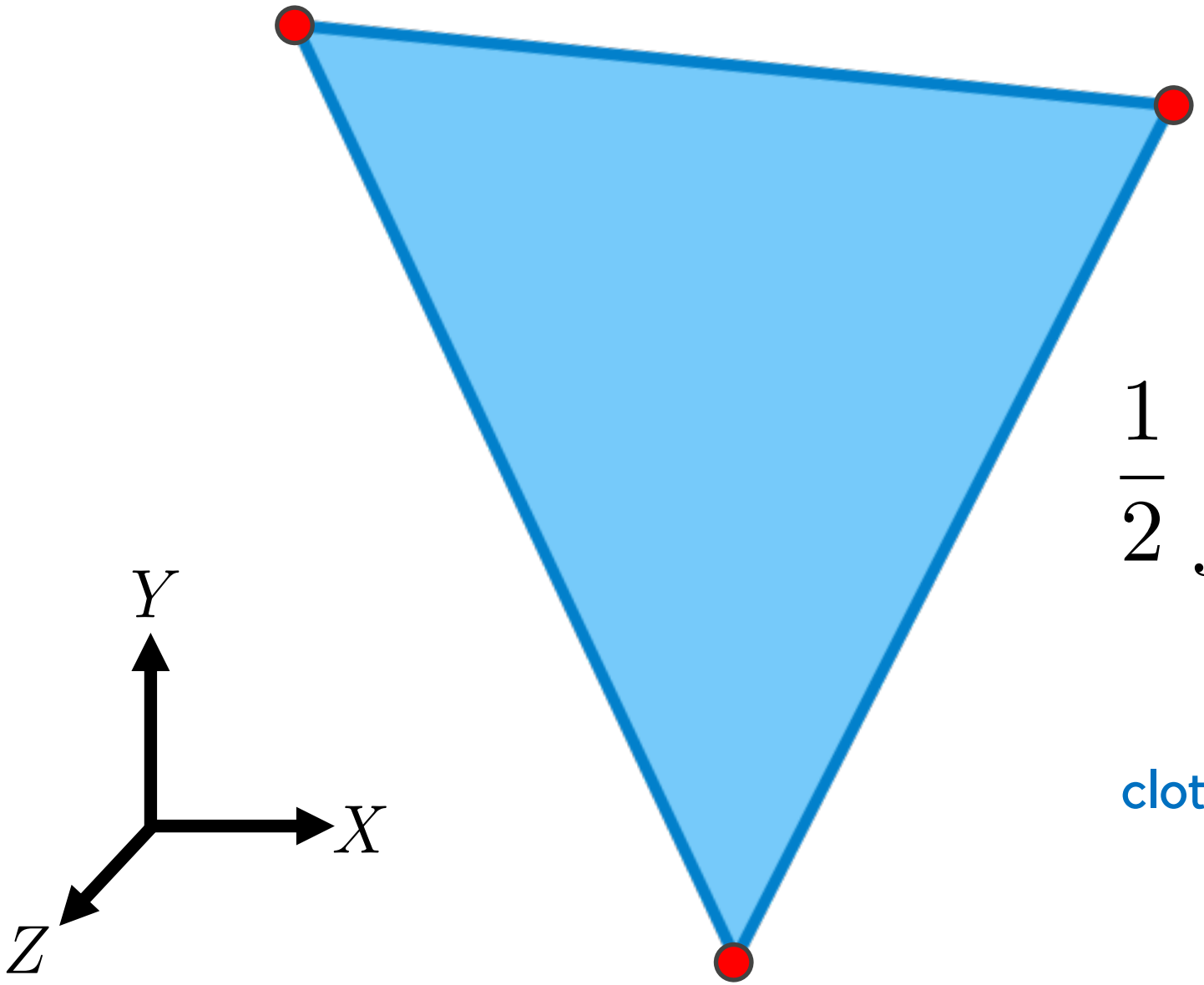
# Generalized Velocities




$$\mathbf{x}(\mathbf{X},t) = \sum_{i=0}^{i=2} \mathbf{x}_i \phi_i(\mathbf{X}) + (\mathbf{X}-\mathbf{X}_0)^T \mathbf{N} \cdot \overbrace{\mathbf{n}}^{\mathbf{n}(\mathbf{q}(t))}$$

$$\dot{\mathbf{q}}(t) = \begin{pmatrix} \dot{\mathbf{x}}_0 \\ \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$$

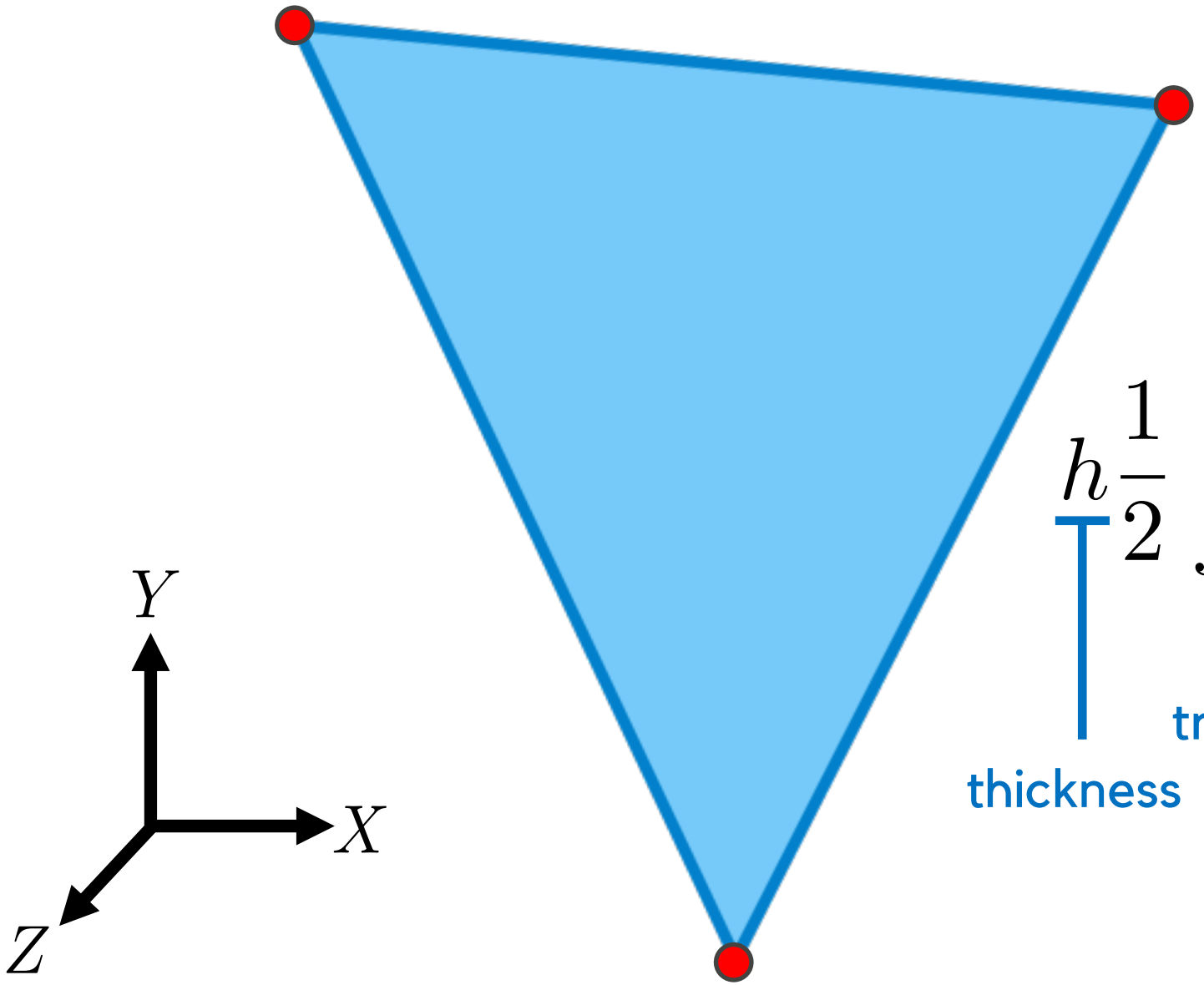
# Kinetic Energy of a Triangle




$$\frac{1}{2}\int_{\Omega} \rho \|\mathbf{v}\left(\mathbf{X}\right)\|_2^2 d\Omega$$

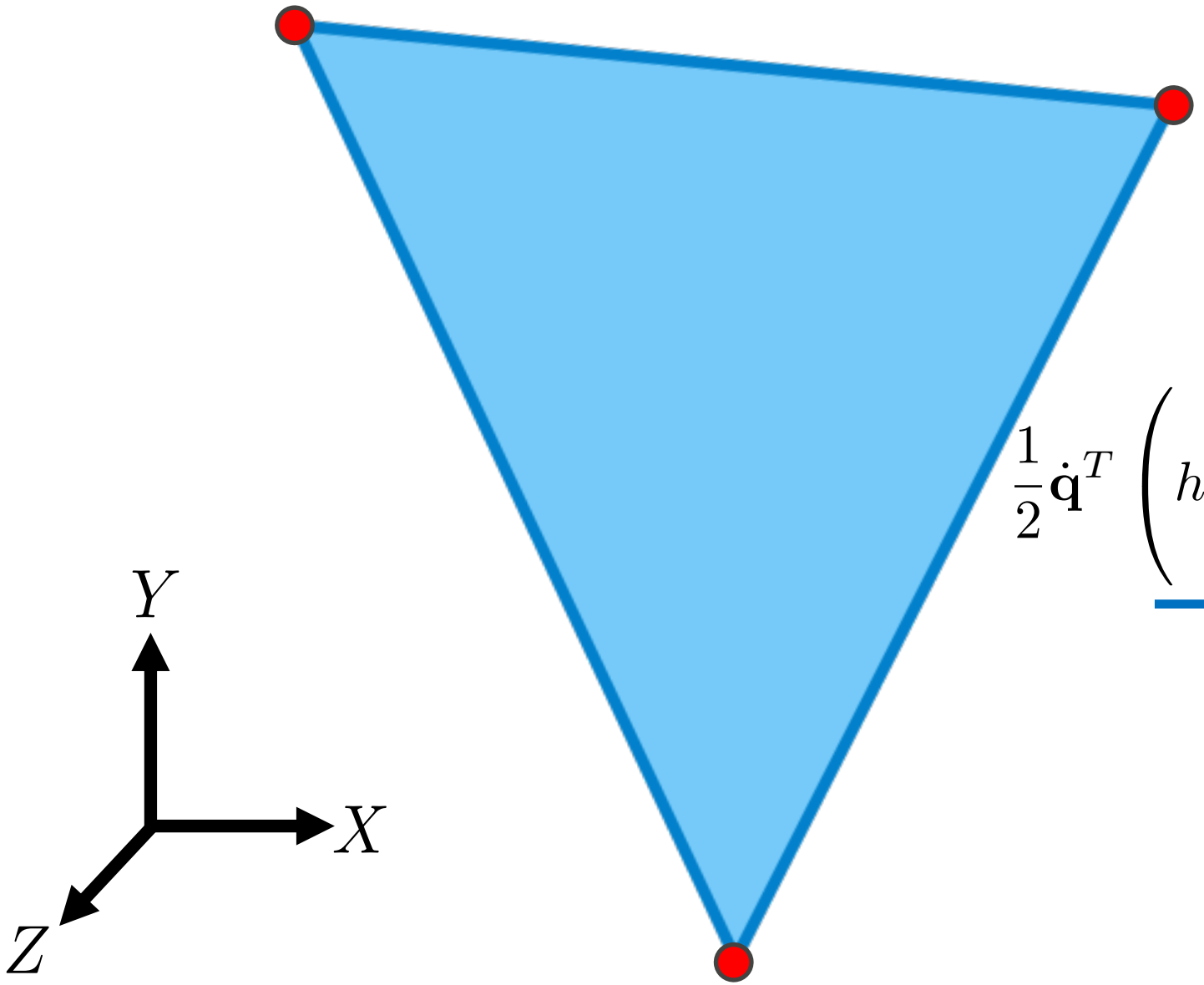
cloth domain

# Kinetic Energy of a Triangle



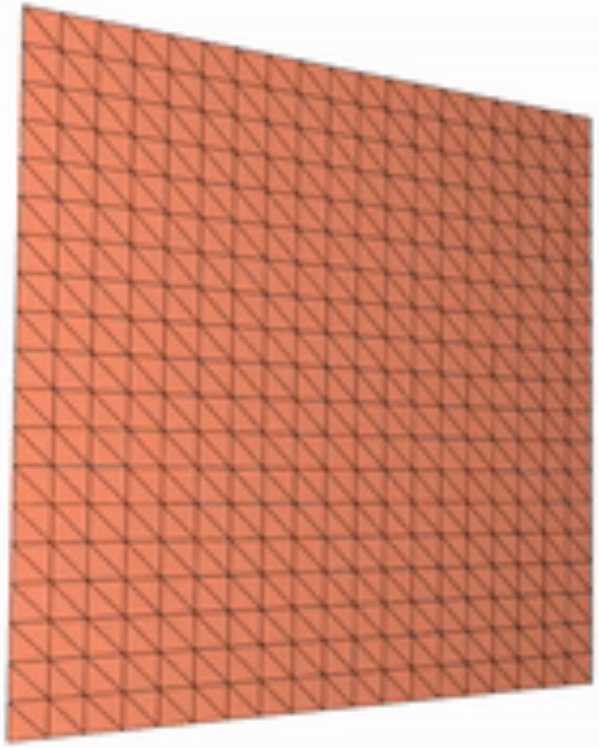

$$\underbrace{h}_{\text{thickness}}\frac{1}{2}\int_{\underbrace{\Gamma}_{\text{triangle}}}\rho\|\mathbf{v}\left(\mathbf{X}\right)\|_2^2 d\Gamma$$

# Kinetic Energy of a Triangle



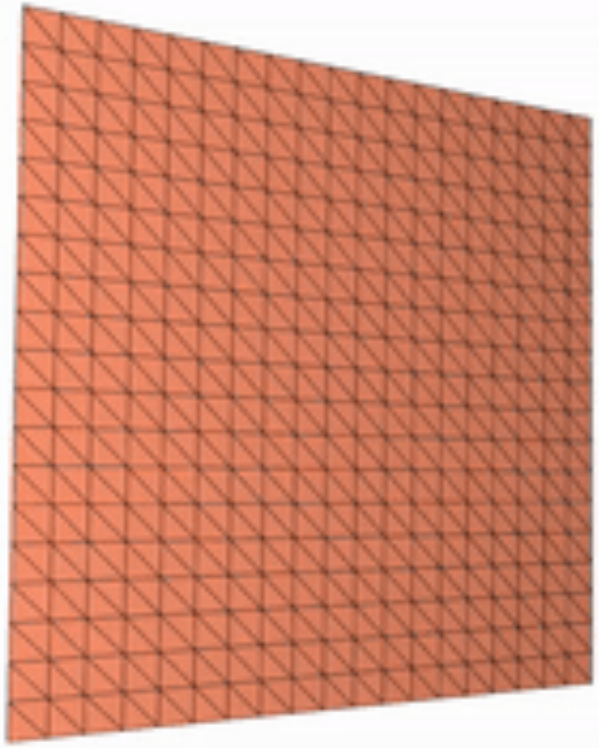

$$\frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\left( h \int_\Gamma \rho \begin{pmatrix} \phi_0\phi_0\mathrm{I} & \phi_0\phi_1\mathrm{I} & \phi_0\phi_2\mathrm{I} \\ \phi_1\phi_0\mathrm{I} & \phi_1\phi_1\mathrm{I} & \phi_1\phi_2\mathrm{I} \\ \phi_2\phi_0\mathrm{I} & \phi_2\phi_1\mathrm{I} & \phi_2\phi_2\mathrm{I} \end{pmatrix} d\Gamma \right)}_{\mathrm{M}_0 \in \mathbb{R}^{9\times 9}} \dot{\mathbf{q}}$$

# Generalized Coordinates for Cloth



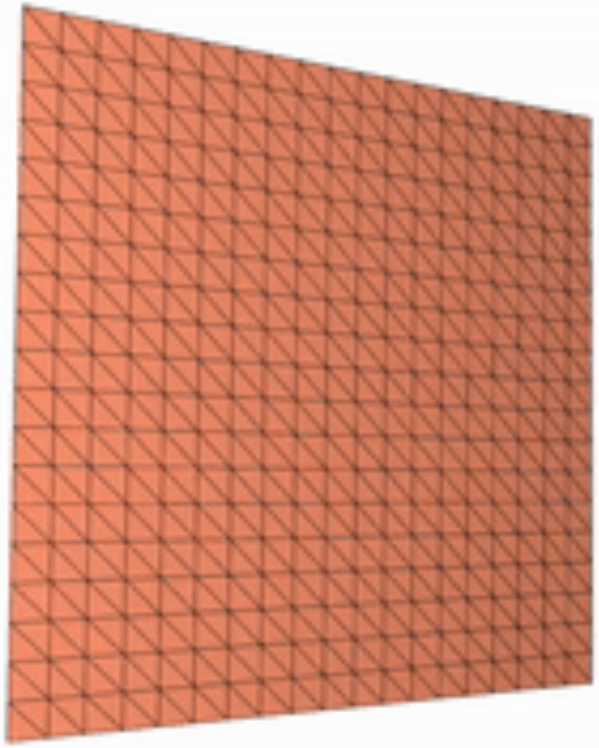
$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

# Generalized Velocities for Cloth



$$\dot{\mathbf{q}} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix}$$

# Kinetic Energy for Cloth



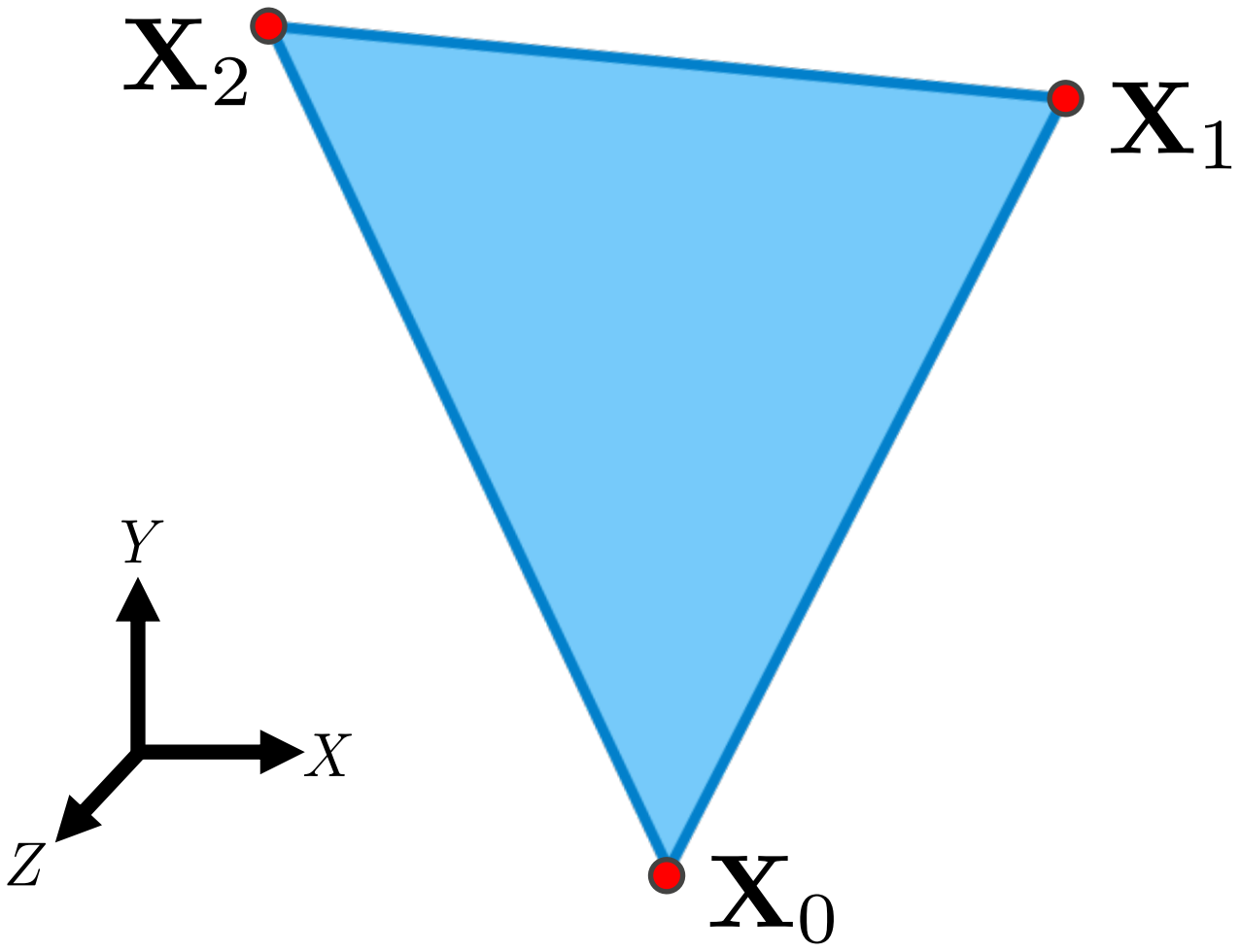
$$\mathrm{E}_j = \begin{pmatrix} \mathrm{S}_0 \\ \mathrm{S}_1 \\ \mathrm{S}_2 \end{pmatrix}$$

triangle selection

$$T = \frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\left( \sum_{j=0}^{j-1} \mathrm{E}_j^T \mathrm{M}_j \mathrm{E}_j \right)}_{\mathrm{M}} \dot{\mathbf{q}}$$

Assemble $\mathrm{M}$ by summing over all triangles

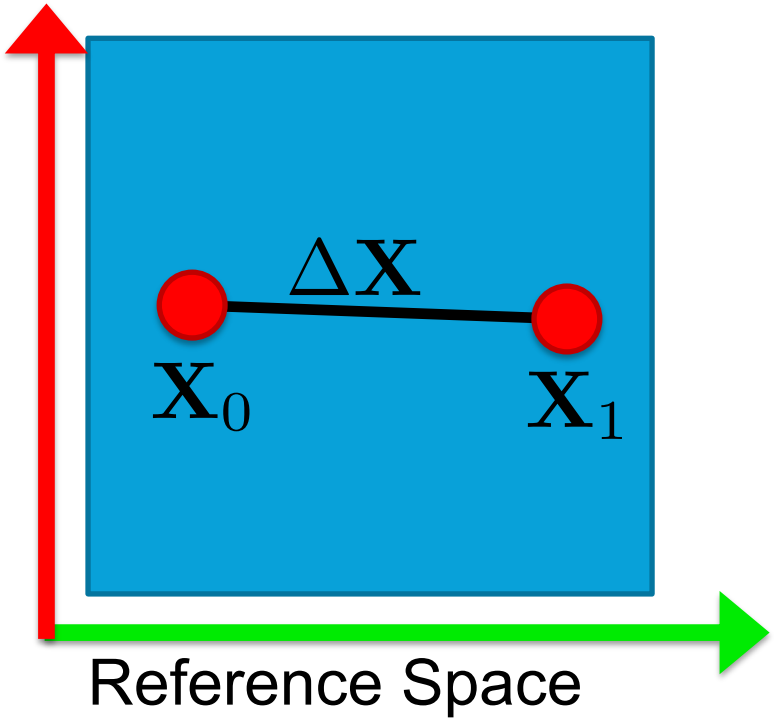
# Potential Energy for a Triangle

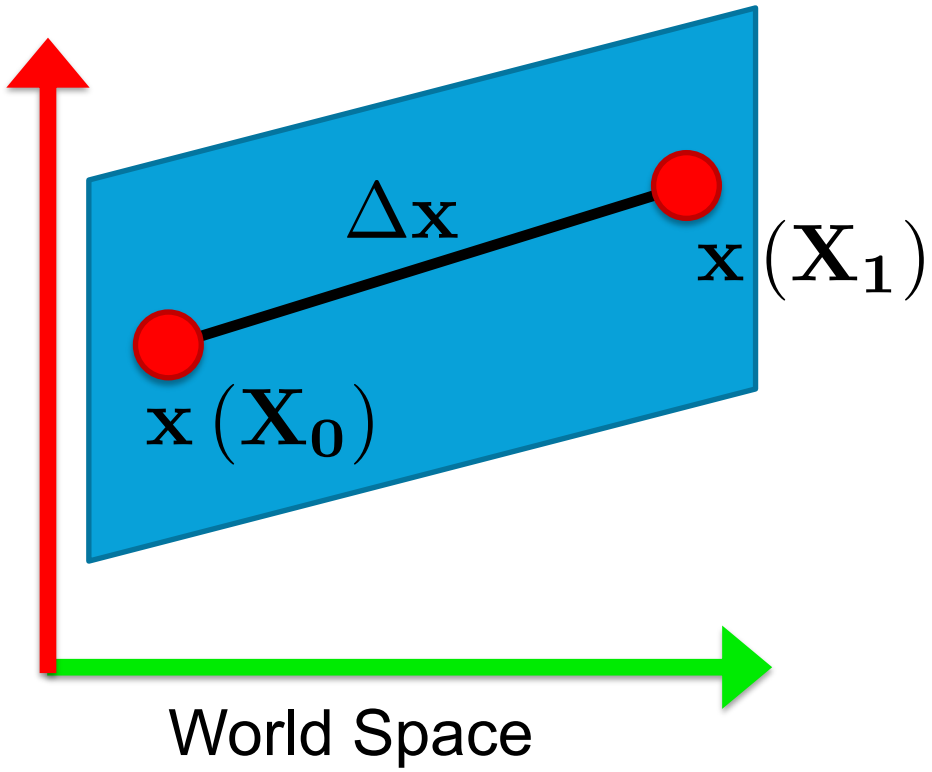



$$V = ?$$

Reference (Undeformed) Space

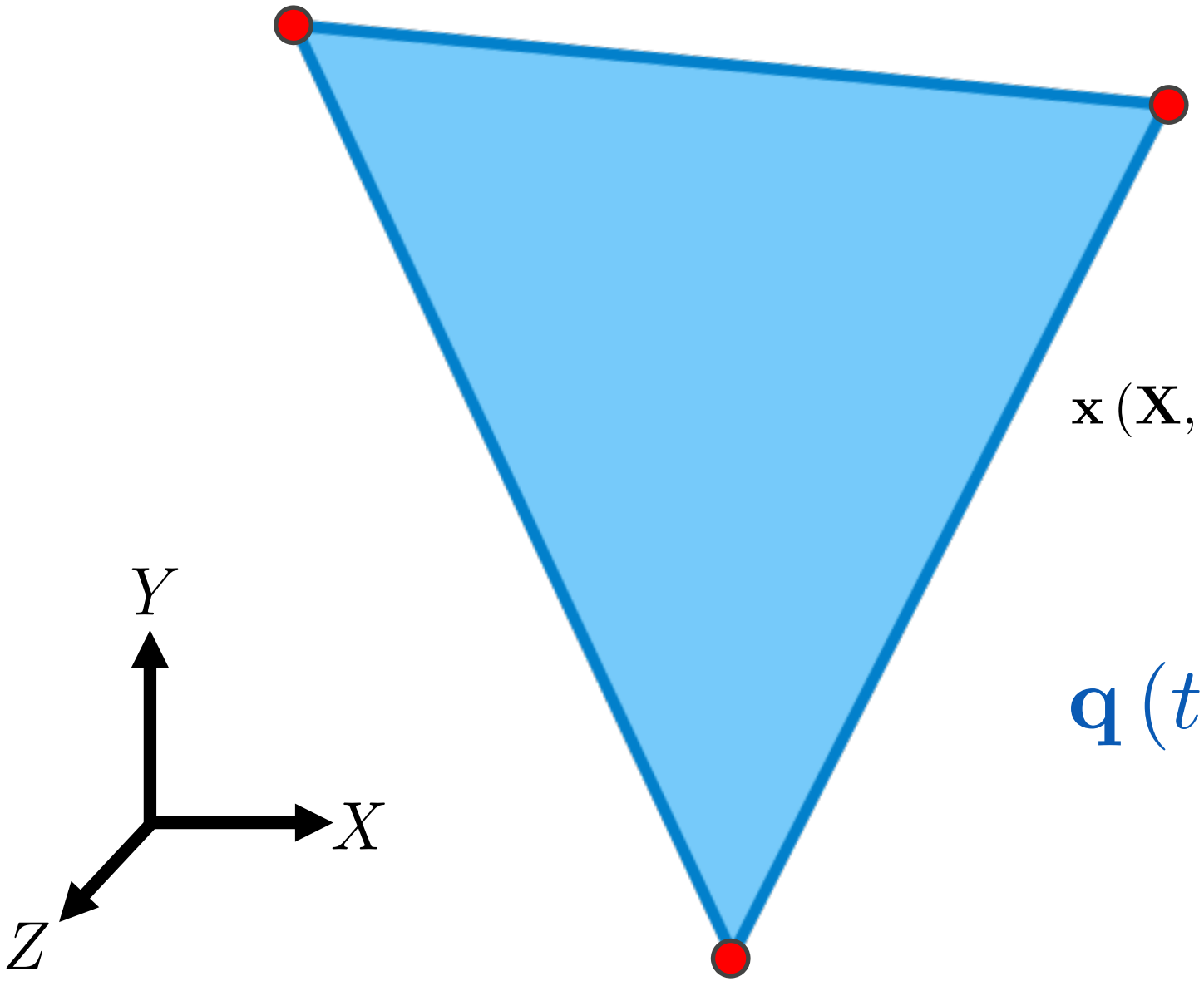
# A Closer Look at Deformation




$$\Delta\mathbf{x} \approx \mathbf{x}(\mathbf{X}_0) + \underbrace{\frac{\partial\mathbf{x}}{\partial\mathbf{X}}}_{\text{deformation gradient } \mathbf{F}}\Delta\mathbf{X} - \mathbf{x}(\mathbf{X}_0)$$

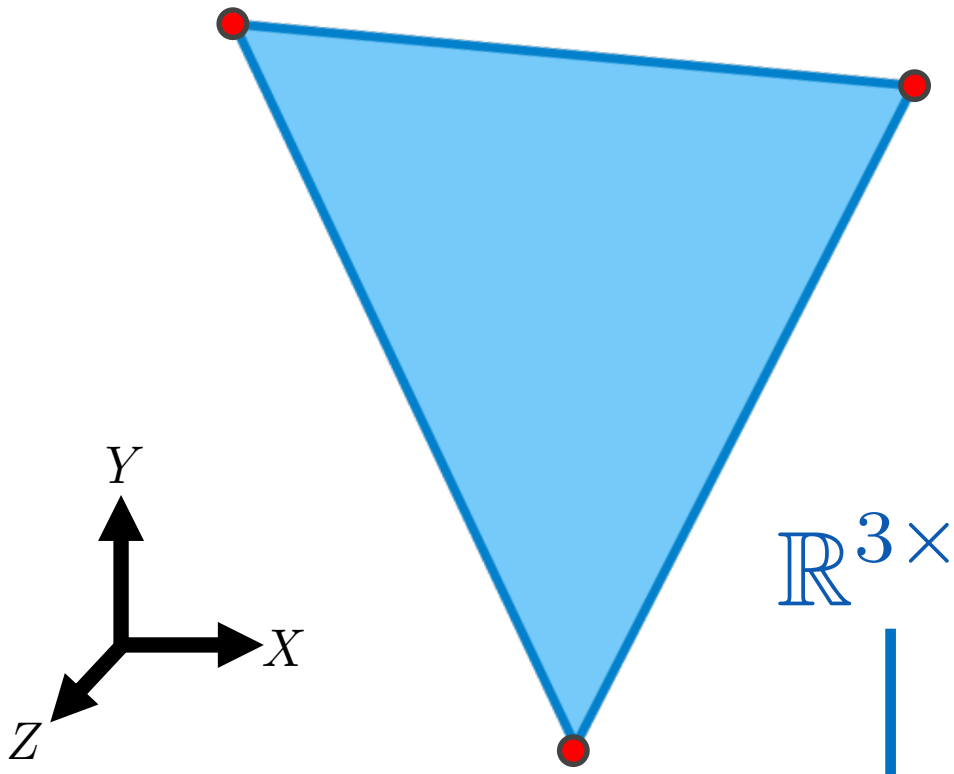
# Deformation Gradient of a Triangle




$$\mathbf{x}(\mathbf{X},t) = \sum_{i=0}^{i=2} \mathbf{x}_i \phi_i(\mathbf{X}) + (\mathbf{X}-\mathbf{X}_0)^T \mathbf{N} \cdot \mathbf{n}(\mathbf{q}(t))$$

$$\mathbf{q}(t) = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$$

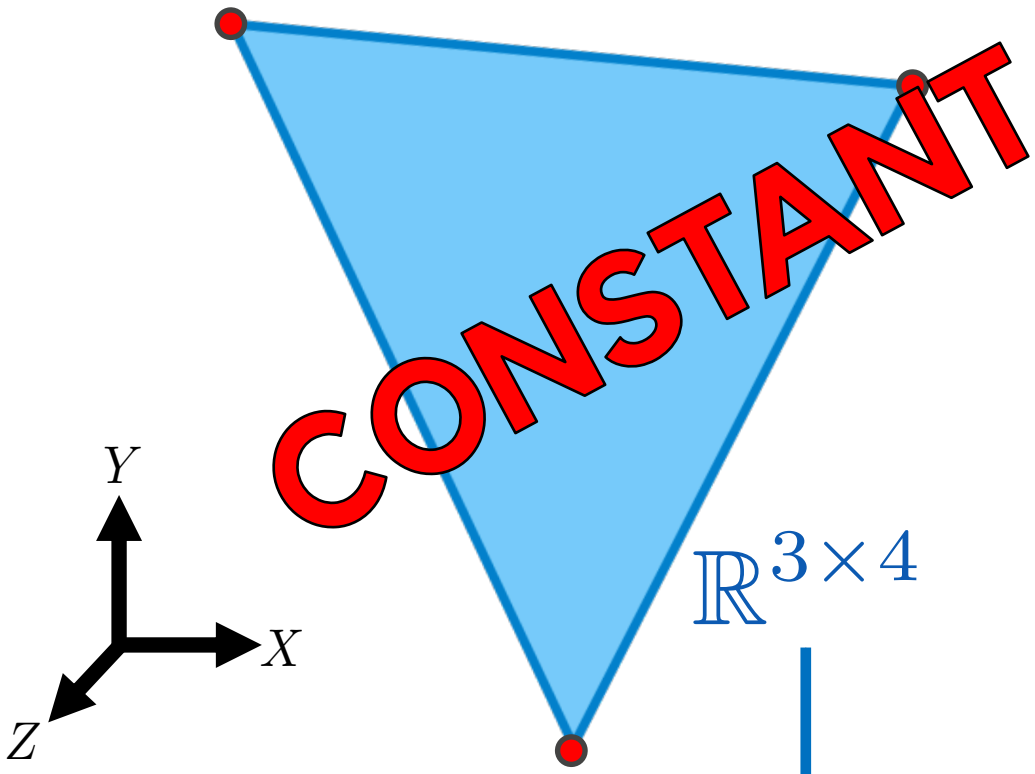
# Deformation Gradient of a Triangle




$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_0 + \underbrace{\begin{pmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{n} \end{pmatrix}}_{\mathbb{R}^{3\times 4}} \underbrace{\begin{pmatrix} -\mathbf{1}^T \left(\mathrm{T}^T\mathrm{T}\right)^{-1} \mathrm{T}^T \\ \left(\mathrm{T}^T\mathrm{T}\right)^{-1} \mathrm{T}^T \\ \mathbf{N}^T \end{pmatrix}}_{\mathbb{R}^{4\times 3}} (\mathbf{X} - \mathbf{X}_0)$$

$$\mathbf{1}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

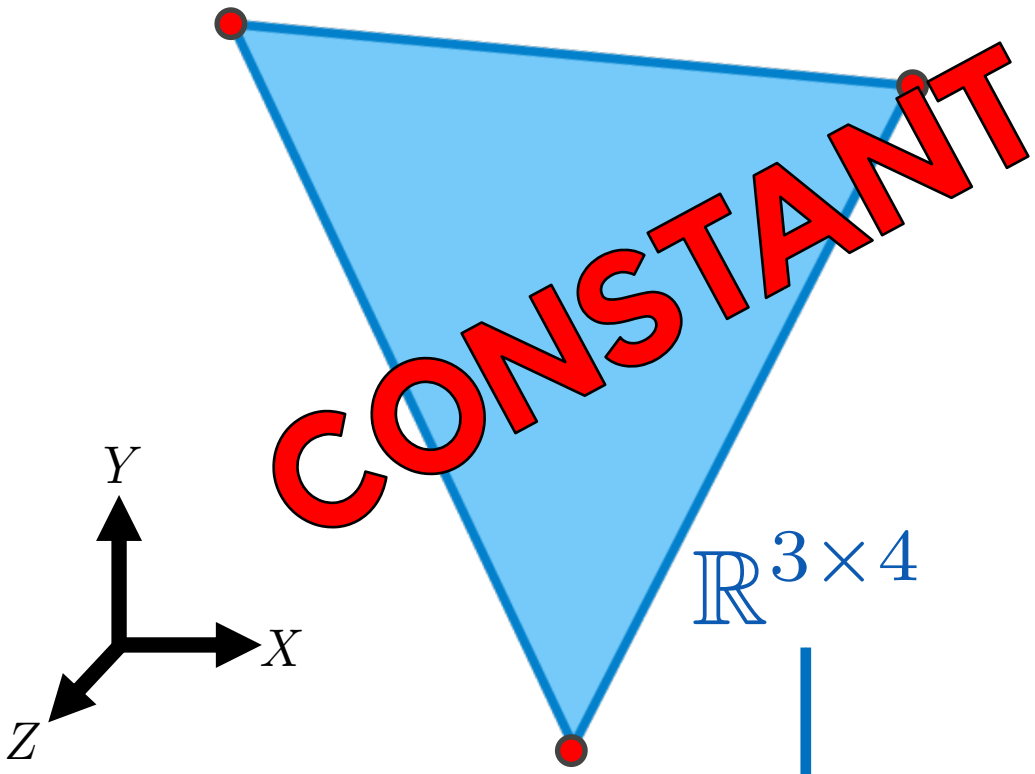
# Deformation Gradient of a Triangle




$$\mathrm{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \underbrace{\begin{pmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{n} \end{pmatrix}}_{\mathbb{R}^{3\times 4}} \underbrace{\begin{pmatrix} -\mathbf{1}^T \left(\mathrm{T}^T\mathrm{T}\right)^{-1} \mathrm{T}^T \\ \left(\mathrm{T}^T\mathrm{T}\right)^{-1} \mathrm{T}^T \\ \mathbf{N}^T \end{pmatrix}}_{\mathbb{R}^{4\times 3}}$$

$$\mathbf{1}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

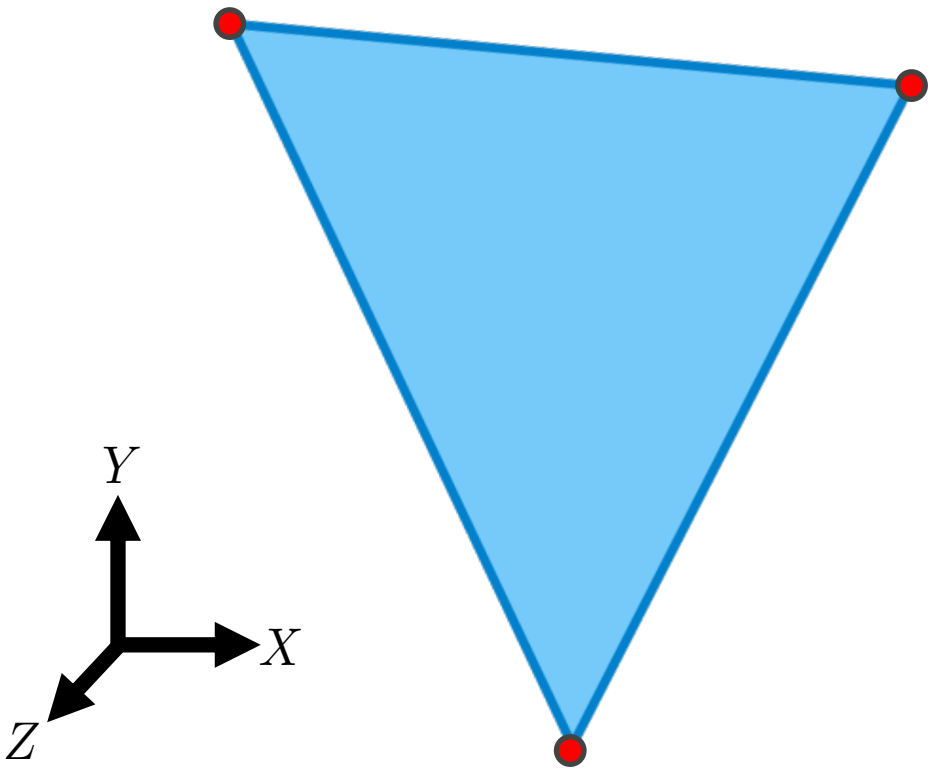
# Deformation Gradient of a Triangle




$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \underbrace{\begin{pmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{n} \end{pmatrix}}_{\mathbb{R}^{3\times 4}} \underbrace{\begin{pmatrix} \frac{\partial \phi}{\partial \mathbf{X}} \\ \mathbf{N}^T \end{pmatrix}}_{\mathbb{R}^{3\times 3}}$$

$$\mathbf{1}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

# From Deformation to Potential Energy

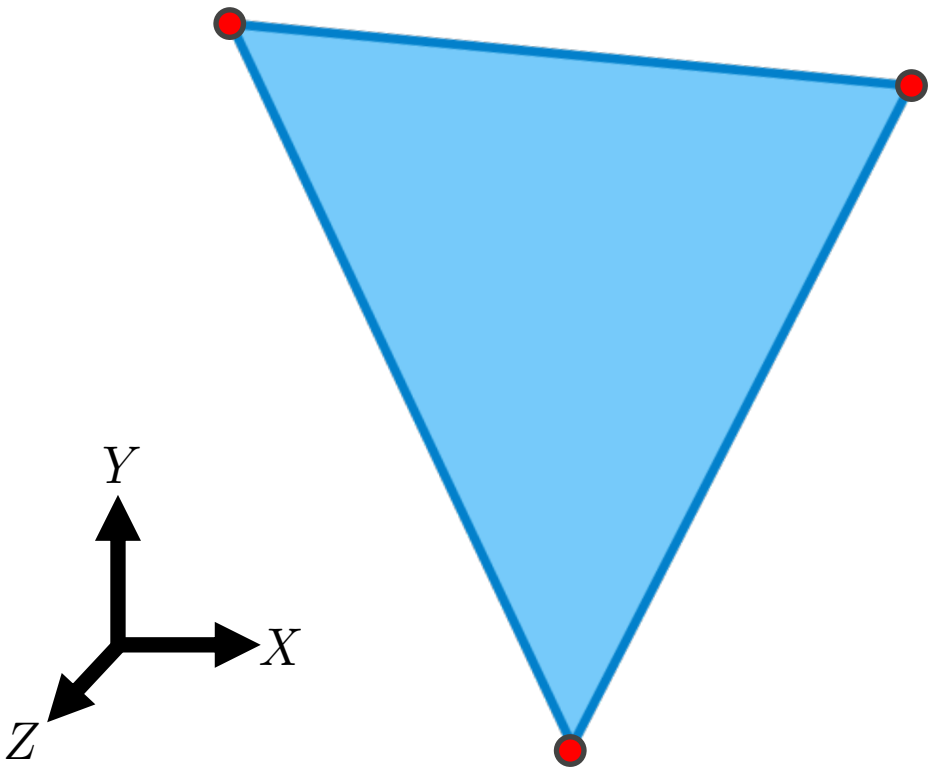



Reference (Undeformed) Space

$$\int_{\Omega} \psi\left(\mathrm{F}\left(\mathbf{X}\right)\right) d\Omega$$

cloth domain

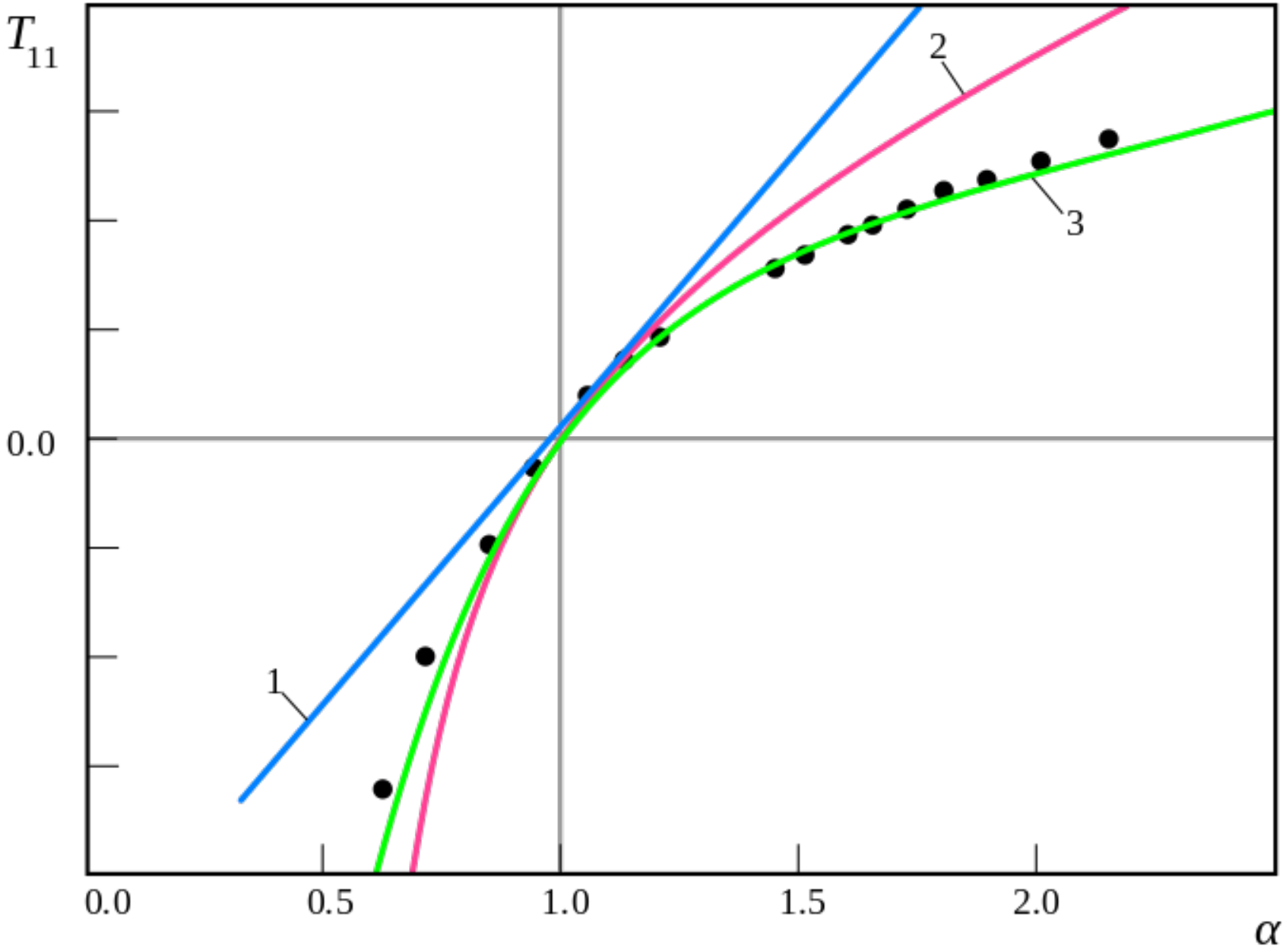
# From Deformation to Potential Energy




Reference (Undeformed) Space

$$\underbrace{h}_{\text{thickness}} \cdot \int_{\underbrace{\Gamma}_{\text{triangle}}} \psi\left(\mathrm{F}\left(\mathbf{X}\right)\right) d\Gamma$$

# Neohookean Strain Energy Density



# A Closer Look at Deformation

**Strain** $\Delta \mathbf{x}^T \Delta \mathbf{x} - \Delta \mathbf{X}^T \Delta \mathbf{X}$

$$\Delta \mathbf{X}^T \mathrm{F}^T \mathrm{F} \Delta \mathbf{X} - \Delta \mathbf{X}^T \Delta \mathbf{X}$$

Right Cauchy Green Deformation

$$\Delta \mathbf{X}^T \left( \mathrm{F}^T \mathrm{F} - \mathrm{I} \right) \Delta \mathbf{X}$$

Green Lagrange Strain

# A Closer Look at Deformation

Right Cauchy Green Deformation

$$\Delta \mathbf{X}^T \left(\mathrm{F}^T \mathrm{F} - \mathrm{I}\right) \Delta \mathbf{X}$$

Eigendecomposition

$$\Delta \mathbf{X}^T \left(\mathrm{V}\mathrm{E}\mathrm{V}^T - \mathrm{I}\right) \Delta \mathbf{X}$$

# A Closer Look at Deformation

Right Cauchy Green Deformation

$$\Delta\mathbf{X}^T \left(\mathrm{F}^T\mathrm{F} - \mathrm{I}\right) \Delta\mathbf{X}$$

$$(\mathrm{V}\Delta\mathbf{X})^T \left( \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} - \mathrm{I} \right) (\mathrm{V}\Delta\mathbf{X})$$

rotated

# A Closer Look at Deformation

Right Cauchy Green Deformation

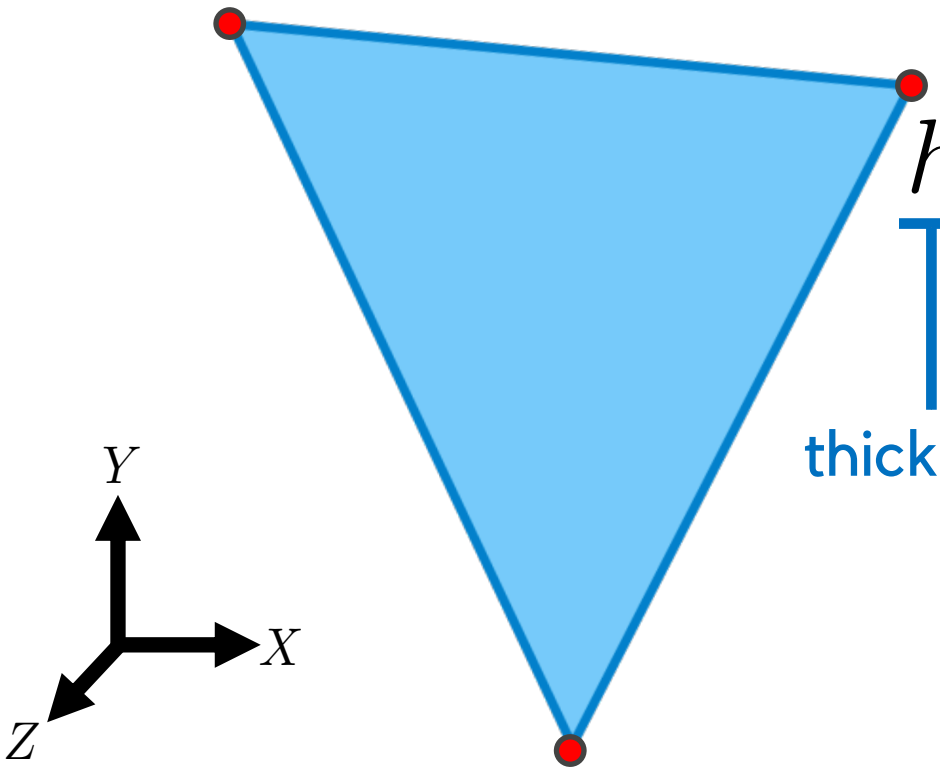
$$\Delta\mathbf{X}^T \left(\mathrm{F}^T\mathrm{F} - \mathrm{I}\right) \Delta\mathbf{X}$$

$$(\mathrm{V}\Delta\mathbf{X})^T \left( \begin{pmatrix} \sigma_0^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{pmatrix} - \mathrm{I} \right) (\mathrm{V}\Delta\mathbf{X})$$

rotated

squared singular values of F

# Potential Energy via Principal Stretches




$$\underbrace{h}_{\text{thickness}} \cdot \underbrace{\int_{\Gamma}}_{\text{triangle}} \psi\left(\underbrace{\sigma_0\left(\mathrm{F}\right), \sigma_1\left(\mathrm{F}\right), \sigma_2\left(\mathrm{F}\right)}_{\text{principal stretches}}\right) d\Gamma$$

Reference (Undeformed) Space

# Strain Energy Density for Cloth

1. Cloth should go back to undeformed shape when all external forces are removed

2. Rigid motion (translation and/or rotation) should not change elastic energy

3. Energy should depend only on vertex positions

**Principal stretches filter out rotations and translations**

**Principal stretches depend on F**

# A Quadratic Energy Model in the Principal Stretches

Lamé Parameters

$$\mu \sum_{i=0}^{2} (\sigma_i - 1)^2 + \frac{\lambda}{2} \left( \sum_{i=0}^{2} \sigma_i - 3 \right)^2$$

deformation

linearized volume preservation

# Co-Rotational Linear Elasticity

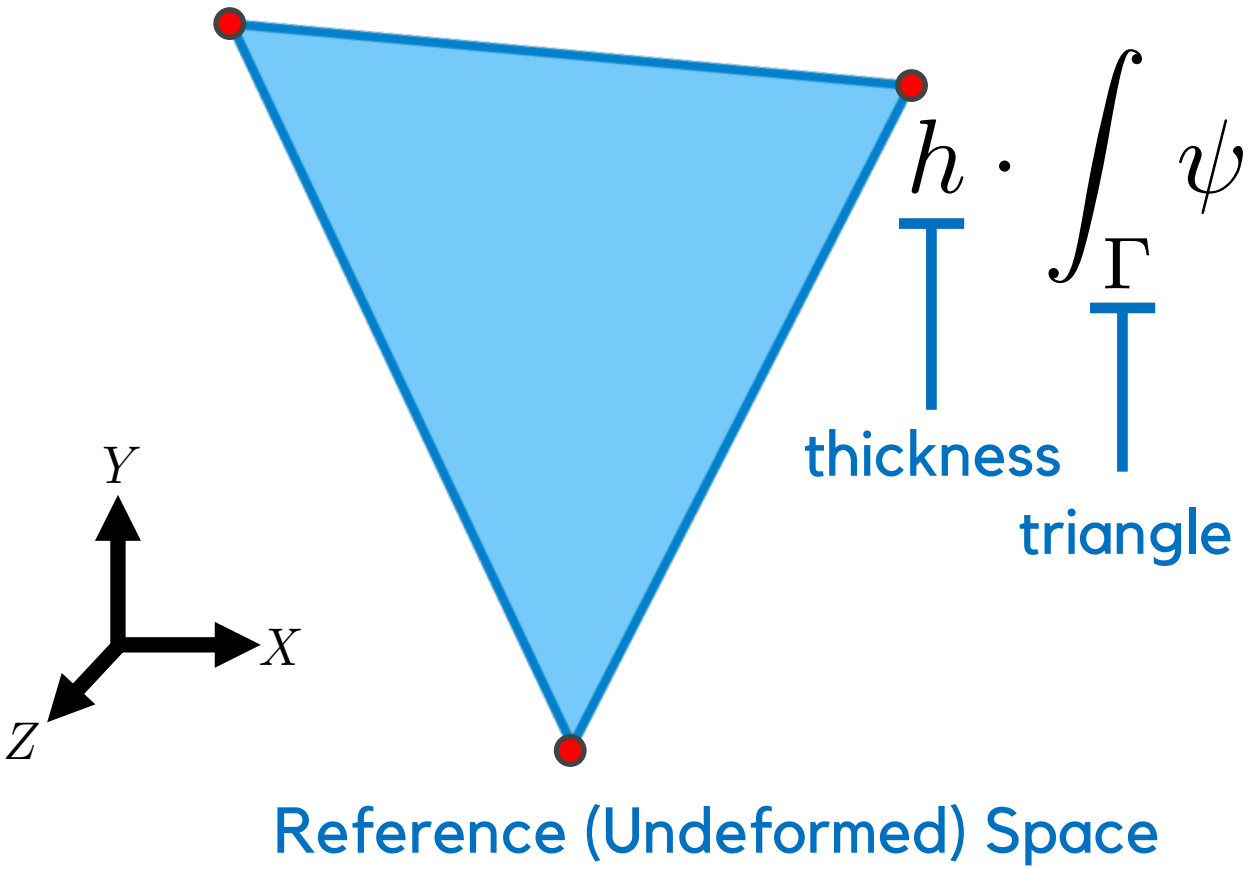
Lamé Parameters

$$\mu \sum_{i=0}^{2} (\sigma_i - 1)^2 + \frac{\lambda}{2} \left( \sum_{i=0}^{2} \sigma_i - 3 \right)^2$$

deformation

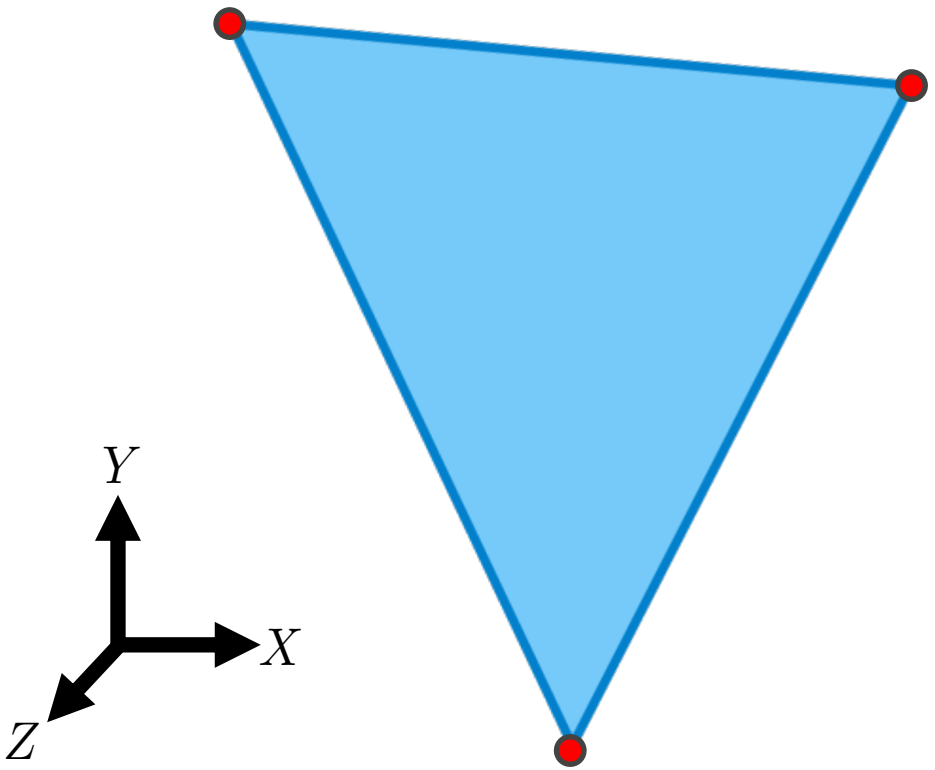
linearized volume preservation

# Potential Energy via Principal Stretches




$$\underbrace{h}_{\text{thickness}} \cdot \underbrace{\int_{\Gamma}}_{\text{triangle}} \psi\left(\sigma_0\left(\mathrm{F}\right), \sigma_1\left(\mathrm{F}\right), \sigma_2\left(\mathrm{F}\right)\right) d\Gamma$$

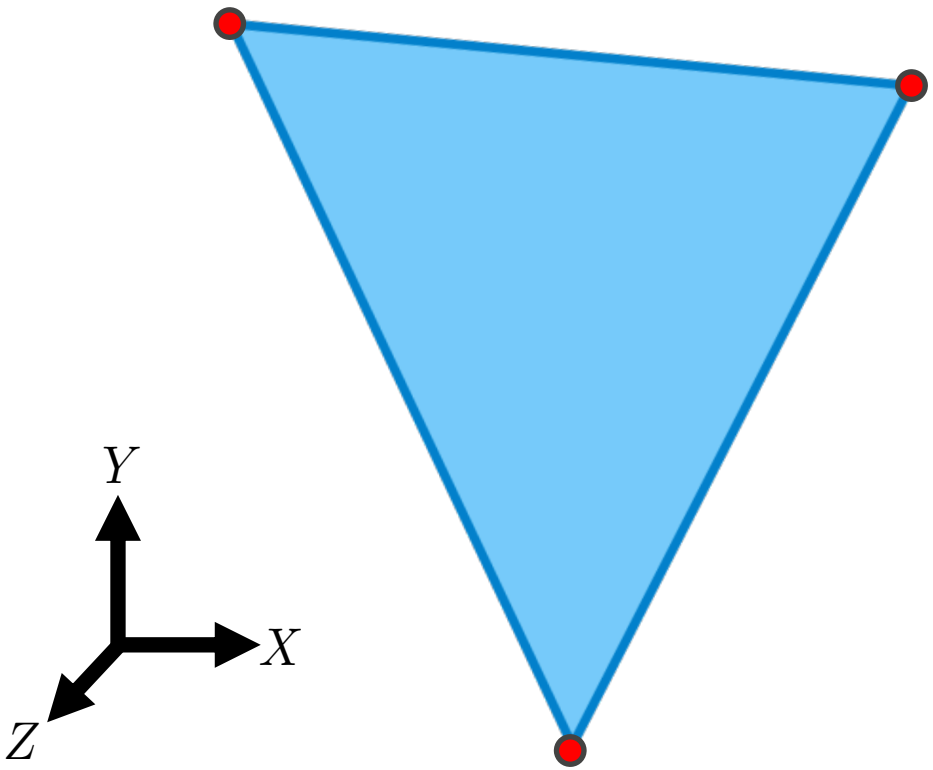
# Potential Energy via Principal Stretches




Reference (Undeformed) Space

$$\underbrace{h}_{\text{thickness}} \cdot \int_{\underbrace{\Gamma}_{\text{triangle}}} \psi\left(\mathrm{F}\left(\mathbf{X}\right)\right) d\Gamma$$

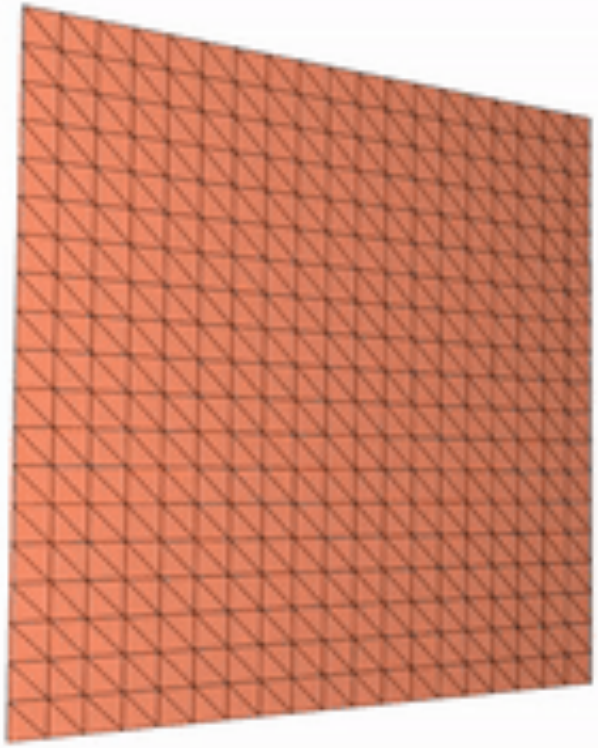
# Potential Energy via Principal Stretches




$$\underbrace{h}_{\text{thickness}} \cdot \text{area} \cdot \psi\left(\mathrm{F}_0\right)$$

Reference (Undeformed) Space

# Potential Energy for Cloth



sum over all triangles

$$V = \sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

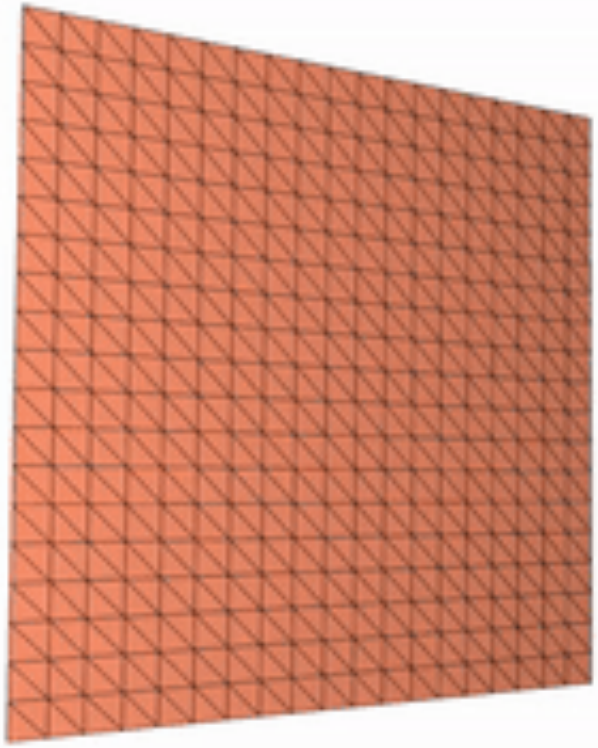
potential energy

triangle selection

$$\mathrm{E}_j = \begin{pmatrix} \mathrm{S}_0 \\ \mathrm{S}_1 \\ \mathrm{S}_2 \end{pmatrix}$$

Energy Plot

# The Lagrangian



$$V = \sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

$$L = T - V$$

$$\frac{1}{2}\dot{\mathbf{q}}^T \mathrm{M} \dot{\mathbf{q}}$$

Energy Plot

# Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \underbrace{-\frac{\partial V}{\partial \mathbf{q}}}_{\text{Generalized Forces } \mathbf{f}}$$

# Equations of Motion

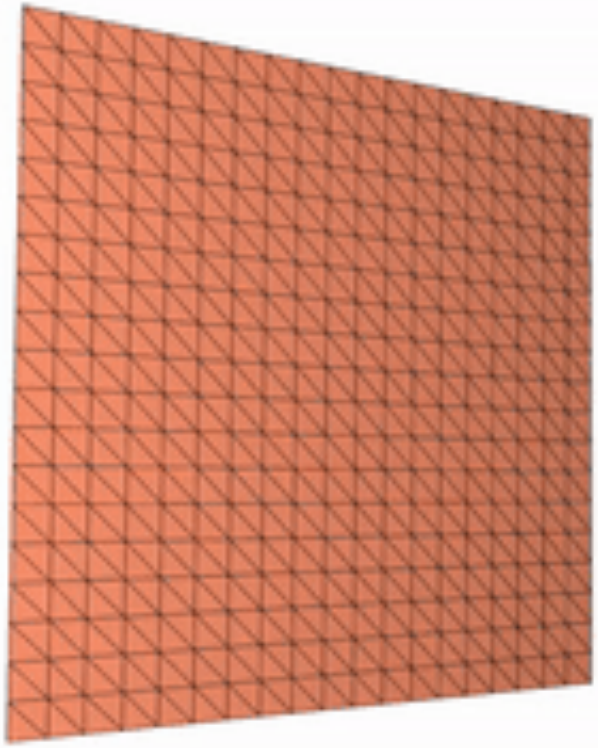
$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

# Time Integration

Usually use an (linearly-) implicit integrator for cloth due to high membrane stiffness

We will need the gradient and the ***Hessian*** and ***Gradient*** of our potential energy function

# Potential Energy for Cloth



sum over all triangles

$$V = \sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

potential energy

triangle selection

$$\mathrm{E}_j = \begin{pmatrix} \mathrm{S}_0 \\ \mathrm{S}_1 \\ \mathrm{S}_2 \end{pmatrix}$$

Energy Plot

# Generalized Forces

$$\mathbf{f} = -\frac{\partial V}{\partial \mathbf{q}} = -\sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \underbrace{\frac{\partial}{\partial \mathbf{q}} \overbrace{\psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)}^{\mathbb{R}^{9\times 1}}}_{\text{chain rule}}$$

$$\mathrm{E}_j^T \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j}^T \frac{\partial \psi}{\partial \mathrm{F}}$$

# Gradient of Principled Stretch Materials

UH OH! Need to take principal stretches into account

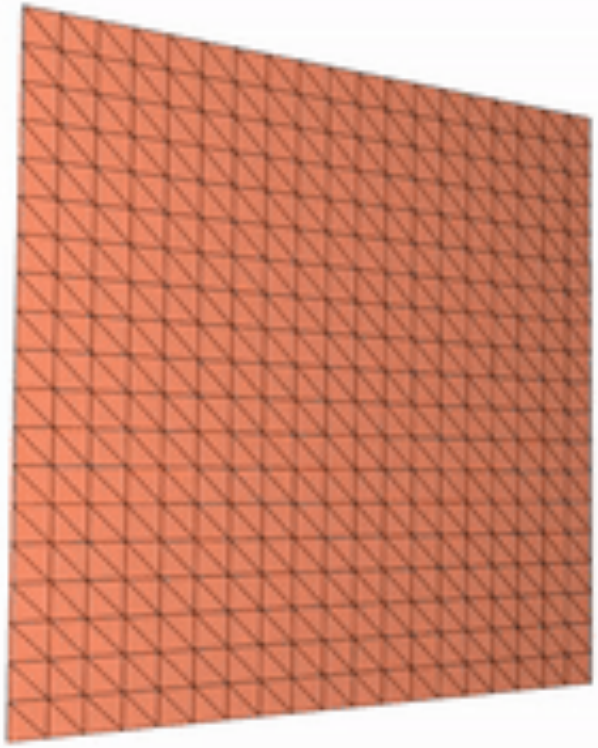
$$\psi\left(\sigma_0\left(\mathrm{F}\right), \sigma_1\left(\mathrm{F}\right), \sigma_2\left(\mathrm{F}\right)\right)$$

***MORE CHAIN RULE***

$$\frac{\partial \psi}{\partial F} = U \begin{pmatrix} \frac{\partial \psi}{\sigma s_0} & 0 & 0 \\ 0 & \frac{\partial \psi}{\partial \sigma_1} & 0 \\ 0 & 0 & \frac{\partial \psi}{\partial \sigma_2} \end{pmatrix} V^T$$

Singular Value Decomposition $\mathrm{F} = USV^T$

# Potential Energy for Cloth



sum over all triangles

$$V = \sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

potential energy

triangle selection

$$\mathrm{E}_j = \begin{pmatrix} \mathrm{S}_0 \\ \mathrm{S}_1 \\ \mathrm{S}_2 \end{pmatrix}$$

Energy Plot

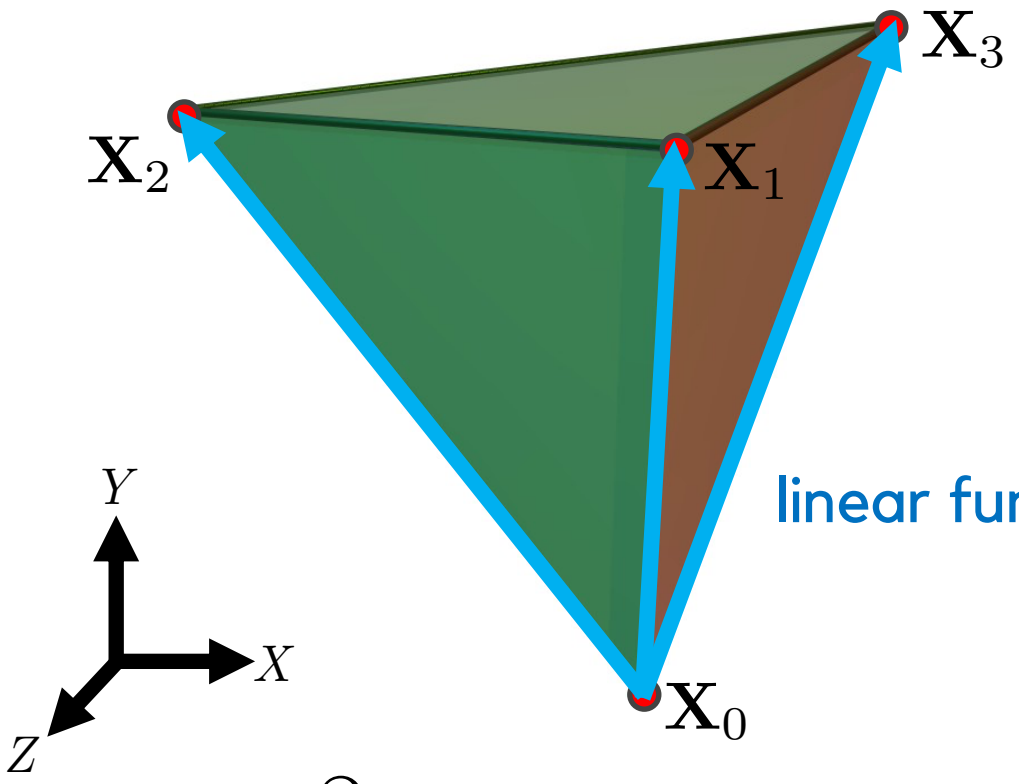
# Stiffness Matrix

$$\mathrm{K} = -\frac{\partial^2 V}{\partial \mathbf{q}^2} = -\sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \frac{\partial^2}{\partial \mathbf{q}^2} \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

chain rule

$$\mathrm{E}_j^T \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j}^T \frac{\partial \psi}{\partial \mathrm{F}} \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j} \mathrm{E}_j + \boxed{\mathrm{E}_j^T \frac{\partial^2 \mathrm{F}}{\partial \mathbf{q}_j^2}^T \cdot \frac{\partial \psi}{\partial \mathrm{F}}}$$

# Finite Elements for Deformation




linear function of **q**

$$\mathrm{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \begin{pmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix} \begin{pmatrix} -\mathbf{1}^T \mathrm{T}^{-1} \\ \mathrm{T}^{-1} \end{pmatrix}$$

$$\mathbf{1}^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

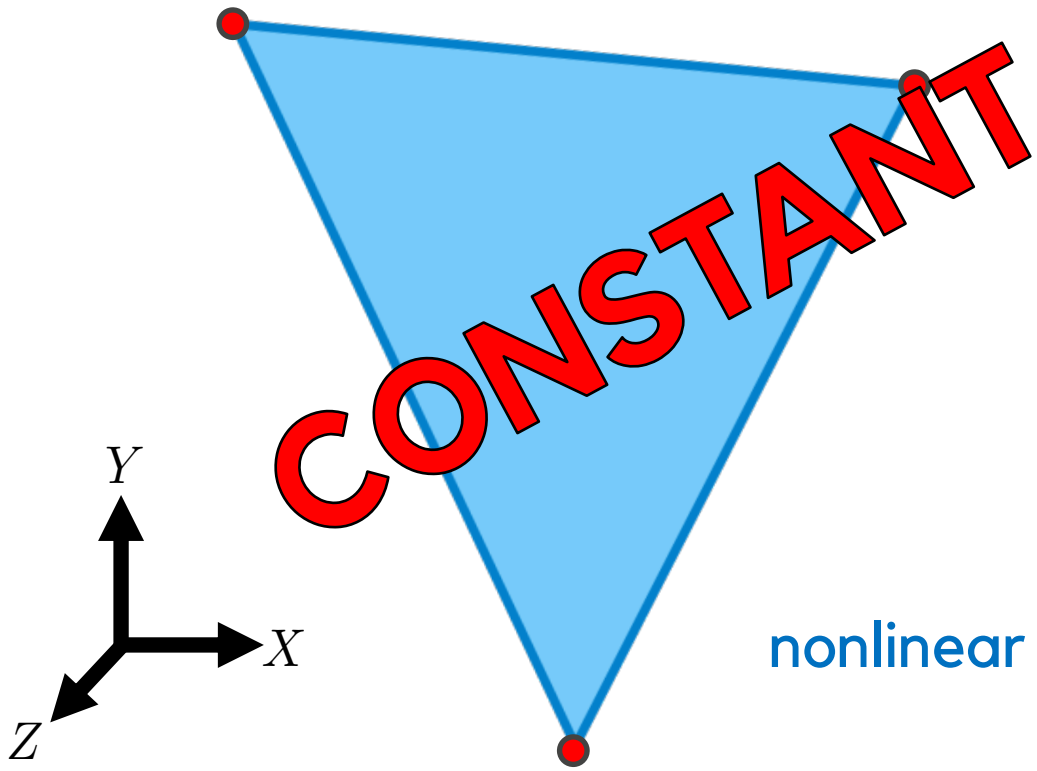
# Stiffness Matrix

$$\mathrm{K} = -\frac{\partial^2 V}{\partial \mathbf{q}^2} = -\sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \frac{\partial^2}{\partial \mathbf{q}^2} \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

chain rule

$$\mathrm{E}_j^T \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j}^T \frac{\partial \psi}{\partial \mathrm{F}} \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j} \mathrm{E}_j + \mathrm{E}_j^T \boxed{\frac{\partial^2 \mathrm{F}}{\partial \mathbf{q}_j^2}}^T \cdot \frac{\partial \psi}{\partial \mathrm{F}}$$

# Deformation Gradient of a Triangle




nonlinear function of **q**

$$\mathrm{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \begin{pmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{n} \end{pmatrix} \begin{pmatrix} \frac{\partial \phi}{\partial \mathbf{X}} \\ \mathbf{N}^T \end{pmatrix}$$

$$\mathbf{1}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

# Stiffness Matrix

$$\mathrm{K} = -\frac{\partial^2 V}{\partial \mathbf{q}^2} = -\sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \frac{\partial^2}{\partial \mathbf{q}^2} \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

chain rule

$$\mathrm{E}_j^T \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j}^T \frac{\partial^2 \psi}{\partial \mathrm{F}^2} \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j} \mathrm{E}_j + \mathrm{E}_j^T \boxed{\frac{\partial^2 \mathrm{F}}{\partial \mathbf{q}_j^2}}^T \cdot \frac{\partial \psi}{\partial \mathrm{F}}$$

# Newton's Method

**Choose an initial guess**

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

**Check for convergence**

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

**Choose search direction**

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Remember $\mathrm{H}^i = -\mathrm{K}^i$

**Use search direction to update current guess**

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

# Quasi-Newton Method

**Choose an initial guess**

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

**Check for convergence**

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

**Choose search direction**

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Use an approximation of $\mathrm{H}^i$

**Use search direction to update current guess**

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

# Stiffness Matrix

$$\mathrm{K} = -\frac{\partial^2 V}{\partial \mathbf{q}^2} = -\sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \frac{\partial^2}{\partial \mathbf{q}^2} \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

chain rule

$$\mathrm{E}_j^T \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j}^T \frac{\partial^2 \psi}{\partial \mathrm{F}^2} \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j} \mathrm{E}_j + \mathrm{E}_j^T \frac{\partial^2 \mathrm{F}}{\partial \mathbf{q}_j^2} \frac{\partial \psi}{\partial \mathrm{F}}$$

# Hessian of Principal Stretch Materials

vector to diagonal matrix

$$\frac{\partial}{\partial \mathrm{F}_{ij}} \frac{\partial \psi}{\partial \mathrm{F}} = \frac{\partial \mathrm{U}}{\partial \mathrm{F}_{ij}} \Delta S \, \mathrm{V}^T + \mathrm{U} \overline{\mathrm{diag}\left(\boldsymbol{\Delta}\mathbf{s}_{ij}\right)} \, \mathrm{V}^T + \mathrm{U} \Delta S \, \frac{\partial \mathrm{V}}{\partial \mathrm{F}_{ij}}^T$$

$(i, j)$ scalar entry in F

$$\underbrace{\begin{pmatrix} \frac{\partial \psi}{\sigma s_0} & 0 & 0 \\ 0 & \frac{\partial \psi}{\partial \sigma_1} & 0 \\ 0 & 0 & \frac{\partial \psi}{\partial \sigma_2} \end{pmatrix}}_{\boldsymbol{\Delta}\mathbf{S}} \underbrace{\begin{pmatrix} \frac{\partial^2 \psi}{\partial s_0^2} & \frac{\partial^2 \psi}{\partial s_0 \partial s_1} & \frac{\partial^2 \psi}{\partial s_0 \partial s_2} \\ \frac{\partial^2 \psi}{\partial s_0 \partial s_1} & \frac{\partial^2 \psi}{\partial s_1^2} & \frac{\partial^2 \psi}{\partial s_1 \partial s_2} \\ \frac{\partial^2 \psi}{\partial s_0 \partial s_2} & \frac{\partial^2 \psi}{\partial s_1 \partial s_2} & \frac{\partial^2 \psi}{\partial s_2^2} \end{pmatrix} \begin{pmatrix} \frac{\partial \sigma_0}{\partial F_{ij}} \\ \frac{\partial \sigma_1}{\partial F_{ij}} \\ \frac{\partial \sigma_2}{\partial F_{ij}} \end{pmatrix}}_{\boldsymbol{\Delta}\mathbf{s}_{ij}}$$

# Aside: Derivative of SVD

Singular Value Decomposition (SVD)

$$\mathrm{A} = \mathrm{USV}^T$$

SVD is parameterized by A

$$\mathrm{U}\left(\mathrm{A}\right)\mathrm{S}\left(\mathrm{A}\right)\mathrm{V}\left(\mathrm{A}\right)^T$$

Can compute derivative of SVD

$$\frac{\partial \mathrm{U}\left(\mathrm{A}\right)\mathrm{S}\left(\mathrm{A}\right)\mathrm{V}\left(\mathrm{A}\right)^T}{\partial \mathrm{A}_{ij}} = \frac{\partial \mathrm{U}}{\partial \mathrm{A}_{ij}}\mathrm{SV}^T + \mathrm{U}\frac{\partial \mathrm{S}}{\partial \mathrm{A}_{ij}}\mathrm{V}^T + \mathrm{US}\frac{\partial \mathrm{V}^T}{\partial A_{ij}}$$

**Normally we use off-the-shelf code to do this**

# Hessian of Principal Stretch Materials

$$\underbrace{\frac{\partial}{\partial \mathrm{F}_{ij}}}_{(i,j)\text{ element of the gradient}} \frac{\partial \psi}{\partial \mathrm{F}} = \frac{\partial \mathrm{U}}{\partial \mathrm{F}_{ij}} \Delta S \, \mathrm{V}^T + \mathrm{U} \overbrace{\mathrm{diag}\left(\boldsymbol{\Delta}\mathbf{s}_{ij}\right)}^{\text{vector to diagonal matrix}} \mathrm{V}^T + \mathrm{U} \Delta S \, \frac{\partial \mathrm{V}}{\partial \mathrm{F}_{ij}}^T$$

$$\underbrace{\begin{pmatrix} \frac{\partial \psi}{\sigma s_0} & 0 & 0 \\ 0 & \frac{\partial \psi}{\partial \sigma_1} & 0 \\ 0 & 0 & \frac{\partial \psi}{\partial \sigma_2} \end{pmatrix}}_{\boldsymbol{\Delta}\mathbf{S}} \underbrace{\begin{pmatrix} \frac{\partial^2 \psi}{\partial s_0^2} & \frac{\partial^2 \psi}{\partial s_0 \partial s_1} & \frac{\partial^2 \psi}{\partial s_0 \partial s_2} \\ \frac{\partial^2 \psi}{\partial s_0 \partial s_1} & \frac{\partial^2 \psi}{\partial s_1^2} & \frac{\partial^2 \psi}{\partial s_1 \partial s_2} \\ \frac{\partial^2 \psi}{\partial s_0 \partial s_2} & \frac{\partial^2 \psi}{\partial s_1 \partial s_2} & \frac{\partial^2 \psi}{\partial s_2^2} \end{pmatrix} \begin{pmatrix} \frac{\partial \sigma_0}{\partial F_{ij}} \\ \frac{\partial \sigma_1}{\partial F_{ij}} \\ \frac{\partial \sigma_2}{\partial F_{ij}} \end{pmatrix}}_{\boldsymbol{\Delta}\mathbf{s}_{ij}}$$

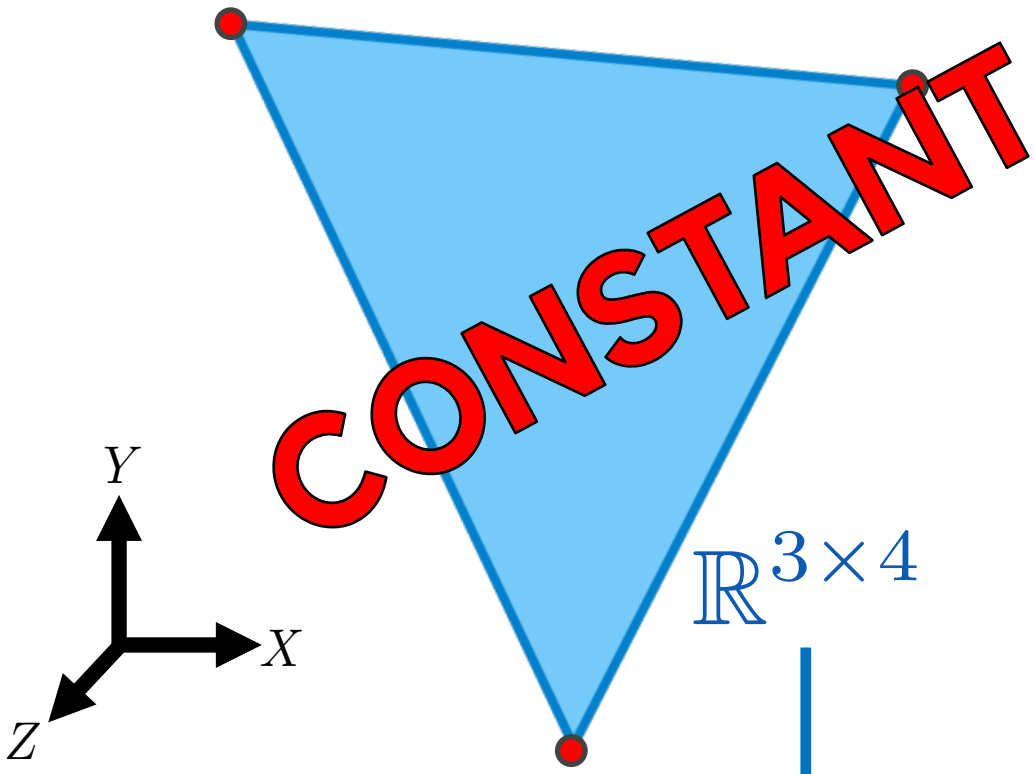
# Stiffness Matrix

$$\mathrm{K} = -\frac{\partial^2 V}{\partial \mathbf{q}^2} = -\sum_{j=0}^{m-1} h \cdot \mathrm{area}_j \cdot \frac{\partial^2}{\partial \mathbf{q}^2} \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

chain rule

$$\mathrm{E}_j^T \frac{\partial \mathrm{F}}{\partial \mathbf{q}_j}^T \frac{\partial^2 \psi}{\partial \mathrm{F}^2} \boxed{\frac{\partial \mathrm{F}}{\partial \mathbf{q}_j}} \mathrm{E}_j + \mathrm{E}_j^T \frac{\partial^2 \mathrm{F}}{\partial \mathbf{q}_j^2} \frac{\partial \psi}{\partial \mathrm{F}}$$

# Deformation Gradient of a Triangle




$$\mathrm{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \overbrace{\begin{pmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{n} \end{pmatrix}}^{\mathbb{R}^{3\times 4}} \overbrace{\begin{pmatrix} \boxed{\begin{matrix} -\mathbf{1}^T \left(\mathrm{T}^T\mathrm{T}\right)^{-1} \mathrm{T}^T \\ \left(\mathrm{T}^T\mathrm{T}\right)^{-1} \mathrm{T}^T \end{matrix}} \\ \mathbf{N}^T \end{pmatrix}}^{\mathrm{D}}$$

$$\mathbf{1}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

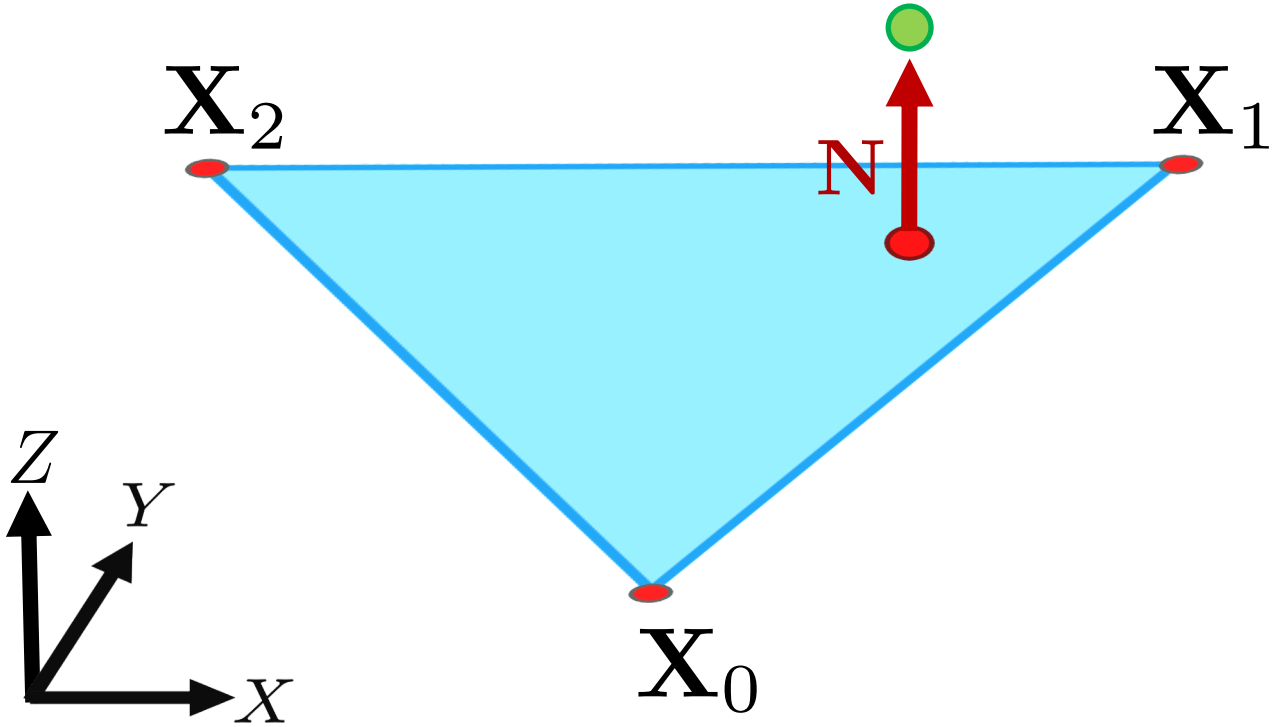
# Vectorized Deformation Gradient

$$
\mathrm{F} = \begin{pmatrix} \frac{\partial x}{\partial X} \\ \frac{\partial x}{\partial Y} \\ \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} \\ \frac{\partial y}{\partial Y} \\ \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} \\ \frac{\partial z}{\partial Y} \\ \frac{\partial z}{\partial Z} \end{pmatrix} = \underbrace{\begin{pmatrix} D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 & 0 \\ D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 & 0 \\ D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 & 0 \\ 0 & D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 \\ 0 & D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 \\ 0 & D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 \\ 0 & 0 & D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} \\ 0 & 0 & D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} \\ 0 & 0 & D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} \end{pmatrix}}_{\mathrm{B}_j} \underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}}_{\mathbf{q}_j} + \underbrace{\begin{pmatrix} N_x & 0 & 0 \\ N_y & 0 & 0 \\ N_z & 0 & 0 \\ 0 & N_x & 0 \\ 0 & N_y & 0 \\ 0 & N_z & 0 \\ 0 & 0 & N_x \\ 0 & 0 & N_y \\ 0 & 0 & N_z \end{pmatrix}}_{\mathrm{N}} \underbrace{\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}}_{\mathbf{n}}
$$

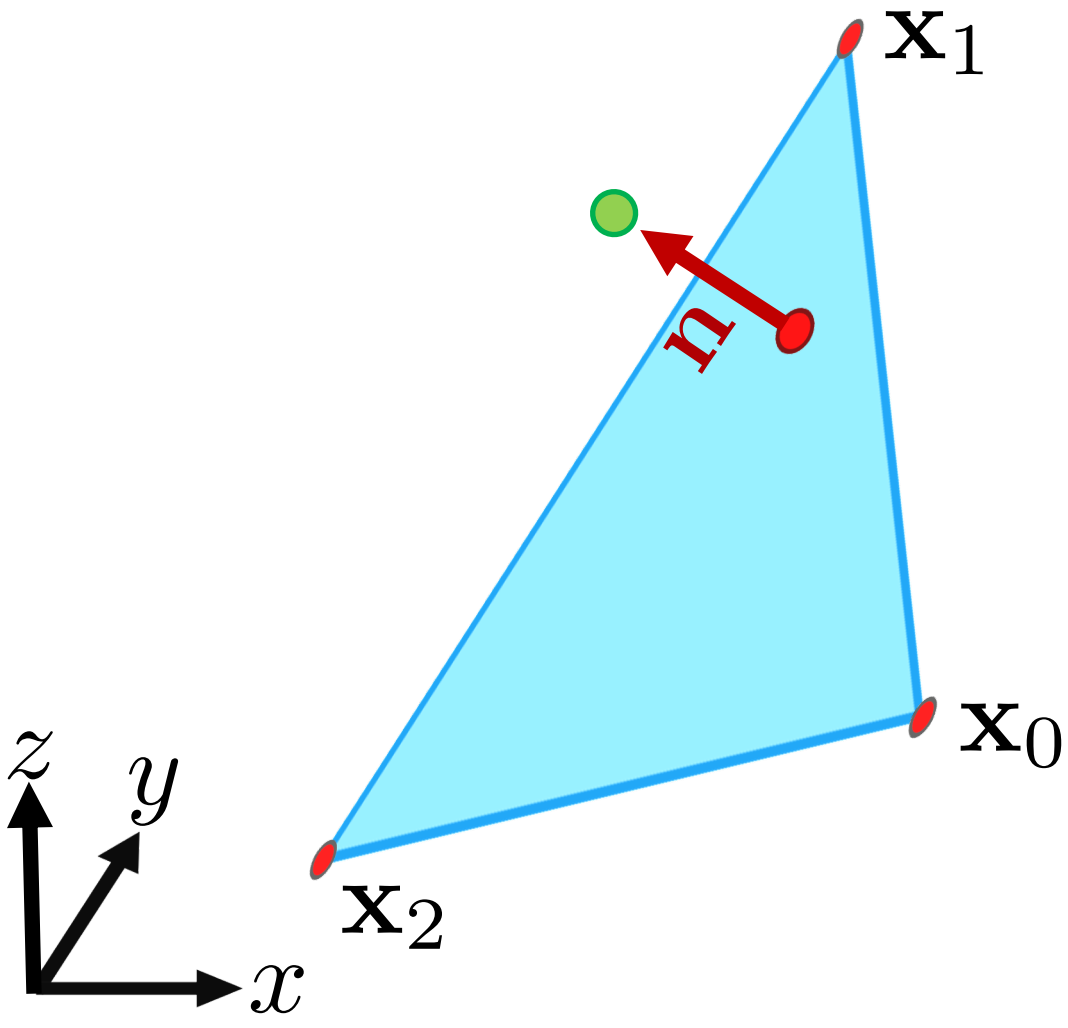
# Deformation Gradient, uh Gradient

$$\frac{\partial \mathrm{F}}{\partial \mathbf{q}_j} = \underbrace{\begin{pmatrix} D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 & 0 \\ D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 & 0 \\ D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 & 0 \\ 0 & D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 \\ 0 & D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 \\ 0 & D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 \\ 0 & 0 & D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} \\ 0 & 0 & D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} \\ 0 & 0 & D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} \end{pmatrix}}_{\mathrm{B}_j} + \underbrace{\begin{pmatrix} N_x & 0 & 0 \\ N_y & 0 & 0 \\ N_z & 0 & 0 \\ 0 & N_x & 0 \\ 0 & N_y & 0 \\ 0 & N_z & 0 \\ 0 & 0 & N_x \\ 0 & 0 & N_y \\ 0 & 0 & N_z \end{pmatrix}}_{\mathrm{N}} \frac{\partial}{\partial \mathbf{q_j}} \underbrace{\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}}_{\mathbf{n}}$$

# Triangular Finite Elements

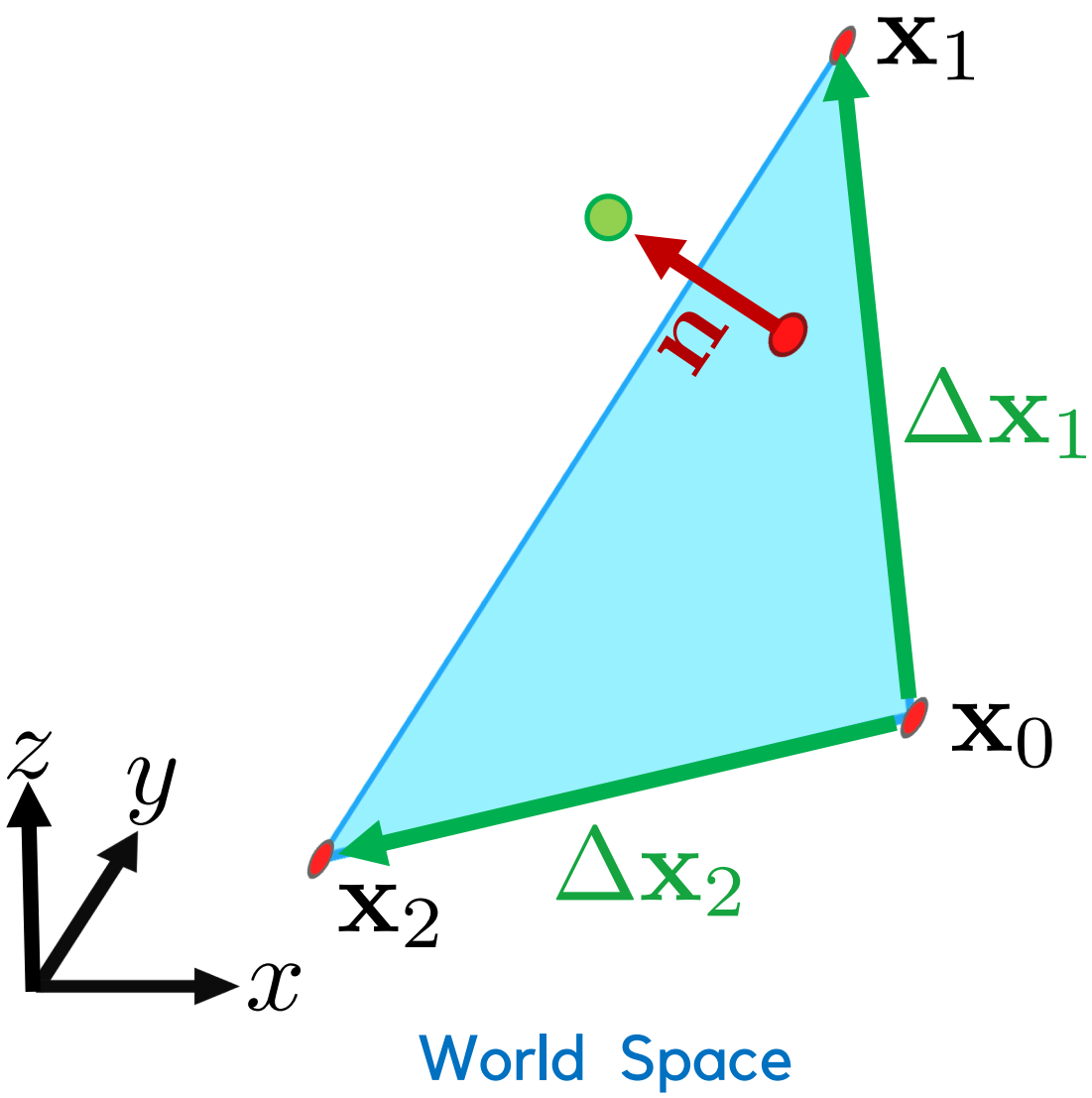



Reference Space




World Space

# Triangular Finite Elements




$$\mathbf{n} = \frac{\tilde{\mathbf{n}}}{|\tilde{\mathbf{n}}|}$$

$$\tilde{\mathbf{n}} = \Delta\mathbf{x}_1 \times \Delta\mathbf{x}_2$$

$$\Delta\mathbf{x}_1 = \begin{pmatrix} -\mathrm{I} & \mathrm{I} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$$

$$\Delta\mathbf{x}_2 = \begin{pmatrix} -\mathrm{I} & 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$$

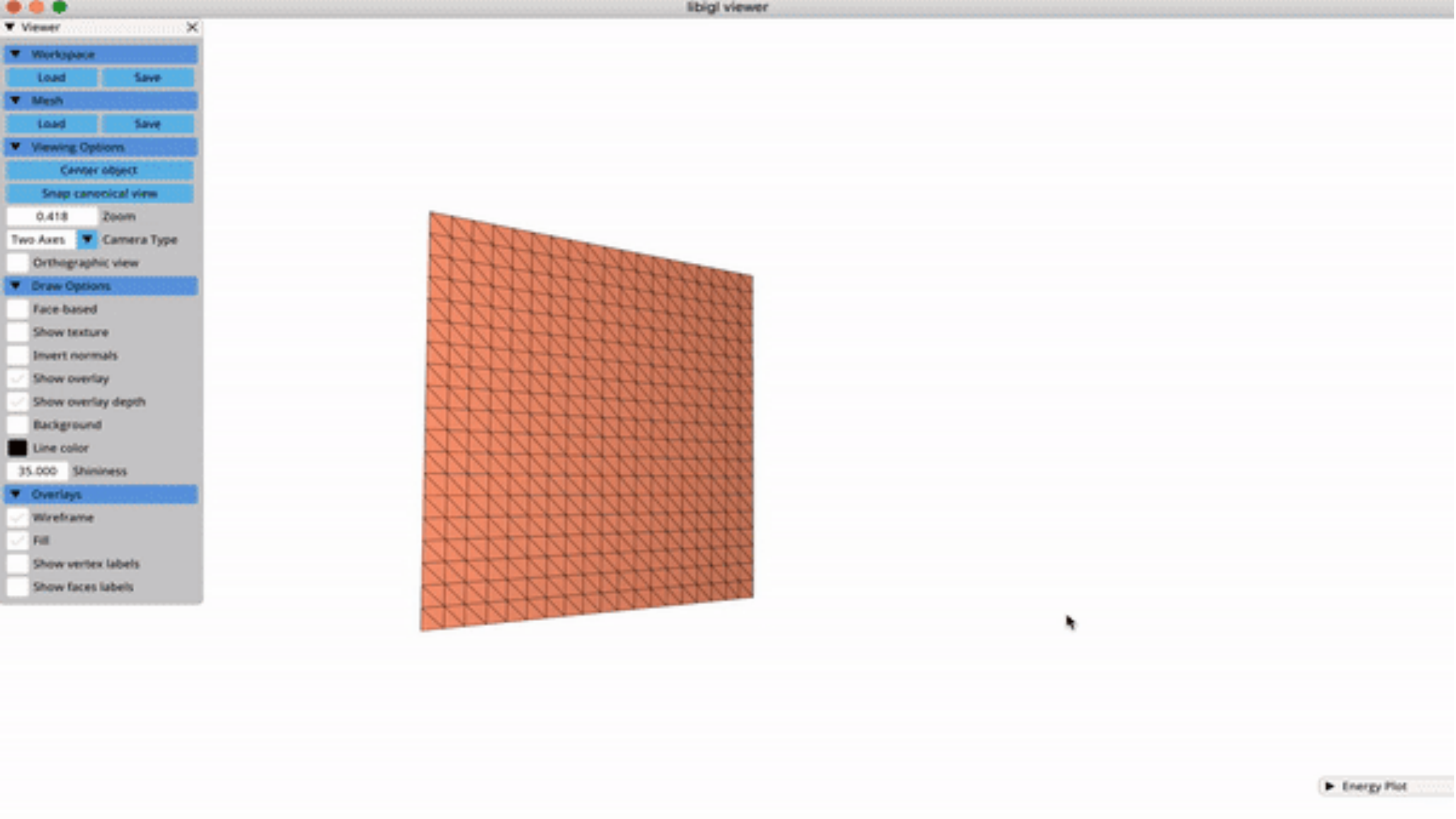
# Normal Gradient

$$\frac{\partial \mathbf{n}}{\partial \mathbf{q}_j} = \overbrace{\frac{1}{|\tilde{\mathbf{n}}|}\left(\mathrm{I} - \mathbf{n}\mathbf{n}^T\right)\left(\underbrace{[\Delta\mathbf{x}_1]}_{\text{Cross Product}}\begin{pmatrix}-\mathrm{I} & 0 & \mathrm{I}\end{pmatrix} - \underbrace{[\Delta\mathbf{x}_2]}_{\text{Cross Product}}\begin{pmatrix}-\mathrm{I} & \mathrm{I} & 0\end{pmatrix}\right)}^{\mathcal{N}}$$

$$[\mathbf{v}] = \begin{pmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{pmatrix}$$

# Deformation Gradient, uh Gradient

$$\frac{\partial \mathrm{F}}{\partial \mathbf{q}_j} = \underbrace{\begin{pmatrix} D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 & 0 \\ D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 & 0 \\ D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 & 0 \\ 0 & D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 \\ 0 & D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 \\ 0 & D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 \\ 0 & 0 & D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} \\ 0 & 0 & D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} \\ 0 & 0 & D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} \end{pmatrix}}_{\mathrm{B}_j} + \underbrace{\begin{pmatrix} N_x & 0 & 0 \\ N_y & 0 & 0 \\ N_z & 0 & 0 \\ 0 & N_x & 0 \\ 0 & N_y & 0 \\ 0 & N_z & 0 \\ 0 & 0 & N_x \\ 0 & 0 & N_y \\ 0 & 0 & N_z \end{pmatrix}}_{\mathrm{N}} \mathcal{N}$$


libigl viewer
Viewer
Workspace
Load
Save
Mesh
Load
Save
Viewing Options
Center object
Snap canonical view
0.418
Zoom
Two Axes
Camera Type
Orthographic view
Draw Options
Face-based
Show texture
Invert normals
Show overlay
Show overlay depth
Background
Line color
35.000
Shininess
Overlays
Wireframe
Fill
Show vertex labels
Show faces labels
Energy Plot

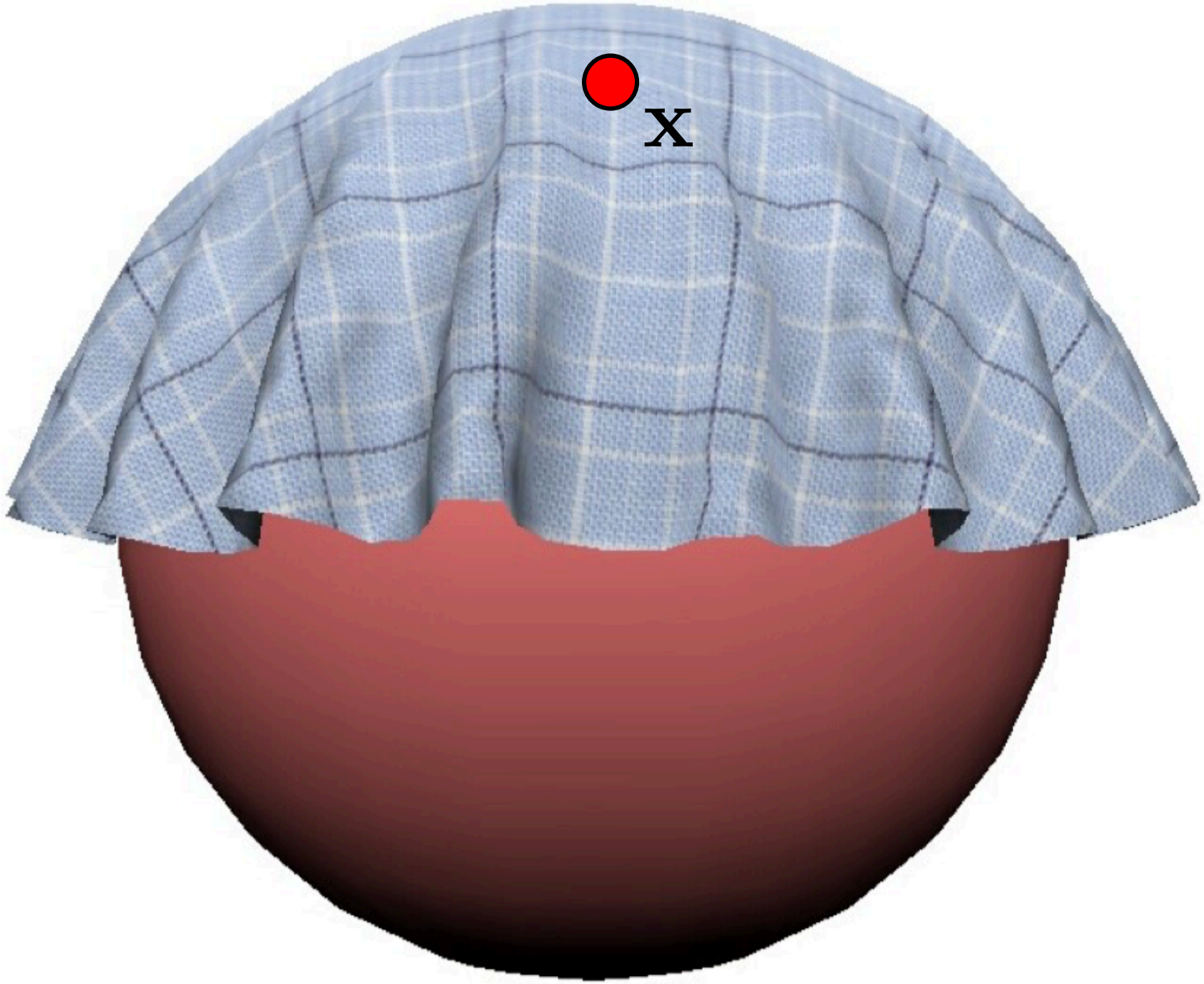
# Collisions in Simulation

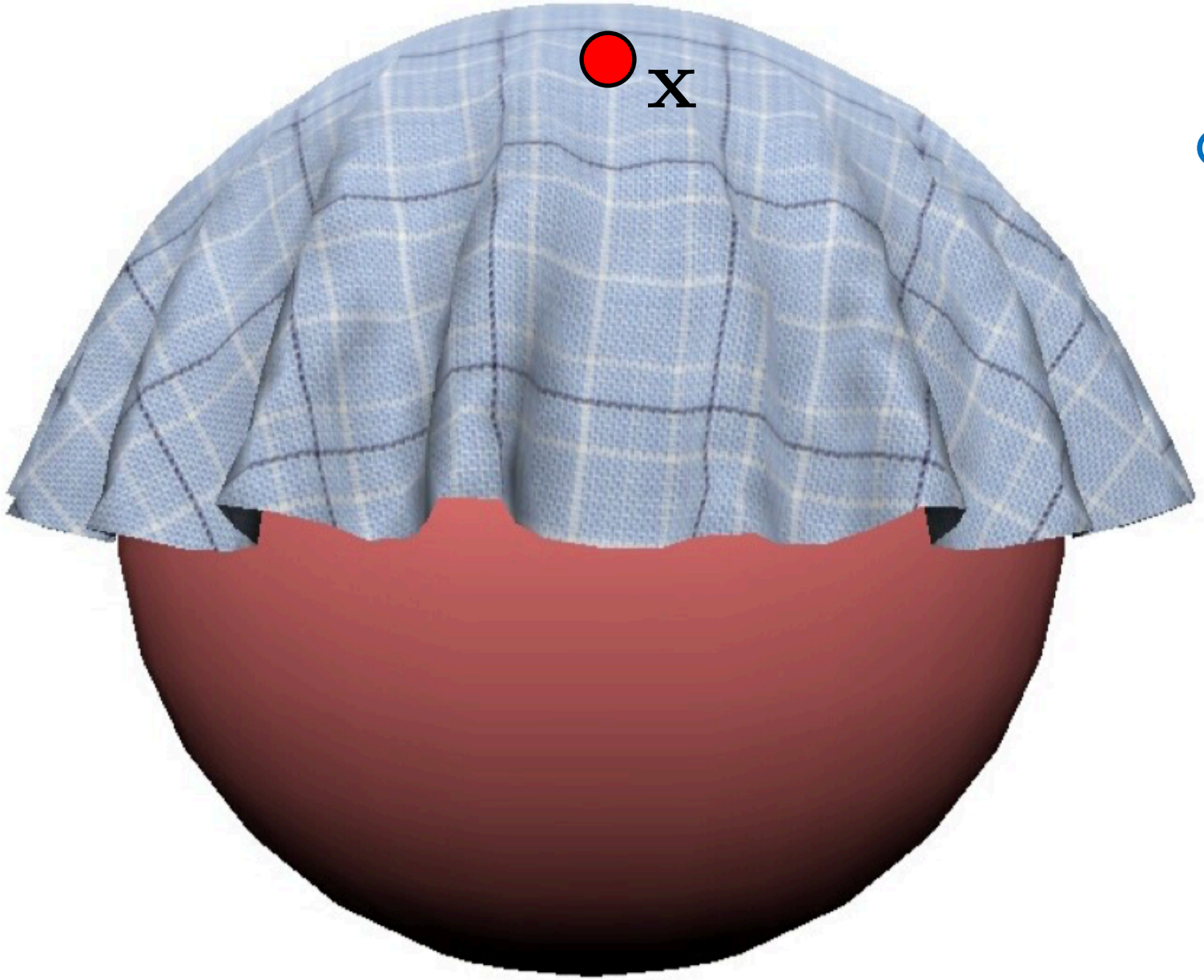
Two phases detection and response

**Detection:** Did I hit anything ?

**Response:** I hit something ! What do I do ?

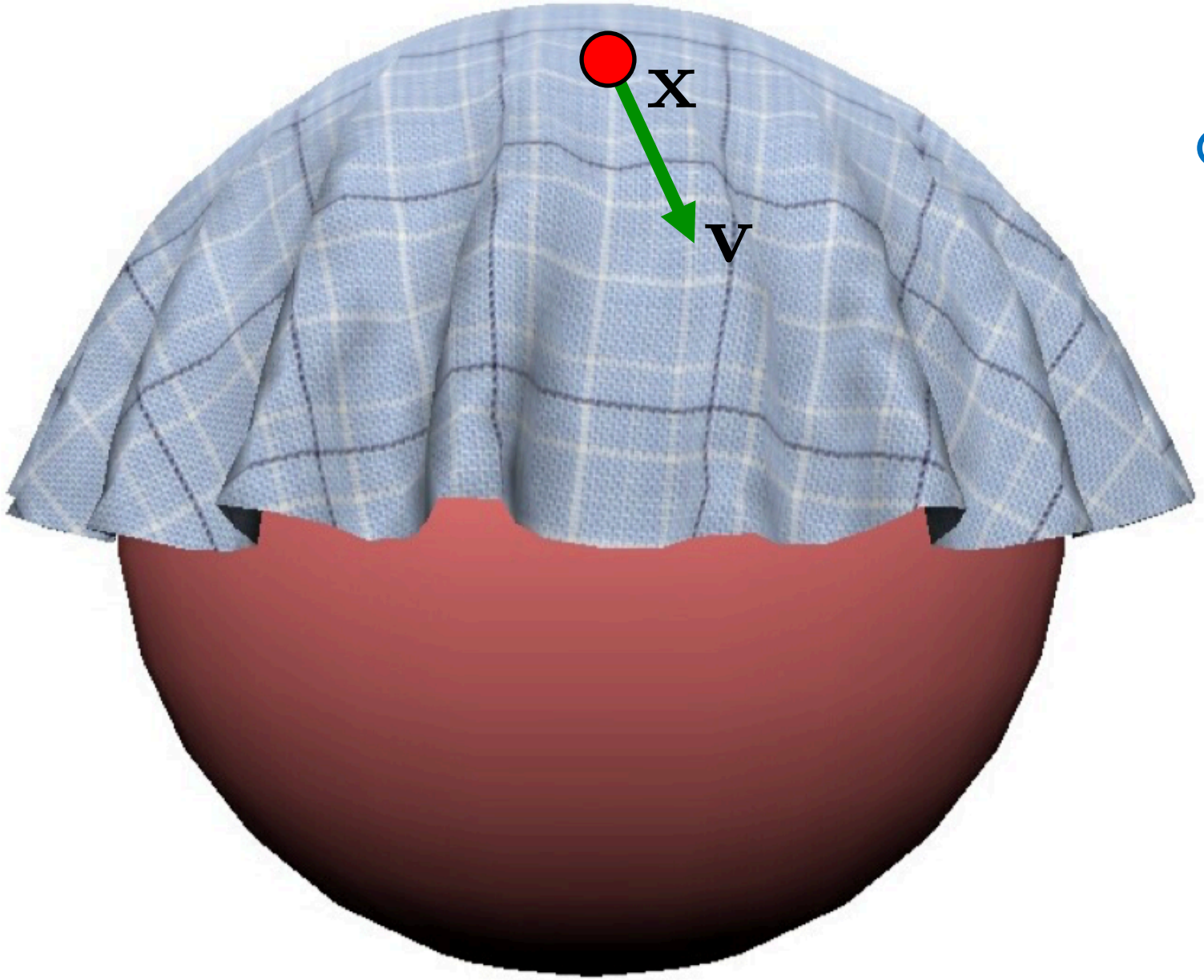
# Collision Detection



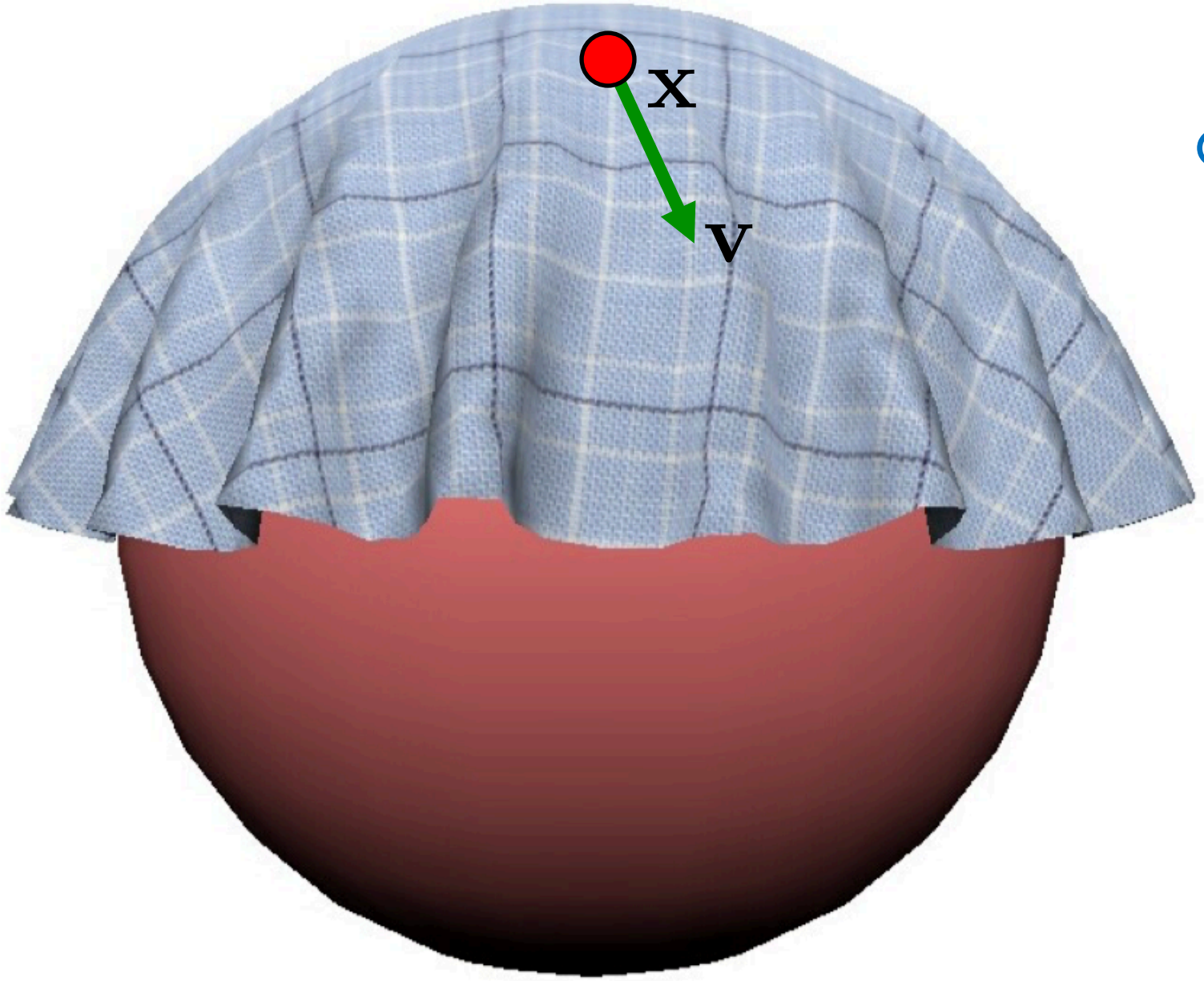



Check if $||\mathbf{x} - \mathbf{c}||_2^2 \leq r^2$

center radius



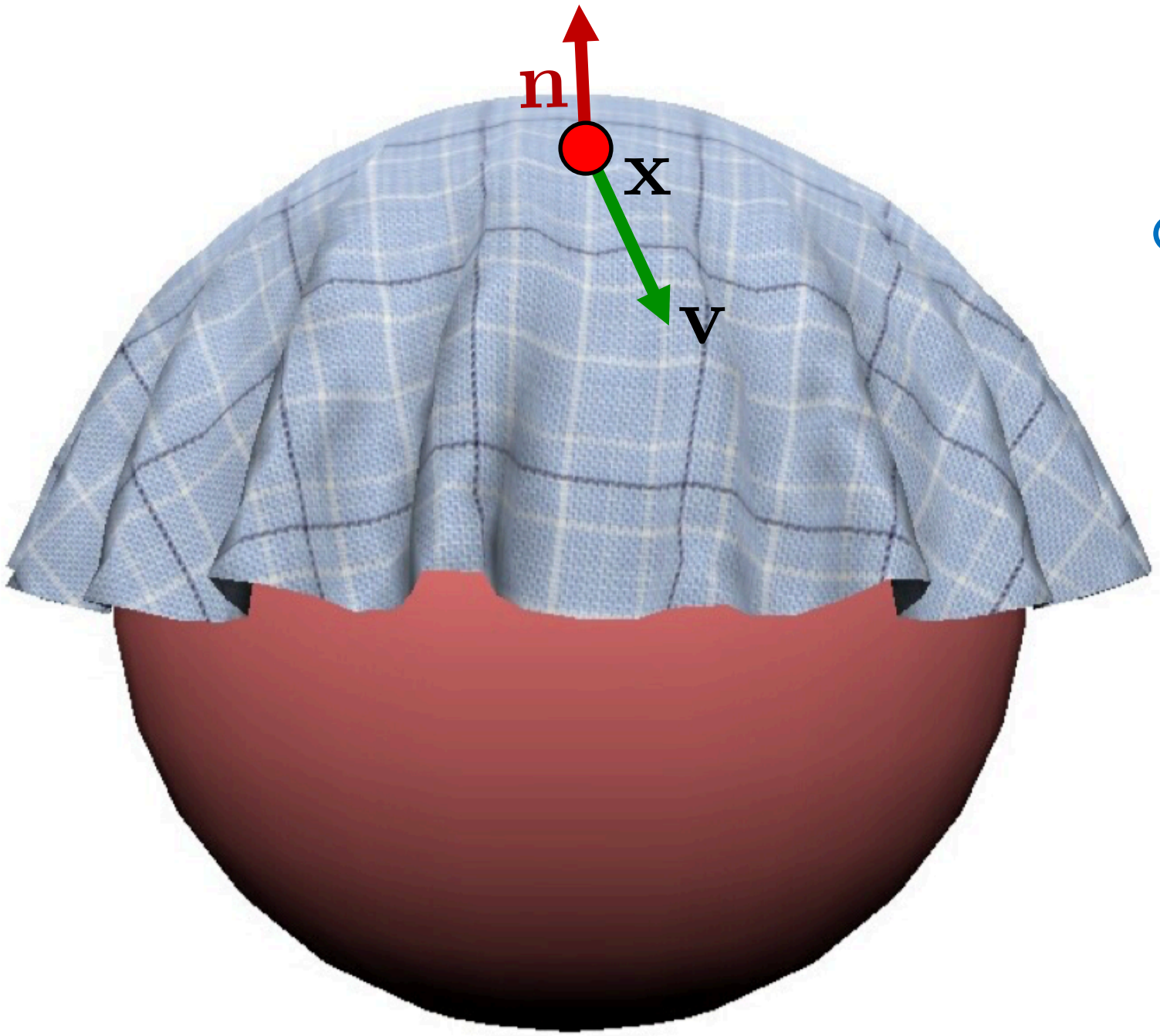

Check if $||\mathbf{x} - \underbrace{\mathbf{c}}_{\text{center}}||_2^2 \leq \underbrace{r^2}_{\text{radius}}$




Check if $||\mathbf{x} - \mathbf{c}||_2^2 \leq r^2$

center radius

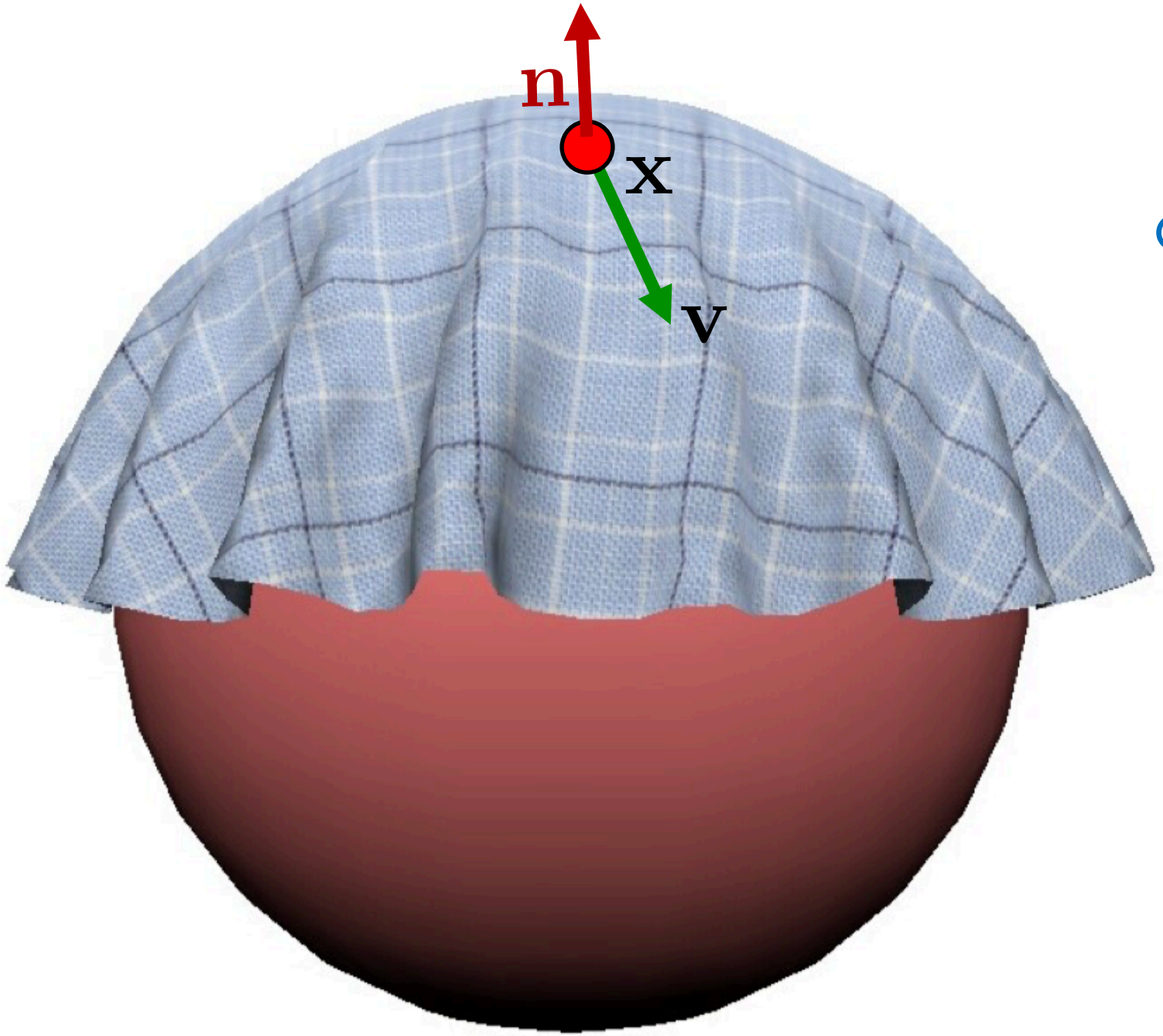
If colliding ...
Prevent it getting worse




Check if $||\mathbf{x} - \mathbf{c}||_2^2 \leq r^2$

center radius

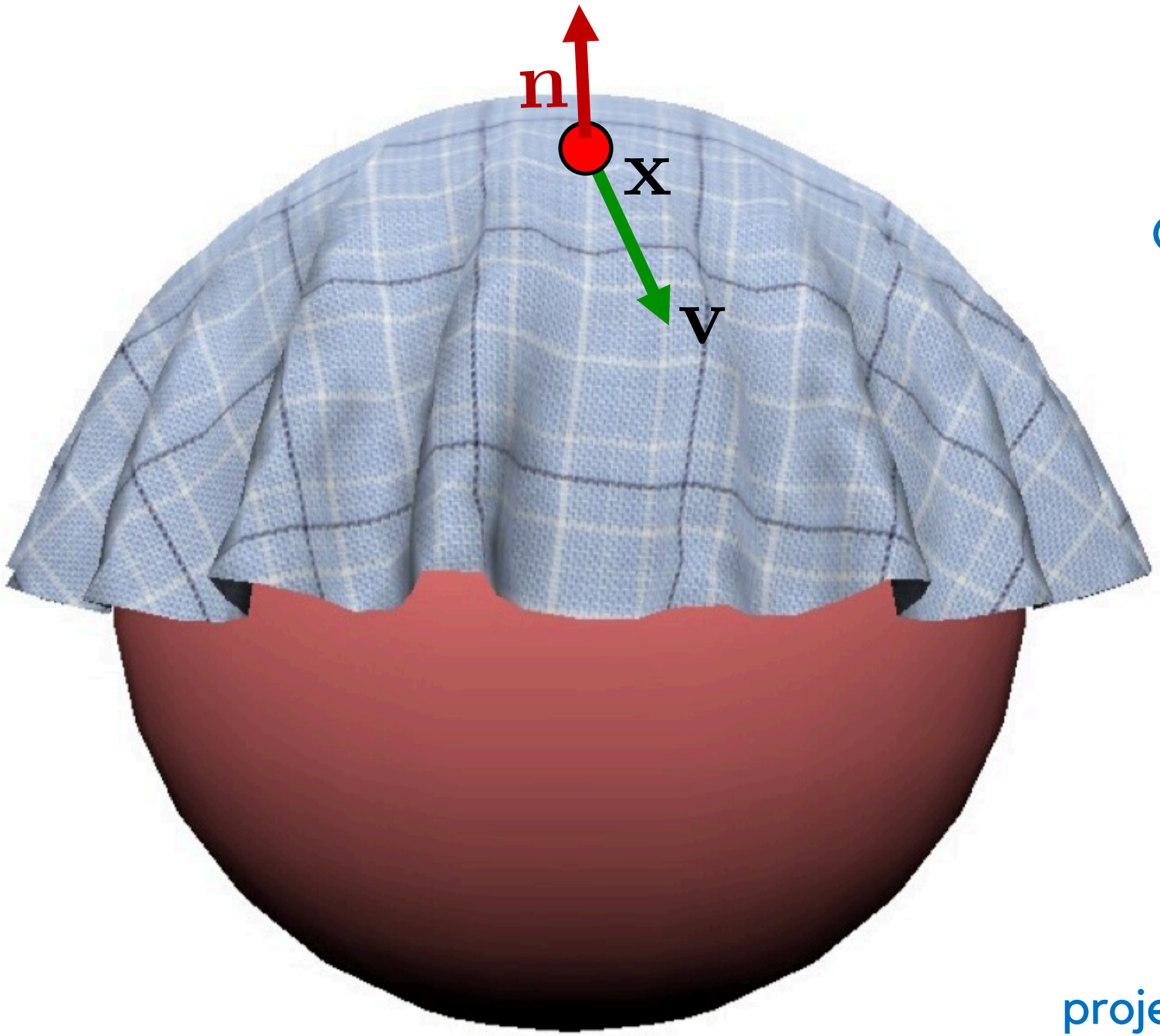
If colliding ...
Stop moving into sphere




Check if $||\mathbf{x} - \mathbf{c}|| \leq r^2$

center radius

If colliding ...
Filter normal component of velocity




Check if $||\mathbf{x} - \mathbf{c}||_2^2 \leq r^2$

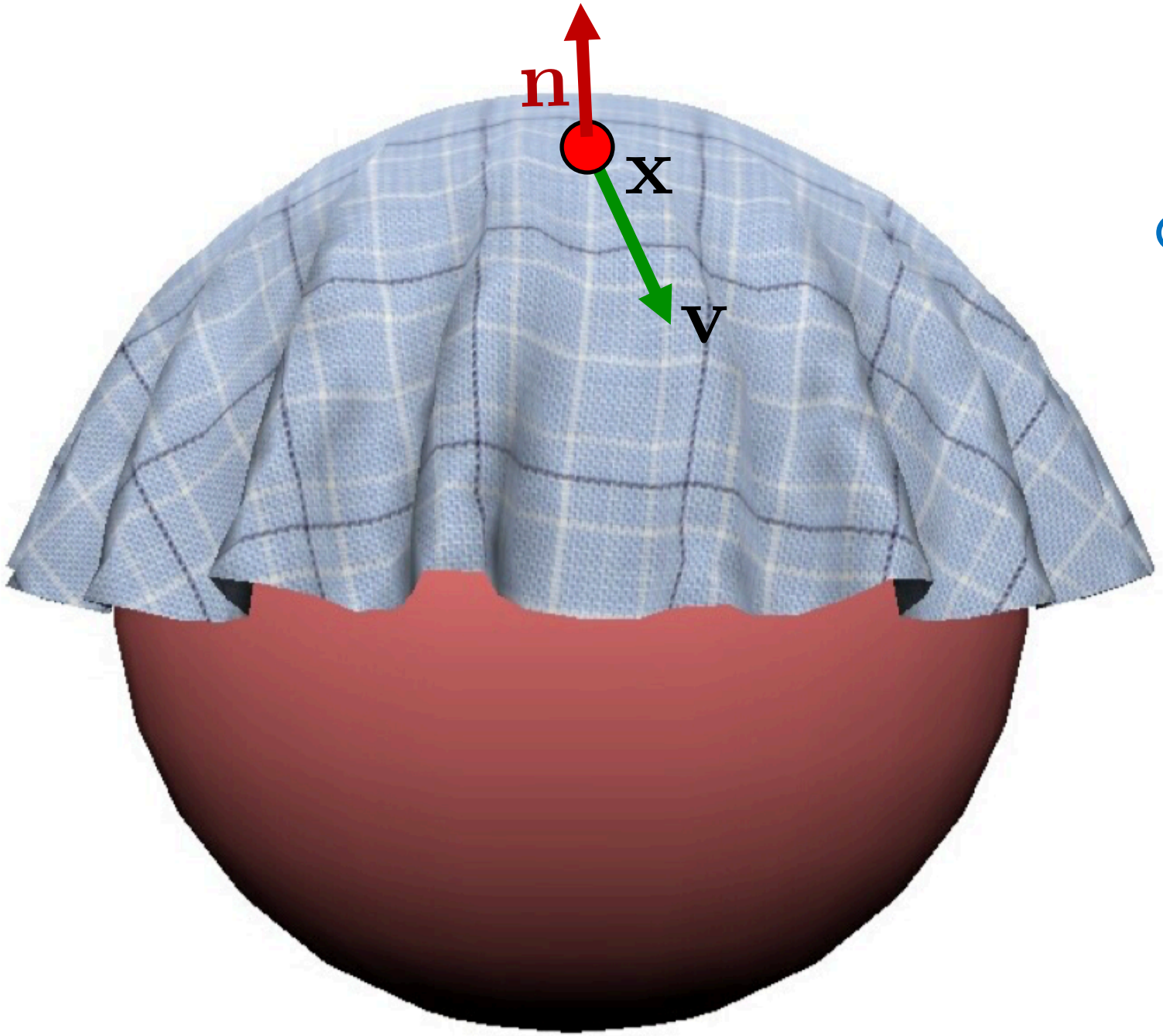
center radius

If colliding ...

Filter normal component of velocity

$$\mathbf{v}_{\text{filtered}} = \mathbf{v} + \alpha \mathbf{n}$$

$$\underbrace{\mathbf{n}^T \mathbf{v}_{\text{filtered}}}_{\text{projection of velocity onto normal}} \geq 0$$




Check if $||\mathbf{x} - \mathbf{c}||_2^2 \leq r^2$

center radius

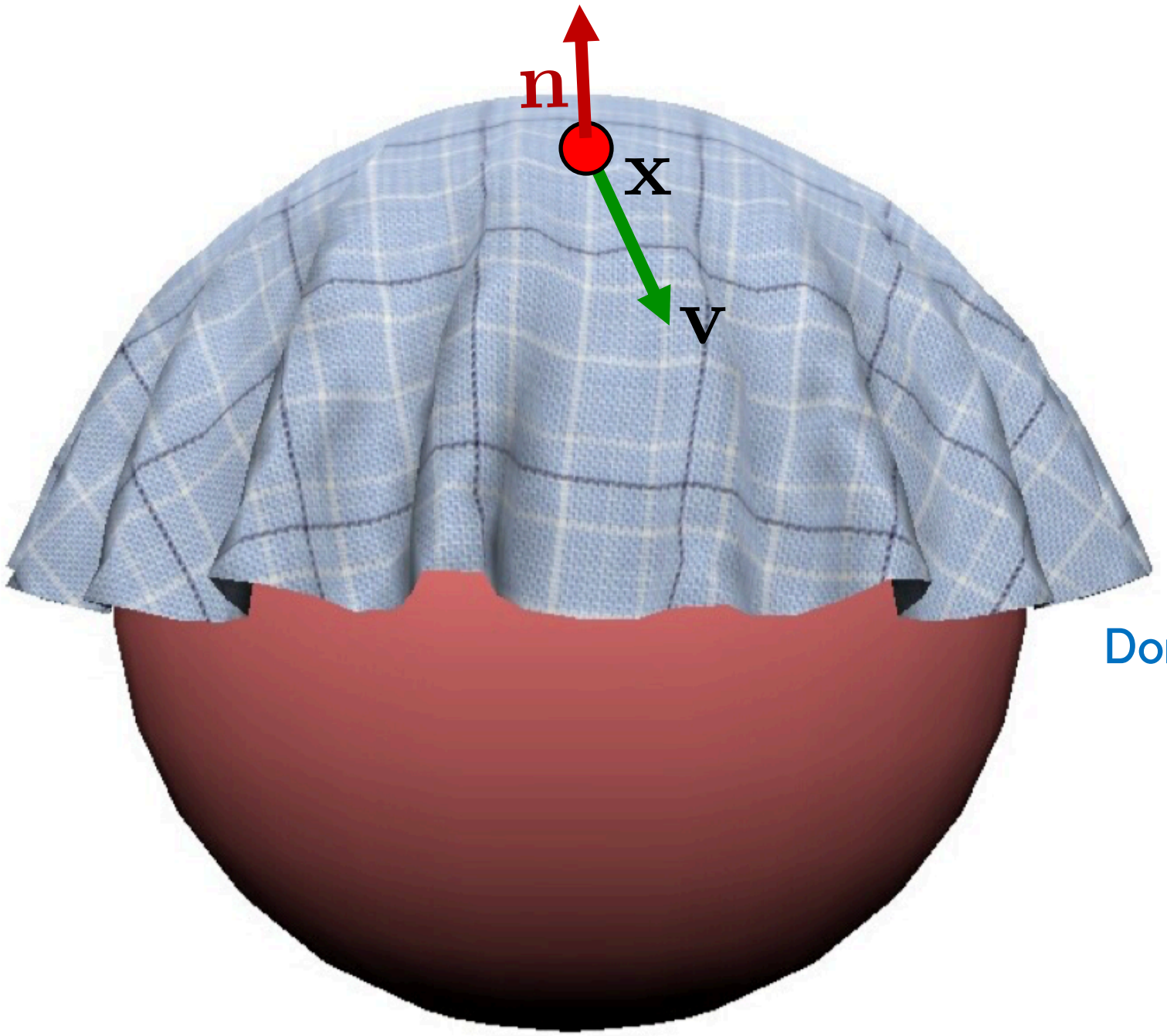
If colliding ...

Filter normal component of velocity

$$\mathbf{v}_{\text{filtered}} = \mathbf{v} + \alpha \mathbf{n}$$

$$\mathbf{n}^T \mathbf{v}_{\text{filtered}} \geq 0$$

$$\alpha \geq -\mathbf{n}^T \mathbf{v}$$




$$\mathbf{v}_{\text{filtered}} = \mathbf{v} + \alpha \mathbf{n}$$

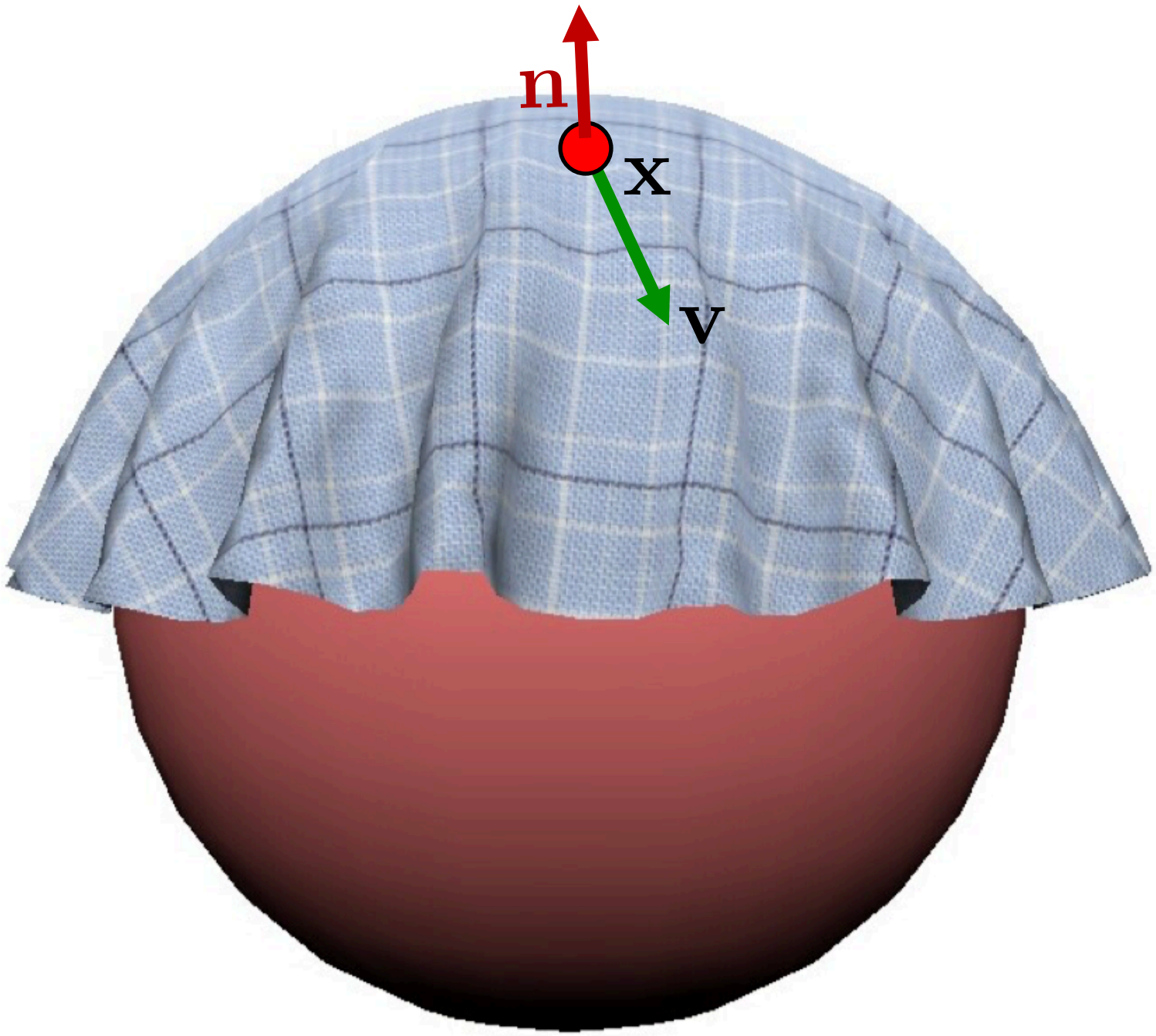
$$\mathbf{n}^T \mathbf{v}_{\text{filtered}} \geq 0$$

$$\alpha \geq -\mathbf{n}^T \mathbf{v}$$

Too many choices !!!

Change velocity as little as needed

Don't do anything if point is separating




$$\mathbf{v}_{\text{filtered}} = \mathbf{v} + \alpha \mathbf{n}$$

$$\mathbf{n}^T \mathbf{v}_{\text{filtered}} \geq 0$$

$$\alpha \geq -\mathbf{n}^T \mathbf{v}$$

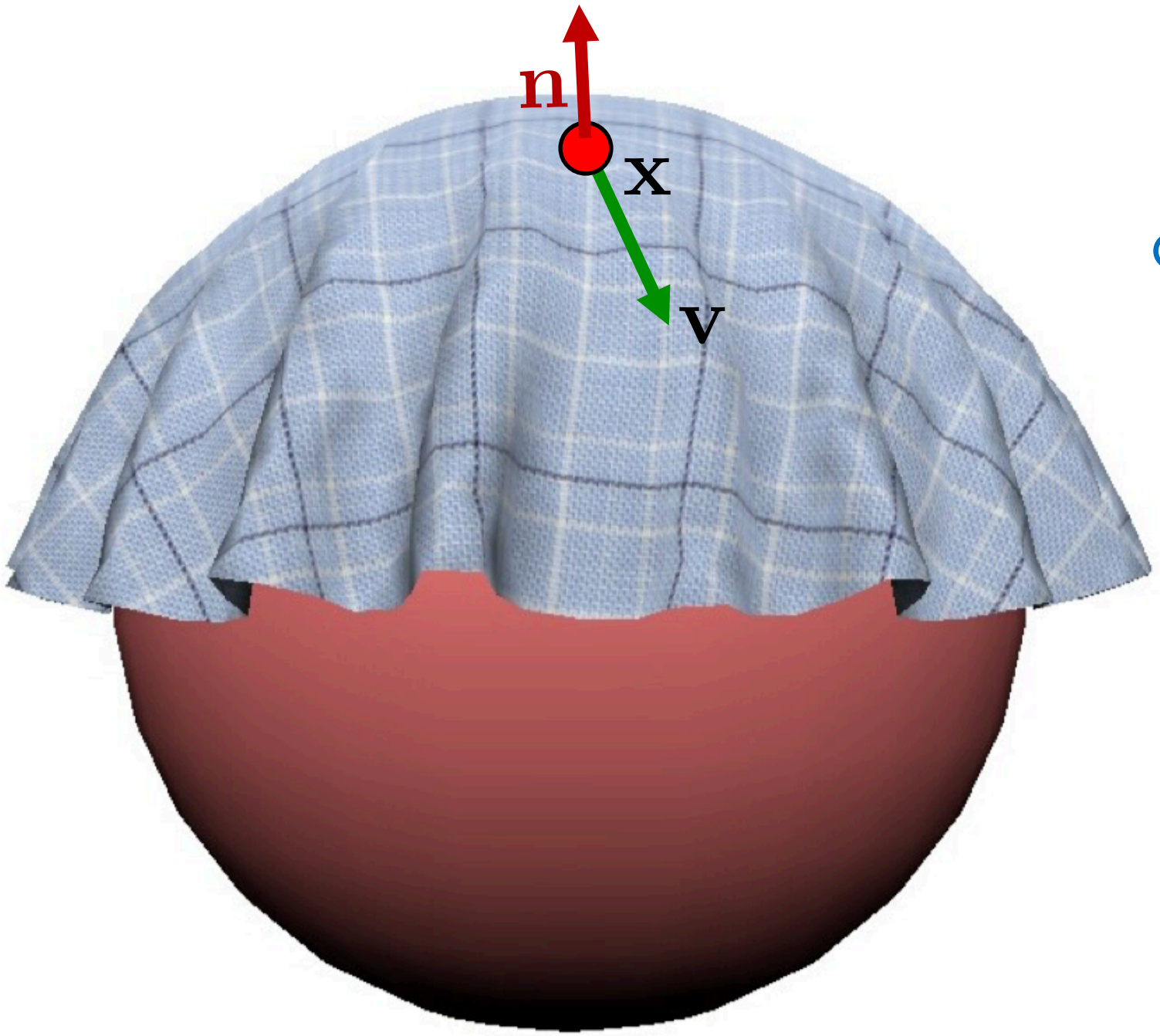
If $\mathbf{n}^T \mathbf{v} \geq 0$

$$\alpha = 0$$

Else

$$\alpha = -\mathbf{n}^T \mathbf{v}$$

$$\alpha = -\min\left(0, \mathbf{n}^T \mathbf{v}\right)$$



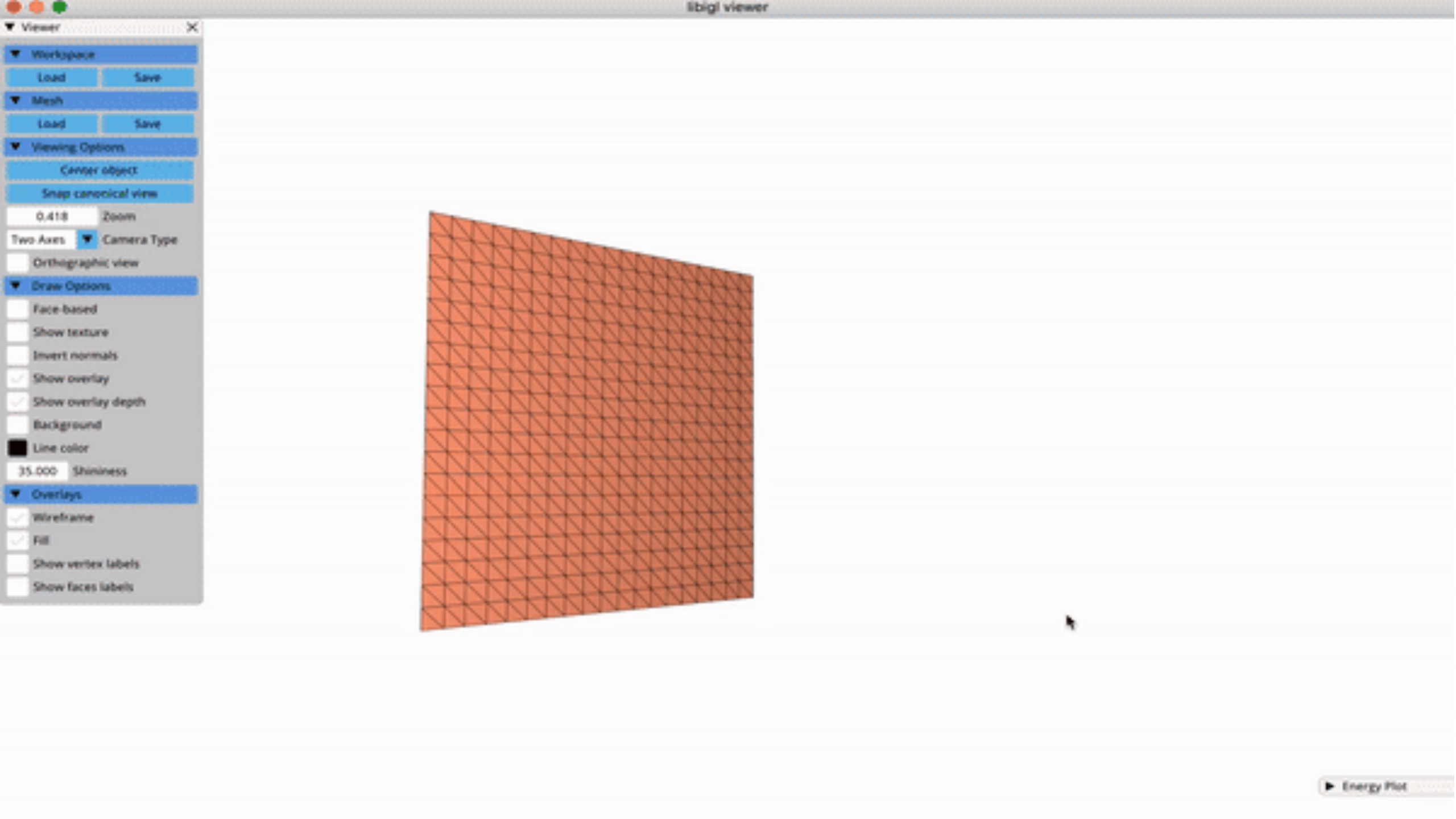
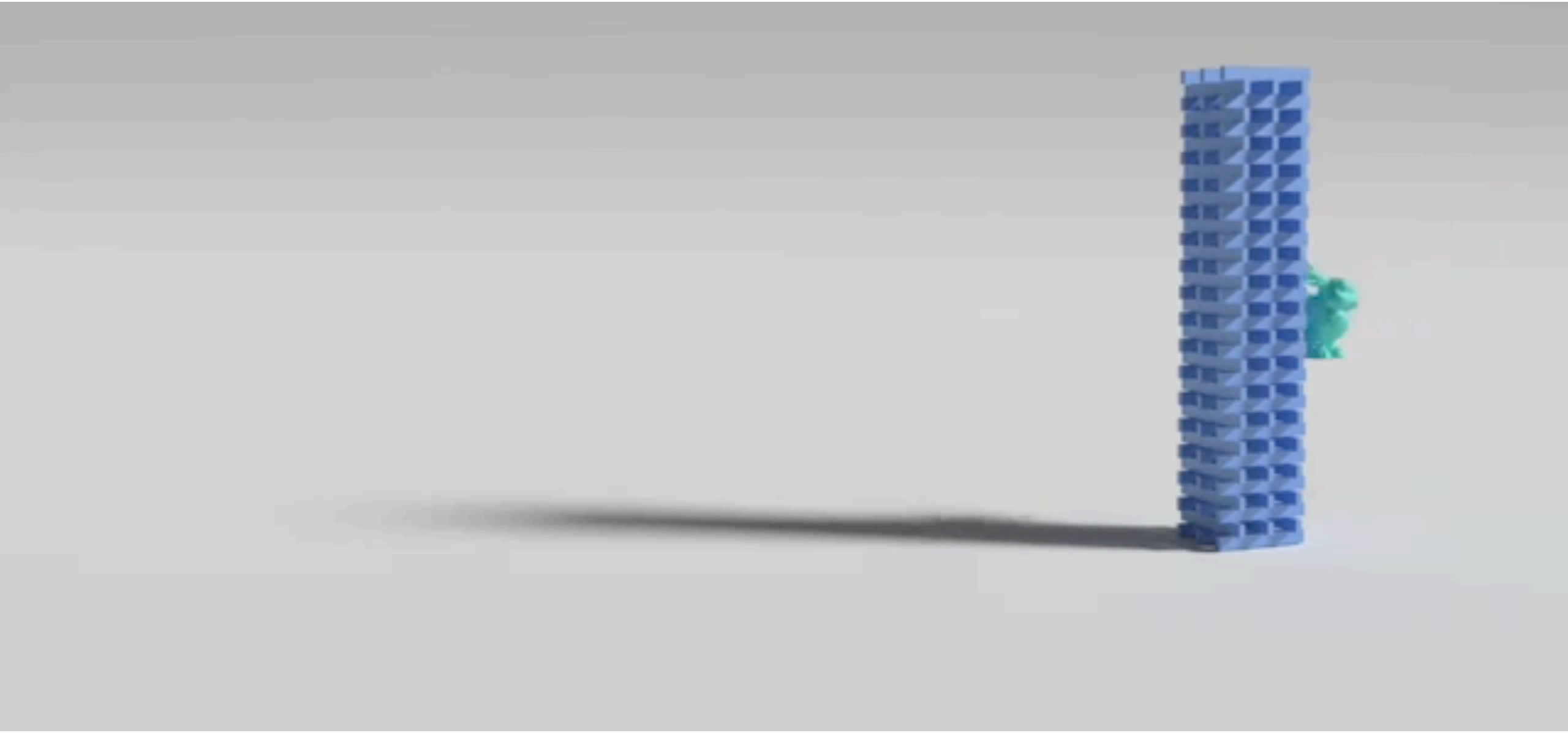

Check if $||\mathbf{x} - \mathbf{c}||_2^2 \leq r^2$

center radius

If colliding ...

$$\alpha = -\min\left(0, \mathbf{n}^T\mathbf{v}\right)$$

$$\mathbf{v}_{\text{filtered}} = \mathbf{v} + \alpha\mathbf{n}$$


libigl viewer
Viewer
Workspace
Load
Save
Mesh
Load
Save
Viewing Options
Center object
Snap canonical view
0.418
Zoom
Two Axes
Camera Type
Orthographic view
Draw Options
Face-based
Show texture
Invert normals
Show overlay
Show overlay depth
Background
Line color
35.000
Shininess
Overlays
Wireframe
Fill
Show vertex labels
Show faces labels
Energy Plot

# Next Video: Rigid Body Dynamics