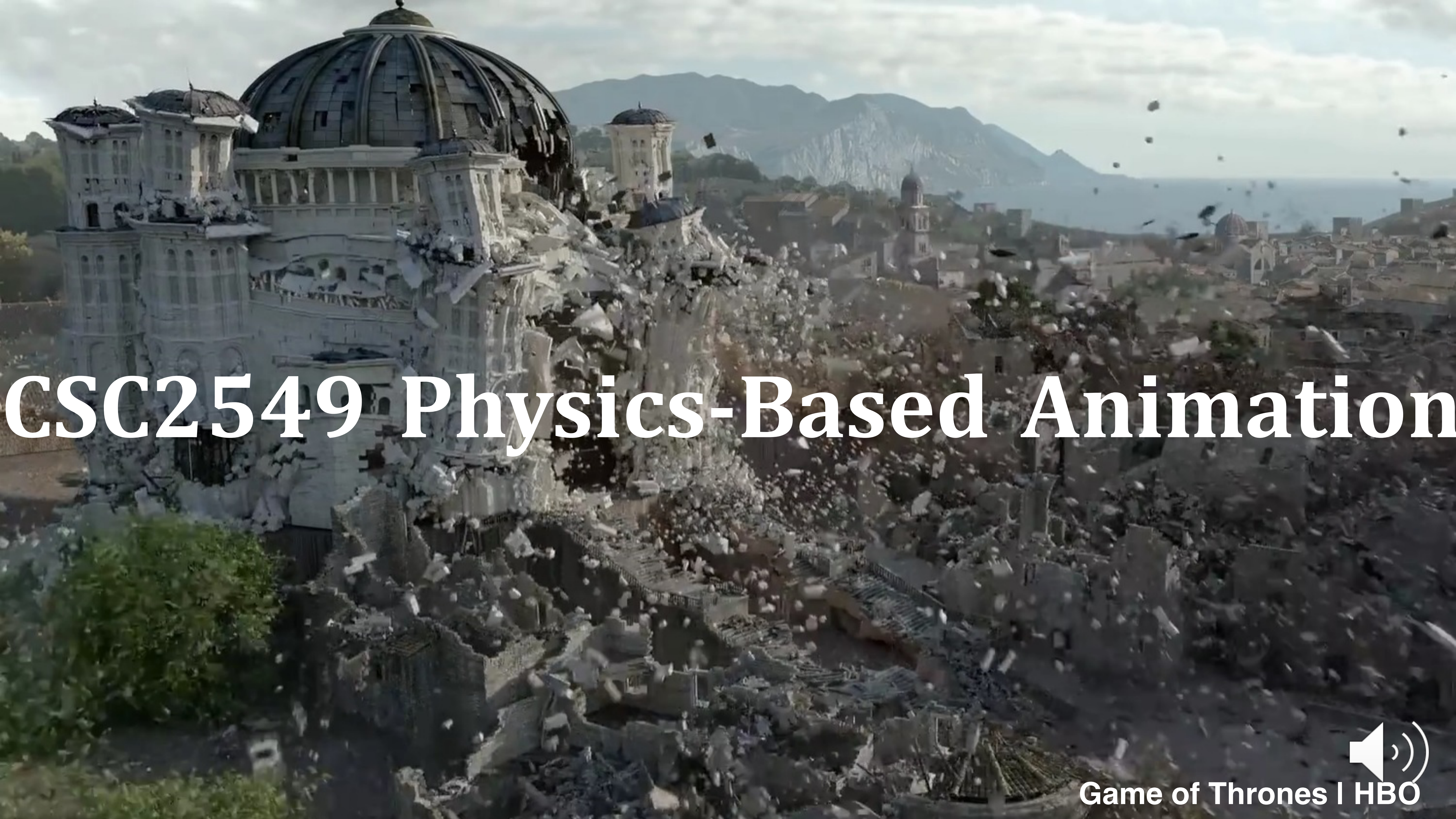
CSC2549 Physics-Based Animation

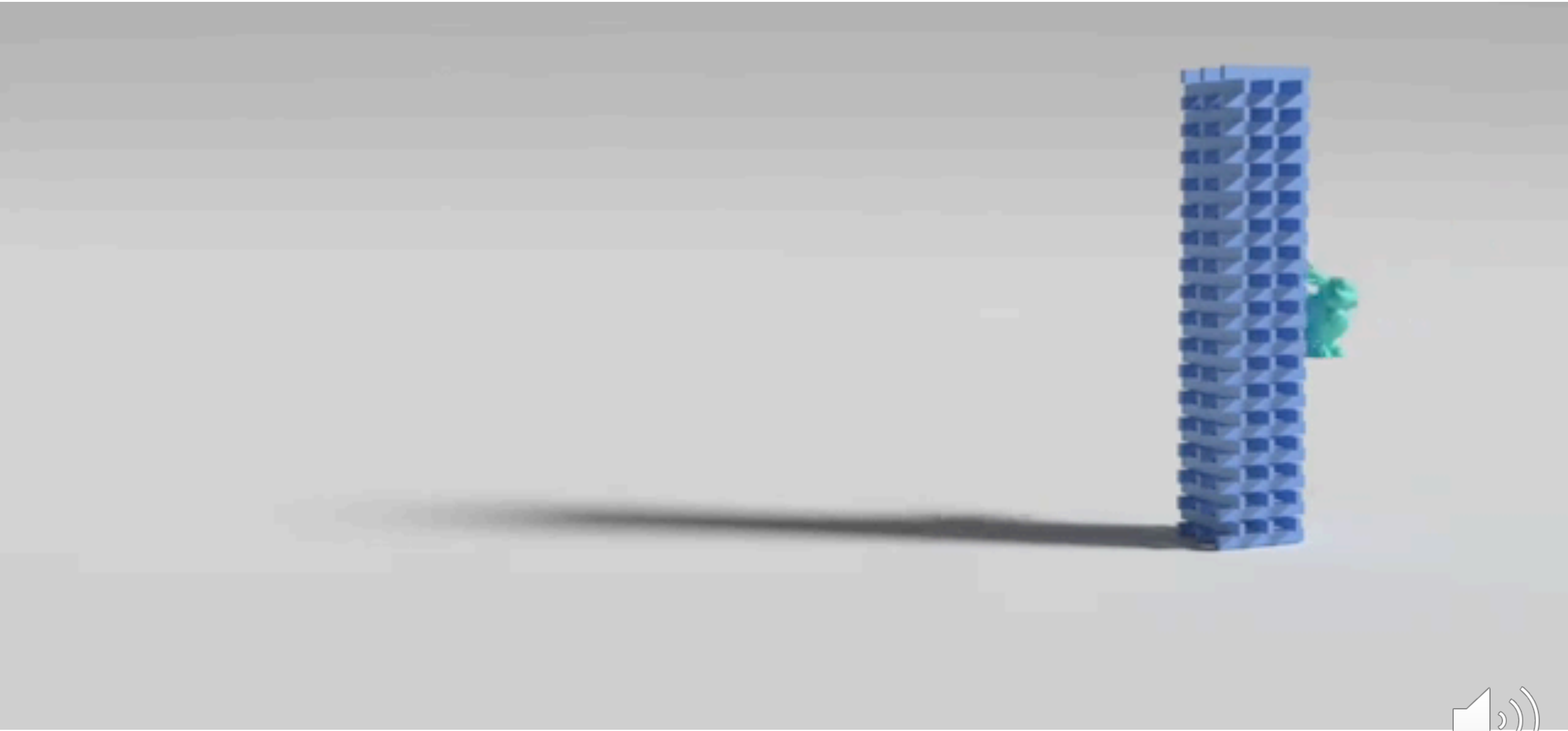


Game of Thrones | HBO

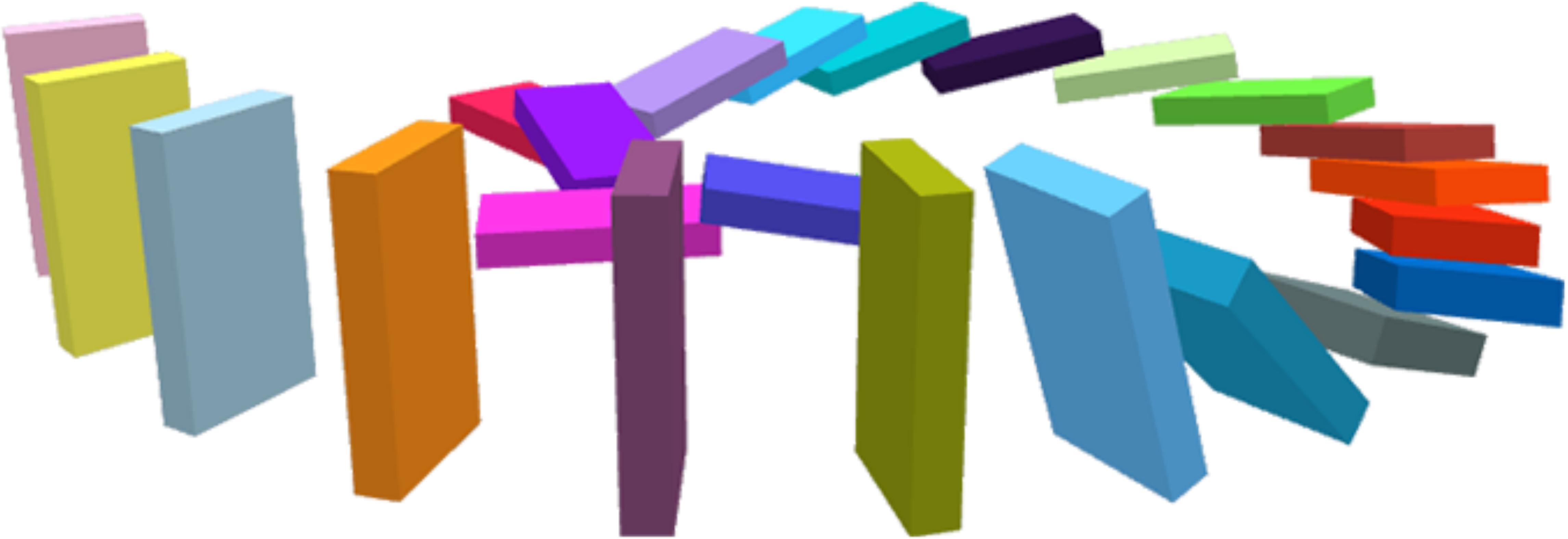
Last Video: Cloth Simulation

Data-Driven Elastic Models for Cloth: Modelling and Measurement | Wang et al.

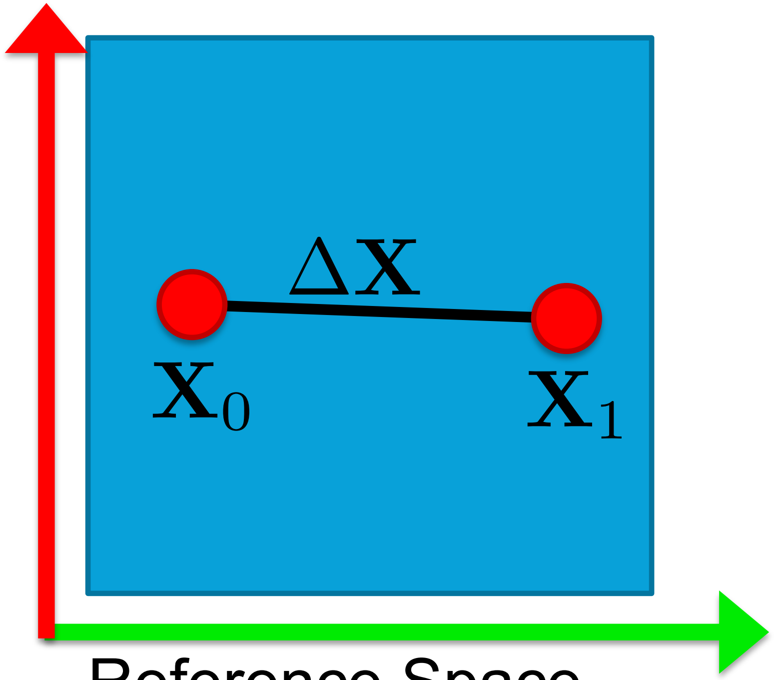
This Video: Rigid Body Simulation



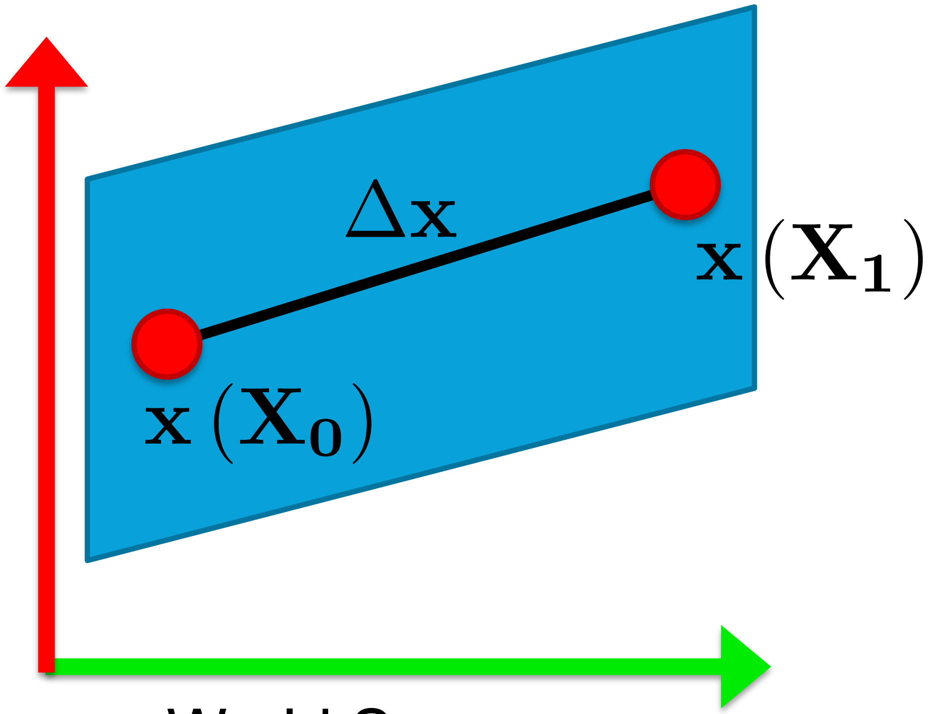
What Makes an Object Rigid ?



What Makes an Object Rigid ?



Reference Space



World Space

$$\Delta \mathbf{x} \approx \cancel{\mathbf{x}(\mathbf{X}_0)} + \underbrace{\frac{\partial \mathbf{x}}{\partial \mathbf{X}}}_{\text{deformation gradient } \mathbf{F}} \Delta \mathbf{X} - \cancel{\mathbf{x}(\mathbf{X}_0)}$$

What Makes an Object Rigid ?

Strain $\Delta\mathbf{x}^T\Delta\mathbf{x} - \Delta\mathbf{X}^T\Delta\mathbf{X}$

$$\Delta\mathbf{X}^T\mathrm{F}^T\mathrm{F}\Delta\mathbf{X} - \Delta\mathbf{X}^T\Delta\mathbf{X}$$

$$\Delta\mathbf{X}^T\underbrace{\left(\overbrace{\mathrm{F}^T\mathrm{F}}^{\text{Right Cauchy Green Deformation}} - \mathrm{I}\right)}_{\text{Green Lagrange Strain}}\Delta\mathbf{X} = \mathbf{0}$$

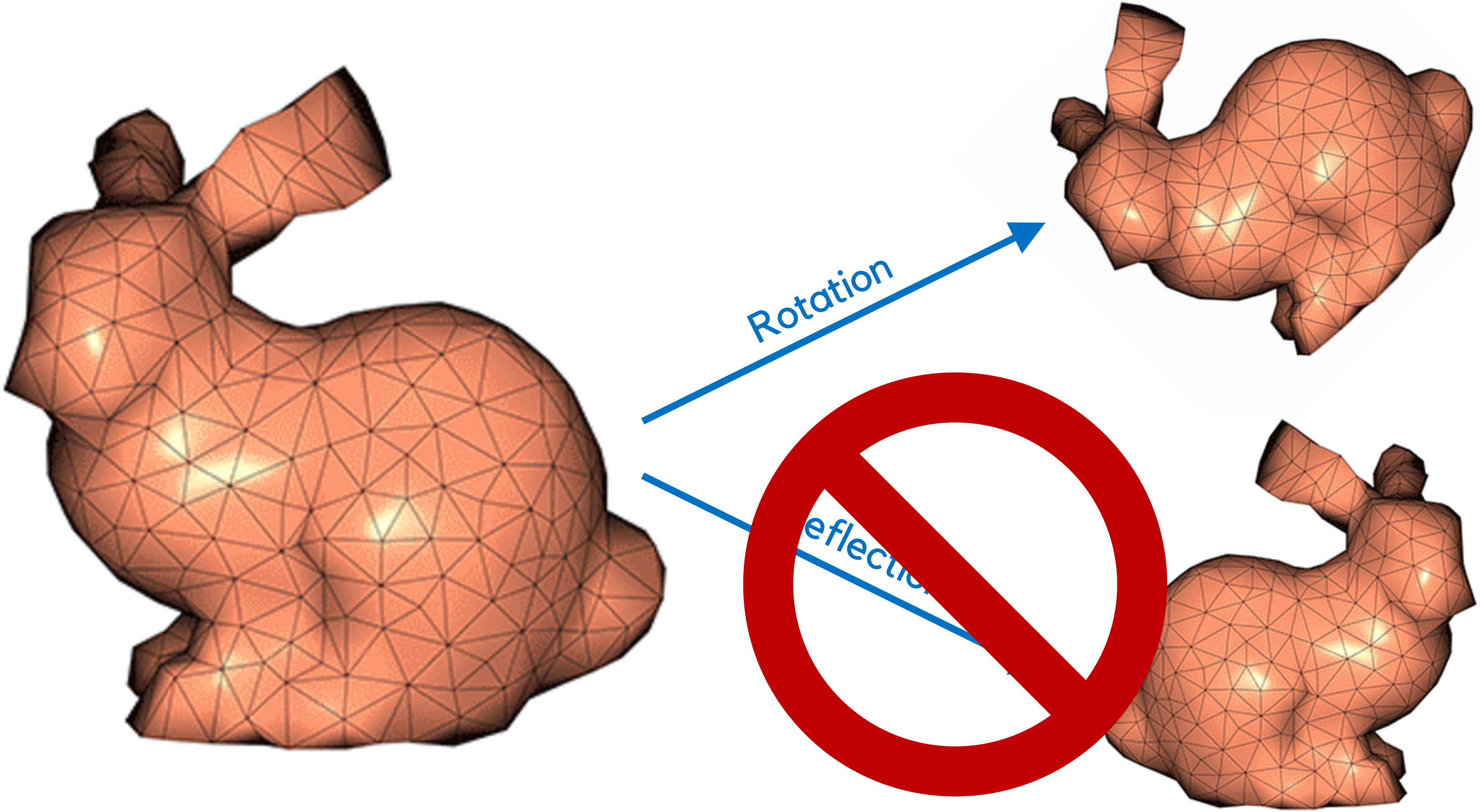
What Makes an Object Rigid ?

$$\Delta\mathbf{X}^T\left(\mathrm{F}^T\mathrm{F}-\mathrm{I}\right)\Delta\mathbf{X}=\mathbf{0}$$

Implies

$$\underbrace{\mathrm{F}^T\mathrm{F}}_{\text{Orthogonal}}=\mathrm{I}$$

What Makes an Object Rigid ?



What Makes an Object Rigid ?

$$\Delta \mathbf{X}^T \left(\mathrm{F}^T \mathrm{F} - \mathrm{I}\right) \Delta \mathbf{X} = \mathbf{0}$$

Implies

$$\mathrm{F}^T \mathrm{F} = \mathrm{I}$$

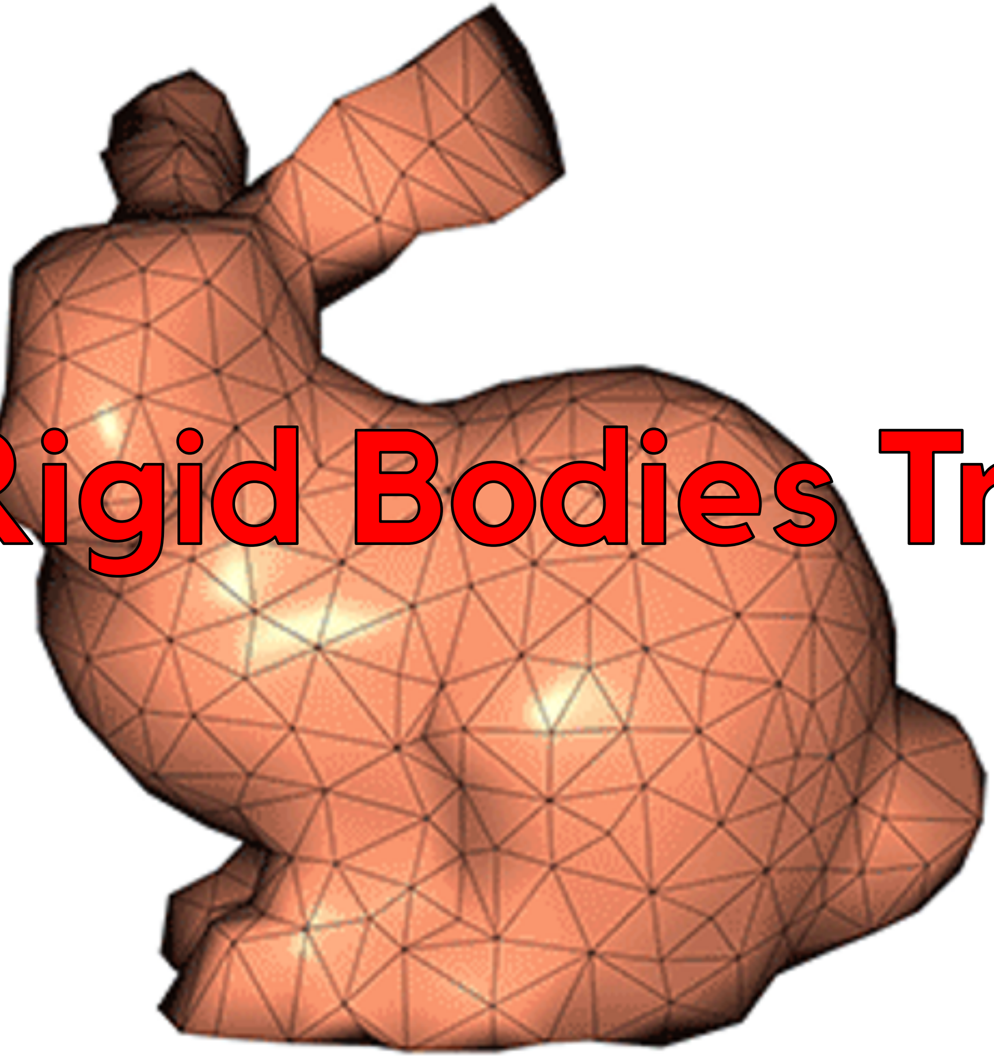
Orthogonal

Rigid Bodies Rotate

$$\mathrm{F} \in SO\left(3\right)$$

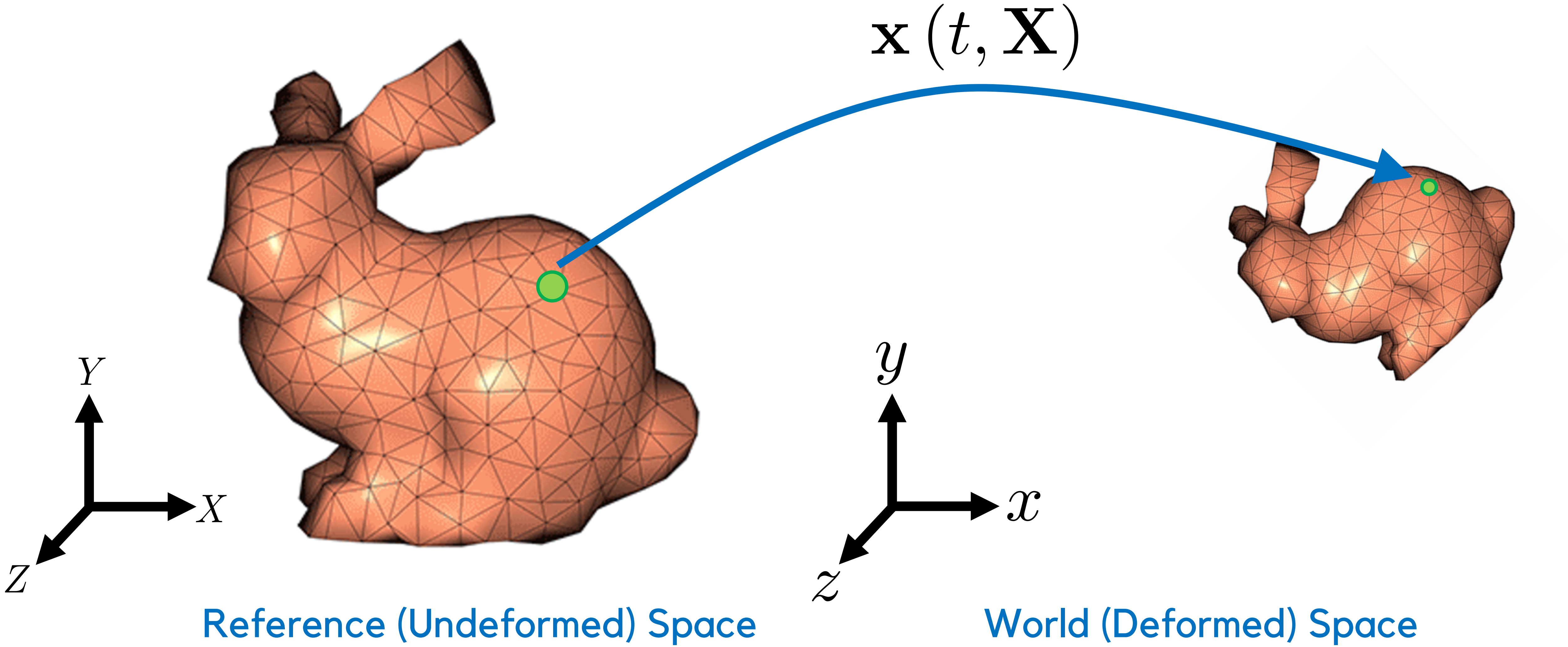
Special Orthogonal Group (Rotations) !

What Makes an Object Rigid ?



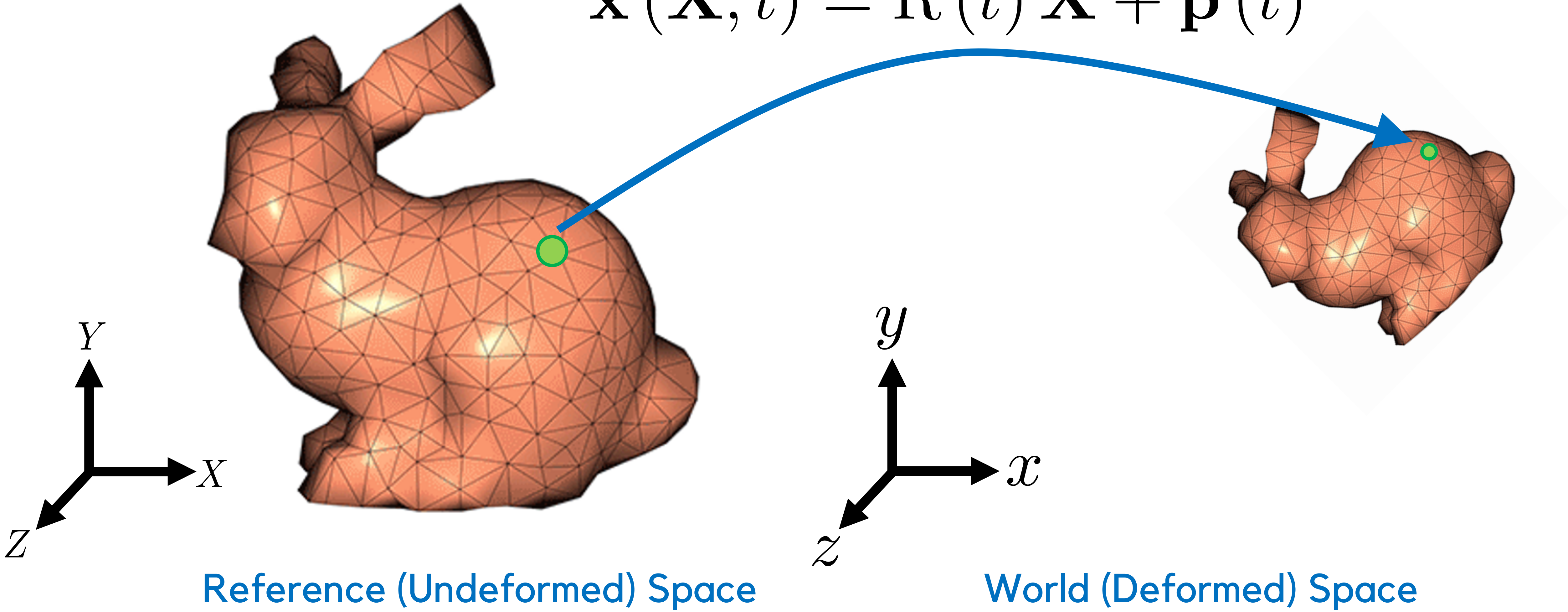
Rigid Bodies Translate

The Rigid Body Mapping

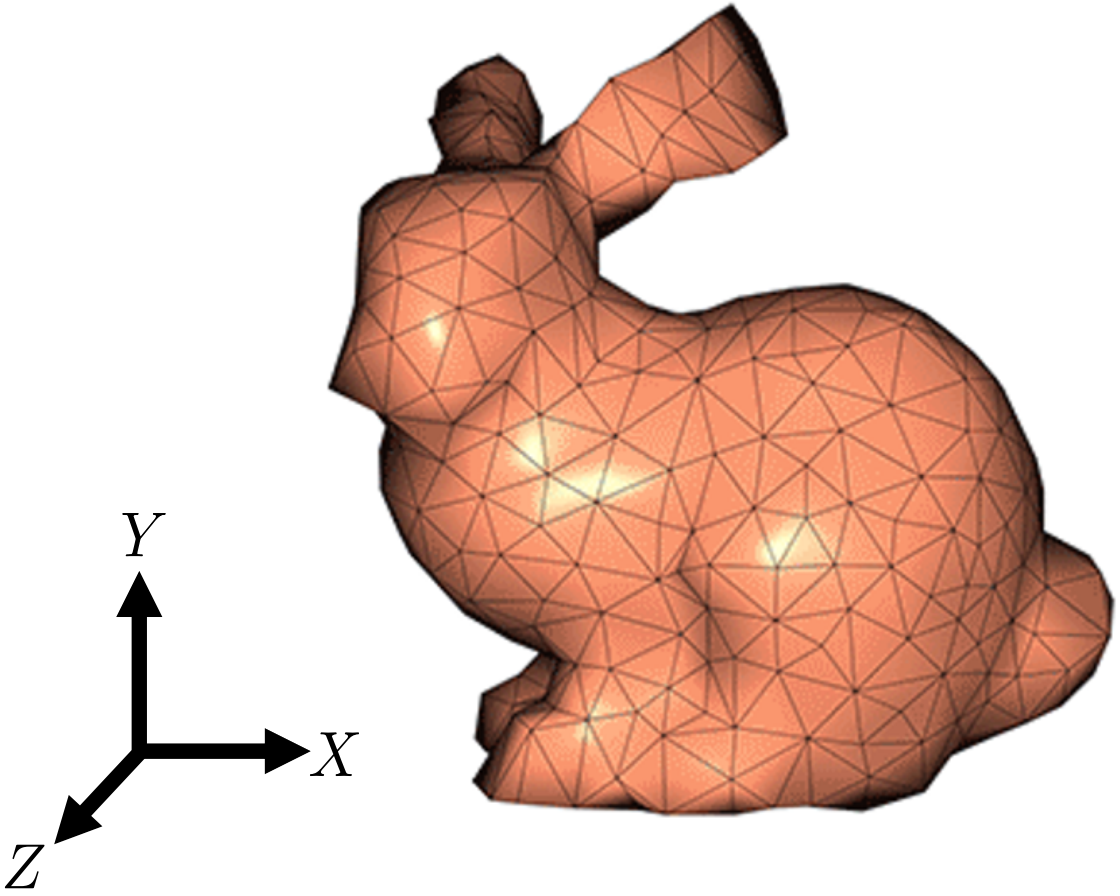


The Rigid Body Mapping

$$\mathbf{x}(\mathbf{X}, t) = \mathrm{R}(t)\,\mathbf{X} + \mathbf{p}(t)$$



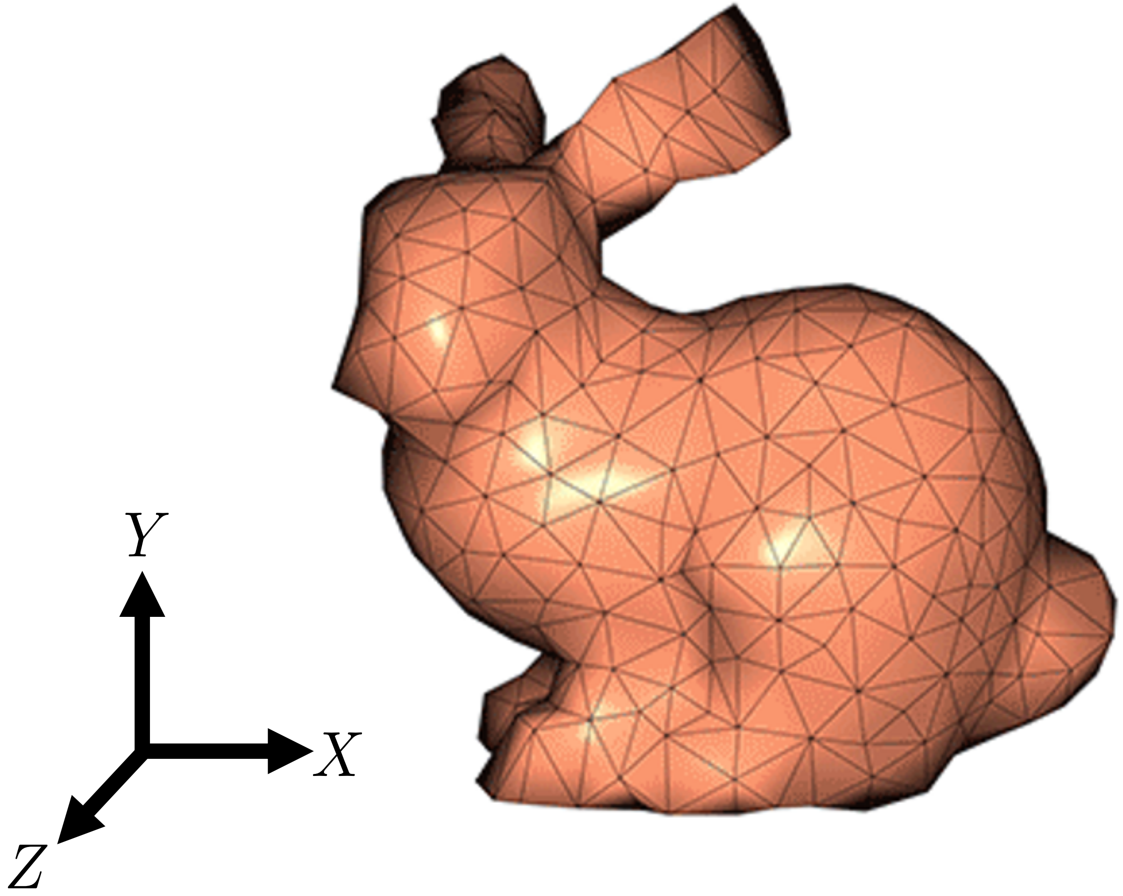
Generalized Coordinates of a Rigid Body



Reference (Undeformed) Space

$$\mathbf{x}\left(\mathbf{X},t\right)=\underbrace{\mathrm{R}\left(t\right)}_{\text{Rotation Matrix}\,\in\,\mathbb{R}^{3\times3}}\mathbf{X}+\overbrace{\mathbf{p}\left(t\right)}^{\text{Translation}\,\in\,\mathbb{R}^{3\times1}}$$

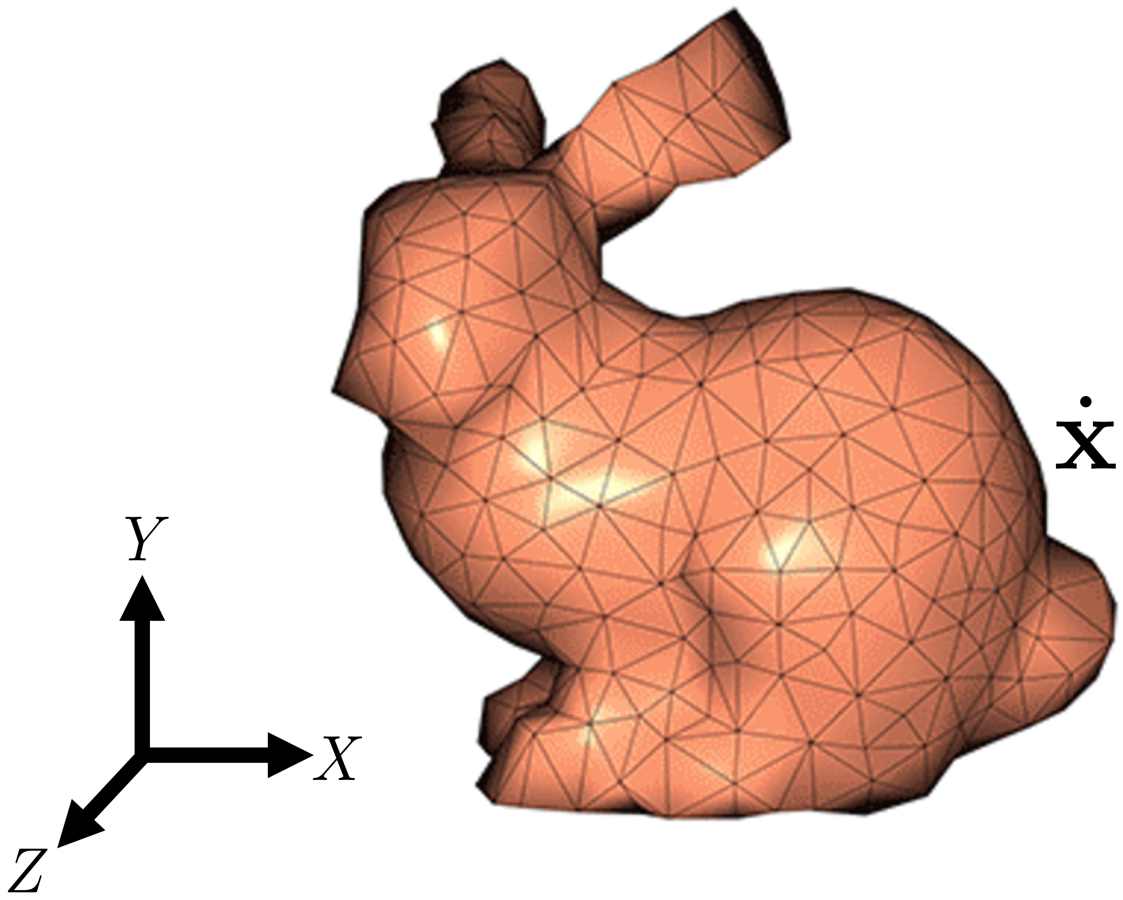
Generalized Coordinates of a Rigid Body



Reference (Undeformed) Space

$$\mathbf{q} = \{\underbrace{\mathrm{R}, \mathbf{p}}_{\text{Set with rotation and translation}}\}$$

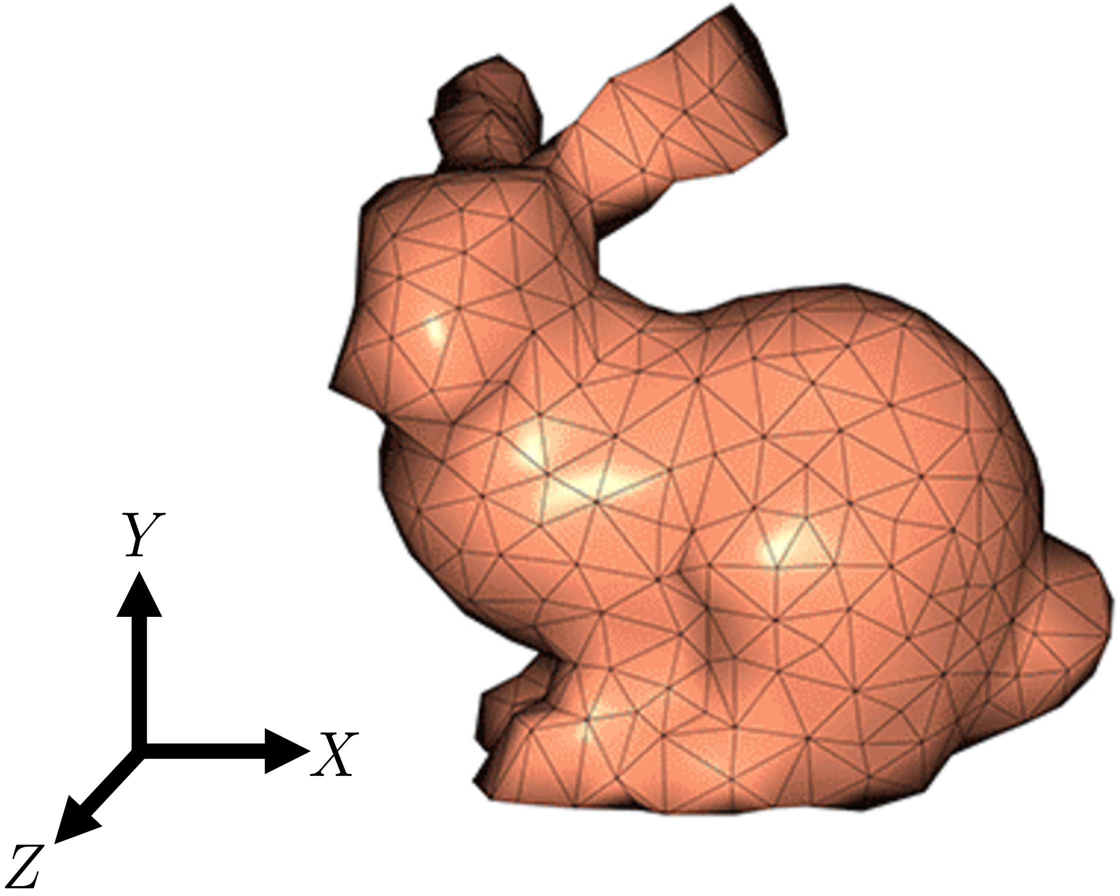
Generalized Velocity of a Rigid Body



$$\dot{\mathbf{x}}\left(\mathbf{X},t\right)=\frac{d}{dt}\left(\underbrace{\mathrm{R}\left(t\right)}_{\text{Rotation Matrix}\,\in\,\mathbb{R}^{3\times3}}\mathbf{X}+\overbrace{\mathbf{p}\left(t\right)}^{\text{Translation}\,\in\,\mathbb{R}^{3\times1}}\right)$$

Reference (Undeformed) Space

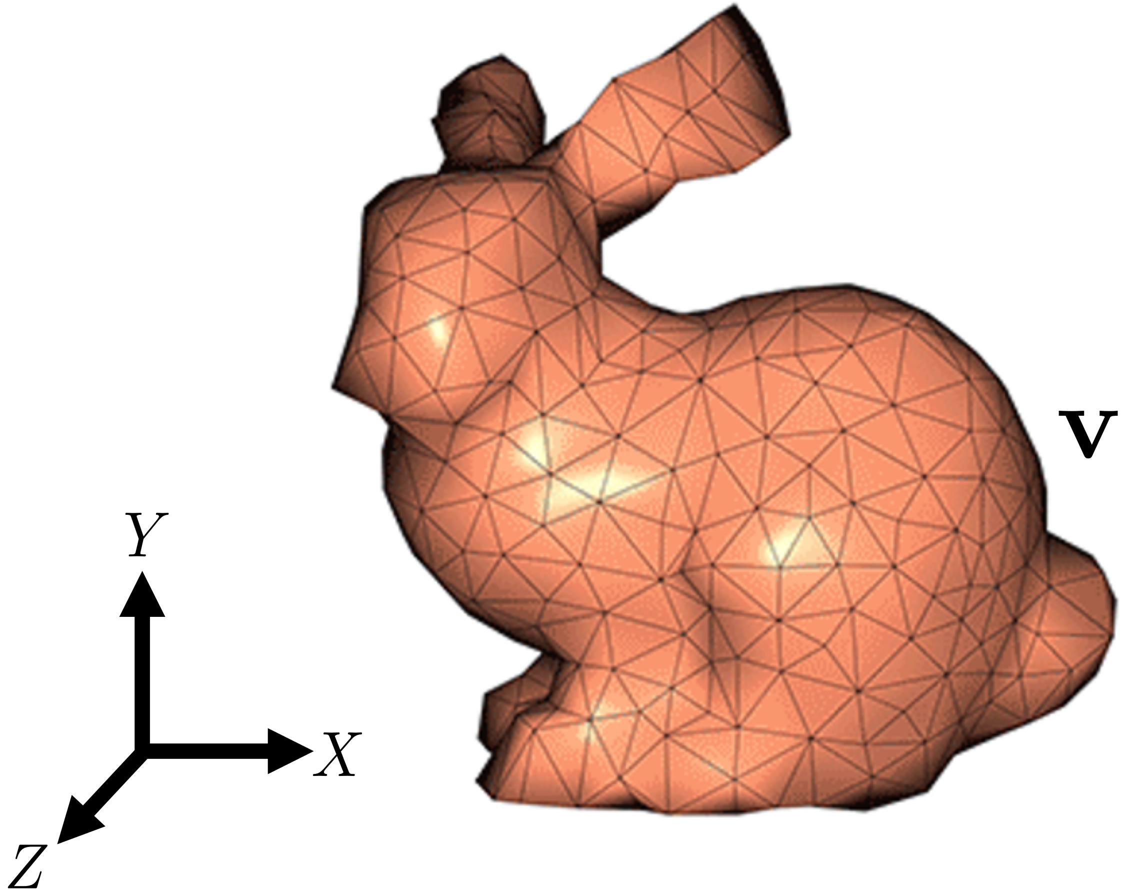
Generalized Velocity of a Rigid Body



Reference (Undeformed) Space

$$\dot{\mathbf{x}}\left(\mathbf{X}, t\right) = \underbrace{\dot{\mathrm{R}}\left(t\right)}_{\text{Time Derivative of Rotation Matrix}}\mathbf{X} + \overbrace{\dot{\mathbf{p}}\left(t\right)}^{\text{Linear Velocity}}$$

Generalized Velocity of a Rigid Body



Linear Velocity

$$\mathbf{v}\left(\mathbf{X},t\right) = \mathrm{R}\left[\mathbf{X}\right]^{T}\mathrm{R}^{T}\omega + \dot{\mathbf{p}}\left(t\right)$$

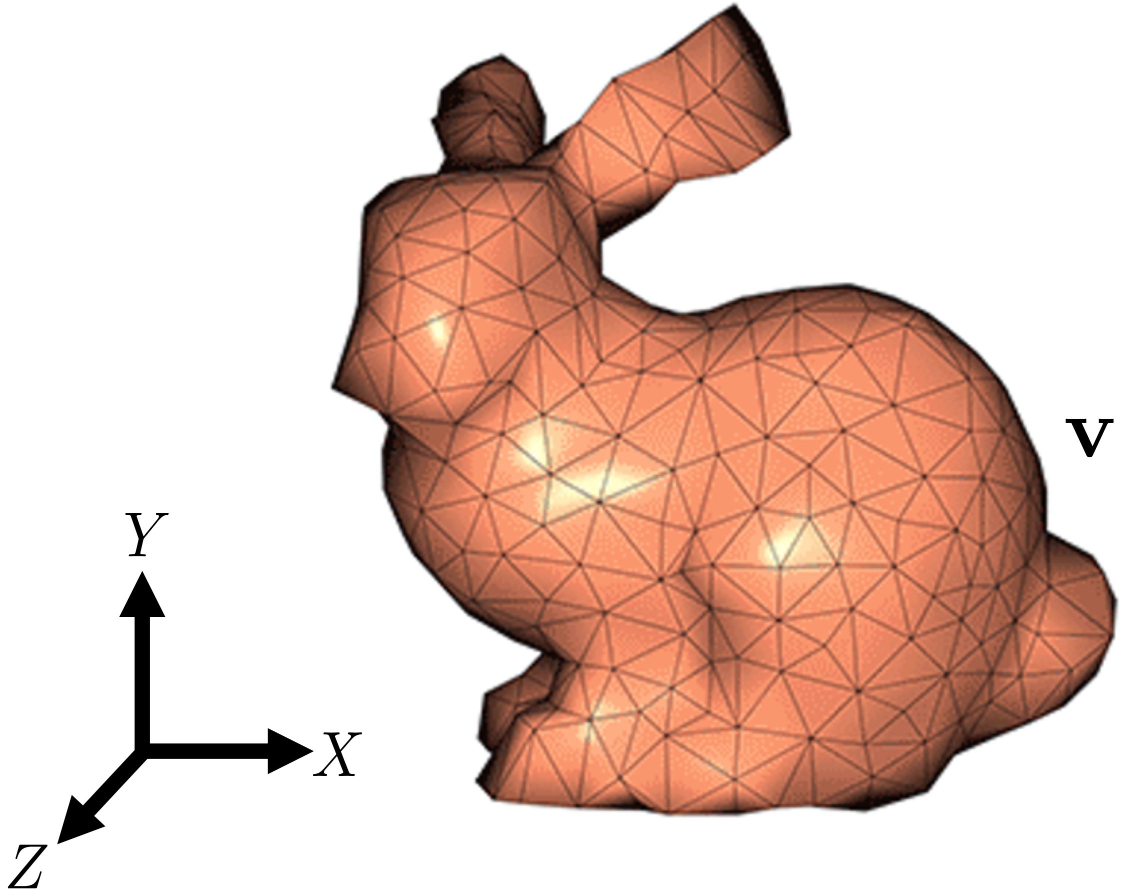
Angular Velocity $\in \mathbb{R}^3$

Reference (Undeformed) Space

$$\left[\mathbf{X}\right] = \begin{pmatrix} 0 & -X_z & X_y \\ X_z & 0 & -X_x \\ -X_y & X_x & 0 \end{pmatrix}$$

Cross Product Matrix

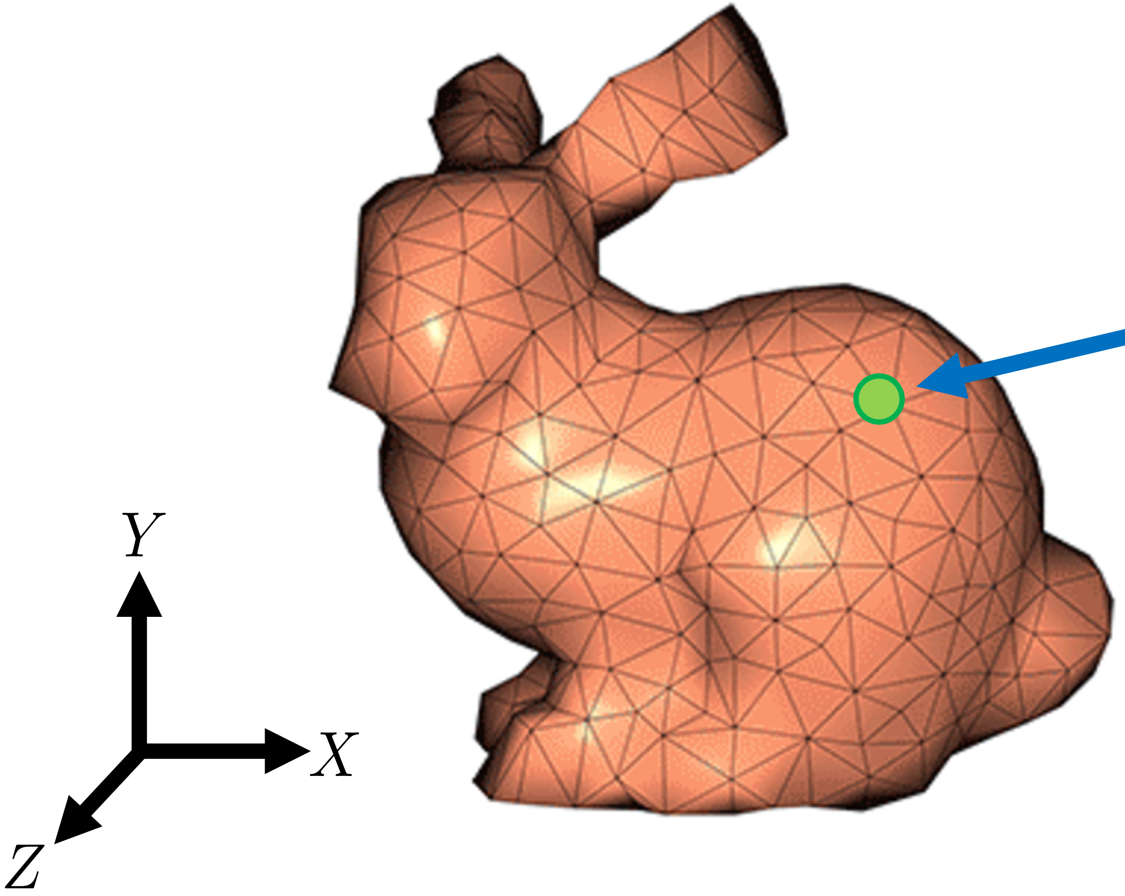
Generalized Velocity of a Rigid Body



Reference (Undeformed) Space

$$\mathbf{v}\left(\mathbf{X}, t\right) = \mathrm{R}\begin{pmatrix}[\mathbf{X}]^T & \mathrm{I}\end{pmatrix}\begin{pmatrix}\mathrm{R}^T & 0 \\ 0 & \mathrm{I}\end{pmatrix}\underbrace{\begin{pmatrix}\omega \\ \dot{\mathbf{p}}\end{pmatrix}}_{\dot{\mathbf{q}} \in \mathbb{R}^6}$$

Kinetic Energy of a Rigid Body



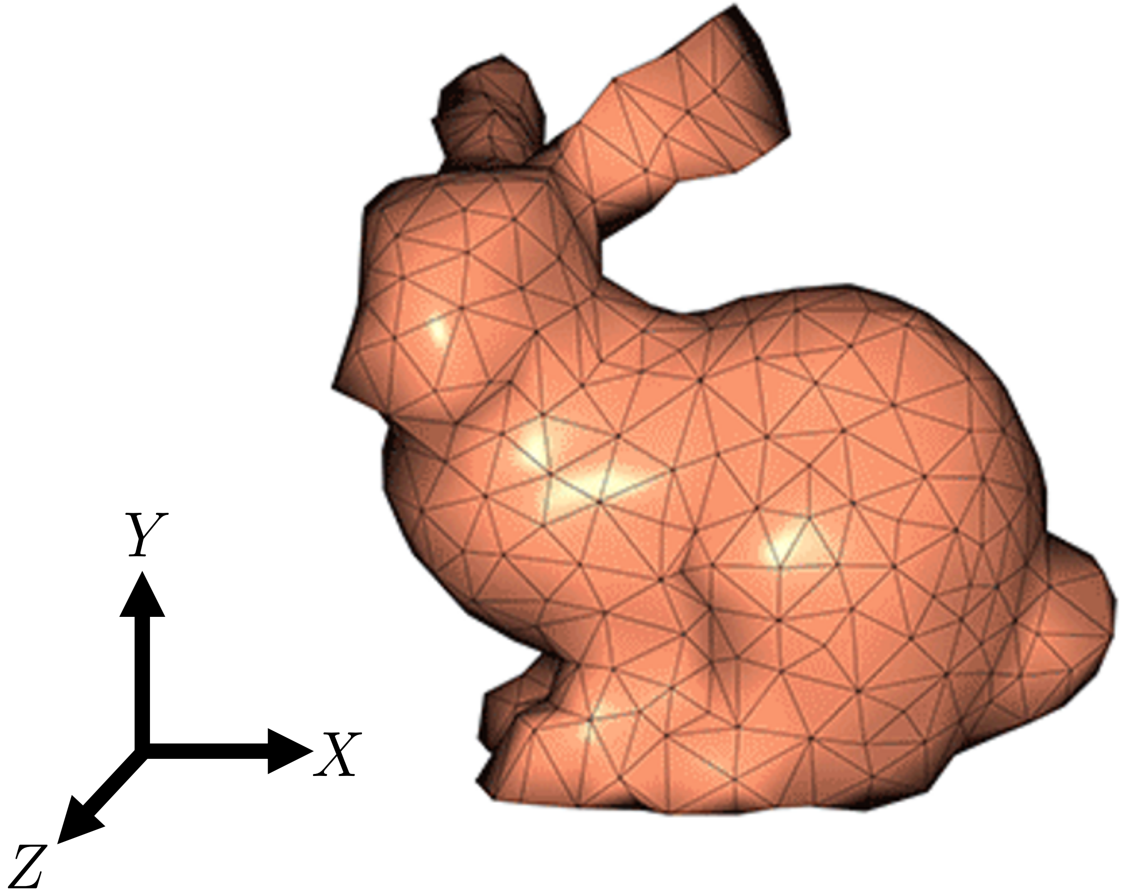
Reference (Undeformed) Space

infinitesimal volume

$$\frac{1}{2}\rho\|\mathbf{v}\left(\mathbf{X}\right)\|_2^2 d\Omega$$

density $\left(\frac{kg}{m^3}\right)$

Kinetic Energy of a Rigid Body



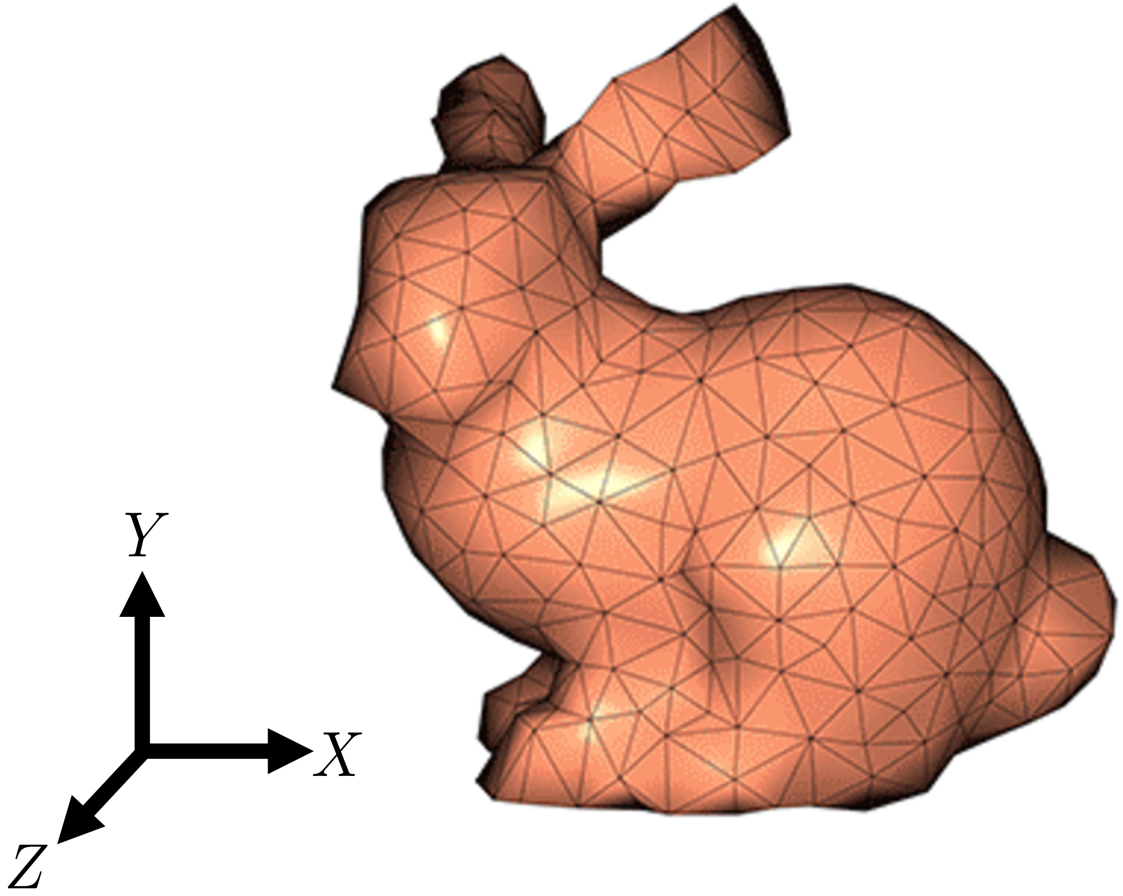
Reference (Undeformed) Space

infinitesimal volume

$$\frac{1}{2}\int_{\Omega}\rho\|\mathbf{v}\left(\mathbf{X}\right)\|_2^2 d\Omega$$

entire rigid body

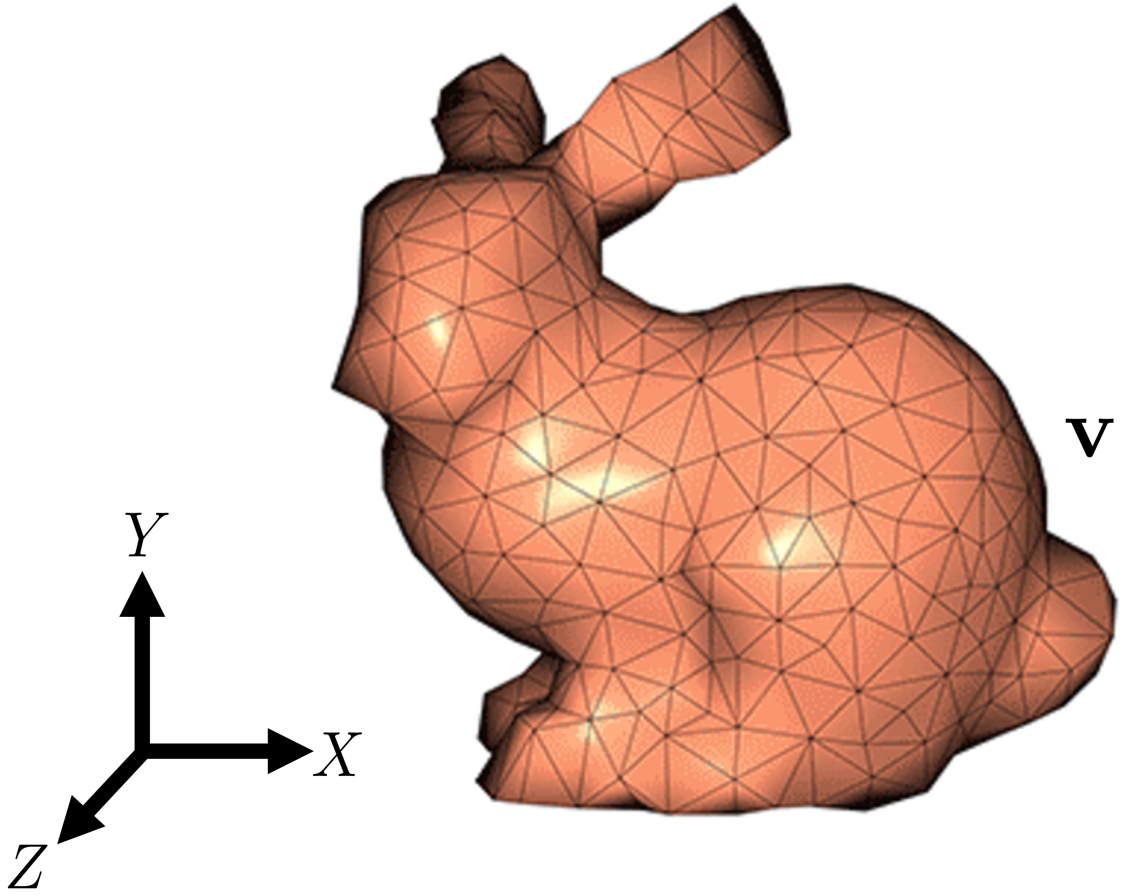
Kinetic Energy of a Rigid Body



Reference (Undeformed) Space

$$\frac{1}{2}\int_{\underbrace{\Omega}_{\text{entire rigid body}}} \rho \mathbf{v}^T \mathbf{v} \, d\Omega$$

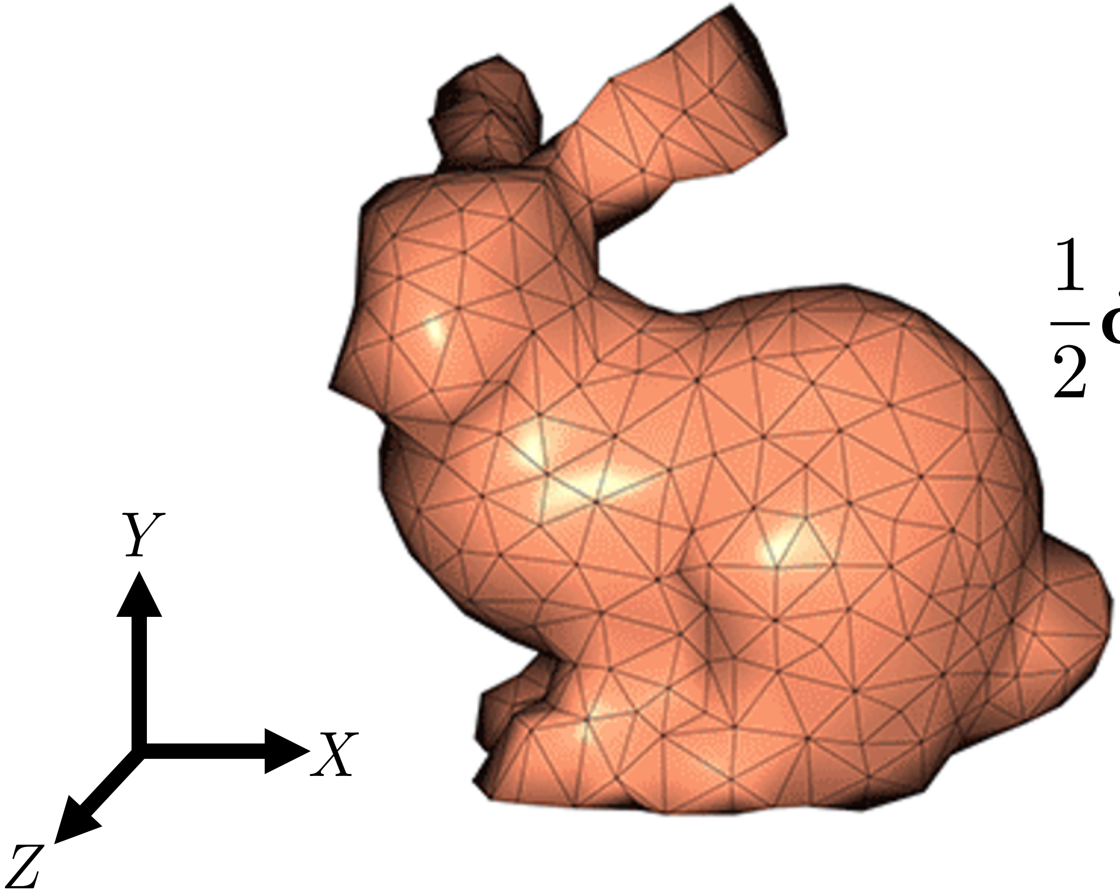
Generalized Velocity of a Rigid Body



Reference (Undeformed) Space

$$\mathbf{v}(\mathbf{X},t) = \mathrm{R}\begin{pmatrix}[\mathbf{X}]^T & \mathrm{I}\end{pmatrix}\overbrace{\begin{pmatrix}\mathrm{R}^T & 0\\ 0 & \mathrm{I}\end{pmatrix}}^{\mathrm{A}}\underbrace{\begin{pmatrix}\omega\\ \dot{\mathbf{p}}\end{pmatrix}}_{\dot{\mathbf{q}}\in\mathbb{R}^6}$$

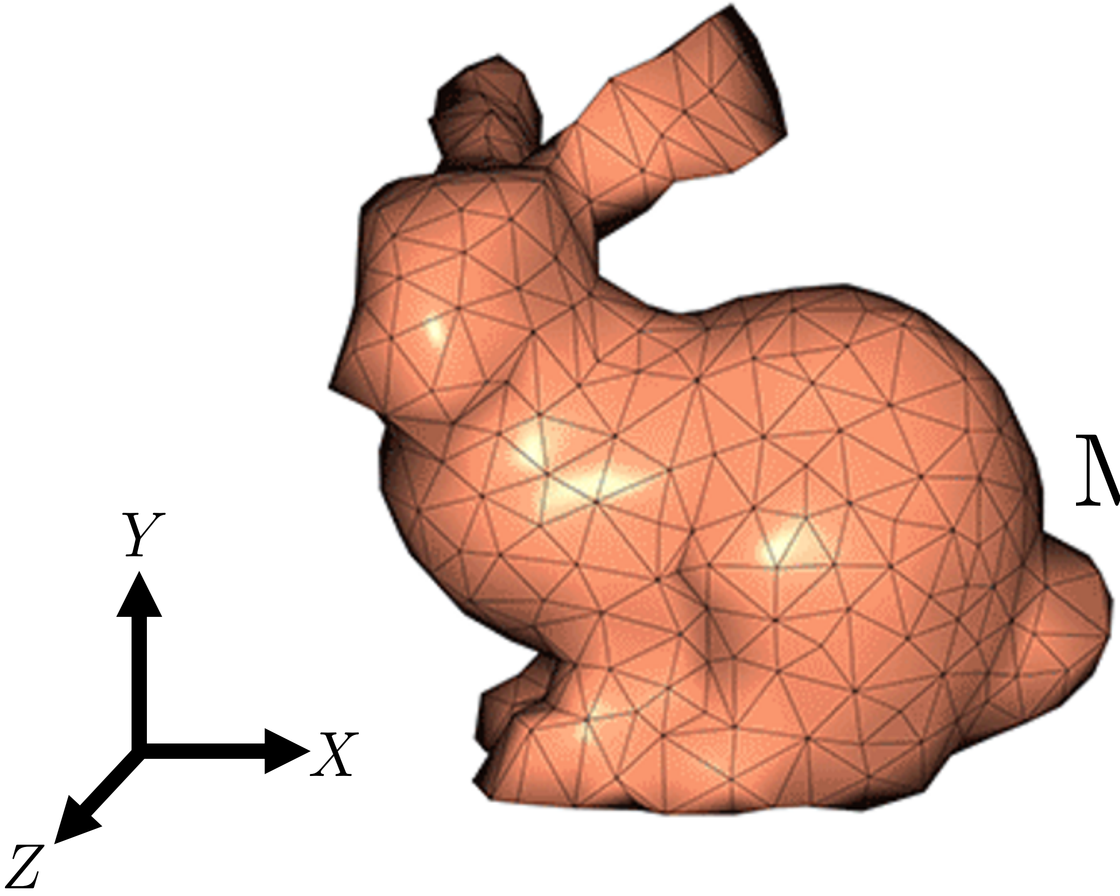
Kinetic Energy of a Rigid Body



Reference (Undeformed) Space

$$\frac{1}{2}\dot{\mathbf{q}}^T A^T \overbrace{\left(\underbrace{\int_\Omega \rho \begin{pmatrix} [\mathbf{X}]\,[\mathbf{X}]^T & [\mathbf{X}] \\ [\mathbf{X}]^T & I \end{pmatrix} d\Omega}_{M_0}\right)}^{\text{Mass Matrix } M} \underbrace{A\dot{\mathbf{q}}}_{\begin{pmatrix} R^T & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \omega \\ \dot{\mathbf{p}} \end{pmatrix}}$$

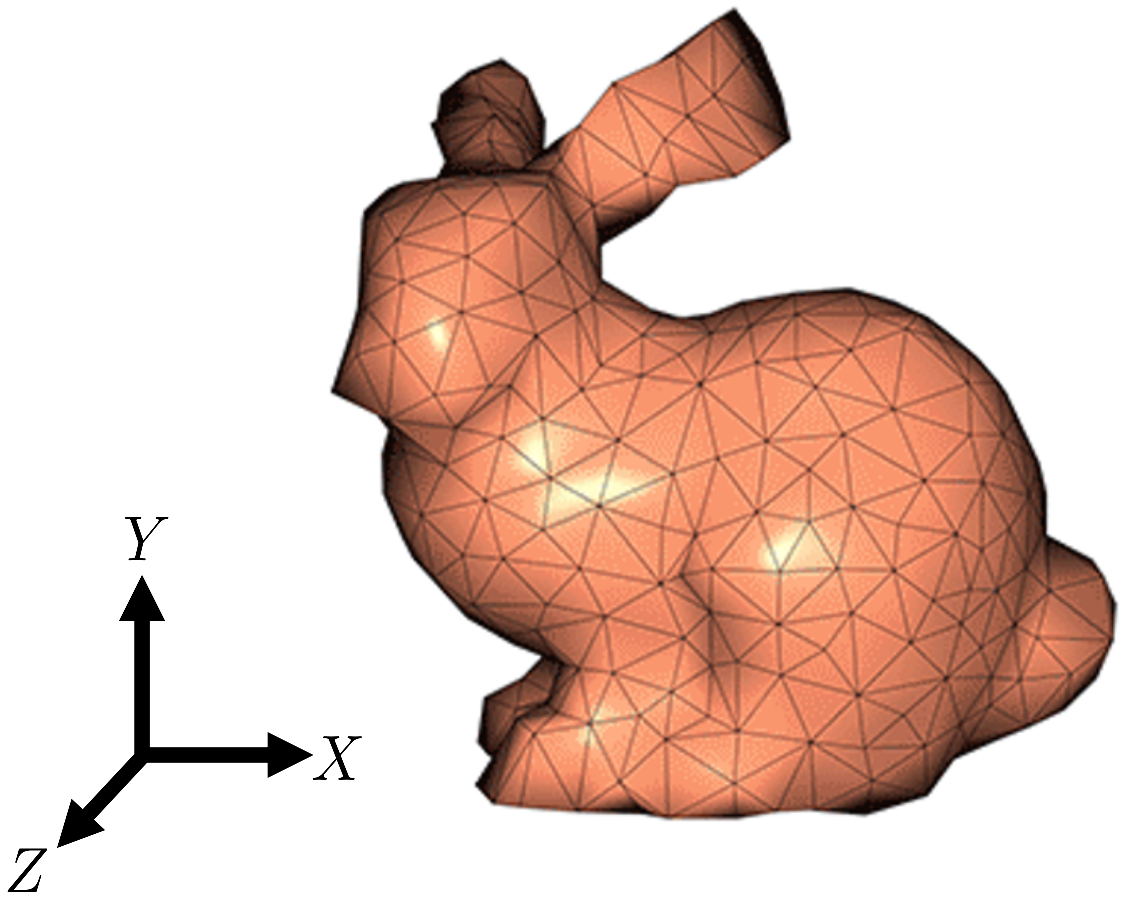
Kinetic Energy of a Rigid Body



Reference (Undeformed) Space

$$\mathrm{M}_0 = \int_{\Omega} \rho \begin{pmatrix} [\mathbf{X}]\,[\mathbf{X}]^T & [\mathbf{X}] \\ [\mathbf{X}]^T & \mathrm{I} \end{pmatrix} d\Omega$$

Aside: The Center-of-Mass

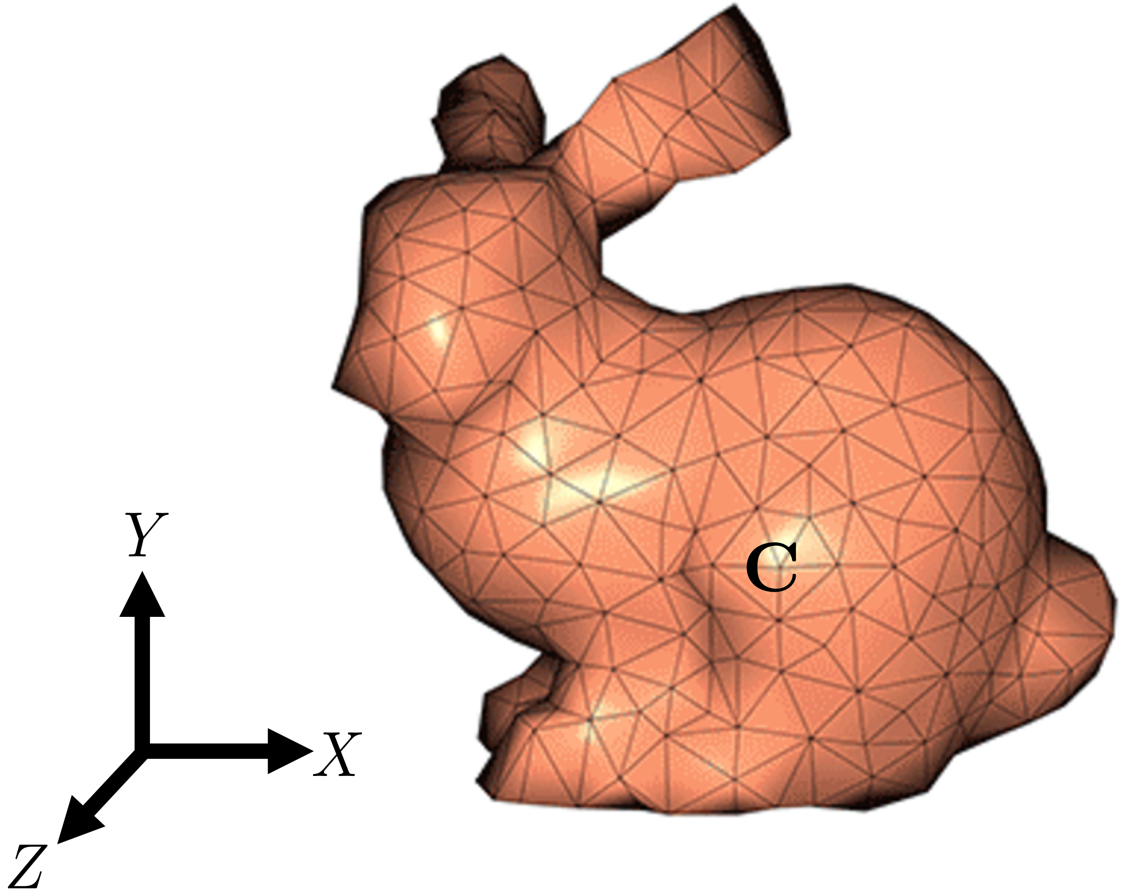


Reference (Undeformed) Space

$$\mathbf{C} = \frac{1}{m}\int_{\Omega} \rho\, \mathbf{X}\, d\Omega$$

$$\underbrace{m = \int_{\Omega} \rho\, d\Omega}_{\text{mass of object}}$$

Center-of-Mass Coordinate System



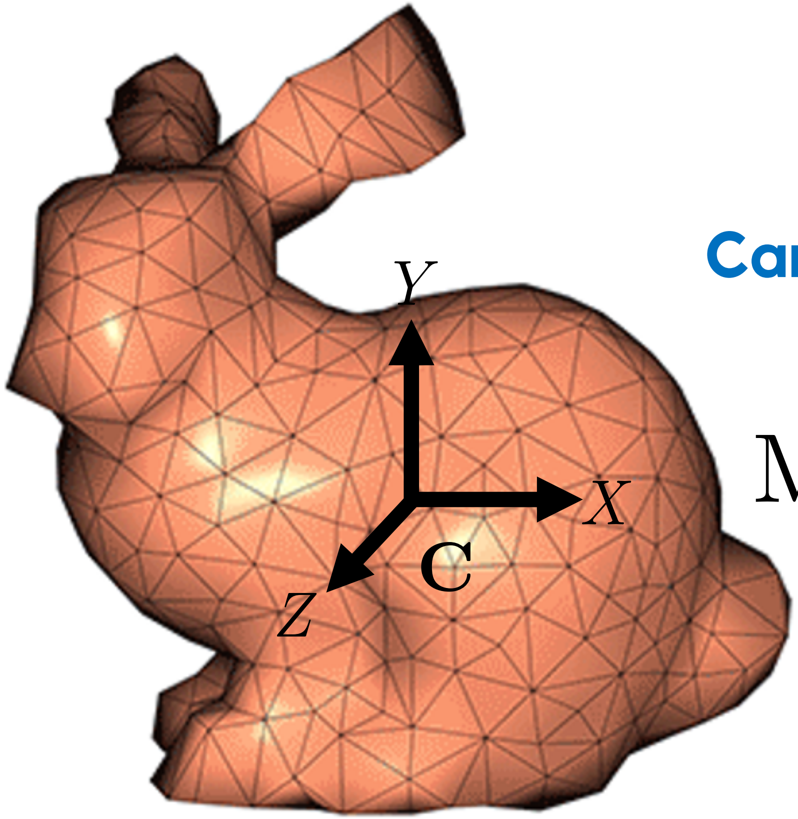
Reference (Undeformed) Space

$$\mathbf{C} = \frac{1}{m}\int_{\Omega} \rho\, \mathbf{X}\, d\Omega$$

$$\bar{\mathbf{X}} = \mathbf{X} - \mathbf{C}$$

$$\underbrace{m = \int_{\Omega} \rho\, d\Omega}_{\text{mass of object}}$$

Kinetic Energy of a Rigid Body

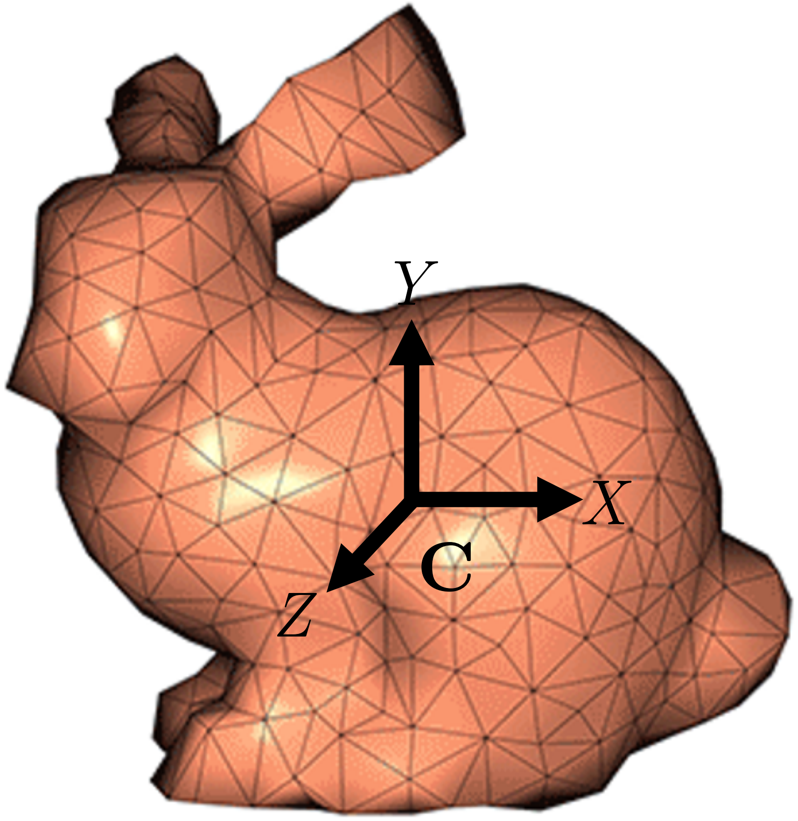


Can Integrate in any Reference Space we Want

$$\mathrm{M}_0 = \int_\Omega \rho \begin{pmatrix} [\bar{\mathbf{X}}]\,[\bar{\mathbf{X}}]^T & [\bar{\mathbf{X}}] \\ [\bar{\mathbf{X}}]^T & \mathrm{I} \end{pmatrix} d\Omega$$

Reference (Undeformed) Space

Kinetic Energy of a Rigid Body



Reference (Undeformed) Space

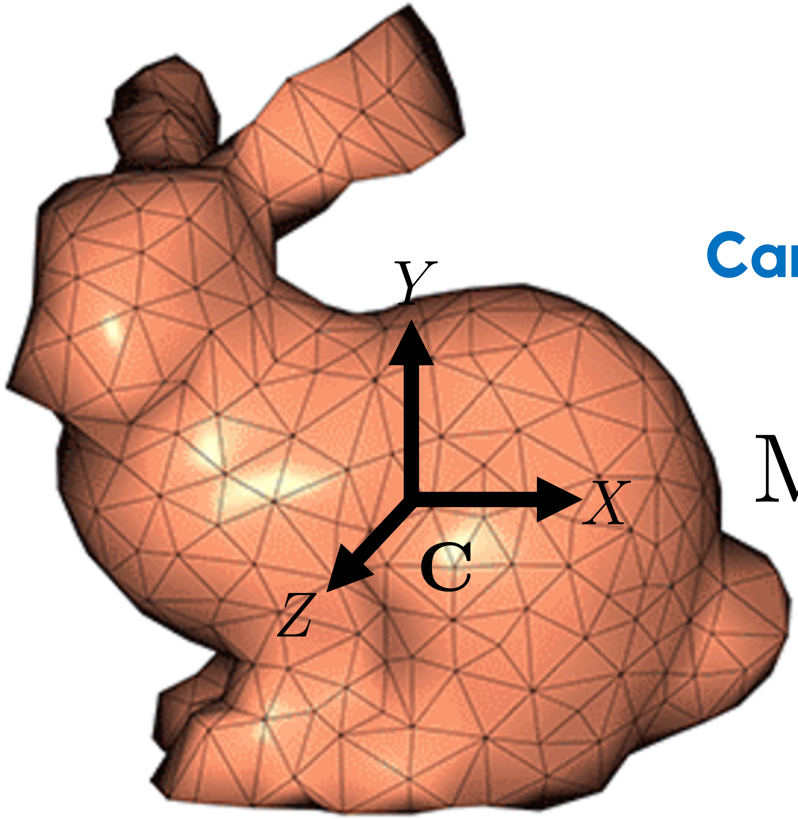
$$\int_{\Omega} \rho \left[\bar{\mathbf{X}}\right] d\Omega$$

$$= \int_{\Omega} \rho \left[\mathbf{X} - \mathbf{C}\right] d\Omega$$

$$= \int_{\Omega} \rho \left[\mathbf{X}\right] d\Omega - \int_{\Omega} \rho \left[\mathbf{C}\right] d\Omega$$

$$= \int_{\Omega} \rho \left[\mathbf{X}\right] d\Omega - m \left[\mathbf{C}\right] = \mathbf{0}$$

Kinetic Energy of a Rigid Body

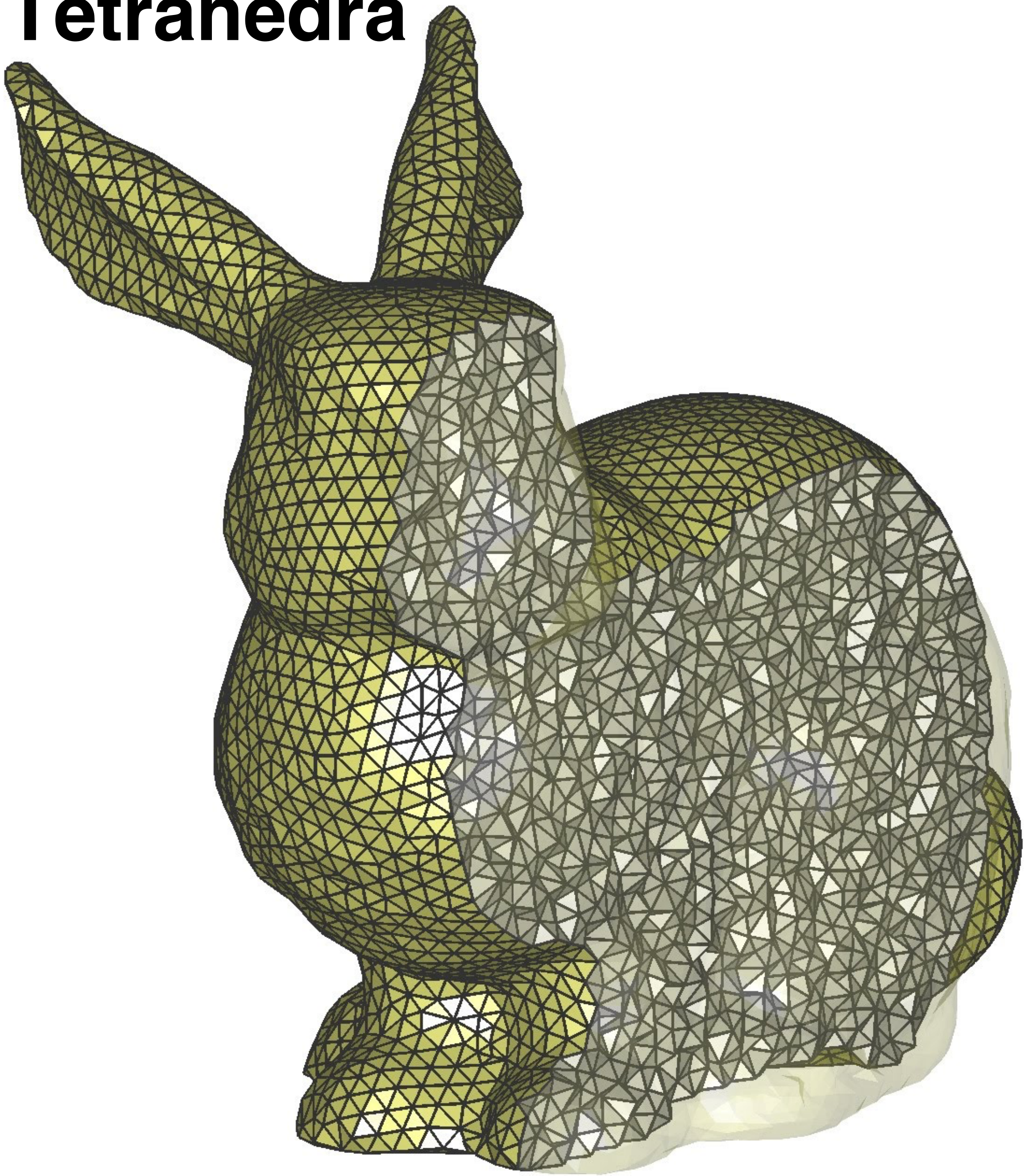


Reference (Undeformed) Space

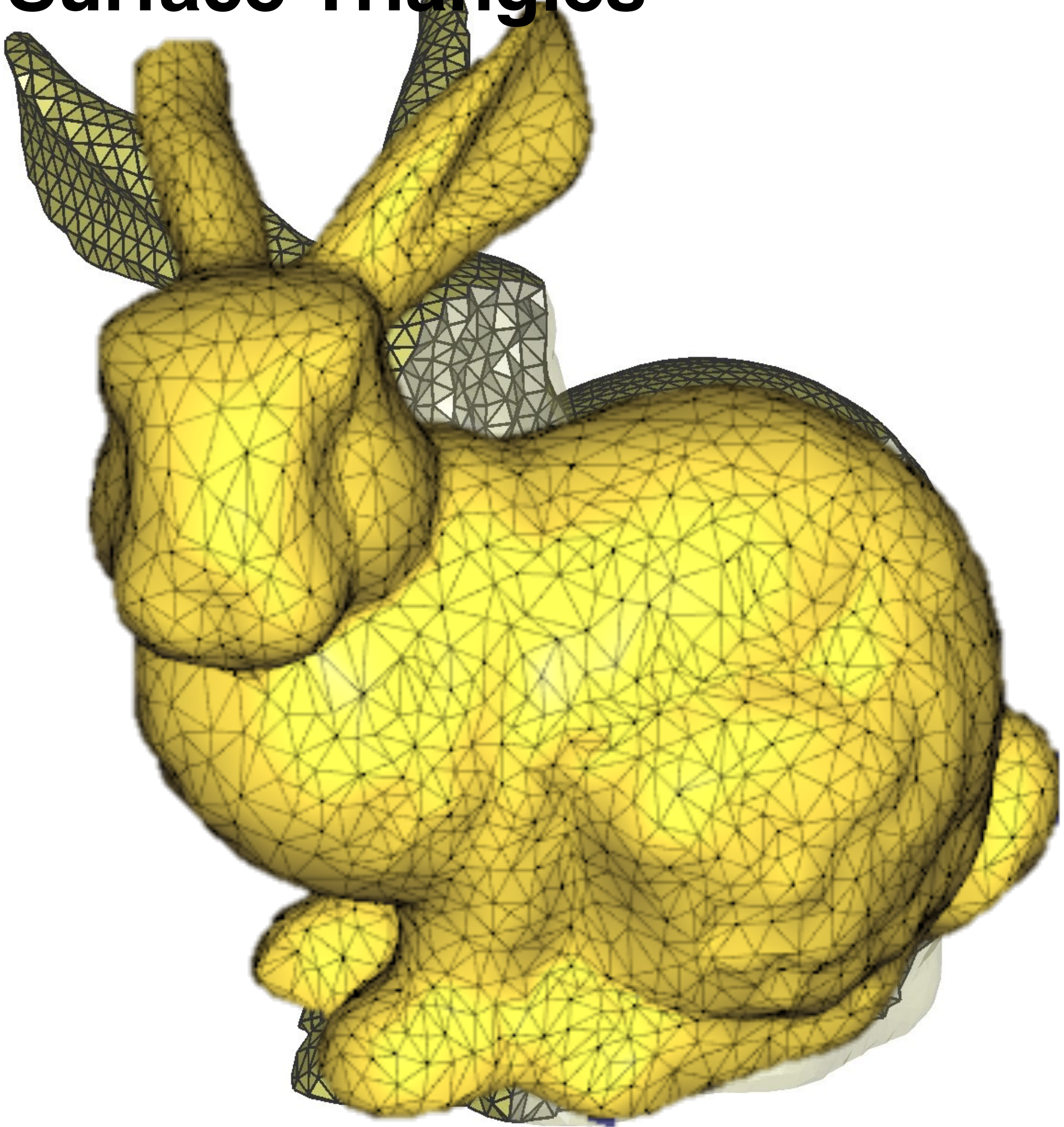
Can Integrate in any Reference Space we Want

$$\mathrm{M}_0 = \int_\Omega \rho \begin{pmatrix} [\bar{\mathbf{X}}]\,[\bar{\mathbf{X}}]^T & \mathbf{0} \\ \mathbf{0} & \mathrm{I} \end{pmatrix} d\Omega$$

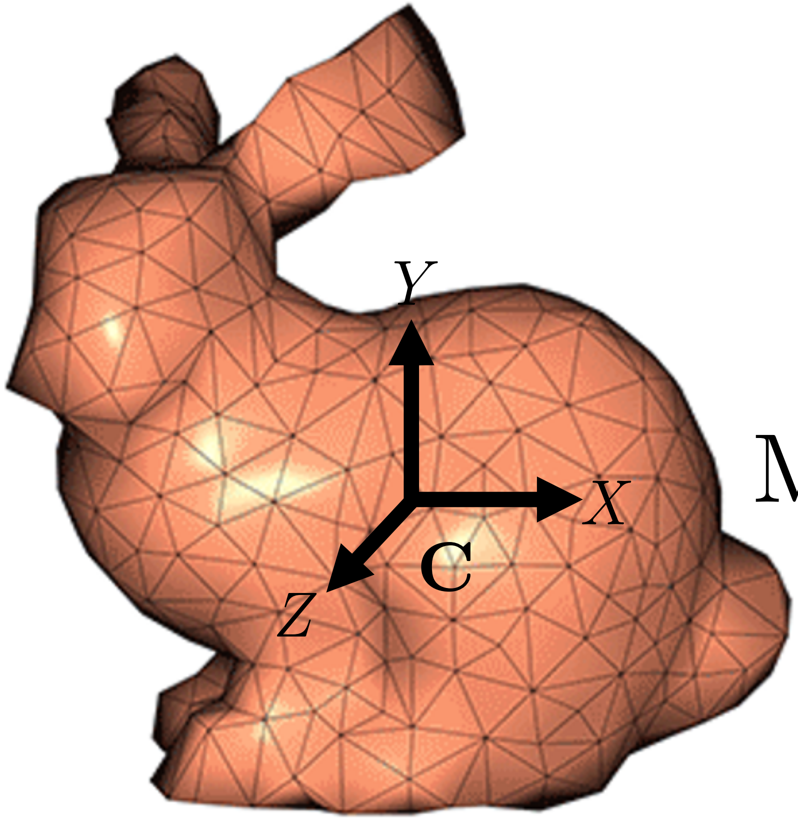
Integrate over Tetrahedra



Integrate over Surface Triangles



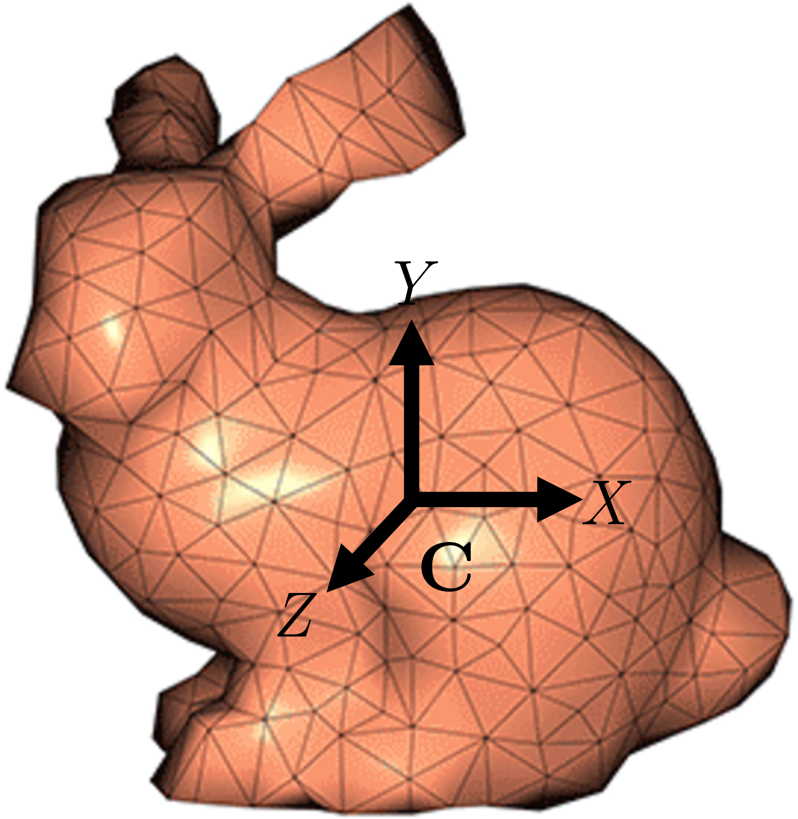
Integrate over Surface Triangles



$$\mathrm{M}_0 = \int_\Omega \rho \begin{pmatrix} [\bar{\mathbf{X}}]\,[\bar{\mathbf{X}}]^T & \mathbf{0} \\ \mathbf{0} & \mathrm{I} \end{pmatrix} d\Omega$$

Reference (Undeformed) Space

Aside: Divergence Theorem



Reference (Undeformed) Space

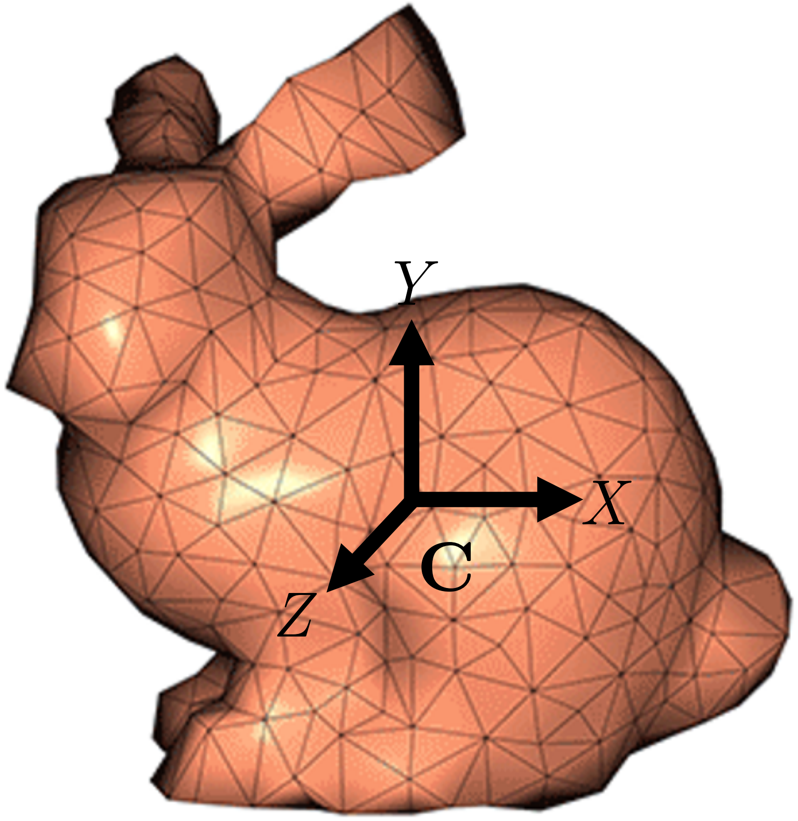
$$\underbrace{\int_{\Omega} \nabla \cdot \overbrace{\mathbf{f}}^{\in \mathbb{R}^3}\left(\bar{\mathbf{X}}\right) d\Omega}_{\text{volume integral}} = \underbrace{\int_{\Gamma} f\left(\bar{\mathbf{X}}\right) \cdot \overbrace{\mathbf{n}}^{\text{surface normal}} d\Gamma}_{\text{surface integral}}$$

Integrate over Surface Triangles

$$\int_\Omega \rho \begin{pmatrix} \bar{\mathbf{X}}_y{}^2 + \bar{\mathbf{X}}_z{}^2 & -\bar{\mathbf{X}}_x\bar{\mathbf{X}}_y & -\bar{\mathbf{X}}_x\bar{\mathbf{X}}_z & 0 & 0 & 0 \\ -\bar{\mathbf{X}}_x\bar{\mathbf{X}}_y & \bar{\mathbf{X}}_x{}^2 + \bar{\mathbf{X}}_z{}^2 & -\bar{\mathbf{X}}_y\bar{\mathbf{X}}_z & 0 & 0 & 0 \\ -\bar{\mathbf{X}}_x\bar{\mathbf{X}}_z & -\bar{\mathbf{X}}_y\bar{\mathbf{X}}_z & \bar{\mathbf{X}}_x{}^2 + \bar{\mathbf{X}}_y{}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} d\Omega$$

terms are all polynomials in reference space

Integrate over Surface Triangles

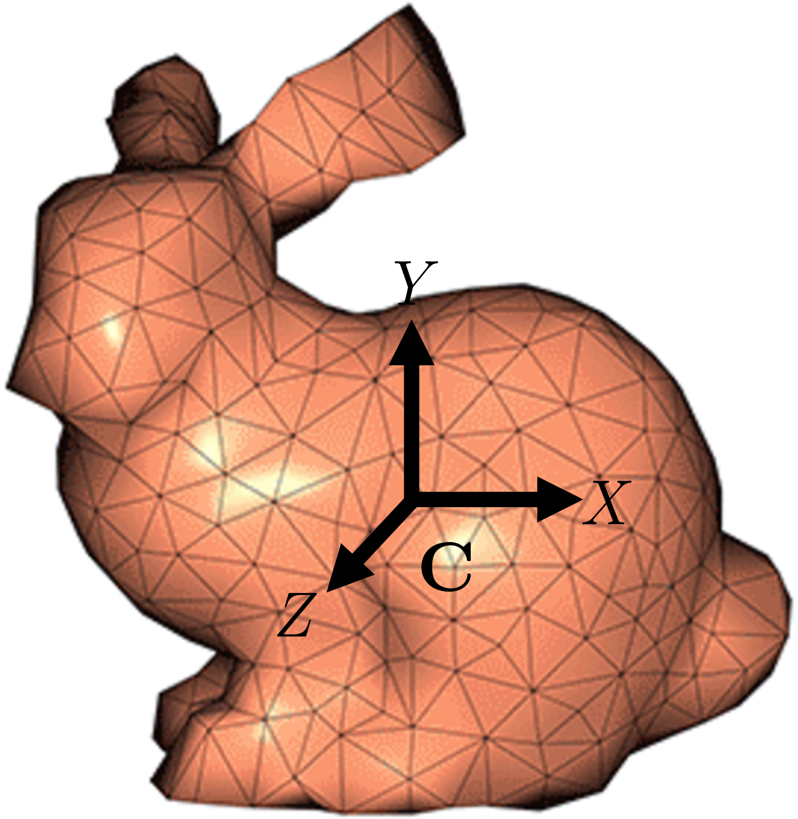


Express using divergence

$$\int_\Omega \rho \cdot 1 \, d\Omega = \int_\Omega \nabla \cdot \begin{pmatrix} \rho \bar{\mathbf{X}}_x \\ 0 \\ 0 \end{pmatrix} d\Omega$$

Reference (Undeformed) Space

Integrate over Surface Triangles

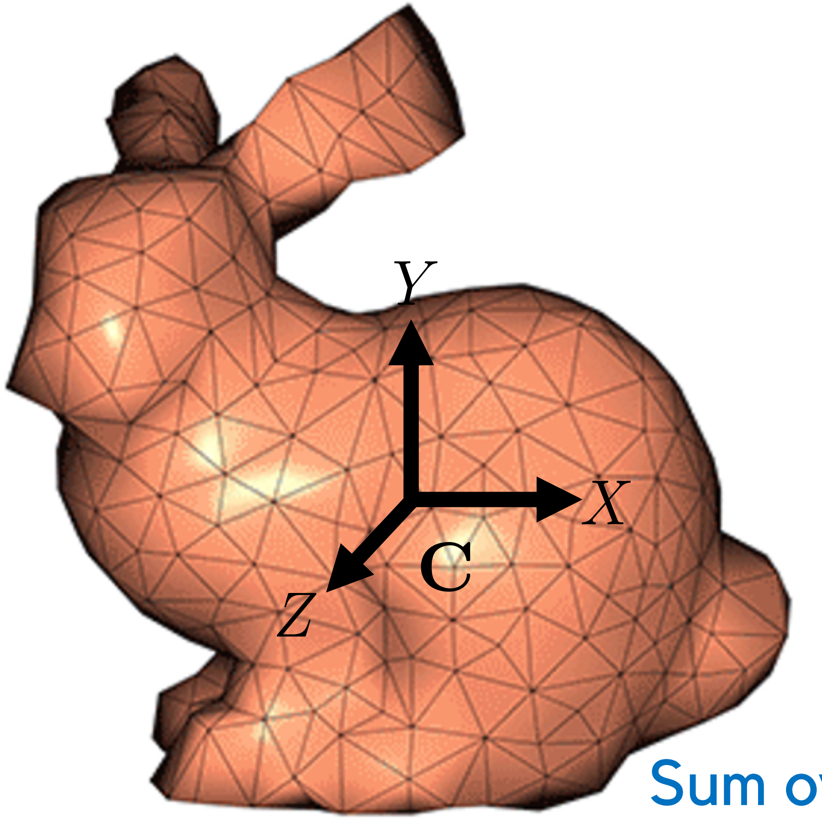


Divergence Theorem

$$\int_{\Omega} \rho \cdot 1 \, d\Omega \; = \int_{\Gamma} \begin{pmatrix} \rho \bar{\mathbf{X}}_x \\ 0 \\ 0 \end{pmatrix} \cdot \mathbf{N} d\Gamma$$

Reference (Undeformed) Space

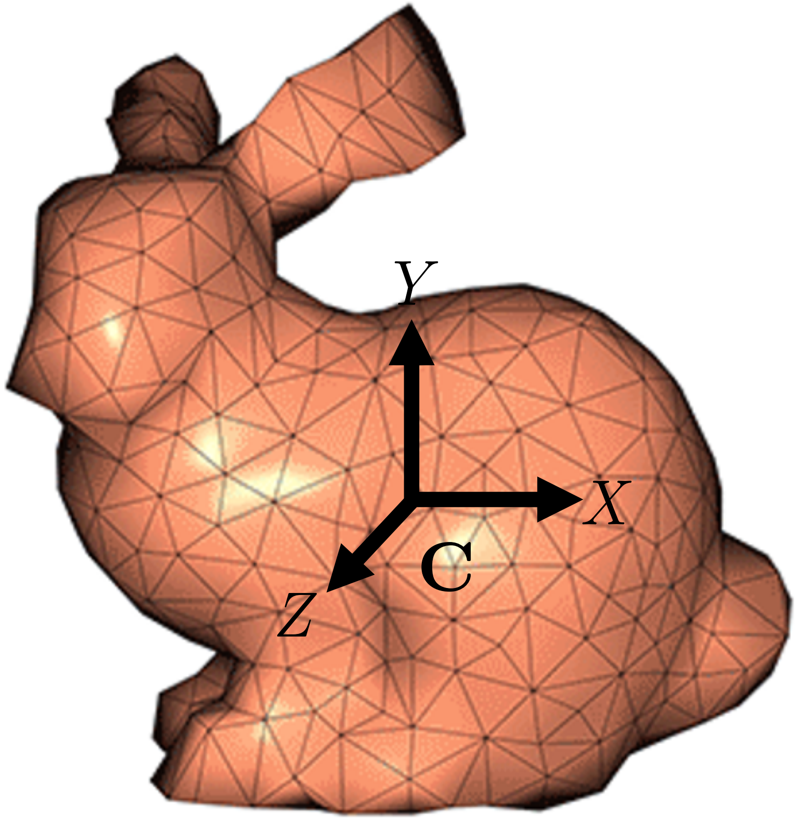
Integrate over Surface Triangles



$$\underbrace{\sum_{j=0}^{m-1}}_{\text{Sum over triangles}} \overbrace{\int_{\Gamma_j} \begin{pmatrix}\rho\bar{\mathbf{X}}_x\\ 0\\ 0\end{pmatrix} \cdot \underbrace{\mathbf{N}_j}_{\text{triangle normal}} d\Gamma_j}^{\text{Integrate over triangle}}$$

Reference (Undeformed) Space

Integrate over Surface Triangles



Reference (Undeformed) Space

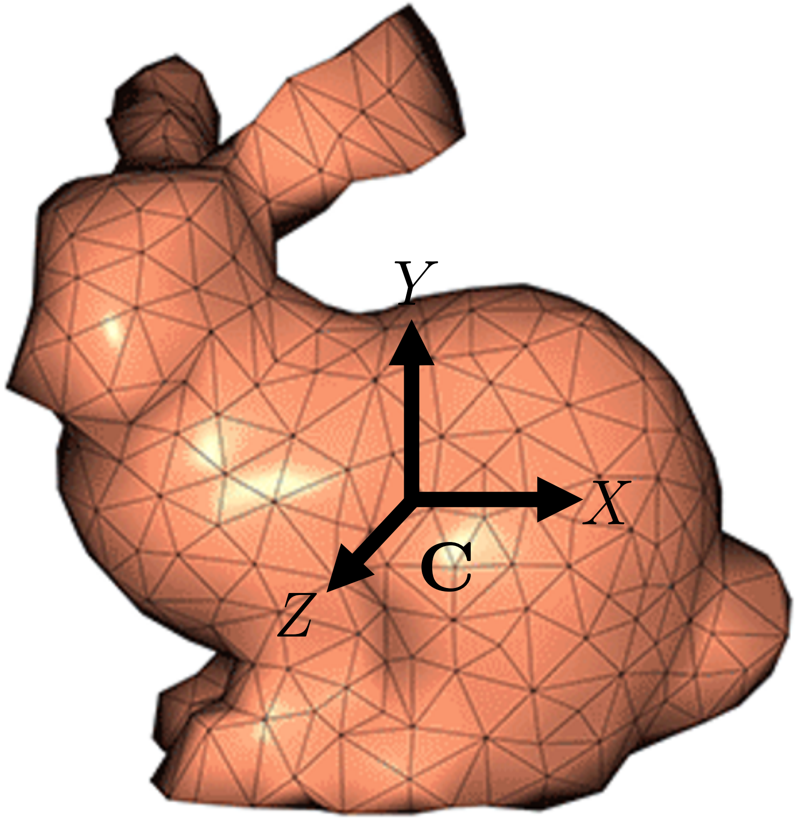
Integrate over triangle

$$\sum_{j=0}^{m-1} \int_{\Gamma_j} \begin{pmatrix} \rho \bar{\mathbf{X}}_x \\ 0 \\ 0 \end{pmatrix} \cdot \mathbf{N}_j d\Gamma_j$$

$$\bar{\mathbf{X}} = \bar{\mathbf{X}}_0 \phi_0 + \bar{\mathbf{X}}_1 \phi_1 + \bar{\mathbf{X}}_2 \phi_2$$

barycentric coordinates

Integrate over Surface Triangles



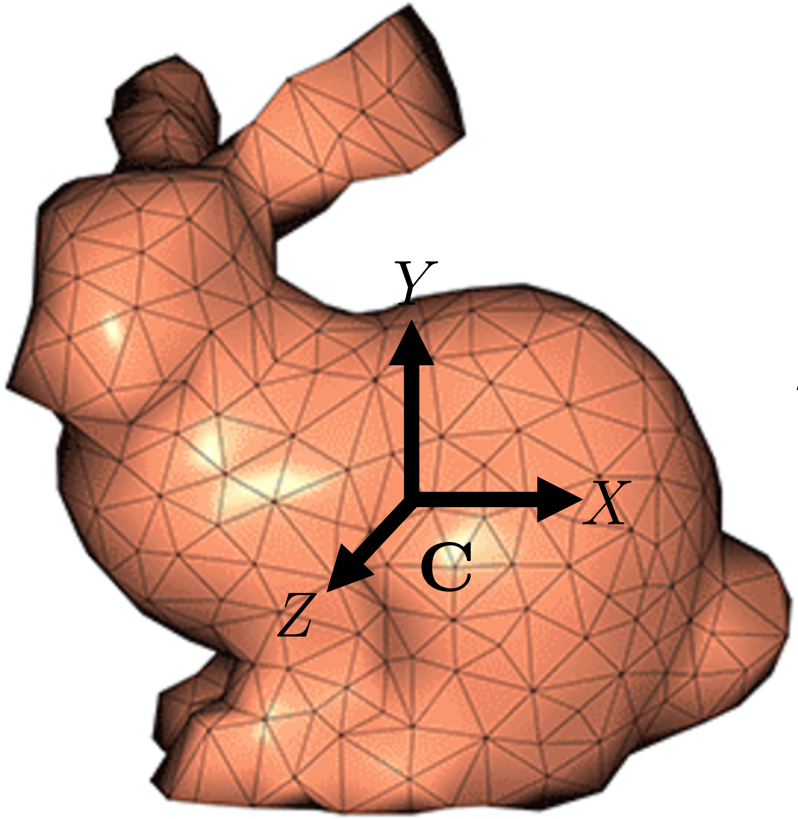
Reference (Undeformed) Space

Use barycentric coordinates

$$\sum_{j=0}^{m-1} \int_{\Gamma_j} \underbrace{h\left(\phi_0, \phi_1, \phi_2\right) d\Gamma_j}$$

h is a polynomial

Integrate over Surface Triangles



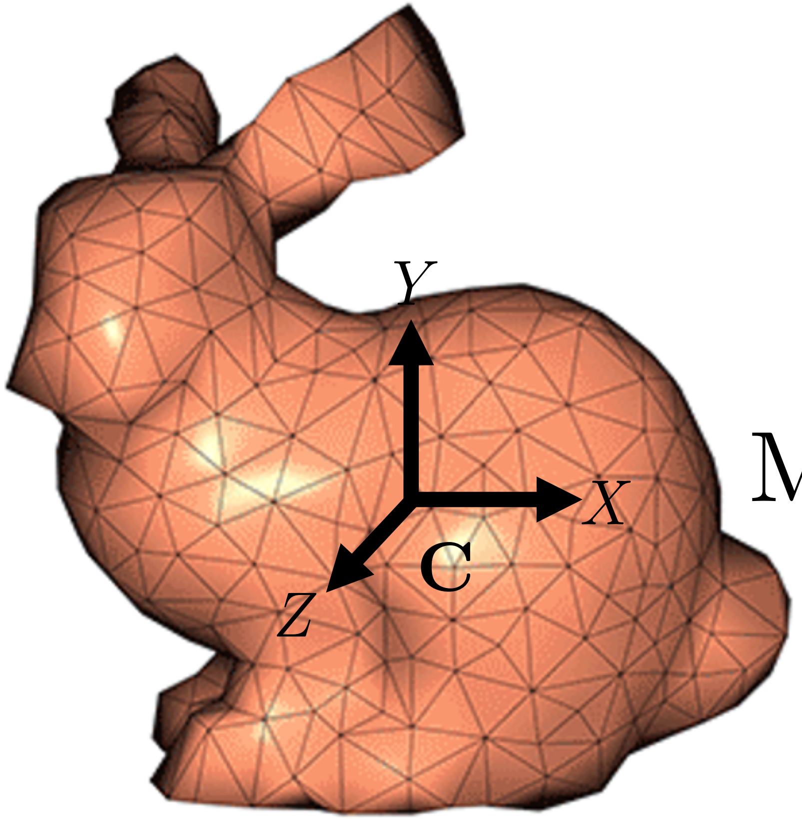
Reference (Undeformed) Space

Use barycentric coordinates

$$\underbrace{\sum_{j=0}^{m-1} \int_0^1 \int_0^{1-\phi_1}} h\left(1-\phi_1-\phi_2, \phi_1, \phi_2\right) d\phi_2 d\phi_1$$

h is a polynomial

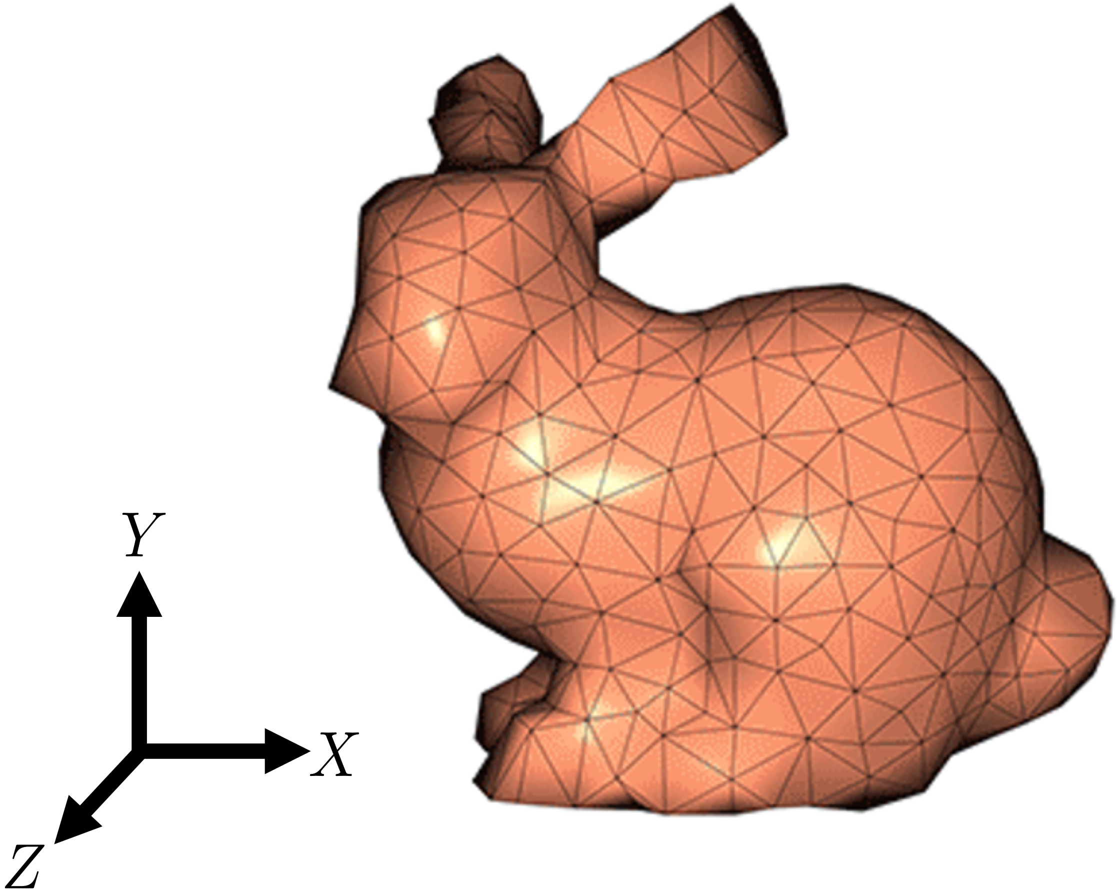
Kinetic Energy of a Rigid Body



$$\mathrm{M}_0 = \begin{pmatrix} \overbrace{\int_\Omega \rho \left[\bar{\mathbf{X}}\right] \left[\bar{\mathbf{X}}\right]^T d\Omega}^{\mathcal{I}} & 0 \\ 0 & \underbrace{m}_{\text{mass of object}}\mathrm{I} \end{pmatrix}$$

Reference (Undeformed) Space

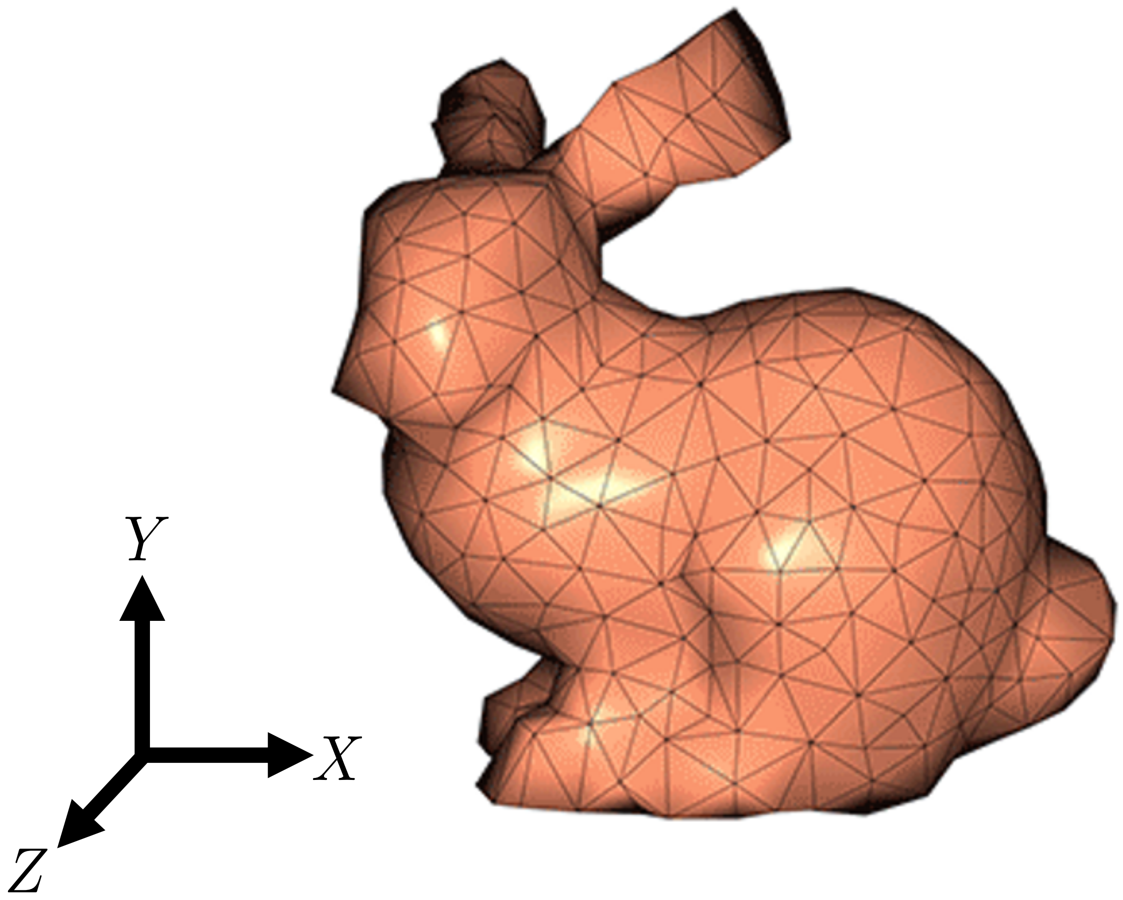
Kinetic Energy of a Rigid Body



Reference (Undeformed) Space

$$T = \frac{1}{2}\dot{\mathbf{q}}^T \mathrm{A}^T \begin{pmatrix} \mathcal{I} & 0 \\ 0 & m\mathrm{I} \end{pmatrix} \underbrace{\mathrm{A}\dot{\mathbf{q}}}_{\begin{pmatrix} \mathrm{R}^T & 0 \\ 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} \omega \\ \dot{\mathbf{p}} \end{pmatrix}}$$

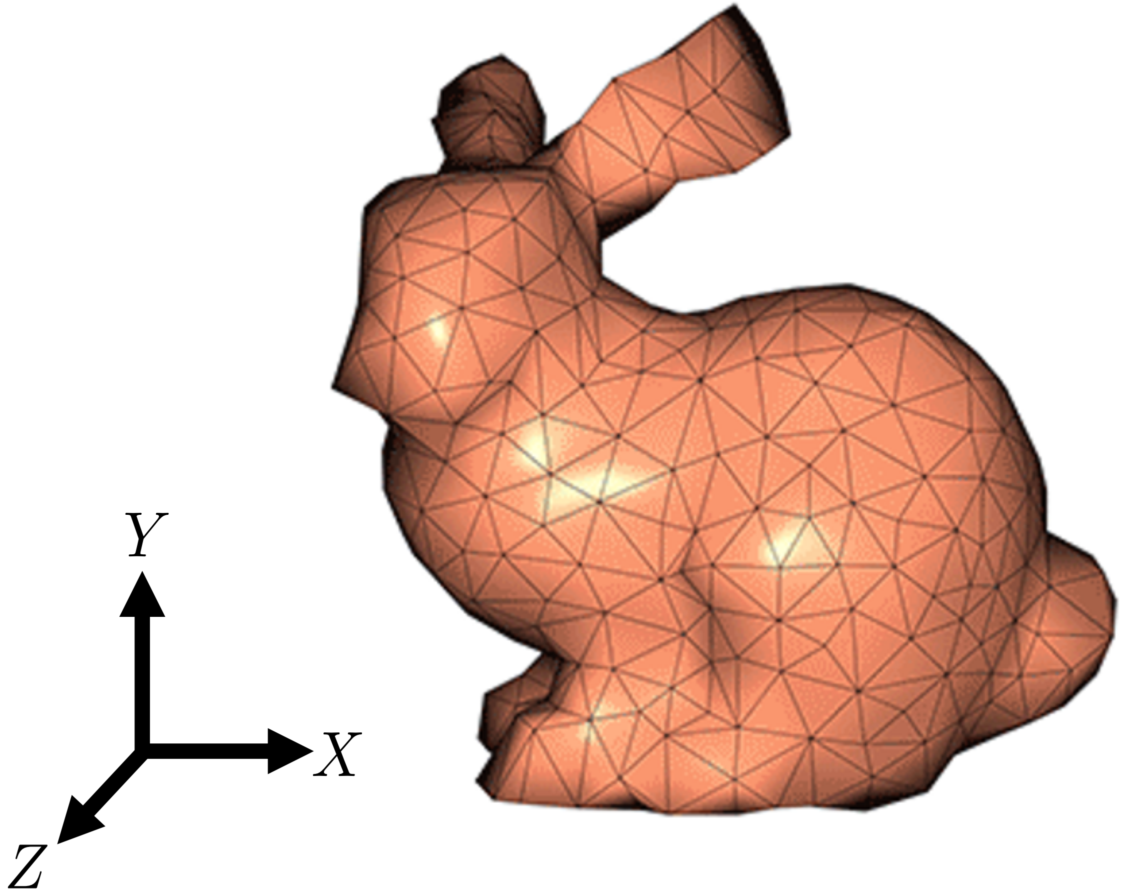
Kinetic Energy of a Rigid Body



Reference (Undeformed) Space

$$T = \frac{1}{2}\dot{\mathbf{q}}^T \begin{pmatrix} \mathrm{R}\mathcal{I}\mathrm{R}^T & 0 \\ 0 & m\mathrm{I} \end{pmatrix} \dot{\mathbf{q}}$$

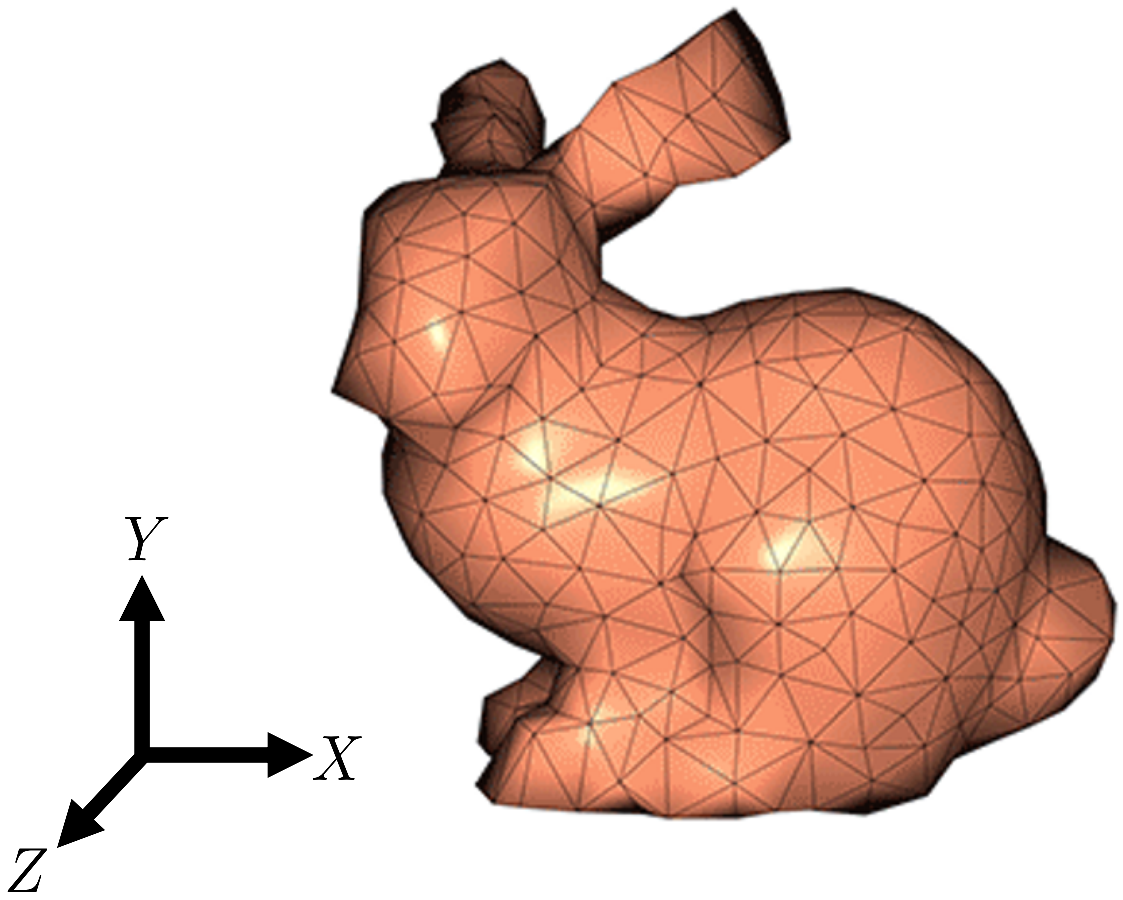
Potential Energy of a Rigid Body



Reference (Undeformed) Space

$$V = ?$$

Potential Energy of a Rigid Body

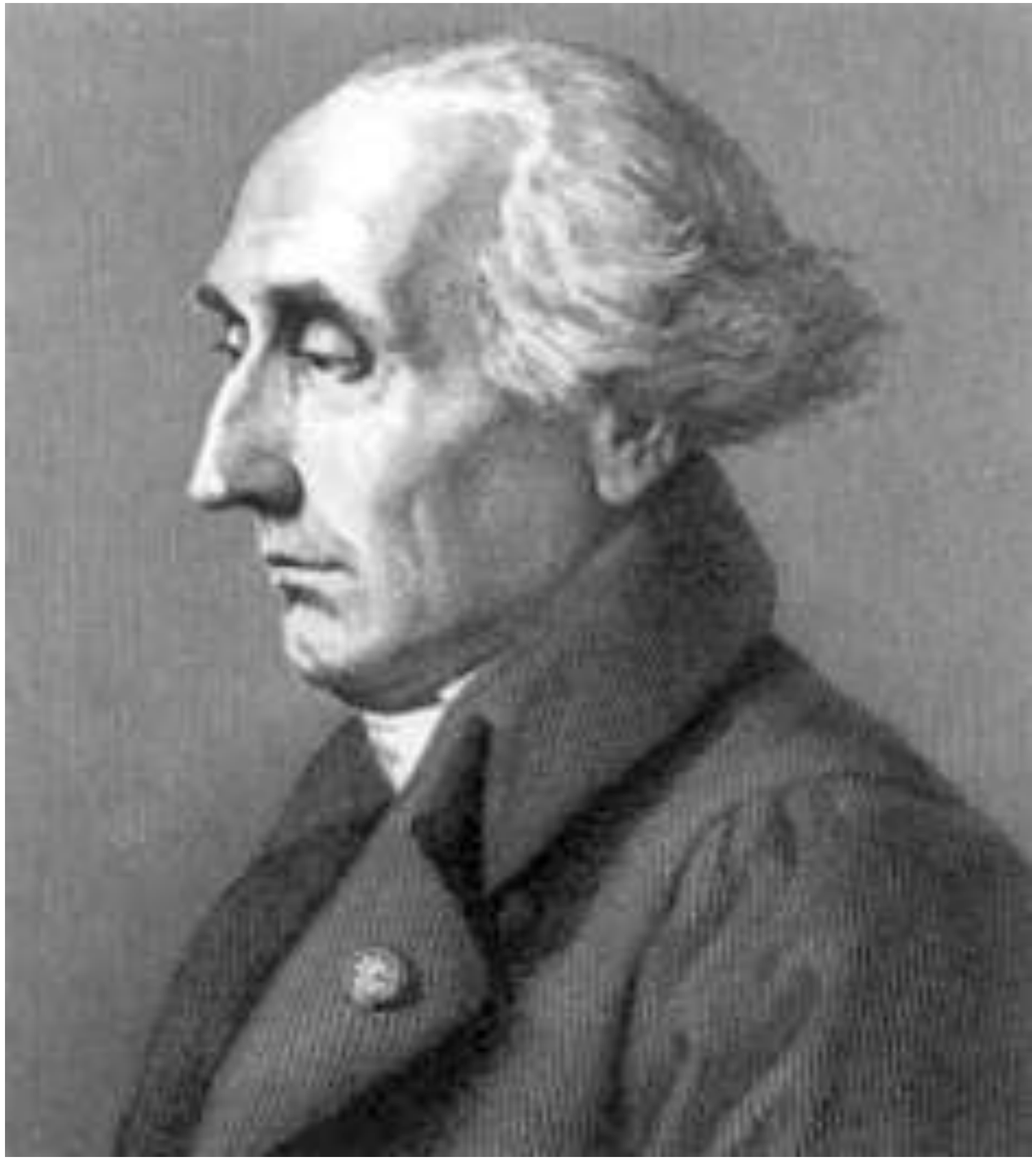


Reference (Undeformed) Space

$$V = 0$$

The Lagrangian

$$L = \underbrace{T}_{\frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\begin{pmatrix} \mathrm{R}\mathcal{I}\mathrm{R}^T & 0 \\ 0 & m\mathrm{I} \end{pmatrix}}_{\text{Rotations mean M is a function of } \mathbf{q} \text{ ☹}} \dot{\mathbf{q}}} - \overbrace{V}^{0}$$



Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}$$

Newton-Euler Equations

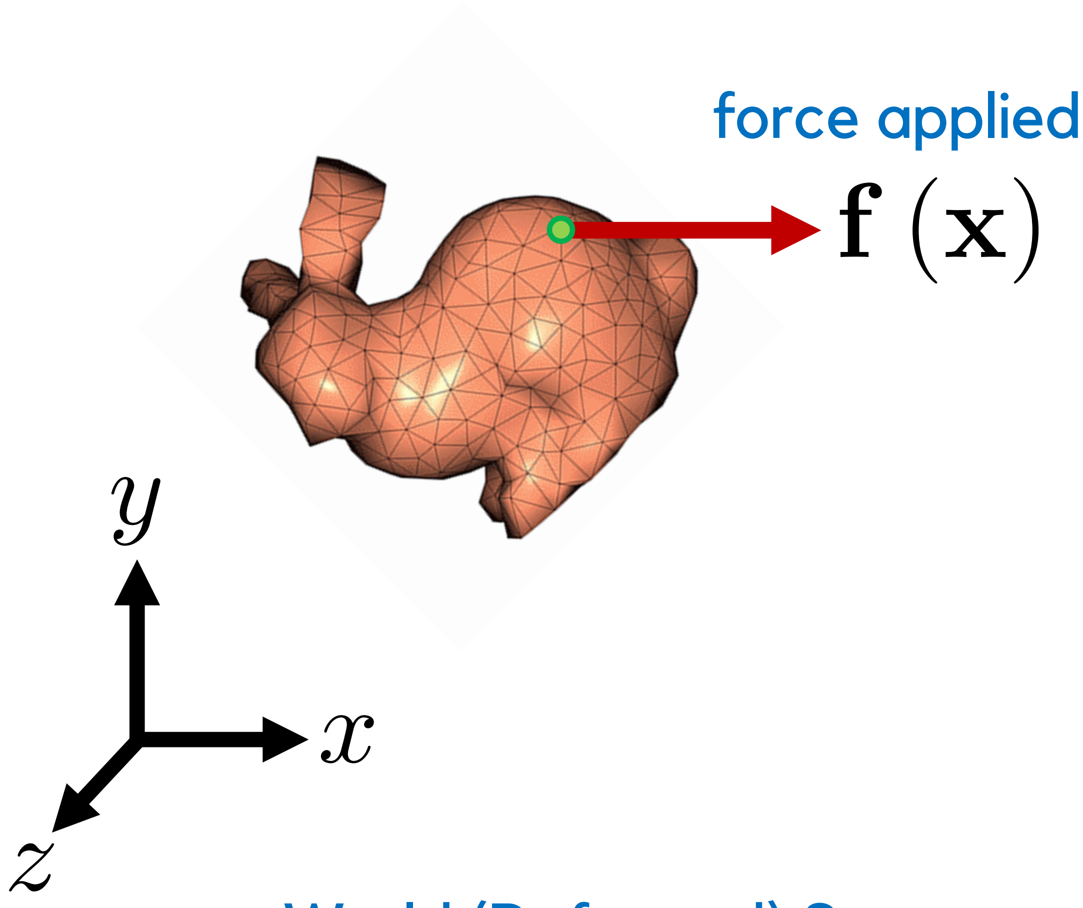
Conservation of Angular Momentum

$$\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\underbrace{\dot{\omega}}_{\text{angular acceleration}} = \underbrace{\omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega\right)}_{\text{quadratic velocity vector}} + \underbrace{\tau_{ext}}_{\text{external torque}}$$

Conservation of Linear Momentum

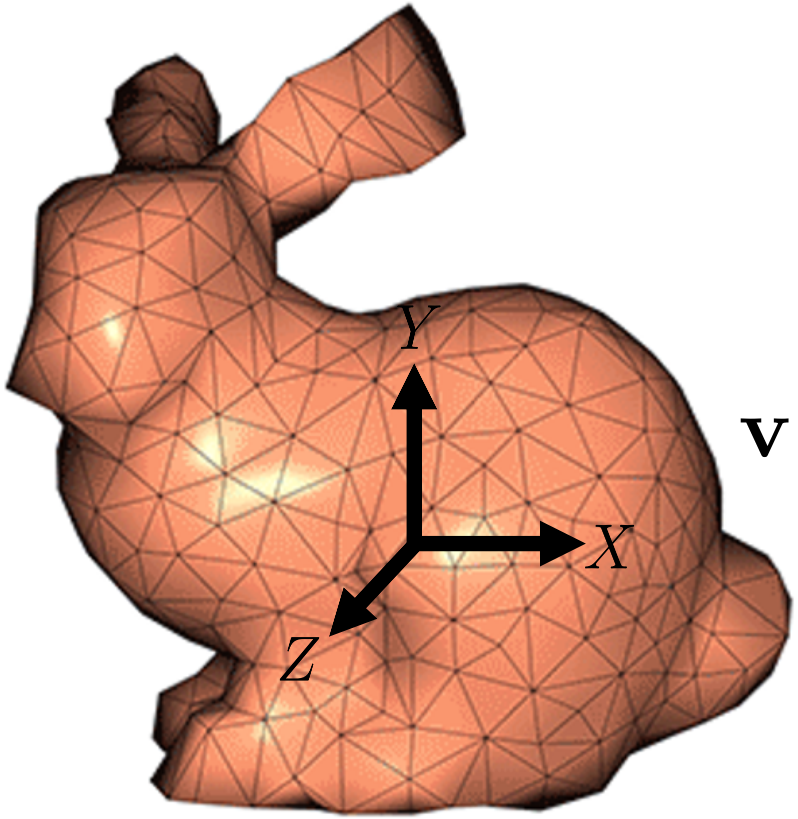
$$m\mathrm{I}\,\underbrace{\ddot{\mathbf{p}}}_{\text{acceleration}} = \underbrace{\mathbf{f}_{ext}}_{\text{external force}}$$

External Torques and Forces



World (Deformed) Space

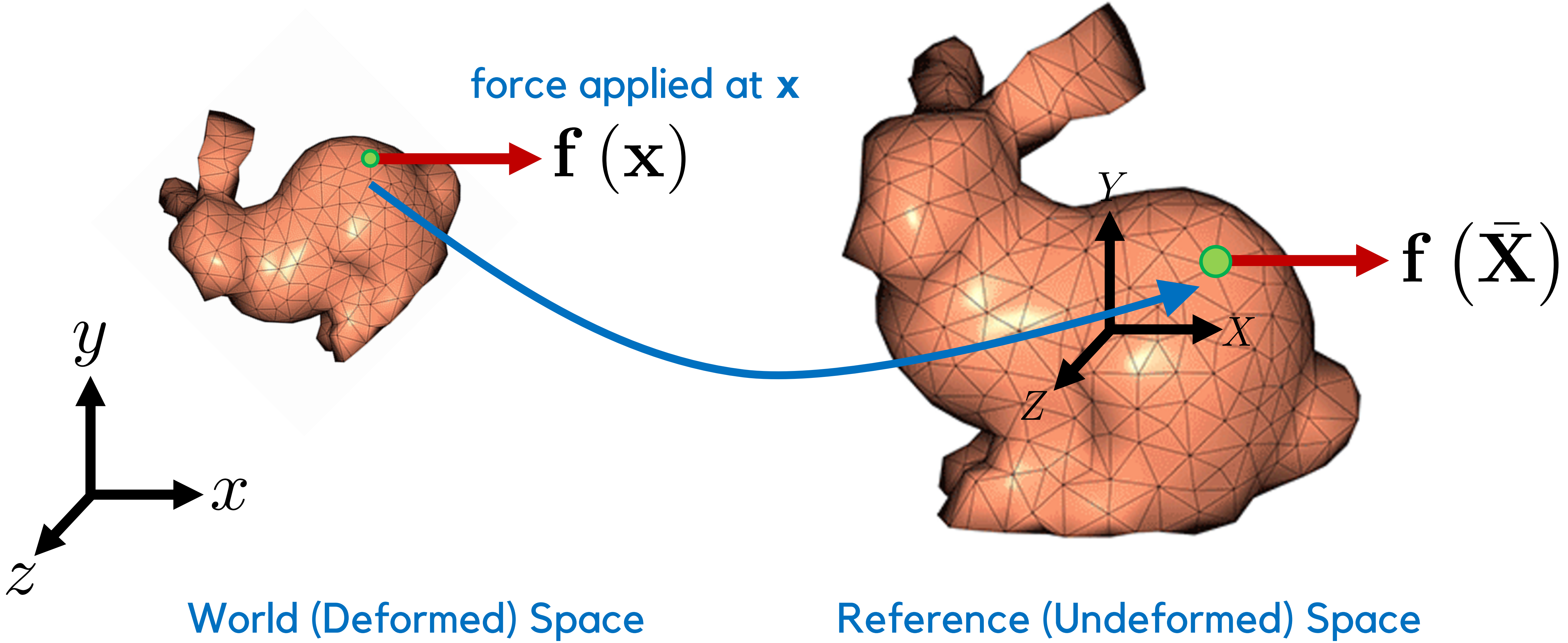
The Rigid Body Jacobian



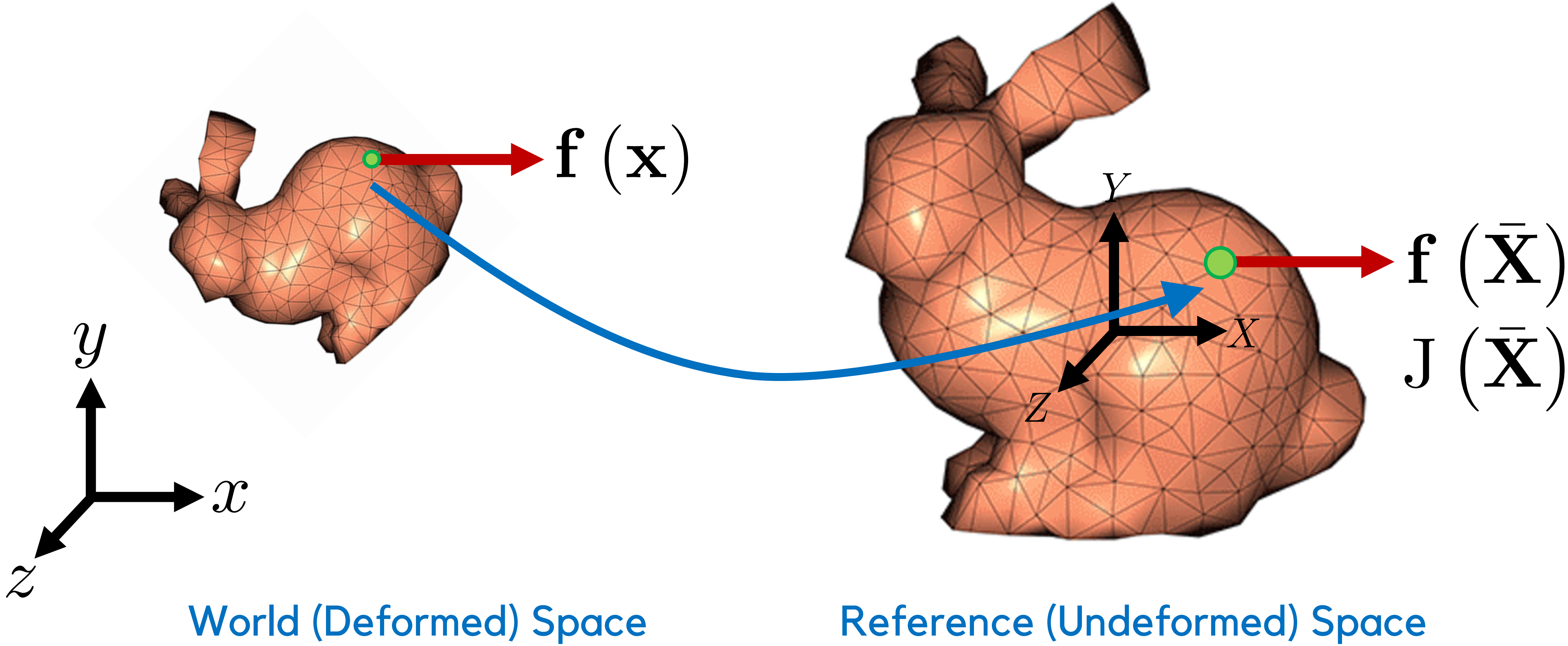
$$\mathbf{v}\left(\bar{\mathbf{X}}, t\right) = \mathrm{R}\underbrace{\begin{pmatrix}[\bar{\mathbf{X}}]^T & \mathrm{I}\end{pmatrix}\begin{pmatrix}\mathrm{R}^T & 0 \\ 0 & \mathrm{I}\end{pmatrix}}_{\substack{\text{Jacobian} \\ \mathrm{J} \in \mathbb{R}^{3\times 6}}}\underbrace{\begin{pmatrix}\omega \\ \dot{\mathbf{p}}\end{pmatrix}}_{\dot{\mathbf{q}} \in \mathbb{R}^6}$$

Reference (Undeformed) Space

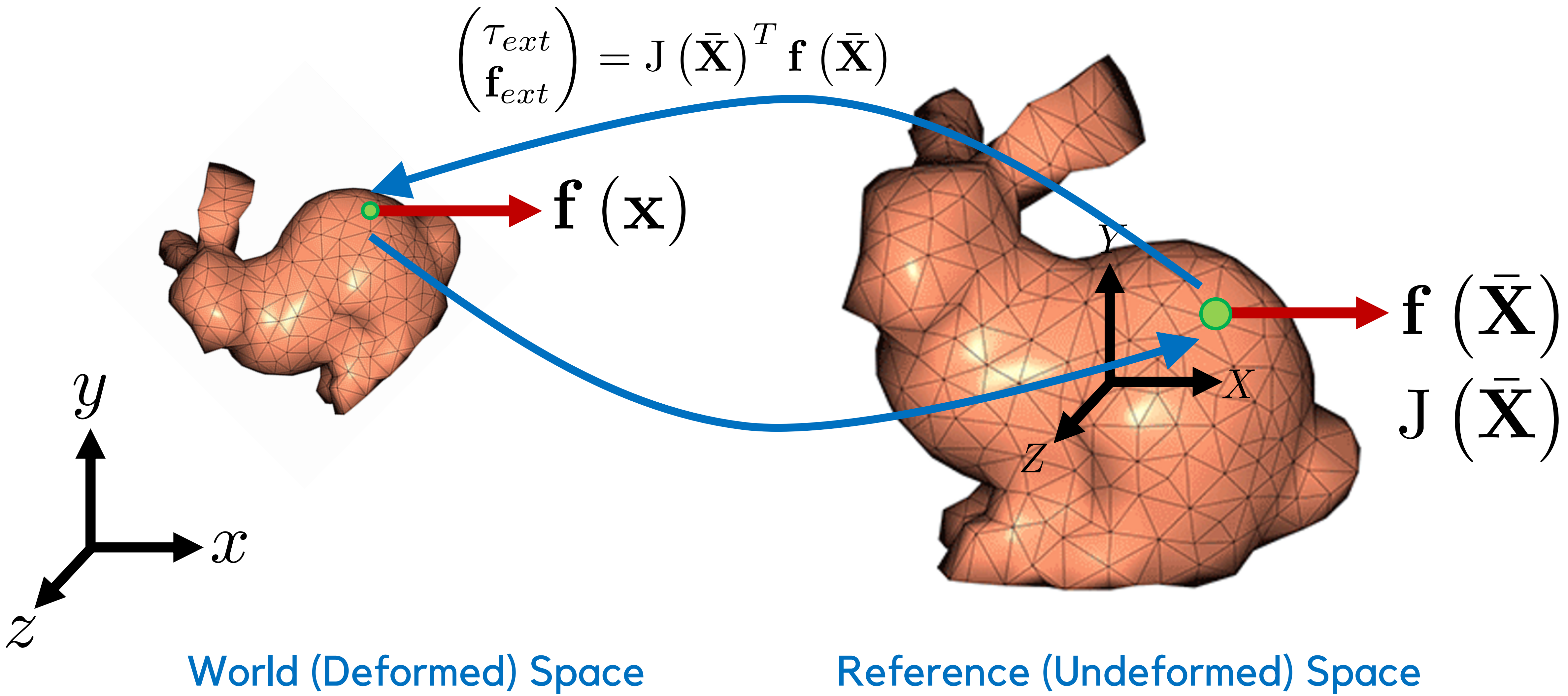
External Torques and Forces



External Torques and Forces



External Torques and Forces



Time Integration for a Rigid Body

$$\left.\begin{aligned}&\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\dot{\omega} = \omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega\right) + \tau_{ext}\\ &m\mathrm{I}\,\ddot{\mathbf{p}} = \mathbf{f}_{ext}\end{aligned}\right\}\text{Two 2}^{\text{nd}}\text{ order ODEs}$$

Derive Update Equations Independently

$$m\mathrm{I}\,\ddot{\mathbf{p}} = \mathbf{f}_{ext}$$

$$m\dot{\mathbf{p}}^{t+1} = m\dot{\mathbf{p}}^{t} + \Delta t\mathbf{f}_{ext}$$

$$\mathbf{p}^{t+1} = \mathbf{p}^{t} + \Delta t\dot{\mathbf{p}}^{t}$$

Time Integration for a Rigid Body

$$\left.\begin{aligned}&\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\dot{\omega} = \omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega\right) + \tau_{ext}\\ &m\mathrm{I}\,\ddot{\mathbf{p}} = \mathbf{f}_{ext}\end{aligned}\right\}\text{Two 2}^{\text{nd}}\text{ order ODEs}$$

Derive Update Equations Independently

$$\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\dot{\omega} = \omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega\right) + \tau_{ext}$$

$$\dot{\omega} \approx \frac{1}{\Delta t}\left(\omega^{t+1} - \omega^t\right)$$

$$\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^{t+1} = \left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^t + \underbrace{\Delta t\,\omega^t \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^t\right) + \Delta t\,\tau^t_{ext}}_{\text{explicit}}$$

Time Integration for a Rigid Body

$$
\left.\begin{aligned}
\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\dot{\omega} &= \omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega\right) + \tau_{ext} \\
m\mathrm{I}\,\ddot{\mathbf{p}} &= \mathbf{f}_{ext}
\end{aligned}\right\} \text{Two 2}^{\text{nd}}\text{ order ODEs}
$$

Derive Update Equations Independently

$$
\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^{t+1} = \left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^t + \underbrace{\Delta t\,\omega^t \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^t\right) + \Delta t\,\tau_{ext}^t}_{\text{explicit}}
$$

$$
\mathrm{R}^{t+1} = ?
$$

Aside: Update Rules from Integration

Standard Forward-Euler update for position

$$\mathbf{x}^{t+1} = \mathbf{x}^t + \Delta t\, \mathbf{v}^t$$

Comes from finite difference approximation of velocity

$$\mathbf{v}^t \approx \frac{1}{\Delta t}\left(\mathbf{x}^{t+1} - \mathbf{x}^t\right)$$

Alternately, can be derived by solving an Initial Value Problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}^t \qquad \mathbf{x}\left(t_0\right) = \mathbf{x}^t$$

Aside: Update Rules from Integration

Alternately, can be derived by solving an Initial Value Problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}^t \qquad \mathbf{x}(t_0) = \mathbf{x}^t$$

Rearrange and integrate both sides in time

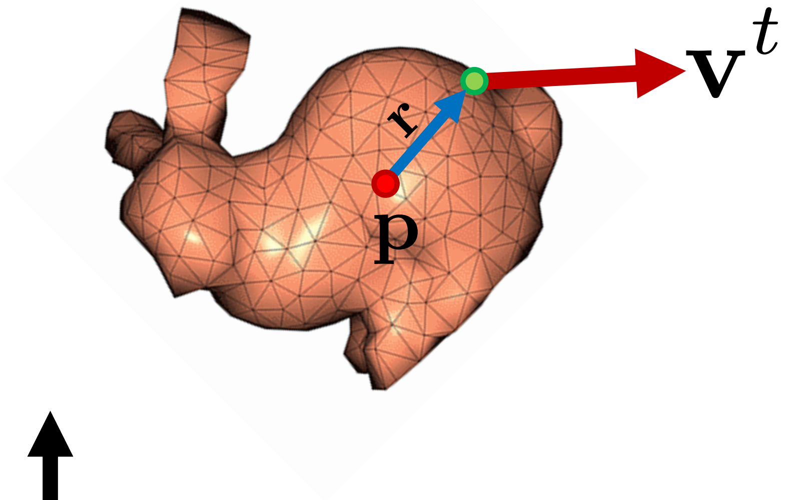
$$\int_{t0}^{t1} d\mathbf{x} = \int_{t0}^{t1} \mathbf{v}^t dt$$

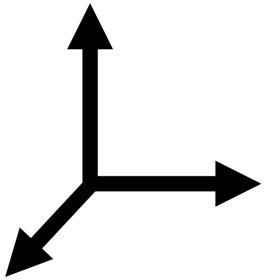
$$\mathbf{x}(t) = \mathbf{v}^t \underbrace{(t_1 - t_0)}_{\Delta t} + \underbrace{\mathbf{c}}_{\text{constant of integration} = \mathbf{x}^t}$$

Aside: Update Rules from Integration

Angular velocity equation

$$\mathbf{v}^t = \underbrace{\omega^t}_{\text{angular velocity vector}} \times \underbrace{\left(\mathrm{R}\bar{\mathbf{X}} + \mathbf{p} - \mathbf{p}\right)}_{\mathbf{r}}$$





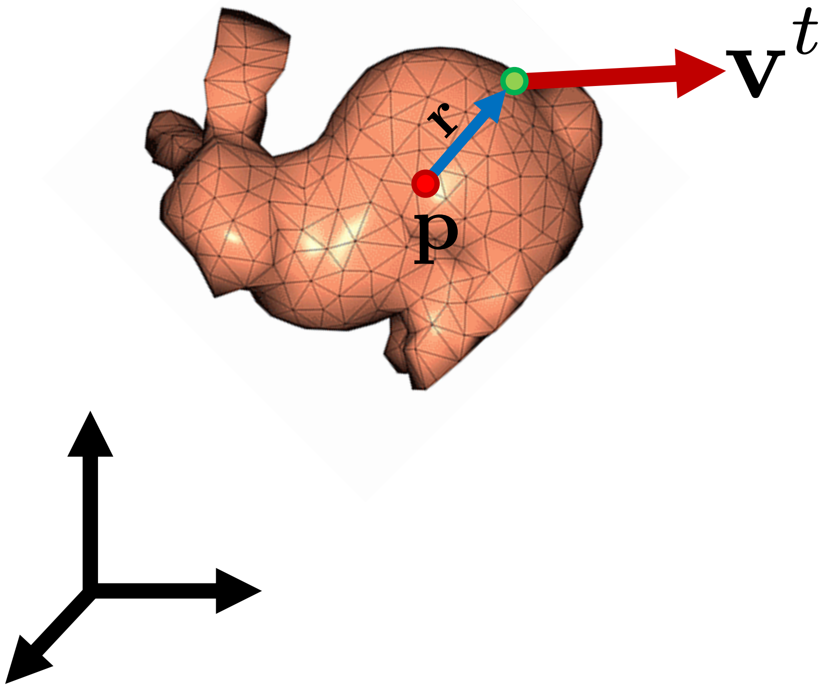
$$\mathbf{y} = \mathrm{R}\bar{\mathbf{X}}$$

$$\mathbf{v}^t = \frac{d\mathbf{y}}{dt} = \omega^t \times \mathbf{y}$$

$$\mathbf{v} = \frac{d\mathbf{y}}{dt} = \underbrace{[\omega]^t}_{\text{cross-product matrix}} \mathbf{y} \qquad \underbrace{\mathbf{y}(t_0) = \mathbf{y}^t}_{\text{initial conditions}}$$

Aside: Update Rules from Integration

$$\mathbf{v} = \frac{d\mathbf{y}}{dt} = \underbrace{[\omega]^t}_{\text{cross-product matrix}}\mathbf{y} \qquad \underbrace{\mathbf{y}\left(t_0\right) = \mathbf{y}^t}_{\text{initial conditions}}$$



Equation is separable, so the solution is

$$\mathbf{y}\left(t_1\right) = \underbrace{\exp\left([\omega]^t \Delta t\right)}_{\text{matrix exponential of skew-symmetric matrix}} \mathbf{y}\left(t_0\right)$$

$$\mathbf{y}\left(t_1\right) = \underbrace{\exp\left([\omega]^t \Delta t\right) \mathrm{R}\left(t_0\right)}_{\text{updated rotation matrix}} \bar{\mathbf{X}}$$

Time Integration for a Rigid Body

$$\left.\begin{aligned}\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\dot{\omega} &= \omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega\right) + \tau_{ext} \\ m\mathrm{I}\,\ddot{\mathbf{p}} &= \mathbf{f}_{ext}\end{aligned}\right\}\text{Two 2}^{\text{nd}}\text{ order ODEs}$$

Derive Update Equations Independently

$$\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^{t+1} = \left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^t + \underbrace{\Delta t\,\omega^t \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^t\right) + \Delta t\,\tau_{ext}^t}_{\text{explicit}}$$

$$\mathrm{R}^{t+1} = ?$$

Time Integration for a Rigid Body

$$\left.\begin{aligned}\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\dot{\omega} &= \omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega\right) + \tau_{ext} \\ m\mathrm{I}\,\ddot{\mathbf{p}} &= \mathbf{f}_{ext}\end{aligned}\right\}\ \text{Two 2nd order ODEs}$$

Derive Update Equations Independently

$$\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^{t+1} = \left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^t + \underbrace{\Delta t\,\omega^t \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega^t\right) + \Delta t\,\tau^t_{ext}}_{\text{explicit}}$$

$$\mathrm{R}^{t+1} = \underbrace{\exp\left([\omega]^t\Delta t\right)}_{\text{Rodrigues' Rotation Formula}}\mathrm{R}^t$$

Time Integration for a Rigid Body

$$\left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\dot{\omega} = \omega \times \left(\left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\omega\right) + \tau_{ext}$$

$$m\mathrm{I}\,\ddot{\mathbf{p}} = \mathbf{f}_{ext}$$

Two 2nd order ODEs

Velocity Update Equations

$$\left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\omega^{t+1} = \left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\omega^t + \Delta t\,\omega^t \times \left(\left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\omega^t\right) + \Delta t\,\tau^t_{ext}$$

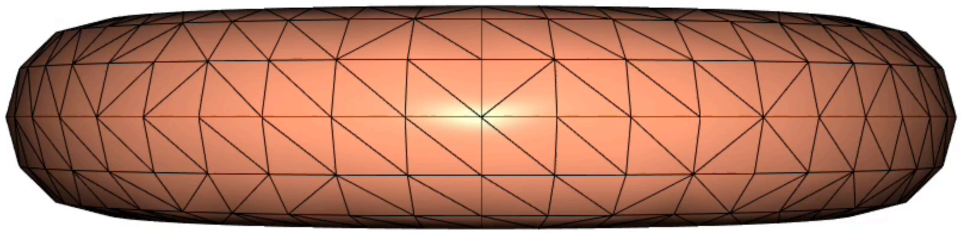
$$m\dot{\mathbf{p}}^{t+1} = m\dot{\mathbf{p}}^t + \Delta t\mathbf{f}_{ext}$$

Position Update Equations

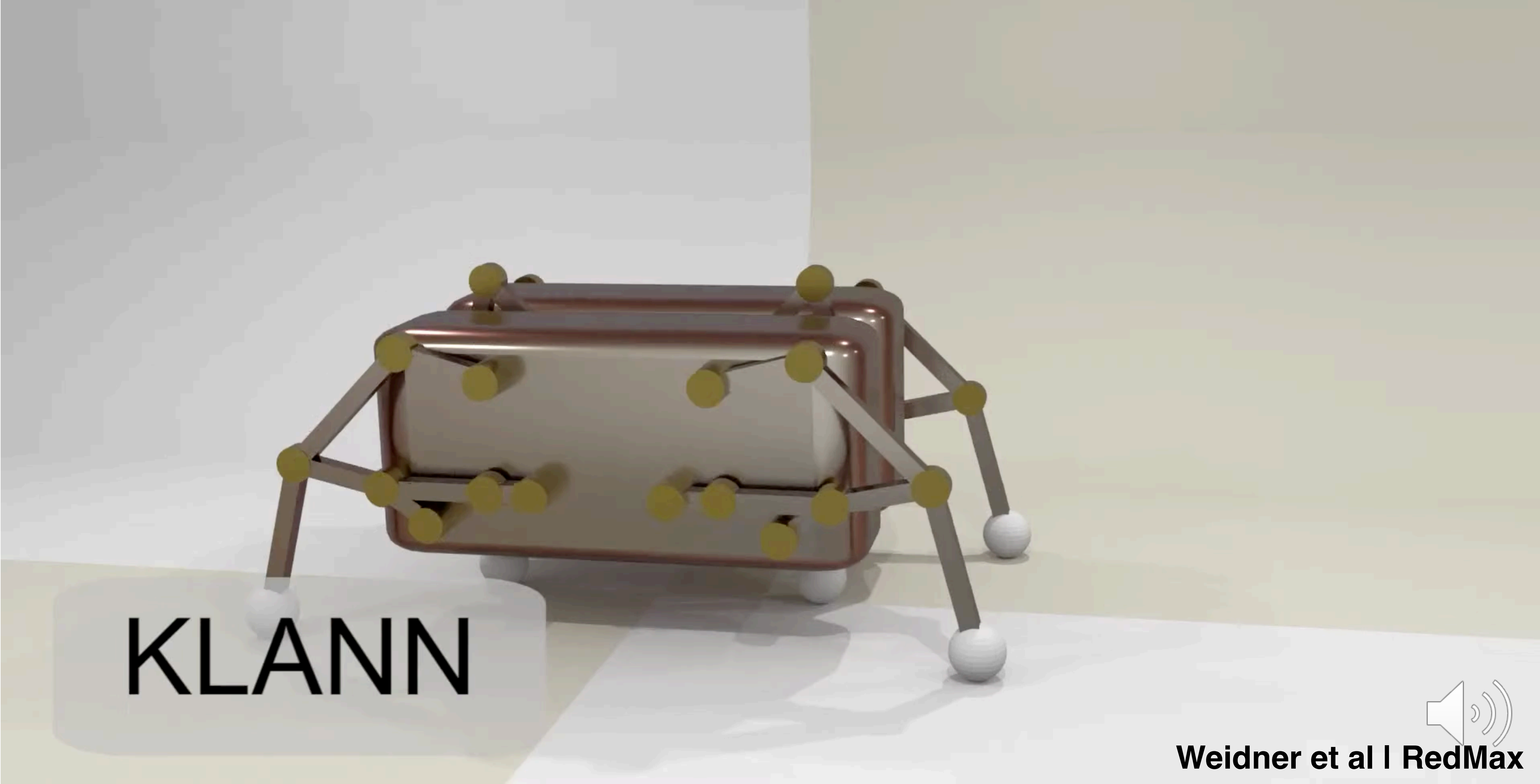
$$\mathrm{R}^{t+1} = \exp\left([\omega]^t\Delta t\right)\mathrm{R}^t$$

$$\mathbf{p}^{t+1} = \mathbf{p}^t + \Delta t\dot{\mathbf{p}}^t$$

Rigid Body Simulation of a Space Donut



Next Video: Jointed Rigid Body Systems



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