Atiyah-MacDonald Solutions

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1 Chapter 2

1. Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\ker \phi$ is finitely generated.

Solution: Because ϕ is surjective, by first iso, $M/\ker\phi \cong A^n$ with the isomorphism being ϕ. Hence every element of M is a sum of something in $\ker\phi$ and something in the preimage of A^n. Then we can see that the kernel and the preimage of A^n have no relations because if $a + b = 0$ for $a \in \ker\phi$ and $b \in \phi^{-1}(A^n)$, then $\phi(a + b) = 0 = \phi(b) \implies b = 0$, showing that there is no relation.

Hence $M = \ker\phi \oplus A^n$. This then implies that $\ker\phi$ is finitely generated because we can project the set of finite generators of M into finite generators of $\ker\phi$.

2. Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution: Let $p(b \otimes y) = by$. We can see that $p(g(n)) = n \forall n \in N$, so p is surjective. Because $p \circ g = \mathrm{Id}$, we can also conclude that g is injective. By first iso, $N/\ker g \cong N \cong \operatorname{Im} g$ and $N_B/\ker p \cong \operatorname{Im} p = N \cong \operatorname{Im} g$. Hence every element of N_B is a sum of an element of $\ker p$ and $\operatorname{Im} g$ (note that the isomorphism $N_B/\ker p \to \operatorname{Im} g$ is via $g \circ p$, outputting things in the right places to let us say that every element of N_B is a sum of the two).

So if we show that $\ker p$ and $\operatorname{Im} g$ have no relations, we can write $N_B = \ker p \oplus \operatorname{Im} g$. Now suppose we have a relation $\sum b_i \otimes y_i + 1 \otimes y \in \ker p \oplus \operatorname{Im} g$ equaling 0. Then $0 = p(\sum b_i \otimes y_i + 1 \otimes y) = 0 + p(1 \otimes y) = y$, making this a trivial relation.

3. In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.
Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Rewritten: If $x \in M_i$ and $\mu_i(x) = 0$, then $\exists j \geq i$ s.t. $\mu_{ij}(x) = 0$ in M_j.

Solution: Because $\mu_i(x) = 0$, $\mu_i(x) = D$. As μ_i is the restriction of μ, which is the quotient map, this tells us that $x \in D$. So $x = \sum_{a \leq b} c_{ab}(x_a - \mu_{ab}(x_a))$ with the sum being finite by definition of D. We can bring the c_{ab} into the x_a, so WLOG we have

$$x = \sum_{a \leq b} x_a - \mu_{ab}(x_a).$$

Now we fix an arbitrary $\ell \geq b$ for all b in the sum. Because $x \in M_i$, we know that all the elements that aren't in M_i must cancel. As bringing them to M_ℓ with $\mu_{b\ell}, b \neq i$ doesn't change the fact that they are 0, we can bring all the terms into M_ℓ as so:

$$\mu_{i\ell}(x) = \mu_{i\ell}(\sum_{a \leq b} x_a - \mu_{ab}(x_a)) = \sum(\mu_{a\ell}(x_a) - \mu_{b\ell}\mu_{ab}(x_a)).$$

As $\mu_{b\ell}\mu_{ab} = \mu_{a\ell}$, we have

$$\mu_{i\ell}(x) = \sum(\mu_{a\ell}(x_a) - \mu_{a\ell}(x_a)) = \sum \mu_{a\ell}(x_a - x_a) = 0.$$

We have hence found such a j as desired ($j = \ell$).