Atiyah-MacDonald Solutions

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1 Chapter 2

In Chapter Exercises:

1. 2.2
   (a) $\mathrm{Ann}(M+N) = \mathrm{Ann}(M) \cap \mathrm{Ann}(N)$.
   (b) $(N : P) = \mathrm{Ann}((N+P)/N)$.

2. 2.15: Let A, B be rings, let M be an A-module, P a B-module and N an (A, B)-bimodule (that is, N is simultaneously an A-module and a B-module and the two structures are compatible in the sense that $a(xb) = (ax)b$ for all $a \in A, b \in B, x \in N$). Then $M \otimes_A N$ is naturally a B-module, $N \otimes_B P$ an A-module, and we have
$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

Solution:
First we construct the B bilinear map
$$(M \otimes_A N) \times P \to M \otimes_A (N \otimes_B P)$$
that sends $(m \otimes n, p) \to m \otimes (n \otimes p)$. The B bilinearity comes from $(b(m \otimes n), p) = (m \otimes nb, p) \mapsto m \otimes (nb \otimes p) = b(m \otimes (n \otimes p)) = m \otimes (b \otimes bp)$, which is also the image of $(m \otimes n, bp)$. Hence this induces a unique B linear map
$$(M \otimes_A N) \otimes_B P \to M \otimes_A (N \otimes_B P).$$
By a symmetric argument, we have a unique A linear map from the other direction, giving us an isomorphism for by tracing where $(m \otimes n) \otimes p$ goes, it goes to $m \otimes (n \otimes p)$ and then to $(m \otimes n) \otimes p$.

3. If $f : A \to B$ is a ring homomorphism and M is a flat A-module, then $M_B = B \otimes_A M$ is a flat B-module.

Solution: The function f makes B an A-algebra. Consider $B \otimes_B N \cong N$ by Proposition 2.14. Obiously B has an (A, B) bimodule structure since B is an algebra.
Then given an exact sequence E, $E \otimes_A N = E \otimes_A (B \otimes_B N) = (E \otimes_A B) \otimes_B N$ by Exercise 2.15 in the Chapter. As B is flat as an A-module and N is flat as a B-module, we are done.

1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Solution: Take a bilinear map $f : (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$. Then by bilinearity, we have $f(mx, y) = f(0, y) + mf(x, y) = f(0, y)$ and $f(x, ny) = f(x, 0) + nf(x, y) = f(x, 0)$, which imply that $mf(x, y) = 0$ and $nf(x, y) = 0$. By Bezout's Lemma, we have that there exists a, b s.t. $am+bn = 1$ as m, n are coprime. Thus $f(x, y) = 0$.

2. Let A be a ring, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. [Tensor the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ with M]

Solution: By tensoring with the exact sequence $\mathfrak{a} \to A \to A/\mathfrak{a} \to 0$, we get

$$0 \to \mathfrak{a} \otimes_A M \to A \otimes_A M \to A/\mathfrak{a} \otimes M \to 0. \qquad \text{(Prop 2.8)}$$

Then by Proposition 2.14, $A \otimes M \to M$ is an isomorphism by $a\otimes$, we have $\mathfrak{a} \otimes M \cong \mathfrak{a}M$ and $A \otimes M \cong M$. Hence by commutativity of

$$\begin{tikzcd} \mathfrak{a} \otimes M \arrow[r] \arrow[d, "\cong"] & A \otimes M \arrow[d, "\cong"] \\ \mathfrak{a}M \arrow[r] & M \end{tikzcd},$$

(for the commutativity, the definitions of the maps down make it obvious) we have that $\mathrm{Im}(\mathfrak{a}M \to M) = \ker(M \to M/\mathfrak{a}M) = \ker(A/\mathfrak{a} \otimes M)$.

So we have this diagram

$$\begin{tikzcd} & & M/\mathfrak{a}M \arrow[dr] & \\ \mathfrak{a}M \arrow[r] & M \arrow[ur] \arrow[dr] & & 0 \\ & & A/\mathfrak{a} \otimes M \arrow[ur] & \end{tikzcd}$$

By some isomorphism theorem and surjectivity of the last maps, we have that $M/\mathfrak{a}M \cong A/\mathfrak{a}\otimes M$.

3. Let A be a local ring, M and N finitely generated A-modules. Prove tha tif $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. [Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2. By Nakayama's lemma, $M_k = 0 \implies M = 0$. But $M \otimes_A N = 0 \implies (Mo \times_A N)_k = 0 \implies M_k \otimes N_k = 0 \implies M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces over a field.]

Solution: We do as the hint suggests: Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field and define $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.

By Nakayama's lemma, $M_k = 0 \implies M = 0$ since $k \subseteq$ the Jacobson radical, $M_k = 0 \implies M = \mathfrak{m}M$, and M is finitely generated.

Then we have that $M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0$ because

As the tensor product of vector spaces is just the direct sum, this implies that $0 = k \otimes k \otimes (M \otimes_A N) = M_k \otimes_A N_k = 0$ by commuting. As A bilinear maps on $M_k \times N_k$ are k linear maps on $M_k \times N_k$, we have $M_k \otimes_k N_k = 0$. Finally, this implies that $M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces.

4. Let $M_i (i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\iff$ each M_i is flat.

Solution: Fix an exact sequence E.

We have that $E \otimes M = E \otimes (\bigoplus M_i) = \bigoplus(E \otimes M_i)$ because each direct sum is finite and hence belongs to a finite direct sum in which we can use Proposition 2.14. Then if $E \otimes M$ is exact, so is each coordinate, which gives us the invidiaul M_i is exact. If each coordinate is exact then so is $E \otimes M$.

5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra. [Use Exercise 4.]

Solution: Clearly $A[x] = \bigoplus_{i=0}^{\infty} Ax^i$. So by the above exercise, it suffices to show that $x^i A$ is flat. Say we have a short exact sequence of A-modules

$$0 \to B \to C \to D \to 0.$$

Then

$$0 \to B \otimes Ax^i \to C \otimes Ax^i \to D \otimes Ax^i \to 0$$

is exact because tensoring with Ax^i is the same as tensoring with A as bilinear maps $B \times Ax^i$ are bilinear on $B \times A$ and likewise for linear maps. So tensoring with Ax^i also induces unique linear maps that make the tensor universal diagram commute, so by uniqueness of the universal property, they are the same.

Finally tensoring with A is the same as the original module by Proposition 2.14. So Ax^i is flat and so is $A[x]$.

6. For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + \cdots + m_r x^r. \qquad (m_i \in M)$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module.
Show that $M[x] \cong A[x] \otimes_A M$.

Solution: First, it is an abelian group by commuting and grouping together terms with the same power of x. Then for the properties of an $A[x]$ module: Distributivity holds by simply defining it as so.

$$\begin{aligned}
(r+s)m &= (r_0 + \cdots + r_j x^j + s_0 + \cdots + s_k x^k)m \\
&= (r_0 + s_0 + \cdots + (r_{j+k} + s_{j+k})x^{j+k})m \\
&= (r_0 + s_0)m + (r_1 + s_1)mx + \cdots \\
&= rm + sm \\
r(sm) &= r(s_0 m + s_1 mx + \cdots + s_k mx^k) \\
&= r(s_0 m) + \cdots + r(s_k mx^k) \\
&= rs_0 m + \cdots + rs_k x^k m \\
&= (rs_0 + \cdots + rs_k x^k)m \\
&= (rs)m \\
1m &= m.
\end{aligned}$$

Hence $M[x]$ is an $A[x]$ module.

We use the universal property. Say we have a bilinear map

$$\begin{tikzcd} A[x] \times M \arrow[r] \arrow[dr, "f"'] & M[x] \\ & B \end{tikzcd}$$

where the top map takes $(a(x), m) \to a(x)m$. Then we have the unique linear map $\hat{f} : M[x] \to B$ that takes $m_0 + m_1 x + \cdots + m_r x^r$ to $f(1, m_0) + f(x, m_1) + \cdots + f(x^r, m_r)$. This is linear because of linearity of f in M. It is unique because we have a basis that uniquely determines the map by linearity, and the bases have to be mapped to the things that generate this map.

Hence by the universal property, $M[x] \cong A[x] \otimes M$.

7. Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ieal in $A[x]$?

Solution: It is clear that $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$. As $A/\mathfrak{p}$ is an integral domain by primality of $\mathfrak{p}$, $(A/\mathfrak{p})[x]$ is an integral domain (look at leading coefficients) and thus $p[x]$ is prime.

Similarly with $\mathfrak{m}$, $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$. Then $A/\mathfrak{m}$ is a field, and clearly $(A/\mathfrak{m})[x]$ is not a field.

8. (a) If M and N are flat A-modules, then so is $M \otimes_A N$.

Solution: Let E be an exact sequence. Then $E \otimes_A M$ is exact by M being flat, and hence $(E \otimes_A) \otimes_A N$ is exact. By Proposition 2.14, this sequence equals $E \otimes_A (M \otimes_A N)$, so $M \otimes_A N$ is flat.

(b) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution: Consider $B \otimes_B N \cong N$ by Proposition 2.14. Then N is flat as an A-module by Exercise 2.20 in the Chapter.

9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Solution: As the last map is surjective, the preimage of M'' is M, so the preimage of the generators of M'' and the kernel generate M. But the kernel is finitely generated as it is the image of M', so M is finitely generated.

10. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Solution: It suffices to show that $N = \mathfrak{a}N + u(M)$ by Corollary 2.7. Clearly $\mathfrak{a}N + u(M) \subseteq N$ by definitions.

Let ϕ_N be the quotient map $N \to N/\mathfrak{a}N$ and ϕ_M be the map $M \to M/\mathfrak{a}M$. Then because $\hat{u}$ is induced by $\phi \circ u$, $\phi_N \circ u = \hat{u}\phi_M$. As both $\hat{u}$ and ϕ_M are surjective, the LHS is too. Hence for every element n of N, by the surjectivity of ϕ_N, there is an element in $u(M)$ s.t. ϕ_N of it equals n. Thus $u(M) + \ker \phi_N = u(M) + \mathfrak{a}N = N$.

11. Let A be a ring $\neq 0$. Show that $A^m \cong A^n \implies m = n$.

> **Solution:** Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes \phi : A/\mathfrak{m} \otimes A^m \to A/\mathfrak{m} \otimes A^n$ is an ismorphism of vector space (Exercise 2 for modules over a field). But then these vector spaces have to have the same dimension over $A/\mathfrak{m}$, which are m and n respectively. So $m = n$.

(a) If $\phi : A^m \to A^n$ is surjective, then $m \geq n$.

> **Solution:** Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be a surjection. Then $1 \otimes \phi : A/\mathfrak{m} \otimes A^m \to A/\mathfrak{m} \otimes A^n$ is a surjection of vector space (Exercise 2) (surjectivity from right exactness of tensoring (Proposition 2.18)). Then note that their dimensions are m, n respectively. As this is a surjective vector space map, $m \geq n$.

(b) If $\phi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?

> **Solution:** No. Consider $A = \mathbb{Z}[x_1, \ldots]$. Then consider the map $A \times A \to A$ that maps $(f(x_1, \ldots), g(x_1, \ldots)) \to f(x_1, x_3, \ldots) + g(x_2, x_4, \ldots)$. Obviously they are injective as they are inclusions under relabelling. Clearly $2 \not\leq 1$.

12. Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\ker \phi$ is finitely generated.

> **Solution:** We can see that M is generated by picking a representative in the preimage of the basis of A^n and the kernel since the quotient is surjective, so every element in M is an element in A^n up to the kernel of the map. This then forms a basis because the preimage of the basis is a basis (otherwise push forward a relation), and the kernel and the basis have no relations because otherwise the basis would be in the kernel.
>
> Hence $M = \ker \phi \oplus A^n$, which implies that $\ker \phi$ is finitely generated, otherwise M wouldn't be finitely generated.

13. Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

> **Solution:** We can see that $p(g(n)) = n \forall n \in N$, which is injective. Hence g is.
>
> To show that it is the direct sum, we do as the hint suggests and realize that $g \circ p$ is the identity map on elements not in $\ker p$ because B is generated as a B-module by 1, so we have generators of N_B being of the form $1 \otimes q$. It easily follows that $gp(1 \otimes q) = 1 \otimes q$, g is B linear.
>
> Thus every element of N_B is either in the image of g or in the kernel of p. Then to show they are independent, suppose we have a non-trivial relation $\sum b_i \otimes y_i + 1 \otimes y \in \ker p + \mathrm{Im} g$ equalling 0. Then $p(\sum b_i \otimes y_i + 1 \otimes y) = 0 + p(1 \otimes y) = y \neq 0$ otherwise it would be a trivial relation.

14. A partially ordered set I is said to be a directed set if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.
Let A be a ring, let I be a directed set and let $(M_i)'_{i \in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

    1. μ_{ii} is the identity mapping of M_i for all $i \in I$;

2. $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

Then the modules M_i and homomorphisms μ_{ij} are said to form a direct system $M = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the `direct limit` of the direct system M. Let C be the direct sum of the M_i, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of M_i.

The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$, is called the direct limit of the direct system M, and is written $\lim_{\to} M_i$ From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.

Solution: Let $Q = \{X_i - \mu_{ij}(X_i), X_i \in M_i, i \leq j\}$ We can see that from definition, for $x_i \in M_i$, $\mu_i(x_i) = x_i + Q = x_i - (x_i - \mu_{ij}(x_i)) + Q = \mu_j(\mu_{ij}(x_i))$.

15. In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.
Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Solution: It suffices to show that $x_j + x_k + Q$ for $x_j \in M_j, x_k \in M_k$ is of the desired form since M are quotient classes of a finite sum. We can see that $x_j + x_k + Q = \mu_{j\ell}(x_i) + \mu_{k\ell}(x_k) + Q$ for $j \leq \ell$ and $k \leq \ell$. Then because $\mu_{j\ell}(x_i) + \mu_{k\ell}(x_k) \in M_k$, $\mu_{j\ell}(x_i) + \mu_{k\ell}(x_k) + Q = \mu_k(\mu_{j\ell}(x_i) + \mu_{k\ell}(x_k))$.

Since $Q = \mu_i(x_i) = x_i + Q$, there is some finite set of $j_\ell \geq i$ s.t. $x_i = \sum (x_{j_\ell} - \mu_{ij_\ell}(x_{j_\ell})), x_{j\ell} \in M_i$. Since the $M_i, M_{j\ell}$ are distinct, $\sum \mu_{ij_\ell}(x_{j_\ell}) = 0$.

16. Show that the direct limit is characterized (up to isomorphism) by the following property. Let N be an A-module and for each $i \in I$, let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Solution: Simply define α to be $(\bigoplus x_i) + Q \mapsto \bigoplus \alpha_i(x_i)$. This is well-defined because for any $m_i - \mu_{ij}(m_i) \in Q$, this gets mapped to $\alpha_i(m_i) - \alpha_j(\mu_{ij}(m_i)) = \alpha_i(m_i) - \alpha_i(m_i) = 0$.

Then this commutes properly because $\alpha(\mu_i(m_i)) = \alpha(m_i + Q) = \alpha_i(m_i)$.

Finally, this is unique because given another α' with these properties and arbitrary $\bigoplus x_i + Q \in M$, $\alpha'(\bigoplus x_i + Q) = \alpha'(\mu_I(x_I))$ given by Exercise 15. Then by definition of α', $\alpha'(\mu_I(x_I)) = \alpha_I(x_I) = \alpha(\mu_I(x_I)) = \alpha(\bigoplus x_i + Q)$. Hence $\alpha' = \alpha$ for all elements of M, and we are done.

The characterizing up to isomorphism is just a classic universal property argument.

17. Let $(M_i)_{i \in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I, there exists $k \in I$ s.t. $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j. Show that

$$\lim_{\to} M_i = \sum M_i = \cup M_i.$$

In particular, any A-module is the direct limit of its finitely generated submodules.

Solution: Obviously this satisfies the conditions for the direct limit as the maps are just embeddings. To show the equality, we can realize $\cup M_i$ as having the properties of the direct limit: Say we have a family of maps α_i into an A-module N that respect the directed system's maps.

Then we have a map $\alpha : \cup M_i \to N$ defined by taking an element m, finding a M_i it is in, and mapping it to $\alpha_i(m)$. This is well-defined because α_i respects the directed system's maps, those being inclusions. Hence it is isomorphic to the direct limit by Exercise 16.

Since $\{M_i\}$ is a poset and we have that for every increasing chain, there is a maximal element (namely the union of all the modules in the chain), there is a maximal element M in this set. This equals $\cup M_i$, and hence $M \subseteq \sum M_i \subseteq M$.

18. Let $\boldsymbol{M} = (M_i, \mu_{ij}), \boldsymbol{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M, \nu_i : N_i \to N$ the associated homomorphisms. A homomorphism $\boldsymbol{\phi} : \boldsymbol{M} \to \boldsymbol{N}$ is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Show that $\boldsymbol{\phi}$ defines a unique homomorphism $\phi = \lim_{\to} \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Solution: We have maps $\psi_i : M_i \to N$ by doing the composition $\nu_i \circ \phi_i$, which commute with the system because $\psi_j(\mu_{ij}) = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_i \circ \phi_i$. Hence there is a unique map $M \to N$ that commutes with the system by the characterizing property of the direct limit. Since this map commutes with the system, $\phi \circ \mu_i = \psi_i = \nu_i \circ \phi_i$.

19. A sequence of direct systems and homomorphisms

$$\boldsymbol{M} \to \boldsymbol{N} \to \boldsymbol{P}$$

is exact if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \to N \to P$ of direct limits is then exact.

Solution: Let $\mu_{ij}, \nu_{ij}, \rho_{ij}$ be the maps in the systems, a_i, b_i be the maps $M_i \to N_i$ and $N_i \to P_i$, and let $\mu_{M\cdot}, \nu_{N\cdot}$ be the maps from $M \to \cdot, N \to \cdot$ that are induced by the direct limit property ($\cdot$ will be N or P).

Fix an element $x \in M$. By Exercise 15, $x = \mu_i(x_i)$ for some i. Then we can see that $\mu_{MP}\mu_i = \rho_i b_i a_i$ for all i by commuting properties of the direct limit. In particular, $\mu_{MP}(x) = \mu_{MP}\mu_i(x) = \rho_i b_i a_i = 0$ since a_i, b_i are in an exact sequence.

Finally, we can show that $\ker \nu_{NP} \subseteq \mathrm{Im}\mu_{MN}$ by supposing $\nu_{NP}(x) = 0$ for $x \in N$. Then by Exercise 15, $x = \nu_i(x_i)$ for some i. By commuting properties, $\nu_{NP}\nu_i = \rho_i b_i$, so $\nu_{NP}(x) = \nu_{NP}\nu_i(x_i) = \rho_i b_i(x_i) = 0$. By exercise 15, if $\rho_i(b_i(x_i)) = 0$, there exists $j \geq i$ s.t. $\rho_{ij}(b_i(x_i)) = 0$. By commutativity of the diagram, this equals $b_j(\nu_{ij}(x_i)) = 0$, which by exactness gives us that $\nu_{ij}(x_i) \in \mathrm{Im}a_j$. By applying ν_j to both sides, we can see that $\nu_i(x_i) \in \mathrm{Im}(M_j \to N)$. Being in the image of $M_j \to N$ is in the image of μ_{MN} since, by being the direct limit, μ_{MN} factors through this map.

Hence $\ker \nu_{NP} = \mathrm{Im}_{MN}$.

To understand this proof clearly, let I just be the naturals and draw the commutative diagrams out.

20. Keeping the same notation as in Exercise 14, let N be any A-module. Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \lim_{\to}(M_i \otimes N)$ be its direct limit. For each $i \in I$, we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$, hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$. Show that ψ is an isomorphism, so that

$$\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N.$$

> **Solution:** We show that $M \otimes N$ satisfies the universal property for direct limits. Suppose we have maps $\{f_i : M_i \otimes N \to Q\}$. Then these lead to bilinear maps $\hat{f}_i : M_i \times N \to Q$ By direct limit properties, we then have a map $M \times N \to Q$. This is blinear because it commutes with bilinear maps. This bilinear map then induces a unique linear map $M \otimes N \to Q$. This is the universal property of direct limits, so $M \otimes N \cong \lim_{\longrightarrow}(M_i \otimes N)$.

21. Let $(A_i)_{i \in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leq j$ in I, let $\alpha_{ij} : A_i \to A_j$ be a ring homomorphism, satisfying conditions (1) and (2) of Exercise 14. Regarding each A_i as a $\mathbb{Z}$-module, we can then form the direct limit $A = \lim_{\longrightarrow} A_i$. Show that A inherits a ring structure from the A_i so that the mappings $A_i \to A$ are ring homomorphisms. The ring A is the direct limit of the system (A_i, α_{ij}).
If $A = 0$ prove that $A_i = 0$ for some $i \in I$.

Solution: For $a, a' \in$ limit A, define $a \cdot a'$ as $\mu_k(\mu_{ik}(a_i)\mu_{jk}(a_j))$ where $a = \mu_i(a_i), a' = \mu_j(a_j)$ and $k \geq i, j$, which is well-defined because for other $k' \geq i, j$, we can find a k'' s.t. $\mu_{k'}(\mu_{ik'}(a_i)\mu_{jk'}(a_j)) = \mu_{k''}(\mu_{k'k''}(\mu_{ik'}(a_i)\mu_{jk'}(a_j))) = \mu_{k''}(\mu_{k'k''}(\mu_{ik'}(a_i))\mu_{k'k''}(\mu_{jk'}(a_j))) = \mu_{k''}(\mu_{ik''}(a_i)\mu_{jk''}(a_j)) = \mu_{k''}(\mu_{kk''}(\mu_{ik}(a_i)\mu_{jk}(a_j))) = \mu_k(\mu_{ik}(a_i)\mu_{jk}(a_j))$. This is obviously commutative and has identity over multiplication because it has the domain of a commutative ring and $\mu_{\cdot,\cdot}$ are homomorphisms.

Next is associativity: Let $a = \mu_i(a_i), b = \mu_j(b_j), c = \mu_k(c_k)$ and $\ell \geq i, j, k$.

$$(a \cdot b) \cdot c = (\mu_\ell(\mu_{i\ell}(a_i)\mu_{j\ell}(b_j))) \cdot \mu_k(c_k)$$
$$\iff$$
$$\mu_\ell((\mu_{i\ell}(a_i)\mu_{j\ell}(b_j))\mu_{k\ell}(c_k)) = a \cdot (b \cdot c).$$

Finally, distributivity: Let $a = \mu_i(a_i), b = \mu_j(b_j), c = \mu_k(c_k)$ and $\ell \geq i, j, k$.

$$a(b+c) = \mu_i(a_i)(\mu_j(b_j) + \mu_k(c_k)) = \mu_i(a_i)(\mu_\ell\mu_{j\ell}(b_j) + \mu_\ell\mu_{k\ell}(c_k)) = \mu_\ell(\mu_{i\ell}(a_i)(\mu_{j\ell}(b_j) + \mu_{k\ell}(c_k))) = \mu_\ell(\mu_{i\ell}(a_i)\mu_{j\ell}$$

If $A = 0$, then $\mu_i(1) = 0 \implies \mu_{ij}(1) = 0$ by Exercise 15. But a ring homomorphism that sends 1 to 0 implies that the ring is 0, so $A_j = 0$.

22. Let (A_i, α_{ij}) be a direct system of rings and let $\mathfrak{N}_i$ be the nilradical of A_i. Show that $\lim_{\rightarrow} \mathfrak{N}_i$ is the nilradical of $\lim_{\rightarrow} A_i$.
If each A_i is an integral domain, then $\lim_{\rightarrow} A_i$ is an integral domain.

> **Solution:** We have the obvious inclusions $\mathfrak{N}_i \to \mathfrak{N}(\lim_{\longrightarrow} A_i)$ since $A_i \to$ limit A is a ring homomorphism ($a^n = 0$ in A_i gets mapped to $a^n = 0$ in limit A).
>
> Next we can map $\mathfrak{N}(\lim_{\longrightarrow} A_i)$ to $\lim_{\longrightarrow} \mathfrak{N}_i$ as so: For any $a^n = 0 \in$ limit A, $a = \mu_i(a_i)$ by Exercise 15, which then gives us $\mu_i(a_i^n) = 0$. By Exercise 15, we then have $\mu_{ij}(a_i^n) = 0$ in A_j. Then $\mu_{ij}(a_i)^n = 0$, giving us an element $\mu_{ij}(a_i)$, which we then map into $\lim_{\longrightarrow} \mathfrak{N}_i$.
>
> This is well-defined because we can always commute any choices to the same, largest index ring. Next this is a homomorphism because given $a = \mu_k(\mu_{ik}(a_i)), b = \mu_k(\mu_{jk}(b_j))$, $a + b = \mu_k(\mu_{ik}(a_i) + \mu_{jk}(a_j)) \to \mu_{k\ell}(\mu_{ik}(a_i) + \mu_{jk}(a_j)) \to \mu_k(\mu_{ik}(a_i) + \mu_{jk}(b_j)) = \mu_i(a_i) + \mu_j(b_j)$, which is what a, b would be mapped to. This is just the identity.
>
> Since there is a homomorphism and an inverse, it is an isomorphism.
>
> If each A_i is an integral domain, then suppose FTSOC that there is $ab = 0$ in $\lim_{\longrightarrow} A_i$, $a, b \neq 0$. Then by Exercise 15, we have $a = \mu_i(a_i), b = \mu_j(b_j)$. Hence $\mu_i(a_i)\mu_j(b_j) = 0 = \mu_k(\mu_{ik}(a_i)\mu_{jk}(b_j))$ for $k \geq i, j$. Then by Exercise 15, there is $\ell \geq k$ s.t. $\mu_{k\ell}(\mu_{ik}(a_i)\mu_{jk}(b_j)) = 0 = \mu_{i\ell}(a_i)\mu_{j\ell}(b_j)$. But then A_j wouldn't be an integral domain (note that $\mu_{\cdot\ell}(\cdot) \neq 0$ because if otherwise, then $\mu_\ell(\mu_{\cdot\ell}(\cdot)) = \mu_\cdot(\cdot) = 0$, contradicting a, b being non-zero).

23. Let $(B_\lambda)_{\lambda\in\Lambda}$ be a family of A-algebras. For each finite subset of Λ, let B_J denote the tensor product (over A) of the B_λ for each $\lambda \in J$. If J' is another finite subset of Λ and $J \subseteq J'$, there is a canonical A-algebra homomorphism $B_J \to B_{J'}$. Let B denote the direct limit of the rings B_J as J runs through all finite subsets of Λ. The ring B has a natural A-algebra structure for which the homomoprhisms $B_J \to B$ are A-algebra homomorphisms. The A-algebra B is the tensor product of the family $(B_\lambda)_{\lambda\in\Lambda}$.

Solution: The canonical A-algebra homomorphism sends $b \in B_J$ to $b \otimes 1 \otimes 1 \otimes \cdots$ ($|J'| - |J|$ times). As A-algebras are also rings, the ring B exists by Exercise 21. Ring homomorphisms that preserve A-module structure are A-algebra homomorphisms.

24. In these Exercises it will be assumed that the reader is familiar with the definitio nand basic properties of the Tor functor.

If M is an A-module, the following are equivalent:

1. M is flat;
2. $\mathrm{Tor}_n^A(M, N) = 0$ for all $n > 0$ and all A-modules N;
3. $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

Solution: (i) $\implies$ (ii): We do as the hint suggests: take a free resolution of N. Tensor this with M. As M is flat, this sequence is then exact, so the homology groups are 0.

Obviously (ii) $\implies$ (iii).

(iii) $\implies$ (i): Take an exact sequence $0 \to N' \to N \to N'' \to 0$. Then $\mathrm{Tor}_1^A(M, N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ is exact. As $\mathrm{Tor}_1^A(M, N'') = 0$, M is flat.

25. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence with N'' flat. Then N' is flat $\iff$ N is flat.

Solution: By the Tor exact sequence, we have

$$\mathrm{Tor}_2^A(M, N'') \to \mathrm{Tor}_1^A(M, N') \to \mathrm{Tor}_1^A(M, N) \to \mathrm{Tor}_1^A(M, N'') \to M\otimes N' \to M\otimes N \to M\otimes N'' \to 0.$$

As $\mathrm{Tor}_2^A(M, N'') = \mathrm{Tor}_1^A(M, N'') = 0$ by flatness of N'' and Exercise 24, $\mathrm{Tor}_1^A(M, N') = \mathrm{Tor}_1^A(M, N)$. By Exercise 24, this means that N is flat iff N' is flat.

26. Let N be an A-module. Then N is flat $\iff$ $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ in A

Solution: $\implies$) is obvious by Exercise 24.

$\Leftarrow$):

Lemma 1. Assuming the RHS, then $\mathrm{Tor}(A/\mathfrak{a}, N) = 0$ for **all** $\mathfrak{b}$.

Proof. $\square$

First we can note that N is flat if $\mathrm{Tor}_1(M, N) = 0$ for all finitely generated A-modules M by Proposition 2.19. Then fix a finitely generated M generated by x_i and define $M_i = \{x_1, \ldots, x_i\}$. Also define the map $f_i : A \to M_i/M_{i-1}$ by sending $a \in A$ to $ax_i + M_{i-1}$. This is surjective as M_i is generated by $x_1, \ldots, x_i$. As such, $\ker f_i$ is an ideal of A. Hence $M_i/M_{i-1} \cong$ Next we show that if $\mathrm{Tor}_1(M, N) = 0$ for all cyclic A-modules M, then N is flat:

27. A ring A is absolutely flat if every A-module is flat. Prove that the following are equivalent:

    1. A is absolutely flat.
    2. Every principal ideal is idempotent.
    3. Every finitely generated ideal is a direct summand of A.

Solution: (i) $\implies$ (ii): Since $A/(x)$ is an A-module, it is flat. Thus the injectivity of $(x) \to A$ makes the map $(x) \otimes A/(x) \to A \otimes A/(x) \cong A/(x)$ injective. This map takes $x \otimes [a] \mapsto x \otimes [a] \mapsto [xa] = 0$ (middle map is due to Proposition 2.19). As it is an injective zero map, $(x) \otimes A/(x) = 0$, and by Exercise 2, $(x) \otimes A/(x) \cong (x)/(x)^2$. Thus $(x) = (x)^2$.

(ii) $\implies$ (iii): As the hint does: Let $x \in A$. Then $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent and we have $(e) = (x)$. For idempotents e, f, $(e, f) = (e + f - ef)$ because $e(e + f - ef) = e + ef - ef = e$ and $f(e + f - ef) = ef + f - ef = f$. Thus every finitely generated ideal is principal by finding idempotents for every generator in the ideal and then reducing them pairwise as so. As such, $A = (e) \oplus (1 - e)$ (note that $(1 - e)^2 = (1 - e)$, so they are independent).

(iii) $\implies$ (i): It suffices to satisfy the conditions in Exercise 26 for all N. Take an exact sequence $0 \to N' \to N \to N'' \to 0$. Then we have the sequence

$$\mathrm{Tor}_1(A/\mathfrak{a}, N'') \to N' \otimes A/\mathfrak{a} \to N \otimes A/\mathfrak{a} \to N'' \otimes A/\mathfrak{a} \to 0.$$

By Exercise 2, $N' \otimes A/\mathfrak{a} \cong N'/\mathfrak{a}N' \cong \mathfrak{b}N'$ as we assume that A is a direct sum of f.g. ideals (namely let $A = \mathfrak{a} \oplus \mathfrak{b}$). Then the map $N' \otimes A/\mathfrak{a} \to N \otimes A/\mathfrak{a}$ is the map $\mathfrak{b}N' \to \mathfrak{b}N$, which is injective as they are simply restrictions of the injective map $N' \to N$. Thus $\mathrm{Tor}_1(A/\mathfrak{a}, N'') = 0$. As we can always realize N as the tail of an exact sequence (simply take $0 \to 0 \to N \to N \to 0$, we are done.

28. A Boolean ring is absolutely flat. The ring of Chapter 1, Exercise 7 is absolutely flat. Every homomorphic image of an absolutely fiat ring is absolutely flat. If a local ring is absolutely flat, then it is a field. If A is absolutely flat, every non-unit in A is a zero-divisor.

Solution: By definition, all principal ideals are idempotent in a Boolean ring, so by Exercise 27 we are done.

The ring in Chapter 1 Exercise 7 is absolutely flat because all principal ideals are idempotent: $(x)^2 = (x^2) = (x)$ because $x^{2n} = x \in (x^2)$.

Say we have f a homomorphism from an absolutely flat ring R. Then every principal ideal in the image is generated by $f(a)$, and $(a^2) = (a)$ by Exercise 27. Hence $(f(a))^2 = (f(a)^2) = (f(a^2)) = (f(a))$.

Fix an absolutely flat local ring R. By Exercise 27, every principal ideal of R is idempotent, so $(x^2) = (x) \forall x \in R$. Hence $x = rx^2, r \in R$. Thus $rx = r^2x^2 = (rx)^2 \implies rx$ is idempotent. But by Exercise 12 of Chapter 1, $rx = 0$ or 1. Thus $(x) = 0$ or 1, which implies that it is a field.

If A is absolutely flat, then take a non-unit x. We have that $(x)^2 = (x)$, so $x \in (x^2) \implies rx^2 = x$ for some r. Thus $x(rx - 1) = 0 \implies x$ is a zero-divisor.