

## Limits

- Products
- Equalizers
- Pullbacks
- General definitions

## Colimits

- Coproducts
- Cequalizers
- Pushouts
- General def

- $\times$  of sets
- $\times$  in a poset coincide w/  $\wedge$ .
- $\text{Vect}$   $\text{Ker}(f)$  is an equalizer.
- 

Morphs  $\lambda$  Epis

- Morphs in Set
- Epis in Set.
- $\mathbb{Z} \xrightarrow{\quad} \oplus$  (in Rings).

Thm

all limits  $\Leftrightarrow$   +  equalizers.

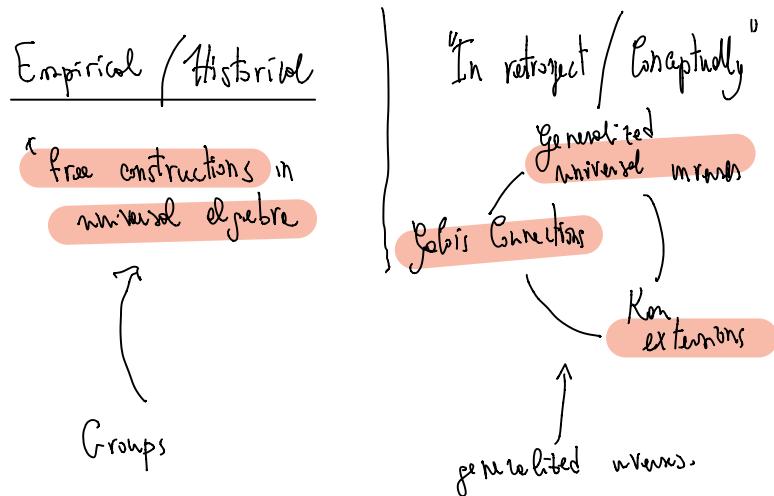
$\omega$  limits  $\Leftrightarrow$   +  coequalizers,

terminal object  $\Leftrightarrow$  

(with) pullbacks.

## Adjunctions

two approaches

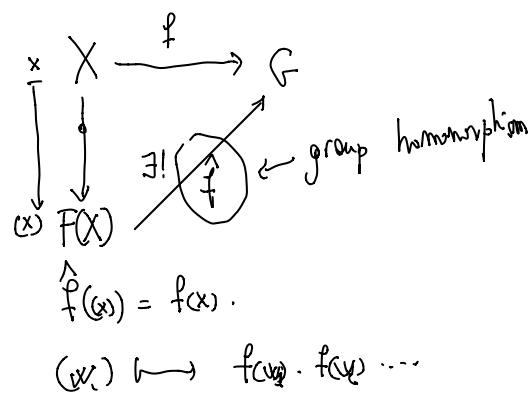


Groups      "the free group on a set  $X$ "

$\{ w_1 w_2 w_3 w_4 \dots \mid w_i \in X \}$

$w_i^{-1} \cdot w = \text{empty word.}$

$F(X) \quad F_X \quad F_n$



So there is a functor

$$\begin{array}{ccc} \text{Set} & \xrightarrow{F} & \text{Grp} \\ X & \longmapsto & F(X) \\ f \downarrow & \sim\!\!\! \sim & \uparrow f \downarrow \\ y & & F(y) \end{array}$$

$$\begin{array}{ccc} \text{Grp} & \xrightarrow{U} & \text{Set} \\ G & \longmapsto & \{G\}. \end{array}$$

the universal property of  $F$

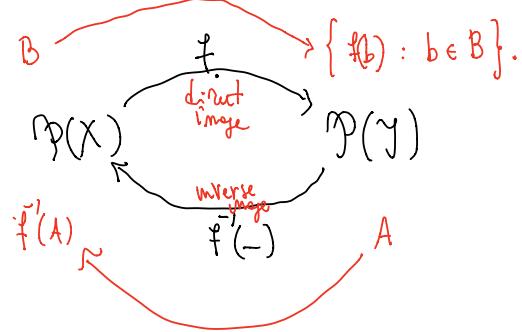
$$(\text{Grp}(FX, G) \underset{\text{Set}(X, U(G))}{\approx} \text{Set}(X, U(G)))$$

↑  
 $f$   
 ↗  $f: X \rightarrow G$   
 ↙ *mt function*

$$\begin{array}{ccc} & & \text{Set} \\ & \swarrow & \searrow \\ A & \xleftarrow{F} & \text{Grp} \\ & \uparrow u & \\ & \text{Grp}(FX, G) & \approx \text{Set}(X, U(G)) \\ & \swarrow & \searrow \\ & A(FX, y) & \approx B(X, uy) \end{array}$$


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$f: X \rightarrow Y$  is a set function.



$P(X)$  is a poset.  
 $\mathcal{F}(X)$  is a category -  $A \rightarrow B \Leftrightarrow A \subseteq B$ .

$f$  and  $f^{-1}$  are functors  
 $\hookrightarrow A \subseteq B \Rightarrow f(A) \subseteq f(B)$ .

Rem

$$\begin{array}{c} \textcircled{f^{-1}A} \downarrow \\ \supseteq B \Leftrightarrow A \supseteq fB \\ \rightsquigarrow \mathcal{P}(X)(A, \textcircled{f^{-1}B}) \cong \mathcal{P}(Y)(f(A), B). \end{array}$$

Grp

$f^{-1}$

Def Let  $L: \mathcal{A} \rightleftarrows \mathcal{B}: R$  be functors between categories. We say that  $L$  is left adjoint to  $R$  ( $R$  is right adjoint to  $F$ ) ( $L \dashv R$ ) if there exist

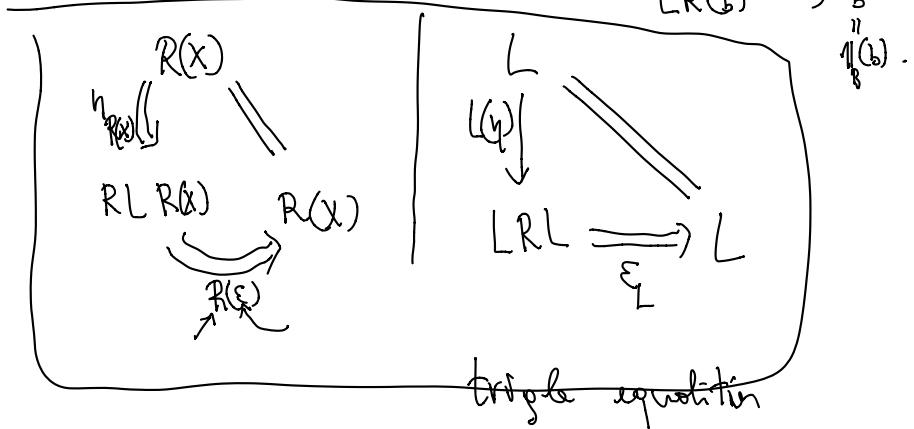
$$\eta : \mathbb{1}_{\mathcal{A}} \Rightarrow RL$$

$$\varepsilon : LR \Rightarrow \mathbb{1}_{\mathcal{B}}$$

such that

$\forall b \in \mathcal{B}$

$$LR(b) \xrightarrow{\varepsilon_b} b \in \mathbb{1}_{\mathcal{B}}(b).$$



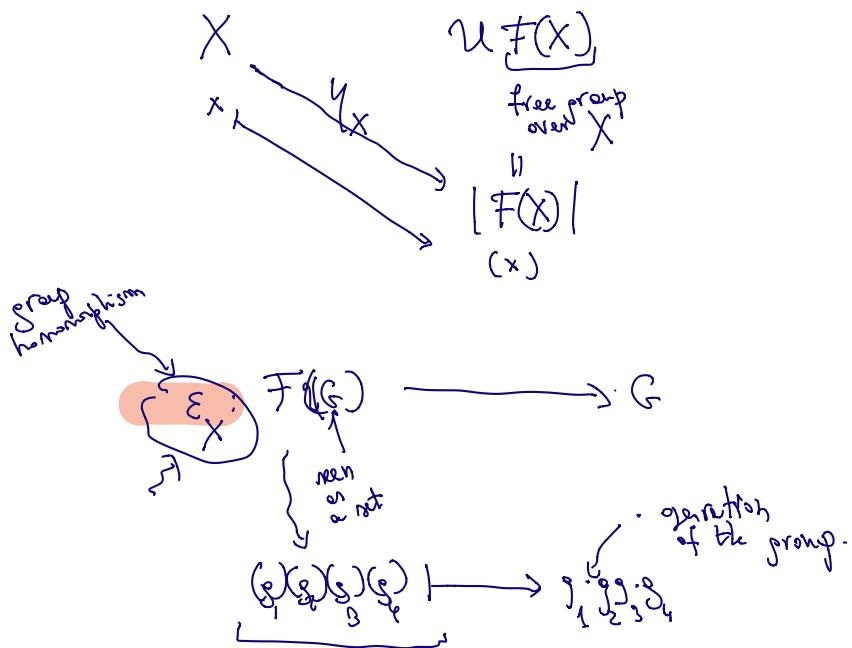
① In the case of groups

$$F \rightarrow UL$$

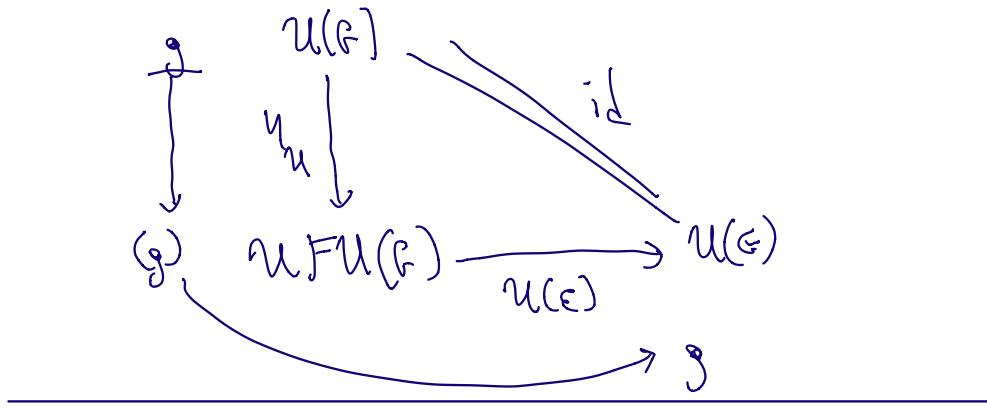
→ ② We try to recover the hom-set definition from the definition above.

1!  $F: \text{Set} \rightleftarrows \text{Grp} : U$

$\sim \gamma_X: X \rightarrow UL(X)$  "maps enter in the right adjoint"  
 $\varepsilon_X: UL(X) \rightarrow X$



We check one triangle equality



then  $L : A \rightleftarrows B : R$ , then

$$\varphi : \underbrace{A(X, Ry)}_{\cong} \xrightarrow{\quad} \underbrace{B(LX, y)}_{\cong} : \varphi^{-1}$$

Prof

$$\varphi : \begin{matrix} X & & L(X) \\ \downarrow f & \nearrow L(\epsilon) & \downarrow \\ Ry & \xrightarrow{\quad} & L(X) \xrightarrow{g \circ -} y \\ & \downarrow & \downarrow \\ & & LR(y) \\ & & \downarrow \varepsilon_y \\ & & y \end{matrix}$$

$$\varphi: \mathcal{A}(X, RY) \longrightarrow \mathcal{B}(LX, Y)$$

$$f \longmapsto \xi_y \circ L(f)$$

$$\varphi^{-1}: \mathcal{B}(L(X), Y) \longrightarrow \mathcal{A}(X, R(OY))$$

$$g \longmapsto R(g) \circ \eta_X.$$

$$\begin{array}{ccccc}
 & X & & X & \\
 & \downarrow \eta_X & & \downarrow \eta_X & \\
 LX & \xrightarrow{RL} & RL(X) & \xrightarrow{R(\xi) \circ \eta_X} & RY \\
 g \downarrow & & R(g) \downarrow & & \\
 Y & & RY & & 
 \end{array}$$

$$\underbrace{\varphi^{-1}\varphi(f)}_f = f.$$

$$\begin{aligned}
 \varphi^{-1}(\xi_y \circ L(f)) &= R(\xi_y \circ L(f)) \circ \eta_X \\
 &= R(\xi_y) \circ L(f) \circ \eta_X = f - \\
 &\quad \underbrace{\text{triangular equat.}}_{\text{triangular equation}}
 \end{aligned}$$

More is true the bijection is  
natural in X and Y

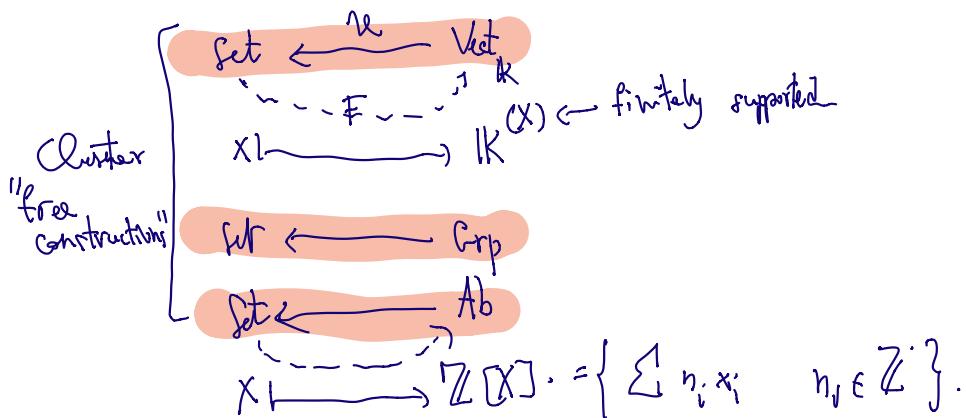
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Adjunctions  $(L, R, \eta, \varepsilon)$ .  
 $\{$   
 $[-, -] \cong [-, R-]$

1 example Groups

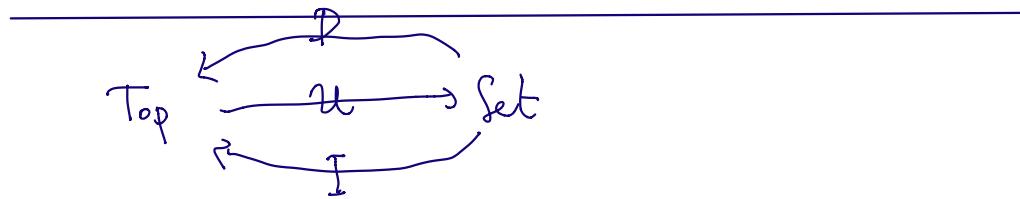
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Examples.



$$\text{Set} \xleftarrow{R\text{-Mod}} \\ X \xrightarrow{\quad} R[X].$$

$$\text{Set} \xleftarrow{\quad} \text{Gra}$$



$$\boxed{D \dashv u \dashv I.}$$

$$\begin{array}{ccc} \text{Top}(D(x), y) & \cong & \text{Set}(X, \underline{u(y)}) \\ \uparrow f & & \uparrow f \\ \text{Set}(u(x), y) & \cong & \text{Top}(X, I(y)) \\ & & \downarrow f \quad \downarrow f. \end{array}$$

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Cartesian closed categories.      8, 5, 7 numbers

$$2^{(5 \times 7)} = (2^5)^7.$$

||

$$\text{Set}(5 \times 7, 2) \cong \text{Set}(7, \text{Set}(5, 2))$$

$\leadsto \text{Set}(X \times Y, Z) \cong \text{Set}(Y, \text{Set}(X, Z))$

$$\begin{array}{ccc}
 & X_x(-) & \\
 \text{Set} & \xrightarrow{\quad} & \text{Set} \\
 y & \longmapsto & X_x y \\
 & (-)^X & \\
 \text{Set} & \xleftarrow{\quad} & \text{Set} \\
 Z^X & \longleftrightarrow & Z
 \end{array}$$

In the category of sets  $X_x(-) \dashv (-)^X$ .

$$\begin{array}{c}
 \text{Def} \quad \uparrow \quad \text{cartesian closed} \\
 \forall X \quad \text{has right adj.}
 \end{array}$$

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Who is the counit in this core?

$$\begin{array}{ccc}
 A \times B & \xrightarrow{\epsilon_B} & B \\
 \overline{(a, f)} & \longmapsto & \overline{f(a)}
 \end{array}$$

evaluation,

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Notice that

$\text{Vect}$  is not cartesian closed!! ( $A \times B = \underline{A \oplus B}$ )

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$$(\text{Vect} \otimes) \circledast: \text{Vect} \longrightarrow \text{Vect}$$

$\mathbb{W} \longmapsto V \otimes W.$

$V \otimes$  has a right adjoint!

$$\begin{array}{ccc} \text{Vect} & \xleftarrow{\quad} & \text{Vect}: [V, -] \\ W^V & \longleftarrow & W \end{array}$$

$$\text{Vect}(A \otimes B, C) \cong \text{Vect}(A, C^B).$$

Morita closed  
category

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Ex not every forgetful functor has a left adjoint

$$\begin{array}{ccc} \text{Set} & \xleftarrow{\quad u \quad} & \text{Fld} \\ \dashv & & \vdash \end{array}$$

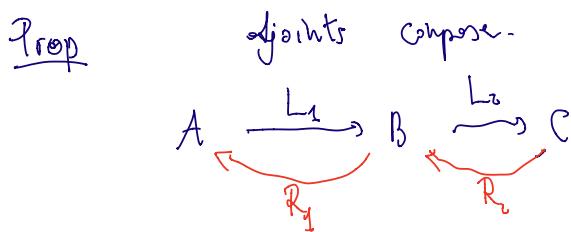
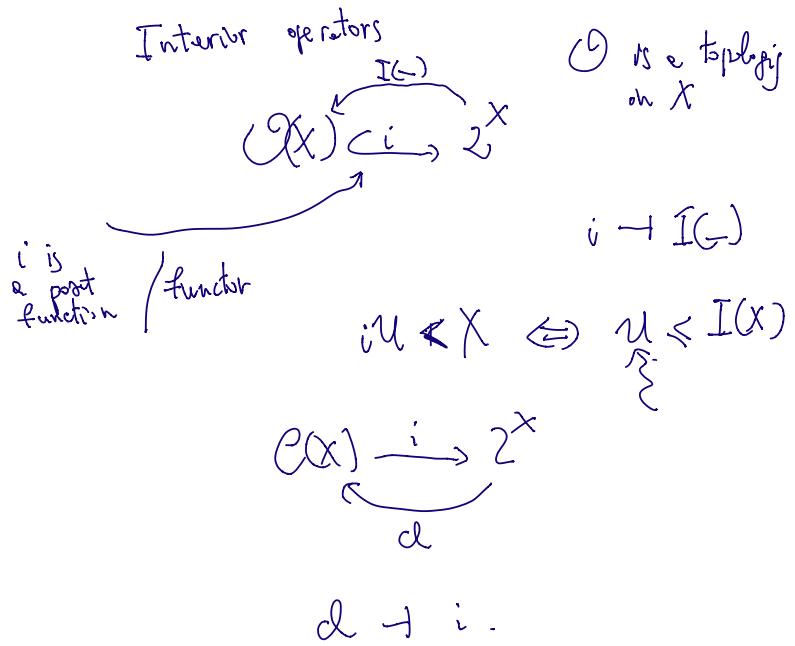
$$\text{Set} \xleftarrow{\quad u \quad} \text{Set}_*$$

$$\begin{array}{ccc} X & \xleftarrow{\quad} & (X, \otimes) \xrightarrow{\quad f \quad} (Y, y_*) \\ & & f(x) = y_* \end{array}$$

is left adjoint to  $u$ .

(L:  $X \longmapsto (X \amalg \{1\}, \uparrow)$ )

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$L_2 \circ L_1$  has a right adjoint and it is

$$R_1 \circ R_2$$

Proof

$$A(-, R_1 R_2 -) \cong B(L_1 -, R_2 -) \cong C(L_2 L_1 -, -)$$

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# CATEGORY THEORY

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## EXERCISES

**Leinster** (□). 2.1.12

**Leinster** (□). 2.1.15

**Leinster** (■). 2.1.16

**Leinster** (□). 2.2.10

**Leinster** (□). 2.2.11

**Leinster** (■). 2.2.12

**Exercise 1** (□). Show that the a right adjoint preserves monomorphisms.

**Leinster** (□). 2.3.12

**Exercise 2** (□). What is a cartesian closed bounded lattice?

**Exercise 3** (□). Show that the inclusion of the category of abelian groups in the category of groups  $\iota : \text{Ab} \hookrightarrow \text{Grp}$  has a left adjoint.

**Exercise 4** (□). Prove that the inclusion  $\text{Haus} \hookrightarrow \text{Top}$  of the full subcategory of Hausdorff spaces into the category of all spaces has a left adjoint.

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.

▲ It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.

The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.