Simulating Deformable Objects

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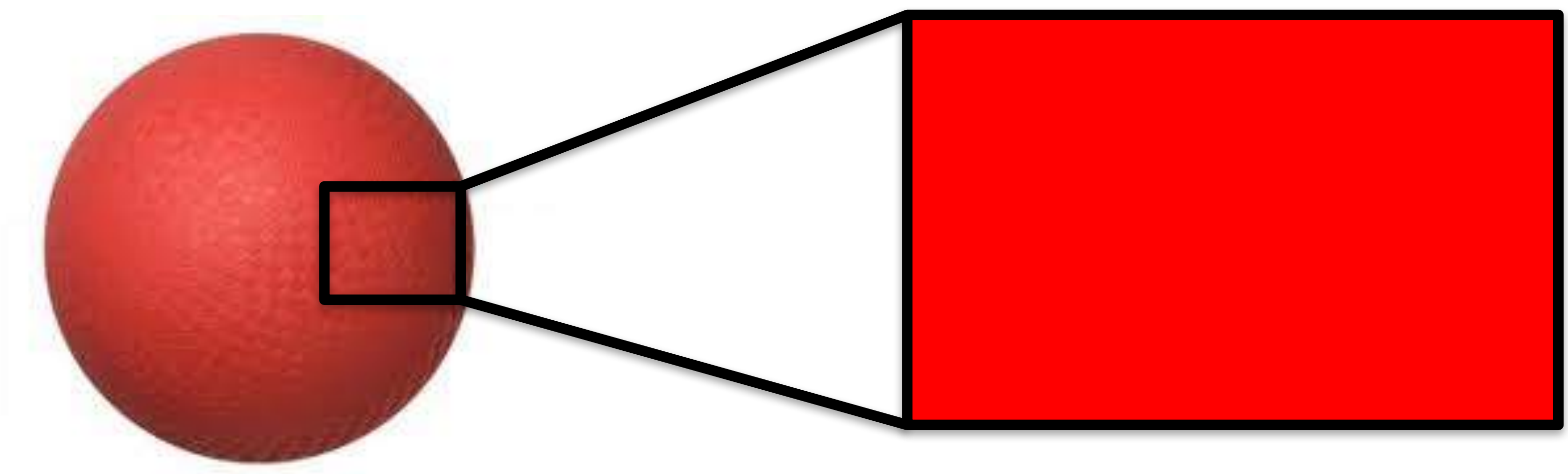
Department of Computer Science

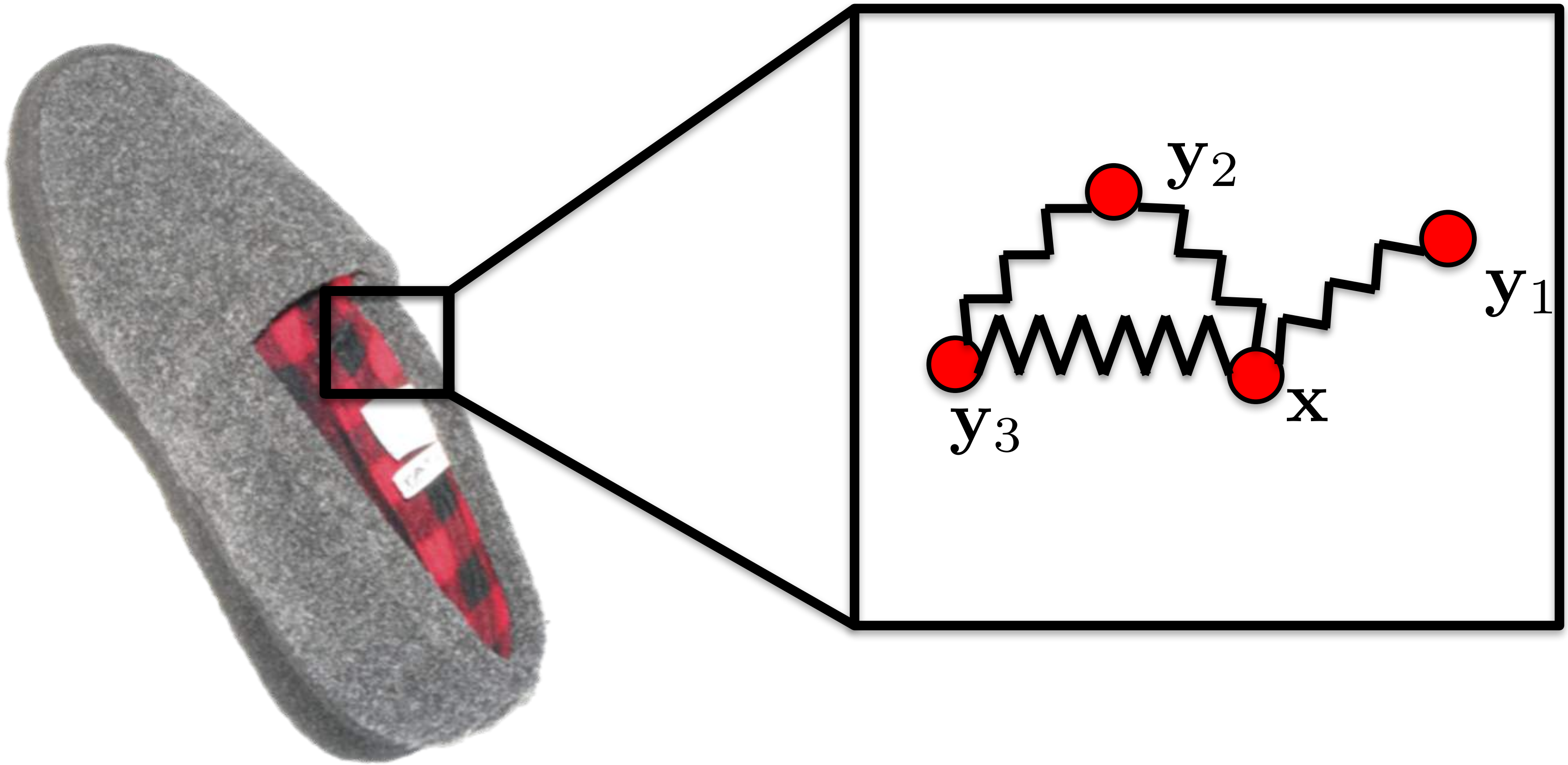
Class Schedule for the Next Few Weeks

- For undergraduates: Graduate courses don't have reading week
- We will have a lecture November 7th.
- No lecture October 24th, I'm travelling.

An Introduction to Continuum Mechanics



An Example



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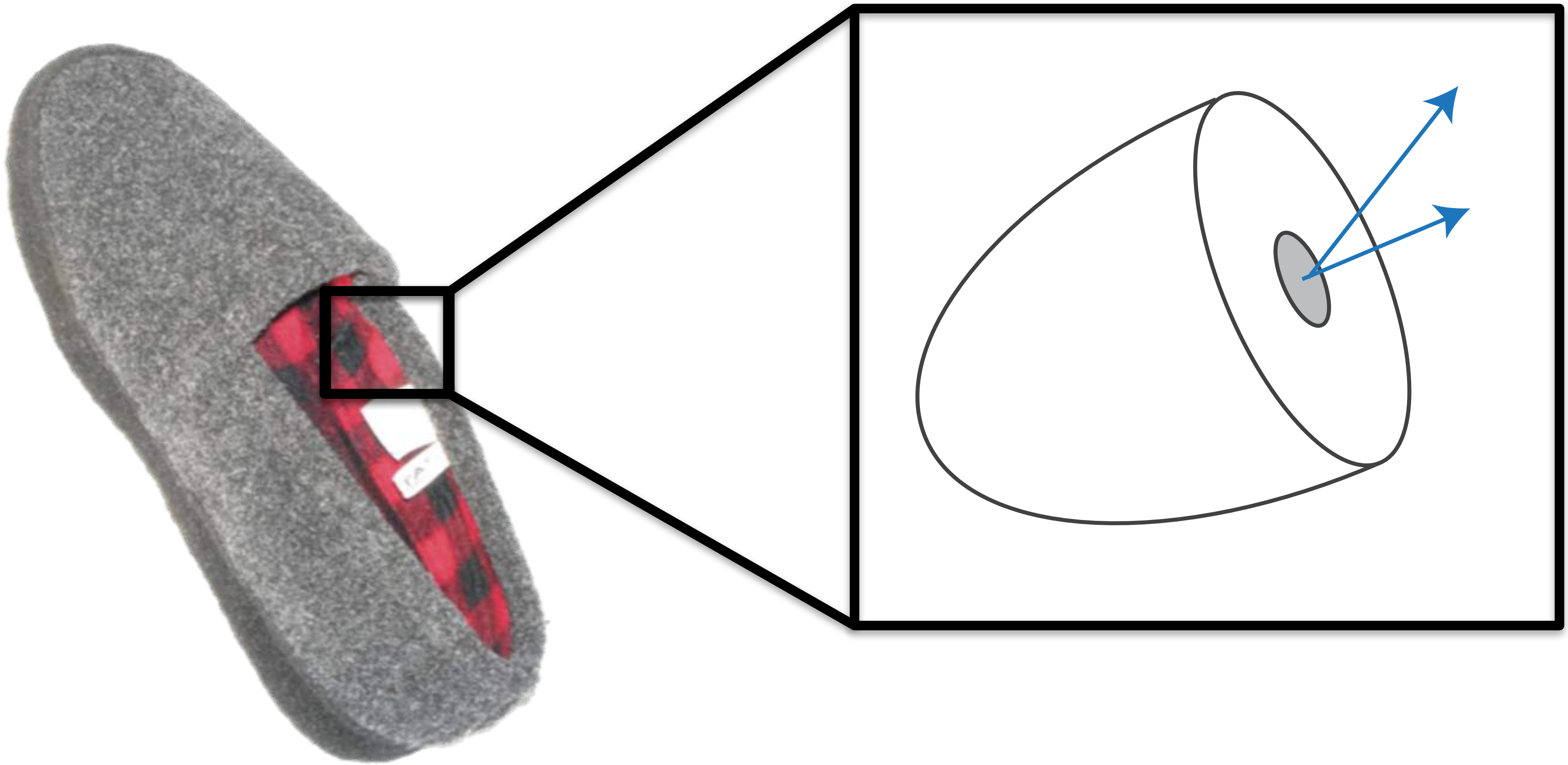
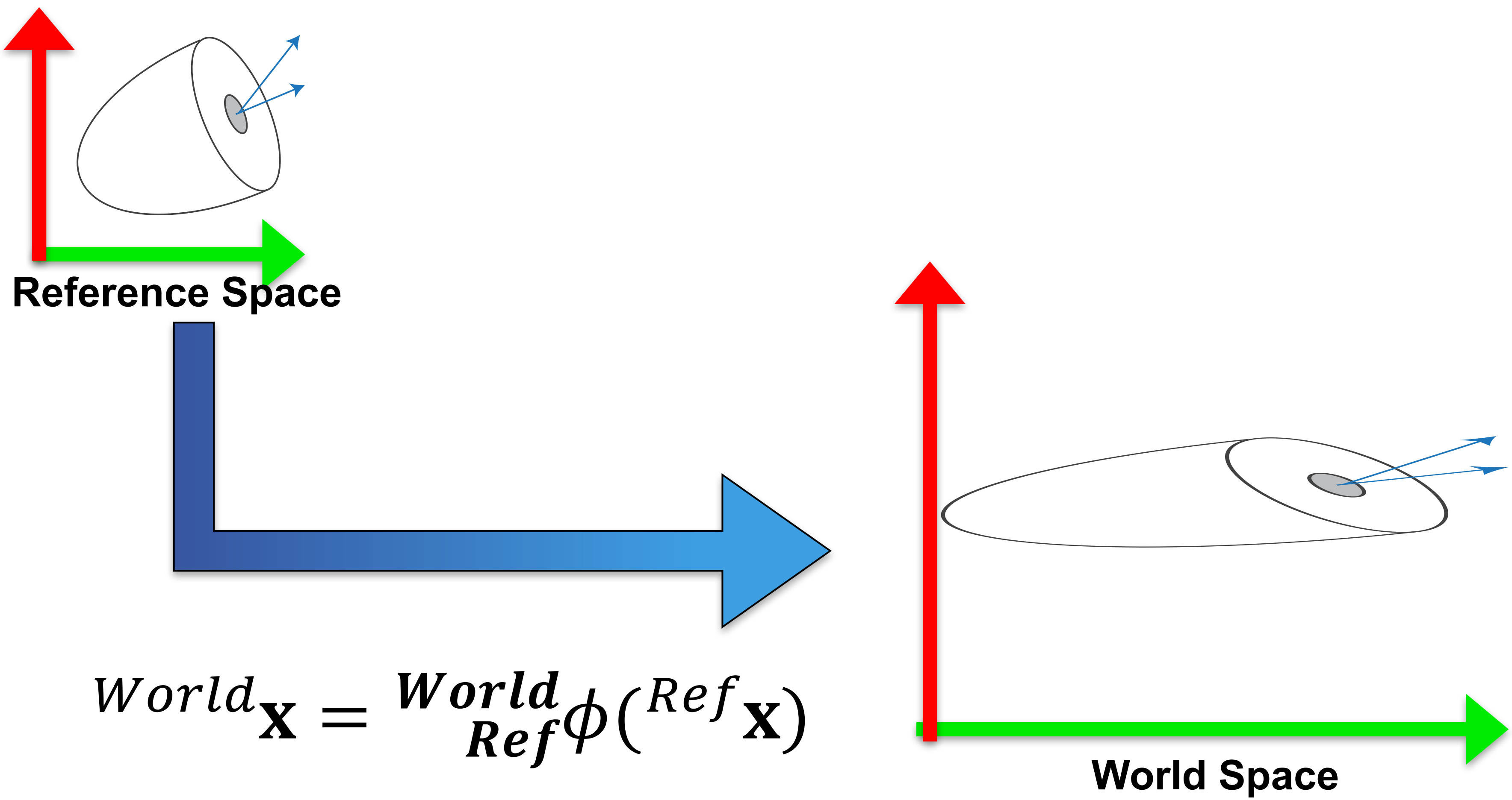


Requirements for Continuum Mechanics

1. Material Model
2. ***Measure of Deformation***

Continuum Mechanics: Deformation

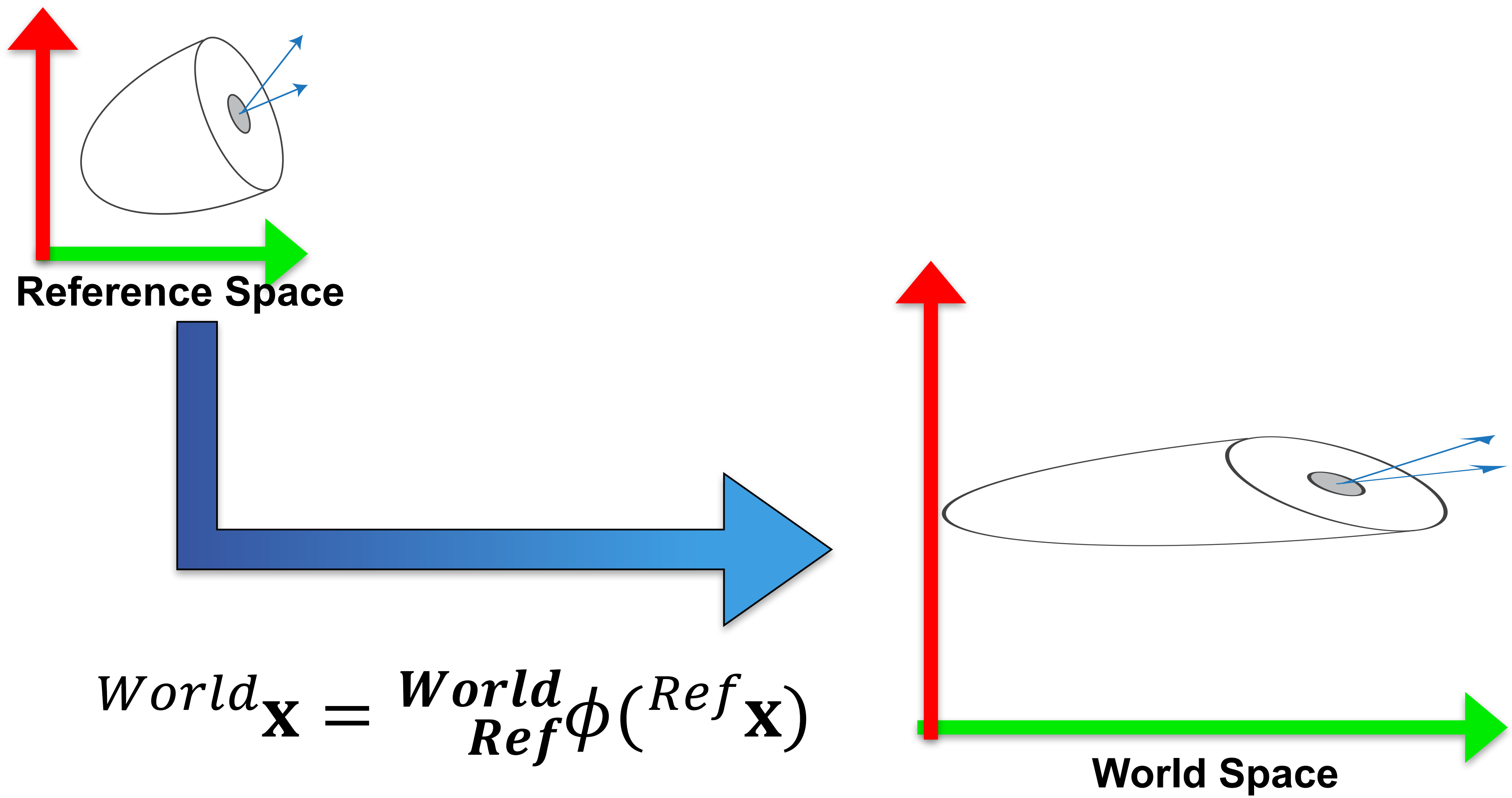
- Defining a measure of deformation:



$$ {}^{World}\mathbf{x} = {}^{\boldsymbol{World}}_{\boldsymbol{Ref}}\phi\left({}^{Ref}\mathbf{x}\right) $$

Continuum Mechanics

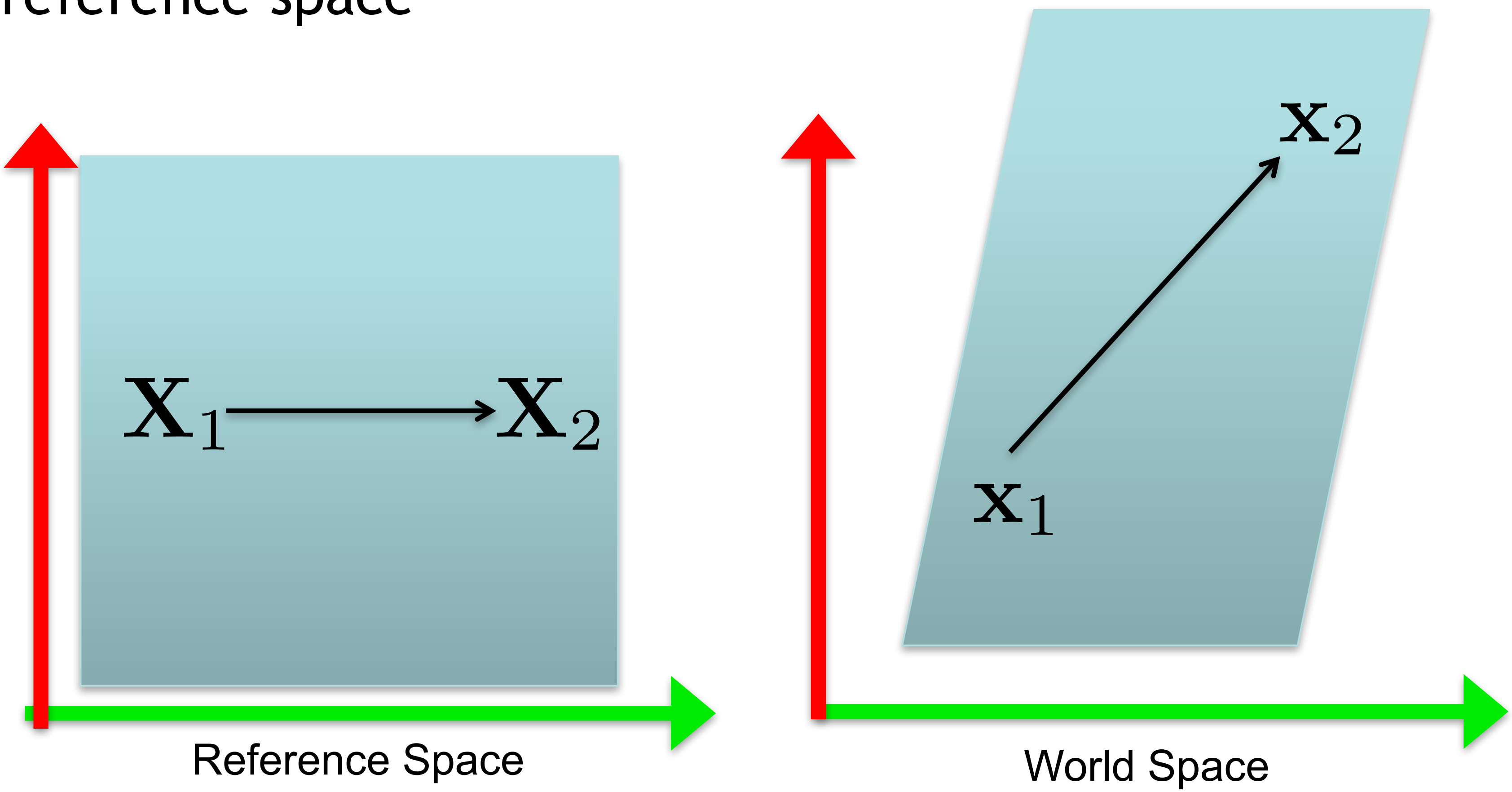
- We will leave our mapping undefined (for now)



$$ {}^{World}\mathbf{x} = {}^{\mathbf{World}}_{\mathbf{Ref}}\phi({}^{Ref}\mathbf{x}) $$

Continuum Mechanics: Deformation

- Consider the effect of ${}^{World}_{Ref}\phi$ on a single vector in our reference space



Continuum Mechanics: Deformation

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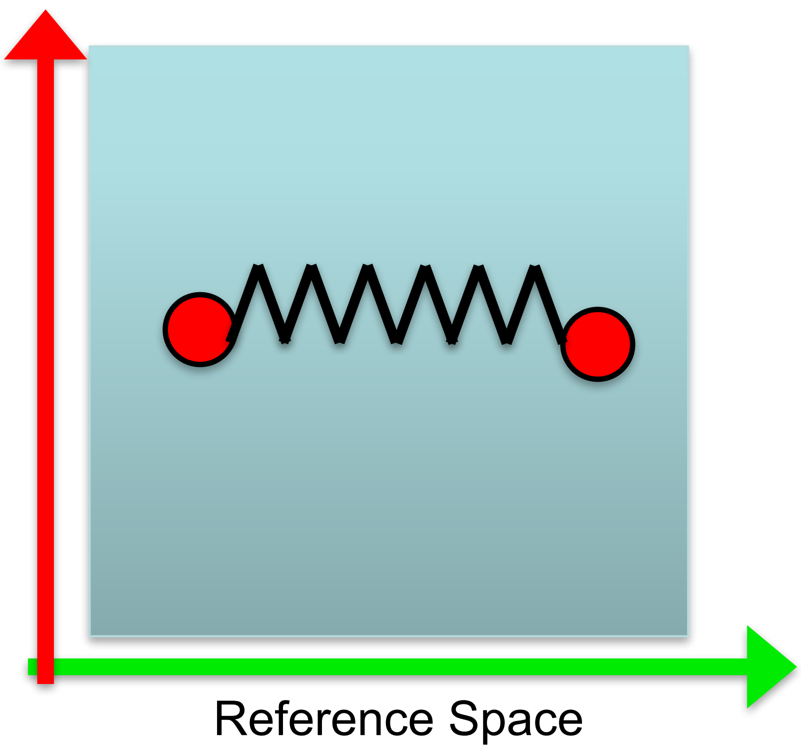
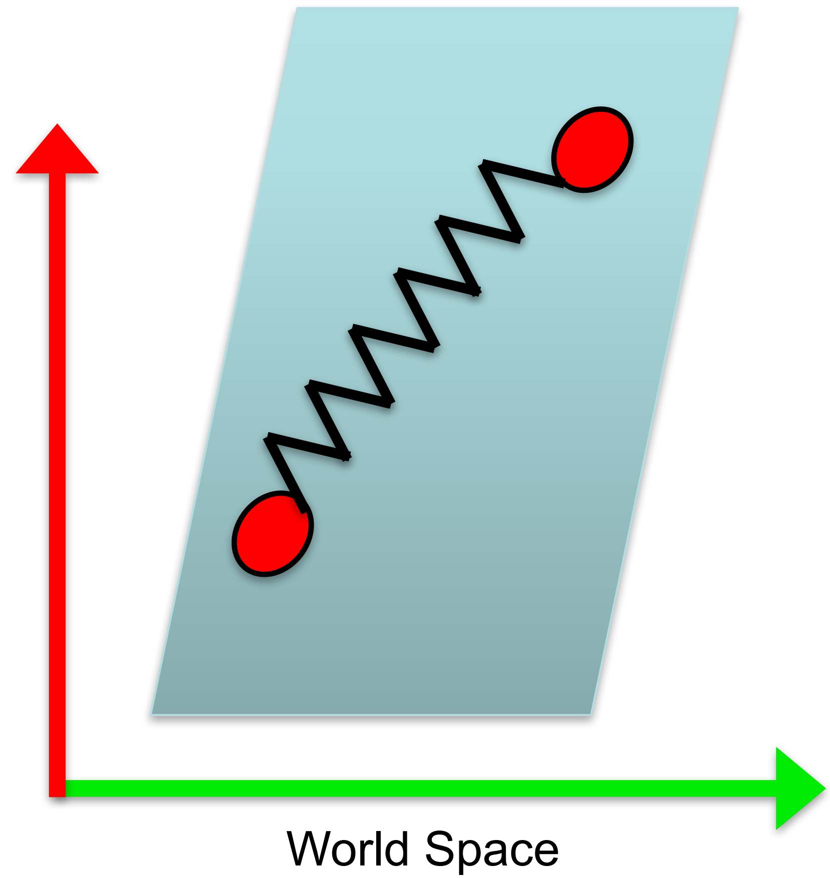
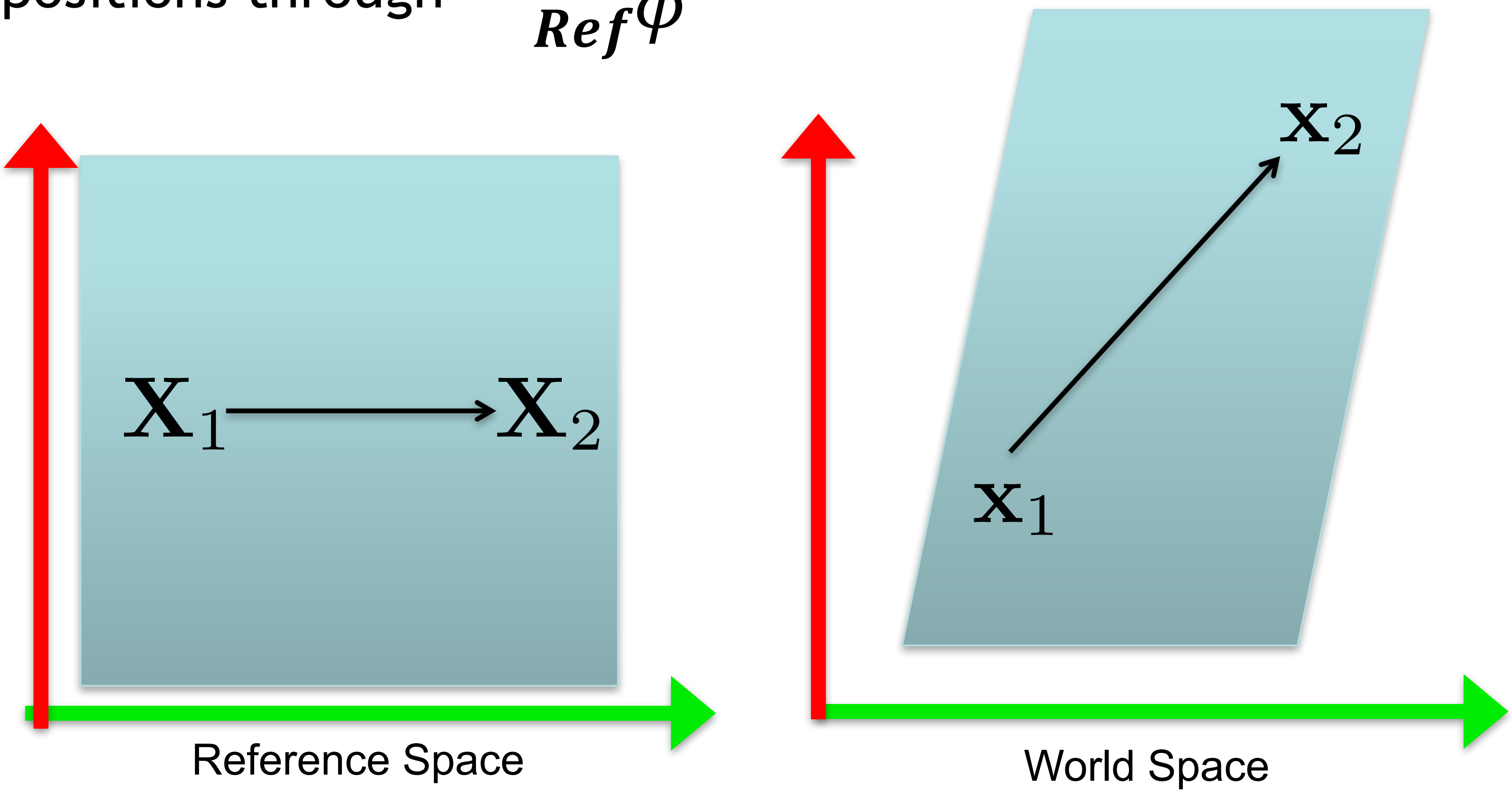




We are pretending we have springs EVERYWHERE in our object!!!

Continuum Mechanics: Deformation

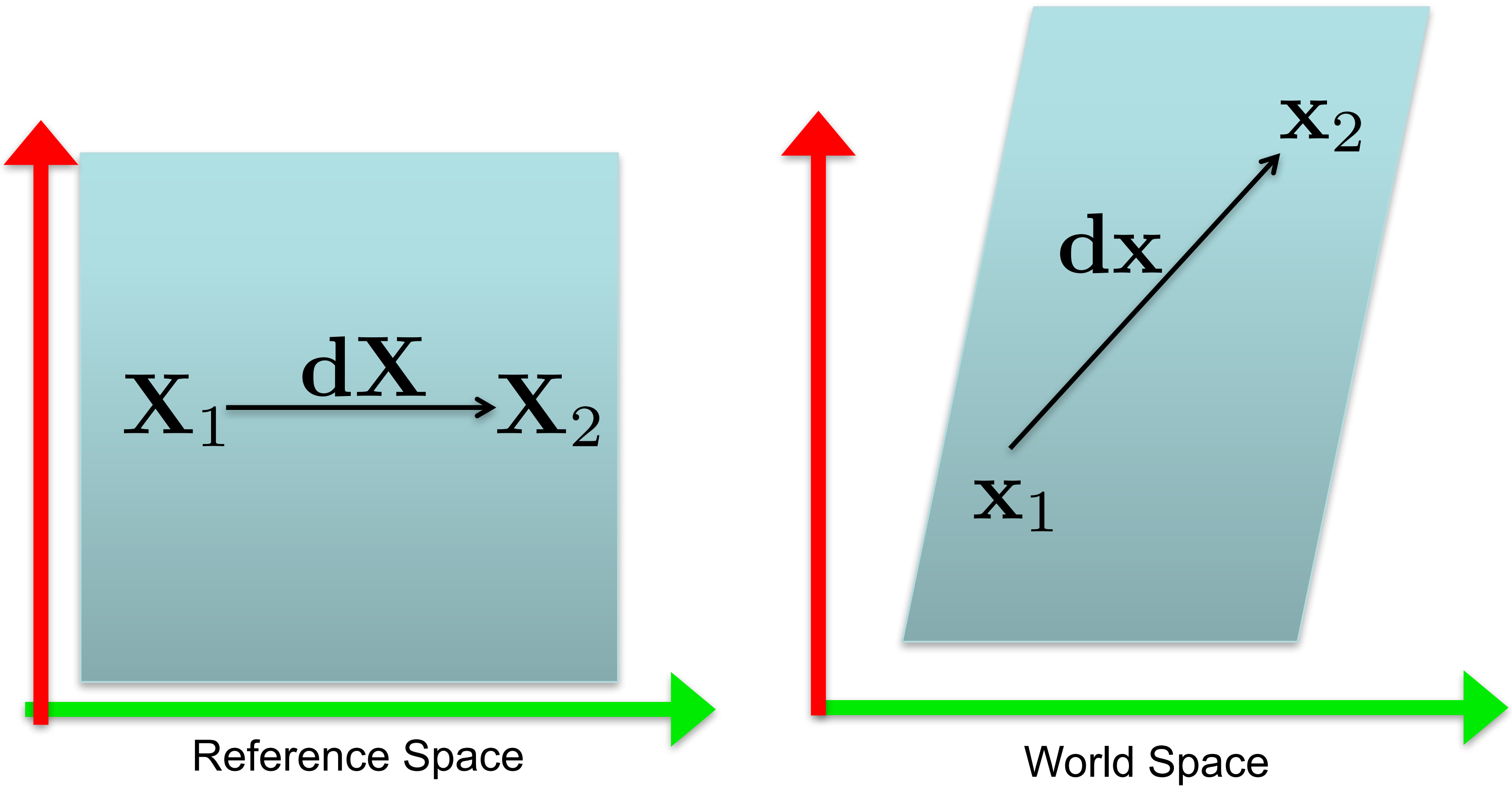
- New positions are given by passing the reference positions through ${}^{World}_{Ref}\phi$



$$\mathbf{x}_2 = \phi\left(\mathbf{X}_2\right)$$

Continuum Mechanics: Deformation

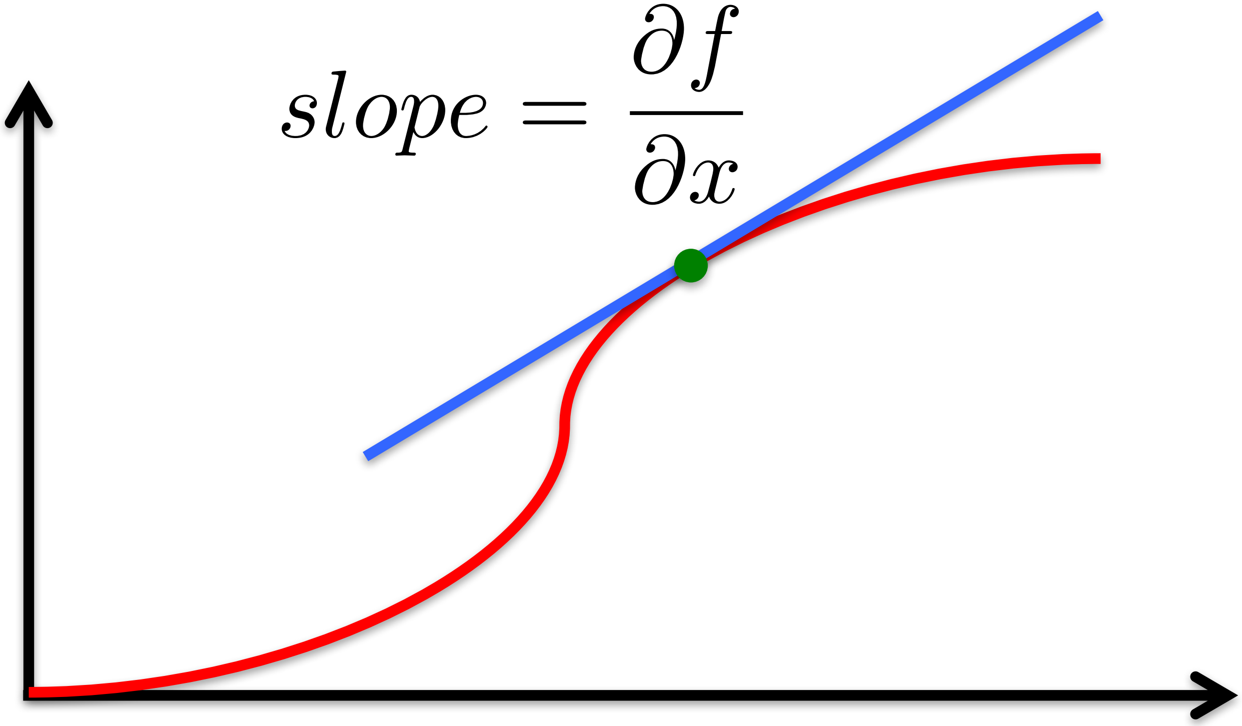
- Just rephrasing so we can see the spring vector



$$\mathbf{x}_1 + \mathbf{dx} = \phi\left(\mathbf{X}_1 + \mathbf{dX}\right)$$

An Aside: Taylor Expansion

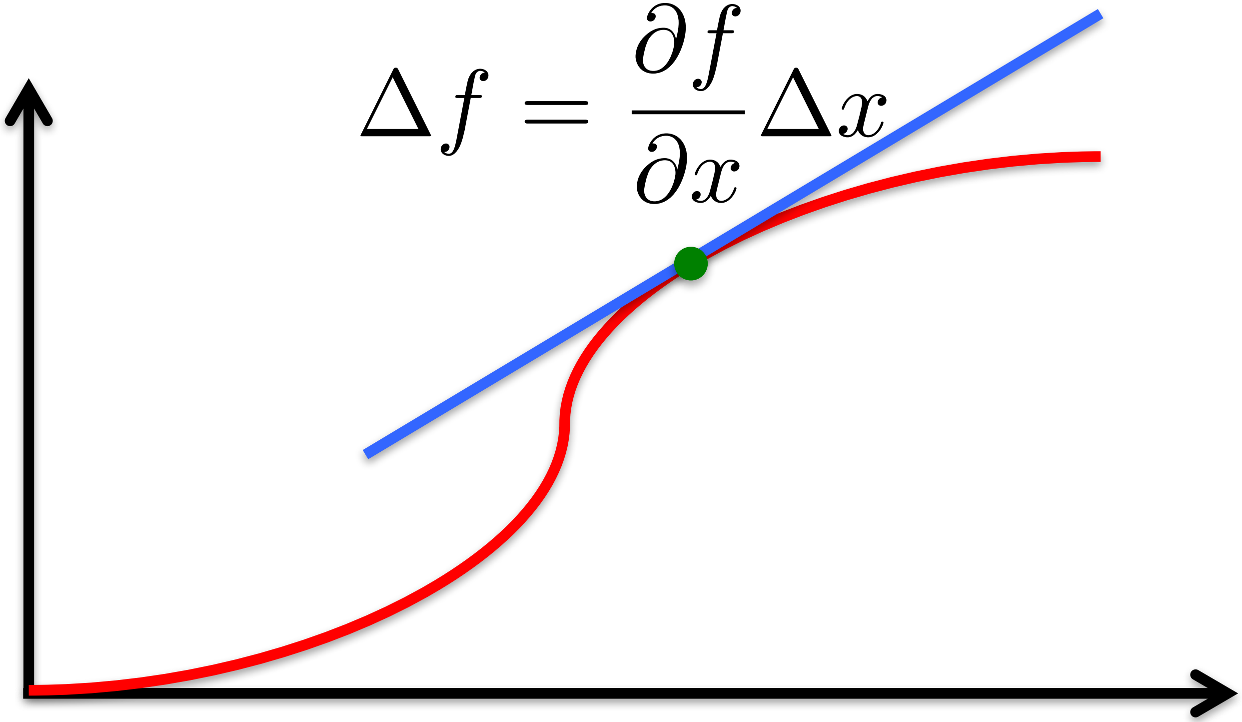
- Approximate small change in a non-linear function



An Aside: Taylor Expansion

- Approximate small change in a non-linear function

$$\Delta f = \frac{\partial f}{\partial x} \Delta x$$



An Aside: Taylor Expansion

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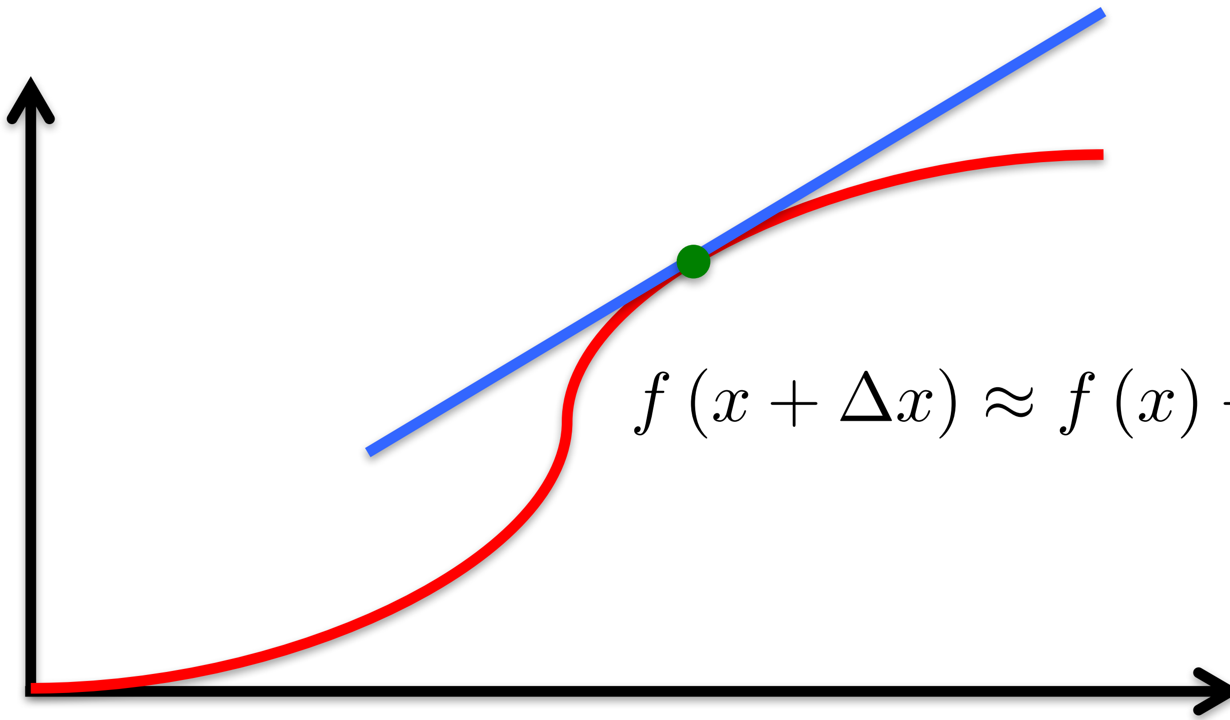


$$f\left(x + \Delta x\right) \approx f\left(x\right) + \frac{\partial f}{\partial x} \Delta x$$

Taylor Expansion: Multi-dimensional Functions

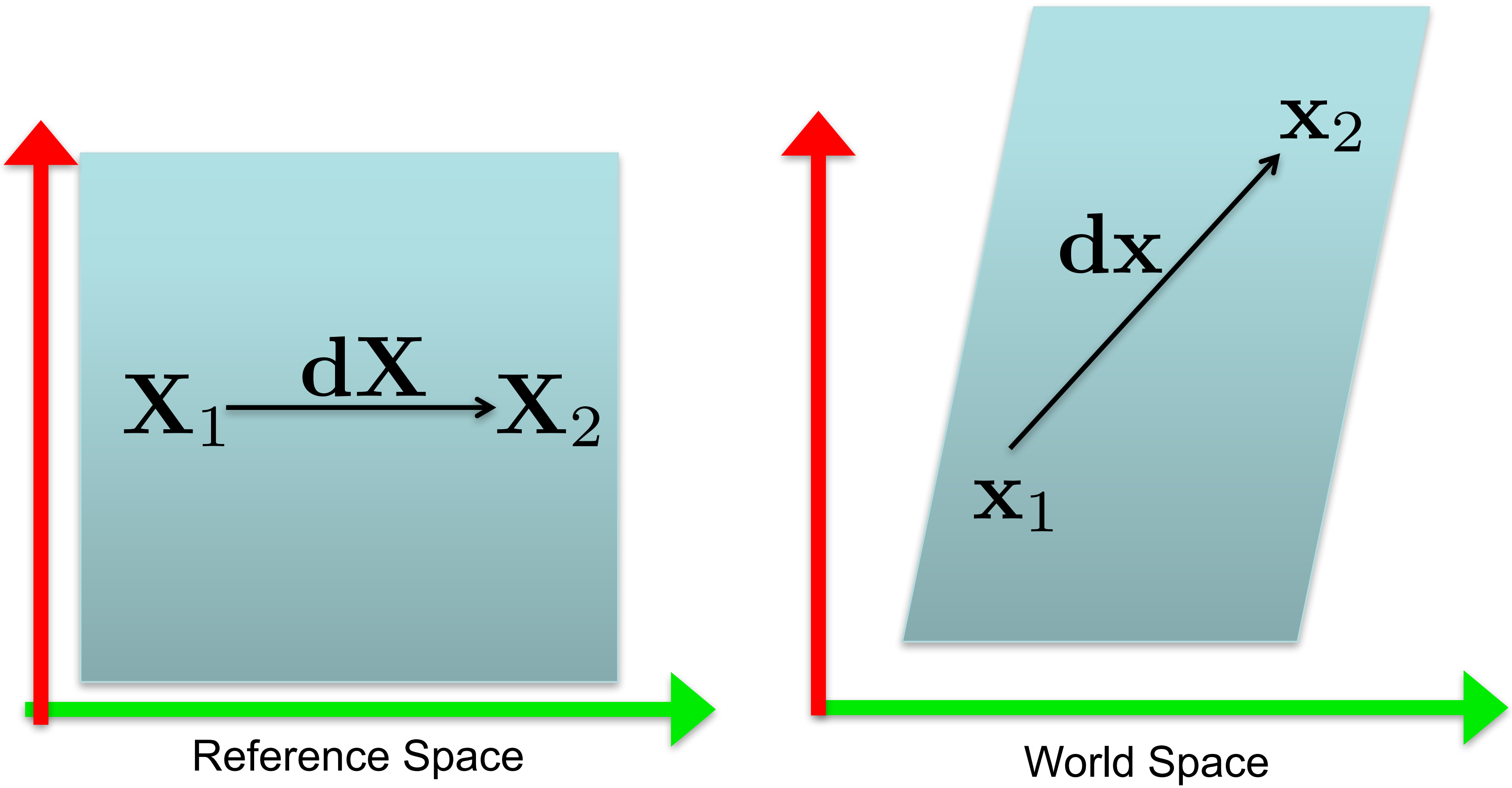
- Almost exactly the same!

$$\mathbf{f}(\mathbf{x} + \Delta \mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + \boxed{\frac{\partial \mathbf{f}}{\partial \mathbf{x}}} \Delta \mathbf{x}$$

Gradient Matrix or "Jacobian"

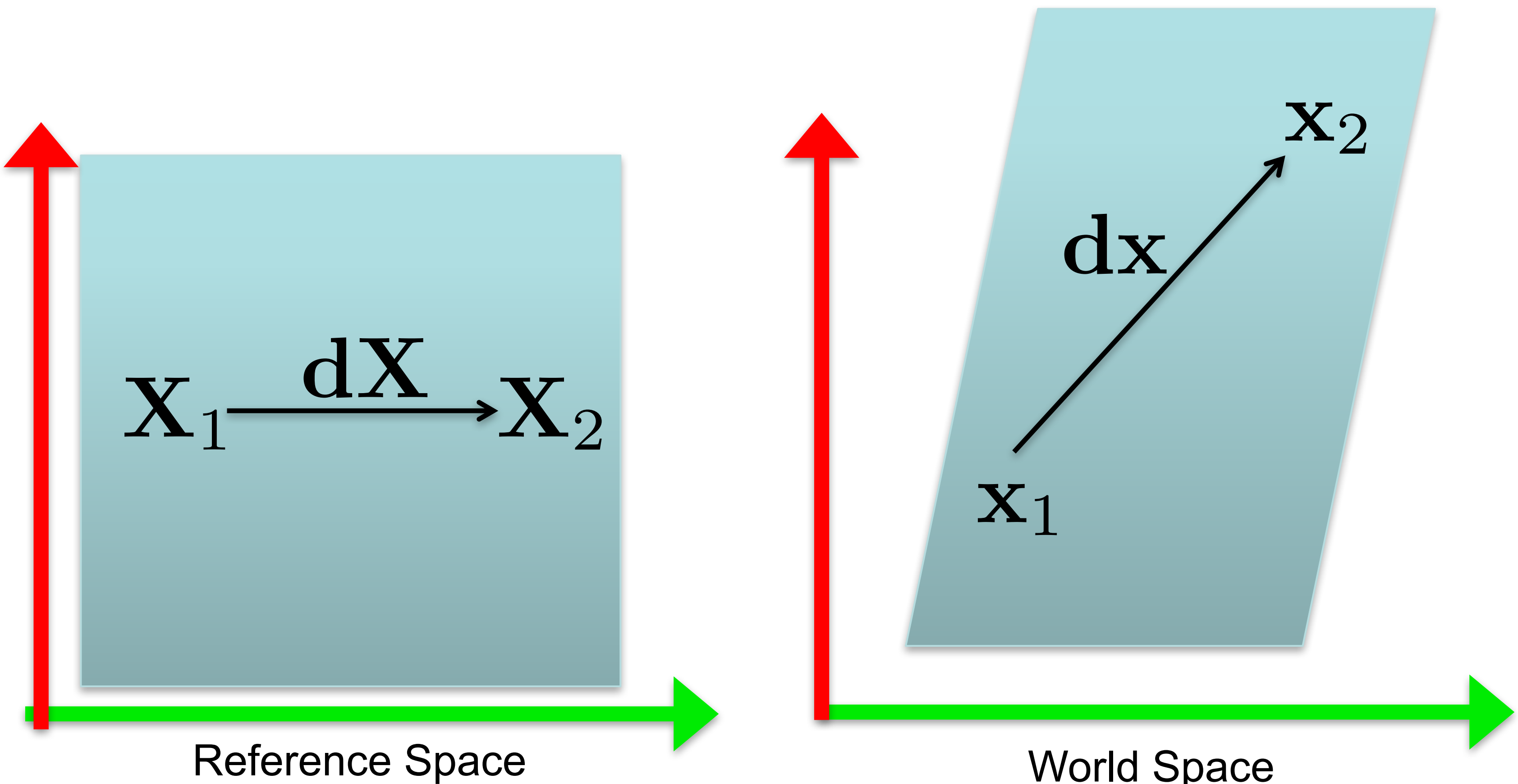
Continuum Mechanics: Deformation

- Apply Taylor Expansion



$$\mathbf{x}_1 + \mathbf{dx} = \phi\left(\mathbf{X}_1 + \mathbf{dX}\right)$$

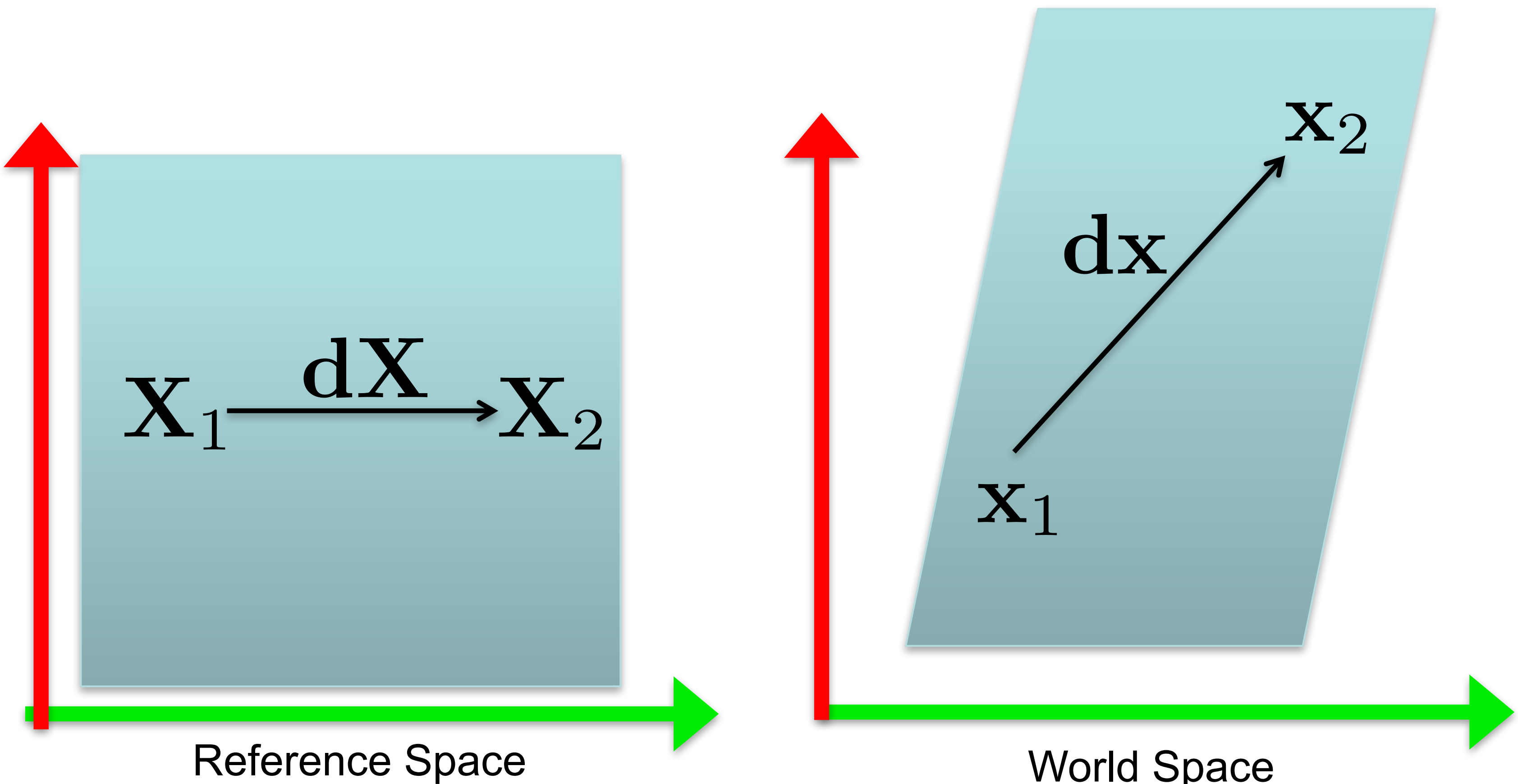
Continuum Mechanics: Deformation



$$\mathbf{x}_1 + \mathbf{dx} \approx \phi\left(\mathbf{X}_1\right) + \frac{\partial \phi}{\partial \mathbf{X}} \mathbf{dX}$$

Continuum Mechanics: Deformation

- $\mathbf{x}_1$ and $\phi(\mathbf{X}_1)$ are the same so we are left with …



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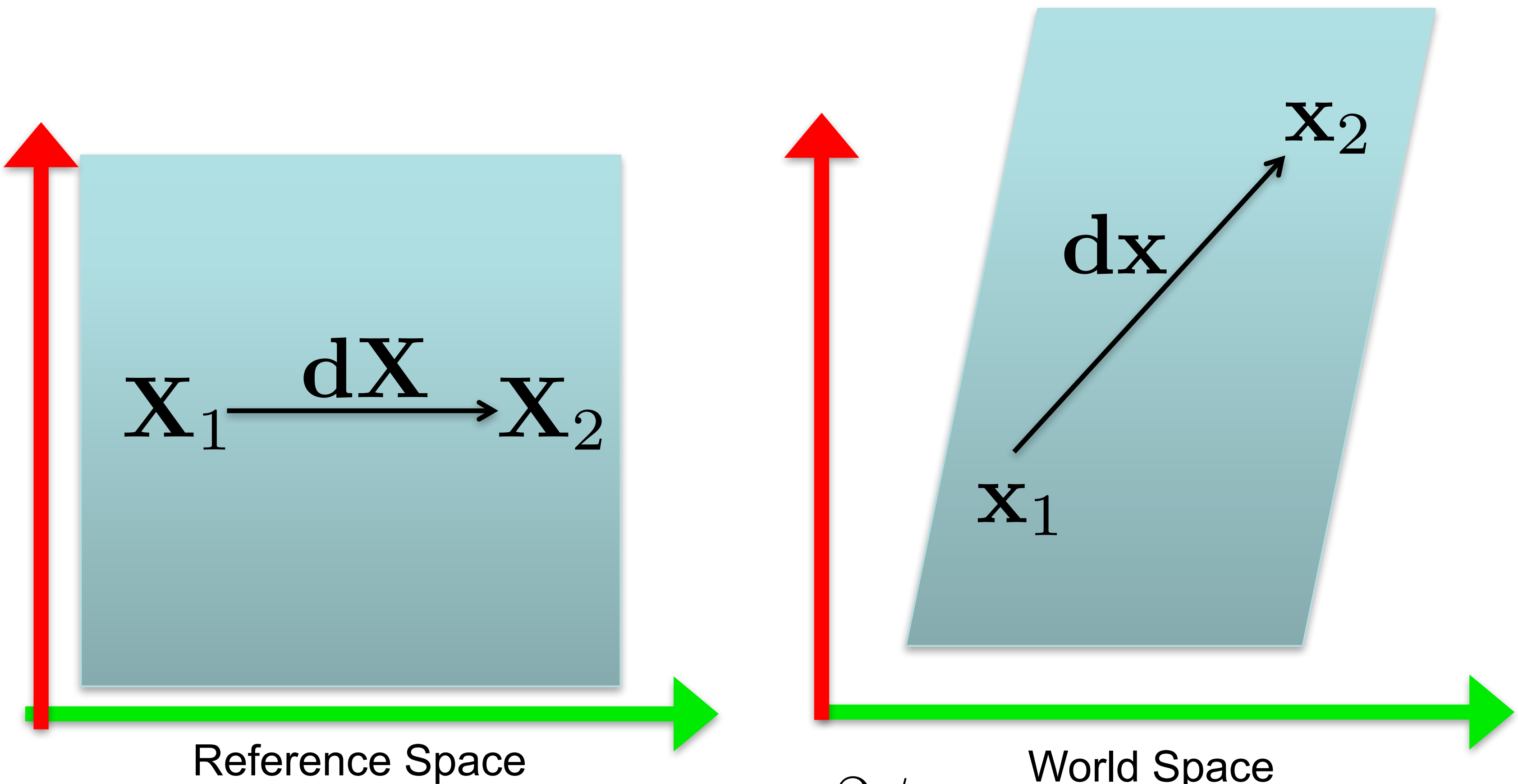
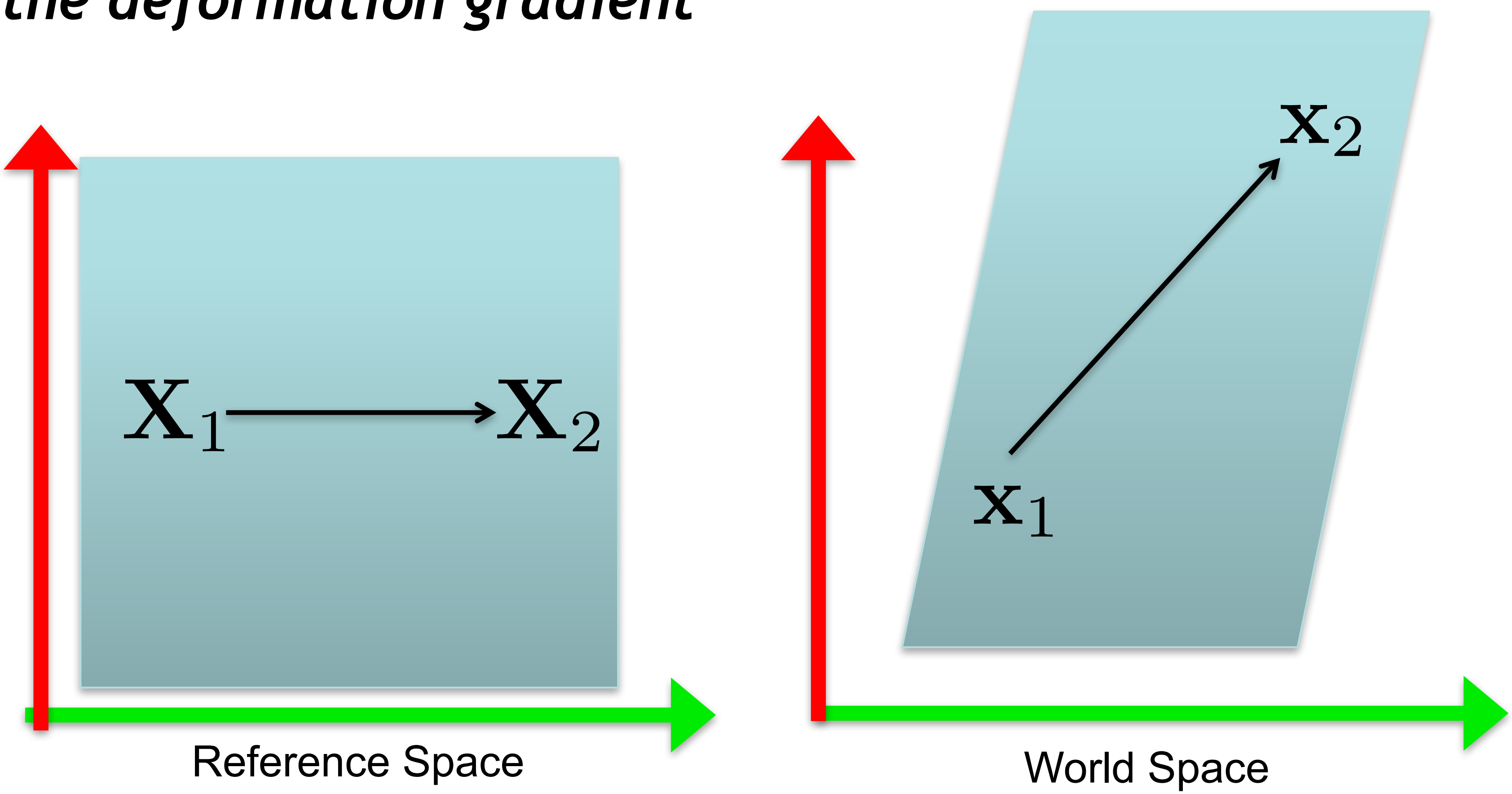
Continuum Mechanics: Deformation



$$\mathbf{dx} \approx \frac{\partial \phi}{\partial \mathbf{X}} \mathbf{dX}$$

Continuum Mechanics: Deformation

- $\mathbf{F}$ is our deformation measure called ***the deformation gradient***



$$\mathbf{dx} \approx \mathbf{F}\mathbf{dX}$$

Continuous Deformation vs. Mass Spring

- Spring Force:

$$-k\left(\left(\frac{l}{l_0}-1\right)\frac{\mathbf{x}-\mathbf{y}_i}{|\mathbf{x}-\mathbf{y}_i|}\right)$$

- Deformation:

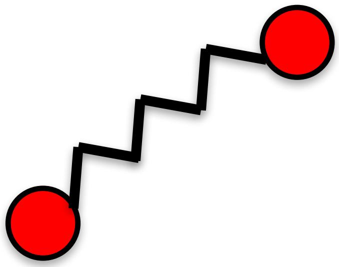
$$\left(\frac{l}{l_0}-1\right)\frac{\mathbf{x}-\mathbf{y}_i}{|\mathbf{x}-\mathbf{y}_i|}$$

Equivalent to $\mathbf{F}$

Continuous Deformation vs. Mass Spring

- Undeformed Spring:

$$\frac{l}{l_0} = ?$$

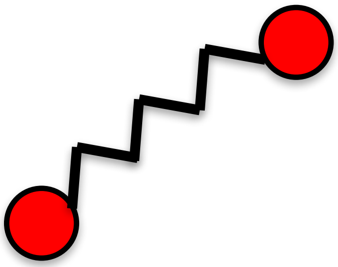


- Undeformed Continuum:

Continuous Deformation vs. Mass Spring

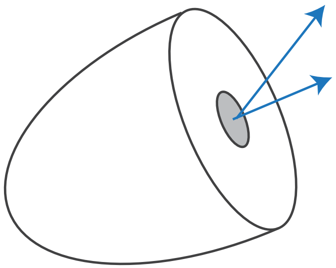
- Undeformed Spring:

$$\frac{l}{l_0} = 1$$



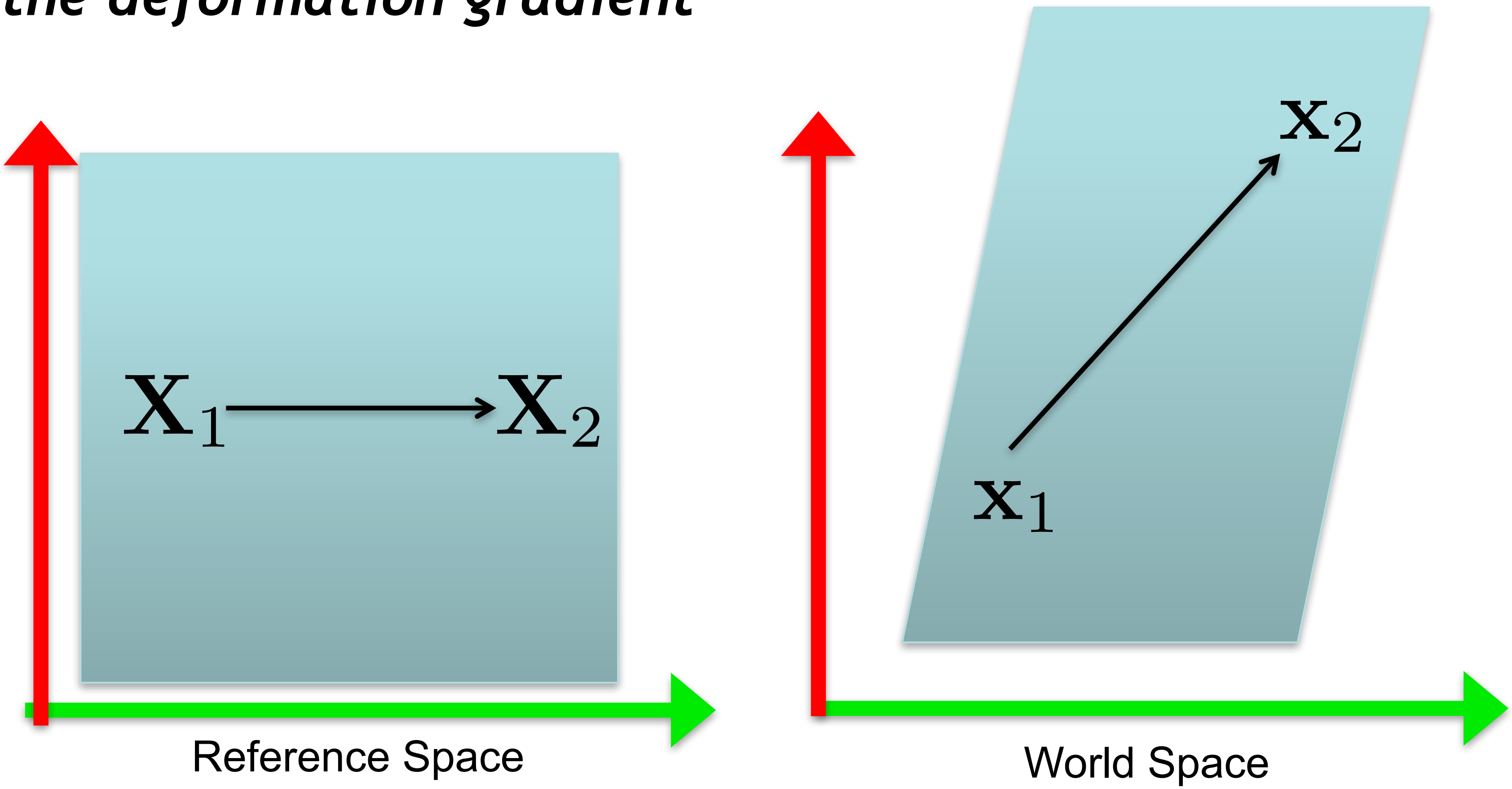
- Undeformed Continuum:

$$\mathbf{F} = ?$$



A Few Slides Ago…

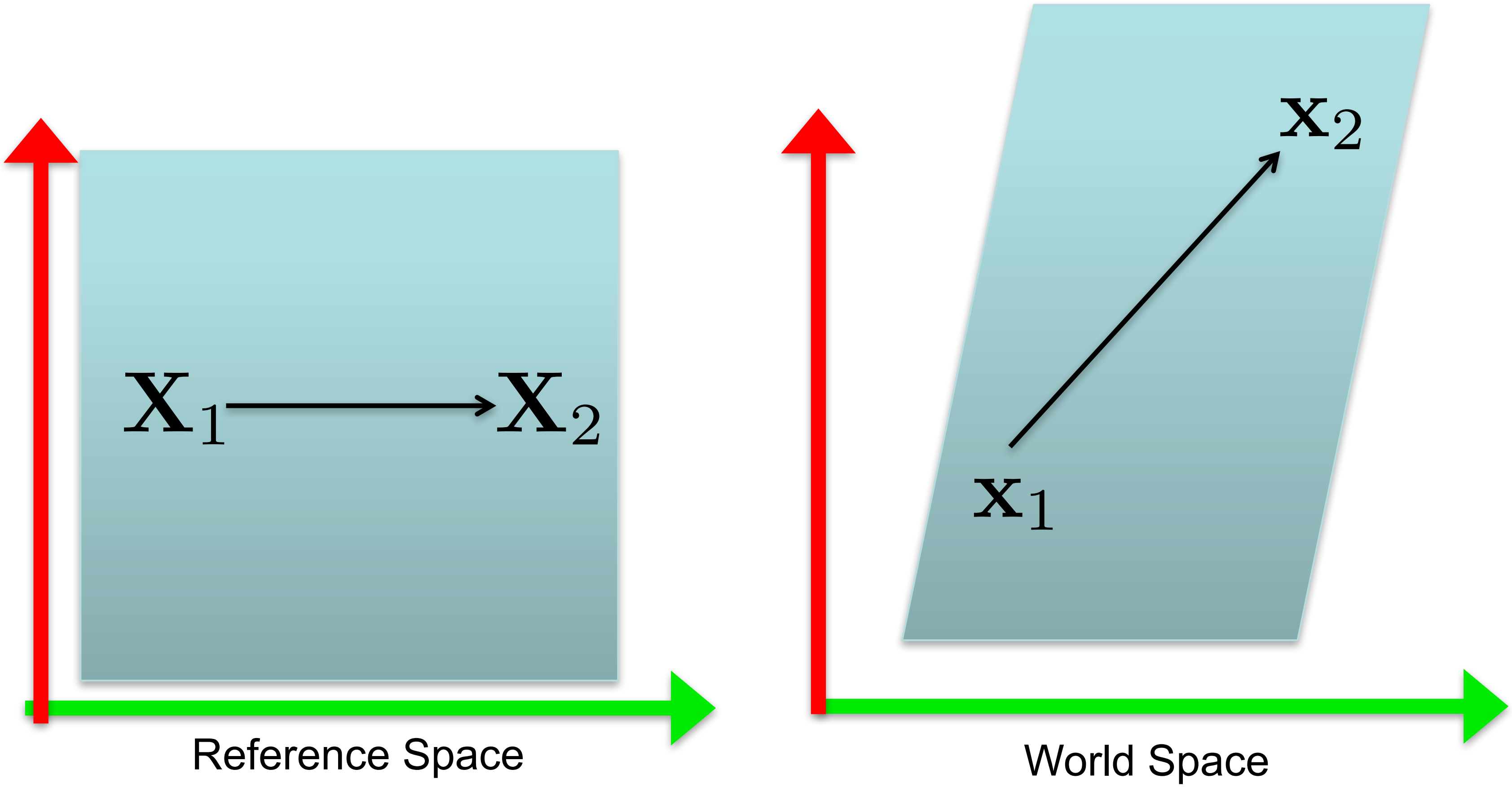
- $\mathbf{F}$ is our deformation measure called *the deformation gradient*



$$\mathbf{dx} \approx \mathbf{F}\mathbf{dX}$$

A Few Slides Ago…

- If there is no deformation $\mathbf{dx} = \mathbf{dX}$



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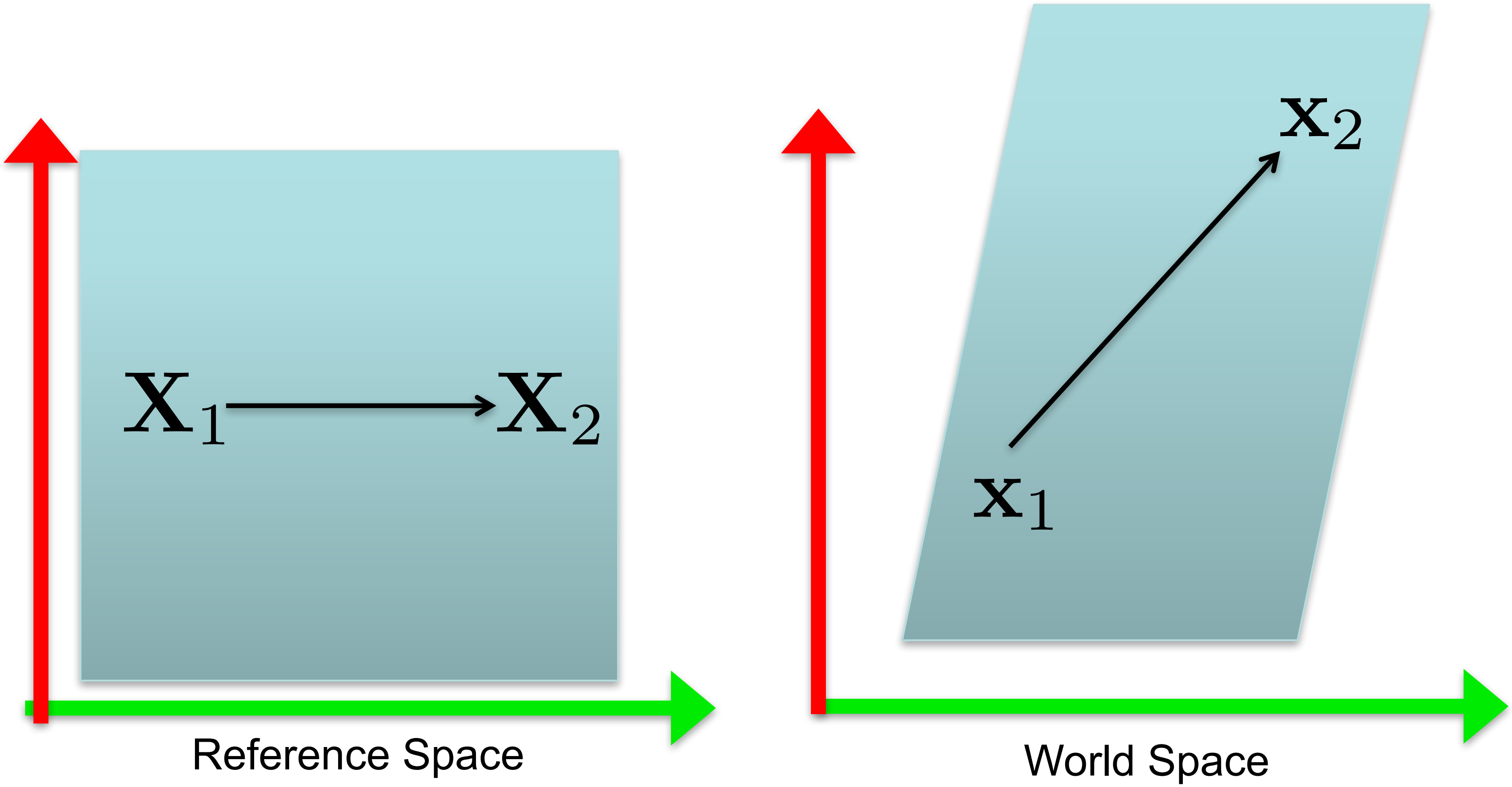

$$\mathbf{dX} \approx \mathbf{FdX}$$

Continuous Deformation vs. Mass Spring

- Undeformed Spring:

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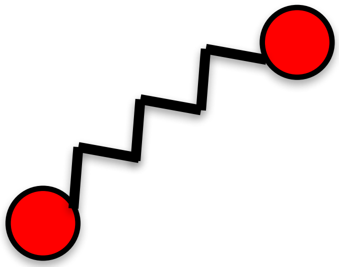
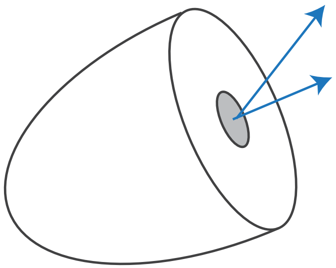


- Undeformed Continuum:

$$\mathbf{F} = \mathrm{I}$$



Continuous Deformation vs. Mass Spring

- Spring Force:

$$-k\left(\left(\frac{l}{l_0}-1\right)\frac{\mathbf{x}-\mathbf{y}_i}{|\mathbf{x}-\mathbf{y}_i|}\right)$$

- Deformation:

$$\boxed{\left(\frac{l}{l_0}-1\right)}\frac{\mathbf{x}-\mathbf{y}_i}{|\mathbf{x}-\mathbf{y}_i|}$$

This is called **strain**

Properties of a Strain

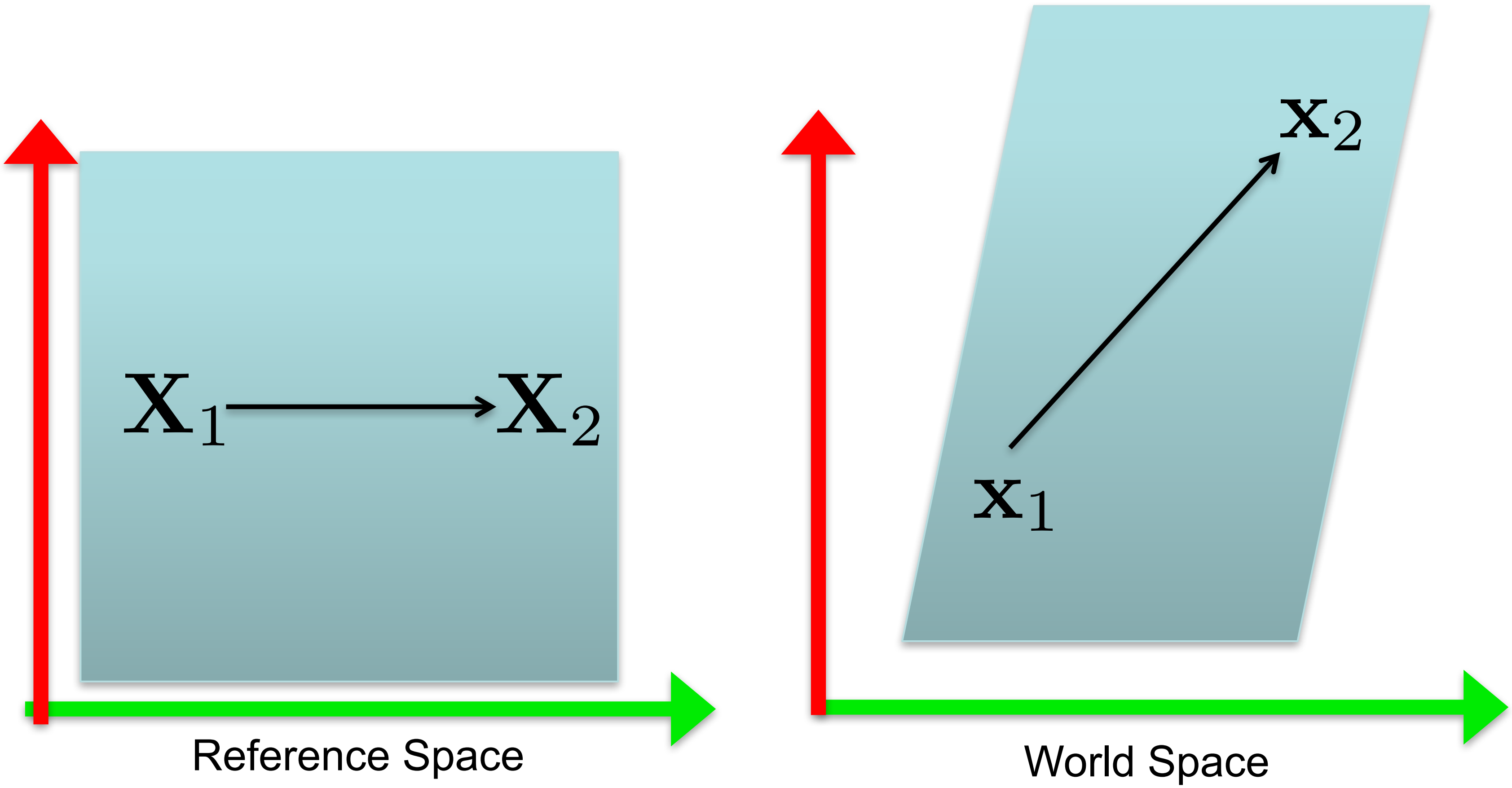
- Spring strain

$$\left(\frac{l}{l_0} - 1\right)$$

- Property 1: 0 if spring is undeformed
- Property 2: invariant to rigid motion
- Can we find a similar measure that would work for an arbitrary volume ?
- Any guesses ?

Proposed Strain Measure

- Let's try and use the difference between the lengths of our deformed vector and its undeformed counterpart



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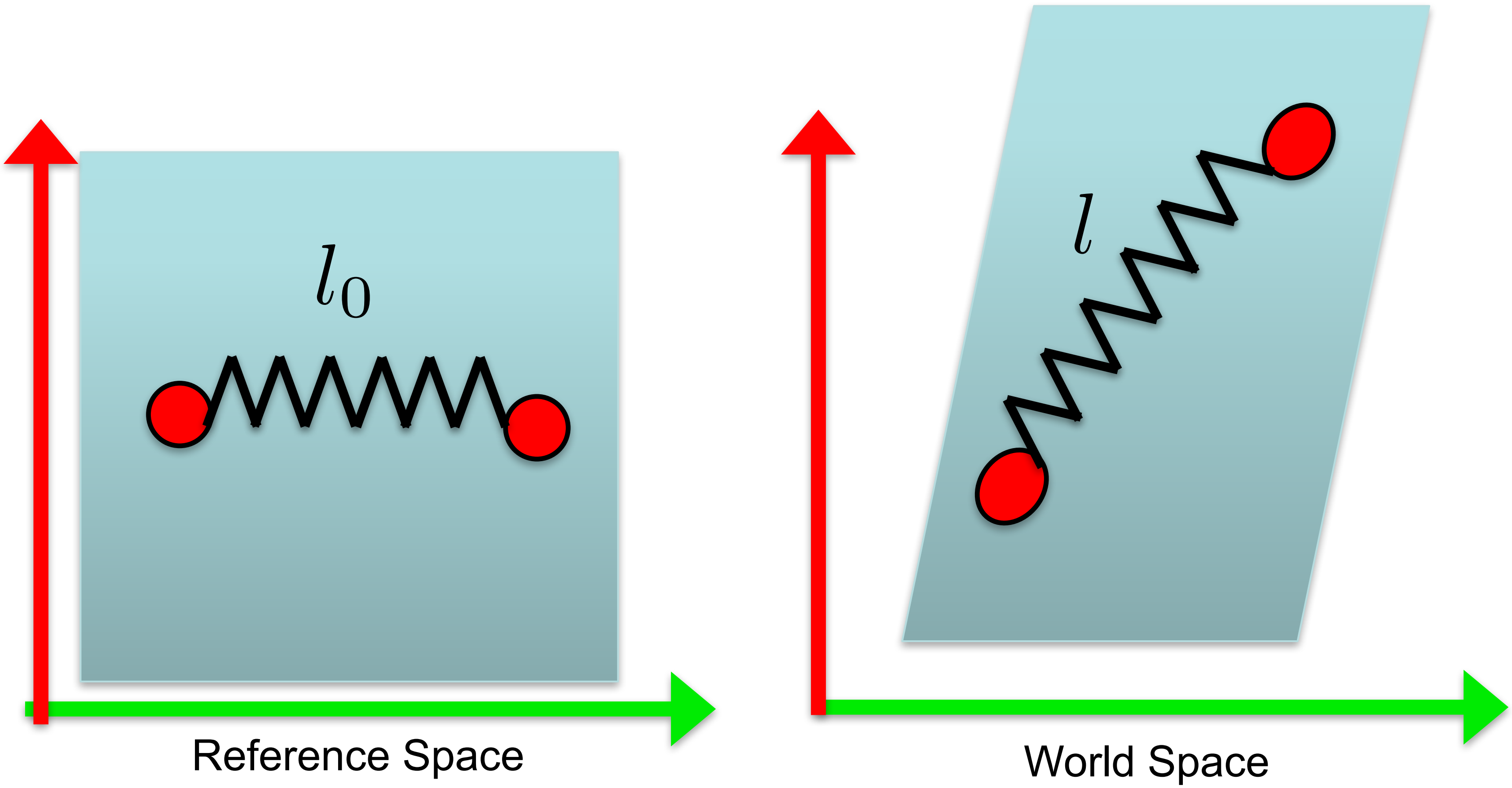

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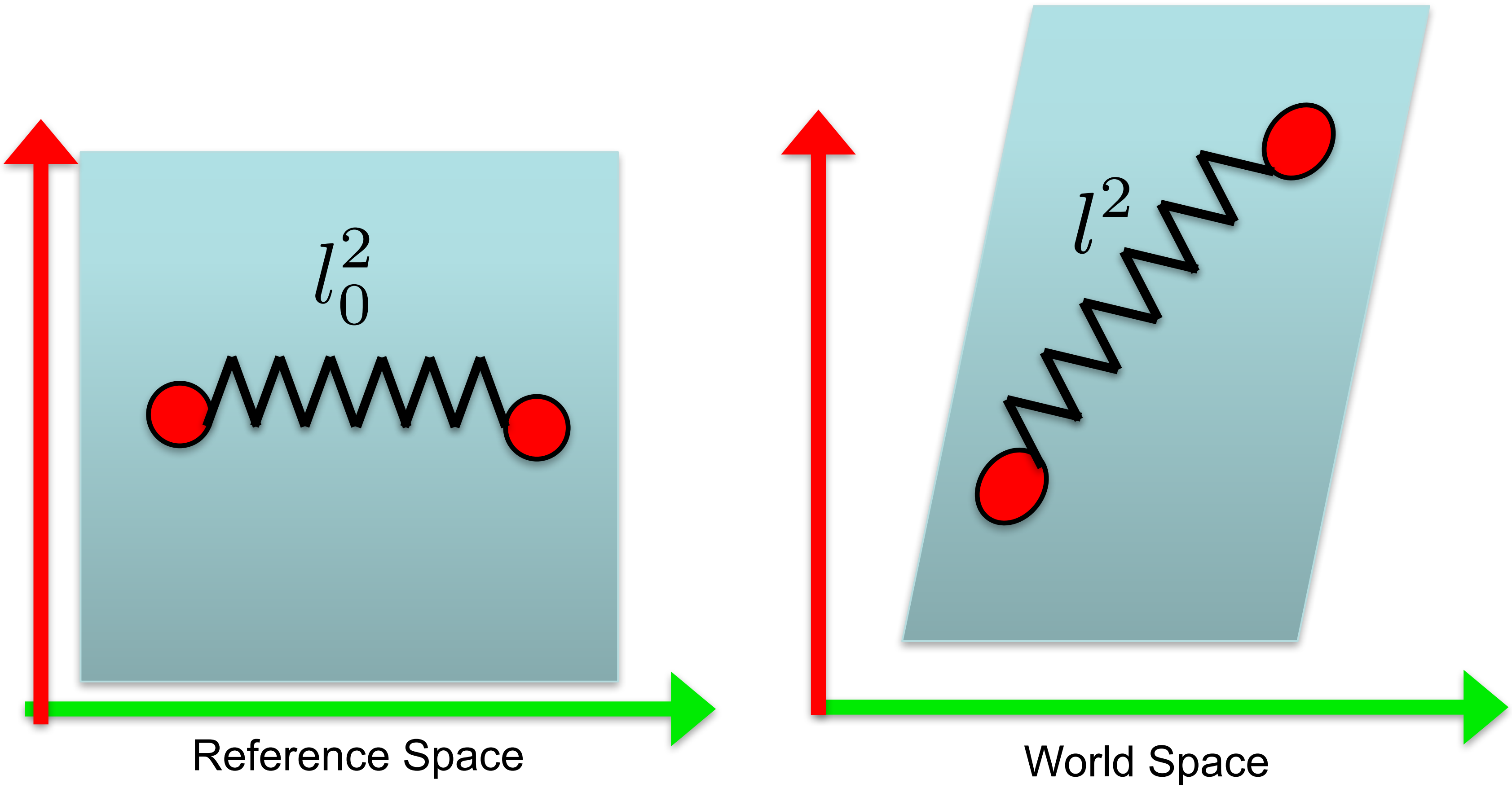
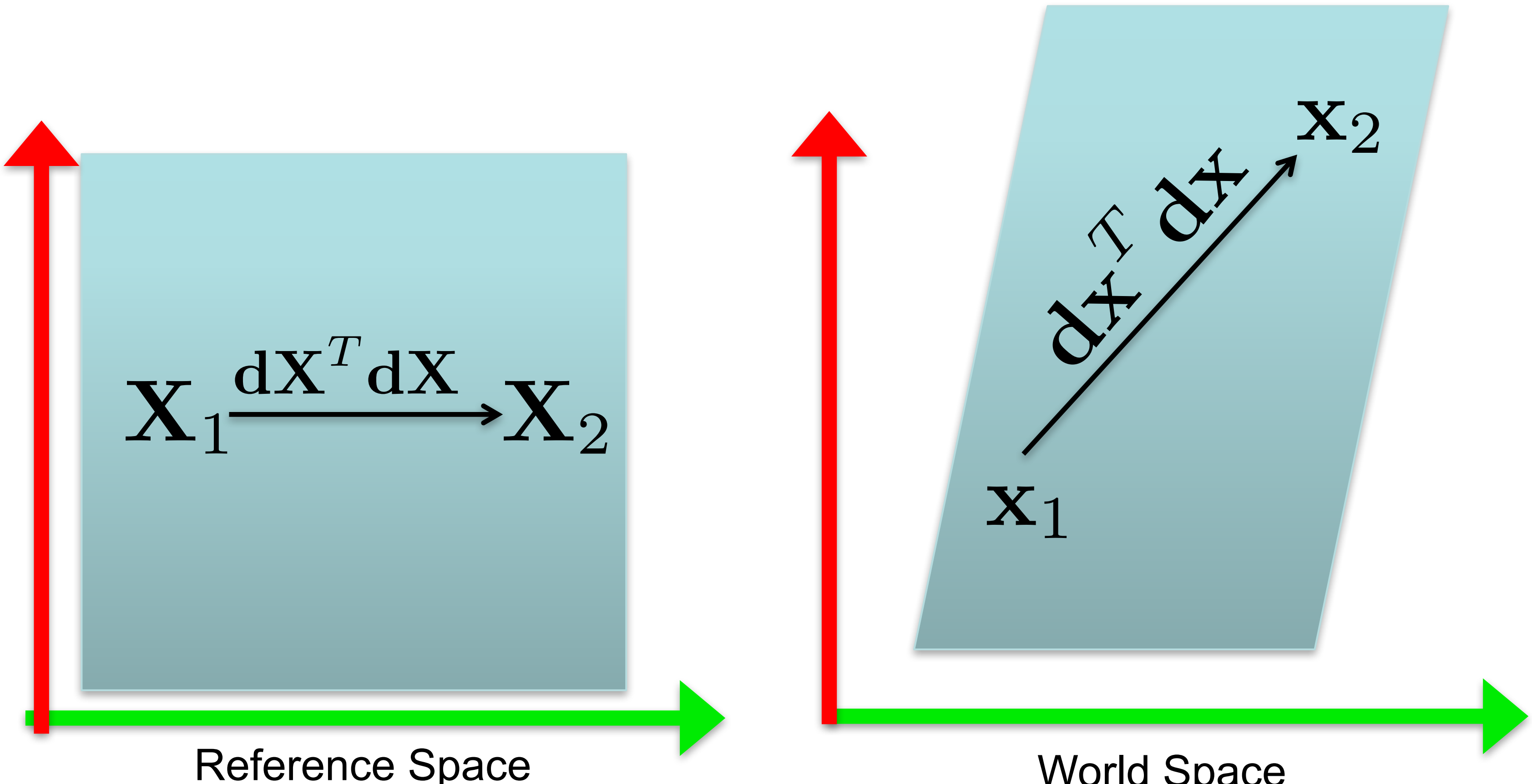


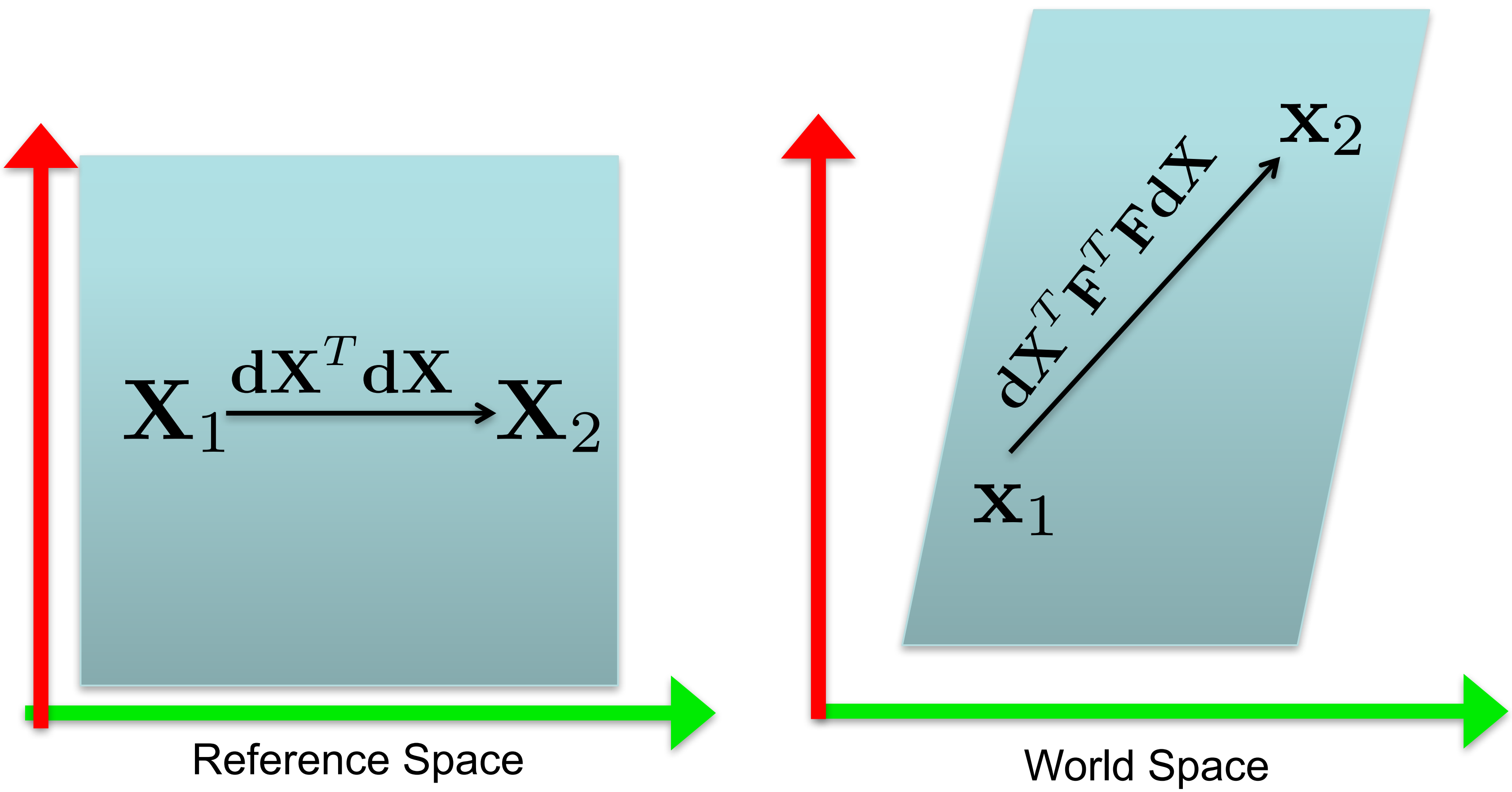
Proposed Strain Measure: Distance Between Points

- Substitute in formula for length squared



Proposed Strain Measure: Distance Between Points

- Use $\mathbf{dx} \approx \mathbf{F}\mathbf{dX}$



Proposed Strain Measures: Distance Between Points

- We want to quantify change in shape so we can take the difference of the original and deformed lengths

$$l^2 - l_0^2$$

$$\mathbf{dX}^T\mathbf{F}^T\mathbf{F}\mathbf{dX} - \mathbf{dX}^T\mathbf{dX}$$

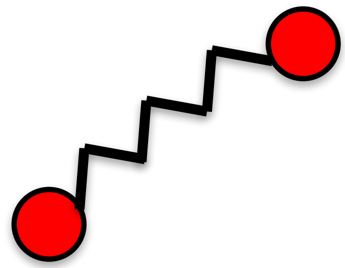
$$\mathbf{dX}^T\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)\mathbf{dX}$$

Green Lagrange Strain: $\boxed{\frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)}$

Continuous Deformation vs. Mass Spring

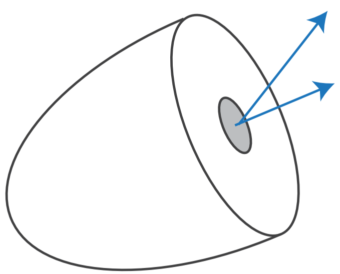
- Undeformed Spring:

$$\frac{l}{l_0} - 1 = ?$$



- Undeformed Continuum:

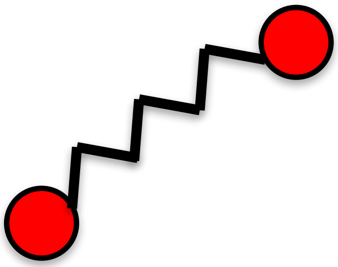
$$\frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = ?$$



Recall...

- Undeformed Spring:

$$\frac{l}{l_0} - 1 = 0$$



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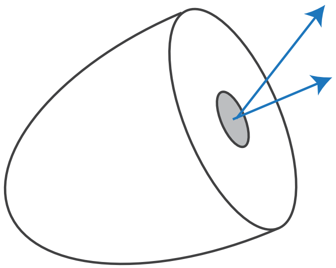

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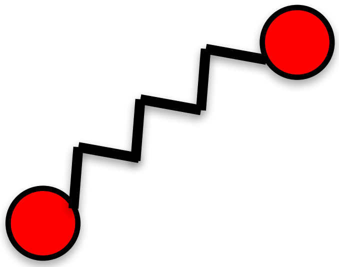

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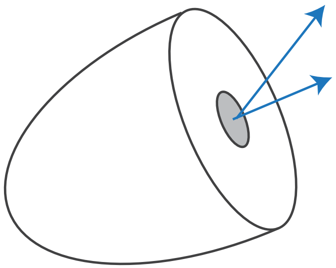
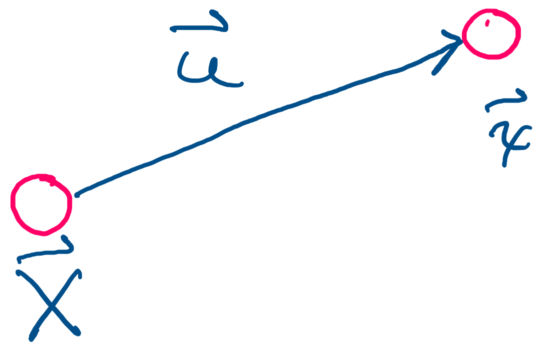


For Small Deformations

Position $\vec{x}(\vec{X}) = \vec{X} + \vec{u}(\vec{X})$

where $\vec{u} \in \mathbb{R}^3$ is the displacement



We can rewrite F using $\vec{u}$

For Small Deformations

$$F = \frac{d\vec{x}}{d\vec{X}} = \frac{d(\vec{X} + \vec{u})}{d\vec{X}} = I + \frac{d\vec{u}}{d\vec{X}}$$

$$\text{Strain: } \frac{1}{2}\left[I - \left(I + \frac{d\vec{u}}{d\vec{X}} \right)^T \left(I + \frac{d\vec{u}}{d\vec{X}} \right) \right]$$

$$\Rightarrow \frac{1}{2}\left[I - I + \frac{d\vec{u}}{d\vec{X}}^T \frac{d\vec{u}}{d\vec{X}} + \frac{d\vec{u}}{d\vec{X}}^T + \frac{d\vec{u}}{d\vec{X}} \right]$$

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$$\Rightarrow \frac{1}{2}\left[\cancel{I - I} + \underbrace{\frac{d\vec{u}}{d\vec{X}}^{T\,T} \frac{d\vec{u}}{d\vec{X}}}_{\text{if } u \text{ is very small} \ldots} + \frac{d\vec{u}}{d\vec{X}}^{T} + \frac{d\vec{u}}{d\vec{X}} \right]$$

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Continuum Mechanics: The Required Stuff

1. **Material Model**
2. Measure of Deformation

$$\frac{1}{2}\left(\frac{\partial \overrightarrow{\mathbf{u}}}{\partial \overrightarrow{\mathbf{X}}}^T + \frac{\partial \overrightarrow{\mathbf{u}}}{\partial \overrightarrow{\mathbf{X}}}\right)$$

- We just need the material model now

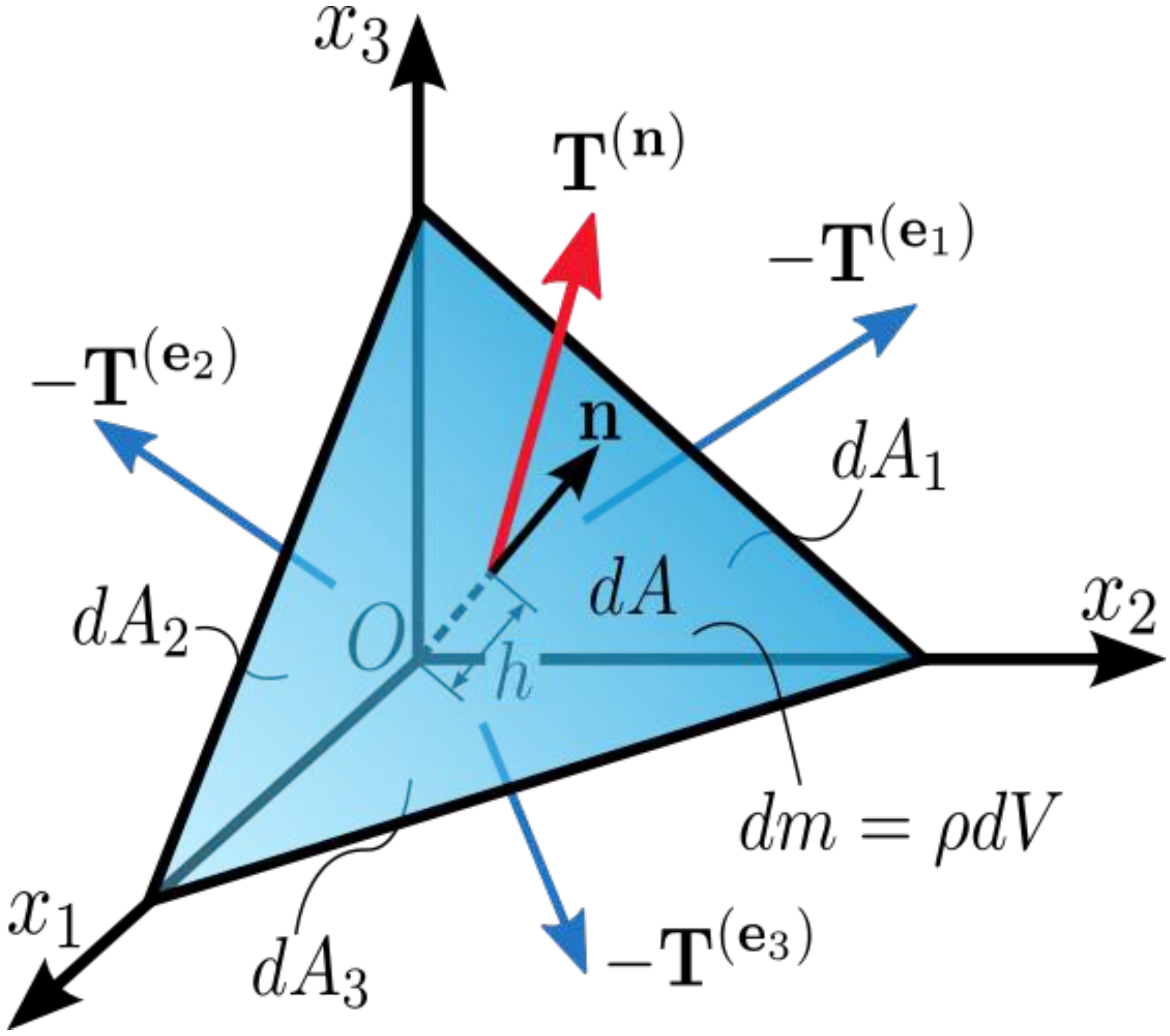
Material Models in Continuum Mechanics

- Materials models in continuum mechanics convert strain into a force per unit area called a stress

$$\sigma = \psi\left(\mathbf{E}\right)$$

Stress, it's
a matrix!

What is Stress ?



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Stress

Strain

Material Models in Continuum Mechanics

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$$\sigma = \psi\left(\mathbf{E}\right)$$

Stress

Strain

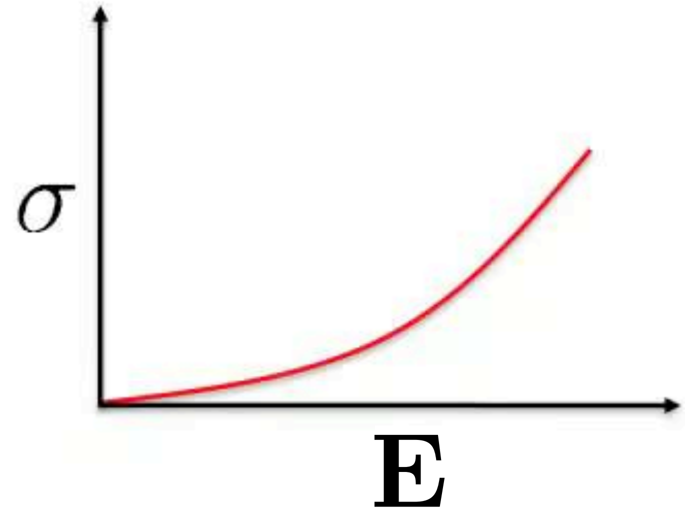
Material
Model

Material Representation

Stress-Strain relationship 1D

- Strain $\mathbf{E} = l/l_0 - 1$
- Stress $\sigma = f/A$

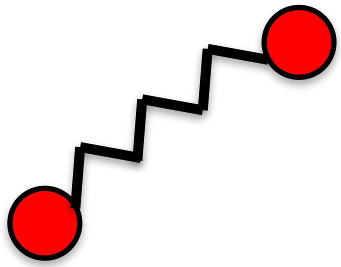
$\rightarrow$



Continuous Deformation vs. Mass Spring

- Mass Spring:

$$\mathbf{f} = \boxed{-k}\left(\left(\frac{l}{l_0} - 1\right)\frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|}\right)$$



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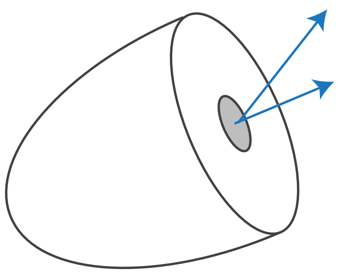
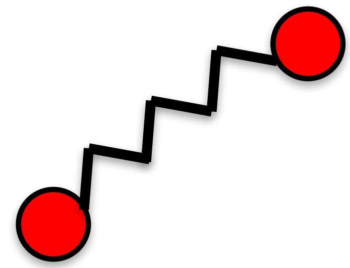


What is ψ ?

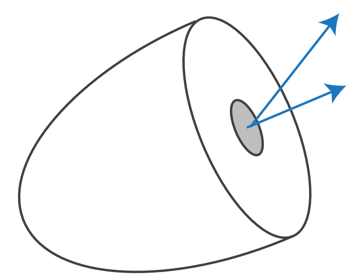
Continuous Deformation vs. Mass Spring

- Mass Spring:



$$\mathbf{f} = -\boxed{k}\left(\left(\frac{l}{l_0} - 1\right)\frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|}\right)$$

- Continuum Mechanics:

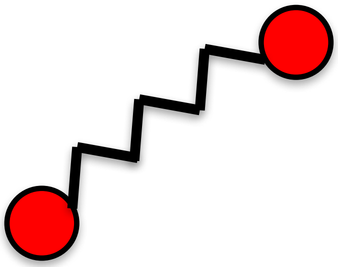


$$\sigma = \mathbf{KE}$$

Continuous Deformation vs. Mass Spring

- Mass Spring:

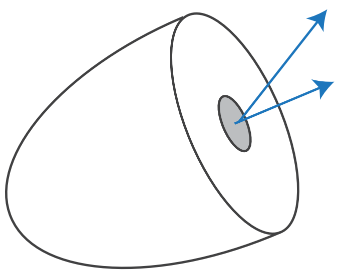
$$\mathbf{f} = -\boxed{k}\left(\left(\frac{l}{l_0} - 1\right)\frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|}\right)$$



- Continuum Mechanics:

$$\sigma = \mathbf{K}\mathbf{E}$$

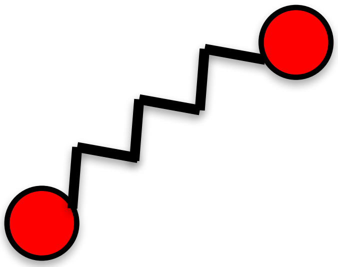
tensor



Continuous Deformation vs. Mass Spring

- Mass Spring:

$$\mathbf{f} = -\boxed{k}\left(\left(\frac{l}{l_0} - 1\right)\frac{\mathbf{x}-\mathbf{y}_i}{|\mathbf{x}-\mathbf{y}_i|}\right)$$

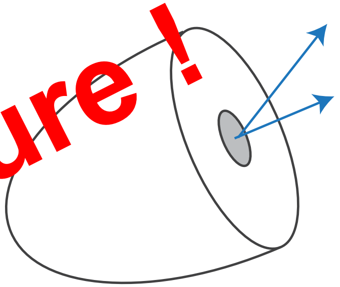


- Continuum Mechanics:

$$\sigma = \mathbf{K}\mathbf{E}$$

tensor

More next lecture !



Continuum Mechanics: The Required Stuff

- Material Model

$$\sigma = \mathbf{KE}$$

- Measure of Deformation

$$\frac{1}{2}\left(\frac{\partial \overrightarrow{\mathbf{u}}}{\partial \overrightarrow{\mathbf{X}}}^{T} + \frac{\partial \overrightarrow{\mathbf{u}}}{\partial \overrightarrow{\mathbf{X}}}\right)$$

- We have all the pieces now

Continuum Mechanics: The Required Stuff

- Material Model

$$\sigma = \mathbf{KE}$$

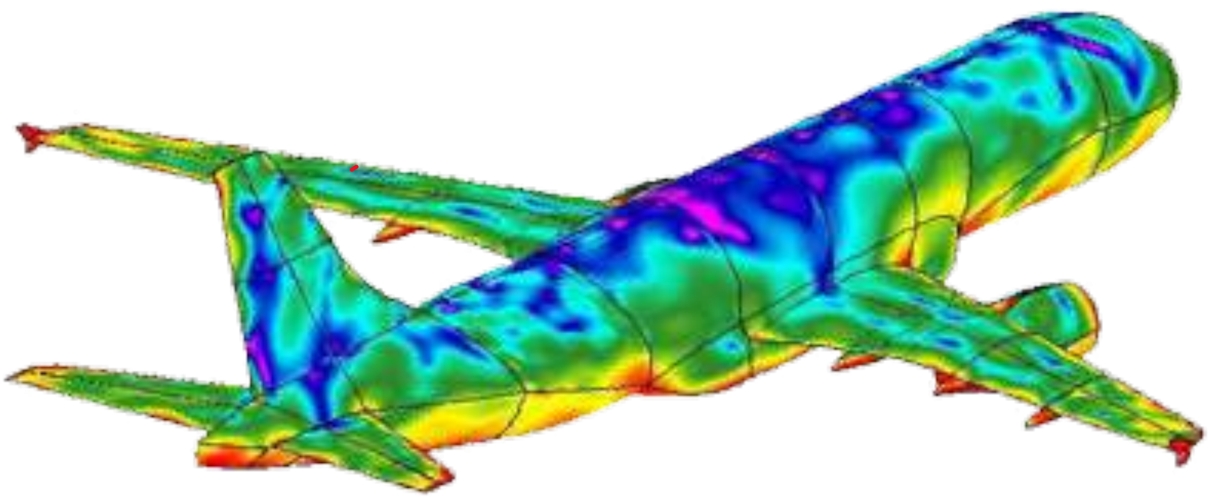
- Measure of Deformation

$$\frac{1}{2}\left(\frac{\partial \overrightarrow{\mathbf{u}}}{\partial \overrightarrow{\mathbf{X}}}^T + \frac{\partial \overrightarrow{\mathbf{u}}}{\partial \overrightarrow{\mathbf{X}}}\right)$$

- One more thing …

Static Structural Analysis

- Solve for the rest shape of the object under some external forces



Static Structural Analysis

$$\underbrace{f_{INT}(u)}_{\text{Internal forces from deformation}} + \underbrace{f_{EXT}}_{\text{External forces like gravity}} = 0$$

Force Balance

Hmmm.... something that equals 0....

Static Structural Analysis

$$f_{INT}(u) + f_{EXT} = 0$$

Optimize!

$$\Downarrow$$

$$u^* = \underset{\Omega}{\operatorname{argmin}} \int \underbrace{\tfrac{1}{2} E^T K E}_{\text{Internal Potential Energy}} - \overbrace{U^T f_{EXT}}^{\text{Work done by external force}} \, d\Omega$$

$$E = \begin{bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \\ E_{yz} \\ E_{xz} \\ E_{xy} \end{bmatrix} \in \mathbb{R}^{6\times 1}$$

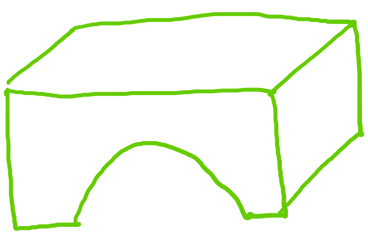
Static Structural Analysis

$$u^* = \operatorname{argmin} \int \frac{1}{2} E^T \underline{K} E - u^T f_{EXT} \, d\Omega$$

Ω

Volume Object

Hooke's Law



$$\frac{a}{(1+b)(1-2b)} \begin{bmatrix} (1-b) & b & b & 0 & 0 & 0 \\ b & (1-b) & b & 0 & 0 & 0 \\ b & b & (1-b) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2b) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2b) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2b) \end{bmatrix}$$

Static Structural Analysis

$$u^* = \underset{\Omega}{\operatorname{argmin}} \int \frac{1}{2} E^T \underline{K} E - u^T f_{EXT} \, d\Omega$$

Hooke's Law

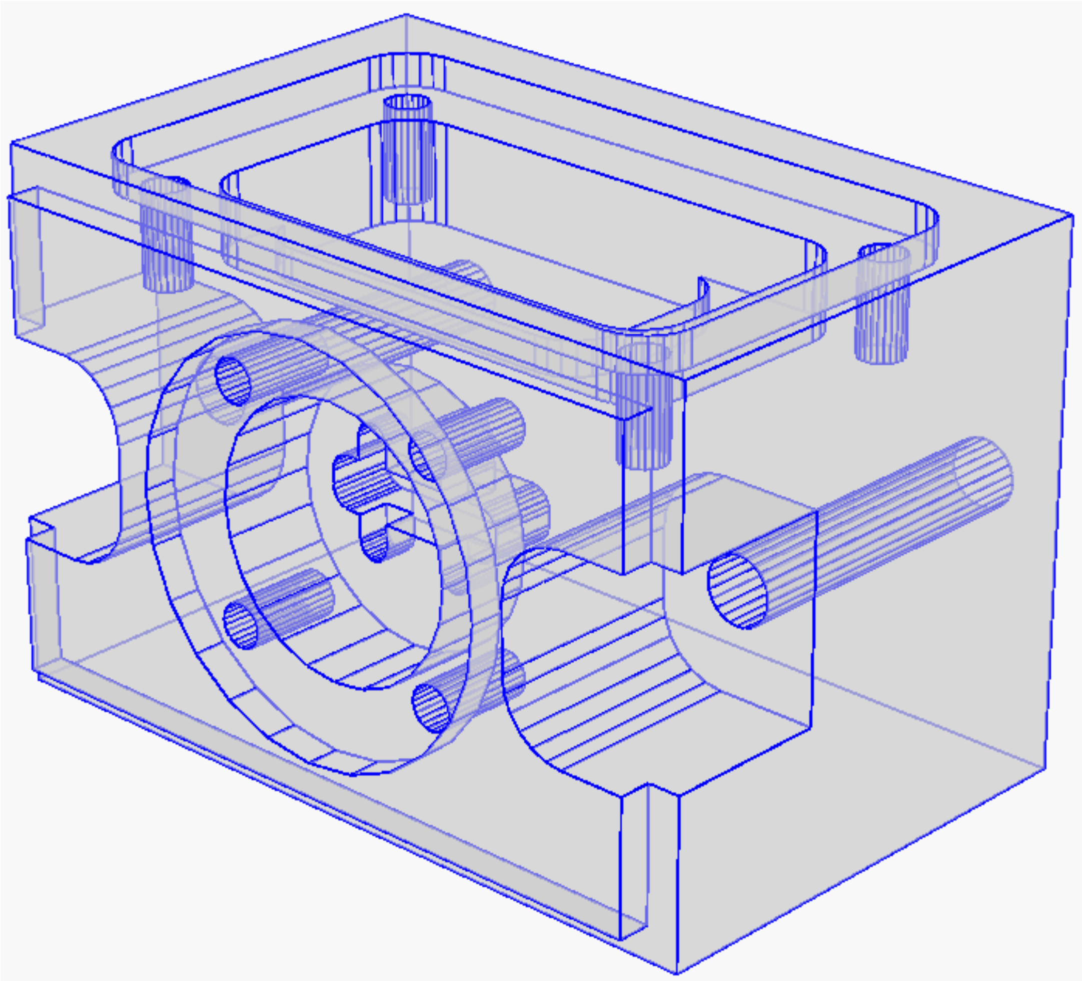
$$\frac{a}{(1+b)(1-2b)} \begin{bmatrix} (1-b) & b & b & 0 & 0 & 0 \\ b & (1-b) & b & 0 & 0 & 0 \\ b & b & (1-b) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2b) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2b) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2b) \end{bmatrix}$$

a = Young's Modulus
($1e6$ is a good value)

b = Poisson's Ratio
(0.45 is good)

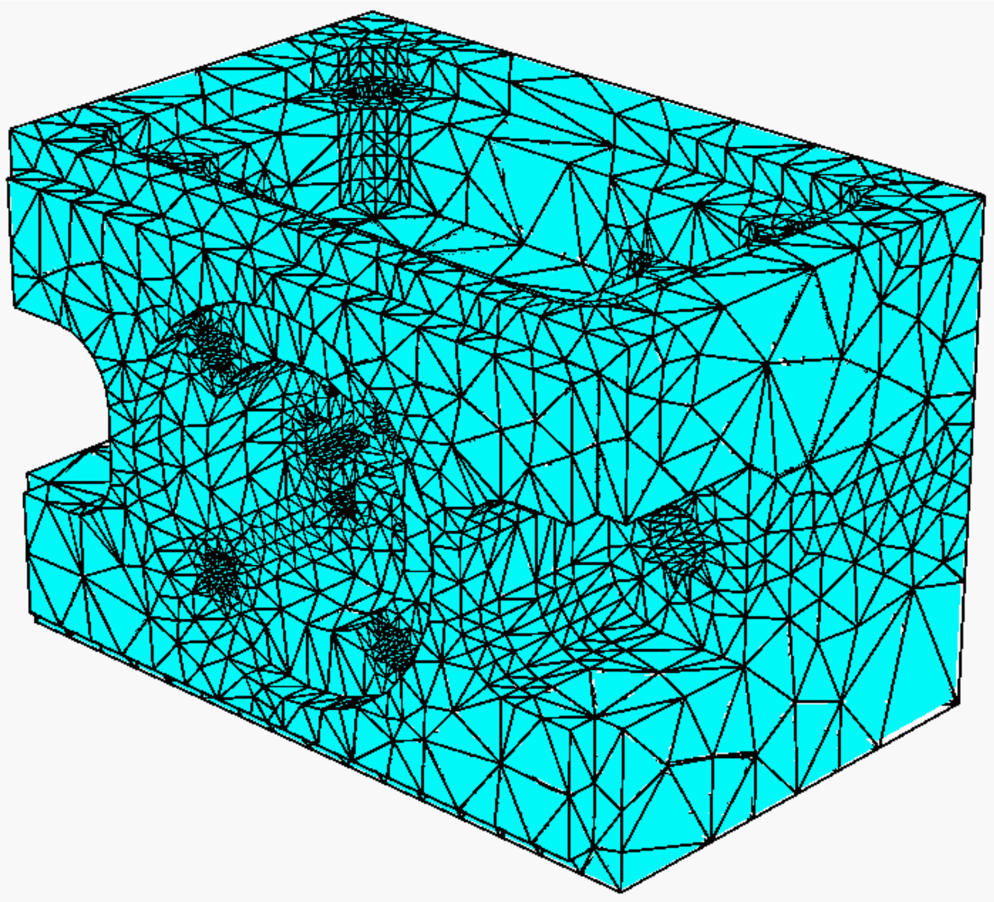
Discretization

1. Input Object



Discretization

2. Divide object into “elements”, small simple volumetric shapes



Back to the Math!

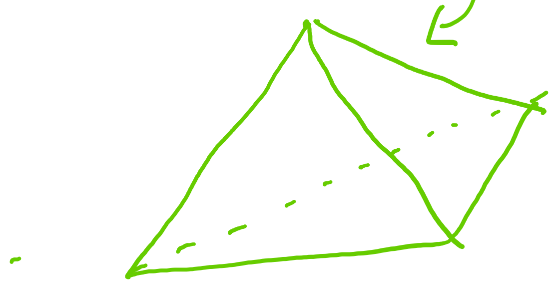
$$u^* = \operatorname{argmin} \int \frac{1}{2} E^T K E - u^T f_{EXT} \, d\Omega$$

$\boxed{\Omega}$ — Divide into tetrahedra

number of tetrahedra

$$u^* = \operatorname{argmin} \sum_{i=1}^{N} \int \frac{1}{2} E^T K E - u^T f_{EXT} \, d\Omega_i$$

Ω_i



Back to the Math!

$$u^* = \underset{}{\mathrm{argmin}} \sum_{i}^{N} \int_{\Omega_i} \frac{1}{2} E^T K E - U^T f_{EXT} \, d\Omega_i$$

What about E?

Back to the Math!

$$u^* = \underset{\Omega_i}{\operatorname{argmin}} \sum_i^N \int \frac{1}{2} E^T K E - U^T f_{EXT} \, d\Omega_i$$

What about E?

$$E = \begin{bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \\ E_{yz} \\ E_{xz} \\ E_{xy} \end{bmatrix}$$

Back to the Math!

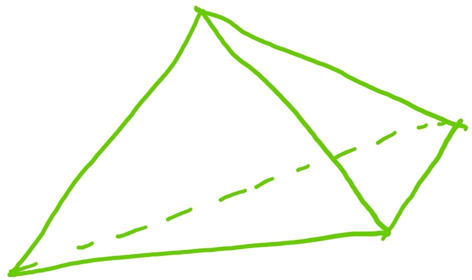
$$u^* = \operatorname{argmin} \sum_i^N \int_{\Omega_i} \tfrac{1}{2} E^T K E - U^T f_{EXT} \, d\Omega_i$$

What about E?

$$E = \begin{bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \\ E_{yz} \\ E_{xz} \\ E_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial X_x} \\ \frac{\partial u_y}{\partial X_y} \\ \frac{\partial u_z}{\partial X_z} \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial X_z} + \frac{\partial u_z}{\partial X_y}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial X_z} + \frac{\partial u_z}{\partial X_x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial X_y} + \frac{\partial u_y}{\partial X_x}\right) \end{bmatrix} = \text{What Next?}$$

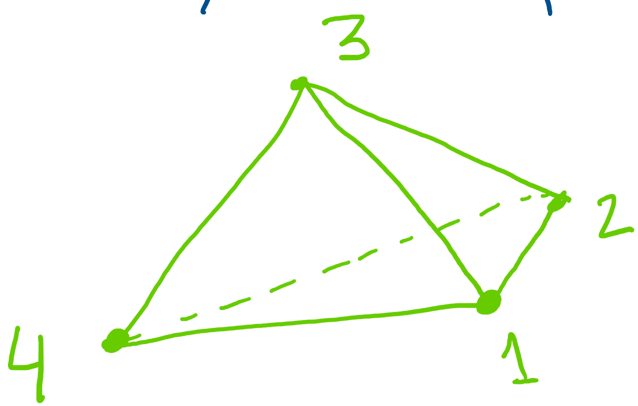
Back to the Math!

We need a way to represent $u(x)$ inside



Back to the Math!

We need a way to represent $u(x)$ inside



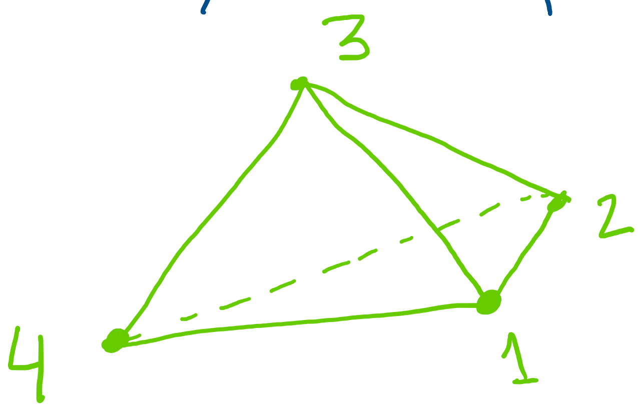
We'll use Barycentric Coordinates

1. One coordinate ϕ_j for each (j^{th}) vertex
2. $\phi_j(x) = 1$ if $x =$ position of vertex j

   $= 0$ else

Back to the Math!

We need a way to represent $u(x)$ inside



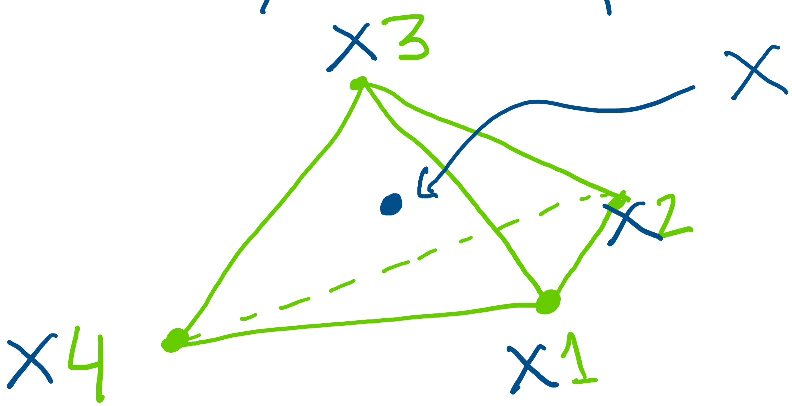
We'll use Barycentric Coordinates

1. One coordinate ϕ_j for each (j^{th}) vertex
2. $\phi_j(x) = 1$ if $x =$ position of vertex j

   $= 0$ else
3. $\sum_{j=1}^{4} \phi_j(x) = 1$

Back to the Math!

We need a way to represent $u(x)$ inside



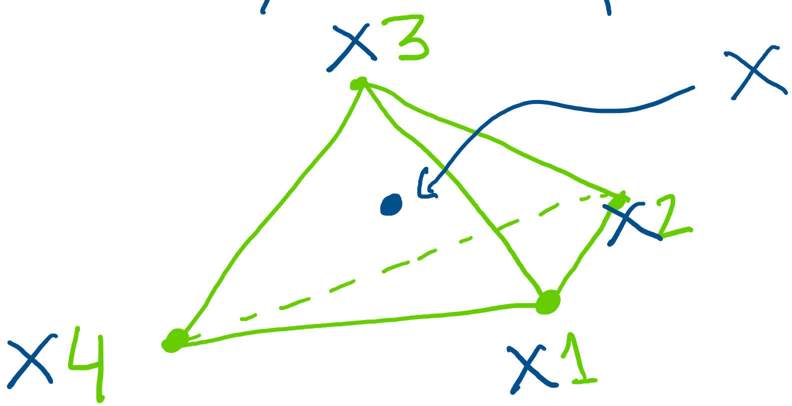
We'll use Barycentric Coordinates

$$x = x_1 \phi_1 + x_2 \phi_2 + x_3 \phi_3 + x_4 \phi_4$$

$\Rightarrow$ 3 equations, 4 unknowns

Back to the Math!

We need a way to represent $u(x)$ inside



We'll use Barycentric Coordinates

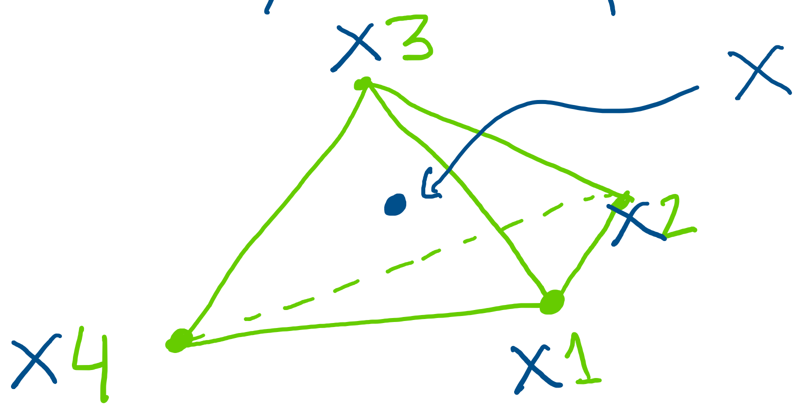
$$x = x_1 \phi_1 + x_2 \phi_2 + x_3 \phi_3 + x_4 \phi_4$$

$\Rightarrow$ 3 equations, 4 unknowns

* Use $\sum \phi_j = 1$ OR $\phi_4 = 1 - \phi_1 - \phi_2 - \phi_3$

Back to the Math!

We need a way to represent $u(X)$ inside



We'll use Barycentric Coordinates

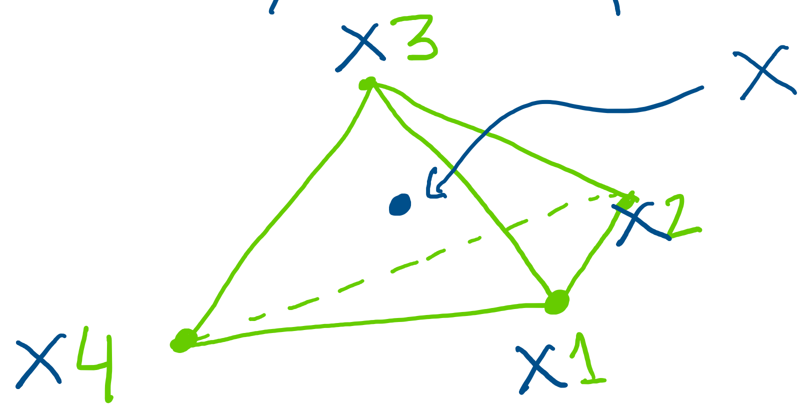
$$\cancel{X = X_1 \phi_1 + X_2 \phi_2 + X_3 \phi_3 + X_4 \phi_4}$$

(1) $$X = X_1 \phi_1 + X_2 \phi_2 + X_3 \phi_3 + X_4 (1 - \phi_1 - \phi_2 - \phi_3)$$

(2) $$X - X_4 = (X_1 - X_4)\phi_1 + (X_2 - X_4)\phi_2 + (X_3 - X_4)\phi_3$$

Back to the Math!

We need a way to represent $u(x)$ inside



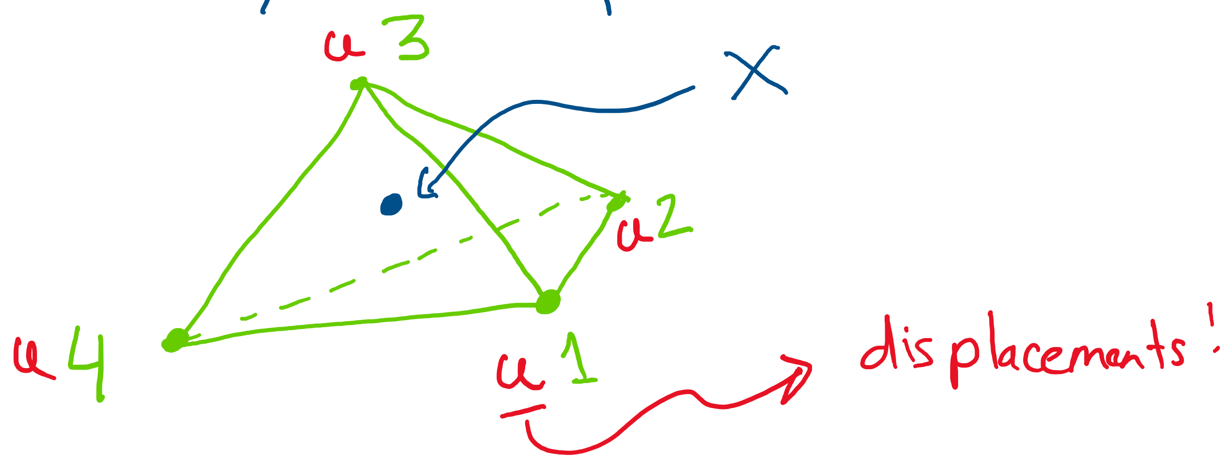
We'll use Barycentric Coordinates

$$x - x_4 = (x_1 - x_4)\phi_1 + (x_2 - x_4)\phi_2 + (x_3 - x_4)\phi_3$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} (x_1 - x_4) & (x_2 - x_4) & (x_3 - x_4) \end{bmatrix}^{-1} \begin{bmatrix} x - x_4 \end{bmatrix}, \quad \phi_4(x) = 1 - \phi_1 - \phi_2 - \phi_3$$

Back to the Math!

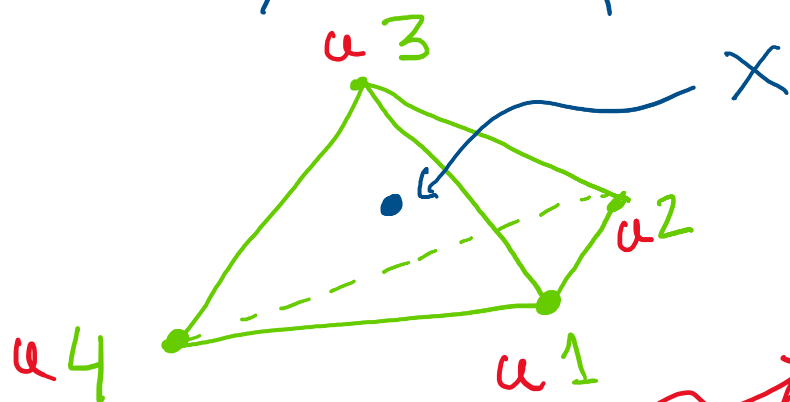
We need a way to represent u(X) inside



$$u(X) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) + u_4 \phi_4(x)$$

Back to the Math!

We need a way to represent u(X) inside



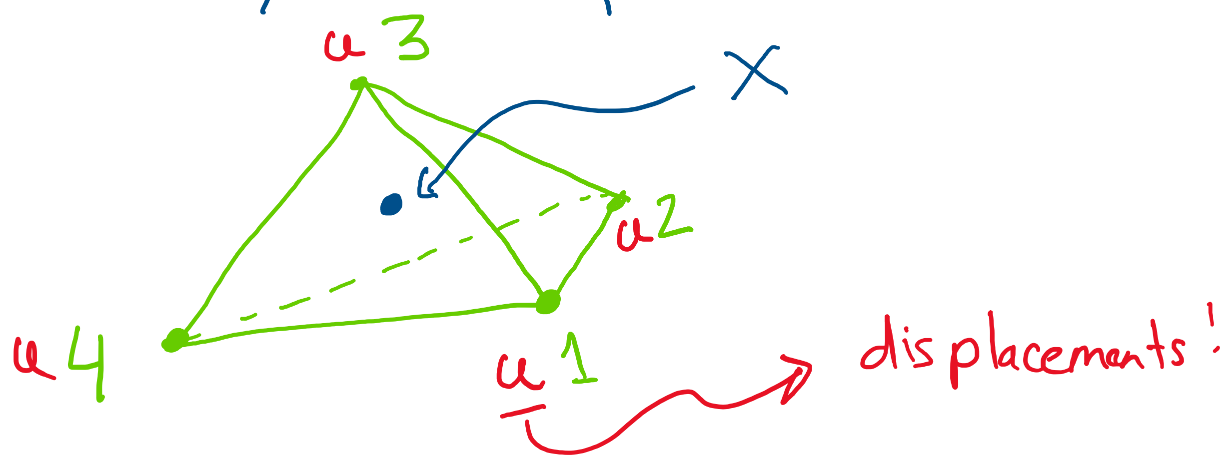
displacements!

$$u(X) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) + u_4 \phi_4(x)$$

Shape Functions

Back to the Math!

We need a way to represent $u(X)$ inside



$$u(X) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) + u_4 \phi_4(x)$$

We can take the derivative of this!

Back to the Math!

$$u^* = \operatorname{argmin} \sum_i^N \int_{\Omega_i} \tfrac{1}{2} E^T K E - U^T f_{EXT} \, d\Omega_i$$

What about E?

$$E = \begin{bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \\ E_{yz} \\ E_{xz} \\ E_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial X_x} \\ \frac{\partial u_y}{\partial X_y} \\ \frac{\partial u_z}{\partial X_z} \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial X_z} + \frac{\partial u_z}{\partial X_y}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial X_z} + \frac{\partial u_z}{\partial X_x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial X_y} + \frac{\partial u_y}{\partial X_x}\right) \end{bmatrix} = \text{What Next?}$$

Use Shape Funcs!

Back to the Math!

$$u(X) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) + u_4 \phi_4(x)$$

$$\frac{\partial u_i}{\partial X_j} = u_{1i} \frac{\partial \phi_1}{\partial X_j} + u_{2i} \frac{\partial \phi_2(x)}{\partial X_j} + U_{3i} \frac{\partial \phi_3(x)}{\partial X_j} + U_{4i} \frac{\partial \phi_4(x)}{\partial X_j}$$

Now we can build E!

Back to the Math!

$$u^* = \operatorname{argmin} \sum_{i}^{N} \int_{\Omega_i} \frac{1}{2} E^T K E - U^T f_{EXT} \, d\Omega_i$$

$$E = \begin{bmatrix} \dfrac{\partial u_x}{\partial x_x} \\ \dfrac{\partial u_y}{\partial x_y} \\ \dfrac{\partial u_z}{\partial x_z} \\ \frac{1}{2}\left(\dfrac{\partial u_y}{\partial x_z} + \dfrac{\partial u_z}{\partial x_y}\right) \\ \frac{1}{2}\left(\dfrac{\partial u_x}{\partial x_z} + \dfrac{\partial u_z}{\partial x_x}\right) \\ \frac{1}{2}\left(\dfrac{\partial u_x}{\partial x_y} + \dfrac{\partial u_y}{\partial x_x}\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \phi^1_{,x} & 0 & 0 & \phi^2_{,x} & 0 & 0 & \phi^3_{,x} & 0 & 0 & \phi^4_{,x} & 0 & 0 \\ 0 & \phi^1_{,y} & 0 & 0 & \phi^2_{,y} & 0 & 0 & \phi^3_{,y} & 0 & 0 & \phi^4_{,y} & 0 \\ 0 & 0 & \phi^1_{,z} & 0 & 0 & \phi^2_{,z} & 0 & 0 & \phi^3_{,z} & 0 & 0 & \phi^4_{,z} \\ 0 & \frac{\phi^1_{,z}}{2} & \frac{\phi^1_{,y}}{2} & 0 & \frac{\phi^2_{,z}}{2} & \frac{\phi^2_{,y}}{2} & & \cdots & & & & \\ & & & & & \cdots & & & & & & \\ & & & & & \cdots & & & & & & \end{bmatrix}}_{B} \overbrace{\begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ u_{2x} \\ u_{2y} \\ u_{2z} \\ u_{3x} \\ u_{3y} \\ u_{3z} \\ u_{4x} \\ u_{4y} \\ u_{4z} \end{bmatrix}}^{u}$$

Back to the Math!

$$u^* = \operatorname{argmin} \sum_i^N \int_{\Omega_i} \frac{1}{2} E^T K E - U^T f_{EXT} \, d\Omega_i$$

Use $E = Bu$

$$u^* = \operatorname{argmin} \sum_i^N \int_{\Omega_i} \frac{1}{2} u_i^T B(x)^T K B(x) u_i - u_i^T f_{EXT} \, d\Omega_i$$

Back to the Math!

$$u^* = \operatorname{argmin} \sum_{i}^{N} \int_{\Omega_i} \frac{1}{2} u_i^T B(x)^T K B(x) u_i - u_i^T f_{EXT} \, d\Omega_i$$

$$\Downarrow$$

$$u^* = \operatorname{argmin} \frac{1}{2} \sum_{i=1}^{N} u_i^T \underbrace{\int_{\Omega_i} B(x)^T K B(x) \, d\Omega_i}_{K_i} u_i \quad \ldots$$

Back to the Math!

$$u^* = \operatorname{argmin} \frac{1}{2} \sum_{i=1}^{N} u_i^T K_i u_i^T - u_i^T f_{EXT}$$

Write as big quadratic function

$$u^* = \operatorname{argmin} \frac{1}{2} u^T K u - u^T f$$

How do you find optimal point?

Solution Procedure

$$\nabla_u \left(\frac{1}{2} u^T K u - u^T f \right) = 0$$

$$K u = f$$

Step 3: Build Per Element Equations

- In all our Elements

$$\mathbf{K}_i\mathbf{y} = \mathbf{f}$$

Step 5: Assembly

- We have a collection of Element "Stiffness" matrices (our $\mathbf{K}_i$'s)

- We will "assemble" these Element matrices into a global matrix for our problem

Step 5: Assembly

- For a particular element we have

$$\begin{pmatrix} 2 & -1 \\ \boxed{-1} & 2 \end{pmatrix} \begin{pmatrix} y_p \\ y_{p+1} \end{pmatrix}$$

Contribution of y_p to “force” at y_{p+1}

Step 5: Assembly

- Now we know how to add our element matrix into the global matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Row p+1, Column p

K

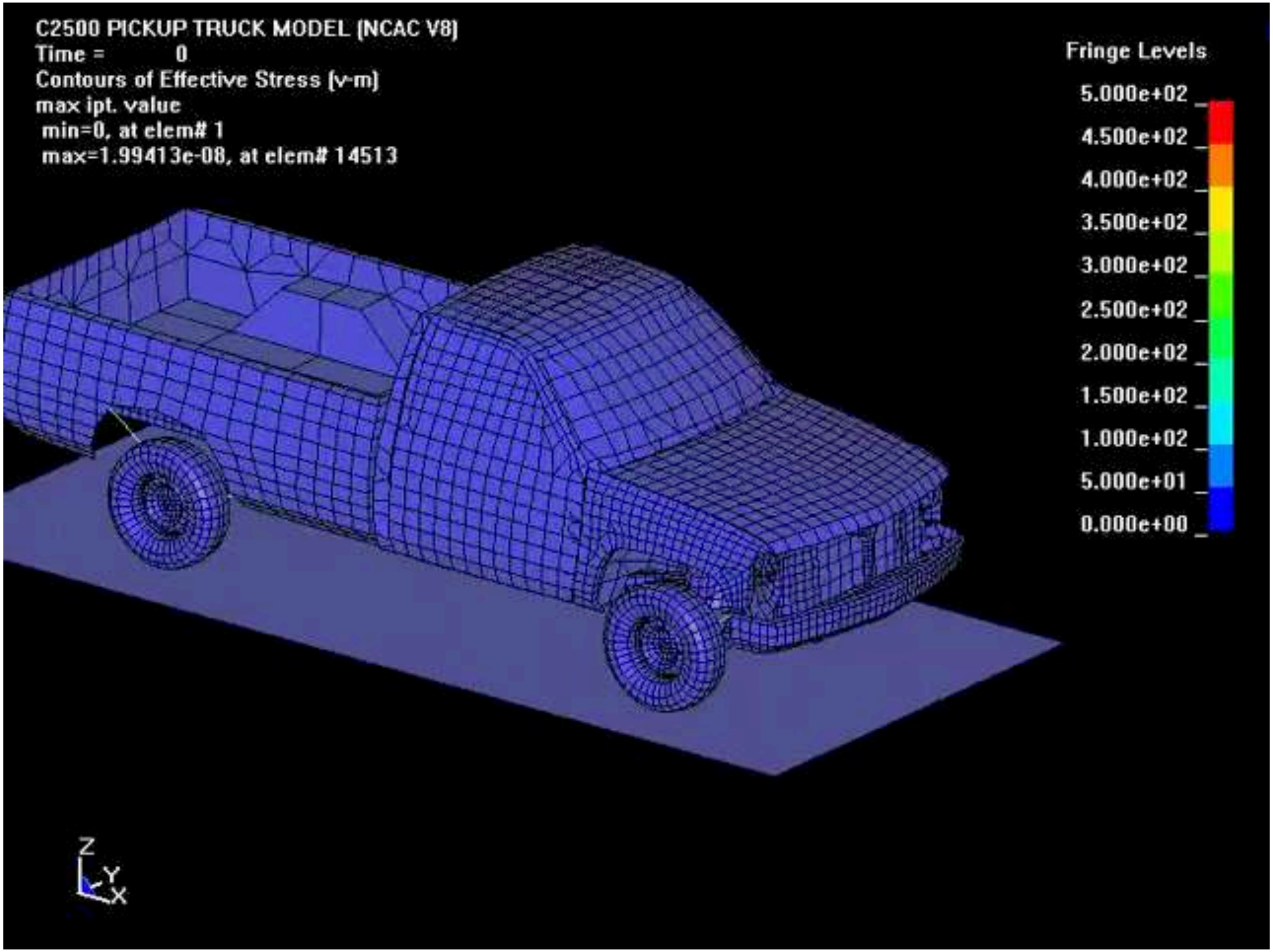
Step 6: Solve

- Now we have a large Linear System

$$\mathbf{K}\mathbf{y} = \mathbf{f}$$

- Which we can solve anyway we like (though some ways are better than others)

So now you to can do this:



Lots of Recent Fabrication Work Uses These Methods

- We’ll talk more about “microstructures” later in the

Elastic Textures for Additive Fabrication

Julian Panetta*, Qingnan Zhou*, Luigi Malomo,
Nico Pietroni, Paolo Cignoni, Denis Zorin

(* Joint First Authors)

Still Lots of Work to Do

- Handling uncertainty (Shameless Self-Promotion)

Next Time

- We'll talk about where the Psi in this equation comes from

$$\sigma = \psi\left(\mathbf{E}\right)$$