CSC417 Physics-Based Animation

… starting at 11:10 am

Doctor Strange | Frame Store

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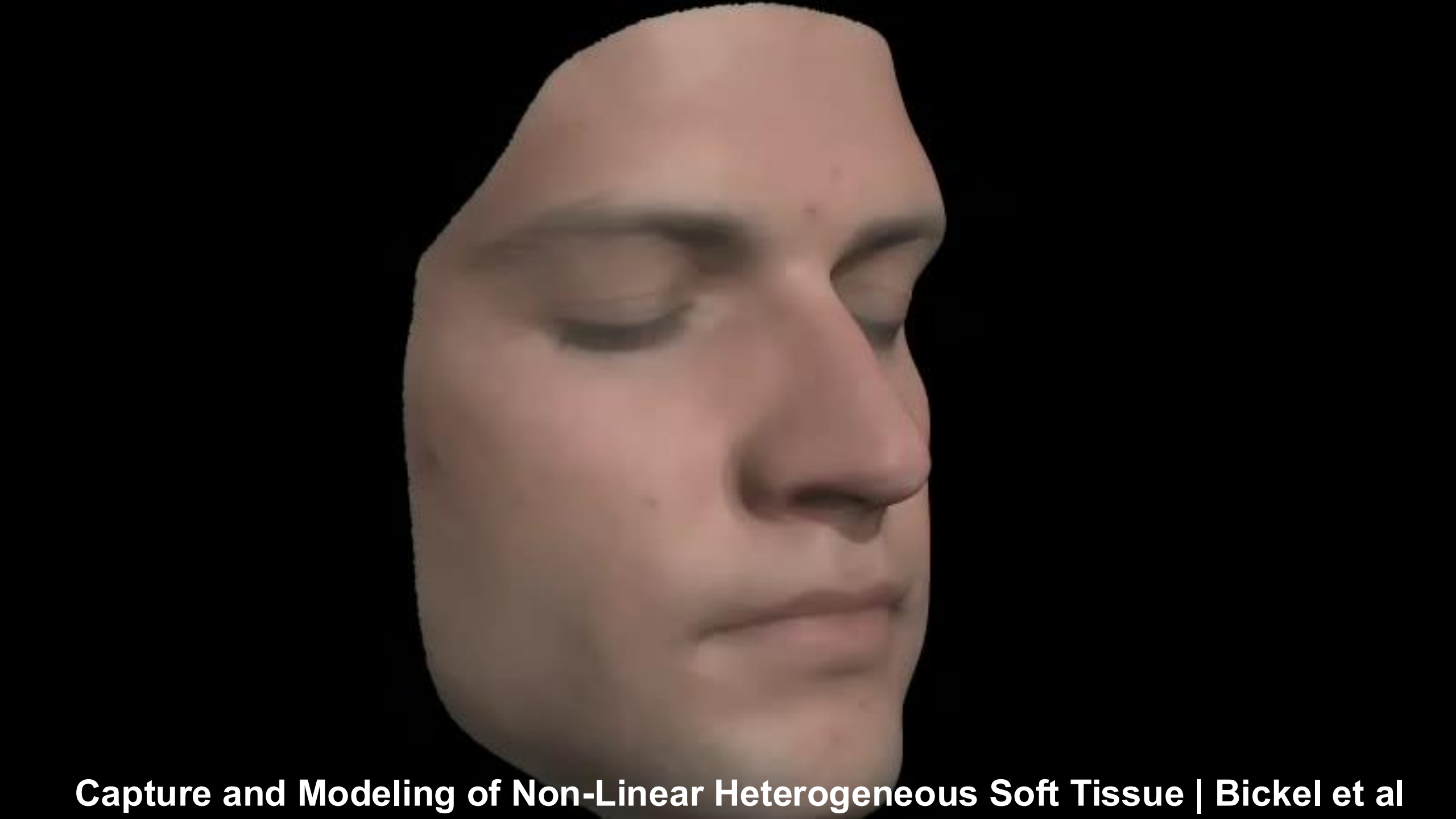


Capture and Modeling of Non-Linear Heterogeneous Soft Tissue | Bickel et al

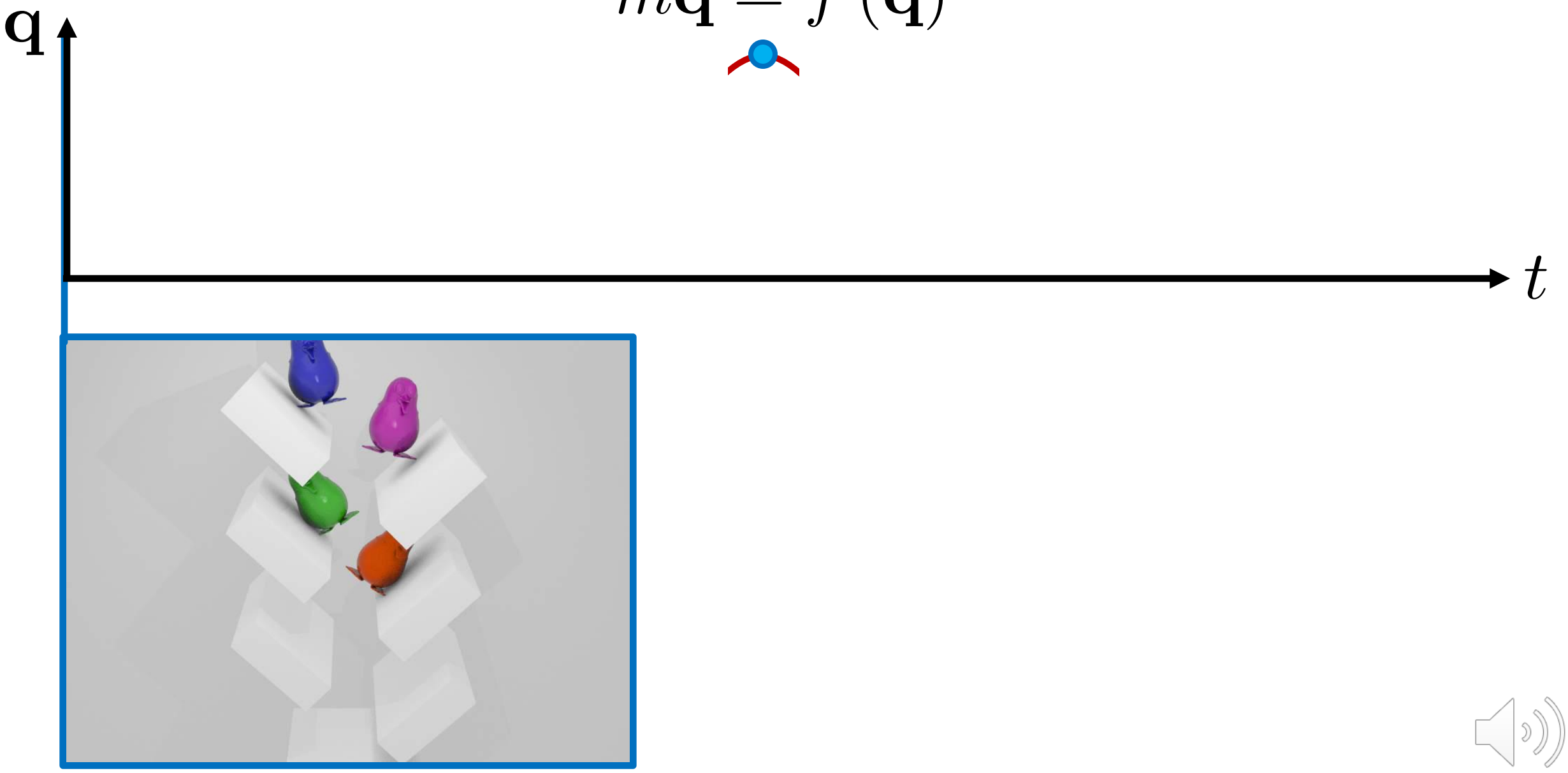
Questions from Previous Lecture ?

Equations of Motion

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Time Integration

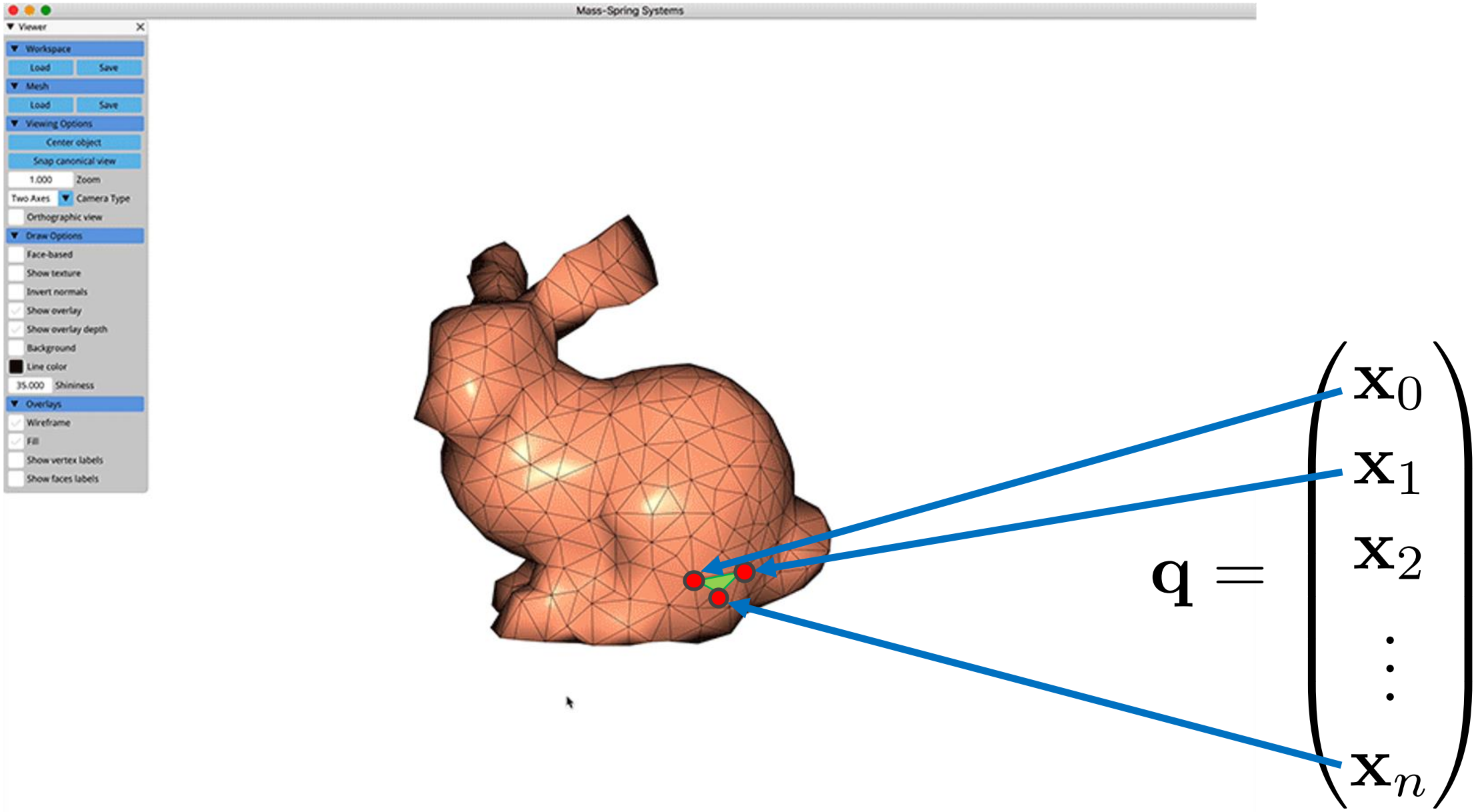
$$m\ddot{\mathbf{q}} = f(\mathbf{q})$$



How to Solve This ?

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

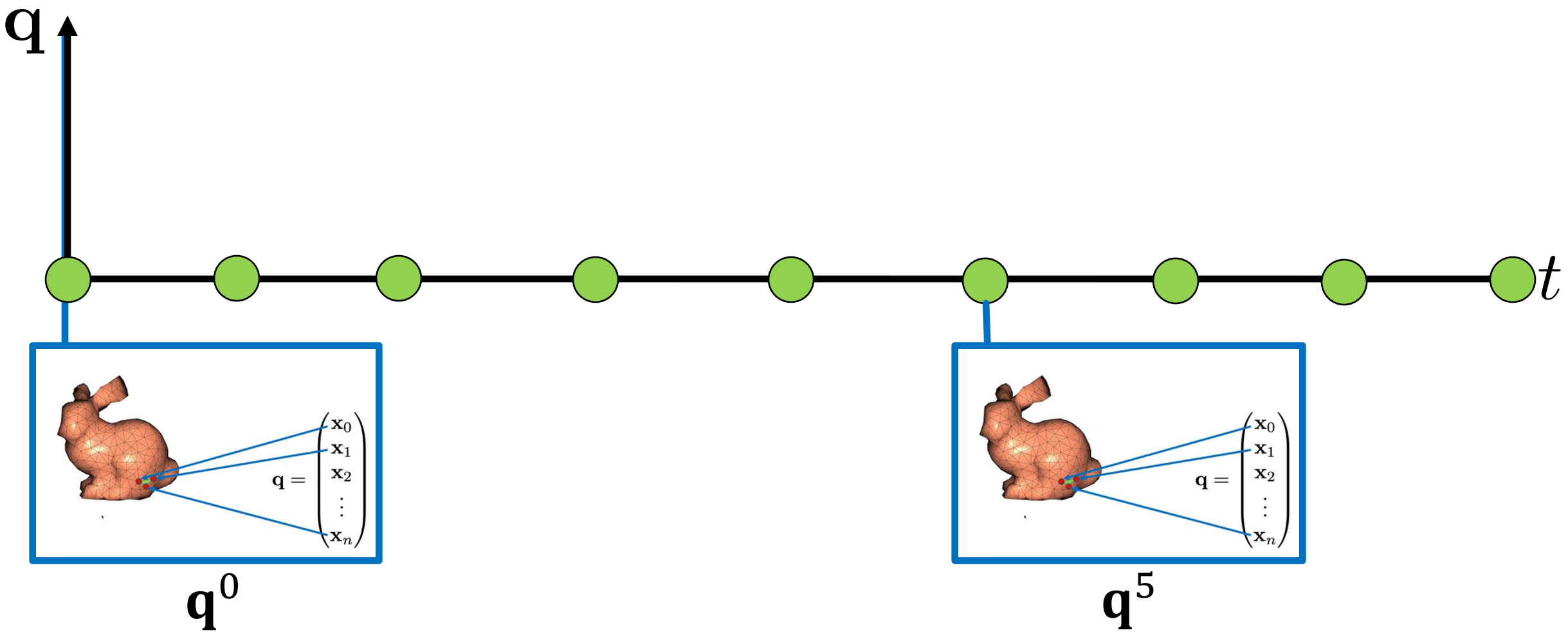
Spatial Discretization -- Finite Elements



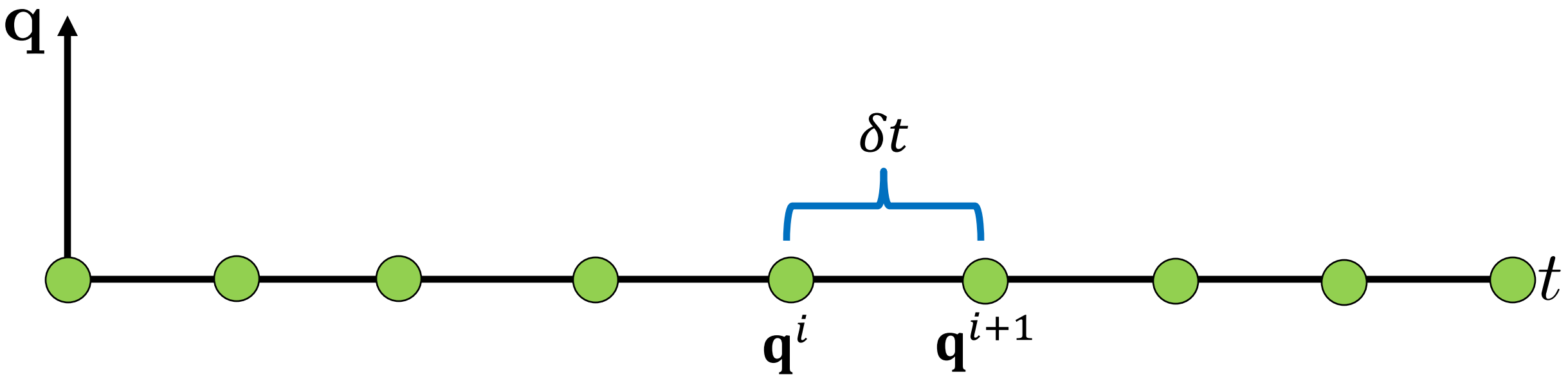
How to Solve This ?

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Time Discretization



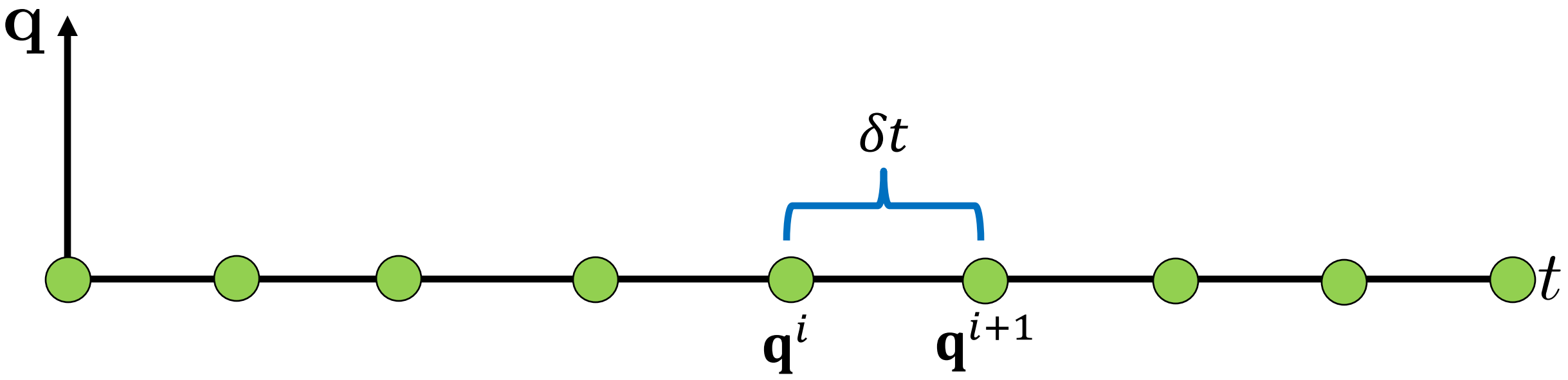
Time Discretization



Continuous Derivative

$$\dot{\boldsymbol{q}} = \lim_{t \to 0} \frac{\mathbf{q}(\mathbf{t} + \delta \mathbf{t}) - \mathbf{q}(\mathbf{t})}{\delta t}$$

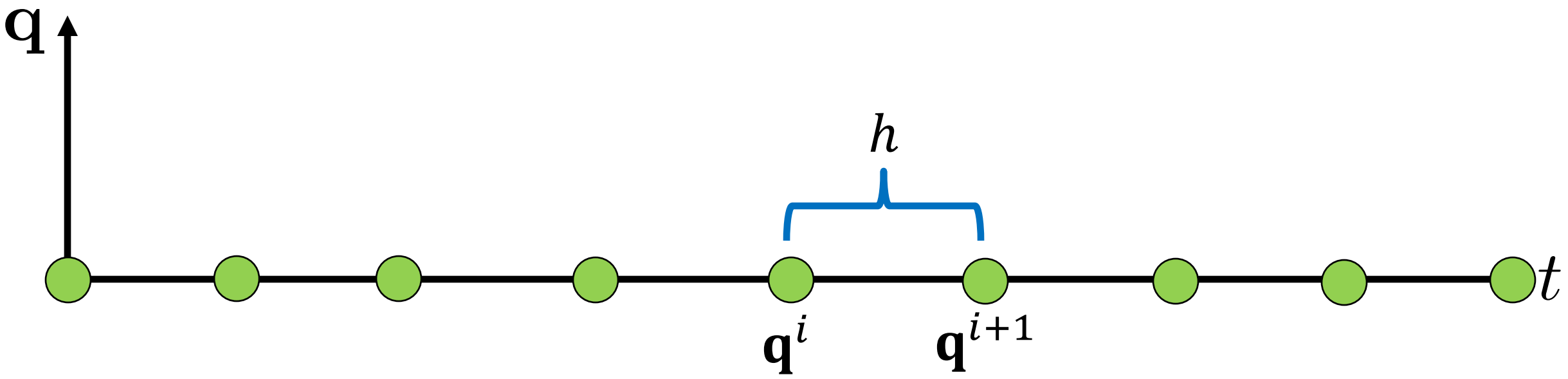
Time Discretization



Finite Difference Derivative

$$\dot{\mathbf{q}} = \frac{\boldsymbol{q}(\mathbf{t} + \delta\mathbf{t}) - \boldsymbol{q}(\mathbf{t})}{\delta t}$$

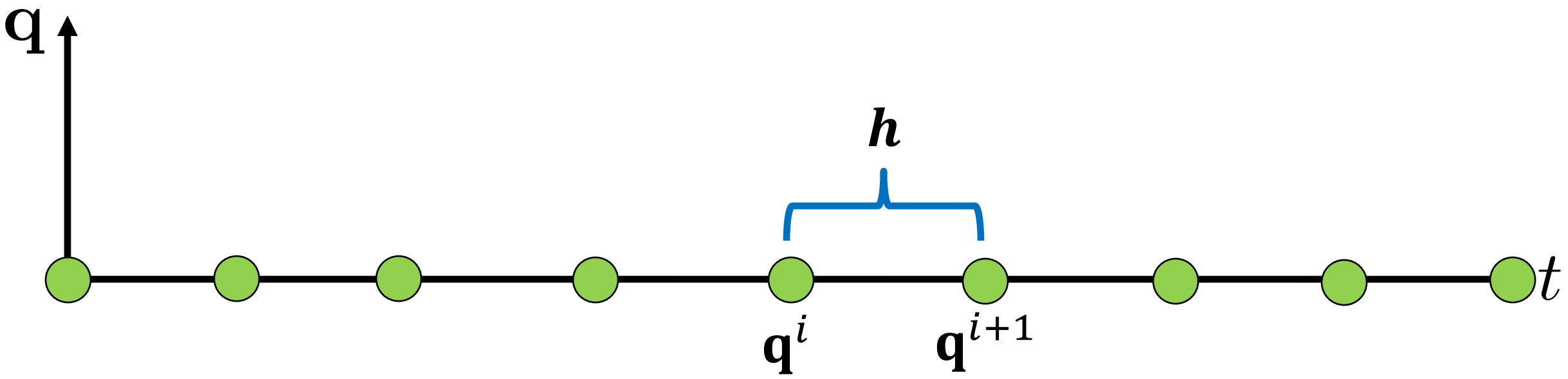
Time Discretization



Finite Difference Derivative

$$\dot{\mathbf{q}}^{i+1} = \frac{\mathbf{q}^{i+1} - \mathbf{q}^i}{h}$$

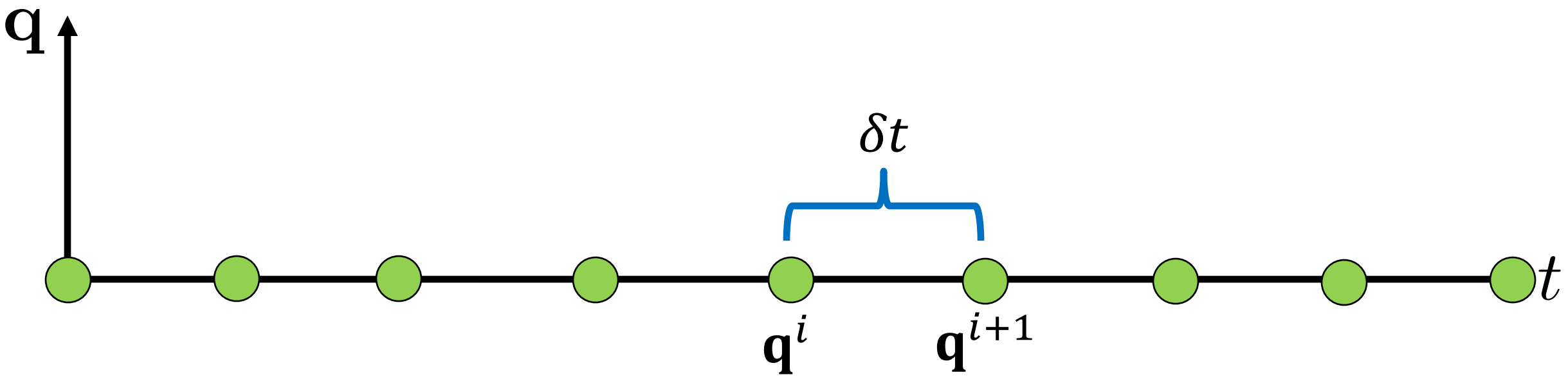
Time Discretization



Finite Difference

$$\ddot{\mathbf{q}} = ?\,?\,?\,?\,?$$

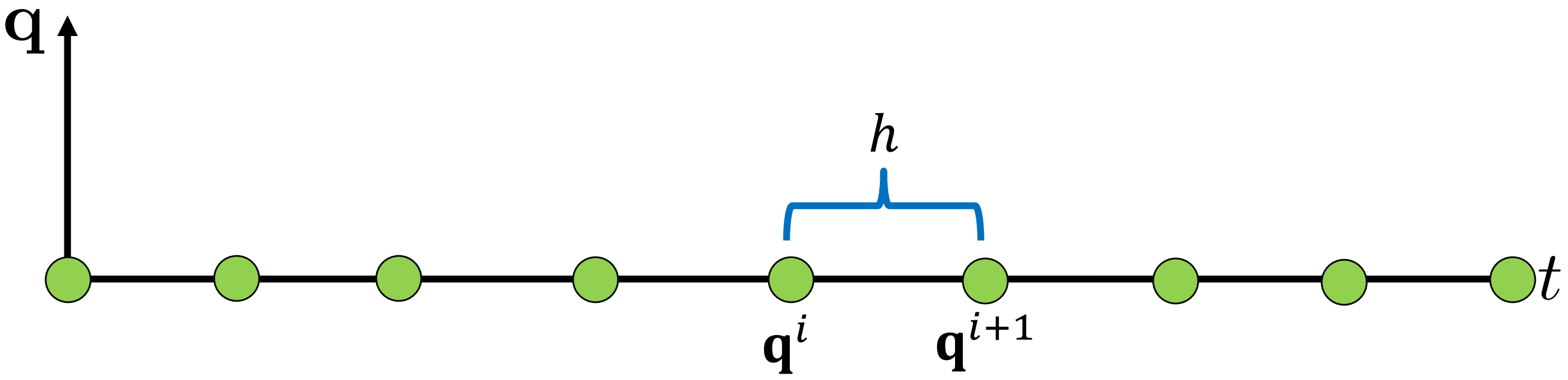
Time Discretization



Continuous Derivative

$$\ddot{\mathbf{q}} = \lim_{t \to 0} \frac{\dot{\mathbf{q}}(\mathbf{t} + \delta \mathbf{t}) - \dot{\mathbf{q}}(\mathbf{t})}{\delta t}$$

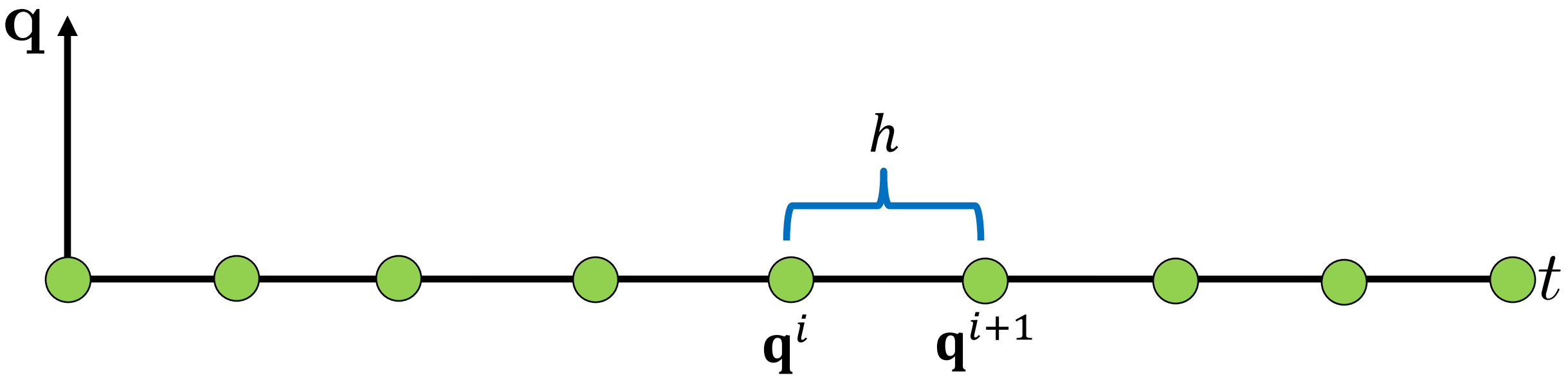
Time Discretization



Finite Difference Derivative

$$\ddot{\mathbf{q}}^{\boldsymbol{i}} = \frac{\dot{\mathbf{q}}^{i+1} - \dot{\mathbf{q}}^{i}}{h}$$

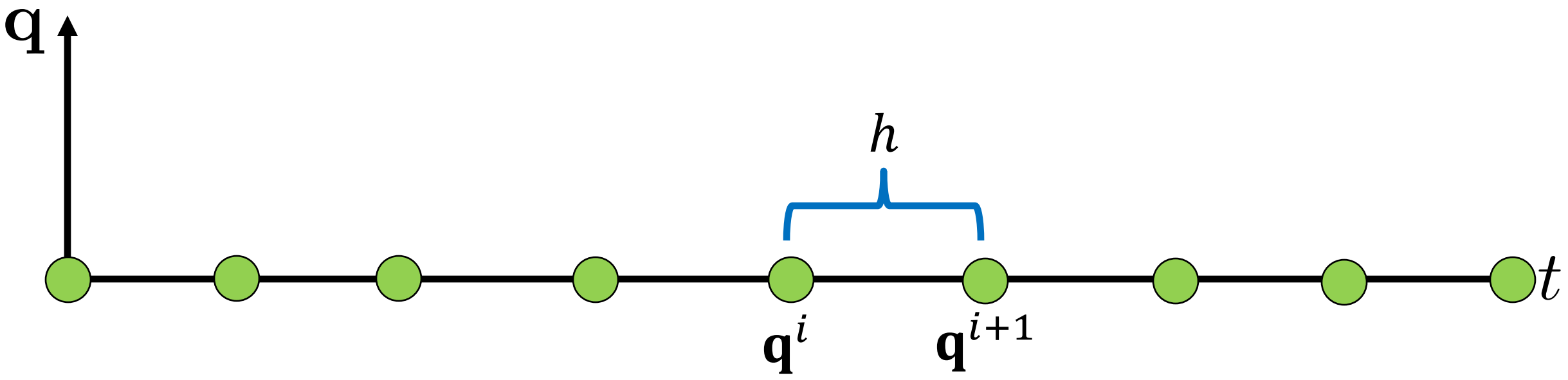
Time Discretization



Finite Difference Derivative

$$\ddot{\mathbf{q}}^i = \frac{1}{h}\left(\frac{\mathbf{q}^{i+1} - \mathbf{q}^i}{h} - \frac{\mathbf{q}^i - \mathbf{q}^{i-1}}{h}\right)$$

Time Discretization



Finite Difference Derivative

$$\ddot{\mathbf{q}}^{\boldsymbol{i}} = \frac{\mathbf{1}}{h^2}\left(\mathbf{q}^{i+1} - \underbrace{\mathbf{2q}^{i} + \mathbf{q}^{i-1}}_{\tilde{\mathbf{q}}^{i}}\right)$$

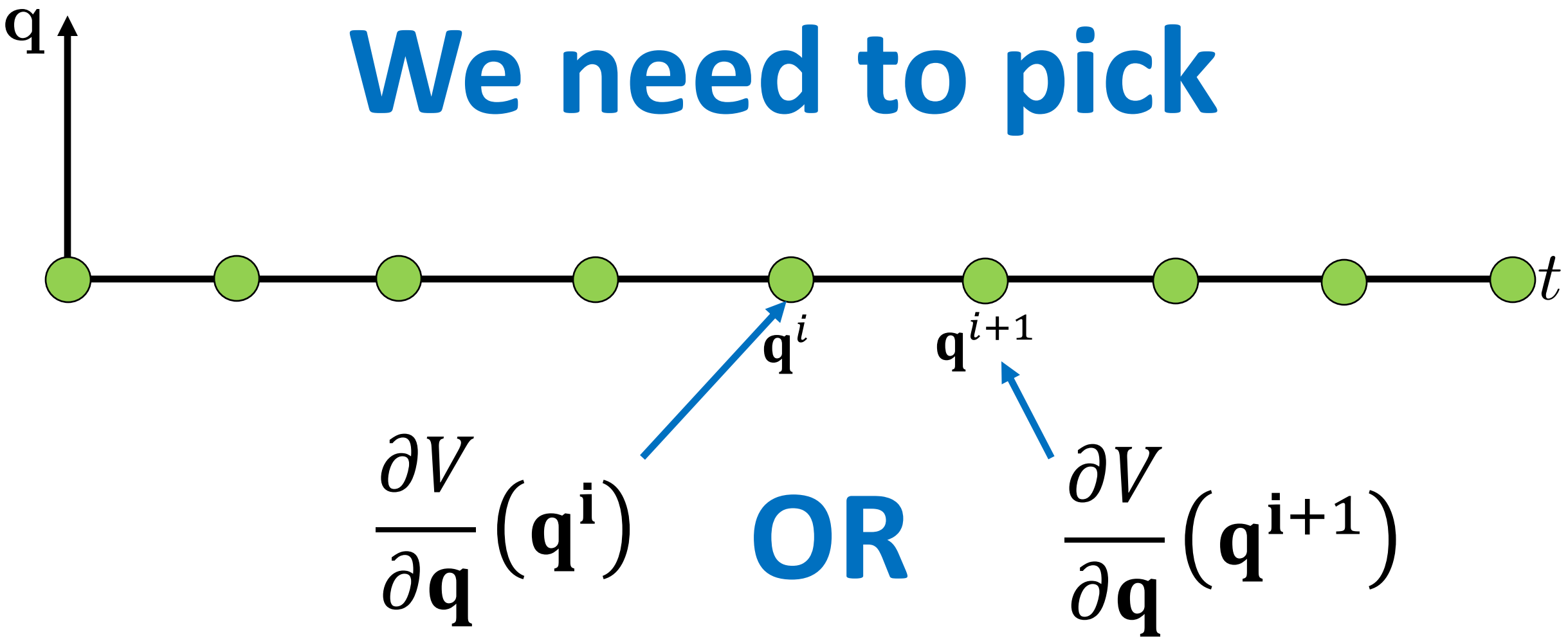
Discretize in Time

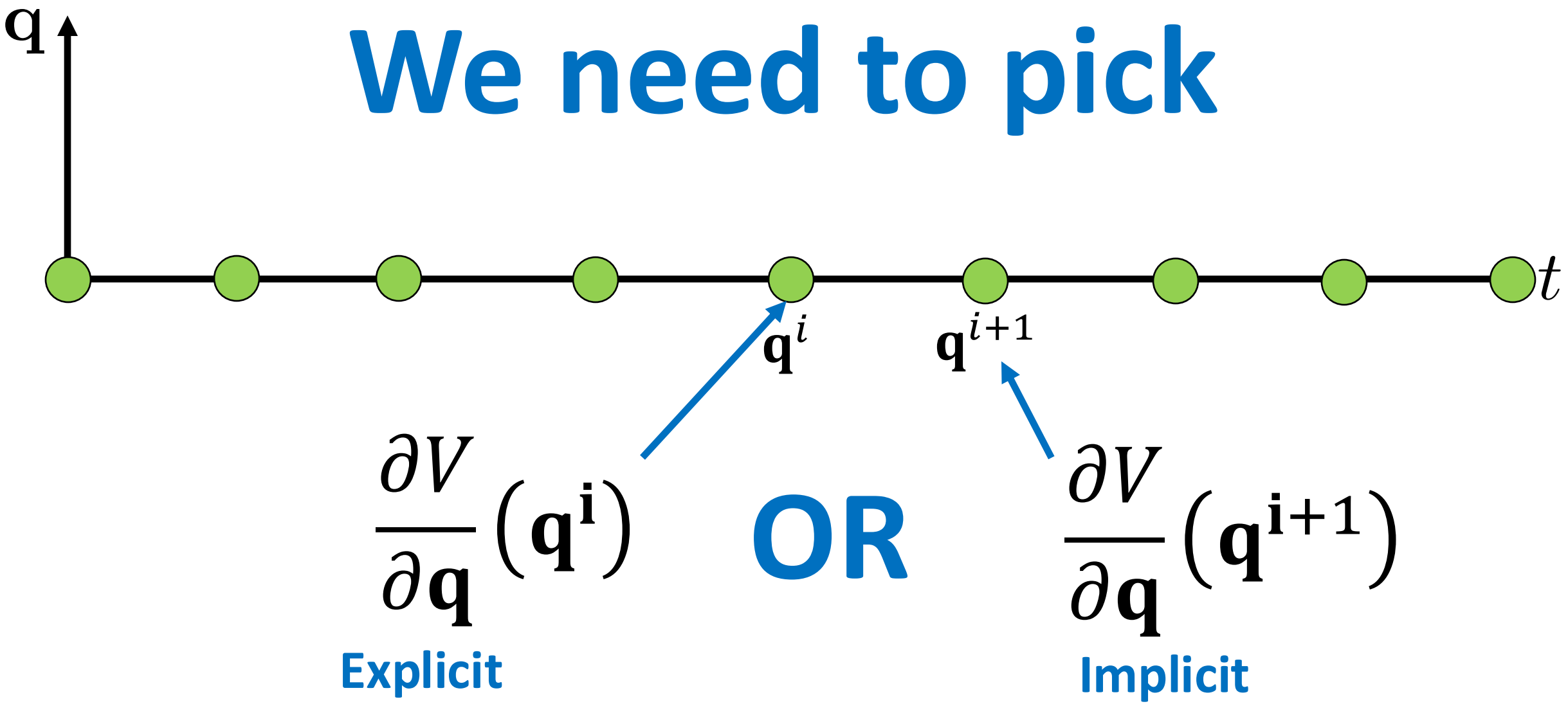
$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Discretize in Time

$$M \frac{1}{h^2}\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) = -\frac{\partial V}{\partial \mathbf{q}}$$

What about this ?


q
We need to pick
t
$\mathbf{q}^{i}$
$\mathbf{q}^{i+1}$
$\frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}^{\mathbf{i}})$
OR
$\frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}^{\mathbf{i+1}})$


q
We need to pick
t
$\mathbf{q}^{i}$
$\mathbf{q}^{i+1}$
$\frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}^{\mathbf{i}})$
OR
$\frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}^{\mathbf{i+1}})$
Explicit
Implicit

Explicit Integrator

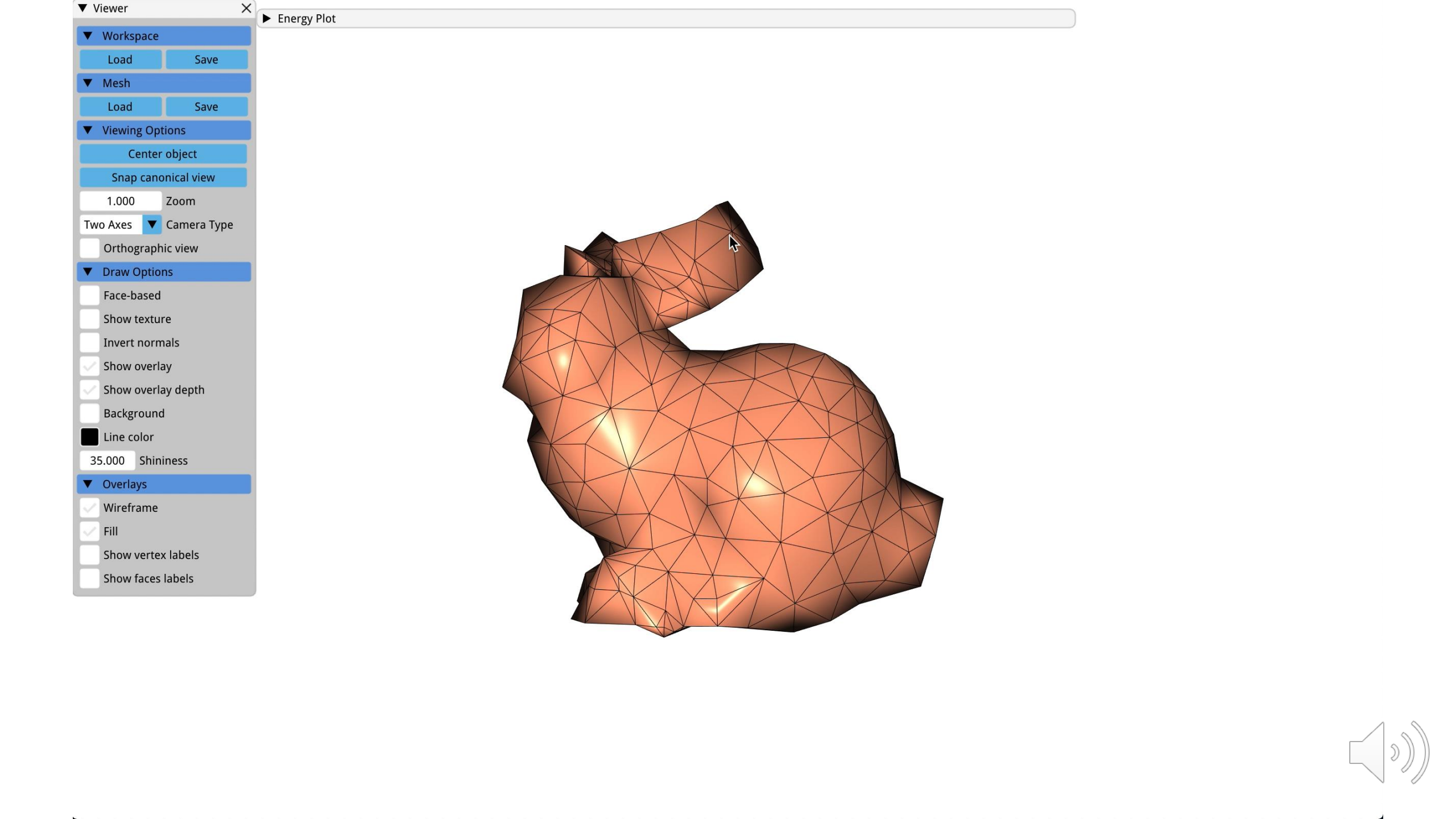
$$M\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) = -h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i}}\right)$$

Forward Euler

Implicit Integrator

$$M\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) = -h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right.$$

Backward Euler: More stable, harder to solve


Viewer
Energy Plot
Workspace
Load
Save
Mesh
Load
Save
Viewing Options
Center object
Snap canonical view
1.000
Zoom
Two Axes
Camera Type
Orthographic view
Draw Options
Face-based
Show texture
Invert normals
Show overlay
Show overlay depth
Background
Line color
35.000
Shininess
Overlays
Wireframe
Fill
Show vertex labels
Show faces labels

How do we Solve This ?

$$M\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) = -h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right.$$

Root Finding ?

$$M\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right) = 0$$

Nonlinear Root Finding is Hard

How do we Solve This ?

$$M\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right) = 0$$

What other type of problem involves finding the zero of a function?

Optimization

$$\nabla \mathbf{g}\left(\mathbf{q}^{\mathbf{i+1}}\right) = 0$$

Find where gradient is zero

How do we Solve This ?

$$M\left(\mathbf{q}^{\mathbf{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}\right) + h^2 \frac{\partial V}{\partial \mathbf{q}}\left(\mathbf{q}^{\mathbf{i+1}}\right) = 0$$

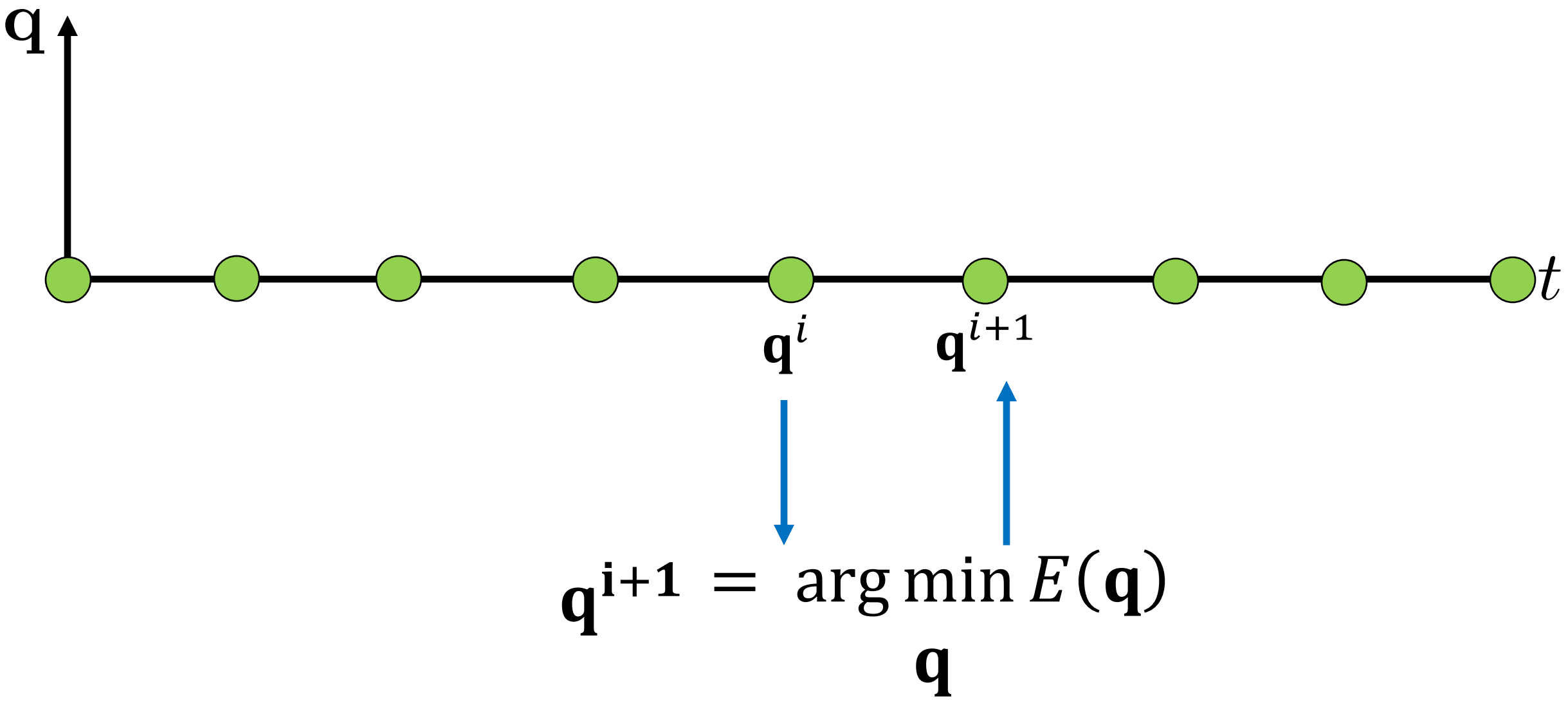
Gradient of what equals this ? Let's guess, then check

How do we Solve This ?

$$E(\mathbf{q^{i+1}}) = \frac{1}{2}(\mathbf{q^{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}})^{\mathrm{T}} M(\mathbf{q^{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}) + h^{2} V(\mathbf{q^{i+1}})$$

Gradient of what equals this ? Let's guess, then check

For each Time Step, Use Optimization !



$$\mathbf{q}^{\mathbf{i+1}} = \arg\min_{\mathbf{q}} E(\mathbf{q})$$

Viewer

Energy Plot

Workspace

Load Save

Mesh

Load Save

Viewing Options

Center object

Snap canonical view

1.000 Zoom

Two Axes Camera Type

Orthographic view

Draw Options

Face-based

Show texture

Invert normals

Show overlay

Show overlay depth

Background

Line color

35.000 Shininess

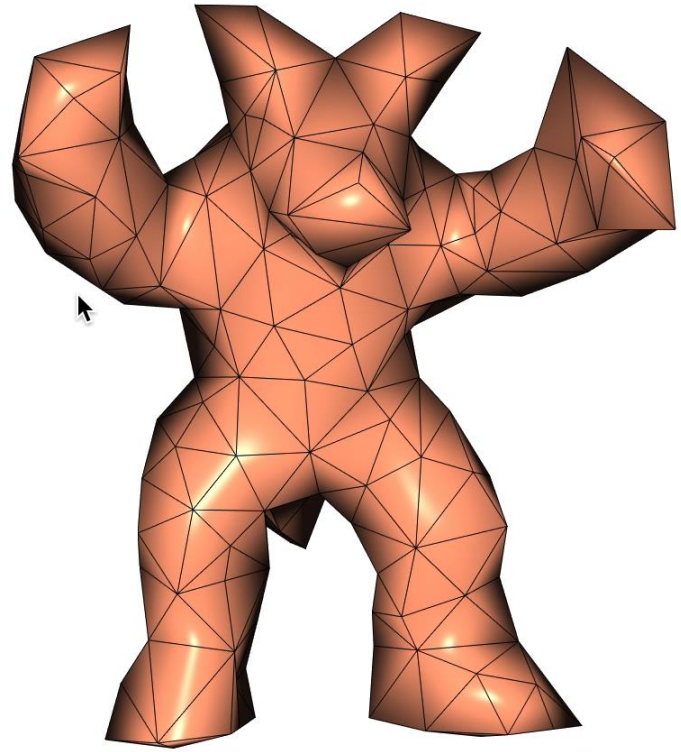
Overlays

Wireframe

Fill

Show vertex labels

Show faces labels



Viewer

Energy Plot

Workspace

Load | Save

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Load | Save

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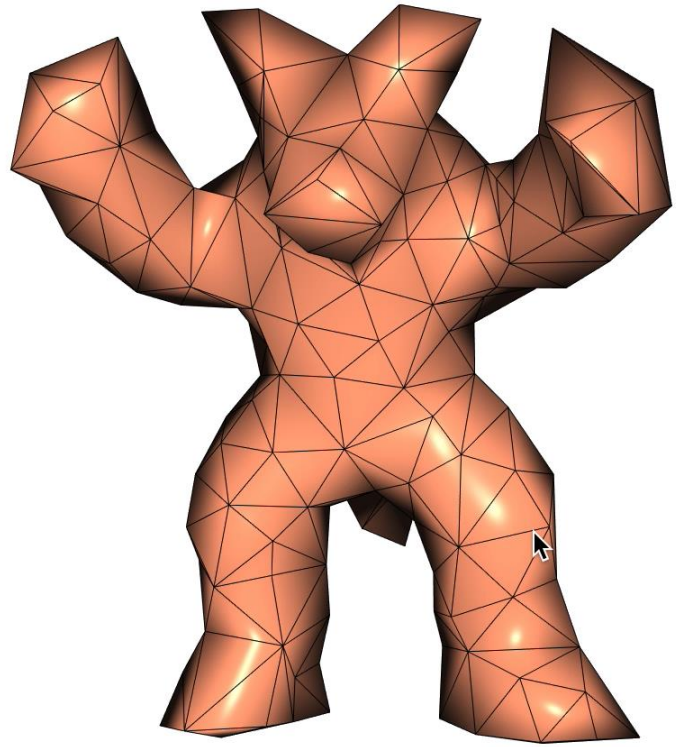


How do we Solve This ?

$$E(\mathbf{q^{i+1}}) = \frac{1}{2}(\mathbf{q^{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}})^{\mathrm{T}} M(\mathbf{q^{i+1}} - \tilde{\mathbf{q}}^{\mathbf{i}}) + h^{2} V(\mathbf{q^{i+1}})$$

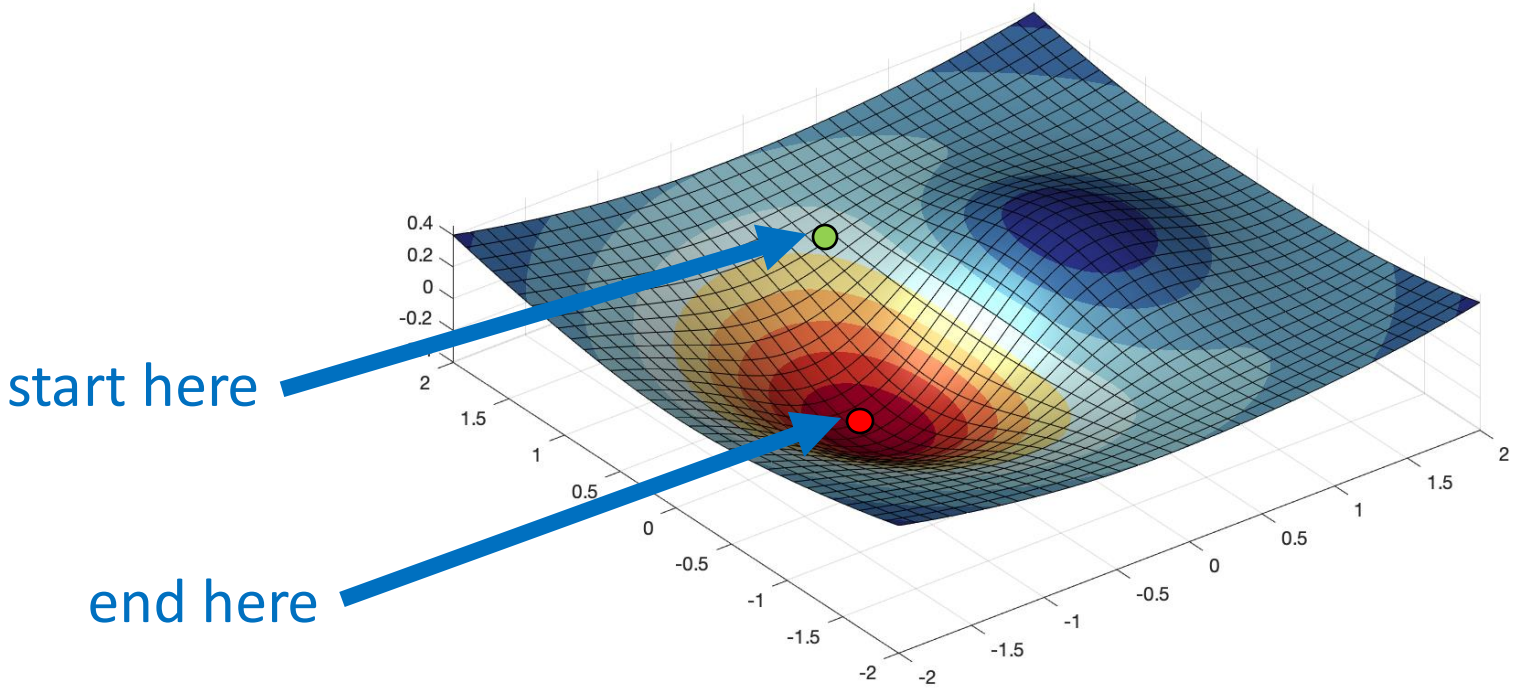
Gradient of what equals this ? Let's guess, then check

WARNING I DID NOT HAVE TIME TO RE-TYPESET THIS MATH SO THE OTPIMIZATION PROBLEM IS SLIGHTLY DIFFERENT FROM THE ONE ABOVE

BUT DON’T PANIC THE ALGORITHM IS STILL THE SAME !!!!!

Backward Euler using Optimization

$$\mathbf{v}^* = \arg\min_{\mathbf{v}} \underbrace{\frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t\mathbf{v}\right)}_{E(\mathbf{v})}$$



Gradient-Based Optimization

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

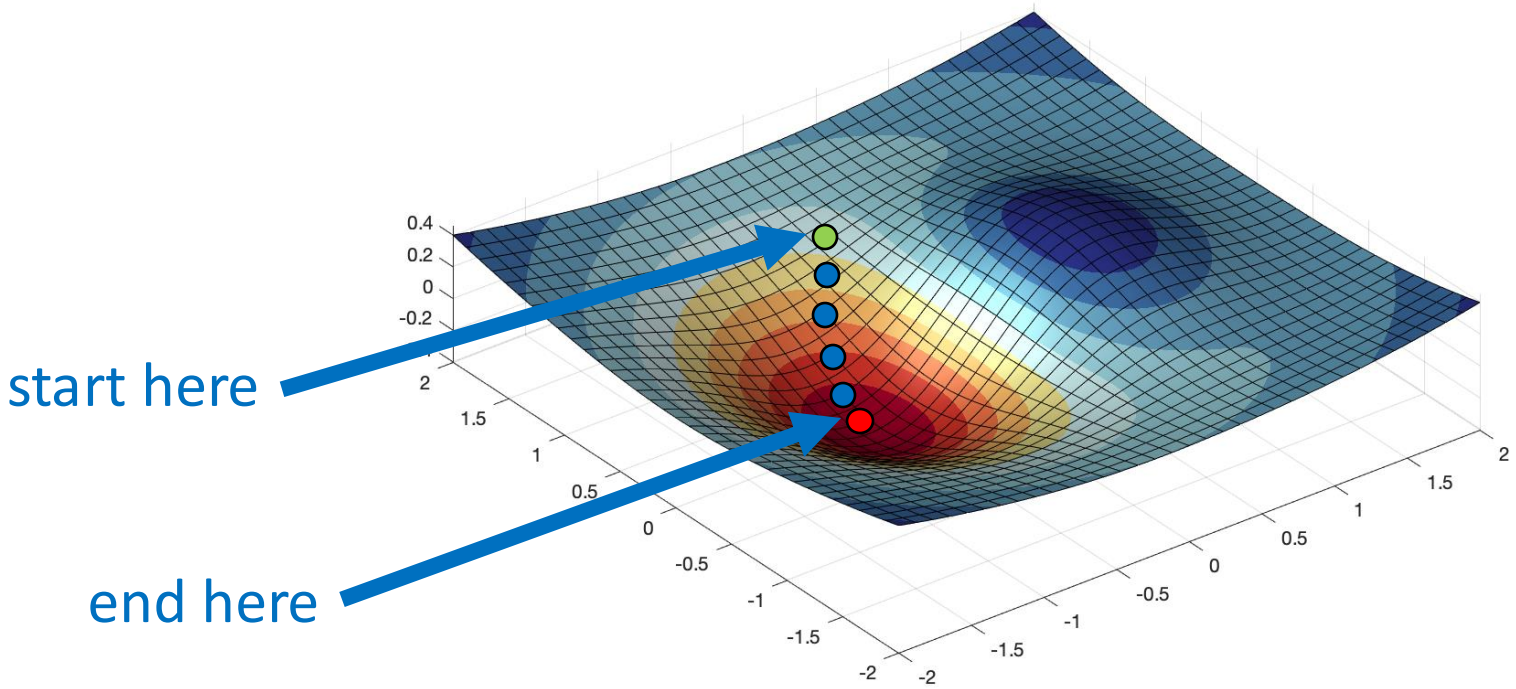
$$\mathbf{d} = \quad ??????$$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Backward Euler using Optimization

$$\mathbf{v}^* = \arg\min_{\mathbf{v}} \underbrace{\frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t\mathbf{v}\right)}_{E(\mathbf{v})}$$



Gradient Descent

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Repeat until converged

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

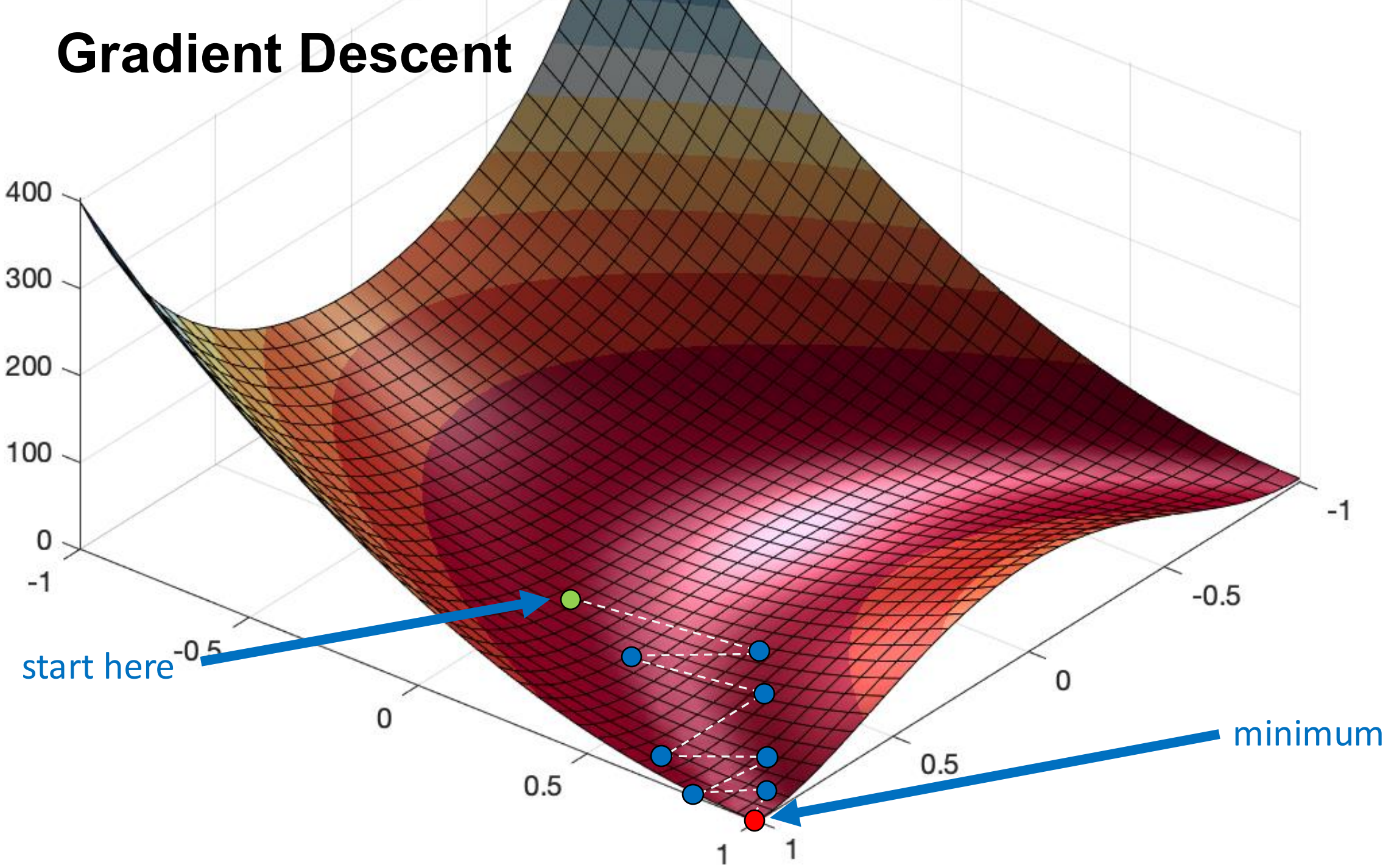
Choose search direction

$$\mathbf{d} = -\frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i}$$

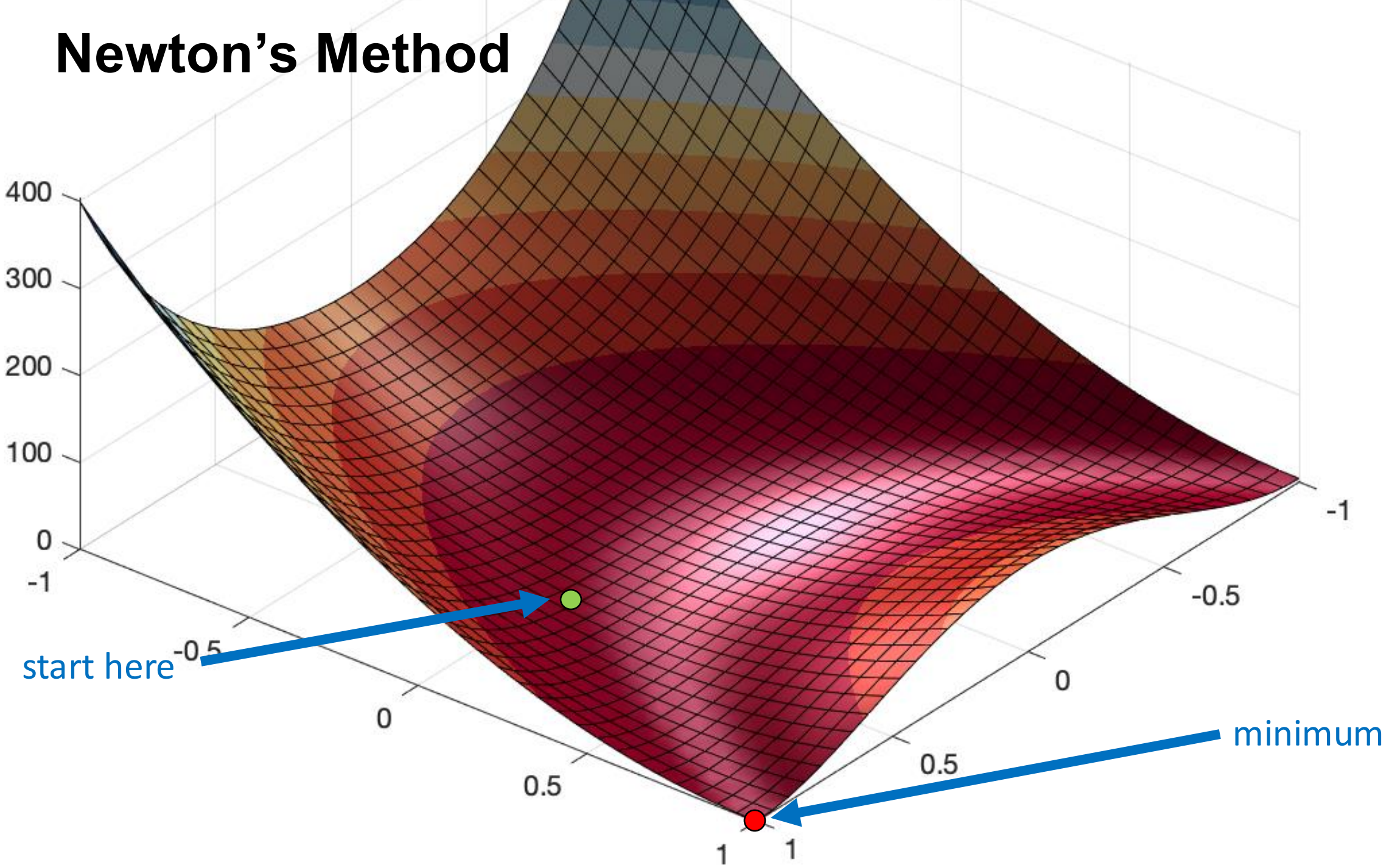
Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

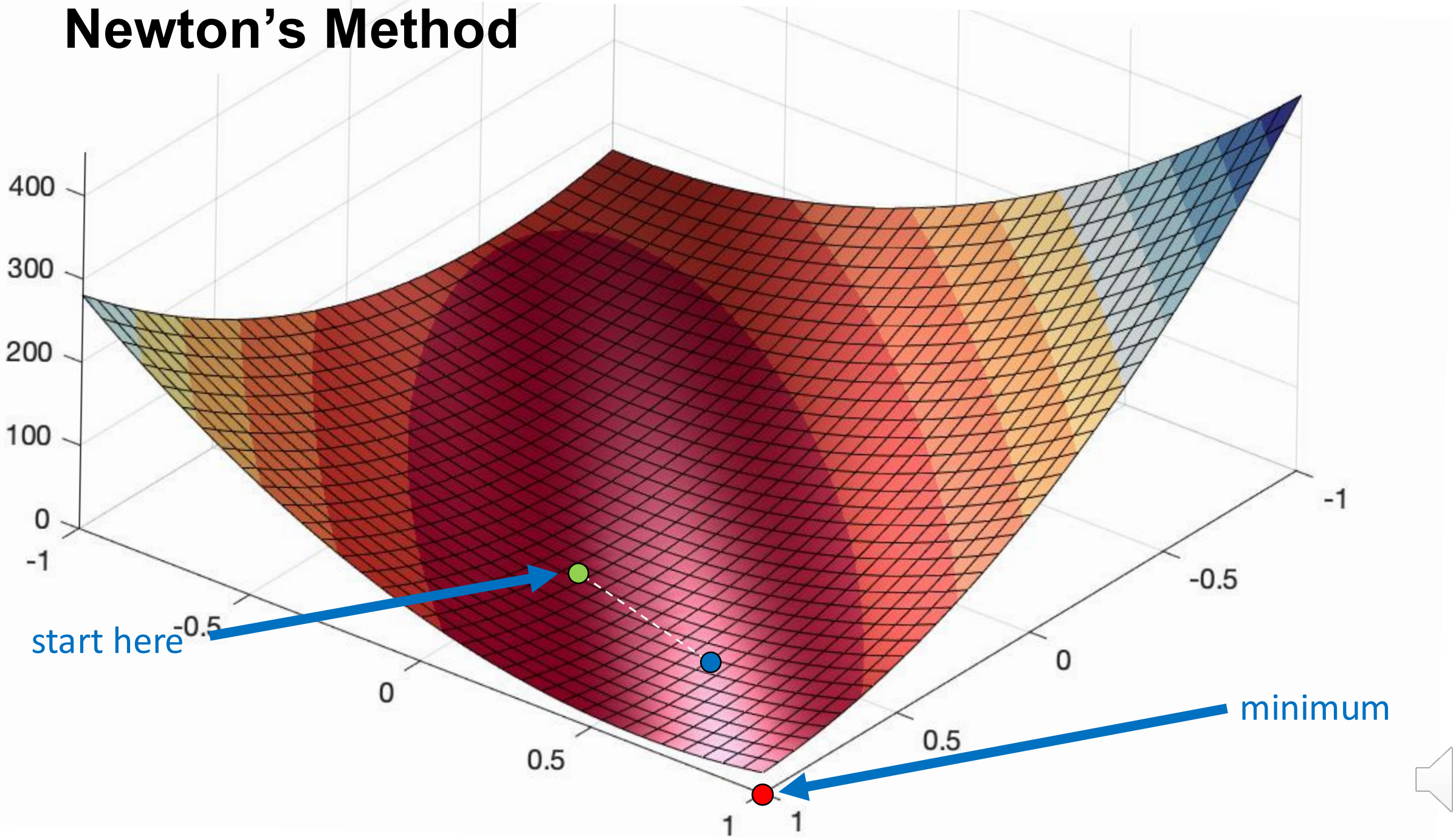
Gradient Descent



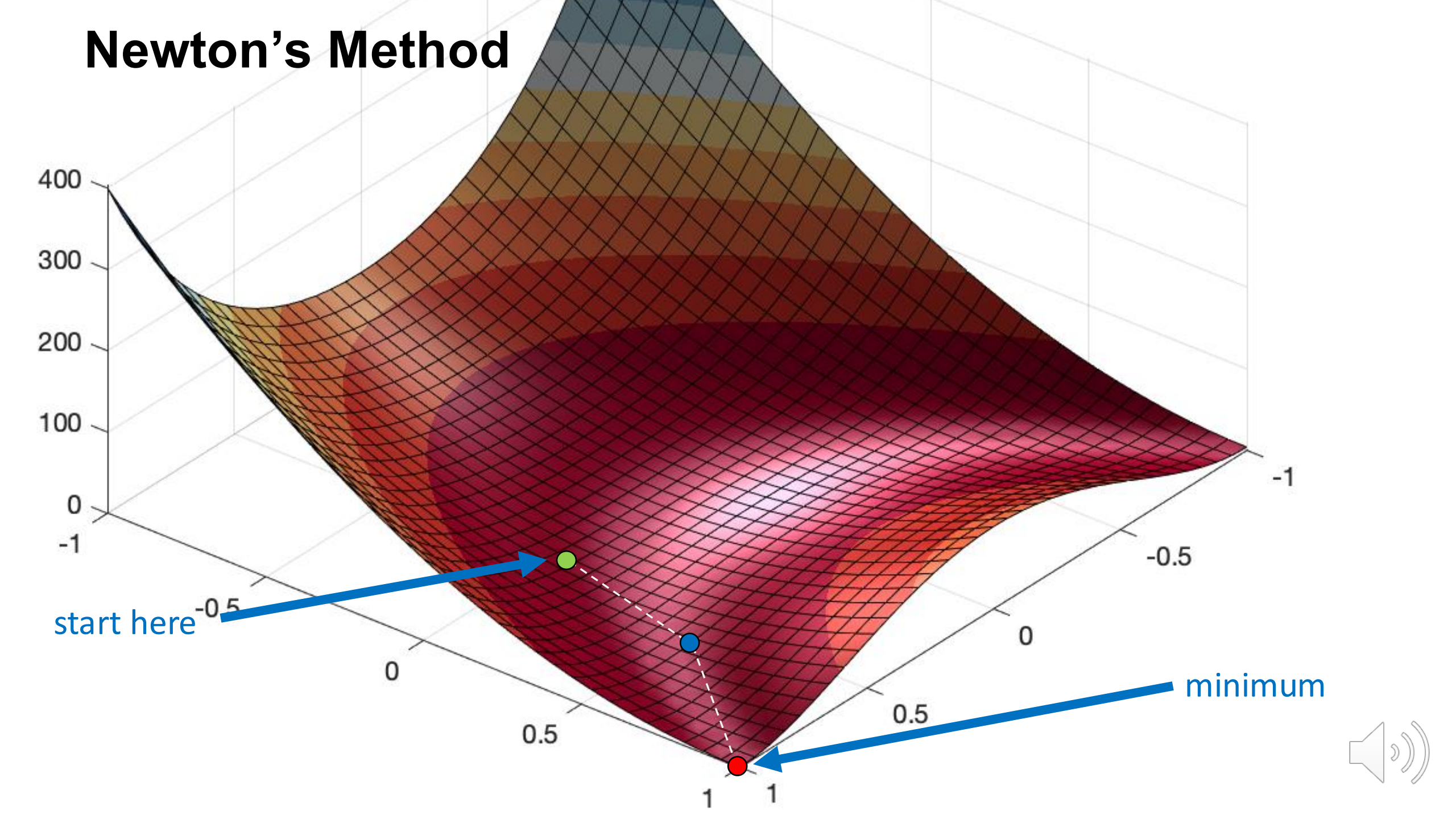
Newton's Method



Newton's Method



Newton's Method



Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{d} = \quad \text{Minimize quadratic approximation}$$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Repeat until converged

Aside: Quadratic Approximations

$$\mathbf{y}^* = \arg\min_{\mathbf{y}} E(\mathbf{y})$$

generic optimization problem

optimal solution

energy/cost function

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} E\left(\mathbf{y}^i + \tilde{\mathbf{d}}\right)$$

Second Order Taylor Approximation

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \frac{1}{2}\tilde{\mathbf{d}}^T \mathrm{H}^i \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \mathbf{g}^i + E^i$$

$$\frac{\partial^2 E}{\partial \mathbf{y}^2}\Big|_{\mathbf{y}^i} \qquad \frac{\partial E}{\partial \mathbf{y}}\Big|_{\mathbf{y}^i} \qquad E\left(\mathbf{y}^i\right)$$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \frac{1}{2}\tilde{\mathbf{d}}^T \mathrm{H}^i \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \mathbf{g}^i \,.$$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Repeat until converged

Aside: Minimizing a Quadratic Function

$$\mathbf{d} = \arg\min_{\tilde{\mathbf{d}}} \frac{1}{2}\tilde{\mathbf{d}}^T \mathrm{H}^i \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \mathbf{g}^i$$

Minimum when gradient is equal to zero

$$\mathrm{H}^i \mathbf{d} + \mathbf{g}^i = \mathbf{0}$$

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Repeat until converged

Gradient and Hessian

$$E(\mathbf{v}) = \frac{1}{2}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right)^T \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + V\left(\mathbf{q}^t + \Delta t\mathbf{v}\right)$$

$$\frac{\partial E}{\partial \mathbf{v}} = \mathrm{M}\left(\mathbf{v} - \dot{\mathbf{q}}^t\right) + \Delta t \frac{\partial V}{\partial \mathbf{q}}\Big|_{\mathbf{q}^t + \Delta t\mathbf{v}}$$

negative generalized forces

$$\frac{\partial^2 E}{\partial \mathbf{v}^2} = \mathrm{M} + \Delta t^2 \frac{\partial^2 V}{\partial \mathbf{q}^2}\Big|_{\mathbf{q}^t + \Delta t\mathbf{v}}$$

negative stiffness matrix

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Repeat until converged

Viewer

Energy Plot

Workspace

Load Save

Mesh

Load Save

Viewing Options

Center object

Snap canonical view

1.000 Zoom

Two Axes Camera Type

Orthographic view

Draw Options

Face-based

Show texture

Invert normals

Show overlay

Show overlay depth

Background

Line color

35.000 Shininess

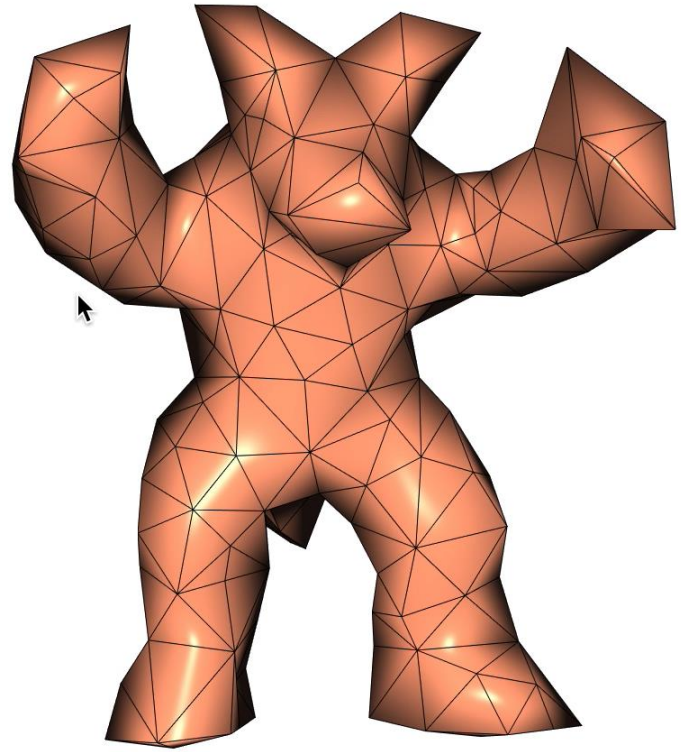
Overlays

Wireframe

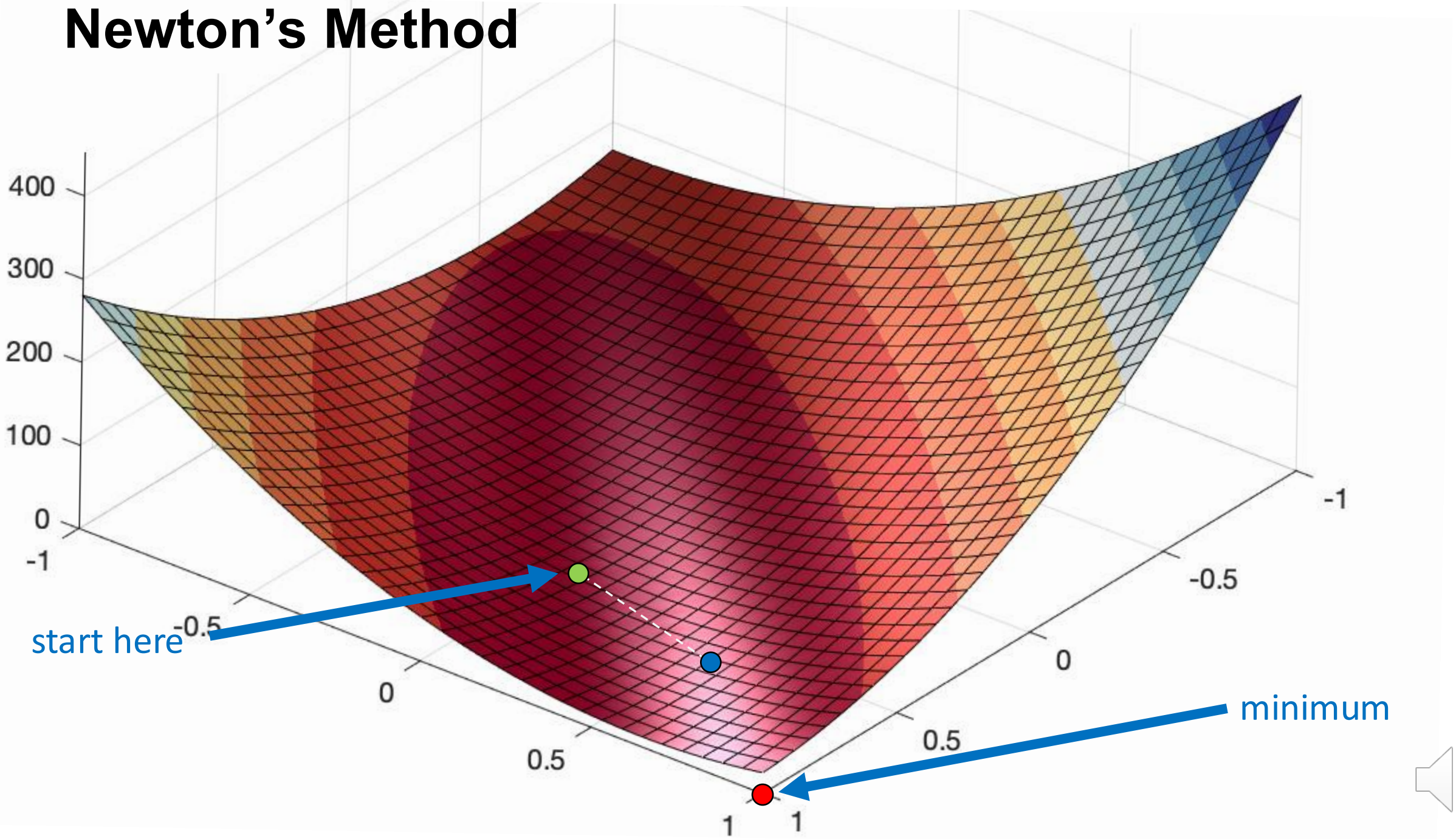
Fill

Show vertex labels

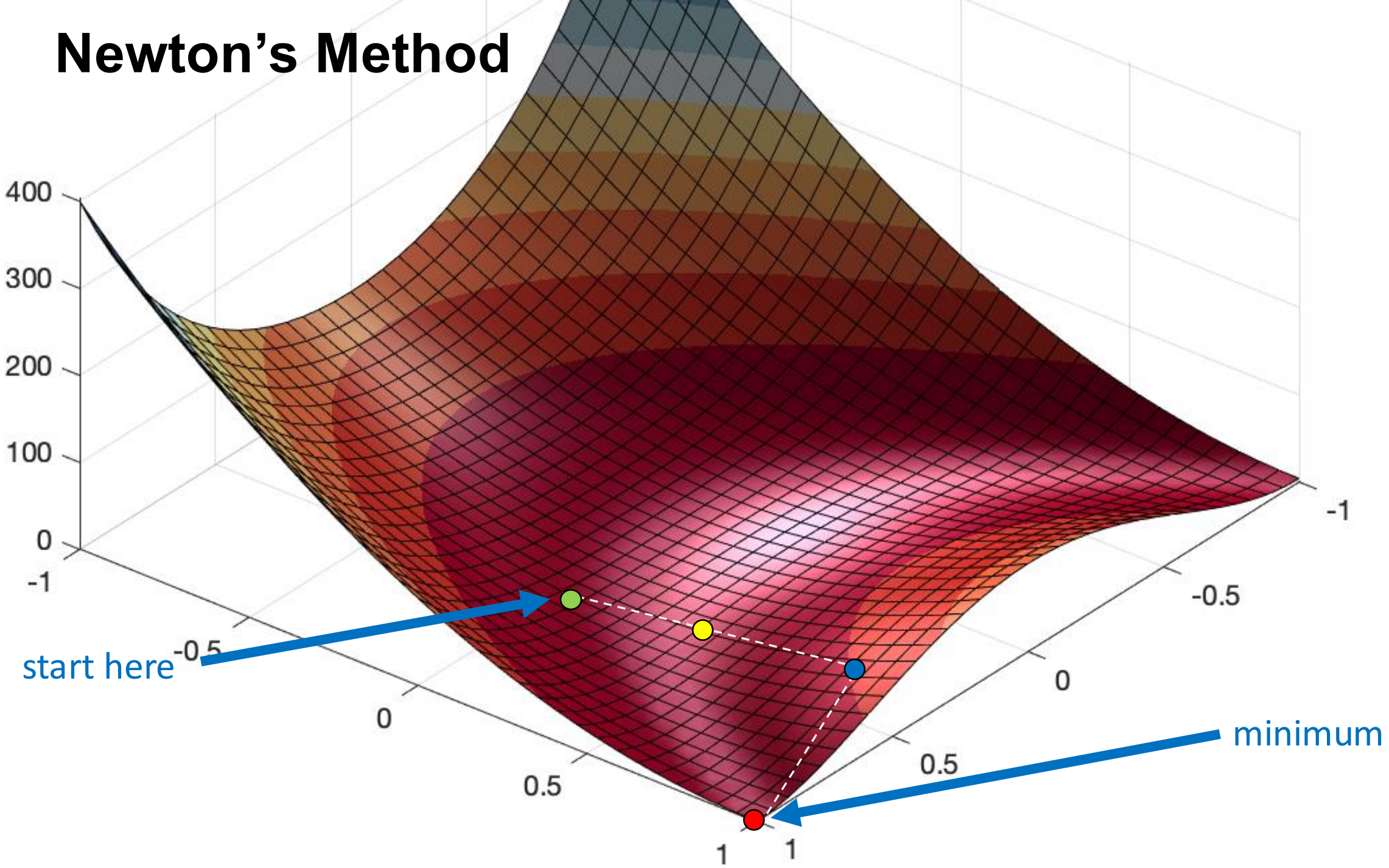
Show faces labels



Newton's Method



Newton's Method



Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$

Solve linear system to get $\mathbf{d}$

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Repeat until converged

Newton's Method

Choose an initial guess

$$i = 0$$
$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}}\Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathrm{H}^i \mathbf{d} = -\mathbf{g}^i$$ Solve linear system to get $\mathbf{d}$

Choose α using line search

Use search direction to update current guess

$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$
$$i = i + 1$$

Repeat until converged

Line Search

$$\mathbf{y}^* = \arg\min_{\mathbf{y}} E(\mathbf{y})$$

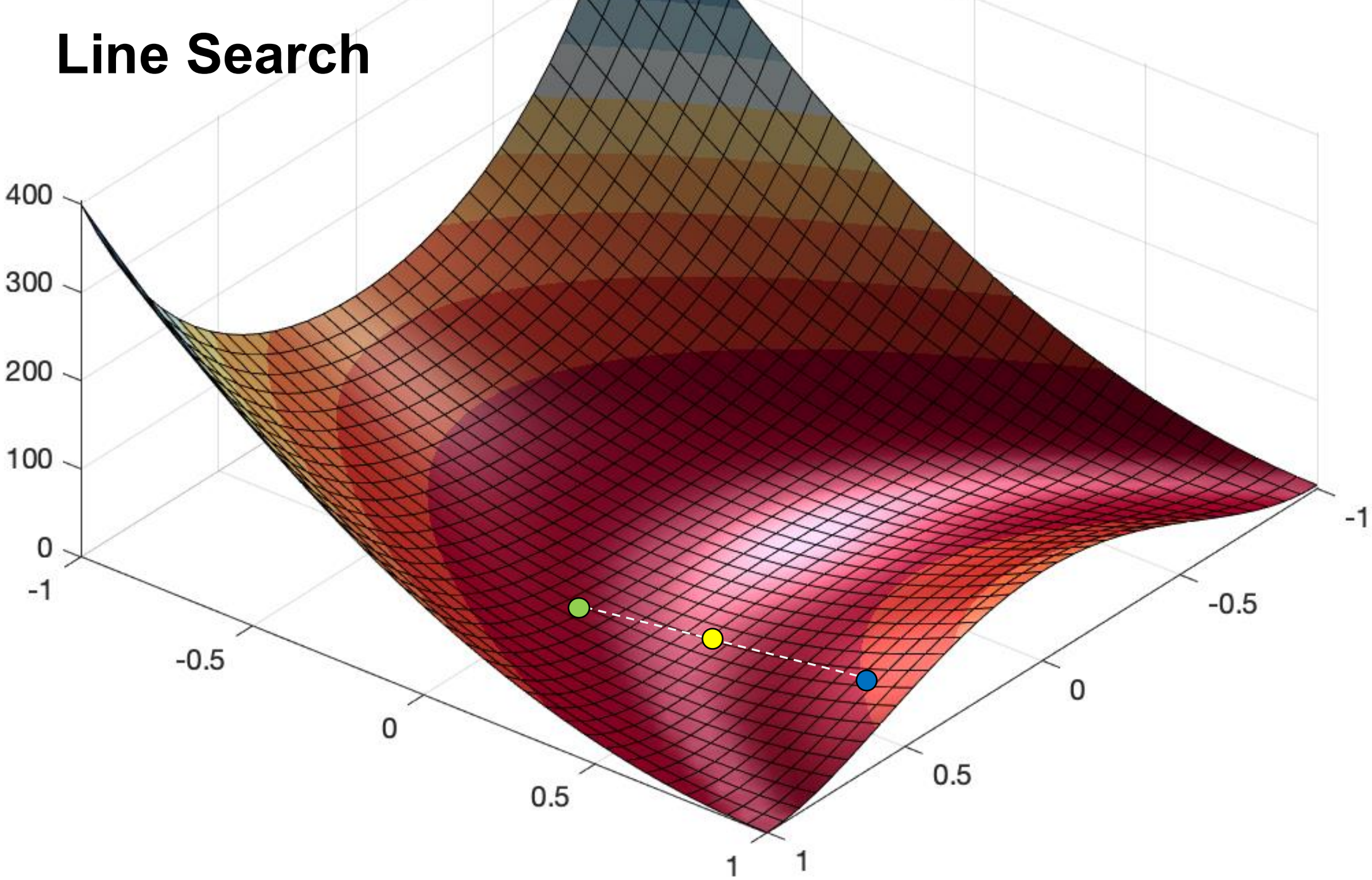
Once we compute $\mathbf{d}$...

One-dimensional optimization

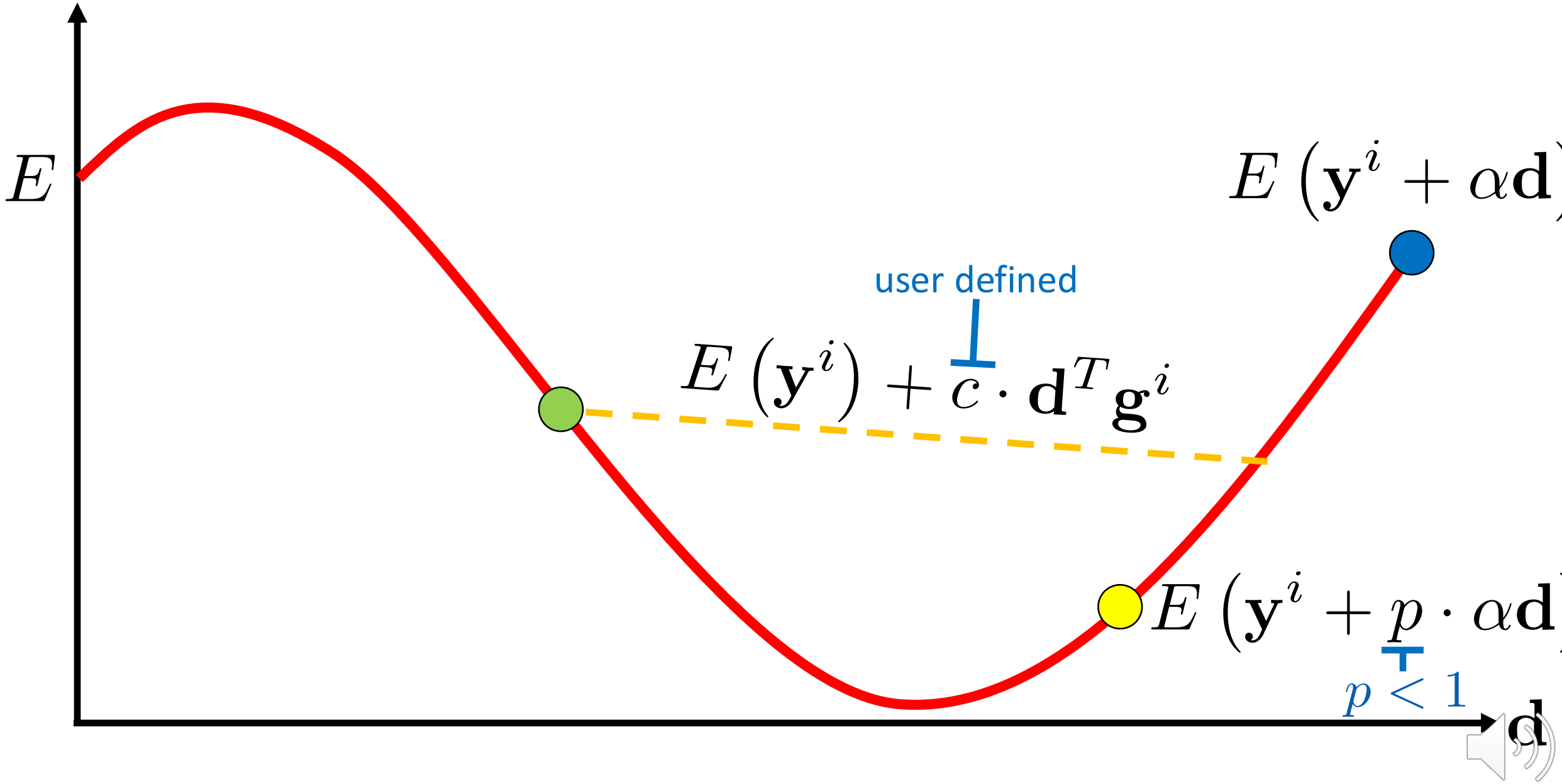
$$\alpha^* = \arg\min_{\alpha} E(\mathbf{y}^i + \alpha\mathbf{d})$$

Solve by 1D Search

Line Search



Line Search


E
$E\left(\mathbf{y}^i + \alpha\mathbf{d}\right)$
user defined
$E\left(\mathbf{y}^i\right) + c \cdot \mathbf{d}^T\mathbf{g}^i$
$E\left(\mathbf{y}^i + p \cdot \alpha\mathbf{d}\right)$
$p < 1$
$\mathbf{d}$

Backtracking Line Search

Choose an initial guess

$$\alpha = \alpha_{max}$$

Repeat

Stop if sufficient decrease achieved or you get stuck

$$E\left(\mathbf{v}^i + \alpha\mathbf{d}\right) <= E\left(\mathbf{v}^i\right) + c \cdot \mathbf{d}^T\mathbf{g}^i \text{ or } \alpha < \text{tol}$$

Reduce α

$$\alpha = p \cdot \alpha$$

Viewer

Energy Plot

Workspace

Load | Save

Mesh

Load | Save

Viewing Options

Center object

Snap canonical view

1.000 Zoom

Two Axes Camera Type

Orthographic view

Draw Options

Face-based

Show texture

Invert normals

Show overlay

Show overlay depth

Background

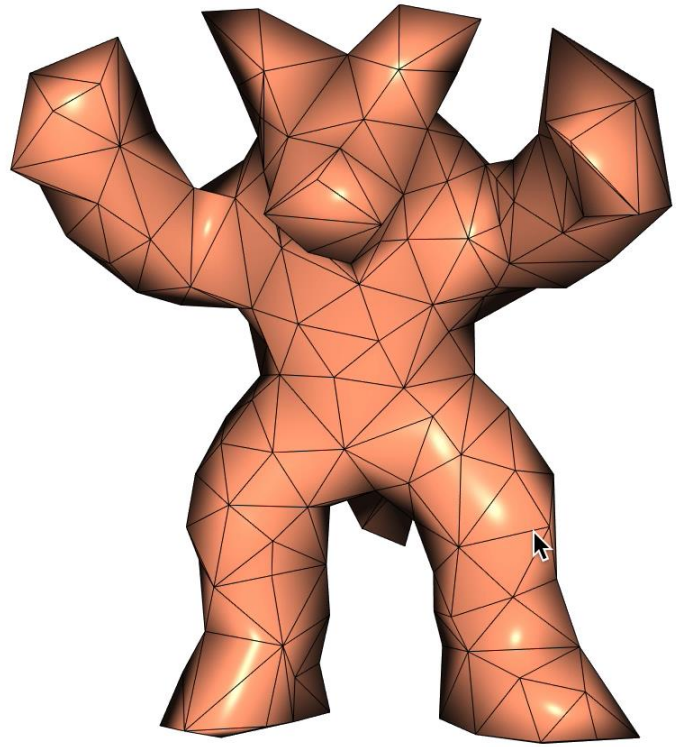
Line color

35.000 Shininess

Overlays

Wireframe

Fill

Show vertex labels

Show faces labels