Limits	Colimits
· Products	· Coproducts
· Equalizers	· Coequalizers
· Pullbacks	· Pushouts
· General definitions	· General def

· $\times$ of sets
· $\times$ in a poset coincide w/ $\wedge$.
· Vect Ker(f) is an equalizer.
·

Monos & Epis

· Monos in Set
· Epis in Set.
· $\mathbb{Z} \hookrightarrow \mathbb{Q}$ (in Rings).

Thm all limits $\Leftrightarrow$ $\prod$ + equalizer.
colimits $\Leftrightarrow$ $\coprod$ + coequalizers.

(with)
terminal object + pullbacks.

Adjunctions

two approaches

Empirical / Historical	"In retrospect / Conceptually"
"free constructions in universal algebra	Generalized universal inverses
↑ Groups	Galois Connections
	Kan extensions
	↑ generalized inverses.

Groups "the free group on a set X"

$F(X)$ $\quad F_X \quad F_n$

$\cong \left\{ w_1 w_2 w_3 w_4 \cdots \quad w_i \in X \right\}.$

w_i^{-1}

$w_i^{-1} \cdot w = \text{empty word}.$

$$
\begin{tikzcd}
x \arrow[d, mapsto] & X \arrow[r, "f"] \arrow[d] & G \\
(x) & F(X) \arrow[ur, "\exists!\, \hat{f}"'] &
\end{tikzcd}
$$

← group homomorphism

$\hat{f}((x)) = f(x).$

$(x_i) \longmapsto f(x_1) \cdot f(x_2) \cdots$

So there is a functor

$$\begin{tikzcd} \mathrm{Set} \arrow[r, "F"] & \mathrm{Grp} \\ X \arrow[r, mapsto] \arrow[d, "f"'] & F(X) \arrow[d, "\hat{f}"'] \\ Y & F(Y) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, "\mathcal{U}"] & \mathrm{Set} \\ G \arrow[r, mapsto] & {|G|}. \end{tikzcd}$$

the universal property of F

$$\mathrm{Grp}(F(X), G) \cong \underline{\mathrm{Set}(X, \mathcal{U}(G))}$$

$$\hat{f} \longleftarrow f : X \to G$$

set function

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend right, "\mathcal{U}"'] & \cancel{\mathcal{B}} \arrow[l, bend right, "F"'] \end{tikzcd} \quad \mathrm{Set}.$$

$$\mathcal{A}(FX, Y) \cong \mathcal{B}(X, \mathcal{U}Y).$$

$f : X \longrightarrow Y$ is a set function

$$
\begin{tikzcd}
B \arrow[rr, mapsto, bend left] & & \{ f(b) : b \in B \}. \\
\mathcal{P}(X) \arrow[rr, bend left, "f_\bullet"{above}, "\substack{\text{direct}\\\text{image}}"{below}] & & \mathcal{P}(Y) \arrow[ll, bend left, "f^{-1}(-)"{below}, "\substack{\text{inverse}\\\text{image}}"{above}] \\
f^{-1}(A) & & A \arrow[ll, mapsto, bend left]
\end{tikzcd}
$$

$\mathcal{P}(X)$ is a poset. $\subseteq$

$\mathcal{P}(X)$ is a category. $A \to B \iff A \subseteq B$.

$f_\bullet$ and f^{-1} are functors

$A \subseteq B \Rightarrow f_\bullet(A) \subseteq f_\bullet(B)$.

Rem

$$f^{-1}A \supseteq B \iff A \supseteq f_\bullet B$$

$$\rightsquigarrow \mathcal{P}(X)(A, f^{-1}B) \cong \mathcal{P}(Y)(f_\bullet A, B).$$

Grp $\qquad\qquad$ f^{-1}.

Def Let $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ be functors between categories. We say that L is left adjoint to R (R is right adjoint to F) ($L \dashv R$) if there exist

$$\eta : \mathbb{1}_{\mathcal{A}} \Longrightarrow RL$$

$$\varepsilon : LR \Longrightarrow \mathbb{1}_{B}$$

$$\begin{tikzcd} B \arrow[r, "R"] \arrow[rr, bend right, "\mathbb{1}_B"'] & A \arrow[r, "L"] \arrow[d, phantom, "\Downarrow"] & B \\ & {} & \end{tikzcd}$$

$\forall\, b \in B$

$$LR(b) \xrightarrow{\varepsilon_b} b = \mathbb{1}_B(b).$$

such that

$$\begin{tikzcd} R(X) \arrow[d, "\eta_{R(X)}"'] \arrow[dr, equal] & \\ RLR(X) \arrow[r, "R(\varepsilon)"'] & R(X) \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, "L(\eta)"'] \arrow[dr, equal] & \\ LRL \arrow[r, "\varepsilon_L"'] & L \end{tikzcd}$$

triangle equalities

1. In the case of groups

$F \dashv U$.

2. We try to recover the hom-set ~~definition~~ from the definition above.

1! $F : \mathrm{Set} \rightleftarrows \mathrm{Grp} : U$

$\rightsquigarrow \eta_X : X \longrightarrow UL(X)$

$\varepsilon_X : LU(X) \longrightarrow X$

" maps enter in the right adjoint ".

$$\begin{tikzcd} X \arrow[dr, "\eta_X"] & U\underbrace{F(X)}_{\text{free group over } X} \\ x \arrow[r, mapsto] & \|\\ & |F(X)| \\ & (x) \end{tikzcd}$$

group homomorphism

$\varepsilon_X : F(G) \longrightarrow G$

seen as a set

$\underbrace{(g_1)(g_2)(g_3)(g_4)} \longmapsto g_1 \cdot g_2 \cdot g_3 \cdot g_4$

· generation of the group.

We check one triangle equality

$$
\begin{tikzcd}
g \arrow[d, mapsto] & U(G) \arrow[d, "\eta_U"'] \arrow[dr, equal, "id"] & \\
(g) \arrow[drr, bend right, mapsto] & UFU(G) \arrow[r, "U(\varepsilon)"'] & U(G) \\
 & & g
\end{tikzcd}
$$

<u>Thm</u> $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$, then

$$\underbrace{\varphi : \mathcal{A}(X, RY) \rightleftarrows \mathcal{B}(LX, Y)}_{} : \varphi^{-1}$$

<u>Proof</u>

$$
\begin{tikzcd}
\varphi : & X \arrow[d, "f"'] & & & L(X) \arrow[d] \\
 & RY \arrow[r, squiggly, "L(-)"] & L(X) \arrow[d] \arrow[rr, squiggly, "\varepsilon \circ -"] & & Y \\
 & & LR(Y) \arrow[d, "\varepsilon_Y"] & & \\
 & & Y & &
\end{tikzcd}
$$

$$\varphi : \mathcal{A}(X, RY) \longrightarrow \mathcal{B}(LX, Y)$$
$$f \longmapsto \varepsilon_Y \circ L(f)$$

$$\varphi^{-1} : \mathcal{B}(L(X), Y) \longrightarrow \mathcal{A}(X, R(Y))$$
$$g \longmapsto R(g) \circ \eta_X .$$

$$\begin{tikzcd}
 & X \arrow[d, "\eta_X"] & \\
LX \arrow[d, "g"'] & RL(X) \arrow[d, "R(g)"'] & X \arrow[d] \\
Y \arrow[ur, "R(-)"] & RY \arrow[ur, "R(-)\circ\eta_X"] & RY
\end{tikzcd}$$

$$\varphi^{-1}\underbrace{\varphi(f)}_{\shortparallel} = f.$$

$$\varphi^{-1}(\varepsilon_Y \circ L(f)) = R(\varepsilon_Y \circ L(f)) \circ \eta_X$$
$$= \underbrace{R(\varepsilon_Y) \circ L(f) \circ \eta_X}_{\text{triangle equality}} = 1 .$$

<u>More is true</u> the bijection is natural in $\underline{X}$ and $\underline{Y}$

Adjunctions $(L, R, \eta, \varepsilon)$.

$$\wr$$

$$[L-, -] \cong [-, R-]$$

1 example <u>Groups</u>

Examples.

Cluster "free constructions"

$$\mathrm{Set} \xleftarrow{\ u\ } \mathrm{Vect}_{\mathbb{K}}$$

$$F$$

$$X \longmapsto \mathbb{K}^{(X)} \leftarrow \text{finitely supported}$$

$$\mathrm{Set} \longleftarrow \mathrm{Grp}$$

$$\mathrm{Set} \longleftarrow \mathrm{Ab}$$

$$X \longmapsto \mathbb{Z}[X] := \left\{ \sum_i n_i x_i \quad n_i \in \mathbb{Z} \right\}.$$

$$\mathrm{Set} \longleftarrow R\text{-}\mathrm{Mod}$$

$$X \longmapsto R[X].$$

$$\mathrm{Set} \longleftarrow \mathrm{Gra}$$

$$
\begin{tikzcd}
\mathrm{Top} \arrow[r, "\mathcal{U}"] & \mathrm{Set} \arrow[l, bend right, "D"'] \arrow[l, bend left, "I"]
\end{tikzcd}
$$

$$\boxed{D \dashv \mathcal{U} \dashv I.}$$

$$\mathrm{Top}(D(X), Y) \cong \mathrm{Set}(X, \mathcal{U}(Y))$$

$$f \longleftarrow\!\shortmid f$$

$$\mathrm{Set}(\mathcal{U}(X), Y) \cong \mathrm{Top}(X, I(Y))$$

$$f \longmapsto f.$$

Cartesian closed categories.

2, 5, 7 numbers

$$2^{(5\times 7)} = \left(2^5\right)^7.$$

$$\|$$

$$\mathrm{Set}(5\times 7, 2) \cong \mathrm{Set}(7, \mathrm{Set}(5, 2))$$

$$\rightsquigarrow \quad \mathrm{Set}(X \times Y, Z) \cong \mathrm{Set}(Y, \mathrm{Set}(X, Z))$$

$$
\begin{array}{ccc}
\mathrm{Set} & \xrightarrow{X \times (-)} & \mathrm{Set} \\
y & \longmapsto & X \times y
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Set} & \xleftarrow{(-)^X} & \mathrm{Set} \\
Z^X & \longleftarrow\!\shortmid & Z
\end{array}
$$

In the category of sets $X \times (-) \dashv (-)^X$.

Def: cartesian closed: $\forall X$ $X \times (-)$ has right adj.

Who is the counit in this case?

evaluation,

$$
\begin{array}{ccc}
A \times B^A & \xrightarrow{\varepsilon_B} & B \\
(a, f) & \longmapsto & f(a)
\end{array}
$$

Notice that

Vect is not cartesian closed !! $(A \times B = \underline{A \oplus B})$

$(\mathrm{Vect}\ \ \otimes)$ $\quad V\otimes \underline{\ \ } : \mathrm{Vect} \longrightarrow \mathrm{Vect}$

$$W \longmapsto V\otimes W.$$

$\forall V$ has a right adjoint!

$$\mathrm{Vect} \longleftarrow \mathrm{Vect} : [V, -]$$

$$\cancel{W^V} \longleftarrow\!\shortmid W$$

$$\mathrm{Vect}(A\otimes B, C) \cong \mathrm{Vect}(A, C^B).$$

Monoidal closed categories

Ex not every forgetful functor has a left adjoint

$$\mathrm{Set} \overset{\nexists}{\underset{U}{\leftrightarrows}} \mathrm{Fld}$$

$$\mathrm{Set} \overset{L}{\underset{U}{\rightleftarrows}} \mathrm{Set}_\bullet$$

$$X \longleftarrow\!\shortmid (X, x_0) \xrightarrow{f} (Y, y_0)$$

$$f(x_0) = y_0.$$

is left adjoint $\longrightarrow$ $L:$ $X \longmapsto (X \amalg \{1\}, 1)$. to U.

Interior operators

$\mathcal{O}$ is a topology on X

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, hook, "i"] & 2^X \arrow[l, bend right, "I(-)"'] \end{tikzcd}$$

i is a poset function / functor

$$i \dashv I(-)$$

$$i\mathcal{U} \leq X \iff \mathcal{U} \leq I(X)$$

$$\begin{tikzcd} \mathcal{C}(X) \arrow[r, "i"] & 2^X \arrow[l, bend left, "cl"] \end{tikzcd}$$

$$cl \dashv i.$$

Prop adjoints compose.

$$\begin{tikzcd} A \arrow[r, "L_1"] & B \arrow[l, bend left, "R_1"] \arrow[r, "L_2"] & C \arrow[l, bend left, "R_2"] \end{tikzcd}$$

$L_2 \circ L_1$ has a right adjoint and it is $R_1 \circ R_2$

Proof

$$\underline{A(-, R_1R_2-)} \cong B(L_1-, R_2-) \cong \underline{C(L_2L_1-, -)}$$

CATEGORY THEORY

IVAN DI LIBERTI

EXERCISES

Leinster (). 2.1.12

Leinster (). 2.1.15

Leinster (). 2.1.16

Leinster (). 2.2.10

Leinster (). 2.2.11

Leinster (). 2.2.12

Exercise 1 (). Show that the a right adjoint preserves monomorphisms.

Leinster (). 2.3.12

Exercise 2 (). What is a cartesian closed bounded lattice?

Exercise 3 (). Show that the inclusion of the category of abelian groups in the category of groups $\iota : \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ has a left adjoint.

Exercise 4 (). Prove that the inclusion Haus $\hookrightarrow$ Top of the full subcategory of Hausdorff spaces into the category of all spaces has a left adjoint.

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
- It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.

Date: October 20, 2020.