Proofs of the Exercises in Chapter 1

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Exercise 1. *Proof.* Let $a, b, c, d, e, f \in \mathbb{Z}$ and $x, y, z \in \mathbb{Z}/m$ such that $(b, m) = (d, m) = (f, m) = 1$ and $xb \equiv a, yd \equiv c, zf \equiv e \pmod{m}$. Then we can show that $\frac{a}{b} + \frac{c}{d} = \frac{e}{f} \implies x + y \equiv z$:

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{e}{f} \\ adf + cbf &= ebd \\ xbdf + ybdf &\equiv zbdf \\ x + y &\equiv z \end{aligned}$$

Furthermore, if $\frac{ac}{bd} = \frac{e}{f}$ for possibly different e, f, then $xy \equiv z$ for possible different z:

$$\begin{aligned} \frac{ac}{bd} &= \frac{e}{f} \\ acf &= bed \\ xbydf &\equiv bzfd \\ xy &\equiv z \end{aligned}$$

$\square$

Exercise 2. *Proof.* Consider the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{p-1}$. We can show that no two elements in this series are equivalent mod odd p. FTOC suppose there are two integers a, b such that $\frac{1}{a} \equiv \frac{1}{b}$, let this be equal to x. Then $ax \equiv 1 \equiv bx$. Since p is prime, all integers in $[1, p-1]$ are relatively prime to p. Thus $ax \equiv bx \implies a \equiv b$, which is impossible since $a \neq b$ and $a + pk > p - 1$ for $k \in \mathbb{N}$ as $a \geq 1$. Thus every fraction is uniquely equivalent to an element in $\mathbb{Z}/p$.

Thus the sum of all the elements in this series is just $1+2+3+4+\cdots+p-1$ as every element of $\mathbb{Z}/p$ must be in this series (there are $p-1$ elements in $\mathbb{Z}/p$ and $p-1$ unique elements in the sequence). Thus the sum is just $(p-1)(p)/2 \equiv 0$. $\square$

Exercise 3. *Proof.* By the hypothesis, $x^4+y^4 = x^7y+1 \equiv 0 \pmod{x^4+y^4}$. Thus $y \equiv \frac{-1}{x^7} \implies x^4 + (\frac{-1}{x^7})^4 = \frac{x^{32}+1}{x^{28}} \equiv 0 \implies x^{32}+1 \equiv 0$. $\square$