Atiyah-MacDonald Solutions

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1 Modules

In Chapter Exercises:

1. 2.2
   (a) $\mathrm{Ann}(M+N) = \mathrm{Ann}(M) \cap \mathrm{Ann}(N)$.

   Solution: $\subseteq$) If $x(M+N) = 0$, then $xM + xN = 0$. This can only happen if xM and xN are 0.
   $\supseteq$) If $xM, xN = 0$, then $xM + xN = x(M+N) = 0$.

   (b) $(N : P) = \mathrm{Ann}((N+P)/N)$.

   Solution: $\subseteq$) If $xP \subseteq N$, then $x((N+P)/N) = ([x]N + [x]P)/N = 0$ because $x \equiv 0 \pmod N$.
   $\supseteq$) If $x((N+P)/N) = 0$, then $([x]N + [x]P)/N = 0 \implies [x]P \subseteq N \implies xP \subseteq N$.

2. 2.15: Let A, B be rings, let M be an A-module, P a B-module and N an (A, B)-bimodule (that is, N is simultaneously an A-module and a B-module and the two structures are compatible in the sense that $a(xb) = (ax)b$ for all $a \in A, b \in B, x \in N$). Then $M \otimes_A N$ is naturally a B-module, $N \otimes_B P$ an A-module, and we have
$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

   Solution:
   First we construct the B bilinear map
$$(M \otimes_A N) \times P \to M \otimes_A (N \otimes_B P)$$
   that sends $(m \otimes n, p) \to m \otimes (n \otimes p)$. The B bilinearity comes from $(b(m \otimes n), p) = (m \otimes nb, p) \mapsto m \otimes (nb \otimes p) = b(m \otimes (n \otimes p)) = m \otimes (b \otimes bp)$, which is also the image of $(m \otimes n, bp)$. Hence this induces a unique B linear map
$$(M \otimes_A N) \otimes_B P \to M \otimes_A (N \otimes_B P).$$
   By a symmetric argument, we have a unique A linear map from the other direction, giving us an isomorphism for by tracing where $(m \otimes n) \otimes p$ goes, it goes to $m \otimes (n \otimes p)$ and then to $(m \otimes n) \otimes p$.

3. If $f : A \to B$ is a ring homomorphism and M is a flat A-module, then $M_B = B \otimes_A M$ is a flat B-module.

> **Solution:** The function f makes B an A-algebra. Consider $B \otimes_B N \cong N$ by Proposition 2.14. Obviously B has an (A, B) bimodule structure since B is an algebra.
>
> Then given an exact sequence E, $E \otimes_A N = E \otimes_A (B \otimes_B N) = (E \otimes_A B) \otimes_B N$ by Exercise 2.15 in the Chapter. As B is flat as an A-module and N is flat as a B-module, we are done.

1. Show that $(\boldsymbol{Z}/m\boldsymbol{Z}) \otimes_{\boldsymbol{Z}} (\boldsymbol{Z}/n\boldsymbol{Z}) = 0$ if m, n are coprime.

> **Solution:** Take a bilinear map $f : (\boldsymbol{Z}/m\boldsymbol{Z}) \times (\boldsymbol{Z}/n\boldsymbol{Z})$. Then by bilinearity, we have $f(mx, y) = f(0, y) + mf(x, y) = f(0, y)$ and $f(x, ny) = f(x, 0) + nf(x, y) = f(x, 0)$, which imply that $mf(x, y) = 0$ and $nf(x, y) = 0$. By Bezout's Lemma, we have that there exists a, b s.t. $am + bn = 1$ as m, n are coprime. Thus $f(x, y) = 0$.

2. Let A be a ring, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. [Tensor the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ with M]

> **Solution:** By tensoring with the exact sequence $\mathfrak{a} \to A \to A/\mathfrak{a} \to 0$, we get
>
> $$0 \to \mathfrak{a} \otimes_A M \to A \otimes_A M \to A/\mathfrak{a} \otimes M \to 0. \tag{Prop 2.8}$$
>
> Then by Proposition 2.14,$A \otimes M \to M$ is an isomorphism by $a\otimes$, we have $\mathfrak{a} \otimes M \cong \mathfrak{a}M$ and $A \otimes M \cong M$. Hence by commutativity of
>
> $$\begin{tikzcd} \mathfrak{a} \otimes M \arrow[r] \arrow[d, "\cong"] & A \otimes M \arrow[d, "\cong"] \\ \mathfrak{a}M \arrow[r] & M \end{tikzcd},$$
>
> (for the commutativity, the definitions of the maps down make it obvious) we have that $\Im(\mathfrak{a}M \to M) = \ker(M \to M/\mathfrak{a}M) = \ker(A/\mathfrak{a} \otimes M)$.
>
> So we have this diagram
>
> $$\begin{tikzcd} & & M/\mathfrak{a}M \arrow[dr] & \\ \mathfrak{a}M \arrow[r] & M \arrow[ur] \arrow[dr] & & 0 \\ & & A/\mathfrak{a} \otimes M \arrow[ur] & \end{tikzcd}$$
>
> By some isomorphism theorem and surjectivity of the last maps, we have that $M/\mathfrak{a}M \cong A/\mathfrak{a} \otimes M$.

3. Let A be a local ring, M and N finitely generated A-modules. Prove tha tif $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. [Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2. By Nakayama's lemma, $M_k = 0 \implies M = 0$. But $M \otimes_A N = 0 \implies (Mo\times_A N)_k = 0 \implies M_k \otimes N_k = 0 \implies M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces over a field.]

Solution: We do as the hint suggests: Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field and define $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.

By Nakayama's lemma,$M_k = 0 \implies M = 0$ since $k \subseteq$ the Jacobson radical, $M_k = 0 \implies M = \mathfrak{m}M$, and M is finitely generated.

Then we have that $M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0$ because

As the tensor product of vector spaces is just the direct sum, this implies that $0 = k \otimes k \otimes (M \otimes_A N) = M_k \otimes_A N_k = 0$ by commuting. As A bilinear maps on $M_k \times N_k$ are k linear maps on $M_k \times N_k$, we have $M_k \otimes_k N_k = 0$. Finally, this implies that $M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces.

4. Let $M_i (i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\iff$ each M_i is flat.

Solution: Fix an exact sequence E.

We have that $E \otimes M = E \otimes (\bigoplus M_i) = \bigoplus (E \otimes M_i)$ because each direct sum is finite and hence belongs to a finite direct sum in which we can use Proposition 2.14. Then if $E \otimes M$ is exact, so is each coordinate, which gives us the invidiaul M_i is exact. If each coordinate is exact then so is $E \otimes M$.

5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra. [Use Exercise 4.]

Solution: Clearly $A[x] = \bigoplus_{i=0}^{\infty} Ax^i$. So by the above exercise, it suffices to show that $x^i A$ is flat. Say we have a short exact sequence of A-modules

$$0 \to B \to C \to D \to 0.$$

Then

$$0 \to B \otimes Ax^i \to C \otimes Ax^i \to D \otimes Ax^i \to 0$$

is exact because tensoring with Ax^i is the same as tensoring with A as bilinear maps $B \times Ax^i$ are bilinear on $B \times A$ and likewise for linear maps. So tensoring with Ax^i also induces unique linear maps that make the tensor universal diagram commute, so by uniqueness of the universal property, they are the same.

Finally tensoring with A is the same as the original module by Proposition 2.14. So Ax^i is flat and so is $A[x]$.

6. For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + \cdots + m_r x^r. \qquad (m_i \in M)$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module.
Show that $M[x] \cong A[x] \otimes_A M$.

Solution: First, it is an abelian group by commuting and grouping together terms with the same power of x. Then for the properties of an $A[x]$ module: Distributivity holds by simply defining it as so.

$$\begin{aligned}
(r+s)m &= (r_0 + \cdots + r_j x^j + s_0 + \cdots + s_k x^k)m \\
&= (r_0 + s_0 + \cdots + (r_{j+k} + s_{j+k})x^{j+k})m \\
&= (r_0 + s_0)m + (r_1 + s_1)mx + \cdots \\
&= rm + sm \\
r(sm) &= r(s_0 m + s_1 mx + \cdots + s_k mx^k) \\
&= r(s_0 m) + \cdots + r(s_k mx^k) \\
&= rs_0 m + \cdots + rs_k x^k m \\
&= (rs_0 + \cdots + rs_k x^k)m \\
&= (rs)m \\
1m = m.
\end{aligned}$$

Hence $M[x]$ is an $A[x]$ module.

We use the universal property. Say we have a bilinear map

$$\begin{tikzcd}
A[x] \times M \arrow[r] \arrow[dr, "f"] & M[x] \\
& B
\end{tikzcd}$$

where the top map takes $(a(x), m) \to a(x)m$. Then we have the unique linear map $\hat{f} : M[x] \to B$ that takes $m_0 + m_1 x + \cdots + m_r x^r$ to $f(1, m_0) + f(x, m_1) + \cdots + f(x^r, m_r)$. This is linear because of linearity of f in M. It is unique because we have a basis that uniquely determines the map by linearity, and the bases have to be mapped to the things that generate this map.

Hence by the universal property,$M[x] \cong A[x] \otimes M$.

7. Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ieal in $A[x]$?

Solution: It is clear that $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$. As $A/\mathfrak{p}$ is an integral domain by primality of $\mathfrak{p}$, $(A/\mathfrak{p})[x]$ is an integral domain (look at leading coefficients) and thus $p[x]$ is prime.

Similarly with $\mathfrak{m}$, $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$. Then $A/\mathfrak{m}$ is a field, and clearly $(A/\mathfrak{m})[x]$ is not a field.

8. (a) If M and N are flat A-modules, then so is $M \otimes_A N$.

Solution: Let E be an exact sequence. Then $E \otimes_A M$ is exact by M being flat, and hence $(E \otimes_A) \otimes_A N$ is exact. By Proposition 2.14, this sequence equals $E \otimes_A (M \otimes_A N)$, so $M \otimes_A N$ is flat.

(b) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution: Consider $B \otimes_B N \cong N$ by Proposition 2.14. Then N is flat as an A-module by Exercise 2.20 in the Chapter.

9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Solution: As the last map is surjective, the preimage of M'' is M, so the preimage of the generators of M'' and the kernel generate M. But the kernel is finitely generated as it is the image of M', so M is finitely generated.

10. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Solution: It suffices to show that $N = \mathfrak{a}N + u(M)$ by Corollary 2.7. Clearly $\mathfrak{a}N + u(M) \subseteq N$ by definitions.

Let ϕ_N be the quotient map $N \to N/\mathfrak{a}N$ and ϕ_M be the map $M \to M/\mathfrak{a}M$. Then because $\hat{u}$ is induced by $\phi \circ u$, $\phi_N \circ u = \hat{u}\phi_M$. As both $\hat{u}$ and ϕ_M are surjective, the LHS is too. Hence for every element n of N, by the surjectivity of ϕ_N, there is an element in $u(M)$ s.t. ϕ_N of it equals n. Thus $u(M) + \ker \phi_N = u(M) + \mathfrak{a}N = N$.

11. Let A be a ring $\neq 0$. Show that $A^m \cong A^n \implies m = n$.

Solution: Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes \phi : A/\mathfrak{m} \otimes A^m \to A/\mathfrak{m} \otimes A^n$ is an ismorphism of vector space (Exercise 2 for modules over a field). But then these vector spaces have to have the same dimension over $A/\mathfrak{m}$, which are m and n respectively. So $m = n$.

(a) If $\phi : A^m \to A^n$ is surjective, then $m \geq n$.

Solution: Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be a surjection. Then $1 \otimes \phi : A/\mathfrak{m} \otimes A^m \to A/\mathfrak{m} \otimes A^n$ is a surjection of vector space (Exercise 2) (surjectivity from right exactness of tensoring (Proposition 2.18)). Then note that their dimensions are m, n respectively. As this is a surjective vector space map,$m \geq n$.

(b) If $\phi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?

Solution: No. Consider $A = \boldsymbol{Z}[x_1, \ldots]$. Then consider the map $A \times A \to A$ that maps $(f(x_1, \ldots), g(x_1, \ldots)) \to f(x_1, x_3, \ldots) + g(x_2, x_4, \ldots)$. Obviously they are injective as they are inclusions under relabelling. Clearly $2 \not\leq 1$.

12. Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\ker \phi$ is finitely generated.

Solution: Because ϕ is surjective, by first iso, $M/\ker \phi \cong A^n$ with the isomorphism being ϕ. Hence every element of M is a sum of something in $\ker \phi$ and something in the preimage of A^n. Then we can see that the kernel and the preimage of A^n have no relations because if $a + b = 0$ for $a \in \ker \phi$ and $b \in \phi^{-1}(A^n)$, then $\phi(a + b) = 0 = \phi(b) \implies b = 0$, showing that there is no relation.

Hence $M = \ker \phi \oplus A^n$. This then implies that $\ker \phi$ is finitely generated because we can project the set of finite generators of M into finite generators of $\ker \phi$.

13. Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution: Let $p(b \otimes y) = by$. We can see that $p(g(n)) = n \forall n \in N$, so p is surjective. Because $p \circ g = \mathrm{id}$, we can also conclude that g is injective. By first iso, $N/\ker g \cong N \cong \Im g$ and $N_B/\ker p \cong \Im p = N \cong \Im g$. Hence every element of N_B is a sum of an element of $\ker p$ and $\Im g$ (note that the isomorphism $N_B/\ker p \to \Im g$ is via $g \circ p$, outputting things in the right places to let us say that every element of N_B is a sum of the two).

So if we show that $\ker p$ and $\Im g$ have no relations, we can write $N_B = \ker p \oplus \Im g$. Now suppose we have a relation $\sum b_i \otimes y_i + 1 \otimes y \in \ker p \oplus \Im g$ equaling 0. Then $0 = p(\sum b_i \otimes y_i + 1 \otimes y) = 0 + p(1 \otimes y) = y$, making this a trivial relation.

14. A partially ordered set I is said to be a directed set if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.
Let A be a ring, let I be a directed set and let $(M_i)'_{i \in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

a. μ_{ii} is the identity mapping of M_i for all $i \in I$;
b. $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

Then the modules M_i and homomorphisms μ_{ij} are said to form a direct system $M = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the `direct limit` of the direct system M. Let C be the direct sum of the M_i, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of M_i.

The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$, is called the direct limit of the direct system M, and is written $\lim_{\to} M_i$ From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.

Solution: Let $Q = \{X_i - \mu_{ij}(X_i), X_i \in M_i, i \leq j\}$ We can see that from definition, for $x_i \in M_i$, $\mu_i(x_i) = x_i + Q = x_i - (x_i - \mu_{ij}(x_i)) + Q = \mu_j(\mu_{ij}(x_i))$.

15. In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.
Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Rewritten: If $x \in M_i$ and $\mu_i(x) = 0$, then $\exists j \geq i$ s.t. $\mu_{ij}(x) = 0$ in M_j.

Solution: Because $\mu_i(x) = 0$, $\mu_i(x) = D$. As μ_i is the restriction of μ, which is the quotient map, this tells us that $x \in D$. So $x = \sum_{a \leq b} c_{ab}(x_a - \mu_{ab}(x_a))$ with the sum being finite by definition of D. We can bring the c_{ab} into the x_a, so WLOG we have

$$x = \sum_{a \leq b} x_a - \mu_{ab}(x_a).$$

Now we fix an arbitrary $\ell \geq b$ for all b in the sum. Because $x \in M_i$, we know that all the elements that aren't in M_i must cancel. As bringing them to M_ℓ with $\mu_{b\ell}, b \neq i$ doesn't change the fact that they are 0, we can bring all the terms into M_ℓ as so:

$$\mu_{i\ell}(x) = \mu_{i\ell}(\sum_{a \leq b} x_a - \mu_{ab}(x_a)) = \sum(\mu_{a\ell}(x_a) - \mu_{b\ell}\mu_{ab}(x_a)).$$

As $\mu_{b\ell}\mu_{ab} = \mu_{a\ell}$, we have

$$\mu_{i\ell}(x) = \sum(\mu_{a\ell}(x_a) - \mu_{a\ell}(x_a)) = \sum \mu_{a\ell}(x_a - x_a) = 0.$$

We have hence found such a j as desired ($j = \ell$).

16. Show that the direct limit is characterized (up to isomorphism) by the following property. Let N be an A-module and for each $i \in I$, let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Solution: Simply define α to be $(\bigoplus x_i) + Q \mapsto \bigoplus \alpha_i(x_i)$. This is well-defined because for any $m_i - \mu_{ij}(m_i) \in Q$, this gets mapped to $\alpha_i(m_i) - \alpha_j(\mu_{ij}(m_i)) = \alpha_i(m_i) - \alpha_i(m_i) = 0$.

Then this commutes properly because $\alpha(\mu_i(m_i)) = \alpha(m_i + Q) = \alpha_i(m_i)$.

Finally, this is unique because given another α' with these properties and arbitrary $\bigoplus x_i + Q \in M$, $\alpha'(\bigoplus x_i + Q) = \alpha'(\mu_I(x_I))$ given by Exercise 15. Then by definition of α', $\alpha'(\mu_I(x_I)) = \alpha_I(x_I) = \alpha(\mu_I(x_I)) = \alpha(\bigoplus x_i + Q)$. Hence $\alpha' = \alpha$ for all elements of M, and we are done.

The characterizing up to isomorphism is just a classic universal property argument.

17. Let $(M_i)_{i \in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I, there exists $k \in I$ s.t. $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j. Show that

$$\lim_{\rightarrow} M_i = \sum M_i = \cup M_i.$$

In particular, any A-module is the direct limit of its finitely generated submodules.

Solution: Obviously this satisfies the conditions for the direct limit as the maps are just embeddings. To show the equality, we can realize $\cup M_i$ as having the properties of the direct limit: Say we have a family of maps α_i into an A-module N that respect the directed system's maps.

Then we have a map $\alpha : \cup M_i \to N$ defined by taking an element m, finding aM_i it is in, and mapping it to $\alpha_i(m)$. This is well-defined because α_i respects the directed system's maps, those being inclusions. Hence it is isomorphic to the direct limit by Exercise 16.

Since $\{M_i\}$ is a poset and we have that for every increasing chain, there is a maximal element (namely the union of all the modules in the chain), there is a maximal element M in this set. This equals $\cup M_i$, and hence $M \subseteq \sum M_i \subseteq M$.

18. Let $\boldsymbol{M} = (M_i, \mu_{ij}), \boldsymbol{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M, \nu_i : N_i \to N$ the associated homomorphisms.
A homomorphism $\boldsymbol{\phi} : \boldsymbol{M} \to \boldsymbol{N}$ is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$

such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Show that $\boldsymbol{\phi}$ defines a unique homomorphism $\phi = \lim_{\rightarrow} \phi_i : M \rightarrow N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

> **Solution:** We have maps $\psi_i : M_i \rightarrow N$ by doing the composition $\nu_i \circ \phi_i$, which commute with the system because $\psi_j(\mu_{ij}) = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_i \circ \phi_i$. Hence there is a unique map $M \rightarrow N$ that commutes with the system by the characterizing property of the direct limit. Since this map commutes with the system,$\phi \circ \mu_i = \psi_i = \nu_i \circ \phi_i$.

19. A sequence of direct systems and homomorphisms

$$\boldsymbol{M} \rightarrow \boldsymbol{N} \rightarrow \boldsymbol{P}$$

is exact if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \rightarrow N \rightarrow P$ of direct limits is then exact.

> **Solution:** Let $\mu_{ij}, \nu_{ij}, \rho_{ij}$ be the maps in the systems, a_i, b_i be the maps $M_i \rightarrow N_i$ and $N_i \rightarrow P_i$, and let $\mu_{M\cdot}, \nu_{N\cdot}$ be the maps from $M \rightarrow \cdot, N \rightarrow \cdot$ that are induced by the direct limit property ($\cdot$ will be N or P).
>
> Fix an element $x \in M$. By Exercise 15,$x = \mu_i(x_i)$ for some i. Then we can see that $\mu_{MP}\mu_i = \rho_i b_i a_i$ for all i by commuting properties of the direct limit. In particular,$\mu_{MP}(x) = \mu_{MP}\mu_i(x) = \rho_i b_i a_i = 0$ since a_i, b_i are in an exact sequence.
>
> Finally, we can show that $\ker \nu_{NP} \subseteq \Im \mu_{MN}$ by supposing $\nu_{NP}(x) = 0$ for $x \in N$. Then by Exercise 15,$x = \nu_i(x_i)$ for some i. By commuting properties,$\nu_{NP}\nu_i = \rho_i b_i$, so $\nu_{NP}(x) = \nu_{NP}\nu_i(x_i) = \rho_i b_i(x_i) = 0$. By exercise 15, if $\rho_i(b_i(x_i)) = 0$, there exists $j \geq i$ s.t. $\rho_{ij}(b_i(x_i)) = 0$. By commutativity of the diagram, this equals $b_j(\nu_{ij}(x_i)) = 0$, which by exactness gives us that $\nu_{ij}(x_i) \in \Im a_j$. By applying ν_j to both sides, we can see that $\nu_i(x_i) \in \Im(M_j \rightarrow N)$. Being in the image of $M_j \rightarrow N$ is in the image of μ_{MN} since, by being the direct limit, μ_{MN} factors through this map.
>
> Hence $\ker \nu_{NP} = \Im_{MN}$.
>
> To understand this proof clearly, let I just be the naturals and draw the commutative diagrams out.

20. Keeping the same notation as in Exercise 14, let N be any A-module. Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \lim_{\rightarrow}(M_i \otimes N)$ be its direct limit. For each $i \in I$, we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \rightarrow M \otimes N$, hence by Exercise 16 a homomorphism $\psi : P \rightarrow M \otimes N$. Show that ψ is an isomorphism, so that

$$\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N.$$

> **Solution:** We show that $M \otimes N$ satisfies the universal property for direct limits. Suppose we have maps $\{f_i : M_i \otimes N \rightarrow Q\}$. Then these lead to bilinear maps $\hat{f}_i : M_i \times N \rightarrow Q$ By direct limit properties, we then have a map $M \times N \rightarrow Q$. This is bilinear because it commutes with bilinear maps. This bilinear map then induces a unique linear map $M \otimes N \rightarrow Q$. This is the universal property of direct limits, so $M \otimes N \cong \varinjlim(M_i \otimes N)$.

21. Let $(A_i)_{i \in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leq j$ in I, let $\alpha_{ij} : A_i \rightarrow A_j$ be a ring homomorphism, satisfying conditions (1) and (2) of Exercise 14. Regarding each A_i as a $\boldsymbol{Z}$-module, we can then form the direct limit $A = \varinjlim A_i$. Show that A inherits a ring structure from the

A_i so that the mappings $A_i \to A$ are ring homomorphisms. The ring A is the direct limit of the system (A_i, α_{ij}).
If $A = 0$ prove that $A_i = 0$ for some $i \in I$.

Solution: For $a, a' \in$ limit A, define $a \cdot a'$ as $\mu_k(\mu_{ik}(a_i)\mu_{jk}(a_j))$ where $a = \mu_i(a_i), a' = \mu_j(a_j)$ and $k \geq i, j$, which is well-defined because for other $k' \geq i, j$, we can find a k'' s.t. $\mu_{k'}(\mu_{ik'}(a_i)\mu_{jk'}(a_j)) = \mu_{k''}(\mu_{k'k''}(\mu_{ik'}(a_i)\mu_{jk'}(a_j))) = \mu_{k''}(\mu_{k'k''}(\mu_{ik'}(a_i))\mu_{k'k''}(\mu_{jk'}(a_j))) = \mu_{k''}(\mu_{ik''}(a_i)\mu_{jk''}(a_j)) = \mu_{k''}(\mu_{kk''}(\mu_{ik}(a_i)\mu_{jk}(a_j)$ $\mu_k(\mu_{ik}(a_i)\mu_{jk}(a_j))$. This is obviously commutative and has identity over multiplication because it has the domain of a commutative ring and $\mu_{\cdot,\cdot}$ are homomorphisms.

Next is associativity: Let $a = \mu_i(a_i), b = \mu_j(b_j), c = \mu_k(c_k)$ and $\ell \geq i, j, k$.

$$(a \cdot b) \cdot c = (\mu_\ell(\mu_{i\ell}(a_i)\mu_{j\ell}(b_j))) \cdot \mu_k(c_k)$$
$$\iff$$
$$\mu_\ell((\mu_{i\ell}(a_i)\mu_{j\ell}(b_j))\mu_{k\ell}(c_k)) = a \cdot (b \cdot c).$$

Finally, distributivity: Let $a = \mu_i(a_i), b = \mu_j(b_j), c = \mu_k(c_k)$ and $\ell \geq i, j, k$.

$a(b + c) = \mu_i(a_i)(\mu_j(b_j) + \mu_k(c_k)) = \mu_i(a_i)(\mu_\ell\mu_{j\ell}(b_j) + \mu_\ell\mu_{k\ell}(c_k)) = \mu_\ell(\mu_{i\ell}(a_i)(\mu_{j\ell}(b_j) + \mu_{k\ell}(c_k))) = \mu_\ell(\mu_{i\ell}(a_i)\mu_{j\ell}(b_j)$

If $A = 0$, then $\mu_i(1) = 0 \implies \mu_{ij}(1) = 0$ by Exercise 15. But a ring homomorphism that sends 1 to 0 implies that the ring is 0, so $A_j = 0$.

22. Let (A_i, α_{ij}) be a direct system of rings and let $\mathfrak{N}_i$ be the nilradical of A_i. Show that $\lim_\to \mathfrak{N}_i$ is the nilradical of $\lim_\to A_i$.
If each A_i is an integral domain, then $\lim_\to A_i$ is an integral domain.

Solution: We have the obvious inclusions $\mathfrak{N}_i \to \mathfrak{N}(\varinjlim A_i)$ since $A_i \to$ limit A is a ring homomorphism ($a^n = 0$ in A_i gets mapped to $a^n = 0$ in limit A).

Next we can map $\mathfrak{N}(\varinjlim A_i)$ to $\varinjlim \mathfrak{N}_i$ as so: For any $a^n = 0 \in$ limit A, $a = \mu_i(a_i)$ by Exercise 15, which then gives us $\mu_i(a_i^n) = 0$. By Exercise 15, we then have $\mu_{ij}(a_i^n) = 0$ in A_j. Then $\mu_{ij}(a_i)^n = 0$, giving us an element $\mu_{ij}(a_i)$, which we then map into $\varinjlim \mathfrak{N}_i$.

This is well-defined because we can always commute any choices to the same, largest index ring. Next this is a homomorphism because given $a = \mu_k(\mu_{ik}(a_i)), b = \mu_k(\mu_{jk}(b_j))$, $a + b = \mu_k(\mu_{ik}(a_i) + \mu_{jk}(a_j)) \to \mu_{k\ell}(\mu_{ik}(a_i) + \mu_{jk}(a_j)) \to \mu_k(\mu_{ik}(a_i) + \mu_{jk}(b_j)) = \mu_i(a_i) + \mu_j(b_j)$, which is what a, b would be mapped to. This is just the identity.

Since there is a homomorphism and an inverse, it is an isomorphism.

If each A_i is an integral domain, then suppose FTSOC that there is $ab = 0$ in $\varinjlim A_i$, $a, b \neq 0$. Then by Exercise 15, we have $a = \mu_i(a_i), b = \mu_j(b_j)$. Hence $\mu_i(a_i)\mu_j(b_j) = 0 = \mu_k(\mu_{ik}(a_i)\mu_{jk}(b_j))$ for $k \geq i, j$. Then by Exercise 15, there is $\ell \geq k$ s.t. $\mu_{k\ell}(\mu_{ik}(a_i)\mu_{jk}(b_j)) = 0 = \mu_{i\ell}(a_i)\mu_{j\ell}(b_j)$. But then A_j wouldn't be an integral domain (note that $\mu_{\cdot\ell}(\cdot) \neq 0$ because if otherwise, then $\mu_\ell(\mu_{\cdot\ell}(\cdot)) = \mu_\cdot(\cdot) = 0$, contradicting a, b being non-zero).

23. Let $(B_\lambda)_{\lambda \in \Lambda}$ be a family of A-algebras. For each finite subset of Λ, let B_J denote the tensor product (over A) of the B_λ for each $\lambda \in J$. If J' is another finite subset of Λ and $J \subseteq J'$, there is a canonical A-algebra homomorphism $B_J \to B_{J'}$. Let B denote the direct limit of the rings B_J as J runs through all finite subsets of Λ. The ring B has a natural A-algebra structure for which the homomoprhisms $B_J \to B$ are A-algebra homomorphisms. The A-algebra B is the tensor product of the family $(B_\lambda)_{\lambda \in \Lambda}$.

Solution: The canonical A-algebra homomorphism sends $b \in B_J$ to $b \otimes 1 \otimes 1 \otimes \cdots$ ($|J'| - |J|$ times). As A-algebras are also rings, the ring B exists by Exercise 21. Ring homomorphisms that preserve A-module structure are A-algebra homomorphisms.

24. In these Exercises it will be assumed that the reader is familiar with the definition and basic properties of the Tor functor.

    If M is an A-module, the following are equivalent:

    a. M is flat;
    b. $\mathrm{Tor}_n^A(M,N) = 0$ for all $n > 0$ and all A-modules N;
    c. $\mathrm{Tor}_1^A(M,N) = 0$ for all A-modules N.

Solution: (i) $\implies$ (ii): We do as the hint suggests: take a free resolution of N. Tensor this with M. As M is flat, this sequence is then exact, so the homology groups are 0.

Obviously (ii) $\implies$ (iii).

(iii) $\implies$ (i): Take an exact sequence $0 \to N' \to N \to N'' \to 0$. Then $\mathrm{Tor}_1^A(M,N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ is exact. As $\mathrm{Tor}_1^A(M,N'') = 0$, M is flat.

25. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence with N'' flat. Then N' is flat $\iff$ N is flat.

Solution: By the Tor exact sequence, we have

$$\mathrm{Tor}_2^A(M,N'') \to \mathrm{Tor}_1^A(M,N') \to \mathrm{Tor}_1^A(M,N) \to \mathrm{Tor}_1^A(M,N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0.$$

As $\mathrm{Tor}_2^A(M,N'') = \mathrm{Tor}_1^A(M,N'') = 0$ by flatness of N'' and Exercise 24, $\mathrm{Tor}_1^A(M,N') = \mathrm{Tor}_1^A(M,N)$. By Exercise 24, this means that N is flat iff N' is flat.

26. Let N be an A-module. Then N is flat $\iff$ $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ in A

Solution: $\implies$) is obvious by Exercise 24.

$\Leftarrow$):

Lemma 1.1. *Assuming the RHS, then* $\mathrm{Tor}(A/\mathfrak{b}, N) = 0$ *for **all*** $\mathfrak{b}$.

Proof. Take the system of finitely generated submodules of $\mathfrak{b}$ call it F, the system associated to that index of just A call it $\boldsymbol{A}$, and the system $A/\mathfrak{b}_i$ for $b_i \in F$ call it Q. As each of these has an exact sequence $0 \to \mathfrak{b} \to A \to A/\mathfrak{b} \to 0$, we have an exact sequence of systems by Exercise 19, giving us an exact sequence $0 \to \varinjlim F \to \varinjlim \boldsymbol{A} \to \varinjlim Q \to 0$. As tensors commute with direct limits (Exercise 20), we have $0 \to \varinjlim F \otimes N \to \varinjlim \boldsymbol{A} \otimes N \to \varinjlim Q \otimes N \to 0$.

By Exercise 17,$\varinjlim F = \mathfrak{b}$. We also have that $\varinjlim Q = A/\mathfrak{b}$ because all the maps in the system Q have kernels contained in B, so by the universal property of the quotient it induces a unique map from $A/\mathfrak{b}$that commutes with the system, so by the universal property of the direct limit, it is the direct limit. So we have the exact sequence

$$0 \to \mathfrak{b} \otimes N \to A \otimes N \to A/\mathfrak{b} \otimes N \to 0.$$

□

First we can note that N is flat if $\mathrm{Tor}_1(M,N) = 0$ for all finitely generated A-modules M by Proposition 2.19. Then fix a finitely generated M generated by x_i and define $M_i = \{x_1, \ldots, x_i\}$. Also define the map $f_i : A \to M_i/M_{i-1}$ by sending $a \in A$ to $ax_i + M_{i-1}$. This is surjective as M_i is generated by $x_1, \ldots, x_i$. As such,$\ker f_i$ is an ideal of A. Hence $M_i/M_{i-1} \cong A/\ker f_i$ So by considering the exact sequence $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \cong A/\ker f_i \to 0$, we can see that we get the Tor sequence

$$\mathrm{Tor}_1(M_{i-1}, N) \to \mathrm{Tor}_1(M_i, N) \to \mathrm{Tor}_1(A/\ker f_i, N).$$

Assume for induction that $\mathrm{Tor}_1(M_{i-1}, N) = 0$. Then by the lemma above,$\mathrm{Tor}_1(A/\ker f_i, N) = 0$. Thus $\mathrm{Tor}_1(M_i, N) = 0$. Obviously $\mathrm{Tor}_1(M_0, N) = 0$. Thus $\mathrm{Tor}_1(M, N) = 0$ for all finitely generated M, allowing us to use Proposition 2.19 to finish.

27. A ring A is absolutely flat if every A-module is flat. Prove that the following are equivalent:

   a. A is absolutely flat.

   b. Every principal ideal is idempotent.

   c. Every finitely generated ideal is a direct summand of A.

Solution: (i) $\implies$ (ii): Since $A/(x)$ is an A-module, it is flat. Thus the injectivity of $(x) \to A$ makes the map $(x) \otimes A/(x) \to A \otimes A/(x) \cong A/(x)$ injective. This map takes $x \otimes [a] \mapsto x \otimes [a] \mapsto [xa] = 0$ (middle map is due to Proposition 2.19). As it is an injective zero map,$(x) \otimes A/(x) = 0$, and by Exercise 2, $(x) \otimes A/(x) \cong (x)/(x)^2$. Thus $(x) = (x)^2$.

(ii) $\implies$ (iii): As the hint does: Let $x \in A$. Then $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent and we have $(e) = (x)$. For idempotents e, f, $(e, f) = (e+f-ef)$ because $e(e+f-ef) = e+ef-ef = e$ and $f(e+f-ef) = ef+f-ef = f$. Thus every finitely generated ideal is principal by finding idempotents for every generator in the ideal and then reducing them pairwise as so. As such,$A = (e) \oplus (1-e)$ (note that $(1-e)^2 = (1-e)$, so they are independent).

(iii) $\implies$ (i): It suffices to satisfy the conditions in Exercise 26 for all N. Take an exact sequence $0 \to N' \to N \to N'' \to 0$. Then we have the sequence

$$\mathrm{Tor}_1(A/\mathfrak{a}, N'') \to N' \otimes A/\mathfrak{a} \to N \otimes A/\mathfrak{a} \to N'' \otimes A/\mathfrak{a} \to 0.$$

By Exercise 2,$N' \otimes A/\mathfrak{a} \cong N'/\mathfrak{a}N' \cong \mathfrak{b}N'$ as we assume that A is a direct sum of f.g. ideals (namely let $A = \mathfrak{a} \oplus \mathfrak{b}$). Then the map $N' \otimes A/\mathfrak{a} \to N \otimes A/\mathfrak{a}$ is the map $\mathfrak{b}N' \to \mathfrak{b}N$, which is injective as they are simply restrictions of the injective map $N' \to N$. Thus $\mathrm{Tor}_1(A/\mathfrak{a}, N'') = 0$. As we can always realize N as the tail of an exact sequence (simply take $0 \to 0 \to N \to N \to 0$, we are done.

28. A Boolean ring is absolutely flat. The ring of Chapter 1, Exercise 7 is absolutely flat. Every homomorphic image of an absolutely fiat ring is absolutely flat. If a local ring is absolutely flat, then it is a field. If A is absolutely flat, every non-unit in A is a zero-divisor.

Solution: By definition, all principal ideals are idempotent in a Boolean ring, so by Exercise 27 we are done.

The ring in Chapter 1 Exercise 7 is absolutely flat because all principal ideals are idempotent:$(x)^2 = (x^2) = (x)$ because $x^{2n} = x \in (x^2)$.

Say we have f a homomorphism from an absolutely flat ring R. Then every principal ideal in the image is generated by $f(a)$, and $(a^2) = (a)$ by Exercise 27. Hence $(f(a))^2 = (f(a)^2) = (f(a^2)) = (f(a))$.

Fix an absolutely flat local ring R. By Exercise 27, every principal ideal of R is idempotent, so $(x^2) = (x) \forall x \in R$. Hence $x = rx^2, r \in R$. Thus $rx = r^2x^2 = (rx)^2 \implies rx$ is idempotent. But by Exercise 12 of Chapter 1,$rx = 0$ or 1. Thus $(x) = 0$ or 1, which implies that it is a field.

If A is absolutely flat, then take a non-unit x. We have that $(x)^2 = (x)$, so $x \in (x^2) \implies rx^2 = x$ for some r. Thus $x(rx - 1) = 0 \implies x$ is a zero-divisor.

2 Rings and Modules of Fractions

1. Verify that these definitions are independent of the choices of representatives (a, s) and (b, t), and that $S^{-1}A$ satisfies the axioms of a commutative ring with identity.

1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution: If there is such an s, then $\forall (m : n) \in S^{-1}M$, $s(m - n) = 0$.

If $S^{-1}M = 0$, then $\forall m \in M$, $(m : 1) = 0 \implies \exists s_m$ s.t. $s_m m = 0$. As M is finitely generated, it suffices to multiply the s_m of all the generators to get a universal annihilator.

2. Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants.

Solution: For every element $(a : 1 + a') \in S^{-1}\mathfrak{a}$, we can show that $1 - (a : 1 + a')y$ is a unit for all $y \in S^{-1}A$ and then use Proposition 1.9 to conclude. Let $y = (\alpha : 1 + a'')$ with $\alpha \in A$ and $a'' \in \mathfrak{a}$. Then $1 - (a : 1 + a')y = 1 - (a\alpha : 1 + a'' + a' + a'a'') = (1 + a'' + a' + a'a'' - a\alpha : 1 + a'' + a' + a'a'')$. By closure properties, $a'' + a' + a'a'' - a\alpha \in \mathfrak{a}$, so the numerator is in $1 + \mathfrak{a}$ and thus invertible.

If $\mathfrak{a}M = M$, then $S^{-1}\mathfrak{a}S^{-1}M = S^{-1}M$ with $S = 1 + \mathfrak{a}$, so Nakayama's lemma can be applied to conclude that $S^{-1}M = 0$. Then by the above exercise, there is $s \in S$ s.t. $sM = 0$. As $s \in S$, $s \equiv 1 \pmod{\mathfrak{a}}$.

3. Let A be a ring, let S and T be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.

Solution: We do this by way of universal property. Suppose we have a homomorphism $f : A \to B$ s.t. $f(s)$ is a unit in B for all $s \in ST$. Then $\hat{f} : U^{-1}(S^{-1}A) \to B$ defined by $\hat{f}((a : b) : (c : 1)) = f(a)f(bc)^{-1}$ is a well-defined homormophism. The inverse in the formula exists because $b \in S, c \in T \implies bc \in ST$. It is annoying to see that it satisfies the homomorphism properties of a well-defined homomorphism, and this then gives well-defined because if It is well-defined because any equivalence $((a : b) : (c : 1)) \equiv ((a' : b') : (c' : 1)) \implies \exists (u : 1) \in U, (u : 1)((a : b)(c' : 1) - (a' : b')(c : 1)) = 0 \in S^{-1}A$. ac':b - a'c:b' = u(ac'b'-a'cb):b'b By computing this out, we have that $(u(ac'b' - a'cb) : b'b) = 0 \in S^{-1}A \implies \exists s, su(ac'b' - acb) = 0$. Apply f to see that

$f(su)f(ac'b' - acb) = 0 \implies f(ac'b') = f(acb)$ because $su \in ST$. As $b, b' \in S, c, c' \in T$, we have that $f(a)f(bc)^{-1} = f(a')f(b'c')^{-1}$.

The commuting diagram is then satisfied as $h : A \to U^{-1}(S^{-1}A)$ takes $a \to ((a : 1) : (1 : 1))$ and $f \circ h = f(a)$. It is unique because where the elements of A and ST get sent uniquely determine where the rest of the elements in $U^{-1}(S^{-1}A)$ get sent (by well-defineness and all elements of $U^{-1}(S^{-1}A)$ being equivalent to something in the form $((a : 1) : (st : 1))$ for $a \in A, s \in S, t \in T$).

Thus by the universal property, they are isomorphic.

4. Let $f : A \to B$ be a homomorphism of rings and let S be a multplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

Solution: We define the map $g : S^{-1}B \to T^{-1}B$ by taking $(b : s) \to (b : f(s))$. This is obviously bijective because $S \cong T$. This is well-defined because if $(b : s) \equiv (b' : s')$, then $\exists s'' \in S, s''(bs' - sb') = 0 \implies f(s'')(bf(s') - f(s)b') = 0 \implies (b : f(s)) \equiv (b : f(s'))$ (note that this is how A acts on B as an A-module. Then this is a homomorphism because $f((b : s) + (b' : s')) = f((f(s')b + f(s)b' : ss')) = (f(s')b + f(s)b' : f(s)f(s')) = f((b : s)) + f((b' : s'))$ and $f((a : s')(b : s)) = f(f(a)b : s's) = (f(a)b : f(s)f(s')) = (a : s')f((b : s))$ (i.e. the way $S^{-1}A$ acts on B is preserved).

As this is a bijective module homomorphism, it is an isomorphism and we are done.

5. Let A be a ring. Suppose that for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent elements $\neq 0$. Show that A ha sno nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

Solution: Suppose FTSOC that A had a non-trivial nilpotent a. Then for some prime ideal $\mathfrak{p}$ (which exists because the case in which A is a field is trivial, and all non-trivial, non-field rings have a maximal ideal by Zorn's Lemma), we can inject a to $A_\mathfrak{p}$ with a homomorphism. But if $a^n = 0$, then $(a : 1)^n = 0$, giving $A_\mathfrak{p}$ a non-trivial nilpotent and giving us a contradiction. Hence A has no non-trivial nilpotents.

6. Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A s.t. $0 \notin S$. Show that Σ has maximal elements, and tha $S \in \Sigma$ is maximal iff $A - S$ is a minimal prime ideal of A.

Solution: Given an ascending chain of elements of Σ, $S_1 \subseteq S_2 \subseteq \cdots$, we can find an upper bound for it by taking the set of finite products across all the S_i. This is closed because a finite product times a finite product is still a finite product, and for any overlapping multiplications of elements in one S_i, that can be replaced by one element of S_i due to S_i being multiplicatively closed. Thus by Zorn's lemma we have a maximal element.

Then S is maximal in Σ implies $A \setminus S$ is a minimal prime ideal because if $A \setminus S$ contained another prime ideal $\mathfrak{p}$, then $A \setminus S \supset \mathfrak{p} \implies A \setminus (A - S) \subseteq A \setminus \mathfrak{p}$. But the LHS is S, a maximal multiplicatively closed subset of A and the RHS is another multiplicatively closed subset of A (that doesn't contain 0) that is in Σ. This is a contradiction, so S contains no other prime ideals.

7. A multiplicatively closed subset S of a ring A is said to be saturated if

$$xy \in S \iff x \in S \text{ and } y \in S.$$

Prove that

a. S is saturated $\iff$ $A - S$ is a union of prime ideals.

> **Solution:** $\implies$) Suppose $x \in A$ was a unit. Then $x \cdot x^{-1}s = s \implies x \in S$ for $s \in S$. Thus $A \setminus S$ has no units.
>
> Then every prime ideal of $S^{-1}A$ is in one-to-one correspondence with prime ideals of A that don't meet S by Proposition 3.11. We can then see that for all non-unit $x \in S^{-1}A$, x isn't a unit and thus is contained in a maximal ideal. This is prime, so there is a prime ideal that contains x that doesn't meet S. Therefore we can write $A \setminus S$ as a union of prime ideals, namely those above.
>
> If $S \setminus A$ is a union of prime ideals, then S is saturated because $\implies$) if $xy \in S$ then if x or y wasn't in S, then $xy \in S \setminus A$ by properties of ideals. $\Leftarrow$) If $x, y \in S$, then if $xy \in S \setminus A$, then $xy \in$ some prime ideal contained in $S \setminus A$, which implies that one of x, y is in $S \setminus A$, a contradiction.

b. If S is any multiplicative closed subset of A, there is a unique smallest saturated multiplicatively closed subset $\overline{S}$ containing S, and that $\overline{S}$ is the complement in A of the union of the prime ideals which do not meet S. ($\overline{S}$ is called the saturation of S.)

> **Solution:** Suppose there are two distinct minimal saturated multiplicated closed subsets $\overline{S}$ and $\overline{S}'$ that contain S. Then $\overline{S} \cap \overline{S}'$ is also saturated ($(\overline{S} \cap \overline{S}')^C = \overline{S}^C \cup \overline{S}'^C$ = a union of prime ideals by the exercise just above). But this is contained in both, so must be equal to both. Hence they are equal and there is a unique one.
>
> For existence, we can show that the complement in A of the union of the prime ideals which don't meet S is a minimal saturated set. It is saturated by part i). It is minimal because if there was a saturated set contained in it, say S', then $A \setminus S'$ would be the union of prime ideals. As $A \setminus S$ is already the union of prime ideals that don't meet S, $A \setminus S'$ must have a prime ideal that meets S, say at x. But if $x \in S$, then $x \in S'$ as $S \subseteq S'$ and S' is saturated. This is a contradiction.

If $S = 1 + \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of A, find $\overline{S}$.

> **Solution:** Solution due to https://math.sci.uwo.ca/~jcarlso6/intro_comm_alg(2019).pdf
>
> It is A. By the above part, $A \setminus \overline{S}$ is the union of prime ideals that don't meet S. If $x \in$ a prime ideal is in S (i.e. meets S), then $1 + a = x$ for some $a \in \mathfrak{a}$. Hence $1 = x - a$ and thus $A \setminus \overline{S}$ is the union of prime ideals not coprime to $\mathfrak{a}$.
>
> As any such prime ideal $\mathfrak{p}$ that is coprime to $\mathfrak{a}$ is contained in a maximal ideal ($\mathfrak{a}+\mathfrak{p} \subsetneq (1)$), it suffices to take the union of the set of maximal ideals that contain $\mathfrak{a}$ ($\mathfrak{a} + \mathfrak{p} \subseteq \mathfrak{m} \implies \mathfrak{p} \subseteq \mathfrak{m}$). This works because , maximal ideals that contain $\mathfrak{a}$ don't meet $1 + \mathfrak{a}$. Hence $\overline{S} = A \setminus \cup\{\mathfrak{m} \in \mathrm{Max}(A) : \mathfrak{a} \subseteq \mathfrak{m}\}$.

8. Let S, T be multiplicatively closed subsets of A s.t. $S \subseteq T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$. Show that the following statements are equivalent:

   a. ϕ is bijective.

   b. For each $t \in T$, $t/1$ is a unit in $S^{-1}A$.

   c. For each $t \in T$, there exists $x \in A$ s.t. $xt \in S$.

   d. T is contained in the saturation of S (Exercise 7).

e. Every prime ideal which meets T also meets S.

Solution: $i) \implies ii)$ If ϕ is bijective, then because ϕ is a homomorphism, $\phi^{-1}(tt^{-1}) = \phi^{-1}(t)\phi^{-1}(t^{-1}) = 1 \implies t$ is a unit in $S^{-1}A$.

$ii) \implies iii)$ As $t/1$ is a unit in $S^{-1}A$, there is some $(a : s)$ s.t. $(ta : s) = 1 \implies \exists s' \in S$ s.t. $s'(ta - s) = 0 \implies tas' = ss' \in S$, so $x = as'$.

$iii) \implies iv)$ The saturation of S is $A \setminus \cup$ prime ideals that don't meet S. Hence it suffices to show that T doesn't meet prime ideals that don't meet S ($T \subseteq A \setminus \cup$ prime ideals that don't meet $S \iff$ no element of T is in such prime ideals). But if T meets one of these prime ideals, then this prime ideal is still prime in $S^{-1}A$ (by Proposition 3.11), say at t. Then by assumption, there is an $x \in A$ s.t. $xt \in S \implies xt \in$ the prime ideal is a unit in $S^{-1}A$, a contradiction.

$iv) \implies v)$ Now suppose FTSOC that there was a prime ideal that meets T that doesn't meet S, say $\mathfrak{p}$ at t. As $T \subseteq \overline{S}$, T doesn't meet any prime ideal that doesn't meet S. This is a blatant contradiction, as $\mathfrak{p}$ is a prime ideal that doesn't meet S (recall that $S \subseteq T$), so T doesn't meet it. Hence $\mathfrak{p}$ does meet S.

$v) \implies i)$ Obviously ϕ is injective, so all we need to show is surjectivity. By assumption and $S \subseteq T$, every prime ideal of $T^{-1}A$ corresponds one-to-one to prime ideals of $S^{-1}A$ by using Proposition 3.11 (by passing through prime ideals that don't meet S) through inclusion. As every non-unit in $T^{-1}A$ is contained in a maximal ideal, which is prime, this corresponds to a prime ideal in $S^{-1}A$, which implies that non-units are Every unit in $T^{-1}A$ is the image of an element of $S^{-1}A$ because

9. The set S_0 of all non-zero divisors in A is a saturated multiplicatively closed subset of A. Hence the set D of zero-divisors in A is a union of prime ideals (see Chapter 1, Exercise 14). Show that every minimal prime ideal of A is contained in D.

Solution: Take a minimal prime ideal $\mathfrak{p}$. Then $A \setminus \mathfrak{p}$ is multiplicatively closed, so if we let $S = A \setminus \mathfrak{p}$, we can apply Exercise 6 to get that $A \setminus \mathfrak{p}$ is maximal in Σ. We can then see that

The ring $S_0^{-1}A$ is call the total ring of fractions of A. Prove that

a. S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \to S_0^{-1}A$ is injective.

Solution: Suppose we had a larger subset with this property, say S'. Then S' contains a zero-divisor, say $ss' = 0$. Hence $f : A \to S'^{-1}A$ maps s and s' to 0, a contradiction.

b. Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.

Solution: Every element in $S_0^{-1}A$ is of the form $(a : s), a \in A, s \in S_0$ by definition. If $a \in S_0$, then it is obviously invertible with inverse $(s : a)$. If $a \notin S_0$, then by definition it is a zero divisor in A and thus a zero-divisor in $S_0^{-1}A$

c. Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e. $A \to S_0^{-1}A$ is bijective).

Solution: As every non-unit is a zero-divisor, D is the set of non-units. Thus S_0 is the set of units, and we have the homomorphism $S_0^{-1}A \to A$ that sends $(a : s) \to as^{-1}$ which is then the inverse of $A \to S_0^{-1}A$ because $a \to (a : 1) \to a$. This gives us an isomorphism.

10. Let A be a ring.

    a. If A is absolutely flat (Chapter 2, Exercise 27) and S is any multiplicatively closed subset of A, then $S^{-1}A$ is absolutely flat.

    Solution: Every $S^{-1}A$ module is also an A-module, so they are flat. Thus $S^{-1}A$ is absolutely flat.

    b. A is absolutely flat $\iff$ $A_{\mathfrak{m}}$ is a field for each maximal ideal $\mathfrak{m}$.

    Solution: As $A_{\mathfrak{m}}$ is $(A \setminus \mathfrak{m})^{-1}A$, it is absolutely flat is A is absolutely flat. As it is local, by Exercise 28, it is a field.

    $\Leftarrow$) Because $A_{\mathfrak{m}}$ is a field for each maximal ideal and we have the correspondence in Proposition 3.11, there are no prime ideals in A that don't meet $A \setminus \mathfrak{m}$ except for the one corresponding to (0). There are no prime ideals other than the one corresponding to (0) because other prime ideal are contained in a non-zero maximal ideal $\mathfrak{m}$, which also doesn't meet $A \setminus \mathfrak{m}$ and thus should correspond to a prime ideal in $A_{\mathfrak{m}}$.

    So the only prime ideal of A is one s.t. $\mathfrak{p}_{\mathfrak{p}} = (0)$ in $A_{(0)}$. Because $\mathfrak{p}$ is the only maximal ideal and $\mathfrak{p}$ is an A-module, by Proposition 3.8, $\mathfrak{p} = 0$. Hence A is a field. Trivially, all principal ideals are idempotent in a field, so by Exercise 27 A is absolutely flat.

    Alternatively, https://math.sci.uwo.ca/~jcarlso6/intro_comm_alg(2019).pdf has a cool solution too!

11. Let A be a ring. Prove that the following are equivalent:

    a. $A/\mathfrak{R}$ is absolutely flat ($\mathfrak{R}$ being the nilradical of A).

    b. Every prime ideal of A is maximal.

    c. Spec(A) is a T_1-space (i.e., every subset consisting of a single point is closed).

    d. Spec(A) is Hausdorff

Solution: *i*) $\iff$ *ii*) Taken from https://math.sci.uwo.ca/~jcarlso6/intro_comm_alg(2019).pdf. All prime ideals of A are maximal iff there are no prime ideals between arbitrary maximal ideal $\mathfrak{m}$ and $\mathfrak{R}$ by definition. By Proposition 1.1, this is in iff with there being no prime ideals between $\mathfrak{m}/\mathfrak{a}$ and 0 in $A/\mathfrak{R}$. Then by Proposition 3.11, this is in iff with there being no non-zero prime ideals in $(A/\mathfrak{R})_{\mathfrak{m}}$. This is true iff $(A/\mathfrak{R})_{\mathfrak{m}}$ is a field, which by Exercise 10 is iff with $A/\mathfrak{R}$ being absolutely flat.

ii) $\iff$ *iii*) Forward: If every prime ideal $\mathfrak{p}$ of A is maximal, then the set of prime ideals that contain $\mathfrak{p}$ is just $\mathfrak{p}$, and by definition these are the closed sets. Thus $\{\mathfrak{p}\}$ is closed.

Then because $\mathfrak{p}$ is closed, it is the set of prime ideals that contain an ideal, which has to be $\mathfrak{p}$ because this set is a singleton, hence it is maximal (gives us reverse direction)

iii) $\iff$ *iv*) Forward: Fix two distinct points $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} A$. By T_1 hypothesis, $\exists U_{\mathfrak{p}}, U_{\mathfrak{q}}$ neighborhoods that separate them. Suppose FTSOC that they had an intersection point $\mathfrak{r}$. Because $U_{\mathfrak{p}}$ is open in the Zariski topology, it is the complement of $V(I)$ for some I, and similarly for $U_{\mathfrak{q}}$ for $V(J)$. Then $(U_{\mathfrak{p}} \cap U_{\mathfrak{q}})^C = V(IJ)$. As $\mathfrak{r}$ is in $U_{\mathfrak{p}} \cap U_{\mathfrak{q}}$, it isn't in $V(IJ)$. Thus it is contained in IJ.

Then $\mathfrak{r}$ is maximal by *ii*). So $IJ = \mathfrak{r}$. Because $IJ \subseteq I$, this implies that $J = I = \mathfrak{r}$. Hence $U_{\mathfrak{p}} = U_{\mathfrak{q}} = V(\mathfrak{r})$, a contradiction.

For reverse, Hausdorff is stronger than T_1.

If these conditions are satisfied, show that Spec(A) is compact and totally disconnected (i.e. the only connected subsets of Spec(A) are those consisting of a single point.

Solution: Because every prime ideal is maximal, the basis generating the Zariski topology is just singletons. As intersections only decrease set size and only finite unions are closed, this topology is the cofinite topology on $\operatorname{Spec} A$.

For point set reasons, this space is then compact (open sets contain all but finitely many points, take the subcover of one open set and finitely many to fill in the gaps).

Fix a subset S of Spec(A). If S is finite, then take two disjoint finite subsets of it. These are closed in Spec(A), so they are a closed partition of S, disconnecting it.

If S is infinite, then take an infinite disjoint partition of it (exists by taking an infinite sequence of distinct points (x_i) and then taking alternating points). These two sets will be open in Spec(A), so they are open in S, disconnecting it.

12. Let A be an integral domain and M an A-module. An element $x \in M$ is a torsion element of M if $\operatorname{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the torsion submodule of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

Proof. $T(M)$ inherits an abelian group operation from M. For closure, if $a, b \in T(M)$ and x, y annihilate them, then $xy(a + b) = xya + xyb = 0$. It is an A-module with the operation inherit from M, and for $a \in A$ and $b \in T(M)$ annihilated by x, $x(ab) = a(xb) = 0$. ☐

a. If M is any A-module, then $M/T(M)$ is torsion-free.

Solution: Take an element $x \in M$. For an element $a \in A$ s.t. $ax = 0 \pmod{T(M)}$, then either $ax = 0$ in M or $ax \in T(M)$. If $ax = 0$, then $x \in T(M)$ and it isn't a non-trivial torsion element.

If $ax \in T(M)$, then it is annihilated by some element $b \in A$ so that $bax = 0$ in M. Then x is annihilated by ab, so $x \in T(M)$ and it isn't a non-trivial torsion element.

Thus $T(M/T(M)) = 0$.

b. If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

Solution: Take an element $m \in T(M)$ annihilated by $a \in A$. Then $am = 0 \implies f(am) = 0 = af(m) \implies f(m) \in T(N)$.

c. If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T(M') \to T(M) \to T(M'')$ is exact.

Solution: Name the maps $m' : M' \to M, m : M \to M''$ and let m'_T, m_T be their respective maps in $T(M_*)$. Because m' was already injective, $T(M') \to T(M)$ is also injective.

$\Im m'_T \subseteq \ker m_T : m_T \circ m'_T = m \circ m'|_{T(M')} \subseteq m \circ m' = 0$.

$\Im m'_T \subseteq \ker m_T$: For any element in $x \in \ker m_T$, $x \in \Im m'$ by exactness. Since $x \in T(M)$, there is an a s.t. $ax = 0$. Let $x = m'(y)$. Then by injectivity of m', $ax = m'(ay) = 0 \implies ay = 0 \implies y \in M(T') \implies x \in \Im m'_T$.

d. If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.

Solution: Name the map t.

$T(M) \subseteq \ker t$) Take $m \in T(M)$, annihilated by $a \in A$. Then $m \mapsto 1 \otimes m$. We can then see that $1 \otimes m = 0$ because $1 \otimes m = a/a \otimes m = 1/a \otimes am = \frac{1}{a} \otimes am = \frac{1}{a} \otimes 0 = 0$.

$T(M) \supseteq \ker t$) By Proposition 3.5, $A_{(0)} \otimes_A M \cong M_{(0)}$ with a map that sends $\frac{a}{b} \otimes m \mapsto \frac{am}{b}$. So if $1 \otimes_A m = 0$, then $\frac{m}{1} = 0 \iff ms = 0$ for $s \in A \setminus (0)$. This is a non-trivially annihilator.

13. Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent:

Solution: Map $\frac{m}{s} \in T(S^{-1}M)$ to $\frac{m}{s}$ in $S^{-1}(T(M))$. This maps into the right place because $\frac{m}{s} \in T(S^{-1}M) \implies \exists \frac{a}{s'} \in S^{-1}A$ s.t. $\frac{am}{ss'} = 0 \implies \frac{am}{1} = 0 \implies m \in T(M) \implies \frac{m}{s} \in S^{-1}(T(M))$. This is obviously injective since A is a domain.

For surjectivity, for any element $\frac{m}{s} \in S^{-1}(T(M))$, map it to $\frac{m}{s} \in T(S^{-1}(M))$. This maps into the right place because if $\frac{m}{s} \in S^{-1}(T(M)), m \in T(M)$. Thus there is an $a \in A$ s.t. $am = 0 \implies \frac{a}{1} \cdot \frac{m}{s} = 0 \implies \frac{m}{s} \in T(S^{-1}(M))$.

a. M is torsion-free.

b. $M_\mathfrak{p}$ is torsion-free for all prime ideals $\mathfrak{p}$.

c. $M_\mathfrak{m}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

Solution: If M is torsion free, then $T(M) = 0$. Since $T(S^{-1}M) = S^{-1}(T(M))$ by Exercise 12, $S^{-1}(T(M)) = 0 = T(S^{-1}(M)) \implies S^{-1}M$ is torsion free for all S. This gives us $ii), ii)$.

$iii) \implies i)$ By Exercise 12, $T(M_\mathfrak{m}) = 0 = T(M)_\mathfrak{m}$. Then by Proposition 3.8, $T(M) = 0$.

14. Let M be an A-module and $\mathfrak{a}$ an ideal of A. Suppose that $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$. Prove that $M = \mathfrak{a}M$.

Solution: By Proposition 1.1, $\mathfrak{m}/\mathfrak{a}$ is a maximal ideal in $A/\mathfrak{a}$. As $M_\mathfrak{m} = 0, (M_\mathfrak{m})/\mathfrak{a}_\mathfrak{m} = 0 = (M/\mathfrak{a})_\mathfrak{m}$ by Corollary 3.4. As this is true for all maximal ideals of $A/\mathfrak{a}$, by Proposition 3.8 $M/\mathfrak{a} = 0$. Hence $M = \mathfrak{a}M$.

15. Let A be a ring, and let F be the A-module A^n. Show that every set of n generators of F is a basis of F.

Solution: Take a set of generators $x_1, \cdots, x_n$ of A^n and let $e_1, \cdots, e_n$ be the canonical basis. Then let $x_i = x_{i1}e_1 + x_{i2}e_2 + \cdots + x_{in}$. Let $\phi(e_i) = x_i$. As x_i are generators, ϕ is automatically surjective. By Proposition 3.9, we localize A at an arbitrary maximal ideal and then show that ϕ_M is injective there.

Fix an arbitrary $\mathfrak{m}$. If ϕ_M isn't injective, there is a relation $a_1x_1 + \cdots + a_nx_n = 0$. This lifts to a relation in $A^n_{\mathfrak{m}}$. This also gives a relation in $\mathfrak{A}_{\mathfrak{m}}/\mathfrak{m}\mathfrak{A}_{\mathfrak{m}} \otimes_{\mathfrak{A}_{\mathfrak{m}}} F_{\mathfrak{m}}$. Let $k = \mathfrak{A}_{\mathfrak{m}}/\mathfrak{m}$. Then this equals k^n. As this is a vector space, the image of the generators $x_1, \cdots, x_n$ become a basis and we then get a contradiction.

Deduce that every set of generators of F has at least n elements.

Solution: Say there was a list of $a < n$ generators. This generates A^a as a submodule of A^n, which then means they form a basis of A^a. But $A^a \neq A_n$.

16. Let B be a flat A-algebra. Then the following conditions are equivalent:

   a. $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of A.
   b. $\text{Spec}(B) \to \text{Spec}(A)$ is surjective.
   c. For every maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}^e \neq (1)$.
   d. If M is any non-zero A-module, then $M_B \neq 0$.
   e. For every A-module M, the mapping $x \mapsto 1 \otimes x$ of M into M_B is injective.

Solution: $i) \iff ii)$ Just Proposition 3.16.

$ii) \implies iii) \implies)$Because $\text{Spec}(B) \to \text{Spec}(A)$ is surjective, $\exists \mathfrak{b}$ s.t. $\mathfrak{b}^c = \mathfrak{m}$. Since ${}^{cec} = \mathfrak{b}^c = \mathfrak{m}$, $\mathfrak{m}^e = \mathfrak{b}^{cece} = \mathfrak{b}^c$ since $\mathfrak{b}^C$ is a prime ideal in A (Proposition 1.17 used many times). Hence $\mathfrak{m}^e \neq (1)$.

$iii) \implies iv)$ Fix a non-zero element $x \in M$ and define $M' = Ax$. Since B is flat, it suffices to show that $B \otimes M' \neq 0$ (because we have the exact sequence $0 \to M' \to M \to M/M' \to 0$ with the first map being an inclusion).

Then we have that $Ax \cong A/\mathfrak{a}$ for some ideal because we can let the ideal be the relations of x to 0. Thus $B \otimes M' = B/\mathfrak{a}^e$ by Exercise 2, Chapter 2. As $\mathfrak{a} \subseteq \mathfrak{m}$, a maximal ideal, we have that $\mathfrak{a}^e \subseteq \mathfrak{m}^e \neq (1)$ by assumption. Hence $B/\mathfrak{a}^e \neq 0 \implies M_B \neq 0$.

$iv) \implies v)$ Let M' be the kernel of $M \to M_B$. Then by flatness of B, $0 \to M'_B \to M_B \to ((M)_B)_B \to 0$ is exact. Then by Chapter 2 Exercise 13 with $N = M_B$, the last map is injective. Thus $M'_B = 0 \implies M' = 0$ by assumption. Hence $\ker(M \to M_B) = 0$, making the map injective.

$v) \implies i)$ Let $M = A/\mathfrak{a}$. Then $M \to B \otimes M = B/\mathfrak{a}B$, by Chapter 2 Exercise 2, is injective. By definition, $B/\mathfrak{a}B = B/f(\mathfrak{a})B$, so this map being injective implies that if an element is in $\mathfrak{a}^e$, then it's preimage, which forms $\mathfrak{a}^{ec}$, is in $\mathfrak{a}$. As $\mathfrak{a}^{ec} \supseteq \mathfrak{a}$, $\mathfrak{a} = \mathfrak{a}^{ec}$.

B is said to be faithfully flat over A.

17. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms. If $g \circ f$ is flat and g is faithfully flat, then f is flat.

Solution: It suffices to show that if $f : M' \to M$ is injective as A-modules, then $f \otimes 1 : M' \otimes B \to M \otimes B$ is injective by Proposition 2.19. Then by flatness of $g \circ f$, $M'_C \to M_C$ is injective, which then gives injectivity of $C \otimes_B B \otimes_A M' \to C \otimes_B B \otimes_A M$ by Proposition 2.14 and Exercise 2.15 in the reading. By the faithful flatness of g, $(M')_B \to (M')_C$, $M_B \to (M_B)_C$ are injective. Thus we have this diagram

$$\begin{tikzcd} M'_B \arrow[r] \arrow[d] & M_B \arrow[d] \\ (M'_B)_C \arrow[r] & (M_B)_C \end{tikzcd}$$

The three edge route from $M'_B \to M_B$ are all injective, so $M'_B \to M_B$ is injective too, proving flatness.

18. Let $f : A \to B$ be a flat homomorphism of rings, let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective.

Solution: We have that $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ by Proposition 3.10. Then by Corollary 3.6, we have that the following is flat $B_\mathfrak{q} = (B_\mathfrak{p})_\mathfrak{q} = (B \setminus \mathfrak{q})^{-1}(f(A \setminus \mathfrak{p}))^{-1}B = (B \setminus \mathfrak{q})^{-1}B$ by Exercise 3 (and noting that $(B \setminus \mathfrak{q})(f(A \setminus \mathfrak{p})) = B \setminus \mathfrak{q}$ because one is in $f(A \setminus \mathfrak{p})$ and everything else is already in $B \setminus \mathfrak{q}$). As $\mathfrak{p}$ is the only maximal ideal of $A_\mathfrak{p}$, $\mathfrak{p}^e \subseteq \mathfrak{q} \neq (1) \implies \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective by Exercise 16.

19. Let A be a ring, M an A-module. The support of M is defined to be the set $\mathrm{Supp}(M)$ of prime ideals $\mathfrak{p}$ of A such that $M_\mathfrak{p} \neq 0$. Prove the following results:

a. $M \neq 0 \iff \mathrm{Supp}(M) \neq \emptyset$

Solution: Instead we prove that $M = 0 \iff \mathrm{Supp}(M) = \emptyset$. By Proposition 3.8, $M = 0 \iff M_\mathfrak{p} = 0$ for all prime ideals of A, so $\mathrm{Supp}(M) = \emptyset$.

b. $V(\mathfrak{a}) = \mathrm{Supp}(A/\mathfrak{a})$

Solution: $\supseteq$) All the prime ideals in $A/\mathfrak{a}$ are those that contain $\mathfrak{a}$ by Proposition 1.1.
$\subseteq$) Take some prime ideal $\mathfrak{p}$ in $V(\mathfrak{a}) = \mathrm{Spec}(A/\mathfrak{a})$. Since $\mathrm{Spec}((A/\mathfrak{a})_\mathfrak{p}) =$ the set of prime ideals contained in $\mathfrak{p}$ and containing $\mathfrak{a}$ by Proposition 3.11 and 1.1, and $\mathfrak{a}$ satisfies that, $(A/\mathfrak{a})_\mathfrak{p} \neq 0$.

c. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

Solution: $\subseteq$) We do contrapositive. Take some prime ideal $\mathfrak{p} \notin \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$. Then by Proposition 3.3,

$$0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$$

is exact. This then equals

$$0 \to 0 \to M_\mathfrak{p} \to 0 \to 0$$

because $\mathfrak{p} \notin \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$. By exactness, $M_\mathfrak{p} = 0$, so $\mathfrak{p} \notin \mathrm{Supp}(M)$.
$\supseteq$) Take some prime $\mathfrak{p} \in \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$. Then by Proposition 3.3,

$$0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$$

is exact. If $\mathfrak{p} \notin \operatorname{Supp}(M)$, then the middle would be 0, forcing the other modules to be 0. This would contradict $\mathfrak{p}$'s presence in at least one of $\operatorname{Supp}(M'), \operatorname{Supp}(M'')$.

d. If $M = \sum M_i$, then $\operatorname{Supp}(M) = \cup \operatorname{Supp}(M_i)$.

Solution: We do induction. This is true for the base case by part *iii*). Then consider we the exact sequence

$$0 \to M_n \to M \to \sum^{n-1} M_i \to 0.$$

By part *iii*), $\operatorname{Supp}(M) = \operatorname{Supp} M_n \cup \operatorname{Supp} \sum^{n-1} M_i$. By inductive hypothesis, $\operatorname{Supp} \sum^{n-1} M_i = \bigcup^{n-1} \operatorname{Supp} M_i$. Hence $\operatorname{Supp} M = \bigcup^{n} \operatorname{Supp} M_i$.

e. If M is finitely generated, then $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$ (and is therefore a closed subset of $\operatorname{Spec}(A)$).

Solution: By Exercise 1, $M_{\mathfrak{p}} = 0 \iff \exists x \in (A \setminus \mathfrak{p}) \cap \operatorname{Ann}(M)$. This is $\iff \mathfrak{p} \notin V(\operatorname{Ann}(M))$ (because $\mathfrak{p} \in V(\operatorname{Ann}(M)) \implies (A \setminus \mathfrak{p}) \cap \operatorname{Ann}(M) = 0$).

f. If M, N are finitely generated, then $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$.

Solution: $0 \neq (M \otimes_A N)_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \iff M_{\mathfrak{p}} \neq 0$ and $N_{\mathfrak{p}} \neq 0$ by Chapter 2, Exercise 3.

g. If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $\operatorname{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \operatorname{Ann}(M))$.

Solution: $\operatorname{Supp}(M/\mathfrak{a}M) = \operatorname{Supp}(A/\mathfrak{a} \otimes_A M)$ by Exercise 2, Chapter 2. By the above exercise, this equals $\operatorname{Supp}(A/\mathfrak{a}) \cap \operatorname{Supp}(M)$. By exercise *v*) and assumption, $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$. Then $V(\mathfrak{a}) = \operatorname{Supp}(A/\mathfrak{a})$ by exercise *ii*), so $\operatorname{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a}) \cap V(\operatorname{Ann}(M)) = V(\mathfrak{a} + \operatorname{Ann}(M))$.

h. If $f : A \to B$ is a ring homomorphism and M is a finitely generated A-module, then $\operatorname{Supp}(B \otimes_A M) = f^{*\,-1}(\operatorname{Supp}(M))$.

Solution: Because M is finitely generated, it is $A^n/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A^n. Then $B \otimes_A M = B \otimes_A A^n/\mathfrak{a} = B/\mathfrak{a}'$ by using Exercise 2, Chapter 2 ($A^n/\mathfrak{a} = A/\mathfrak{a}_1 \oplus A/\mathfrak{a}_2 \oplus \cdots \oplus A/\mathfrak{a}_n$). As such, by *ii*), $\operatorname{Supp}(B/\mathfrak{a}') = V(f(\mathfrak{a}'))$.
Finally, note that $f^{*\,-1}(\operatorname{Supp}(M))$ is the set of prime ideals in B that contain $f(\operatorname{Ann}(M))$ ($f^{*\,-1}$ is the preimage of contraction, so elements in it must contract to a prime ideal containing $\operatorname{Ann}(M)$. As such they contain $f(\operatorname{Ann}(M))$, and if a prime ideal in B contains $f(\operatorname{Ann}(M))$ then the contraction is a prime ideal in A containing $\operatorname{Ann}(M)$).

20. Let $f : A \to B$ be a ring homomorphism, $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ the associated mapping. Show that

a. Every prime ideal of A is a contracted ideal $\iff f^*$ is surjective.

Solution: Definition of comtracting, surjectivity, and the map f^*.

b. Every prime ideal of B is an extended ideal $\implies f^*$ is injective.

Solution: Suppose FTSOC that we had two prime ideas $\mathfrak{p} = p^e, \mathfrak{q} = q^e$ s.t. $\mathfrak{p}^c = \mathfrak{q}^c$. Then $p^{ec} = q^{ec} \in \mathrm{Spec}(A)$. But then $p = p^{ece} = q^{ece} = q$ by Proposition 1.17, so $\mathfrak{p} = \mathfrak{q}$, which is the statement for injectivity.

Is the converse of $ii)$ true?

Solution: No it isn't. Consider $A = \boldsymbol{Z}/4\boldsymbol{Z}, B = (\boldsymbol{Z}/4\boldsymbol{Z})[i]$. The only prime ideal in B is $(2, i)$ because we need to eliminate 2 to make it an integral domain, and allowing i means allowing $(1+i)^2 = 0$, a zero-divisor. The only prime ideal in A is (2). So f^* is injective, but $(2, i)$ is not an extension of (2).

21. a. Let A be a ring, S a multiplicatively closed subset of A, and $\phi : A \to S^{-1}A$ the canonical homomorphism. Show that $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a homeomorphism of $\mathrm{Spec}(S^{-1}A)$ onto its image in $X = \mathrm{Spec}(A)$. Let this image be denoted by $S^{-1}X$.

Solution: By Proposition 3.11, this is injective. Then for continuity, we can note that localization is is an order preserving functor because of exactness of localization (injections become injections) and if $\mathfrak{a}_\mathfrak{p} \subseteq \mathfrak{b}_\mathfrak{p}$, then $\mathfrak{a} = \mathfrak{a}_\mathfrak{p}^c = (\mathfrak{a}_\mathfrak{p} \cap \mathfrak{b}_\mathfrak{p})^c = \mathfrak{a}_\mathfrak{p}^c \cap \mathfrak{b}_\mathfrak{p}^c = \mathfrak{a} \cap \mathfrak{b}$ by Exercise 1.18 in the reading. Hence $\mathfrak{a} \subseteq \mathfrak{b}$. Thus closed basis sets become and are from closed basis sets, making this a homeomorphism.

In particular, if $f \in A$, the image of $\mathrm{Spec}(A_f)$ in X is the basic open set X_f (Chapter 1, Exercise 17).

Solution: The prime ideals in A_f are those that don't contain f. By Proposition 3.11, these correspond to prime ideals that don't meet $f, f^2, \cdots$. This is just $V(f)$, as any prime ideal that meets $f, f^2 \cdots$ contains f by primeness and any ideal that doesn't meet any of $f, f^2, \cdots$ clearly doesn't contain the ideal (f) (as these contain $f, f^2, \cdots$).

b. Let $f : A \to B$ be a ring homomorphism. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, and let $f^* : Y \to X$ be the mapping associate with f. Identifying $\mathrm{Spec}(S^{-1}A)$ with its canonical image $S^{-1}X$ in X, and $\mathrm{Spec}(S^{-1}B)$ $(= \mathrm{Spec}(f(S)^{-1}B))$ with its canonical image $S^{-1}Y$ in Y, show that $S^{-1}f^* : \mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$, and that $S^{-1}Y = f^{*\,-1}(S^{-1}X)$.

Solution: Let $\phi_A : A \to S^{-1}A, \phi_B : B \to S^{-1}B$. Then take $\mathfrak{q} \in \phi_B^*(\mathrm{Spec}(S^{-1}B))$. Let $\mathfrak{q}' \in \mathrm{Spec}(S^{-1}B)$ be s.t. $\phi_B^{-1}(\mathfrak{q}') = \mathfrak{q}$.
Thus $f^*(\mathfrak{q}) = f^{-1}(\mathfrak{q}) = f^{-1}(\phi_B^{-1}(\mathfrak{q}')) = (\phi_B \circ f)^{-1}(\mathfrak{q}')$. By diagram chasing ($S^{-1}f$ makes this diagram commute by definition

$$\left.\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ S^{-1}A & \longrightarrow & S^{-1}B \end{array}\right)$$

this equals $(S^{-1}f \circ \phi_A)^{-1}(\mathfrak{q}') = \phi_A^{-1} S^{-1} f^{-1}(\mathfrak{q}') = \phi_A^* S^{-1} f^*(\mathfrak{q}')$, which is in $\phi_A^*(\mathrm{Spec}(S^{-1}A)) = S^{-1}X$.

What the second part is asking is that f^* doesn't map anything else into $S^{-1}X$. Suppose that there was an ideal I s.t. $f^*(I) = f^{-1}(I) \in \phi_A^{-1}(\operatorname{Spec}(S^{-1}A))$. Then by Proposition 3.11, $f^{-1}(I)$ is a prime ideal that doesn't meet S. If I met $f(S)$, then $f^{-1}(I)$ would meet S, a contradiction. Thus I doesn't meet $f(S)$, and hence is in $\phi_B^{-1}(\operatorname{Spec}(S^{-1}B))$.

c. Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B. Let $\overline{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the homomorphism induced by f. If $\operatorname{Spec}(A/\mathfrak{a})$ is identified with its canonical image $V(\mathfrak{a})$ in X, and $\operatorname{Spec}(B/\mathfrak{b})$ with its image $V(\mathfrak{b})$ in Y, show that $\overline{f}^*$ is the restriction of f^* to $V(\mathfrak{b})$.

Solution: The map $\overline{f}^*(b') = \overline{f}^{-1}(b')$. Let $\phi : B \to B/\mathfrak{b}$. Then by definition, $\overline{f}^{-1}(b') = f^{-1}(b)$ for $b \in \operatorname{Spec} B$, but only for $b \in V(\mathfrak{b})$. Thus $\overline{f}^*$ is the restriction of f^* to $V(\mathfrak{b})$.

d. Let $\mathfrak{p}$ be a prime ideal of A. Take $S = A - \mathfrak{p}$ in $ii)$ and then reduce mod $S^{-1}\mathfrak{p}$ as in $iii)$. Deduce that the subspace $f^{*\ -1}(\mathfrak{p})$ of Y is naturally homeomorphic to $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_\mathfrak{p}$.

Solution: Let ϕ_A, ϕ_B be the localization maps. Using $ii)$, we get
$\phi_B^*(\operatorname{Spec} B_\mathfrak{p}) = f^{*\ -1}(\phi_A^*(\operatorname{Spec} A_\mathfrak{p}))$. Then using $iii)$, we set $A' = A_\mathfrak{p}$, $\mathfrak{a} = \mathfrak{p}_\mathfrak{p}$, $B' = B_\mathfrak{p}$, to conclude that $\overline{f}^*$ is the restriction of f^* to $\operatorname{Spec}(B_{\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}})$. Note that $\operatorname{Spec} A_\mathfrak{p}$ = the set of prime ideals contained in $\mathfrak{p}$ by Proposition 3.12. Then the inclusion diagram below commutes by $ii), iii)$

$$\begin{tikzcd}
\operatorname{Spec} B_\mathfrak{p}/\mathfrak{p} \arrow[r, "(\overline{f}_\mathfrak{p})^*"] \arrow[d, hook] & \operatorname{Spec} A_\mathfrak{p}/\mathfrak{p} \arrow[d, hook] \\
\operatorname{Spec} B_\mathfrak{p} \arrow[r, "f_\mathfrak{p}^*"] \arrow[d, hook] & \operatorname{Spec} A_\mathfrak{p} \arrow[d, hook] \\
\operatorname{Spec} B \arrow[r, "f^*"] & \operatorname{Spec} A
\end{tikzcd}$$

This then gives us that $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}) = f^{-1}(\mathfrak{p})$ by commuting down in the right column and seeing that (0) in $k(\mathfrak{p})$ corresponds to prime ideal in $A_\mathfrak{p}$ that contain $\mathfrak{p}$, which correspond to prime ideals in A that are contained in $\mathfrak{p}$, i.e. only $\mathfrak{p}$.
Then $B_\mathfrak{p}/\mathfrak{p} = (A/\mathfrak{p} \otimes_A B)_\mathfrak{p}$ by exercise 2, Chapter 2 and commutativity of localization and tensor. Finally, $k(\mathfrak{p}) \otimes_A B = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ because (thanks https://math.sci.uwo.ca/~jcarlso6/intro_comm_alg(2019).pdf for this)

$$\begin{aligned}
k(\mathfrak{p}) \otimes_A B = (A/\mathfrak{p})_{(0)} \otimes_A B &= (A/\mathfrak{p})_{(0)} \otimes_{A/\mathfrak{p}} A/\mathfrak{p} \otimes_A B && \text{(Proposition 3.5)} \\
&= (A/\mathfrak{p})_{(0)} \otimes_{A/\mathfrak{p}} B/\mathfrak{p} && \text{(Exercise 2, Chapter 2)} \\
&= (B/\mathfrak{p})_{(0)} && \text{(Proposition 3.5)} \\
&= B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}.
\end{aligned}$$

$\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$ is called the fiber of f^* over $\mathfrak{p}$.

22. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then the canonical image of $\operatorname{Spec}(A_\mathfrak{p})$ in $\operatorname{Spec}(A)$ is equal to the intersection of all the open neighborhoods of $\mathfrak{p}$ in $\operatorname{Spec}(A)$.

Solution: $\mathrm{Spec}(A_{\mathfrak{p}})$ = set of prime ideals contained in $\mathfrak{p}$ by Proposition 3.11. Every open neighborhood of $\mathfrak{p}$ in $\mathrm{Spec}(A)$ is the complement of $V(\cdot)$ that doesn't contain $\mathfrak{p}$. Thus $\cdot$ can't contain any prime ideal contained in $\mathfrak{p}$, as otherwise $V(\cdot)$ would contain $\mathfrak{p}$. Let $\{N_i\}$ be the open neighborhoods of $\mathfrak{p}$. Then $\cap N_i = (\cup N_i^C)^C$. As none of the N_i^C contain prime ideals contained in $\mathfrak{p}$ by above, their union doesn't either. Hence the complement does. So $\mathrm{Spec}(A_{\mathfrak{p}}) \subseteq \cap N_i$.

Now suppose we had some element in $\mathfrak{a} \in \cap N_i$ that wasn't in $\mathrm{Spec}(A_{\mathfrak{p}})$. Then $\mathfrak{a}$ isn't contained in $\mathfrak{p}$ As such, $\mathfrak{a}$ then meets $A \setminus \mathfrak{p}$, so $\exists \ell \in \mathfrak{a} \setminus \mathfrak{p}$. But then $\mathfrak{p} \in X_\ell$ and $\mathfrak{a}$ is not. Thus we can cut out out non-elements of $\mathrm{Spec}(A_{\mathfrak{p}})$).

23. Let A be a ring, let $X = \mathrm{Spec}(A)$ and let U be a basic open set in X (i.e., $U = X_f$ for some $f \in A$: Chapter 1, Exercise 17).

   a. If $U' = X_g$ be another basic open set such that $U' \subseteq U$. Show that there is an equation of the form $g^n = uf$ for some integer $n > 0$ and some $u \in A$, and use this to define a homomorphism $\rho : A(U) \to A(U')$ (i.e., $A_f \to A_g$) by mapping a/f^m to au^m/g^{mn}. Show that ρ depends only on U and U'. This homomorphism is called the restriction homomorphism.

Solution: Let $U = X_g$. Then Because $V(g) = X_g^C = X_f^C = V(f)$, f and g generate the same ideal. Hence there is $a \in A$ s.t. $g = fa$ and $a' \in A$ s.t. $f = a'g$. So

$$A_f \cong A_g$$

because $A_f \subseteq A_g$ by canceling out the a term in elements of A_g and vice versa.

   b. Let $U' = X_g$ be another basic open set such that $U' \subseteq U$. Show that there is an equation of the form $g^n = uf$ for some integer $n > 0$ and some $u \in A$, and use this to define a homomorphism $\rho : A(U) \to A(U')$ (i.e. $A_f \to A_g$) by mapping $\frac{a}{f^m}$ to $\frac{au^m}{g^{mn}}$. Show that ρ depends only on U and U'. This homomorphism is called the restriction homomorphism.

Solution: To show the existence of an n, we have

$$\begin{aligned} X_g \subseteq X_f &\iff V(g)^C \subseteq V(f)^C \\ &\iff V(g) \supseteq V(f) \\ &\iff I(V(g)) \subseteq I(V(f)) \\ &\iff \sqrt{g} \subseteq \sqrt{f} && \text{(Nullstellensatz)} \\ &\implies (g) \subseteq \sqrt{f}. \end{aligned}$$

Hence $g \in \sqrt{f} \implies \exists n$ s.t. $g^n \in (f) \implies g^n = fu$.
This is a homomorphism because $\rho((1:1)) = (1u^0 : g^0) = (1:1)$, $(au^m : g^{mn})(bu^{n'} : g^{n'n}) = g^{\rho}((a:f^m)(b:f^{n'})) = \rho((ab : f^{m+n'})) = (abu^{m+n'} : g^{mn+nn'}) = (au^m : g^{mn})(bu^{n'} : g^{nn'})$, and $\rho((a:f^m)+(b:f^{n'})) = \rho((af^{n'}+bf^m : f^{m+n'}) = ((af^{n'}+bf^m)u^{m+n'} : g^{(m+n')n}) = (af^{n'}u^{m+n'} : g^{mn+nn'}) + (bf^m u^{m+n'} : g^{mn+nn'}) = (ag^{n'n}u^m : g^{mn+nn'}) + (bg^{mn}u^{n'} : g^{mn+nn'}) = (au^m : g^{mn}) + (bu^{n'} : g^{nn'}) = \rho(a:f^m) + \rho(b:f^{n'})$.
Suppose we had other choices of f, g being f', g' respectively. Then we have a, b s.t. $f = f'b$ and $gc = g'$. As $g^n = uf$, $(g')^n = g^n c^n = ufbc^n = uf'c^n$. Then $\rho' : (a : (f')^m) = (a(uc^n)^m : (g')^{mn}) = (a(uc^n)^m : (gc)^{mn}) = (au : g^{mn})$. This is the same as ρ.

   c. If $U = U'$, then ρ is the identity map.

Solution: If $U = U'$, then we can WLOG let $U = X_f = U'$. Hence $n = 1, u = 1 \to \rho(\frac{a}{f^m}) = \frac{a}{f^m}$, which induces the identity map.

d. If $U \supseteq U' \supseteq U''$ are basic open sets in X, show that the diagram

$$\begin{tikzcd} A(U) \arrow[rr] \arrow[dr] & & A(U'') \\ & A(U') \arrow[ur] & \end{tikzcd}$$

(in which the arrows are restriction homomorphisms) is commutative.

Solution: By composing $\rho_{UU'}$ and $\rho_{U'U''}$, we get that $\frac{a}{f}$ gets mapped to $\frac{a(u_{UU'}u_{U'U''})^m}{(g_{UU'}g_{U'U''})^m}$. Since the subscript doesn't depend on choice, we can let $g_{UU''} = g_{UU'}g_{U'U''}$ and $u_{UU''} = u_{UU'}u_{U'U''}$. This obviously commutes.

e. Let x $(= \mathfrak{p})$ be a point of X. Show that

$$\varinjlim_{U \ni x} A(U) \cong A_{\mathfrak{p}}.$$

Solution: For a U with $x \in U$, $A(U) \subseteq A_{\mathfrak{p}}$ because $V(f) = U^C$ implies that $(f) \not\subseteq \mathfrak{p}$. As such, we have the map $A(U) \to A_{\mathfrak{p}}$ defined by $(a : f^m) \mapsto (a : f^m)$ (f^m is in $A \setminus \mathfrak{p}$). Then if we have a collection of maps $\{f_U : A(U) \to P\}$, we can show that this induces a unique homomorphism $A_{\mathfrak{p}} \to P$ that makes the diagram commute.
If we have a map as such, then we can define a map $A_{\mathfrak{p}} \to P$ by taking an element $(a : b) \in A_{\mathfrak{p}}$ and mapping it to the image of $(a : b)$ of a map f_{X_b}. This is well-defined because there were no choices. It commutes properly because given $\rho : A(U) \to A(U')$, an element $(a : u) \in A(U)$ gets mapped to ρ_{U,X_u} and then to $f_{X_u}(a : u)$. Because the family $\{f_U\}$ commuted a priori, $f_{X_u}(a : u) = f_u(a : u) = f_{U'}\rho(a : u)$. So we have the diagram

$$\begin{tikzcd} & & & A(X_u) \arrow[dll] \\ A(U) \arrow[urrr] \arrow[rr] \arrow[dr] \arrow[ddr] & & A(U') \arrow[ddl] & \\ & A_{\mathfrak{p}} \arrow[d] & & \\ & P & & \end{tikzcd}$$

commutes. Finally, to show that $A(U') \to A_{\mathfrak{p}} \to P$ commutes as well, draw this diagram (let b' be the denominator of $\rho(a : u)$)

$$\begin{tikzcd} & & & A(X_{b'}) \arrow[dll] \\ A(U) \arrow[urrr] \arrow[rr] \arrow[ddr] & & A(U') \arrow[dl] \arrow[ddl] & \\ & A_{\mathfrak{p}} \arrow[d] & & \\ & P & & \end{tikzcd}$$

that commutes for similar reason to above. Combine the two to get what is needed.

The assignment of the ring $A(U)$ to each basic open set U of X and the restriction homomorphisms ρ, satisfying the conditions $iii)$ and $iv)$ above, constitutes a presheaf or rings on the basis of open sets $(X_f)_{f\in A}$. $v)$ says that the stalk of this presheaf at $x \in X$ is the corresponding local ring $A_{\mathfrak{p}}$.

24. Show that the presheaf of Exercise 23 has the following property. Let $(U_i)_{i\in I}$ be a covering of X by basic open sets. For each $i \in I$, let $s_i \in A(U_i)$ be such that, for each pair of indices i, j, the images of s_i and s_j in $A(U_i \cap U_j)$ are equal. Then there exists a unique $s \in A$ $(= A(X))$ whose image in $A(U_i)$ is s_i, for all $i \in I$. (This essentially implies that the presheaf is a sheaf).

Solution: I tried a really long proof and I don't think it works at the last step. I'm too depressed right now to finish the proof, so just look at https://math.sci.uwo.ca/~jcarlso6/intro_comm_alg(2019).pdf.

25. Let $f : A \to B$, $g : A \to C$ be ring homomorphisms and let $h : A \to B \otimes_A C$ be defined by $h(x) = f(x) \otimes g(x)$. Let X, Y, Z, T be the prime spectra of $A, B, C, B \otimes_A C$ respectively. Then $h^*(T) = f^*Y \cap g^*(Z)$.

Solution: Consider $h^{*\ -1}(\mathfrak{p})$ for $\mathfrak{p} \in X$. Let $k = k(\mathfrak{p})$. By Exercise 21 iv, this is $\operatorname{Spec}(k \otimes_A (B \otimes_A C))$. By Proposition 2.14, this is $\operatorname{Spec}(k \otimes_k k \otimes_A (B \otimes_A C))$. Further use of this Proposition gives us that this is equal to $\operatorname{Spec}((B \otimes_A k) \otimes_k (C \otimes_A k)$.

Finally, if we have $\mathfrak{p} \in h^*(T)$, then $h^{*\ -1}(\mathfrak{p})$ is non-empty. As such (in an iff), $(B \otimes_A k) \otimes_k (C \otimes_A k) \neq 0 \iff B \otimes_A k \neq 0$ and $C \otimes_A k \neq 0$. As these are $f^{*\ -1}(\mathfrak{p})$ and $g^{*\ -1}(\mathfrak{p})$ respectively, $\mathfrak{p} \in f^*(Y) \cap g^*(Z)$. The reverse direction holds.

26. Let $(B_\alpha, g_{\alpha\beta})$ be a direct system of rings and B the direct limit. For each α, let $f_\alpha : A \to B_\alpha$ be a ring homomorphism such that $g_{\alpha\beta} \circ f_\alpha = f_\beta$ whenever $\alpha \le \beta$ (i.e. the B_α form a direct system of A-algebras). The f_α induce $f : A \to B$. Show that

$$f^*(\operatorname{Spec}(B)) = \bigcap_\alpha f_\alpha^*(\operatorname{Spec}(B_\alpha)).$$

Solution: Take $\mathfrak{p} \in f^*(\operatorname{Spec}(B))$. Then $f^{*\ -1}(\mathfrak{p}) \neq \emptyset \iff \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$ by Exercise 21 iv). This happens iff $k(\mathfrak{p}) \otimes_A B \neq 0 \iff B \neq 0 \iff B_\alpha \neq 0 \forall\alpha$ by Chapter 2 Exercise 21. This happens iff $\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B_\alpha) \neq 0 \iff f_\alpha^{*\ -1}(\mathfrak{p}) \neq 0 \forall\alpha$. Thus $\mathfrak{p} \in f^*(\operatorname{Spec}(B)) \iff \mathfrak{p} \in f^*(\operatorname{Spec}(B_\alpha))\forall\alpha$.

27.

a. Let $f_\alpha : A \to B_\alpha$ be any family of A-algebras and let $f : A \to B$ be their tensor product over A (Chapter 2, Exercise 23). Then

$$f^*(\operatorname{Spec}(B)) = \bigcap_\alpha f_\alpha^*(\operatorname{Spec}(B_\alpha)).$$

Solution: The tensor product over A forms a directed system of rings (let the indexing set be $\mathcal{A}$), so we can use Exercise 26 to get that

$$f^*(\mathrm{Spec}(B)) = \bigcap_{a \in \mathcal{A}} f_a^*(\mathrm{Spec}(B_a))$$

where A is the indexing set of the tensor product over A. For any $a \in A$ s.t. $f_a = \bigotimes_{i \in I \subseteq \{\alpha\}} f_i$ with I finite (this is the form of a map $A \to B_a$ in the tensor product over A). By Exercise 25, we have that $f_a^*(\mathrm{Spec}(B_a)) = \bigcap f_i^*(\mathrm{Spec}(B_i))$. As each of these composite ones are already present in the basic intersection of $\bigcap_{a \in \{\alpha\}} f_a^*(\mathrm{Spec}(B_a))$, we have that

$$f^*(\mathrm{Spec}(B)) = \bigcap_{a \in \{\alpha\}} f_a^*(\mathrm{Spec}(B_a)).$$

This is what we wanted.

b. Let $f_\alpha : A \to B$ be any finite family of A-algebras and let $B = \sqcup_\alpha B_\alpha$. Define $f : A \to B$ by $f(x) = (f_\alpha(x))$. Then $f^*(\mathrm{Spec}(B)) = \prod_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$.

Solution: I think there is a categorical proof of this, but I don't know enough about the category that $\mathrm{Spec}(-)$ would live in for us to pushout.
For any $\mathfrak{p} \in f^*(\mathrm{Spec}(B))$, $f^{*\ -1}(\mathfrak{p}) \neq \emptyset \iff \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B) \neq \emptyset \iff B \neq 0 \iff$ one of $B_\alpha \neq 0$ (Exercise 21 spam). This is iff $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A B_\alpha) \neq \emptyset \iff f_\alpha^{*\ -1}(\mathfrak{p}) \neq \emptyset \implies \mathfrak{p} \in f_\alpha^*(\mathrm{Spec}(B_\alpha)) \iff \mathfrak{p} \in \bigcup_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$.

c. Hence the subsets of $X = \mathrm{Spec}(A)$ of the form $f^*(\mathrm{Spec}(B))$, where $f : A \to B$ if a ring homomorphism, satisfy the axioms for closed sets in a topological space. The associated topology is the constructible topology on X. It is finer than the Zariski topology (i.e., there are more open sets, or equivalently more closed sets).

Solution: We can see the basis for the Zariski topology are closed in the constructible topology because for any $V(I)$ with I an ideal (suffices by Chapter 1 Exercise 15), we have a map $A \to A/I$, who's spectrum contract to primes that contain I.

d. Let X_C denote the set X endowed with the constructible topology. Show that X_C is quasi-compact.

Solution: Take an open cover of X_C $\mathcal{U} = \{U_k\}$ with $U_k^C = f_k^*(\mathrm{Spec}\, B_k)$. Take the tensor product over $f_k : A \to B_k$. Then by Exercise 27, with B the limit,

$$f^*(\mathrm{Spec}(B)) = \bigcap_\alpha f_k^*(\mathrm{Spec}(B_k)) = \emptyset.$$

Thus $\mathrm{Spec}(B) = \emptyset$, and hence B is 0. By Chapter 2 Exercise 21, one of $\bigotimes_I B_i = 0$ where I is finite.
Finally, take the subsystem of $\bigotimes_{I' \subseteq I} B_i$. The direct limit of this is $\bigotimes_I B_i$ because this equals the limit (which is 0 by Chapter 2 Exercise 21). Let $g : A \to \bigotimes_I B_i$. By Exercise 27, $g^*(\mathrm{Spec}(B)) = \cap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$ with $\{alpha\} \subseteq I$. As $g^*(\mathrm{Spec}(B)) = \emptyset = \cap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$, so $f_\alpha^*(\mathrm{Spec}(B_\alpha))^C$ forms an open cover of X.

28. (Continuation of Exercise 27.)

    a. For each $g \in A$, the set X_g (Chapter 1, Exercise 17) is both open and closed in the constructible topology.

    > **Solution:** The openness of X_g is inherited from the Zariski topology. It being closed is because $X_g \cong \mathrm{Spec}(A_g)$, as the preimage of $\mathrm{Spec}(A_g)$ are the prime ideals that don't meet (g) (Proposition 3.11), which is what X_g is.

    b. Let C' denote the smallest topology on X for which the set X_g are both open and closed, and let $X_{C'}$ denote the set X endowed with this topology. Show that $X_{C'}$ is Hausdorff.

    > **Solution:** We can see that $\forall \mathfrak{p} \in X_{C'}$, $\mathfrak{p} \cong \mathrm{Spec}(k(\mathfrak{p}))$ because $k(\mathfrak{p})$ is a field and hence only has the prime (0), which preimages to $\mathfrak{p}$ (Proposition 3.11 needed). Thus $\mathfrak{p}$ is closed.

    c. Deduce that the identity mapping $X_C \to X_{C'}$ is homeomorphism. Hence a subset E of X is of the form $f^*(\mathrm{Spec}(B))$ for some $f : A \to B$ if and only if it is closed in the topology C'.

    > **Solution:** Proof from `https://math.sci.uwo.ca/~jcarlso6/intro_comm_alg(2019).pdf`
    > By 28i), and definition of C', $C \supseteq C'$. So $\mathrm{id} : X_C \to X_{C'}$ is continuous. Obviously the identity is bijective. Then because any continuous bijection between Hausdorff, compact spaces are homeos, we are done.
    > The image of an open set U has a closed complement, which is then compact by compactness of C. It thus has compact image, which is then closed by Hausdorffness, which is then the complement of $\mathrm{id}(U)$, showing that $\mathrm{id}(U)$ is closed in $X_{C'}$.

    d. The topological space X_C is compact, Hausdorff and totally disconnected.

    > **Solution:** X_C is compact by 27iv. It is Hausdorff by 28ii,iii. It is totally disconnected because any non-trivial subset has a disconnection by taking a point p in it and intersecting X_p with the subset. X_p is clopen, so we are done.

29. Let $f : A \to B$ be a ring homomorphism. Show that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a continuous closed mapping (i.e., maps closed sets to closed sets) for the constructible topology.

    > **Solution:** The image of closed sets is closed because a subset S of $\mathrm{Spec}(B)$ that is the contraction of $\mathrm{Spec}(C)$ along a map $c : B \to C$ makes $f^*(S) = f^{-1}c^{-1}(\mathrm{Spec}(C)) = (c \circ f)^{-1}(\mathrm{Spec}(C)$ and $c \circ f$ is a map $A \to C$, making this closed in the constructible topology.
    >
    > To show continuity, it suffices to show that the preimage of X_g is open, as it is a basis of $C' = C$ by 28ii,iii. As f^* is continuous on these sets by Chapter 1 Exercise 21, we are done.

30. Show that the Zariski topology and the constructible topology on $\mathrm{Spec}(A)$ are the same if and only if $A/\mathfrak{R}$ is absolutely flat (where $\mathfrak{R}$ is the nilradical of A).

Solution: If they are the same, then the Zariski topology is Hausdorff. This implies that $A/\mathfrak{R}$ is absolutely flat by Exercise 11.

If $A/\mathfrak{R}$ is absolutely flat, then by Exercise 11 the Zariski topology totally disconnected, and Hausdorff. By total disconnectedness and Hausdorffness, we have that every set of two points is disconnected. But the only disconnection of it has to be by singletons, which are both closed, implying the other is open. Thus the basic open sets are open and closed, and because X_C is homeomorphic to $X_{C'}$, which is the smallest topology generated by the basic sets being open and closed, the Zariski topology contains C' (Exercise 28). But as X_C is finer than the Zariski topology, the Zariski topology is contained by C'. Hence the two are the same, and hence the topologies are the same.

3 Primary Decomposition

Exercise 3.0.1. If an ideal $\mathfrak{a}$ has a primary decomposition, then $\operatorname{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components.

Proof. By the First Uniqueness Theorem, the associated primes to $\mathfrak{a}$ is independent of the decomposition. As $\mathfrak{a}$ being decomposable implies that $\mathfrak{a}$ is a finite intersection, the number of associated primes must be finite. Finally, because irreducible components of $\operatorname{Spec}(A/\mathfrak{a})$ correspond to minimal primes belonging to $\mathfrak{a}$ by the remark after Prop 4.6, this implies that there are a finite number of irreducible components. $\square$

Exercise 3.0.2. If $\mathfrak{a} = r(\mathfrak{a})$, then $\mathfrak{a}$ has no embedded prime ideals.

Proof. I'm going to assume that $\mathfrak{a}$ is decomposable, because otherwise the definition in the chapter doesn't make sense. Thus decompose $\mathfrak{a}$ as a minimal primary decomposition $\cap^n \mathfrak{q}_i$ and let $r(\mathfrak{q}_i) = \mathfrak{p}_i$. Then $\mathfrak{a} = r(\mathfrak{a}) = r(\cap^n \mathfrak{q}_i) = \cap^n r(\mathfrak{q}_i) = \cap^n \mathfrak{p}_i$. If there were any embedded prime ideals among the $\mathfrak{p}_i$, then we could eliminate the term and see that $\cap^n r(\mathfrak{q}_i)$ is not minimal, a contradiction. Thus there are no embedded primes. $\square$

Exercise 3.0.3. If A is absolutely flat, every primary ideal is maximal.

Proof. Let $\mathfrak{q}$ be a primary ideal. Our strategy will be to show that $A/\mathfrak{q}$ is a field. Then $A/\mathfrak{q}$ is absolutely flat by Exercise 2.28 (homomorphic image).

By Exercise 2.28, all non-units are zero divisors, and because $\mathfrak{q}$ is primary, all zero-divisors are nilpotents. By Exercise 1.10, a ring where every element is either nilpotent or a unit is local. By Exercise 2.28, this implies that $A/\mathfrak{q}$ is a field. $\square$

Exercise 3.0.4. In the polynomial ring $\boldsymbol{Z}[t]$, the ideal $\mathfrak{m} = (2,t)$ is maximal and the ideal $\mathfrak{q} = (4,t)$ is $\mathfrak{m}$-primary, but is not a power of $\mathfrak{m}$.

Proof. First we can see that $\mathfrak{m}$ is maximal because $\boldsymbol{Z}[t]/\mathfrak{m} \cong \boldsymbol{Z}_2$, which is a field. Then, $\mathfrak{q}$ is $\mathfrak{m}$-primary because

(1) it is a primary ideal for if $ab = 0 \in \boldsymbol{Z}[t]/\mathfrak{q}$, then because $\boldsymbol{Z}[t]/\mathfrak{q} \cong \boldsymbol{Z}_4$, we either have one of the terms equivalent to 4 in $\boldsymbol{Z}_4$ or both terms are equivalent to 2. The former case is $ab \in \mathfrak{q} \implies a \in \mathfrak{q}$ and the latter is when $a \notin \mathfrak{q} \implies b^n \in \mathfrak{q}$, with $n = 2$ here.

(2) to check that $\sqrt{\mathfrak{q}} = \mathfrak{m}$, it suffices to show that $(\boldsymbol{Z}[t]/\mathfrak{q})/\mathfrak{N} \cong \boldsymbol{Z}[t]/\mathfrak{m}$ with $\mathfrak{N}$ the nilradical of $\boldsymbol{Z}[t]/\mathfrak{q}$. We can see that the nilradical of $\boldsymbol{Z}[t]/\mathfrak{q} \cong \boldsymbol{Z}_4$ is isomorphic to the ideal (2) in $\boldsymbol{Z}_4$, and $(\boldsymbol{Z}[t]/\mathfrak{q})/\mathfrak{N} \cong \boldsymbol{Z}_4/(2) \cong \boldsymbol{Z}_2 \cong \boldsymbol{Z}[t]/\mathfrak{m}$.

Finally, suppose FTSOC that $(4,t) = (2,t)^n$. If $n > 2$, $4 \notin (2,t)$. Thus $n = 2$ as $n = 1$ is obviously eliminated. But $(2,t)^2 = (4, 2t, t^2)$, which doesn't contain t. $\square$

Exercise 3.0.5. In the polynomial ring $K[x,y,z]$ where K is a field and x,y,z are independent indeterminates, let $\mathfrak{p}_1 = (x,y)$, $\mathfrak{p}_2 = (x,z)$, $\mathfrak{m} = (x,y,z)$; $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, and $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Which components are isolated and which are embedded?

Proof. Obviously $\mathfrak{a} \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ as $\mathfrak{a} = (x^2, xz, yx, yz)$. Now suppose we have $a \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Then because $a \in \mathfrak{p}_1$, $a = xp(x,y,z) + yq(x,y,z)$. Because $a \in \mathfrak{p}_2$ and $xp(x,y,z) \in \mathfrak{p}_2$ and y is prime in $K[x,y,z]$, we must have that either $x|q(x,y,z)$ or $z|q(x,y,z)$. In either case, x^2 or $yz|yq(x,y,z)$, so that term is in $\mathfrak{p}_1\mathfrak{p}_2$.

Thus all we need to show now is that $xp(x,y,z) \in \mathfrak{p}_1\mathfrak{p}_2$. From the above, we can also conclude that $yq(x,y,z) \in \mathfrak{m}^2$. Thus we know that $xp(x,y,z) \in \mathfrak{m}^2$, hence either x, y, z divides p. In all three cases, this puts xp in $\mathfrak{p}_1\mathfrak{p}_2$. Hence $a \in \mathfrak{a}$.

Because $\sqrt{m^2} = \mathfrak{m}$ as $\mathfrak{m}$ is prime, $\mathfrak{m}$ is an associated prime. As $\mathfrak{p}_1, \mathfrak{p}_2$ are prime, they are associated primes. Clearly $\mathfrak{p}_1, \mathfrak{p}_2$ are minimal and contain $\mathfrak{m}$, so $\mathfrak{p}_i$ are isolated and $\mathfrak{m}$ is embedded. □

Exercise 3.0.6. Let X be an infinite compact Hausdorff space, $C(X)$ the ring of real-valued continuous functions on X (Chapter 1, Exercise 26). Is the zero ideal decomposable in this ring?

Proof. No. Recall that by Exercise 1.16 that every maximal ideal is of the form $\mathfrak{m}_x = \{f \in C(X) | f(x) = 0\}$. First we show that every primary ideal is contained in exactly one maximal ideal. Suppose we have primary $\mathfrak{p} \subseteq \mathfrak{m}_x \cap \mathfrak{m}_y$.

Because X is Hausdorff, there is open disjoint neighborhoods U, V such that $x \in U$ and $y \in V$. By Urysohn's Lemma, we have $f_x, f_y \in C(X)$ such that $f_x(x) = 1, f_x(U^c) = 0, f_y(y) = 1, f_y(V^c) = 0$. Then $f_x f_y = 0$ because f_x is non-zero on U and f_y is non-zero on V, so their product is non-zero on $U \cap V = \emptyset$. Because $f_x(x) = 1$ and $f_y(y) = 1$, neither are in $\mathfrak{p}$. But this contradicts $\mathfrak{p}$ being primary.

Now suppose FTSOC that $(0) = \cap^n \mathfrak{q}_i$. Let $\mathfrak{q}_i \subseteq \mathfrak{m}_{x_i}$. Now take a point $x \notin \{x_i\}$. By Urysohn's Lemma, there is δ_i that vanishes on x_i and not x. Let f_i be the product of an element in $\mathfrak{q}_i$ that doesn't vanish on x (because $\mathfrak{q}_i \not\subseteq \mathfrak{m}_x$) with δ_i. Then $\prod f_i \in \cap \mathfrak{q}_i$ but isn't 0 as it doesn't vanish on x. This contradicts $(0) = \cap^n \mathfrak{q}_i$. □

Exercise 3.0.7. Let A be a ring an let $A[x]$ denote the ring of polynomials in one indeterminate over A. For each ideal $\mathfrak{a}$ of A, let $\mathfrak{a}[x]$ denote the set of all polynomials in $A[x]$ with coefficients in $\mathfrak{a}$.

a. $\mathfrak{a}[x]$ is the extension of $\mathfrak{a}$ to $A[x]$.

b. If $\mathfrak{p}$ is a prime ideal in A, then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

c. If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$.

d. If $\mathfrak{a} = \cap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition in A, then $\mathfrak{a}[x] = \cap_{i=1}^n \mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.

e. If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.

Proof. i) The image of $\mathfrak{a}$ in $A[x]$ is $\mathfrak{a}$. The ideal generated by it includes ax^n for $a \in \mathfrak{a}$ and natural n. Thus all polynomials with coefficients in $\mathfrak{a}$ are in the ideal. Finally, all elements of the ideal generated by $\mathfrak{a}$ in $A[x]$ have coefficients in $\mathfrak{a}$ by definition (finite sums of products of elements of $A[x]$ with elements of $\mathfrak{a}$).

ii) We have that $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$ because each element of $A[x]/\mathfrak{p}[x]$ is a polynomial with coefficients in $A/\mathfrak{p}$. A polynomial ring over an integral domain $(A/\mathfrak{p})[x]$ is an integral domain.

iii) Consider $A[x]/\mathfrak{q}[x]$. By the above, this is isomorphic to $(A/\mathfrak{q})[x]$. Now suppose we have a zero divisor in this ring, say $fg = 0$. Then by chapter 1 exercise 2, we have $a \in A/\mathfrak{q}$ such that $af = 0$. Thus every coefficient of f is a zero divisor in $A/\mathfrak{q}$. Therefore every coefficient is nilpotent, as $\mathfrak{q}$ is primary. By Chapter 1 Exercise 2 again, this makes f nilpotent.

Next we can see that $\mathfrak{p}[x] \subseteq r(\mathfrak{q}[x])$ because $\mathfrak{q} \subseteq \mathfrak{q}[x]$ and $r(\mathfrak{q}[x])$ is an ideal in $A[x]$. Finally, we can note that $\mathfrak{q}[x] \subseteq \mathfrak{p}[x]$ because $\mathfrak{q} \subseteq \mathfrak{p}$, and as $r(\mathfrak{q}[x])$ is the smallest prime ideal containing $\mathfrak{q}[x]$, it must equal $\mathfrak{p}[x]$.

Lemma 3.1. $\mathfrak{p} \subseteq \mathfrak{q} \iff \mathfrak{p}[x] \subseteq \mathfrak{q}[x]$

Proof. If $\mathfrak{p} \subseteq \mathfrak{q}$, any polynomial with coefficients in $\mathfrak{p}$ have coefficients in $\mathfrak{q}$.

If $\mathfrak{p}[x] \subseteq \mathfrak{q}[x]$, then $\mathfrak{p} \subseteq \mathfrak{p}[x]$ gives us that $\mathfrak{p} \subseteq \mathfrak{q}$ ($\mathfrak{q}$ is the degree zero component). □

iv) We can see that $\cap \mathfrak{q}_i[x] = (\cap \mathfrak{q}_i)[x]$ because a polynomial in all $\mathfrak{q}_i[x]$ has coefficients in all $\mathfrak{q}_i$. By iii), all we need to show is that $\cap \mathfrak{q}_i[x]$ is a minimal primary decomposition. Condition i) is clearly met because the degree 0 components are distinct. The conditions for ii) are satisfied because of the above lemma and the fact that $(\mathfrak{q}_i)[x] = \cap \mathfrak{q}_i[x]$.

v) This is because of iv) and the lemma above. $\square$

Exercise 3.0.8. Let k be a field. Show that in the polynomial ring $k[x_1, \ldots, x_n]$ the ideals $\mathfrak{p}_i = (x_1, \ldots, x_i)$ $(1 \leq i \leq n)$ are prime and all their powers are primary. Use Exercise 7.

Proof. Clearly $k[x_1, \ldots, x_n]/\mathfrak{p}_i$ is prime because this is isomorphic to $k[x_{i+1}, \ldots, x_n]$, an integral domain. Let $\mathfrak{q}_i^m = \mathfrak{p}_i^m \cap k[x_1, \ldots, x_i]$. Then $\mathfrak{p}_i^m = (\mathfrak{q}_i^m)^e = \mathfrak{q}_i^m[x_{i+1}, \ldots, x_n]$. By 7iii), it then suffices to show that $\mathfrak{q}_i^m$ is $\mathfrak{q}_i$ primary (as this will then make $\mathfrak{q}_i^m[x_{i+1}, \ldots, x_n]$ primary). By discussion in the chapter, $r(\mathfrak{q}_i^m) = \mathfrak{q}_i$. So all we need to do is to show that $\mathfrak{q}_i^m$ is primary in $k[x_1, \ldots, x_i]$.

Suppose we have $ab = 0$ in $k[x_1, \ldots, x_i]/\mathfrak{q}_i^m$. We can see that a or b has no degree 0 component, because otherwise we would have a degree 0 element that can't be cancelled out ($\mathfrak{q}_i^m$ is a homogenous ideal, so we still get a direct sum decomposition of $k[x_1, \ldots, x_i]/\mathfrak{q}_i^m$). Finally, because all monomials are nilpotent in this ring, this implies that a or b is in the ideal of nilpotents, showing that all non-units are nilpotent. Hence $\mathfrak{q}_i^m$ is primary. $\square$

Exercise 3.0.9. In a ring A, let $D(A)$ denote the set of prime ideals $\mathfrak{p}$ which satisfy the following condition: there exists $a \in A$ such that $\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : a)$. Show that $x \in A$ is a zero divisor $\iff$ $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$.
Let S be a multiplicatively closed subset of A, and identify $\operatorname{Spec}(S^{-1}A)$ with its image in $\operatorname{Spec}(A)$ (Chapter 3, Exercise 21). Show that

$$D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A).$$

If the zero ideal has a primary decomposition, show that $D(A)$ is the set of associated prime ideals of 0.

Proof. Suppose that $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$. Then because $a\mathfrak{p} \subseteq (0)$, $ax = 0$ and x is a zero divisor.

If x is a zero divisor: Suppose $ax = 0$. Then $x \in (0 : a)$, and $(0 : a) \neq (1)$ because A can be assumed to be non-zero. Thus there is $\mathfrak{p} \supseteq (0 : a)$. Because the intersection of a descending chain of prime ideals is prime, we can find a minimal such prime among those containing $(0 : a)$.

$D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A)$:

$\subseteq$: The left hand side is the set of minimal prime ideals that contain some $(0 : \frac{a}{b})$. By the inclusion respecting bijection from Proposition 3.11, these correspond to minimal prime ideals in A that don't meet S. Because each $S^{-1}\mathfrak{p}$ contains $(0 : \frac{a}{b})$, the contraction contains $(0 : a)$ for if $ac = 0$ for $c \in A$, then $\frac{a}{b} \cdot \frac{c}{1} = 0$ in $S^{-1}A$.

$\supseteq$: The right hand side is the set of minimal prime ideal that doesn't contain S by Proposition 3.11 yet contain $(0 : a)$ for some a (it is minimal among these primes because the bijection $\operatorname{Spec} S^{-1}A$ to a subset of $\operatorname{Spec} A$ respects inclusions). Thus these biject to minimal prime ideals of S^{-1}, and they contain $(0 : \frac{a}{1})$ because $\forall \frac{p}{q} \in S^{-1}\mathfrak{p}$, $\frac{a}{1}\frac{p}{q} = \frac{ap}{q} = 0$ because $p \in \mathfrak{p}$ and $ap = 0$.

Finally, assume that there is a primary decomposition, say $0 = \cap \mathfrak{q}_i$ with associated primes $\mathfrak{p}_i = r(0 : a_i)$. Take $\mathfrak{p} \in D(A)$ and let $\mathfrak{p}$ be minimal among prime ideals containing $(0 : a)$. Then $\mathfrak{p} \supseteq r(0 : a)$ because $r(0 : a)$ is the intersection of prime ideals containing $(0 : a)$. Thus

$$\mathfrak{p} \supseteq r(\cap \mathfrak{q}_i : a) = \cap r(\mathfrak{q}_i : a) \supseteq r(0 : a) \supseteq (0 : a).$$

The latter containment is due to 0 being in all the $\mathfrak{q}_i$ so that $(\mathfrak{q}_i : a) \supseteq (0 : a)$.

We can see that $\cap r(\mathfrak{q}_i : a) = \cap_{\mathfrak{n}_i \in S \subseteq \{\mathfrak{p}_i\}} \mathfrak{n}_i$ where S is some subset of the associated primes. This is because for each i, $a \in \mathfrak{q}_i$ implies that $(\mathfrak{q}_i : a) = A$, allowing us to remove it from the intersection, and if $a \notin \mathfrak{q}_i$, then because $\mathfrak{q}_i$ is primary, $x \in (\mathfrak{q}_i : a) \implies ax \in \mathfrak{q}_i \implies x^a \in \mathfrak{q}_i$ for some a. Thus by taking the radical of $(\mathfrak{q}_i : a)$, we see that $x \in \mathfrak{p}_i$.

Next, by Proposition 1.11, $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. The minimality of $\mathfrak{p}$ then implies that $\mathfrak{p} = \mathfrak{p}_i$, an associated prime.

Now conversely, an associated prime $\mathfrak{p}_i = r(0 : a_i)$ is minimal among primes containing $(0 : a_i)$ by definition of the radical. $\square$

Exercise 3.0.10. For any prime ideal $\mathfrak{p}$ in a ring A, let $S_\mathfrak{p}(0)$ denote the kernel of the homomorphism $A \to A_\mathfrak{p}$. Prove that

a. $S_\mathfrak{p}(0) \subseteq \mathfrak{p}$.

b. $r(S_\mathfrak{p}(0)) = \mathfrak{p} \iff \mathfrak{p}$ is a minimal prime ideal of A.

c. If $\mathfrak{p} \supseteq \mathfrak{p}'$, then $S_\mathfrak{p}(0) \subseteq S_{\mathfrak{p}'}(0)$.

d. $\cap_{\mathfrak{p} \in D(A)} S_\mathfrak{p}(0) = 0$, where $D(A)$ is defined in Exercise 9.

Proof. i) Any $a \in S_\mathfrak{p}(0)$ has the property that $\frac{a}{1} = 0$, i.e. $\exists s \in A \setminus \mathfrak{p}$ such that $as = 0$ in A. Mapping this into $A/\mathfrak{p}$, we have that $as \equiv 0$. Because $A/\mathfrak{p}$ is an integral domain and $s \notin \mathfrak{p}$, $a \equiv 0 \implies a \in \mathfrak{p}$.

iii) Let $a \in S_\mathfrak{p}(0)$. Then there is $s \in A \setminus \mathfrak{p} \subseteq A \setminus \mathfrak{p}'$ such that $as = 0$ in A. But then $\frac{a}{1} = 0 \in A_{\mathfrak{p}'}$ because $A \setminus \mathfrak{p} \subseteq A \setminus \mathfrak{p}'$, so we can use the same element to prove that it vanishes.

ii)

$\implies$) Suppose we have $\mathfrak{q} \subseteq \mathfrak{p}$. Then by iii) $S_\mathfrak{p}(0) \subseteq S_\mathfrak{q}(0)$. So we then have

$$S_\mathfrak{p}(0) \subseteq S_\mathfrak{q}(0) \subseteq \mathfrak{q} \subseteq \mathfrak{p}.$$

But because $r(S_\mathfrak{p}(0)) = \mathfrak{p}$ and the radical is the intersection of the prime ideals containing it, this implies that $\mathfrak{p} \subseteq \mathfrak{q}$. Thus $\mathfrak{p} = \mathfrak{q}$ and $\mathfrak{p}$ is minimal.

$\Leftarrow$) Because $\mathfrak{p}$ is minimal, $S \setminus \mathfrak{p}$ is maximal among multiplicatively closed subsets of A that don't contain 0. Thus for any $x \in \mathfrak{p}$, $0 \in$ the multiplicatively closed subset spanned by $S \cup \{x\}$. Hence there is $s \in S$ and n such that $sx^n = 0$. Thus $x \in r(0 : s)$. But $S_\mathfrak{p}(0) = \cup_{s \in A\setminus\mathfrak{p}}(0 : s)$ by Proposition 3.11. So by the remark on page 9, $r(S_\mathfrak{p}(0)) = \cup r(0 : s) \implies x \in r(S_\mathfrak{p}(0))$. As x was arbitrary, $\mathfrak{p} \subseteq r(S_\mathfrak{p}(0))$, which by exercise i) gives us that $\mathfrak{p} \subseteq r(S_\mathfrak{p}(0)) \subseteq r(\mathfrak{p}) = \mathfrak{p}$. Thus $r(S_\mathfrak{p}(0)) = \mathfrak{p}$.

iv) Obviously 0 is in the intersection. Now suppose we have $x \neq 0$. Then $(0 : x) \neq (1)$, so we can take a minimal prime ideal $\mathfrak{q}$ containing $(0 : x)$. This is in $D(A)$ by definition. Because $(0 : x) \subseteq \mathfrak{q}$, there is no $s \in A \setminus \mathfrak{q}$ such that $xs = 0$ by definition of $(0 : x)$. Thus $x \notin S_\mathfrak{q}(0)$. Hence $x \notin \cap S_\mathfrak{p}(0)$. □

Exercise 3.0.11. If $\mathfrak{p}$ is a minimal prime ideal of a ring A, show that $S_\mathfrak{p}(0)$ (Exercise 10) is the smallest $\mathfrak{p}$-primary ideal.
Let $\mathfrak{a}$ be the intersection of the ideals $S_\mathfrak{p}(0)$ as $\mathfrak{p}$ runs through the minimal prime ideals of A. Show that $\mathfrak{a}$ is contained in the nilradical of A.
Suppose that the zero ideal is decomposable. Prove that $\mathfrak{a} = 0$ if and only if every prime ideal of 0 is isolated.

Proof. If $\mathfrak{p}$ is a minimal prime, then $r(S_\mathfrak{p}(0)) = \mathfrak{p}$ by Exercise 4.10ii. Further, $S_\mathfrak{p}(0)$ is a primary ideal for if we have $ab \in S_\mathfrak{p}(0)$, then there is $c \in A \setminus \mathfrak{p}$ such that $abc = 0$. As $0 \in \mathfrak{p}$ and $\mathfrak{p}$ is prime, $c \notin \mathfrak{p} \implies ab \in \mathfrak{p}$. Thus either a or b are in $\mathfrak{p} = r(S_\mathfrak{p}(0))$. Finally, $S_\mathfrak{p}(0)$ is the smallest $\mathfrak{p}$-primary ideal because given a $\mathfrak{p}$-primary ideal I, $I \subseteq r(I) = \mathfrak{p}$. Thus $A \setminus \mathfrak{p} \cap I = \emptyset$, so $I_\mathfrak{p}$ is $\mathfrak{p}_\mathfrak{p}$-primary by Proposition 4.8, and the contraction of $I_\mathfrak{p}$ is I. But the contraction contains $S_\mathfrak{p}(0)$.

Because $S_\mathfrak{p}(0) \subseteq r(S_\mathfrak{p}(0))$, we have that $\mathfrak{a} \subseteq r(\mathfrak{a}) = \cap r(S_{\mathfrak{p}_i}(0))$ where the $\mathfrak{p}_i$ range over minimal prime ideals. But by Exercise 4.10ii, $r(S_{\mathfrak{p}_i}(0)) = \mathfrak{p}_i$. Thus $\mathfrak{a} \subseteq \cap \mathfrak{p}_i$. The RHS equals the nilradical because the nilradical is the intersection of all prime ideals, and we can just remove non-minimal ones.

Assume that the zero ideal is decomposable into $\cap \mathfrak{q}_i$ with $r(\mathfrak{q}_i) = \mathfrak{p}_i$.

Assume every prime ideal of 0 is isolated. Then no By Exercise 4.9, $D(A)$ is the set of associated primes of 0, which by assumption are isolated. So by Exercise 4.10iv, $\cap_{\mathfrak{q} \in D(A)} S_\mathfrak{q}(0) = 0$.

Suppose that $\mathfrak{a} = 0$. For each minimal prime $\mathfrak{p}^j$, let $\{\mathfrak{p}_{j,i}\}$ meet $A \setminus \mathfrak{p}^j$. Let $\mathfrak{q}_{j,i}$ be the corresponding primary components. By Proposition 4.9, $\mathfrak{a} = \cap_j \cap_{j,i} \mathfrak{q}_{j,i}$. As this contains Then because of Exercise 4.10iii), $\cap S_\mathfrak{q}(0) \subseteq \cap S_\mathfrak{p}(0)$ where $\mathfrak{q}$ range over associated primes of 0 as each $\mathfrak{q}$ contains a minimal prime ideal. Thus $\cap S_\mathfrak{q}(0) = 0$. □

Exercise 3.0.12. Let A be a ring, S a multiplicatively closed subset of A. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the contraction of $S^{-1}\mathfrak{a}$ in A. The ideal $S(\mathfrak{a})$ is called the *saturation* of $\mathfrak{a}$ with respect to S. Prove that

a. $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$.

b. $S(r(\mathfrak{a})) = r(S(\mathfrak{a}))$.

c. $S(\mathfrak{a}) = (1) \iff \mathfrak{a}$ meets S.

d. $S_1(S_2(\mathfrak{a})) = (S_1S_2)(\mathfrak{a})$.
If $\mathfrak{a}$ has a primary decomposition, prove that the set of ideals $S(\mathfrak{a})$ (where S runs through all multiplicatively closed subsets of A) is finite.

Proof. i) Because of Exercise 1.18, $S(\mathfrak{a}) \cap S(\mathfrak{b}) = \mathfrak{a}^c \cap \mathfrak{b}^c = (\mathfrak{a} \cap \mathfrak{b})^c = S(\mathfrak{a} \cap \mathfrak{b})$.

ii) Because of Exercise 1.18, $S(r(\mathfrak{a})) = r(\mathfrak{a})^c = r(\mathfrak{a}^c) = r(S(\mathfrak{a}))$.

iii) Proposition 3.11.

iv) By Proposition 3.11ii), $S_2(\mathfrak{a}) = \mathfrak{a}^{ec} = (S_2^{-1}\mathfrak{a})^c = \cup_{s_2 \in S_2}(\mathfrak{a} : s_2)$.. Thus $S_1(S_2(\mathfrak{a})) = \cup_{s_1 \in S_1} \cup_{s_2 \in S_2} ((\mathfrak{a} : s_2) : s_1)$. By Exercise 1.12, this equals $\cup_{s_1 \in S_1} \cup_{s_2 \in S_2} (\mathfrak{a} : s_2 s_1)$. This then equals $\cup_{s \in S_1 S_2}(\mathfrak{a} : s) = ((S_1S_2)^{-1}\mathfrak{a})^c = S_1S_2(\mathfrak{a})$.

Now suppose that $\mathfrak{a}$ has a decomposition $\cap \mathfrak{q}_i$. Then by Proposition 4.9, $S(\mathfrak{a})$ is an intersection of a finite subset of the $\mathfrak{q}_i$'s. There are only finitely many possibilities. □

Exercise 3.0.13. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then *nth symbolic power of* $\mathfrak{p}$ is defined to be the ideal (in the notation of Exercise 12)

$$\mathfrak{p}^{(n)} = S_\mathfrak{p}(\mathfrak{p}^n)$$

where $S_\mathfrak{p} = A \setminus \mathfrak{p}$. Show that

a. $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal;

b. if $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component;

c. If $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is its $\mathfrak{p}$-primary component

d. $\mathfrak{p}^{(n)} = \mathfrak{p}^n \iff \mathfrak{p}^n$ is $\mathfrak{p}$-primary.

Proof. i) Suppose we have $ab \in \mathfrak{p}^{(n)}$ and $a \notin \mathfrak{p}^{(n)}$. Then by Proposition 3.11, $S_\mathfrak{p}(\mathfrak{p}^n) = \cup_{s \in A\setminus\mathfrak{p}}(\mathfrak{p}^n : s)$. Now let s be such that $ab \in (\mathfrak{p}^n : s)$. Then $abs \in \mathfrak{p}^n$.

If $bs \in \mathfrak{p}$, then because $s \notin \mathfrak{p}$, $b \in \mathfrak{p} \implies b^n \in \mathfrak{p}^{(n)}$, the requirement for $\mathfrak{p}^{(n)}$ to be primary. If $bs \notin \mathfrak{p}$, then $a \in (\mathfrak{p}^n : bs) \subseteq \cup_{s \in A\setminus\mathfrak{p}}(\mathfrak{p}^n : s)$.

Finally, because $r((S_\mathfrak{p}\mathfrak{p}^n)^c) = r(S_\mathfrak{p}\mathfrak{p}^n)^c = (S_\mathfrak{p}\mathfrak{p})^c$ because of Proposition 4.8. Then by Proposition 3.11 this equals $\mathfrak{p}$.

Alternatively, one can observe that $\mathfrak{p}$ is minimal in $A/\mathfrak{p}^n$ because $r(\mathfrak{p}^n) = \mathfrak{p}$ and $r(\mathfrak{p}^n)$ is contained in every prime ideal containing $\mathfrak{p}^n$ and every ideal of $A/\mathfrak{p}^n$ corresponds to an ideal of A containing $\mathfrak{p}^n$. Thus Exercise 1.11 applied to $A/\mathfrak{p}^n$ with the minimal prime $\mathfrak{p}$ gives us that $S_p(0)$ is the smallest $\mathfrak{p}$ primary ideal in $A/\mathfrak{p}^n$. This corresponds to the smallest $\mathfrak{p}$ primary ideal in A containing $\mathfrak{p}^n$.

ii) Take a minimal decomposition of $\mathfrak{p}^n$. Because $r(\mathfrak{p}^n : 1) = \mathfrak{p}$ and of the uniqueness theorem, there are $\mathfrak{p}$-primary ideals in the decomposition. Recall that the alternate proof above tells us that $\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$. As any primary ideal in the decomposition must contain $\mathfrak{p}^n$, $\mathfrak{p}^{(n)}$ must be contained in any $\mathfrak{p}$-primary associated ideal to $\mathfrak{p}^n$. But this contradicts minimality unless the $\mathfrak{p}$-primary associated ideal was $\mathfrak{p}^{(n)}$.

iii) Because $\mathfrak{p}^m \subseteq \mathfrak{p}^{(m)}$, $\mathfrak{p}^{m+n} \subseteq \mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$. Thus all the $\mathfrak{p}$-primary ideals in the decomposition contain $\mathfrak{p}^{m+n}$, and thus contain $\mathfrak{p}^{(m+n)}$ as it is the smallest such.

Finally, we can show that $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^{(m+n)}$, which completes the problem because we can then intersect a minimal decomposition of $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ with $\mathfrak{p}^{(m+n)}$ and collect the $\mathfrak{p}$-primary ideals together via Lemma 4.3 to get that $\mathfrak{p}^{(m+n)}$ is the $\mathfrak{p}$-primary component. Now to show this: by definition, $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)} = (\mathfrak{p}^m)^{ec}(\mathfrak{p}^n)^{ec}$. Thus by taking extensions and using Exercise 1.18 and Proposition 1.17, we have that $(\mathfrak{p}^{m+n})^e = (\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})^e$. Contracting, we get that

$$\mathfrak{p}^{(m+n)} = ((\mathfrak{p}^{(m)})^e(\mathfrak{p}^{(n)})^e))^c \supseteq (\mathfrak{p}^{(m)})^{ec}(\mathfrak{p}^{(n)})^{ec} \supseteq \mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$$

via assorted uses of Proposition 1.17 and Exercise 1.18.

iv) If $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, then we are done by part i. If $\mathfrak{p}^n$ is $\mathfrak{p}$-primary, then because $\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$ primary ideal containing $\mathfrak{p}^n$ by the alternate proof of i) and $\mathfrak{p}^n$ is a $\mathfrak{p}$-primary ideal, $\mathfrak{p}^{(n)} \subseteq \mathfrak{p}$. But clearly $\mathfrak{p}^{(n)} \supseteq \mathfrak{p}$. □

Exercise 3.0.14. Let $\mathfrak{a}$ be a decomposable ideal in a ring A and let $\mathfrak{p}$ be a maximal element of the set of ideals $(\mathfrak{a} : x)$, where $x \in A$ and $x \notin \mathfrak{a}$. Show that $\mathfrak{p}$ is a prime ideal belonging to $\mathfrak{a}$.

Proof. First we can see that $\mathfrak{p}$ is prime: suppose we have $ab \in \mathfrak{p} = (\mathfrak{a} : x)$. Because $\mathfrak{a}$ is decomposable, $(\mathfrak{a} : x) = (\cap \mathfrak{q}_i : x) = \cap(\mathfrak{q}_i : x)$. If $a \notin \mathfrak{p}$, then $(\mathfrak{p} : a) \supseteq \mathfrak{p}$. But then $(\mathfrak{p} : a) = (\cap(\mathfrak{q}_i : x) : a) = \cap((\mathfrak{q}_i : x) : a) = \cap(\mathfrak{q}_i : ax) = (\cap \mathfrak{q}_i : ax) = (\mathfrak{a} : ax)$ by Exercise 1.12 in the chapter.

If $ax \notin \mathfrak{a}$, then because $\mathfrak{p}$ is maximal $(\mathfrak{p} : a) = \mathfrak{p}$, and $b \in (\mathfrak{p} : a)$. If $ax \in \mathfrak{a}$, then by definition, $a \in (\mathfrak{a} : x)$.

Finally, $\mathfrak{p}$ is a prime ideal belonging to $\mathfrak{a}$ because it is of the right form for the Uniqueness Theorem: $\mathfrak{p} = (\mathfrak{a} : x) = r(\mathfrak{a} : x)$. □

Exercise 3.0.15. Let $\mathfrak{a}$ be a decomposable ideal in a ring A, let Σ be an isolated set of prime ideals belonging to $\mathfrak{a}$, and let $\mathfrak{q}_\Sigma$ be the intersection of the corresponding primary components. Let f be an element of A such that, for each prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$, we have $f \in \mathfrak{p} \iff \mathfrak{p} \notin \Sigma$, and let S_f be the set of all powers of f. Show that $\mathfrak{q}_\Sigma = S_f(\mathfrak{a}) = (\mathfrak{a} : f^n)$ for all large n.

Proof. First we can show that $S_f(\mathfrak{a}) = (\mathfrak{a} : f^n)$ We use Proposition 3.11 to get that $\mathfrak{a}^{ec} = \cup_n(\mathfrak{a} : f^n)$. Clearly $(\mathfrak{a} : f^n) \subseteq (\mathfrak{a} : f^{n+1})$. Thus if we show that $(\mathfrak{a} : f^n)$ stabilizes for some large enough n, we have shown this part. Now

Take some $q \in \mathfrak{q}_\Sigma$. Then $q \in \mathfrak{q}_i$ for some isolated component □

Exercise 3.0.16. If A is a ring in which every ideal has a primary decomposition, show that every ring of fractions $S^{-1}A$ has the same property.

Proof. Just Proposition 4.9 and Proposition 3.11. □

Exercise 3.0.17. Let A be a ring with the following property.

(L1) For every ideal $\mathfrak{a} \neq (1)$ in A and every prime ideal $\mathfrak{p}$, there exists $x \notin \mathfrak{p}$ such that $S_\mathfrak{p}(\mathfrak{a}) = (\mathfrak{a} : x)$, where $S_\mathfrak{p} = A \setminus \mathfrak{p}$.

Then every ideal in A is an intersection of (possibly infinitely many) primary ideals. [Let $\mathfrak{a}$ be an ideal $\neq (1)$ in A, and let $\mathfrak{p}_1$ be a minimal element of the set of prime ideals containing $\mathfrak{a}$. Then $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$ is $\mathfrak{p}_1$-primary (by Exercise 11), and $\mathfrak{q}_1 = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{p}_1$. Show that $\mathfrak{a} = \mathfrak{q}_1 \cap (\mathfrak{a} + ((x))$.

Now let $\mathfrak{a}_1$ be a maximal element of the set of ideals $\mathfrak{b} \supseteq \mathfrak{a}$ such that $\mathfrak{q}_1 \cap \mathfrak{b} = \mathfrak{a}$, and choose $\mathfrak{a}_1$ so that $x \in \mathfrak{a}_1$, and therefore $\mathfrak{a}_1 \not\subseteq \mathfrak{p}_1$. Repeat the construction starting with $\mathfrak{a}_1$ and so on. At the nth stage we have $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{a}_n$ where the $\mathfrak{q}_1$ are primary ideals, $\mathfrak{a}_n$ is maximal among the ideals $\mathfrak{b}$ containing $\mathfrak{a}_{n-1} = \mathfrak{a}_n \cap \mathfrak{q}_n$ such that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{b}$, and $\mathfrak{a}_n \not\subseteq \mathfrak{p}_n$. If at any stage we have $\mathfrak{a}_n = (1)$, the process stops, and $\mathfrak{a}$ is a finite intersection of primary ideals. If not, continue by transfinite induction, observing that each $\mathfrak{a}_n$ strictly contains $\mathfrak{a}_{n-1}$.]

Proof. □

4 Integral Dependence and Valuations

Exercise 4.0.1. Let $f\colon A \to B$ be an integral homomorphism of rings. Show that $f^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a *closed* mapping, i.e. that it maps closed sets to closed sets. (This is a geometrical equivalent of (5.10).)

Proof. We want to show that for all ideals $\mathfrak{a} \subseteq B$, $f^*(V(\mathfrak{a})) = V(\mathfrak{b})$ for some ideal $\mathfrak{b} \subseteq A$. I propose that we let $\mathfrak{b} = f^{-1}(\mathfrak{a})$ (which is an ideal because preimages take ideals to ideals).

$\subseteq$: For $\mathfrak{p} \in V(\mathfrak{a})$, $\mathfrak{a} \subseteq \mathfrak{p}$. Thus $\mathfrak{b} = f^{-1}(\mathfrak{a}) \subseteq f^{-1}(\mathfrak{p}) = f^*(\mathfrak{p}) \implies f^*(\mathfrak{p}) \in V(\mathfrak{b})$.

$\supseteq$: Take some $\mathfrak{p} \in V(\mathfrak{b})$. We can first note that because $\mathfrak{b} = f^{-1}(\mathfrak{a})$ and $0 \in \mathfrak{a}$, $\ker f \subseteq \mathfrak{b} \subseteq \mathfrak{p}$. Then $f(\mathfrak{p})$ is prime as an ideal of $f(A)$, because if $f(a)f(b) \in f(\mathfrak{p})$, then $f(ab) \in f(\mathfrak{p})$ implies that there is some $c \in \ker f$ such that $ab - c \in \mathfrak{p}$. Because $\ker f \subseteq \mathfrak{p}$, $ab \in \mathfrak{p} \implies$ either a or b is in $\mathfrak{p}$, showing that $f(\mathfrak{p})$ is prie.

Then by Theorem 5.10, as B is integral over $f(A)$, $\exists \mathfrak{q} \in \operatorname{Spec} B$ such that $\mathfrak{q} \cap f(A) = f(\mathfrak{p})$. Finally, $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ shows that this map is surjective. □

Exercise 4.0.2. Let A be a subring of a ring B such that B is integral over A, and let $f\colon A \to \Omega$ be a homomorphism of A into an algebraically closed field Ω. Show that f can be extended to a homomorphism of B into Ω.

Proof. Let Σ be the set of all rings R such that $A \subseteq R \subseteq B$ and R has an extension of f to R. Give it a partial order of $R_1 \leq R_2$ iff $R_1 \subseteq R_2$ and f_{R_2} is an extension of f_{R_1}. Every ascending chain has bounded above since given a sequence $R_1 \subseteq \cdots \subseteq R_2 \subseteq \cdots$ in Σ, their union is a subring of B and f can be extended to the union by mapping $x \in R_i$ to $f_{R_i}(x)$. This is well-defined because the partial order ensures the output doesn't depend on the choice of i. Thus by Zorn's Lemma, we have a maximal element R.

This maximal element is B, because given an element $x \in B$, $R[x]$ a subring of B and has an extension of f to $R[x]$ as follows: Suppose x satisfies the equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

Then map x to a root of this polynomial in Ω and extend f_R in this way. Because R is maximal, $R = R[x] \implies x \in R$. Thus $B \subseteq R \subseteq B \implies R = B$. $\square$

Exercise 4.0.3. Let $f\colon B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, prove that $f \otimes 1\colon B \otimes_A C \to B' \otimes_A C$ is integral. (This includes (5.6) ii) as a special case.)

Proof. It suffices to show that pure tensors are integral over $\operatorname{im}(f \otimes 1)$ because they generate $B' \otimes_A C$ and sums and products of integral elements are integral (i.e. Corollary 5.3). Now suppose we have $b' \otimes c$ and

$$(b')^n + a_1(b')^{n-1} + \cdots + a_n = 0 \qquad a_i \in f(A)$$

because f is integral.

Then we will show that

$$(b' \otimes c)^n + a_1(1 \otimes c)(b' \otimes c)^{n-1} + \cdots + a_n(1 \otimes c^n)(1 \otimes 1)$$

will be an integral equation for $b' \otimes c$ over $f(B \otimes_A C)$. First, each $a_i(1 \otimes c^i)$ is in $f(B \otimes_A C)$ as $a_i \in f(B)$. Then we expand:

$$\begin{aligned}
(b' \otimes c)^n &+ a_1(1 \otimes c)(b' \otimes c)^{n-1} + \cdots + a_n(1 \otimes c^n)(1 \otimes 1) \\
&= ((b')^n \otimes c^n) + (a_1(b')^{n-1} \otimes c^{n-1}) + \cdots + (a_n \otimes c^n) \\
&= ((b')^n + a_1(b')^{n-1} + \cdots + a_n) \otimes c^n \\
&= 0 \otimes c^n = 0
\end{aligned}$$

$\square$

Exercise 4.0.4. Let A be a subring of B such that B is integral over A. Let $\mathfrak{n}$ be a maximal ideal of B and let $\mathfrak{m} = \mathfrak{n} \cap A$ be the corresponding maximal ideal of A (see (5.8)). Is $B_{\mathfrak{n}}$ necessarily integral over $A_{\mathfrak{m}}$?

Proof. This is not true. Fix a field k and consider $B = k[x]$ and $A = k[x^2 - 1]$. Then x is a root of $y^2 - (1 - (x^2 - 1)) \in A[y]$, so B is integral over A .

Finally, let $\mathfrak{n} = (x - 1)$. Then $\mathfrak{m} = (x^2 - 1)$. Now I claim that $\frac{1}{x+1}$ is not integral in $B_{\mathfrak{n}}$ over $A_{\mathfrak{m}}$. Suppose FTSOC that it was. Then we have $f(y) \in A_{\mathfrak{m}}[y]$ such that f vanishes at $\frac{1}{x+1}$.

Let $F = A_{(0)}$ and $E = B_{(0)}$. Then by field theory, the minimal polynomial of $\frac{1}{x+1}$ divides $f(y)$ in $F[y]$. Its minimal polynomial is $y^2 + \frac{2}{x^2-1}y - \frac{1}{x^2-1}$. So suppose that

$$f = (y^2 + \frac{2}{x^2 - 1}y - \frac{1}{x^2 - 1})g \tag{4.1}$$

for $g \in F[y]$.

If g has $x^2 - 1$ terms in its (simplified) denominator, then we get an obvious contradiction with f being in $A_{\mathfrak{m}}[y]$. If g has no $x^2 - 1$ terms in its (simplified) numerator, then we also get an obvious contradiction. Thus $g = (x^2 - 1)h$ for some $h \in F[y]$. But then the leading coefficient of f (in y) will have an $x^2 - 1$ term in it, contradicting the definition of f being an integral relation for $\frac{1}{x+1}$. $\square$

Exercise 4.0.5. Let $A \subseteq B$ be rings, B integral over A.

a. If $x \in A$ is a unit in B then it is a unit of A.

b. The Jacobson radical of A is the contraction of the Jacobson radical of B.

Proof. i) Suppose we have $xb = 1$ with $b \in B$. Then we have some integral relation of lowest degree

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0$$

with $a_i \in A$. By multiplying it by x, we get

$$b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1} + a_n x = 0$$

But this is an integral relation of lower degree, implying that b is the root of a polynomial over A with degree 1. But this just implies that $b \in A$.

ii) The contraction of the Jacobson radical of B is $(\bigcap_{\mathfrak{m} \in \operatorname{Specm} B} \mathfrak{m}) \cap A$. By Corollary 5.8, $\mathfrak{m} \in \operatorname{Specm} B \iff \mathfrak{m} \cap A$ is Maximal. Thus $(\bigcap_{\mathfrak{m} \in \operatorname{Specm} B} \mathfrak{m}) \cap A = \bigcap_{\mathfrak{n} \in \operatorname{Specm} A} \mathfrak{n}$, which is the Jacobson radical of A. □

Exercise 4.0.6. Let $B_1, \ldots, B_n$ be integral A-algebras. Show that $\prod_{i=1}^n B_i$ is an integral A-algebra.

Proof. Suppose we have $(b_1, b_2, b_n) \in \prod B_i$ with

$$f_i(b_i) = b_i^{n_i} + a_{1i}^{n_i - i} + \cdots + a_{n_i i} = 0.$$

Then let $f(x) = \prod f_i(x)$. This is a polynomial over A, and by considering it as a polynomial in $\prod B_i$, we can see that $(b_1, b_2, \ldots, b_n)$ is a root of it: the ring operation is done coordinate wise, so $f((b_1, b_2, \ldots, b_n)) = (f(b_1), f(b_2), \ldots, f(b_n)) = (0, 0, \ldots, 0)$. □

Exercise 4.0.7. Let A be a subring of a ring B, such that the set $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Proof. Suppose we have an integral relation for $b \in B \setminus A$ of least degree,

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then I claim that we get a contradiction from this:

$$b(b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1}) + a_n = 0.$$

Suppose that $b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1} \in A$. Then by subtracting the element of A, we get a lower degree integral relation, a contradiction. Thus $b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1} \in B \setminus A$. Because $B \setminus A$ is multiplicatively closed, $b(b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1}) \in B \setminus A$. But $-a_n$ is in A, a contradiction. □

Exercise 4.0.8.

a. Let A be a subring of an integral domain B, and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ such that $fg \in C[x]$. Then f, g are in $C[x]$.

b. Prove the same result without assuming that B (or A) is an integral domain.

Proof. ii) We use induction on the degree of fg.

First we show it for fg of degree 2. Then $f(x) = x + a$ and $g(x) = x + b$ for some $a, b \in B$. If $fg \in C$, then $f(x)g(x) = x^2 + (a+b)x + ab$.

Using the quadratic formula shows us that $-a, -b$ are roots of this polynomial, so $-a, -b$ are integral over $A[a+b, ab]$. Because $a+b, ab \in C$, $A[a+b, ab]$ is finitely generated as an A-module. As $-a, -b$ are integral over $A[a+b, ab]$, $A[a+b, ab, -a, -b]$ is finitely generated as an A-module. Hence $-a, -b \in C$. Thus $f, g \in C[x]$.

Finally, assume it is true up to fg of degree $n-1$. Let $f(x) = f_1(x)x + b_1$ and $g(x) = g_1(x)x + b_2$. Note that if $fg \in C[x]$, $(b_1 g_1(x) + b_2 f_1(x))x + b_1 b_2 \in C[x]$, being the last two terms of fg. Thus by subtracting off $(b_1 g_1(x) + b_2 f_1(x))x + b_1 b_2$, the result is also in $C[x]$. Thus $f_1(x) g_1(x) x^2 \in C[x]$, so $f_1(x) g_1(x) \in C[x]$. By our induction hypothesis, $f_1, g_1 \in C[x]$. Thus $f_1(x)x + b_1, g_1(x)x + b_2$ are in $C[x]$. □

Exercise 4.0.9. Let A be a subring of a ring B and let C be the integral closure of A in B. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$.

Proof. We use induction on the degree to first show that $C[x]$ is integral over $A[x]$. Obviously degree 0 terms of $C[x]$ are integral over $A[x]$. We can also show that x is integral over $A[x]$, since it is the root of $y - x$.

Now assume that all terms of $C[x]$ up to degree $n-1$ are integral over $A[x]$. Say we have $f = c_0x^n + \cdots + c_n$. By induction hypothesis, $c_1x^{n-1} + \cdots + c_n$ is integral over $A[x]$. Thus if we show that $f - (c_1x^{n-1} + \cdots + c_n) = c_0x^n$ is integral over $A[x]$, we are done. Finally, because x and c_0 are integral over $A[x]$ and products of integral elements are integral, c_0x^n is integral.

Now FTSOC suppose we have $f = b_0x^n + \cdots + b_n \in B[x] \setminus C[x]$ integral over $A[x]$. We can pick one with the least number of non-zero coefficients. Then any integral relation of f will produce an integral relation for b_n by focusing on the degree 0 component. Thus $b_n \in C$. Hence $f - b_n$ is also integral over $A[x]$ and in $B[x] \setminus C[x]$. But $f - b_n$ has fewer non-zero coefficients than f, contradicting our assumption, allowing us to conclude that $C[x]$ is the integral closure of $A[x]$. □

Exercise 4.0.10. A ring homomorphism $f\colon A \to B$ is said to have the *going-up property* (resp. the *going-down property*) if the conclusion of the going-up theorem (5.11) (resp. the going-down theorem (5.16)) holds for B and its subring $f(A)$.

Let $f^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the mapping associated with f.

a. Consider the following three statements:

  (i) f^* is a closed mapping.

  (ii) f has the going-up property.

  (iii) Let $\mathfrak{q}$ be any prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^*\colon \operatorname{Spec}(B/\mathfrak{q}) \to \operatorname{Spec}(A/\mathfrak{p})$ is surjective.

  Prove that (a) $\implies$ (b) $\iff$ (c) (See also Chapter 6, Exercise 11.)

b. Consider the following three statements:

  (a') f^* is an open mapping.

  (b') f has the going-down property.

  (c') For any prime ideal $\mathfrak{q}$ of B, if $\mathfrak{p} = \mathfrak{q}^c$, then $f^*\colon \operatorname{Spec}(B_{\mathfrak{q}}) \to \operatorname{Spec}(A_{\mathfrak{p}})$ is surjective.

  Prove that $(a') \implies (b') \iff (c')$. (See also Chapter 7, Exercise 23).

Proof. $(a) \implies (b)$ By induction, it suffices to show the going up property for $n = 2, m = 1$. So suppose we have $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ a chain of prime ideals of $f(A)$ and $\mathfrak{q}_1$ a prime ideal in B with $\mathfrak{q}_1 \cap f(A) = \mathfrak{p}_1$.

First note that $A/\ker f \cong \operatorname{im} f = f(A)$ by the first isomorphism theorem and $f : A \to B$ factors through $A/\ker f$. This gives us that $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ factors through $\operatorname{Spec} A/\ker f$ because Spec is a functor. Then because $\operatorname{Spec} A/\ker f$ is a closed subset of $\operatorname{Spec} A$ (Proposition 1.1), f^* being a closed map gives us that the factored map $\operatorname{Spec} B \to \operatorname{Spec} A/\ker f$ is closed.

Finally, this then tells us that $f^*(V(\mathfrak{q}_1)) = V(\mathfrak{a})$. Because $f^*(\mathfrak{q}_1) = \mathfrak{p}_1$, $\mathfrak{p}_1 \in V(\mathfrak{a})$. Because $\mathfrak{p}_2 \supseteq \mathfrak{p}_1$, $\mathfrak{p}_2 \in V(\mathfrak{a})$. Thus there is some $\mathfrak{q}_2 \in V(\mathfrak{q}_1)$ such that $f^*(\mathfrak{q}_2) = \mathfrak{p}_2$. This is the condition we need to go up.

$(b) \iff (c)$ First we show the fowards direction. Take some prime $\mathfrak{n}' \in \operatorname{Spec}(A/(\mathfrak{p})$. This corresponds to a prime ideal $\mathfrak{n} \in \operatorname{Spec} A$ that contains $\mathfrak{p}$. Thus it suffices to show that there is a prime ideal in B that contains $\mathfrak{q}$ that contracts to $\mathfrak{n}$.

Next we can see that $\mathfrak{p}$ and $\mathfrak{n}$ correspond to prime ideals p, n in $f(A)$ as $\mathfrak{q}^c = \mathfrak{p} \implies \ker f \subseteq \mathfrak{p}$. So we have this set up:

$$p \subseteq n$$

with $\mathfrak{q} \cap f(A) = p$, so by going up, we have a prime ideal $\mathfrak{t} \in \operatorname{Spec} B$ such that $\cap f(A) = n$. Thus $\mathfrak{t}$ is a prime ideal in B that contains $\mathfrak{q}$ and contracts to $\mathfrak{n}$, showing surjectivity.

Now to show the reverse direction. It suffices to show the going up with $n = 2, m = 1$, so take $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ prime ideals in $f(A)$ and $\mathfrak{q}_1$ a prime ideal in B with $\mathfrak{q}_1 \cap f(A) = \mathfrak{p}_1$. Then let $\mathfrak{q} = \mathfrak{q}_1$ and $\mathfrak{p} = f^{-1}(\mathfrak{p}_1)$. By hypothesis, $\operatorname{Spec} B/\mathfrak{q} \to \operatorname{Spec} A/\mathfrak{p}$ is surjective.

Because $f^{-1}(\mathfrak{p}_2) \supseteq f^{-1}(\mathfrak{p}_1)$, by Proposition 1.1, $f^{-1}(\mathfrak{p}_2)$ corresponds to a prime ideal in $A/\mathfrak{p}$, denote it be p_2. Now let q_2 be the preimage of it in $\operatorname{Spec} B/\mathfrak{q}$. This corresponds to a prime ideal $\mathfrak{q}_2$ in B containing $\mathfrak{q}$ by Proposition 1.1. This is the desired extension as $\mathfrak{q}_2 \supseteq \mathfrak{q} = \mathfrak{q}_1$ and $\mathfrak{q}_2 \cap f(A) = f(f^{-1}(\mathfrak{q}_2)) = f(f^*(\mathfrak{q}_2)) = f(f^{-1}(\mathfrak{p}_2)) = \mathfrak{p}_2$.

$(a') \implies (b')$ It suffices to show the going down property for $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$ and $\mathfrak{q}_1$ prime ideals of $f(A)$ and B respectively such that $\mathfrak{q}_1 \cap f(A) = \mathfrak{p}_1$. $\square$