

A close-up shot from the movie Toy Story 3 showing a hand holding a small, glowing, orange, textured object, possibly a piece of wood or a small fire, against a dark background. The object is emitting a bright, warm light, and the hand is positioned in the lower left corner of the frame.

CSC417 Physics-Based **Animation**

Last Week: Affine Body Simulation with Contact



Questions from Previous Lecture ?



We Solve This Every Time Step

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

Gradient of what equals this ? Let's guess, then check

Newton's Method

Choose an initial guess

$$i = 0$$

$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{H}^i \mathbf{d} = -\mathbf{g}^i \quad \text{Solve linear system to get } \mathbf{d}$$

Choose α using line search

Use search direction to update current guess

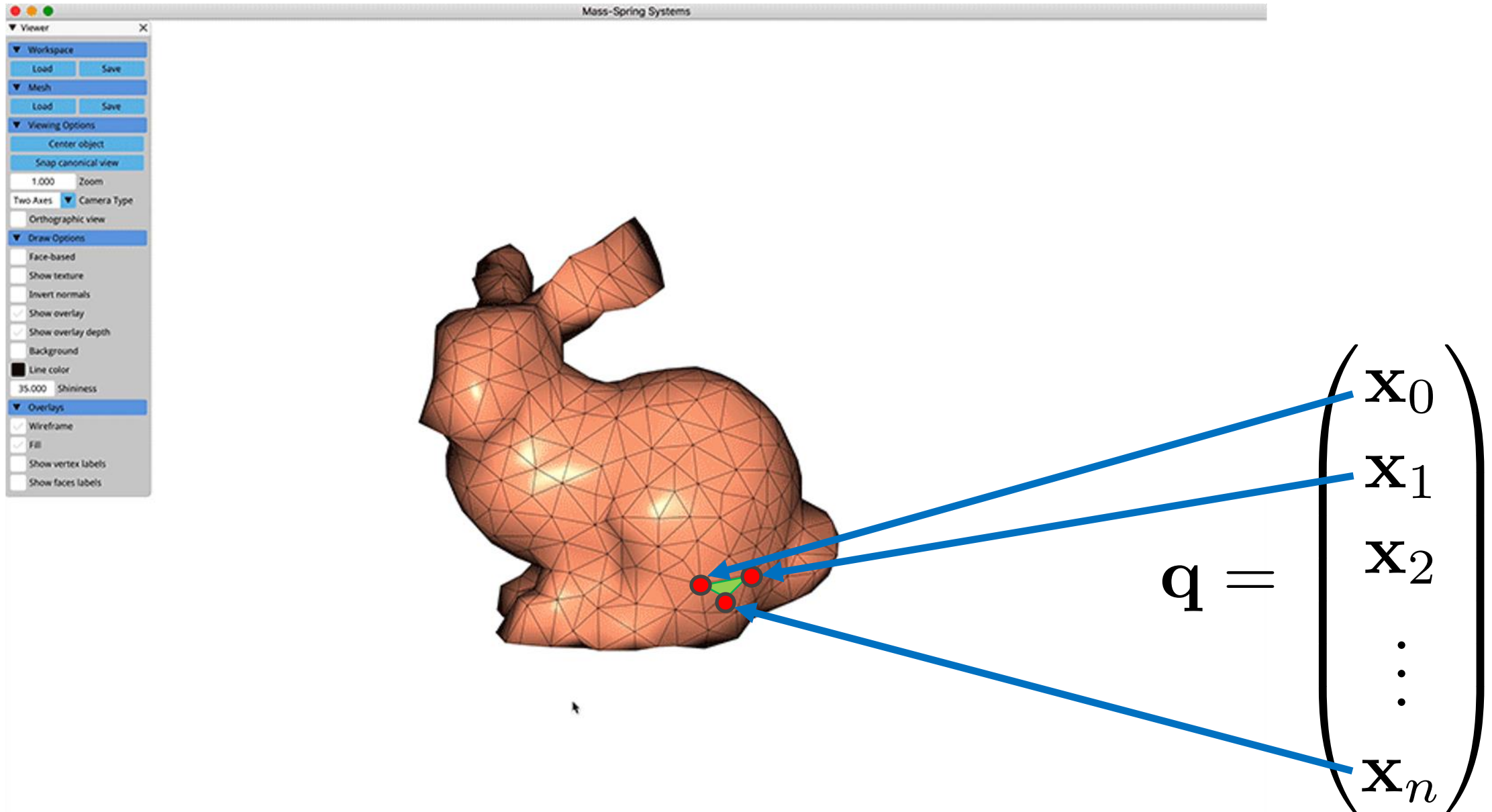
$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

$$i = i + 1$$

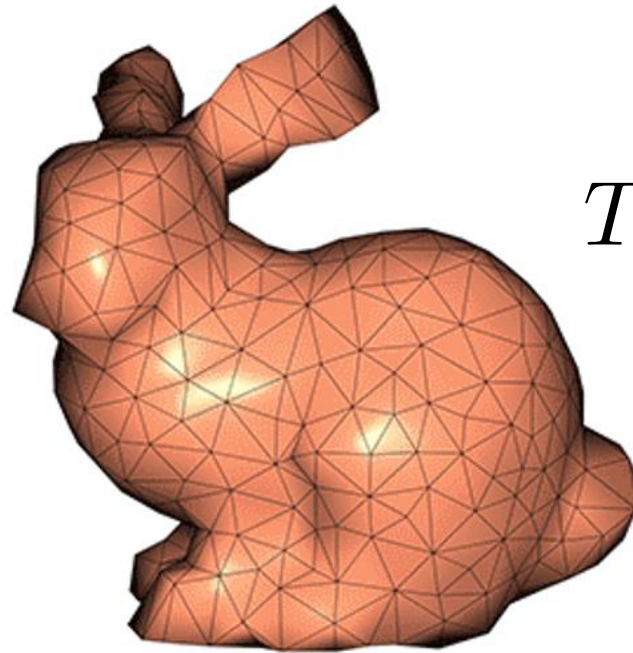
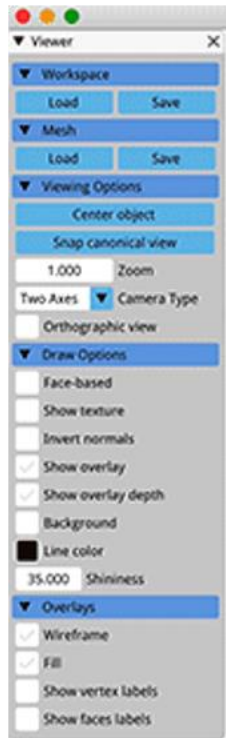
Repeat until converged



Spatial Discretization -- Finite Elements



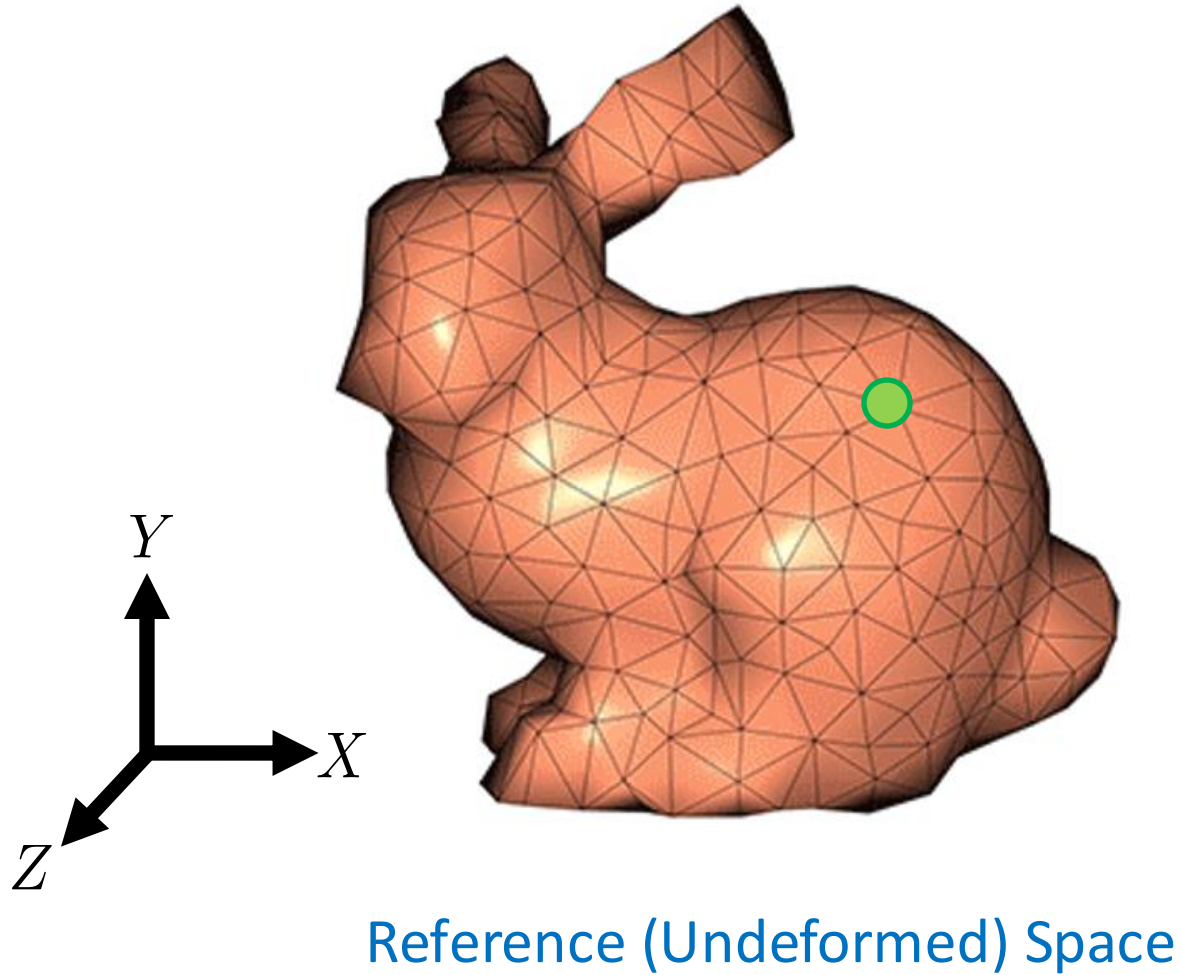
Kinetic Energy for a Bunny



$$T = \frac{1}{2} \dot{\mathbf{q}}^T \underbrace{\left(\sum_{j=0}^{j-1} \mathbf{E}_j^T \mathbf{M}_j \mathbf{E}_j \right)}_{\mathbf{M}} \dot{\mathbf{q}}$$

Assemble \mathbf{M} by summing over all tetrahedra

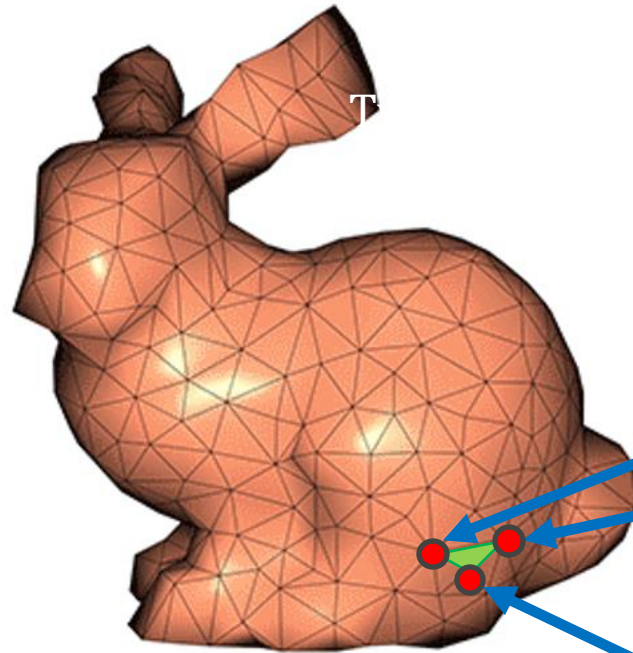
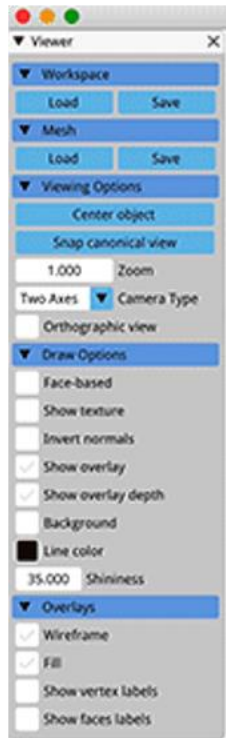
Affine Body Dynamics – Reduced-Order Elasticity



$$\mathbf{x}(\mathbf{X}, t) = \mathbf{J}(\mathbf{X})\mathbf{q}(t)$$



Spatial Discretization -- Finite Elements



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Reduced
Space

$$= \mathbf{U} \mathbf{r}$$

$|r| \ll |q|$

Pros and Cons of Model Reduction

Pros

- Runtime depends on smaller number of degrees of freedom
- Many calculations become independent of mesh resolution

Cons

- Loss of expressivity (reduced model can't represent all the motions the full space model can)

**How Do We Build a Reduced-Order Model
That's Compact and Expressive ?**

How Do We Build a Reduced-Order Model That's Compact and Expressive ?

1. Modal Analysis
2. Data-Driven Deformation Modes

Modal Analysis Starts Here

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Modal Analysis Starts Here

$$M\ddot{\mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = 0$$

Linearize Around Rest Shape

$$M(\boldsymbol{q}(0)) \ddot{\boldsymbol{u}}(t) + \frac{\partial V(\boldsymbol{q}(0) + \boldsymbol{u}(t))}{\partial \boldsymbol{q}} = 0$$

Linearize Around Rest Shape

$$M\ddot{\mathbf{u}}(t) + \frac{\partial^2 V}{\partial \mathbf{q}^2} \mathbf{u}(t) = 0$$

T
H

Linearize Around Rest Shape

$$M\ddot{\boldsymbol{u}}(t) + H\boldsymbol{u}(t) = 0$$

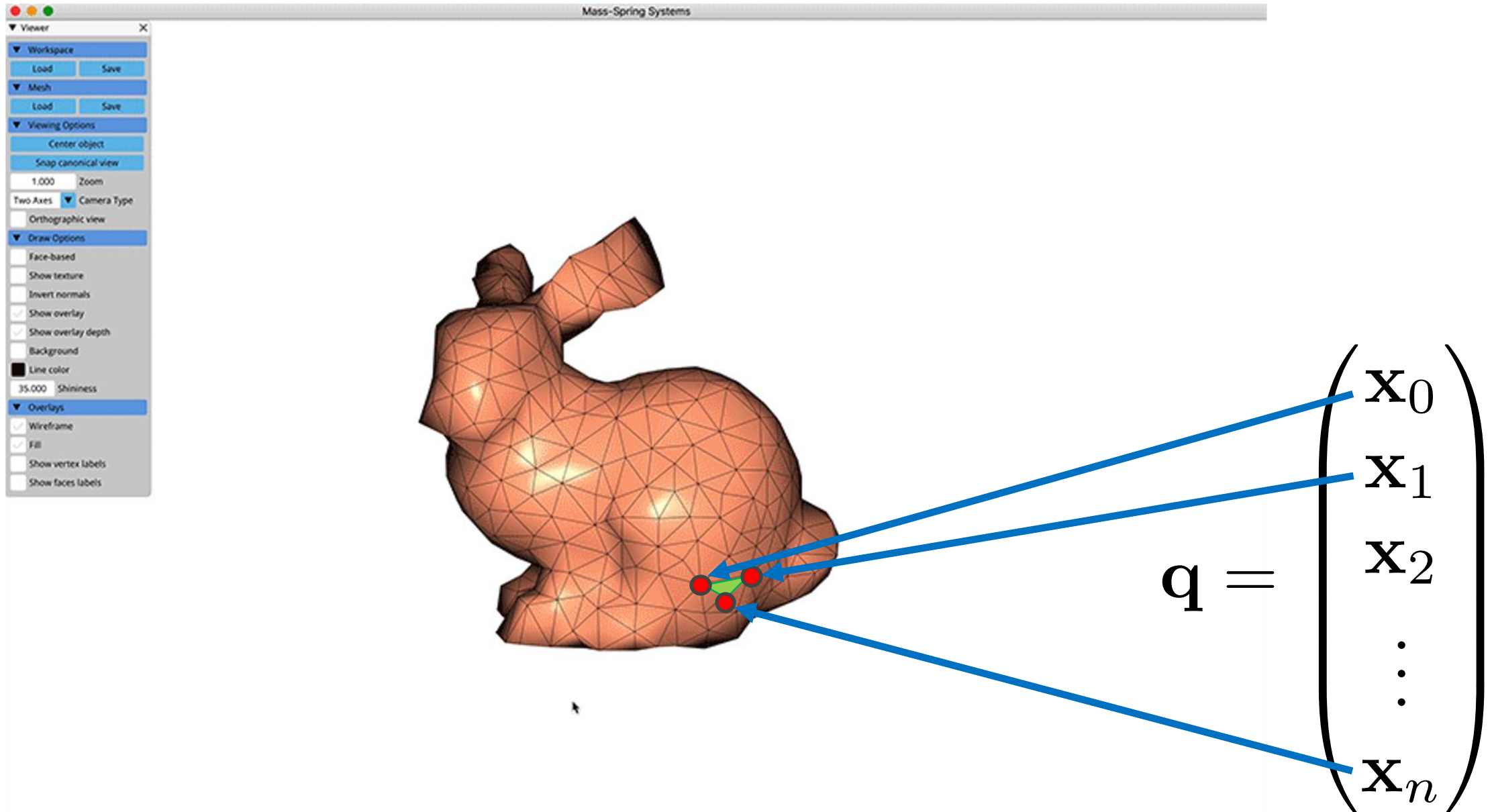
A linear, homogenous ODE!

How can we solve it ?

Assume we know the solution !

$$\mathbf{u} = \overset{\mathcal{R}^{3n} \text{ vector}}{\overline{\mathbf{v}}} e^{\lambda t}$$
$$\ddot{\mathbf{u}} = \lambda^2 \underset{\text{Complex constant}}{\mathbf{v}} e^{\lambda t}$$

Spatial Discretization -- Finite Elements



Assume we know the solution !

$$\mathbf{u} = \overbrace{v}^{\mathcal{R}^{3n} \text{ vector}} e^{\lambda t}$$
$$\ddot{\mathbf{u}} = \lambda^2 \underbrace{v}_{\text{Complex constant}} e^{\lambda t}$$

Substitute our “guess” into the equation

$$\lambda^2 M \mathbf{u} e^{\lambda t} - K \mathbf{u} e^{\lambda t} = \mathbf{0}$$

Substitute our “guess” into the equation

$$\lambda^2 M \cancel{ue^{\lambda t}} - K \cancel{ue^{\lambda t}} = 0$$

Generalized Eigenvector Problem

$$\lambda^2 M \mathbf{u} - K \mathbf{u} = \mathbf{0}$$

λ
Eigenvalue

\mathbf{u}
Eigenvector

Modal Analysis

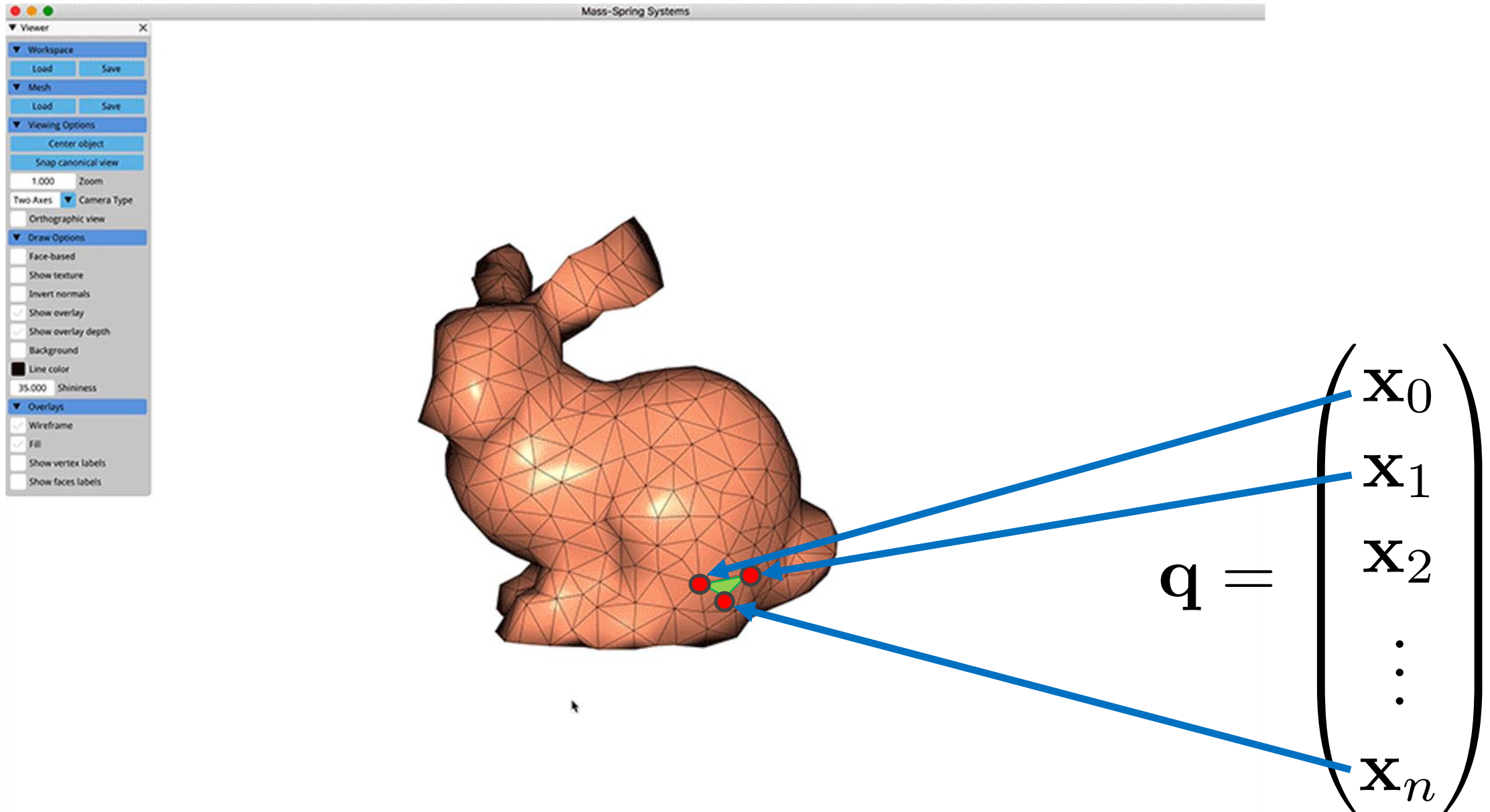
So a good reduced space is a vector of the k lowest order Eigenvectors

$$U \in \mathcal{R}^{n \times k}$$

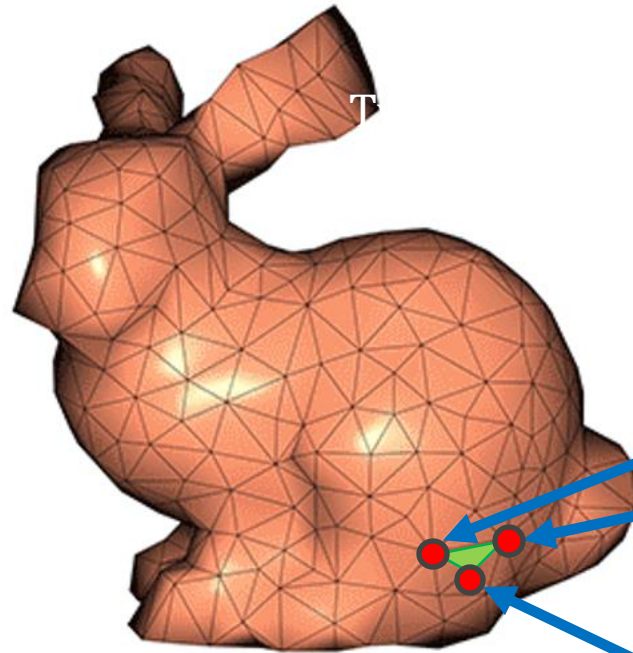
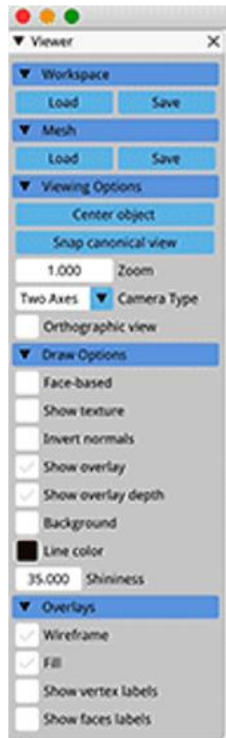
Reduced Space



Spatial Discretization -- Finite Elements



Spatial Discretization -- Finite Elements



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} : \mathbf{q}(0) + \mathbf{U}\mathbf{r}$$

|
Generalized
Coordinates

An Aside: Variational Modal Analysis

$$U^* = \arg \min_U \operatorname{tr}(U^T H U)$$

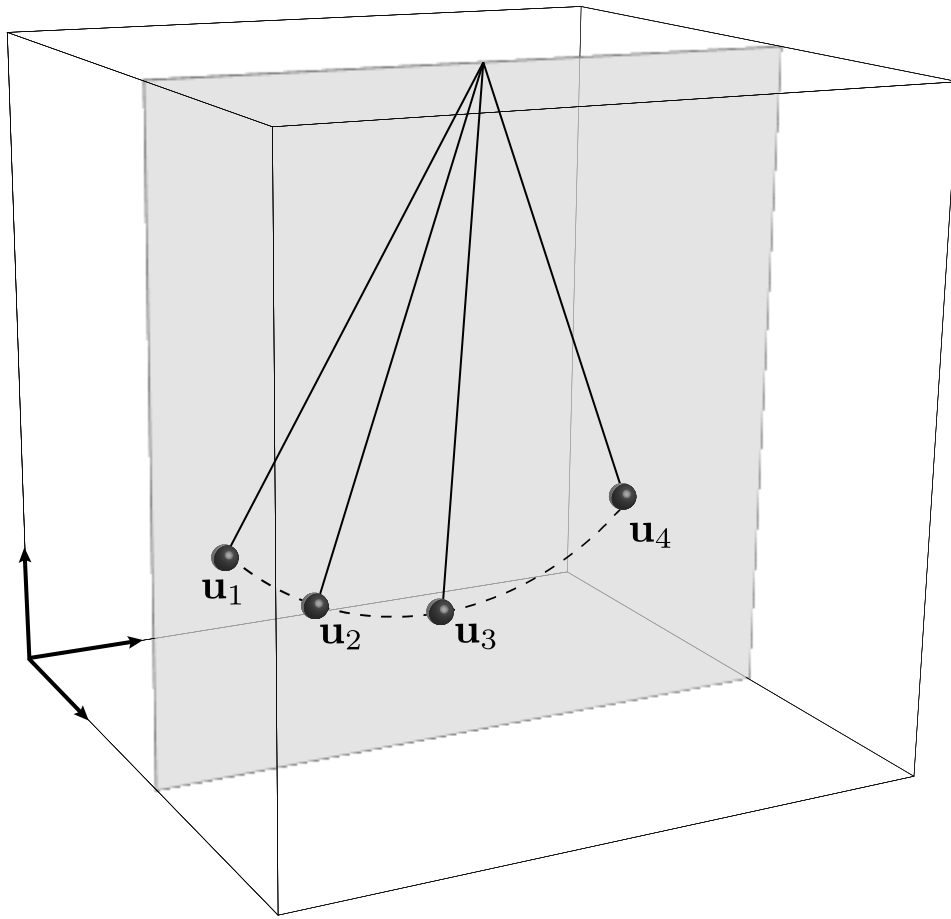
$$s.t. \ U^T M U = I$$

How Do We Build a Reduced-Order Model That's Compact and Expressive ?

~~1. Modal Analysis~~

2. Data-Driven Deformation Modes

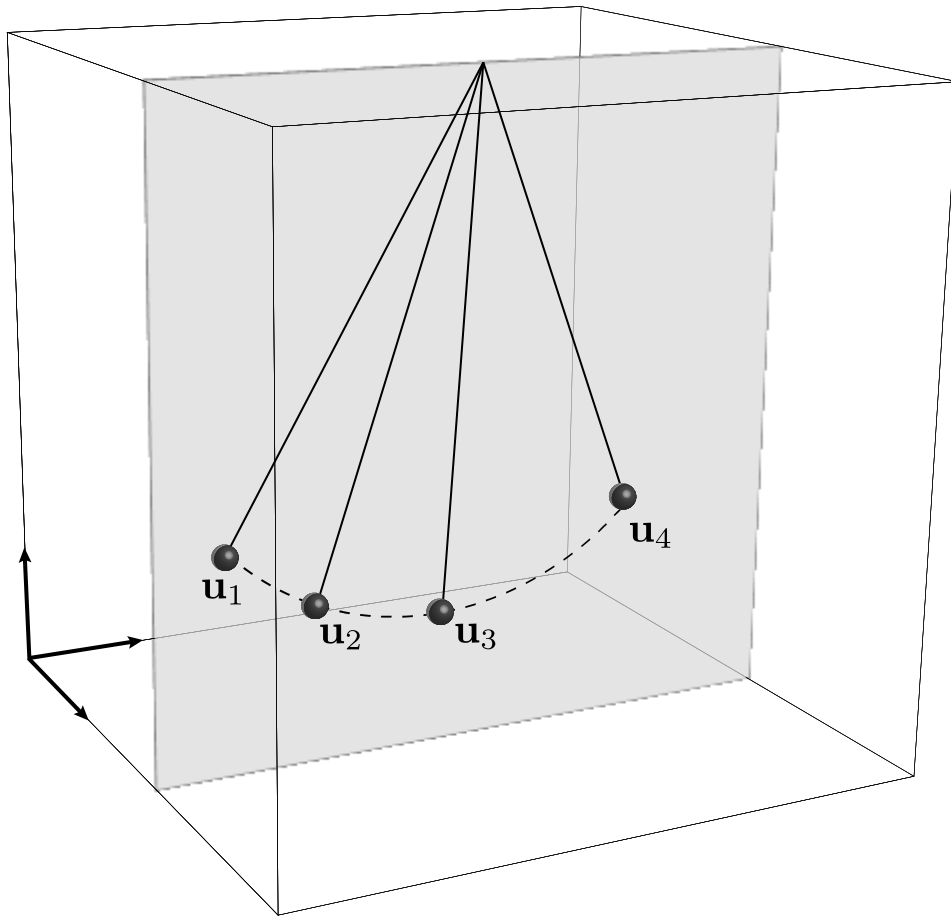
Data-Driven Deformation Modes



Collect Snapshots

$$Q = [q_0, q_1, q_2 \dots]$$

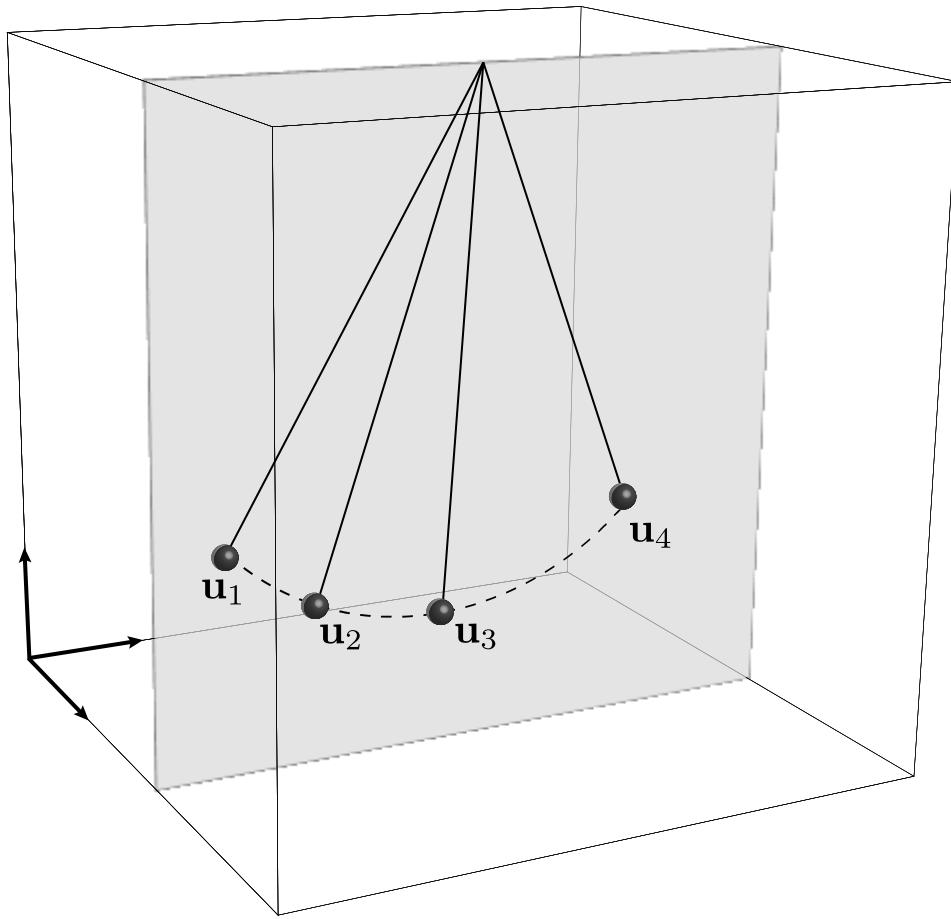
Data-Driven Deformation Modes



Collect Snapshots

$$Q = [q_0, q_1, q_2 \dots]$$

Probabilistic Reconstruction

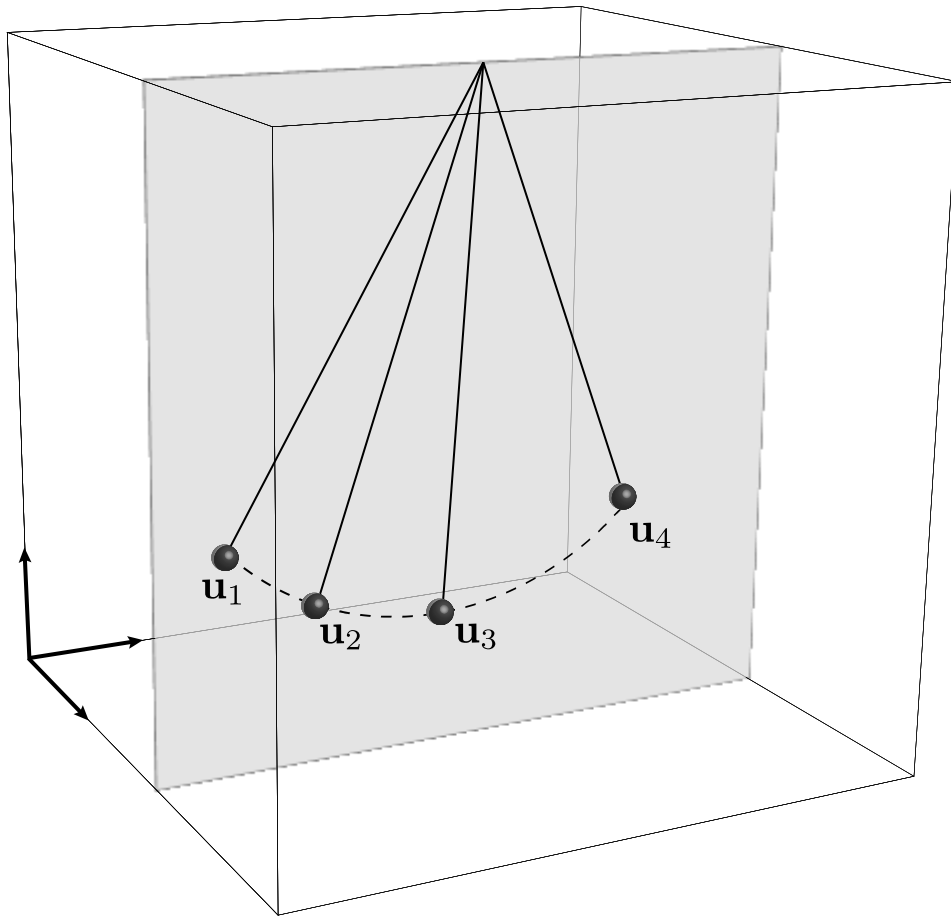


Given a **bunch of states**, assume their **displacements** from the **mean state** can be modelled linearly

$$s.t \mathbf{q}_i = \mu(\mathbf{0}) + U\mathbf{r}_i$$

Mean pose from data

Probabilistic Reconstruction



Given a **bunch of states**, assume their **displacements** from the **mean state** can be modelled linearly

$$s.t. \frac{(\mathbf{q}_i - \mu(\mathbf{0}))}{U \mathbf{r}_i}$$

Displacement u_i

Probabilistic Reconstruction

Find U that minimizes expected reconstruction error

$$U^* = \arg \min \sum_i ||\mathbf{u}_i - U \mathbf{r}_i||_2^2 \underbrace{p_{data}(\mathbf{u}_i)}$$

Data distribution (unknown)

Probabilistic Reconstruction

Find U that minimizes expected reconstruction error

$$U^* = \arg \min \sum_i ||\mathbf{u}_i - U \mathbf{r}_i||_2^2 \underbrace{p_{data}(\mathbf{u}_i)}$$

Data distribution (unknown)

$$s.t. U^T U = I$$

Prevent collapse of reduced space

Some useful identities

Projection onto an orthogonal subspace

$$\mathbf{r}_i = U^T \mathbf{u}_i$$

Vector norms as traces

$$||\mathbf{v} - A\mathbf{v}||_2^2 = \text{trace}(A\mathbf{v}\mathbf{v}^T)$$

Linearity of Expectation

$$\sum_i \text{trace}(AB_i) \mathbf{p}_{data} = \text{trace}(A \sum_i B_i \mathbf{p}_{data})$$

Probabilistic Reconstruction

Ok let's apply some identities

$$U^* = \arg \min \sum_i ||\mathbf{u}_i - U \mathbf{r}_i||_2^2 \underbrace{p_{data}(\mathbf{u}_i)}$$

Data distribution (unknown)

$$s.t. U^T U = I$$

Prevent collapse of reduced space

Probabilistic Reconstruction

Step 1: Get rid of r

$$U^* = \arg \min \sum_i \left\| \mathbf{u}_i - UU^T \mathbf{u}_i \right\|_2^2 p_{data}(\mathbf{u}_i)$$

$$s.t. U^T U = I$$

Prevent collapse of reduced space

Probabilistic Reconstruction

Step 2: Get rid of norm

$$\begin{aligned} & \mathbf{U}^* \\ &= \arg \min \sum_i \text{trace}((\mathbf{I} - \mathbf{U}\mathbf{U}^T)\mathbf{u}_i\mathbf{u}_i^T)p_{data}(\mathbf{u}_i) \end{aligned}$$

$$s.t. \mathbf{U}^T\mathbf{U} = \mathbf{I}$$

Prevent collapse of reduced space

Probabilistic Reconstruction

Step 3: Apply linearity of expectation

$$U^* = \arg \min \text{trace}((I - UU^T) \underbrace{\sum_i \mathbf{u}_i \mathbf{u}_i^T p_{data}(\mathbf{u}_i)})$$

$$s.t. U^T U = I$$

This just the covariance Σ

Probabilistic Reconstruction

Step 4: Get rid of terms independent of U

$$\begin{aligned} U^* = \arg \min & -\text{trace}(UU^T \Sigma) \\ \text{s.t. } & U^T U = I \end{aligned}$$

Probabilistic Reconstruction

Step 5: Rearrange and get rid of minus sign

$$\begin{aligned} \mathbf{U}^* &= \arg \max \text{trace}(\mathbf{U}^T \Sigma \mathbf{U}) \\ s.t. \quad &\mathbf{U}^T \mathbf{U} = \mathbf{I} \end{aligned}$$

An Aside: Variational Modal Analysis

$$U^* = \arg \min_U \operatorname{tr}(U^T H U)$$

$$s.t. \ U^T M U = I$$

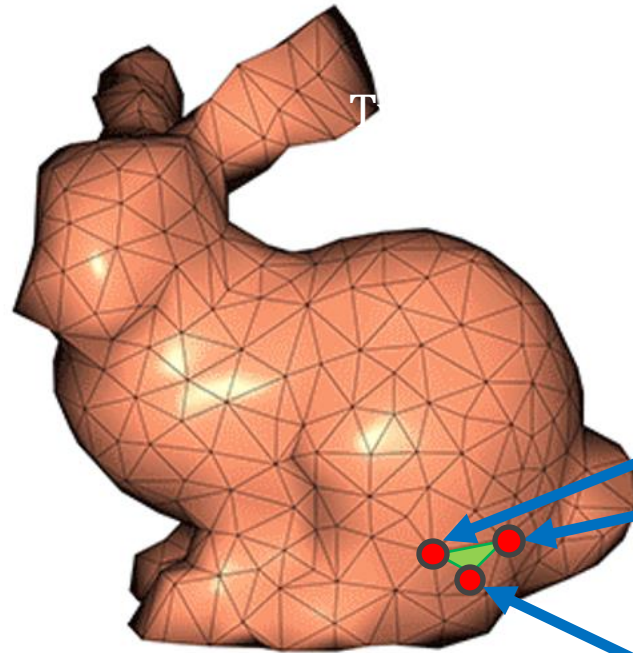
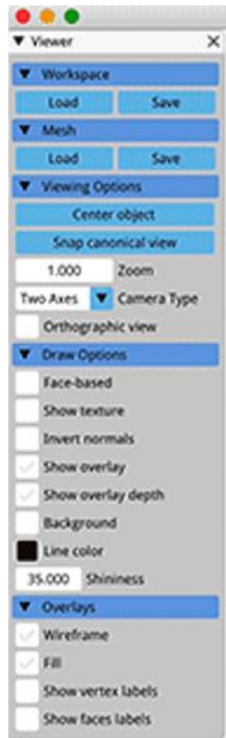
Probabilistic Reconstruction

$$\begin{aligned} \boldsymbol{U}^* &= \arg \max \text{trace}(\boldsymbol{U}^T \boldsymbol{\Sigma} \boldsymbol{U}) \\ &\text{s.t. } \boldsymbol{U}^T \boldsymbol{U} = \boldsymbol{I} \end{aligned}$$

Principal Component Analysis

$$\begin{aligned} \boldsymbol{U}^* &= \arg \max \operatorname{trace}(\boldsymbol{U}^T \boldsymbol{\Sigma} \boldsymbol{U}) \\ &\text{s.t. } \boldsymbol{U}^T \boldsymbol{U} = \boldsymbol{I} \end{aligned}$$

Spatial Discretization -- Finite Elements



$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} : \mathbf{q}(0) + \mathbf{U}\mathbf{r}$$

|

Generalized Coordinates

How do we use this reduced space ?

Answer, direct substitution

$$E(\mathbf{q}^{i+1}) = \frac{1}{2} (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i)^T M (\mathbf{q}^{i+1} - \tilde{\mathbf{q}}^i) + h^2 V(\mathbf{q}^{i+1})$$

Gradient of what equals this ? Let's guess, then check

How do we use this reduced space ?

Answer, direct substitution

$$E(r^{i+1}) = \frac{1}{2} (r^{i+1} - \tilde{r}^i)^T \underbrace{U^T M U}_{\text{Reduced Mass Matrix}} (r^{i+1} - \tilde{r}^i) + h^2 V(q(0) + U^t r^{i+1})$$

Reduced Mass Matrix

What are the reduced gradient and Hessian ?

Newton's Method

Choose an initial guess

$$i = 0$$

$$\mathbf{v}^0 = \text{something}$$

Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

$$\mathbf{H}^i \mathbf{d} = -\mathbf{g}^i$$

\mathbf{H} and \mathbf{g} are now reduced so smaller and faster to solve

Choose α using line search

Use search direction to update current guess

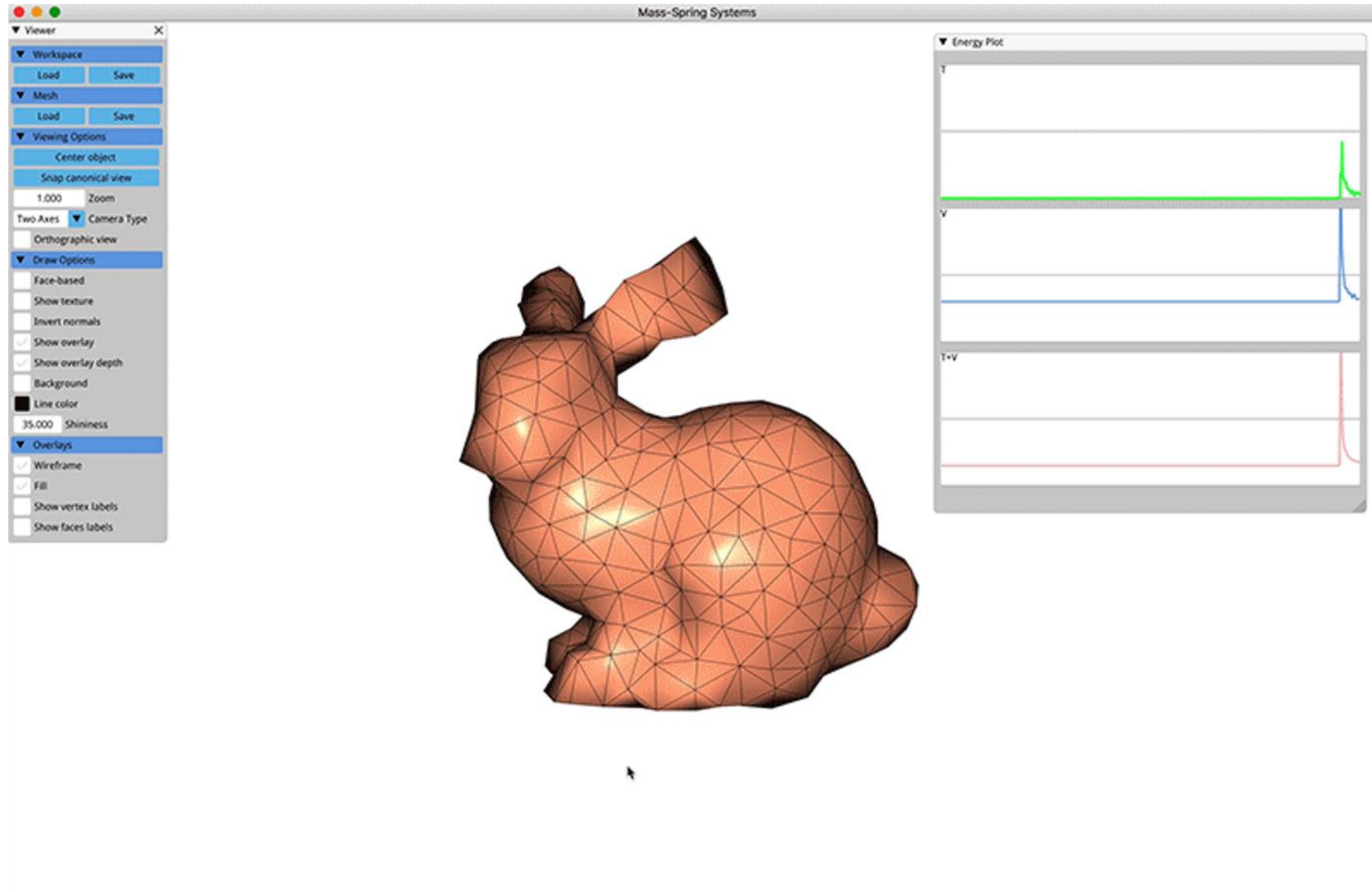
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$$i = i + 1$$

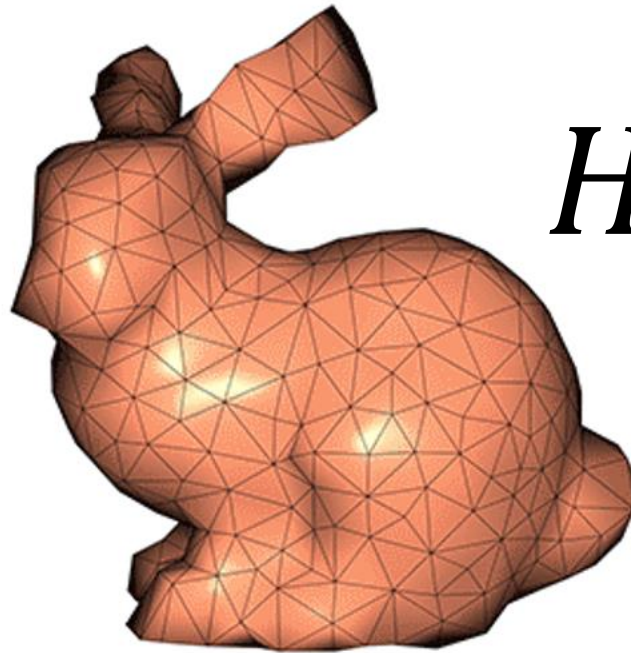
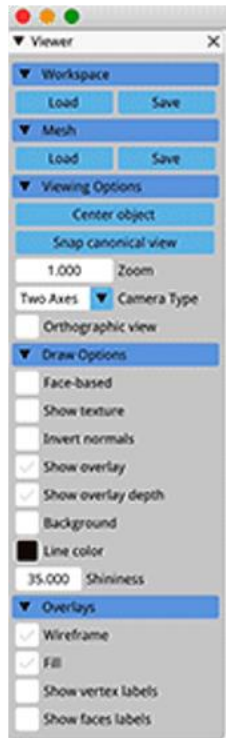
Repeat until converged



Problem, it's still slow! Why ?



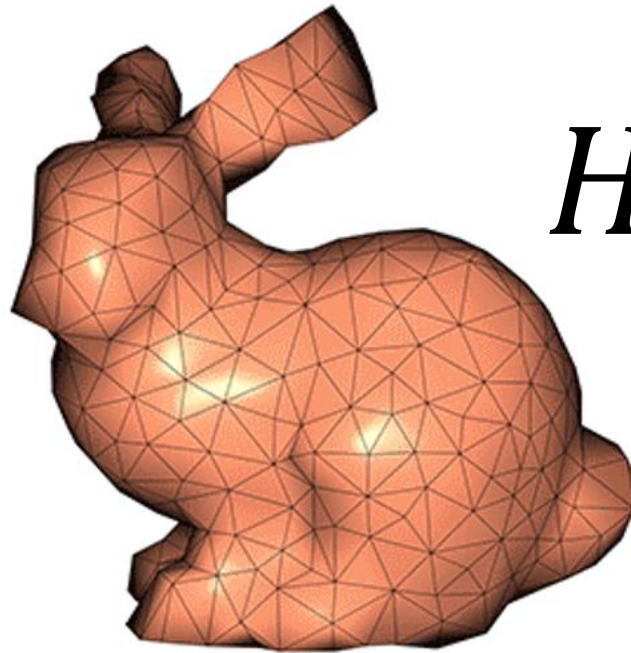
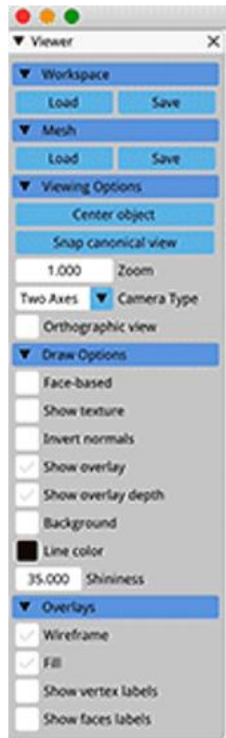
Assembly still visits every element ☹



$$H = \sum_t U^T \underbrace{H_i U}_T$$

Small, but there's a lot of them

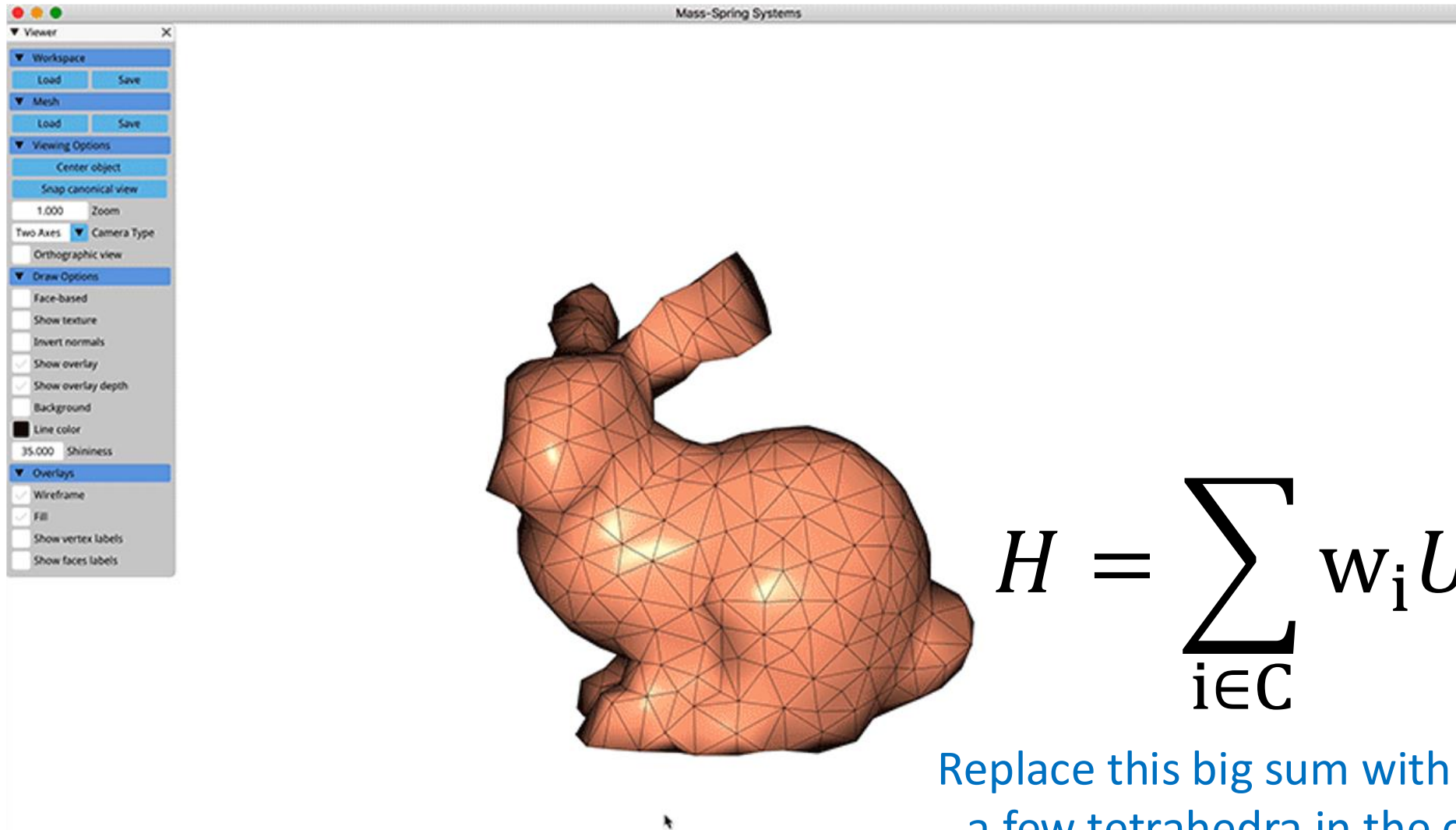
Optimal Quadrature



$$H = \sum_t U^T H_i U$$

Replace this big sum with a small sum

Optimal Quadrature

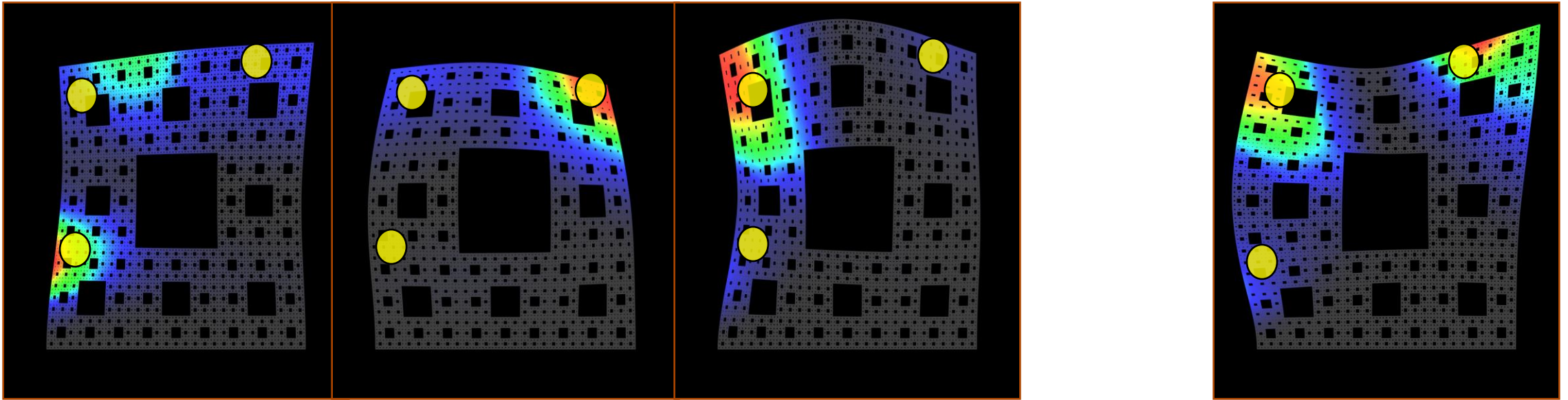


$$H = \sum_{i \in C} w_i U^T H_i U$$

Replace this big sum with a small sum over a few tetrahedra in the cubature set C.

One Method: Data Driven

$$\{\mathbf{q}_i, \mathbf{g}_i\}$$



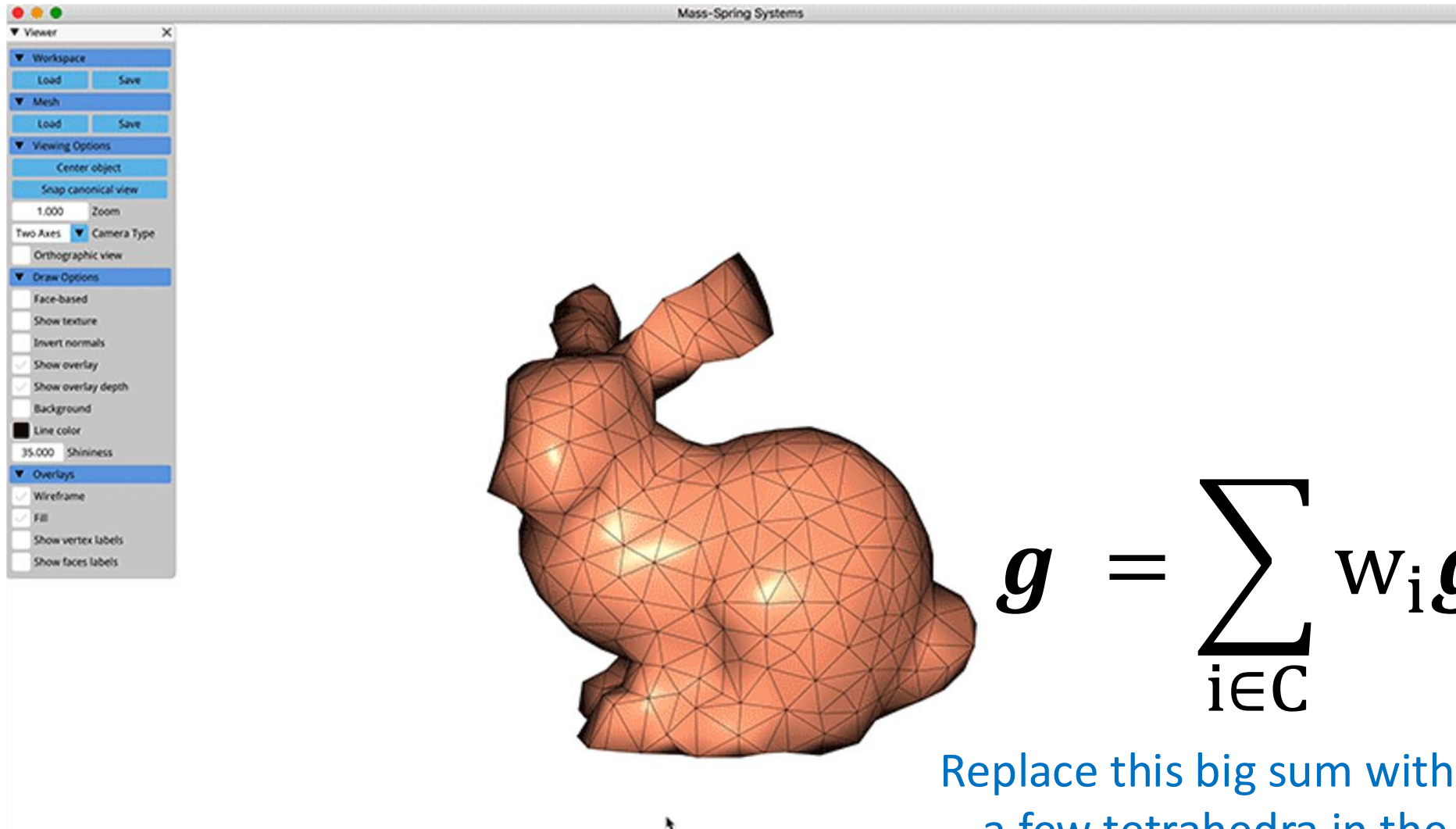
$\mathbf{q}^{(1)}$

$\mathbf{q}^{(2)}$

$\mathbf{q}^{(3)}$

$\mathbf{q}^{(T)}$

Optimal Quadrature



$$\mathbf{g} = \sum_{i \in C} w_i \mathbf{g}_i$$

Replace this big sum with a small sum over a few tetrahedra in the cubature set C.

Cubature for all poses

$$\underbrace{\begin{pmatrix} \mathbf{g}_0^0 & \cdots & \mathbf{g}_n^0 \\ \vdots & \ddots & \vdots \\ \mathbf{g}_0^T & \cdots & \mathbf{g}_n^T \end{pmatrix}}_{\text{Per-tetrahedron gradient values}} \underbrace{\begin{pmatrix} \mathbf{w}_0 \\ \vdots \\ \mathbf{w}_n \end{pmatrix}}_{\text{Weights}} = \underbrace{\begin{pmatrix} \mathbf{g}_0 \\ \vdots \\ \mathbf{g}_n \end{pmatrix}}_{\text{Per-pose value}}$$

Cubature for all poses

$$\begin{array}{ccc} & W & \\ & \perp & \\ \begin{pmatrix} \mathbf{g}_0^0 & \cdots & \mathbf{g}_n^0 \\ \vdots & \ddots & \vdots \\ \mathbf{g}_0^T & \cdots & \mathbf{g}_n^T \end{pmatrix} & \begin{pmatrix} \mathbf{w}_0 \\ \vdots \\ \mathbf{w}_n \end{pmatrix} & = \begin{pmatrix} \mathbf{g}_0 \\ \vdots \\ \mathbf{g}_n \end{pmatrix} \\ \hline G & & \mathbf{g} \end{array}$$

Solve using Non-linear least squares

Non-Linear Least Squares

$$\begin{aligned} \mathbf{U}^* = \arg \min & ||\mathbf{G}\mathbf{w} - \mathbf{g}||_2^2 \\ \text{s.t } & \mathbf{w} \geq 0 \end{aligned}$$

Newton's Method

Choose an initial guess

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Check for convergence

$$\left\| \frac{\partial E}{\partial \mathbf{v}} \Big|_{\mathbf{v}^i} \right\| < \text{tol}$$

Choose search direction

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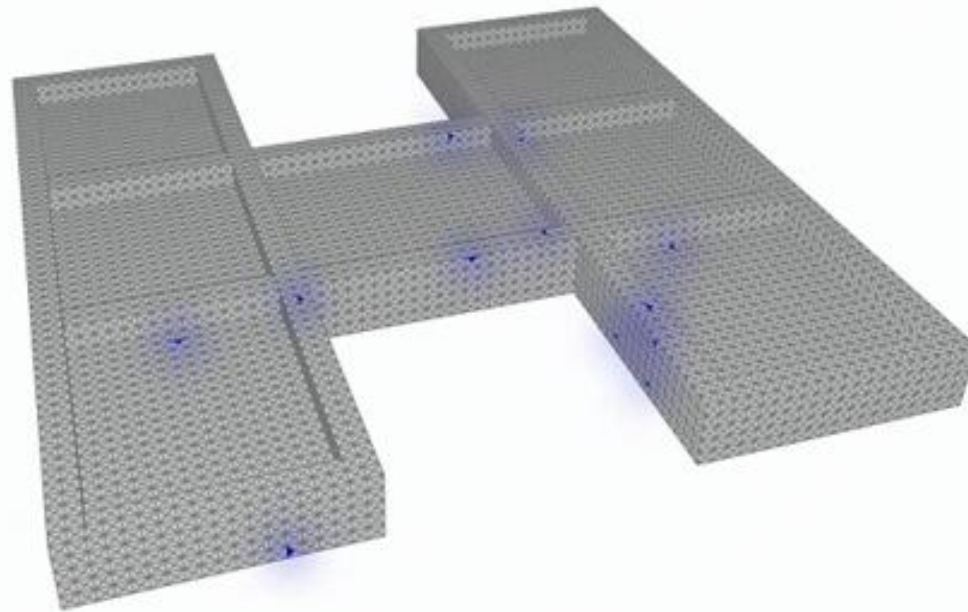
$$\mathbf{v}^{i+1} = \mathbf{v}^i + \alpha \mathbf{d}$$

$$i = i + 1$$

Repeat until converged



PCA Reanalysis of a Balloon



Next Video: Fast Solvers for Elastodynamics !

