ATIYAH-MACDONALD SOLUTIONS

VINCENT TRAN

1. Primary Decomposition

Exercise 1.0.1. If an ideal $\mathfrak{a}$ has a primary decomposition, then $\mathrm{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components.

Proof. By the First Uniqueness Theorem, the associated primes to $\mathfrak{a}$ is independent on the decomposition. As $\mathfrak{a}$ being decomposable implies that $\mathfrak{a}$ is a finite intersection, the number of associated primes must be finite. Finally, because irreducible components of $\mathrm{Spec}(A/\mathfrak{a})$ correspond to minimal primes belonging to $\mathfrak{a}$ by the remark after Prop 4.6, this implies that there are a finite number of irreducible components. □

Exercise 1.0.2. If $\mathfrak{a} = r(\mathfrak{a})$, then $\mathfrak{a}$ has no embedded prime ideals.

Proof. I'm going to assume that $\mathfrak{a}$ is decomposable, because otherwise the definition in the chapter doesn't make sense. Thus decompose $\mathfrak{a}$ as a minimal primary decomposition $\cap^n \mathfrak{q}_i$ and let $r(\mathfrak{q}_i) = \mathfrak{p}_i$. Then $\mathfrak{a} = r(\mathfrak{a}) = r(\cap^n \mathfrak{q}_i) = \cap^n r(\mathfrak{q}_i) = \cap^n \mathfrak{p}_i$. If there were any embedded prime ideals among the $\mathfrak{p}_i$, then we could eliminate the term and see that $\cap^n r(\mathfrak{q}_i)$ is not minimal, a contradiction. Thus there are no embedded primes. □

Exercise 1.0.3. If A is absolutely flat, every primary ideal is maximal.

Proof. Let $\mathfrak{q}$ be a primary ideal. Our strategy will be to show that $A/\mathfrak{q}$ is a field. Then $A/\mathfrak{q}$ is absolutely flat by Exercise 2.28 (homomorphic image).

By Exercise 2.28, all non-units are zero divisors, and because $\mathfrak{q}$ is primary, all zero-divisors are nilpotents. By Exercise 1.10, a ring where every element is either nilpotent or a unit is local. By Exercise 2.28, this implies that $A/\mathfrak{q}$ is a field. □

Exercise 1.0.4. In the polynomial ring $\boldsymbol{Z}[t]$, the ideal $\mathfrak{m} = (2, t)$ is maximal and the ideal $\mathfrak{q} = (4, t)$ is $\mathfrak{m}$-primary, but is not a power of $\mathfrak{m}$.

Proof. First we can see that $\mathfrak{m}$ is maximal because $\boldsymbol{Z}[t]/\mathfrak{m} \cong \boldsymbol{Z}_2$, which is a field. Then, $\mathfrak{q}$ is $\mathfrak{m}$-primary because

(1) it is a primary ideal for if $ab = 0 \in \boldsymbol{Z}[t]/\mathfrak{q}$, then because $\boldsymbol{Z}[t]/\mathfrak{q} \cong \boldsymbol{Z}_4$, we either have one of the terms equivalent to 4 in $\boldsymbol{Z}_4$ or both terms are equivalent to 2. The former case is $ab \in \mathfrak{q} \implies a \in \mathfrak{q}$ and the latter is when $a \notin \mathfrak{q} \implies b^n \in \mathfrak{q}$, with $n = 2$ here.

(2) to check that $\sqrt{\mathfrak{q}} = \mathfrak{m}$, it suffices to show that $(\boldsymbol{Z}[t]/\mathfrak{q})/\mathfrak{N} \cong \boldsymbol{Z}[t]/\mathfrak{m}$ with $\mathfrak{N}$ the nilradical of $\boldsymbol{Z}[t]/\mathfrak{q}$. We can see that the nilradical of $\boldsymbol{Z}[t]/\mathfrak{q} \cong \boldsymbol{Z}_4$ is isomorphic to the ideal (2) in $\boldsymbol{Z}_4$, and $(\boldsymbol{Z}[t]/\mathfrak{q})/\mathfrak{N} \cong \boldsymbol{Z}_4/(2) \cong \boldsymbol{Z}_2 \cong \boldsymbol{Z}[t]/\mathfrak{m}$.

Finally, suppose FTSOC that $(4,t) = (2,t)^n$. If $n > 2$, $4 \notin (2,t)$. Thus $n = 2$ as $n = 1$ is obviously eliminated. But $(2,t)^2 = (4, 2t, t^2)$, which doesn't contain t. $\square$

Exercise 1.0.5. In the polynomial ring $K[x,y,z]$ where K is a field and x,y,z are independent indeterminates, let $\mathfrak{p}_1 = (x,y)$, $\mathfrak{p}_2 = (x,z)$, $\mathfrak{m} = (x,y,z)$; $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, and $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Which components are isolated and which are embedded?

Proof. Obviously $\mathfrak{a} \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ as $\mathfrak{a} = (x^2, xz, yx, yz)$. Now suppose we have $a \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Then because $a \in \mathfrak{p}_1$, $a = xp(x,y,z) + yq(x,y,z)$. Because $a \in \mathfrak{p}_2$ and $xp(x,y,z) \in \mathfrak{p}_2$ and y is prime in $K[x,y,z]$, we must have that either $x|q(x,y,z)$ or $z|q(x,y,z)$. In either case, x^2 or $yz|yq(x,y,z)$, so that term is in $\mathfrak{p}_1\mathfrak{p}_2$.

Thus all we need to show now is that $xp(x,y,z) \in \mathfrak{p}_1\mathfrak{p}_2$. From the above, we can also conclude that $yq(x,y,z) \in \mathfrak{m}^2$. Thus we know that $xp(x,y,z) \in \mathfrak{m}^2$, hence either x,y,z divides p. In all three cases, this puts xp in $\mathfrak{p}_1\mathfrak{p}_2$. Hence $a \in \mathfrak{a}$.

Because $\sqrt{m^2} = \mathfrak{m}$ as $\mathfrak{m}$ is prime, $\mathfrak{m}$ is an associated prime. As $\mathfrak{p}_1, \mathfrak{p}_2$ are prime, they are associated primes. Clearly $\mathfrak{p}_1, \mathfrak{p}_2$ are minimal and contain $\mathfrak{m}$, so $\mathfrak{p}_i$ are isolated and $\mathfrak{m}$ is embedded. $\square$

Exercise 1.0.6. Let X be an infinite compact Hausdorff space, $C(X)$ the ring of real-valued continuous functions on X (Chapter 1, Exercise 26). Is the zero ideal decomposable in this ring?

Proof. No. Recall that by Exercise 1.16 that every maximal ideal is of the form $\mathfrak{m}_x = \{f \in C(X) | f(x) = 0\}$. First we show that every primary ideal is contained in exactly one maximal ideal. Suppose we have primary $\mathfrak{p} \subseteq \mathfrak{m}_x \cap \mathfrak{m}_y$.

Because X is Hausdorff, there is open disjoint neighborhoods U, V such that $x \in U$ and $y \in V$. By Urysohn's Lemma, we have $f_x, f_y \in C(X)$ such that $f_x(x) = 1, f_x(U^c) = 0, f_y(y) = 1, f_y(V^c) = 0$. Then $f_x f_y = 0$ because f_x is non-zero on U and f_y is non-zero on V, so their product is non-zero on $U \cap V = \emptyset$. Because $f_x(x) = 1$ and $f_y(y) = 1$, neither are in $\mathfrak{p}$. But this contradicts $\mathfrak{p}$ being primary.

Now suppose FTSOC that $(0) = \cap^n \mathfrak{q}_i$. Let $\mathfrak{q}_i \subseteq \mathfrak{m}_{x_i}$. Now take a point $x \notin \{x_i\}$. By Urysohn's Lemma, there is δ_i that vanishes on x_i and not x. Let f_i be the product of an element in $\mathfrak{q}_i$ that doesn't vanish on x (because $\mathfrak{q}_i \not\subseteq \mathfrak{m}_x$) with δ_i. Then $\prod f_i \in \cap \mathfrak{q}_i$ but isn't 0 as it doesn't vanish on x. This contradicts $(0) = \cap^n \mathfrak{q}_i$. $\square$

Exercise 1.0.7. Let A be a ring an let $A[x]$ denote the ring of polynomials in one indeterminate over A. For each ideal $\mathfrak{a}$ of A, let $\mathfrak{a}[x]$ denote the set of all polynomials in $A[x]$ with coefficients in $\mathfrak{a}$.

(i) $\mathfrak{a}[x]$ is the extension of $\mathfrak{a}$ to $A[x]$.
(ii) If $\mathfrak{p}$ is a prime ideal in A, then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.
(iii) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$.

(iv) If $\mathfrak{a} = \cap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition in A, then $\mathfrak{a}[x] = \cap_{i=1}^n \mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.
(v) If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.

Proof. i) The image of $\mathfrak{a}$ in $A[x]$ is $\mathfrak{a}$. The ideal generated by it includes ax^n for $a \in \mathfrak{a}$ and natural n. Thus all polynomials with coefficients in $\mathfrak{a}$ are in the ideal. Finally, all elements of the ideal generated by $\mathfrak{a}$ in $A[x]$ have coefficients in $\mathfrak{a}$ by definition (finite sums of products of elements of $A[x]$ with elements of $\mathfrak{a}$).

ii) We have that $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$ because each element of $A[x]/\mathfrak{p}[x]$ is a polynomial with coefficients in $A/\mathfrak{p}$. A polynomial ring over an integral domain $(A/\mathfrak{p})[x]$ is an integral domain.

iii) Consider $A[x]/\mathfrak{q}[x]$. By the above, this is isomorphic to $(A/\mathfrak{q})[x]$. Now suppose we have a zero divisor in this ring, say $fg = 0$. Then by chapter 1 exercise 2, we have $a \in A/\mathfrak{q}$ such that $af = 0$. Thus every coefficient of f is a zero divisor in $A/\mathfrak{q}$. Therefore every coefficient is nilpotent, as $\mathfrak{q}$ is primary. By Chapter 1 Exercise 2 again, this makes f nilpotent.

Next we can see that $\mathfrak{p}[x] \subseteq r(\mathfrak{q}[x])$ because $\mathfrak{q} \subseteq \mathfrak{q}[x]$ and $r(\mathfrak{q}[x])$ is an ideal in $A[x]$. Finally, we can note that $\mathfrak{q}[x] \subseteq \mathfrak{p}[x]$ because $\mathfrak{q} \subseteq \mathfrak{p}$, and as $r(\mathfrak{q}[x])$ is the smallest prime ideal containing $\mathfrak{q}[x]$, it must equal $\mathfrak{p}[x]$.

Lemma 1.1. $\mathfrak{p} \subseteq \mathfrak{q} \iff \mathfrak{p}[x] \subseteq \mathfrak{q}[x]$

Proof. If $\mathfrak{p} \subseteq \mathfrak{q}$, any polynomial with coefficients in $\mathfrak{p}$ have coefficients in $\mathfrak{q}$.

If $\mathfrak{p}[x] \subseteq \mathfrak{q}[x]$, then $\mathfrak{p} \subseteq \mathfrak{p}[x]$ gives us that $\mathfrak{p} \subseteq \mathfrak{q}$ ($\mathfrak{q}$ is the degree zero component). □

iv) We can see that $\cap \mathfrak{q}_i[x] = (\cap \mathfrak{q}_i)[x]$ because a polynomial in all $\mathfrak{q}_i[x]$ has coefficients in all $\mathfrak{q}_i$. By iii), all we need to show is that $\cap \mathfrak{q}_i[x]$ is a minimal primary decomposition. Condition i) is clearly met because the degree 0 components are distinct. The conditions for ii) are satisfied because of the above lemma and the fact that $(\mathfrak{q}_i)[x] = \cap \mathfrak{q}_i[x]$.

v) This is because of iv) and the lemma above. □

Exercise 1.0.8. Let k be a field. Show that in the polynomial ring $k[x_1, \ldots, x_n]$ the ideals $\mathfrak{p}_i = (x_1, \ldots, x_i)$ $(1 \leq i \leq n)$ are prime and all their powers are primary. Use Exercise 7.

Proof. Clearly $k[x_1, \ldots, x_n]/\mathfrak{p}_i$ is prime because this is isomorphic to $k[x_{i+1}, \ldots, x_n]$, an integral domain. Let $\mathfrak{q}_i^m = \mathfrak{p}_i^m \cap k[x_1, \ldots, x_i]$. Then $\mathfrak{p}_i^m = (\mathfrak{q}_i^m)^e = \mathfrak{q}_i^m[x_{i+1}, \ldots, x_n]$. By 7iii), it then suffices to show that $\mathfrak{q}_i^m$ is $\mathfrak{q}_i$ primary (as this will then make $\mathfrak{q}_i^m[x_{i+1}, \ldots, x_n]$ primary). By discussion in the chapter, $r(\mathfrak{q}_i^m) = \mathfrak{q}_i$. So all we need to do is to show that $\mathfrak{q}_i^m$ is primary in $k[x_1, \ldots, x_i]$.

Suppose we have $ab = 0$ in $k[x_1, \ldots, x_i]/\mathfrak{q}_i^m$. We can see that a or b has no degree 0 component, because otherwise we would have a degree 0 element that can't be cancelled out ($\mathfrak{q}_i^m$ is a homogenous ideal, so we still get a direct sum decomposition of $k[x_1, \ldots, x_i]/\mathfrak{q}_i^m$). Finally, because all monomials are nilpotent in this ring, this implies that a or b is in the ideal of nilpotents, showing that all non-units are nilpotent. Hence $\mathfrak{q}_i^m$ is primary. □

Exercise 1.0.9. In a ring A, let $D(A)$ denote the set of prime ideals $\mathfrak{p}$ which satisfy the following condition: there exists $a \in A$ such that $\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : a)$. Show that $x \in A$ is a zero divisor $\iff$ $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$.
Let S be a multiplicatively closed subset of A, and identify $\operatorname{Spec}(S^{-1}A)$ with its image in $\operatorname{Spec}(A)$ (Chapter 3, Exercise 21). Show that

$$D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A).$$

If the zero ideal has a primary decomposition, show that $D(A)$ is the set of associated prime ideals of 0.

Proof. Suppose that $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$. Then because $a\mathfrak{p} \subseteq (0)$, $ax = 0$ and x is a zero divisor.

If x is a zero divisor: Suppose $ax = 0$. Then $x \in (0 : a)$, and $(0 : a) \neq (1)$ because A can be assumed to be non-zero. Thus there is $\mathfrak{p} \supseteq (0 : a)$. Because the intersection of a descending chain of prime ideals is prime, we can find a minimal such prime among those containing $(0 : a)$.

$D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A)$:

$\subseteq$: The left hand side is the set of minimal prime ideals that contain some $(0 : \frac{a}{b})$. By the inclusion respecting bijection from Proposition 3.11, these correspond to minimal prime ideals in A that don't meet S. Because each $S^{-1}\mathfrak{p}$ contains $(0 : \frac{a}{b})$, the contraction contains $(0 : a)$ for if $ac = 0$ for $c \in A$, then $\frac{a}{b} \cdot \frac{c}{1} = 0$ in $S^{-1}A$.

$\supseteq$: The right hand side is the set of minimal prime ideal that doesn't contain S by Proposition 3.11 yet contain $(0 : a)$ for some a (it is minimal among these primes because the bijection $\operatorname{Spec} S^{-1}A$ to a subset of $\operatorname{Spec} A$ respects inclusions). Thus these biject to minimal prime ideals of S^{-1}, and they contain $(0 : \frac{a}{1})$ because $\forall \frac{p}{q} \in S^{-1}\mathfrak{p}$, $\frac{a}{1}\frac{p}{q} = \frac{ap}{q} = 0$ because $p \in \mathfrak{p}$ and $ap = 0$.

Finally, assume that there is a primary decomposition, say $0 = \cap \mathfrak{q}_i$ with associated primes $\mathfrak{p}_i = r(0 : a_i)$. Take $\mathfrak{p} \in D(A)$ and let $\mathfrak{p}$ be minimal among prime ideals containing $(0 : a)$. Then $\mathfrak{p} \supseteq r(0 : a)$ because $r(0 : a)$ is the intersection of prime ideals containing $(0 : a)$. Thus

$$\mathfrak{p} \supseteq r(\cap \mathfrak{q}_i : a) = \cap r(\mathfrak{q}_i : a) \supseteq r(0 : a) \supseteq (0 : a).$$

The latter containment is due to 0 being in all the $\mathfrak{q}_i$ so that $(\mathfrak{q}_i : a) \supseteq (0 : a)$.

We can see that $\cap r(\mathfrak{q}_i : a) = \cap_{\mathfrak{n}_i \in S \subseteq \{\mathfrak{p}_i\}} \mathfrak{n}_i$ where S is some subset of the associated primes. This is because for each i, $a \in \mathfrak{q}_i$ implies that $(\mathfrak{q}_i : a) = A$, allowing us to remove it from the intersection, and if $a \notin \mathfrak{q}_i$, then because $\mathfrak{q}_i$ is primary, $x \in (\mathfrak{q}_i : a) \implies ax \in \mathfrak{q}_i \implies x^a \in \mathfrak{q}_i$ for some a. Thus by taking the radical of $(\mathfrak{q}_i : a)$, we see that $x \in \mathfrak{p}_i$.

Next, by Proposition 1.11, $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. The minimality of $\mathfrak{p}$ then implies that $\mathfrak{p} = \mathfrak{p}_i$, an associated prime.

Now conversely, an associated prime $\mathfrak{p}_i = r(0 : a_i)$ is minimal among primes containing $(0 : a_i)$ by definition of the radical. $\square$

Exercise 1.0.10. For any prime ideal $\mathfrak{p}$ in a ring A, let $S_{\mathfrak{p}}(0)$ denote the kernel of the homomorphism $A \to A_{\mathfrak{p}}$. Prove that

(i) $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$.
(ii) $r(S_{\mathfrak{p}}(0)) = \mathfrak{p} \iff \mathfrak{p}$ is a minimal prime ideal of A.
(iii) If $\mathfrak{p} \supseteq \mathfrak{p}'$, then $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{p}'}(0)$.

(iv) $\cap_{\mathfrak{p}\in D(A)} S_{\mathfrak{p}}(0) = 0$, where $D(A)$ is defined in Exercise 9.

Proof. i) Any $a \in S_{\mathfrak{p}}(0)$ has the property that $\frac{a}{1} = 0$, i.e. $\exists s \in A \setminus \mathfrak{p}$ such that $as = 0$ in A. Mapping this into $A/\mathfrak{p}$, we have that $as \equiv 0$. Because $A/\mathfrak{p}$ is an integral domain and $s \notin \mathfrak{p}$, $a \equiv 0 \implies a \in \mathfrak{p}$.

iii) Let $a \in S_{\mathfrak{p}}(0)$. Then there is $s \in A \setminus \mathfrak{p} \subseteq A \setminus \mathfrak{p}'$ such that $as = 0$ in A. But then $\frac{a}{1} = 0 \in A_{\mathfrak{p}'}$ because $A \setminus \mathfrak{p} \subseteq A \setminus \mathfrak{p}'$, so we can use the same element to prove that it vanishes.

ii)

$\implies$) Suppose we have $\mathfrak{q} \subseteq \mathfrak{p}$. Then by iii) $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{q}}(0)$. So we then have

$$S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{q}}(0) \subseteq \mathfrak{q} \subseteq \mathfrak{p}.$$

But because $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$ and the radical is the intersection of the prime ideals containing it, this implies that $\mathfrak{p} \subseteq \mathfrak{q}$. Thus $\mathfrak{p} = \mathfrak{q}$ and $\mathfrak{p}$ is minimal.

$\Leftarrow$) Because $\mathfrak{p}$ is minimal, $S \setminus \mathfrak{p}$ is maximal among multiplicatively closed subsets of A that don't contain 0. Thus for any $x \in \mathfrak{p}$, $0 \in$ the multiplicatively closed subset spanned by $S \cup \{x\}$. Hence there is $s \in S$ and n such that $sx^n = 0$. Thus $x \in r(0 : s)$. But $S_{\mathfrak{p}}(0) = \cup_{s\in A\setminus\mathfrak{p}}(0 : s)$ by Proposition 3.11. So by the remark on page 9, $r(S_{\mathfrak{p}}(0)) = \cup r(0 : s) \implies x \in r(S_{\mathfrak{p}}(0))$. As x was arbitrary, $\mathfrak{p} \subseteq r(S_{\mathfrak{p}}(0))$, which by exercise i) gives us that $\mathfrak{p} \subseteq r(S_{\mathfrak{p}}(0)) \subseteq r(\mathfrak{p}) = \mathfrak{p}$. Thus $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$.

iv) Obviously 0 is in the intersection. Now suppose we have $x \neq 0$. Then $(0 : x) \neq (1)$, so we can take a minimal prime ideal $\mathfrak{q}$ containing $(0 : x)$. This is in $D(A)$ by definition. Because $(0 : x) \subseteq \mathfrak{q}$, there is no $s \in A \setminus \mathfrak{q}$ such that $xs = 0$ by definition of $(0 : x)$. Thus $x \notin S_{\mathfrak{q}}(0)$. Hence $x \notin \cap S_{\mathfrak{p}}(0)$. □

Exercise 1.0.11. If $\mathfrak{p}$ is a minimal prime ideal of a ring A, show that $S_{\mathfrak{p}}(0)$ (Exercise 10) is the smallest $\mathfrak{p}$-primary ideal.
Let $\mathfrak{a}$ be the intersection of the ideals $S_{\mathfrak{p}}(0)$ as $\mathfrak{p}$ runs through the minimal prime ideals of A. Show that $\mathfrak{a}$ is contained in the nilradical of A.
Suppose that the zero ideal is decomposable. Prove that $\mathfrak{a} = 0$ if and only if every prime ideal of 0 is isolated.

Proof. If $\mathfrak{p}$ is a minimal prime, then $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$ by Exercise 4.10ii. Further, $S_{\mathfrak{p}}(0)$ is a primary ideal for if we have $ab \in S_{\mathfrak{p}}(0)$, then there is $c \in A\setminus\mathfrak{p}$ such that $abc = 0$. As $0 \in \mathfrak{p}$ and $\mathfrak{p}$ is prime, $c \notin \mathfrak{p} \implies ab \in \mathfrak{p}$. Thus either a or b are in $\mathfrak{p} = r(S_{\mathfrak{p}}(0))$. Finally, $S_{\mathfrak{p}}(0)$ is the smallest $\mathfrak{p}$-primary ideal because given a $\mathfrak{p}$-primary ideal I, $I \subseteq r(I) = \mathfrak{p}$. Thus $A \setminus \mathfrak{p} \cap I = \emptyset$, so $I_{\mathfrak{p}}$ is $\mathfrak{p}_{\mathfrak{p}}$-primary by Proposition 4.8, and the contraction of $I_{\mathfrak{p}}$ is I. But the contraction contains $S_{\mathfrak{p}}(0)$.

Because $S_{\mathfrak{p}}(0) \subseteq r(S_{\mathfrak{p}}(0))$, we have that $\mathfrak{a} \subseteq r(\mathfrak{a}) = \cap r(S_{\mathfrak{p}_i}(0))$ where the $\mathfrak{p}_i$ range over minimal prime ideals. But by Exercise 4.10ii, $r(S_{\mathfrak{p}_i}(0)) = \mathfrak{p}_i$. Thus $\mathfrak{a} \subseteq \cap\mathfrak{p}_i$. The RHS equals the nilradical because the nilradical is the intersection of all prime ideals, and we can just remove non-minimal ones.

Assume that the zero ideal is decomposable into $\cap\mathfrak{q}_i$ with $r(\mathfrak{q}_i) = \mathfrak{p}_i$.

Assume every prime ideal of 0 is isolated. Then no By Exercise 4.9, $D(A)$ is the set of associated primes of 0, which by assumption are isolated. So by Exercise 4.10iv, $\cap_{\mathfrak{q}\in D(A)} S_{\mathfrak{q}}(0) = 0$.

Suppose that $\mathfrak{a} = 0$. For each minimal prime $\mathfrak{p}^j$, let $\{\mathfrak{p}_{j,i}\}$ meet $A \setminus \mathfrak{p}^j$. Let $\mathfrak{q}_{j,i}$ be the corresponding primary components. By Proposition 4.9, $\mathfrak{a} = \cap_j \cap_{j,i} \mathfrak{q}_{j,i}$. As this contains Then because of Exercise 4.10iii), $\cap S_{\mathfrak{q}}(0) \subseteq \cap S_{\mathfrak{p}}(0)$ where $\mathfrak{q}$ range over associated primes of 0 as each $\mathfrak{q}$ contains a minimal prime ideal. Thus $\cap S_{\mathfrak{q}}(0) = 0$. □

Exercise 1.0.12. Let A be a ring, S a multiplicatively closed subset of A. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the contraction of $S^{-1}\mathfrak{a}$ in A. The ideal $S(\mathfrak{a})$ is called the *saturation* of $\mathfrak{a}$ with respect to S. Prove that

(i) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$.
(ii) $S(r(\mathfrak{a})) = r(S(\mathfrak{a}))$.
(iii) $S(\mathfrak{a}) = (1) \iff \mathfrak{a}$ meets S.
(iv) $S_1(S_2(\mathfrak{a})) = (S_1S_2)(\mathfrak{a})$.
If $\mathfrak{a}$ has a primary decomposition, prove that the set of ideals $S(\mathfrak{a})$ (where S runs through all multiplicatively closed subsets of A) is finite.

Proof. i) Because of Exercise 1.18, $S(\mathfrak{a}) \cap S(\mathfrak{b}) = \mathfrak{a}^c \cap \mathfrak{b}^c = (\mathfrak{a} \cap \mathfrak{b})^c = S(\mathfrak{a} \cap \mathfrak{b})$.

ii) Because of Exercise 1.18, $S(r(\mathfrak{a})) = r(\mathfrak{a})^c = r(\mathfrak{a}^c) = r(S(\mathfrak{a}))$.

iii) Proposition 3.11.

iv) By Proposition 3.11ii), $S_2(\mathfrak{a}) = \mathfrak{a}^{ec} = (S_2^{-1}\mathfrak{a})^c = \cup_{s_2 \in S_2}(\mathfrak{a} : s_2)$.. Thus $S_1(S_2(\mathfrak{a})) = \cup_{s_1 \in S_1} \cup_{s_2 \in S_2} ((\mathfrak{a} : s_2) : s_1)$. By Exercise 1.12, this equals $\cup_{s_1 \in S_1} \cup_{s_2 \in S_2} (\mathfrak{a} : s_2 s_1)$. This then equals $\cup_{s \in S_1 S_2}(\mathfrak{a} : s) = ((S_1S_2)^{-1}\mathfrak{a})^c = S_1S_2(\mathfrak{a})$.

Now suppose that $\mathfrak{a}$ has a decomposition $\cap \mathfrak{q}_i$. Then by Proposition 4.9, $S(\mathfrak{a})$ is an intersection of a finite subset of the $\mathfrak{q}_i$'s. There are only finitely many possibilities. □

Exercise 1.0.13. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then *nth symbolic power of* $\mathfrak{p}$ is defined to be the ideal (in the notation of Exercise 12)

$$\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n)$$

where $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$. Show that

(i) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal;
(ii) if $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component;
(iii) If $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is its $\mathfrak{p}$-primary component
(iv) $\mathfrak{p}^{(n)} = \mathfrak{p}^n \iff \mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary.

Proof. i) Suppose we have $ab \in \mathfrak{p}^{(n)}$ and $a \notin \mathfrak{p}^{(n)}$. Then by Proposition 3.11, $S_{\mathfrak{p}}(\mathfrak{p}^n) = \cup_{s \in A \setminus \mathfrak{p}}(\mathfrak{p}^n : s)$. Now let s be such that $ab \in (\mathfrak{p}^n : s)$. Then $abs \in \mathfrak{p}^n$.

If $bs \in \mathfrak{p}$, then because $s \notin \mathfrak{p}$, $b \in \mathfrak{p} \implies b^n \in \mathfrak{p}^{(n)}$, the requirement for $\mathfrak{p}^{(n)}$ to be primary. If $bs \notin \mathfrak{p}$, then $a \in (\mathfrak{p}^n : bs) \subseteq \cup_{s \in A \setminus \mathfrak{p}}(\mathfrak{p}^n : s)$.

Finally, because $r((S_{\mathfrak{p}}\mathfrak{p}^n)^c) = r(S_{\mathfrak{p}}\mathfrak{p}^n)^c = (S_{\mathfrak{p}}\mathfrak{p})^c$ because of Proposition 4.8. Then by Proposition 3.11 this equals $\mathfrak{p}$.

ii)

iii) □

Exercise 1.0.14. Let $\mathfrak{a}$ be a decomposable ideal in a ring A and let $\mathfrak{p}$ be a maximal element of the set of ideals $(\mathfrak{a} : x)$, where $x \in A$ and $x \notin \mathfrak{a}$. Show that $\mathfrak{p}$ is a prime ideal belonging to $\mathfrak{a}$.

Proof. First we can see that $\mathfrak{p}$ is prime: suppose we have $ab \in \mathfrak{p} = (\mathfrak{a} : x)$. Because $\mathfrak{a}$ is decomposable, $(\mathfrak{a} : x) = (\cap \mathfrak{q}_i : x) = \cap(\mathfrak{q}_i : x)$. If $a \notin \mathfrak{p}$, then $(\mathfrak{p} : a) \supseteq \mathfrak{p}$. But then $(\mathfrak{p} : a) = (\cap(\mathfrak{q}_i : x) : a) = \cap((\mathfrak{q}_i : x) : a) = \cap(\mathfrak{q}_i : ax) = (\cap \mathfrak{q}_i : ax) = (\mathfrak{a} : ax)$ by Exercise 1.12 in the chapter.

If $ax \notin \mathfrak{a}$, then because $\mathfrak{p}$ is maximal $(\mathfrak{p} : a) = \mathfrak{p}$, and $b \in (\mathfrak{p} : a)$. If $ax \in \mathfrak{a}$, then by definition, $a \in (\mathfrak{a} : x)$.

Finally, $\mathfrak{p}$ is a prime ideal belonging to $\mathfrak{a}$ because it is of the right form for the Uniqueness Theorem: $\mathfrak{p} = (\mathfrak{a} : x) = r(\mathfrak{a} : x)$. □

Exercise 1.0.15. Let $\mathfrak{a}$ be a decomposable ideal in a ring A, let Σ be an isolated set of prime ideals belonging to $\mathfrak{a}$, and let $\mathfrak{q}_\Sigma$ be the intersection of the corresponding primary components. Let f be an element of A such that, for each prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$, we have $f \in \mathfrak{p} \iff \mathfrak{p} \notin \Sigma$, and let S_f be the set of all powers of f. Show that $\mathfrak{q}_\Sigma = S_f(\mathfrak{a}) = (\mathfrak{a} : f^n)$ for all large n.

Proof. First we can show that $S_f(\mathfrak{a}) = (\mathfrak{a} : f^n)$ We use Proposition 3.11 to get that $\mathfrak{a}^{ec} = \cup_n (\mathfrak{a} : f^n)$. Clearly $(\mathfrak{a} : f^n) \subseteq (\mathfrak{a} : f^{n+1})$. Thus if we show that $(\mathfrak{a} : f^n)$ stabilizes for some large enough n, we have shown this part. Now

Take some $q \in \mathfrak{q}_\Sigma$. Then $q \in \mathfrak{q}_i$ for some isolated component □

Exercise 1.0.16. If A is a ring in which every ideal has a primary decomposition, show that every ring of fractions $S^{-1}A$ has the same property.

Proof. Just Proposition 4.9 and Proposition 3.11. □

Exercise 1.0.17. Let A be a ring with the following property.

(L1) For every ideal $\mathfrak{a} \neq (1)$ in A and every prime ideal $\mathfrak{p}$, there exists $x \notin \mathfrak{p}$ such that $S_\mathfrak{p}(\mathfrak{a}) = (\mathfrak{a} : x)$, where $S_\mathfrak{p} = A \setminus \mathfrak{p}$.

Then every ideal in A is an intersection of (possibly infinitely many) primary ideals. [Let $\mathfrak{a}$ be an ideal $\neq (1)$ in A, and let $\mathfrak{p}_1$ be a minimal element of the set of prime ideals containing $\mathfrak{a}$. Then $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$ is $\mathfrak{p}_1$-primary (by Exercise 11), and $\mathfrak{q}_1 = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{p}_1$. Show that $\mathfrak{a} = \mathfrak{q}_1 \cap (\mathfrak{a} + ((x))$.

Now let $\mathfrak{a}_1$ be a maximal element of the set of ideals $\mathfrak{b} \supseteq \mathfrak{a}$ such that $\mathfrak{q}_1 \cap \mathfrak{b} = \mathfrak{a}$, and choose $\mathfrak{a}_1$ so that $x \in \mathfrak{a}_1$, and therefore $\mathfrak{a}_1 \not\subseteq \mathfrak{p}_1$. Repeat the construction starting with $\mathfrak{a}_1$ and so on. At the nth stage we have $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{a}_n$ where the $\mathfrak{q}_1$ are primary ideals, $\mathfrak{a}_n$ is maximal among the ideals $\mathfrak{b}$ containing $\mathfrak{a}_{n-1} = \mathfrak{a}_n \cap \mathfrak{q}_n$ such that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{b}$, and $\mathfrak{a}_n \not\subseteq \mathfrak{p}_n$. If at any stage we have $\mathfrak{a}_n = (1)$, the process stops, and $\mathfrak{a}$ is a finite intersection of primary ideals. If not, continue by transfinite induction, observing that each $\mathfrak{a}_n$ strictly contains $\mathfrak{a}_{n-1}$.]

Proof. □

2. Integral Dependence and Valuations

Exercise 2.0.1. Let $f \colon A \to B$ be an integral homomorphism of rings. Show that $f^* \colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a *closed* mapping, i.e. that it maps closed sets to closed sets. (This is a geometrical equivalent of (5.10).)

Proof. We want to show that for all ideals $\mathfrak{a} \subseteq B$, $f^*(V(\mathfrak{a})) = V(\mathfrak{b})$ for some ideal $\mathfrak{b} \subseteq A$. I propose that we let $\mathfrak{b} = f^{-1}(\mathfrak{a})$ (which is an ideal because preimages take ideals to ideals).

$\subseteq$: For $\mathfrak{p} \in V(\mathfrak{a})$, $\mathfrak{a} \subseteq \mathfrak{p}$. Thus $\mathfrak{b} = f^{-1}(\mathfrak{a}) \subseteq f^{-1}(\mathfrak{p}) = f^*(\mathfrak{p}) \implies f^*(\mathfrak{p}) \in V(\mathfrak{b})$.

$\supseteq$: Take some $\mathfrak{p} \in V(\mathfrak{b})$. We can first note that because $\mathfrak{b} = f^{-1}(\mathfrak{a})$ and $0 \in \mathfrak{a}$, $\ker f \subseteq \mathfrak{b} \subseteq \mathfrak{p}$. Then $f(\mathfrak{p})$ is prime as an ideal of $f(A)$, because if $f(a)f(b) \in f(\mathfrak{p})$, then $f(ab) \in f(\mathfrak{p})$ implies that there is some $c \in \ker f$ such that $ab - c \in \mathfrak{p}$. Because $\ker f \subseteq \mathfrak{p}$, $ab \in \mathfrak{p} \implies$ either a or b is in $\mathfrak{p}$, showing that $f(\mathfrak{p})$ is prie.

Then by Theorem 5.10, as B is integral over $f(A)$, $\exists \mathfrak{q} \in \operatorname{Spec} B$ such that $\mathfrak{q} \cap f(A) = f(\mathfrak{p})$. Finally, $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ shows that this map is surjective. □

Exercise 2.0.2. Let A be a subring of a ring B such that B is integral over A, and let $f \colon A \to \Omega$ be a homomorphism of A into an algebraically closed field Ω. Show that f can be extended to a homomorphism of B into Ω.

Proof. We can extend f by mapping $x \in B$ as follows. Suppose x satisfies the minimal monic A relation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

□

Exercise 2.0.3. Let $f \colon B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, prove that $f \otimes 1 \colon B \otimes_A C \to B' \otimes_A C$ is integral. (This includes (5.6) ii) as a special case.)

Proof. It suffices to show that pure tensors are integral over $\operatorname{im}(f \otimes 1)$ because they generate $B' \otimes_A C$ and sums and products of integral elements are integral (i.e. Corollary 5.3). Now suppose we have $b' \otimes c$ and

$$(b')^n + a_1 (b')^{n-1} + \cdots + a_n = 0 \qquad a_i \in f(A)$$

because f is integral.

Then we will show that

$$(b' \otimes c)^n + a_1(1 \otimes c)(b' \otimes c)^{n-1} + \cdots + a_n(1 \otimes c^n)(1 \otimes 1)$$

will be an integral equation for $b' \otimes c$ over $f(B \otimes_A C)$. First, each $a_i(1 \otimes c^i)$ is in $f(B \otimes_A C)$ as $a_i \in f(B)$. Then we expand:

$$\begin{aligned}
&(b' \otimes c)^n + a_1(1 \otimes c)(b' \otimes c)^{n-1} + \cdots + a_n(1 \otimes c^n)(1 \otimes 1) \\
&\quad = ((b')^n \otimes c^n) + (a_1 (b')^{n-1} \otimes c^{n-1}) + \cdots + (a_n \otimes c^n) \\
&\quad = ((b')^n + a_1 (b')^{n-1} + \cdots + a_n) \otimes c^n \\
&\quad = 0 \otimes c^n = 0
\end{aligned}$$

□

Exercise 2.0.4. Let A be a subring of B such that B is integral over A. Let $\mathfrak{n}$ be a maximal ideal of B and let $\mathfrak{m} = \mathfrak{n} \cap A$ be the corresponding maximal ideal of A (see (5.8)). Is $B_{\mathfrak{n}}$ necessarily integral over $A_{\mathfrak{m}}$?

Proof. □

Exercise 2.0.5. Let $A \subseteq B$ be rings, B integral over A.

(i) If $x \in A$ is a unit in B then it is a unit of A.
(ii) The Jacobson radical of A is the contraction of the Jacobson radical of B.

Proof. i) Suppose we have $xb = 1$ with $b \in B$. Then we have some integral relation of lowest degree

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0$$

with $a_i \in A$. By multiplying it by x, we get

$$b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1} + a_n x = 0$$

But this is an integral relation of lower degree, implying that b is the root of a polynomial over A with degree 1. But this just implies that $b \in A$.

ii) The contraction of the Jacobson radical of B is $(\bigcap_{\mathfrak{m} \in \operatorname{Specm} B} \mathfrak{m}) \cap A$. By Corollary 5.8, $\mathfrak{m} \in \operatorname{Specm} B \iff \mathfrak{m} \cap A$ is Maximal. Thus $(\bigcap_{\mathfrak{m} \in \operatorname{Specm} B} \mathfrak{m}) \cap A = \bigcap_{\mathfrak{n} \in \operatorname{Specm} A} \mathfrak{n}$, which is the Jacobson radical of A. □

Exercise 2.0.6. Let $B_1, \ldots, B_n$ be integral A-algebras. Show that $\prod_{i=1}^n B_i$ is an integral A-algebra.

Proof. Suppose we have $(b_1, b_2, b_n) \in \prod B_i$ with

$$f_i(b_i) = b_i^{n_i} + a_{1i}^{n_i - i} + \cdots + a_{n_i i} = 0.$$

Then let $f(x) = \prod f_i(x)$. This is a polynomial over A, and by considering it as a polynomial in $\prod B_i$, we can see that $(b_1, b_2, \ldots, b_n)$ is a root of it: the ring operation is done coordinate wise, so $f((b_1, b_2, \ldots, b_n)) = (f(b_1), f(b_2), \ldots, f(b_n)) = (0, 0, \ldots, 0)$. □

Exercise 2.0.7. Let A be a subring of a ring B, such that the set $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Proof. Suppose we have an integral relation for $b \in B \setminus A$ of least degree,

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then I claim that we get a contradiction from this:

$$b(b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1}) + a_n = 0.$$

Suppose that $b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1} \in A$. Then by subtracting the element of A, we get a lower degree integral relation, a contradiction. Thus $b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1} \in B \setminus A$. Because $B \setminus A$ is multiplicatively closed, $b(b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1}) \in B \setminus A$. But $-a_n$ is in A, a contradiction. □

Exercise 2.0.8.

(i) Let A be a subring of an integral domain B, and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ such that $fg \in C[x]$. Then f, g are in $C[x]$.
(ii) Prove the same result without assuming that B (or A) is an integral domain.

Proof. ii) We use induction on the degree of fg.

First we show it for fg of degree 2. Then $f(x) = x + a$ and $g(x) = x + b$ for some $a, b \in B$. If $fg \in C$, then $f(x)g(x) = x^2 + (a+b)x + ab$.

Using the quadratic formula shows us that $-a, -b$ are roots of this polynomial, so $-a, -b$ are integral over $A[a+b, ab]$. Because $a+b, ab \in C$, $A[a+b, ab]$ is finitely generated as an A-module. As $-a, -b$ are integral over $A[a+b, ab]$, $A[a+b, ab, -a, -b]$ is finitely generated as an A-module. Hence $-a, -b \in C$. Thus $f, g \in C[x]$.

Finally, assume it is true up to fg of degree $n-1$. Let $f(x) = f_1(x)x + b_1$ and $g(x) = g_1(x)x + b_2$. Note that if $fg \in C[x]$, $(b_1g_1(x) + b_2f_1(x))x + b_1b_2 \in C[x]$, being the last two terms of fg. Thus by subtracting off $(b_1g_1(x)+b_2f_1(x))x+b_1b_2$, the result is also in $C[x]$. Thus $f_1(x)g_1(x)x^2 \in C[x]$, so $f_1(x)g_1(x) \in C[x]$. By our induction hypothesis, $f_1, g_1 \in C[x]$. Thus $f_1(x)x + b_1, g_1(x)x + b_2$ are in $C[x]$. □

Exercise 2.0.9. Let A be a subring of a ring B and let C be the integral closure of A in B. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$.

Proof. We use induction on the degree to first show that $C[x]$ is integral over $A[x]$. Obviously degree 0 terms of $C[x]$ are integral over $A[x]$. We can also show that x is integral over $A[x]$, since it is the root of $y - x$.

Now assume that all terms of $C[x]$ up to degree $n-1$ are integral over $A[x]$. Say we have $f = c_0x^n + \cdots + c_n$. By induction hypothesis, $c_1x^{n-1} + \cdots + c_n$ is integral over $A[x]$. Thus if we show that $f - (c_1x^{n-1} + \cdots + c_n) = c_0x^n$ is integral over $A[x]$, we are done. Finally, because x and c_0 are integral over $A[x]$ and products of integral elements are integral, c_0x^n is integral.

Now FTSOC suppose we have $f = b_0x^n + \cdots + b_n \in B[x] \setminus C[x]$ integral over $A[x]$. We can pick one with the least number of non-zero coefficients. Then any integral relation of f will produce an integral relation for b_n by focusing on the degree 0 component. Thus $b_n \in C$. Hence $f - b_n$ is also integral over $A[x]$ and in $B[x] \setminus C[x]$. But $f - b_n$ has fewer non-zero coefficients than f, contradicting our assumption, allowing us to conclude that $C[x]$ is the integral closure of $A[x]$. □