

EXERCISES FROM HARTSHORNE

VINCENT TRAN

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1. SCHEMES

1.1. Sheaves.

Exercise 1.1.1. Let A be an abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf \mathcal{A} defined in the text is the sheaf associated to this presheaf.

Proof. Let the constant presheaf be \mathcal{C} . Fix a connected, open U . Then for a fixed but arbitrary $P \in U$, we have A choices for $s(P)$. We can then see that by the second condition of \mathcal{C}^+ and making V small enough that it falls in the connected open set, there exists t in $\mathcal{C}(V) = A$ such that for all $Q \in V, t_Q = t = s(Q)$. Thus s is constant on V as t is in A .

Finally, take the collection of these neighborhoods. Pick one. Because U is connected, there must be non-empty intersection between this neighborhood and the union of the other neighborhoods. By doing the above argument for a point in their intersection, s is constant on all of U .

By construction, s satisfies conditions (1) and (2) on U , putting $s \in \mathcal{C}^+(U)$. Thus $\mathcal{C}^+(U) = A$. By picking constants on connected components, we can see that such a function satisfies (1) and (2), making $\mathcal{C}^+(U) = \mathcal{A}(U)$. \square

Exercise 1.1.2.

- (a) For any morphism of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$, show that for any point P , $(\ker \phi)_P = \ker(\phi_P)$ and $(\operatorname{im} \phi)_P = \operatorname{im}(\phi_P)$.
- (b) Show that ϕ is injective (respectively, surjective) if and only if the induced map on the stalks ϕ_P is injective (respectively, surjective) for all P .
- (c) Show that a sequence

$$\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \rightarrow \dots$$

of sheaves and morphisms is exact if and only if for each $P \in X$, the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof. a) We have the following commutative diagram by definition:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \ker \phi(U) & \xrightarrow{\iota} & \ker \phi(V) & \longrightarrow & \cdots \\
 & & \searrow & & \swarrow & & \\
 & & & \lim \ker \phi & & & \\
 & & \vdots & & \vdots & & \\
 \cdots & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \cdots \\
 & & \searrow & & \swarrow & & \\
 & & & \mathcal{F}_P & & & \\
 & & \vdots & & \vdots & & \\
 \cdots & \longrightarrow & \mathcal{G}(U) & \xrightarrow{\phi_P} & \mathcal{G}(V) & \longrightarrow & \cdots \\
 & & \searrow & & \swarrow & & \\
 & & & \mathcal{G}_P & & & \\
 & & & \rho_{G,V} & & &
 \end{array}$$

We have a map $\lim \ker \phi \rightarrow \ker \phi_P$ because any element $u \in \lim \ker \phi$ can be represented by $(v, V), v \in \ker \phi(V)$, and v maps to 0 in \mathcal{G}_P (so map u to v_P). Further this is injective, because each map $\ker \phi(U) \rightarrow \mathcal{F}(U)$ is injective, making the induced map injective (Proposition 1.1). Hence if we show it is surjective, we have a bijective homomorphism and thus are the same.

Then for any element $\ell \in \ker \phi_P \subseteq \mathcal{F}_P$, let (v_1, V_1) represent it. Let $(0, V_2)$ represent $\phi_P(\ell)$. Then $\rho_{G,V}(\phi(V)(v)) = 0$ by commutativity. Next restrict 0 to $V_1 \cap V_2$ to get $0 \in \mathcal{G}(V_1 \cap V_2)$, implying that via commuting that $v_1|_{V_1 \cap V_2}$ is mapped to 0 under $\phi(V_1 \cap V_2)$.

So $v_1|_{V_1 \cap V_2} \in \ker \phi(V_1 \cap V_2)$. We have thus found an element $\iota(v_1|_{V_1 \cap V_2})$ that gets mapped to ℓ as $\iota(v_1|_{V_1 \cap V_2})$ can be represented by $(v_1|_{V_1 \cap V_2}, V_1 \cap V_2)$ and $(v_1|_{V_1 \cap V_2})_P \in \ker \phi_P$.

I don't really want to make another large diagram to trace it for the image, so I'll just trust that it is very similar.

b) Follows from a).

c) If $\ker \phi^i = \text{im } \phi^{i-1}$, then $(\ker \phi^i)_P = (\text{im } \phi^{i-1})_P$. By a), we then have $\ker \phi_P^i = \text{im } \phi_P^{i-1}$, showing exactness of stalks. \square

Exercise 1.1.3.

- (a) Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on X . Show that ϕ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U , and there are elements $t_i \in \mathcal{F}(U_i)$, such that $\phi(t_i) = s|_{U_i}$ for all i .
- (b) Give an example of a surjective morphism of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$, and an open set U such that $\phi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is not surjective.

Proof. a) \implies : Take an open cover of U , $\{U_i\}$. Because ϕ is surjective, there is an element t such that $\phi(t) = s$. Then $\phi(t|_{U_i}) = \phi(t)|_{U_i} = s|_{U_i}$. Thus let $t_i = t|_{U_i}$.

\Leftarrow : Take an arbitrary open set U . We want to show that $(\text{im}^+ \phi)(U) = \mathcal{G}(U)$. As $\text{im}^+ \phi$ is a subsheaf of \mathcal{G} , all we need to show is surjectivity. Take some $s \in \mathcal{G}(U)$. Let U_i be the open cover given by hypothesis. Next we will want to use the gluing property of sheaves to get an element $t \in \mathcal{F}(U)$, so we have to check that t_i from the hypothesis agree on intersections.

Take t_i, t_j in $\mathcal{F}(U_i), \mathcal{F}(U_j)$ respectively and let $U_{ij} = U_i \cap U_j$. Then by the definition of a sheaf morphism, $\phi(t_i|_{U_{ij}}) = \phi(t_i)|_{U_{ij}} = s_{U_{ij}}$ and $\phi(t_j|_{U_{ij}}) = \phi(t_j)|_{U_{ij}} = s_{U_{ij}}$. So $t_i|_{U_{ij}} - t_j|_{U_{ij}} \in \ker \phi(U_{ij})$. Hence there is a unique element $t \in \mathcal{F}(U)$ such that $t|_{U_i} = t_i$.

b) Let \mathcal{F} be the constant sheaf of $A = \mathbb{Z}/2\mathbb{Z}$ and \mathcal{G} be \mathcal{F} . Then we have the sheaf morphism $\mathcal{F} \rightarrow \mathcal{G}$ by mapping $\mathcal{F}(\emptyset) = 0$ to 0 , $\mathcal{F}(0) = \mathcal{F}(1) = A$ to A and $\mathcal{F}(A) = A^2$ to A via quotienting. Then the sheaf associated to the image presheaf is the constant sheaf, but $\phi(\mathcal{F}(A)) \neq \mathcal{G}(A) = A^2$. \square

Exercise 1.1.4.

- (a) Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves such that $\phi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is injective for each U . Show that the induced map $\phi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}^+$ of associated sheaves is injective.
- (b) Use part (a) to show that if $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then $\text{im } \phi$ can be naturally identified with a subsheaf of \mathcal{G} , as mentioned in the text.

Proof. a) It suffices to check that the map is injective on stalks. Because $\phi(U)$ is injective, ϕ_p is injective for all $p \in X$ by 1.2a. As $\mathcal{F}_p^+ = \mathcal{F}_p$ and $\mathcal{G}_p^+ = \mathcal{G}_p$ and $\phi_p^+ = \phi_p$, the induced map is injective on all stalks. Hence it is injective.

b) We have injective maps $\iota(U) : \text{im } \phi(U) \rightarrow \mathcal{G}(U)$ via inclusion. Then by part a, the induced map $\iota^+ : (\text{im } \phi)^+ \rightarrow \mathcal{G}^+$ is injective. But because \mathcal{G} is a sheaf, $\mathcal{G}^+ = \mathcal{G}$, so $(\text{im } \phi)^+$ can be naturally identified with a subsheaf. \square

Exercise 1.1.5. Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Proof. It suffices to check that the stalks are isomorphic by Proposition 1.1. First we have that $\text{im } \phi = \mathcal{G}$, so $(\text{im } \phi)_p = \mathcal{G}_p$. Because the stalks of the associated sheaf and presheaf are the same, $(\text{im } \phi)_p$ equals the stalk of the presheaf $U \mapsto \text{im } \phi$ at p , denote it by $\text{im } \phi_p^-$. But since $\ker \phi = 0$, $\mathcal{F}(U)/\ker \phi(U) \cong \mathcal{F}(U)$. By the first isomorphism theorem, the LHS is isomorphic to $\text{im } \phi(U)$ (not as a sheaf). Hence $\text{im } \phi_p^- = \lim_U \text{im } \phi(U) \cong \lim_U \mathcal{F}(U)/\ker \phi(U) \cong \mathcal{F}_p$. So they are isomorphic as stalks. \square

Exercise 1.1.6.

- (a) Let \mathcal{F}' be a subsheaf of a sheaf \mathcal{F} . Show that the natural map of \mathcal{F} to the quotient sheaf \mathcal{F}/\mathcal{F}' is surjective, and has kernel \mathcal{F}' . Thus there is an exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{F}' \rightarrow 0.$$

- (b) Conversely, if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence, show that \mathcal{F}' is isomorphic to a subsheaf of \mathcal{F} , and that \mathcal{F}'' is isomorphic to the quotient of \mathcal{F} by this subsheaf.

Proof. a) The natural quotient map is $\phi : U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$. Hence ϕ obviously has kernel \mathcal{F}' . To show that it is surjective, we have to show that the sheaf associated to $U \mapsto \text{im } \phi(U)$, call the sheaf $\text{im } \phi^+$ and the presheaf $\text{im } \phi^-$, equals \mathcal{F}/\mathcal{F}' . To do this, we can check it on stalks. As the stalks of $\text{im } \phi^+$ equal the stalks of the presheaf $U \mapsto \text{im } \phi(U)$, we just have to verify that $\text{im } \phi_p^- \cong \mathcal{F}/\mathcal{F}'_p$. This is a consequence of ϕ being surjective on each section and limits of isomorphic objects are isomorphic.

b) Label the non-zero maps a, b . Then first we can show that $\mathcal{F}' \cong \text{im } a$ via the forward map of a with restricted codomain. We can do this via checking the map on stalks. It is injective on stalks because $0 = (\ker a)_p = \ker a_p$ (1.2a). Because the stalks of the image presheaf and image sheaf are the same, we simply have to check that $\text{im } a_p = (\text{im } a)_p$. This is true by 1.2a. \square

Exercise 1.1.7. Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves.

- (a) Show that $\text{im } \phi \cong \mathcal{F}/\ker \phi$.
- (b) Show that $\text{coker } \phi \cong \mathcal{G}/\text{im } \phi$.

Proof. a) It suffices to check isomorphism on stalks as we have the obvious morphism $\mathcal{F}/\ker \phi \rightarrow \text{im } \phi$ (as the former is isomorphic as presheaves to the image presheaf). We have that $(\text{im } \phi)_p$ is the stalk of the image presheaf, so $(\text{im } \phi)_p \cong (\text{im } \phi^-)_p \cong \text{im } (\phi^-)_p \cong \mathcal{F}_p/\ker \phi_p$ by the first isomorphism theorem. This is just $(\mathcal{F}/\ker \phi)_p$.

b) It suffices to check isomorphism on stalks, with the morphism being $\mathcal{G}/\text{im } \phi$ to the cokernel presheaf. As the stalks of the cokernel presheaf and the sheaf are the same, all we have to check is that $(\text{coker } \phi)_p \cong \mathcal{G}_p/\text{im } \phi_p$. This is true because $(\text{coker } \phi)_p = \text{coker } \phi_p$ by 1.2a, and $\text{coker } \phi_p = \mathcal{G}_p/\text{im } \phi_p$ by definition. \square

Exercise 1.1.8. For any open subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e. if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of sheaves, then $0 \rightarrow \Gamma(U, \mathcal{F}') \rightarrow \Gamma(U, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F}'') \rightarrow 0$ is an exact sequence of groups. The functor $\Gamma(U, \cdot)$ need not be exact; see (Ex. 1.21) below.

Proof. Let the maps in the exact sequence of sheaves be a, b . The injectivity of $\Gamma(U, \mathcal{F}') \rightarrow \Gamma(U, \mathcal{F})$ is easy, since $\ker a = 0$, so $\ker a(U) = 0$. Next we show that $\text{im } a(U) = \ker b(U)$.

Lemma 1.1. For an injective morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves on X , the image presheaf is a sheaf.

Proof. We verify the two properties:

- i) If we have an open cover of $\text{im } f$ $\{U_i\}$ and an element $s \in \Gamma(\mathcal{G}, X)$ such that $s|_{U_i} = 0$, then $t_i := f^{-1}(s|_{U_i})$, which exists uniquely by definition of image presheaf and injectivity, equals 0 and is in $\Gamma(\mathcal{F}, U_i)$. Thus we have a set of elements of an open cover obviously compatible on intersections (zero elements always restrict to 0), lifting to a unique global zero section t by property i) of \mathcal{F} being a sheaf. But $f^{-1}(s)$ also has the property of being a global section that restricts to the right

elements on sections of the open cover (once again using injectivity), i.e.

$$\begin{array}{ccc} f^{-1}(s) & \longrightarrow & s \\ \downarrow & & \downarrow \\ f^{-1}(s)|_{U_i} & \longrightarrow & s|_{U_i} \end{array}$$

commutes because $f^{-1}(s)|_{U_i} = f^{-1}(s|_{U_i})$. Hence $f^{-1}(s) = t = 0 \implies s = 0$.

ii) If we have an open cover $\{U_i\}$ and sections s_i of it that agree on intersections, then by bijectivity of f on the image, $f^{-1}(s_i|_{U_i \cap U_j}) = f^{-1}(s_i)|_{U_i \cap U_j}$. Thus $f^{-1}(s_i)$ are sections of an open cover of the sheaf \mathcal{F} , so they lift to a global section. By bijectivity of f on the image (we need this for commuting to work right), this global section is a global section of the image. \square

By the above lemma, the image presheaf is a sheaf, and as both the image presheaf and image sheaf are subsheaves of \mathcal{G} , they are the same. Thus $\text{im } a(U) = (\text{im } a)(U) = \ker a(U)$. \square

Exercise 1.1.9 (Direct Sum). Let \mathcal{F} and \mathcal{G} be sheaves on X . Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called the *direct sum* of \mathcal{F} and \mathcal{G} , and is denoted by $\mathcal{F} \oplus \mathcal{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X .

Proof. We first verify the first property of a sheaf: If we have a section $f \in \mathcal{F}(U) \oplus \mathcal{G}(U)$ and an open cover $\{U_i\}$ such that $f|_{U_i} = 0$, then by projecting into the $\mathcal{F}(U)$ and $\mathcal{G}(U)$ coordinates, we get a section of $\mathcal{F}(U)$ and $\mathcal{G}(U)$ with restrictions to an open cover equaling 0. By \mathcal{F} and \mathcal{G} being sheaves, the components of f are both 0, hence $f = 0$.

Second property: Use the same notation as above. Then we have $f_i \in \mathcal{F}(U_i) \oplus \mathcal{G}(U_i)$. By projecting to each component, we get sections on U of each sheaf, say f', f'' . Then I propose $f := f' \oplus f''$ as the needed element of $\mathcal{F}(U) \oplus \mathcal{G}(U)$. This is because the restriction map on this presheaf is $\rho_{\mathcal{F}} \oplus \rho_{\mathcal{G}}$, so by \mathcal{F}, \mathcal{G} being sheaves, the restriction of each component of f to U_i equals f_i , showing that f has the desired property.

It satisfies the direct product and direct sum because projecting is a morphism:

$$\begin{array}{ccc} \mathcal{F}(U) \oplus \mathcal{G}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) \oplus \mathcal{G}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

commutes because the restriction morphism is $\rho_{\mathcal{F}} \oplus \rho_{\mathcal{G}}$, hence the projection of a restriction is the restriction of a projection. This implies the categorical definition because \oplus in the category of abelian groups is the direct product and direct sum, hence the projection morphisms being sheaf morphisms gives us the categorical stuff for free. \square

Exercise 1.1.10 (Direct Limit.). Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X . We define the *direct limit* of the system $\{\mathcal{F}_i\}$, denoted $\varinjlim \mathcal{F}_i$ to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X , i.e. that it has the following universal

property: given a sheaf \mathcal{G} , and a collection of morphisms $\mathcal{F}_i \rightarrow \mathcal{G}$, compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathcal{F}_i \rightarrow \mathcal{G}$ such that for each i , the original map $\mathcal{F}_i \rightarrow \mathcal{G}$ is obtained by composing the maps $\mathcal{F}_i \rightarrow \varinjlim \mathcal{F}_i \rightarrow \mathcal{G}$.

Proof. Because each open set U has a morphism $\mathcal{F}_i(U) \rightarrow \mathcal{G}(U)$, there is a morphism $\varinjlim \mathcal{F}_i(U) \rightarrow \mathcal{G}(U)$ by definition of direct limits in category of abelian groups which has the desired commuting property. This is a presheaf morphism because the compatibility of $\mathcal{F}_i(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{G}(V)$ with $\mathcal{F}_i(U) \rightarrow \mathcal{F}_i(V) \rightarrow \mathcal{G}(V)$ gives us one map $\varinjlim \mathcal{F}_i(U) \rightarrow \mathcal{G}(U)$ that commutes with the upper triangle and lower triangle.

$$\begin{array}{ccc} \varinjlim \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & \searrow & \downarrow \\ \varinjlim \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \end{array}$$

Because there is a presheaf morphism $f^- : \varinjlim \mathcal{F}_i \rightarrow \mathcal{G}$, there is a unique morphism from the sheafification $f : \varinjlim \mathcal{F} \rightarrow \mathcal{G}$ such that $f^- = f \circ \theta$. This gives us the uniqueness, and the commuting properties desired. \square

Exercise 1.1.11 (1.11). Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on a noetherian topological space X . In this case, show that the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf. In particular,

$$\Gamma(X, \varinjlim \mathcal{F}_i) = \varinjlim \Gamma(X, \mathcal{F}_i).$$

Proof. We verify this directly.

Property 1: Suppose we have U , open cover $\{U_i\}$, and a section $s \in \varinjlim \mathcal{F}_i(U)$ s.t. $s|_{U_i} = 0$. Because X is Noetherian, we can let $\{U_i\}$ be finite. By definition of the direct limit presheaf, restriction commutes with the limit. So $0 = s|_{U_j} \in \varinjlim (\mathcal{F}_i(U_j))$.

Recall that every element of the direct limit can be represented by a pair (s_{ij}, j) with $s_{ij} \in \mathcal{F}_j(U_i)$. Because $s|_{U_i} = 0$, there is a k_i such that $s_{ik_i} = 0 \in \mathcal{F}_{k_i}(U_i)$. As we have a direct system, we can find $I \geq i \forall i$. Hence $s_{ik_i} = 0 \in \mathcal{F}_I(U_i)$. By \mathcal{F}_I being a sheaf, this implies that $s = 0$ in $\mathcal{F}_I(U)$. But this implies that $s = 0 \in \varinjlim \mathcal{F}_i(U)$.

Property 2: Suppose we have a finite open cover $\{U_i\}$ and sections $s_i \in \varinjlim \mathcal{F}_i(U_i)$ that agree on intersections. As above, we can represent s_i by (s_{ij}, j) with $s_{ij} \in \mathcal{F}_j(U_i)$. Because $\{U_i\}$ is a finite cover, we can find $I \geq i$ for all i . Then by mapping s_{ij} into $\mathcal{F}_I(U_i)$, we get sections of an open cover of a sheaf that agree on intersections. Thus we lift to a global section $s \in \mathcal{F}_I(U)$. Then by mapping this into $\varinjlim \mathcal{F}_i(U_i)$, we have found an element that restricts properly. \square

Exercise 1.1.12 (Inverse Limit). Let $\{\mathcal{F}_i\}$ be an inverse system of sheaves on X . Show that the presheaf $U \mapsto \varprojlim \mathcal{F}_i(U)$ is a sheaf. It is called the *inverse limit* of the system $\{\mathcal{F}_i\}$, and is denoted by $\varprojlim \mathcal{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

Proof. We verify the desired properties:

i) If we have an open cover $\{U_i\}$ of X and a global section s of $\varprojlim \mathcal{F}_j(U)$ that restricts to 0 on the open cover. Then we can represent s as a pair (s_j, j) with $s_j \in \mathcal{F}_j(U)$.

$$\begin{array}{ccc} \varprojlim \mathcal{F}_j(U) & \longrightarrow & \mathcal{F}_j(U) \\ \downarrow & & \downarrow \\ \varprojlim \mathcal{F}_j(U_i) & \longrightarrow & \mathcal{F}_j(U_i) \end{array}$$

Because s restricts to 0 on $\varprojlim \mathcal{F}_j(U_i)$ for all i , the morphism from the top left to bottom right sends s to 0. Now suppose FTSOC that there was some j such that s in $\mathcal{F}_j(U)$ wasn't zero. Then for all i , the image of s in $\mathcal{F}_j(U_i)$ is 0. As \mathcal{F}_j are sheaves, this means that the image of s in $\mathcal{F}_j(U)$ is 0 for all j , a contradiction. Hence $s = 0 \in \mathcal{F}_j(U)$ for all j , so $s = 0$ in $\varprojlim \mathcal{F}_j(U)$.

ii) If we have an open cover $\{U_i\}$ of X and sections s_i of $\varprojlim \mathcal{F}_j(U_i)$ that agree on intersections, then for all j , we have an element $S_j \in \mathcal{F}_j(U)$ that agree on intersections and restrict to $\pi_j(s_i)$ on the open cover. Then $(S_j)_{j \in I}$ with I the index set is an element of the projective limit, since the projections and morphisms of the inverse system commute. Because the projection and restriction commute properly by construction, $(S_j)_{j \in I}$ is the lift we are looking for. \square

Exercise 1.1.13 (Espace Etalé of a Presheaf). (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [1, Ch. II, §1.2].) Given a presheaf \mathcal{F} on X , we define a topological space $\text{Spé}(\mathcal{F})$, called the *espace étalé* of \mathcal{F} , as follows. As a set, $\text{Spé}(\mathcal{F}) = \bigcup_{P \in X} \mathcal{F}_P$. We define a projection map $\pi : \text{Spé}(\mathcal{F}) \rightarrow X$ by sending $s \in \mathcal{F}_P$ to P . For each open set $U \subseteq X$ and each section $s \in \mathcal{F}(U)$, we obtain a map $\bar{s} : U \rightarrow \text{Spé}(\mathcal{F})$ by sending $P \mapsto s_P$, its germ at P . This map has the property that $\pi \circ \bar{s} = \text{id}_U$, in other words, it is a “section” of π over U . We now make $\text{Spé}(\mathcal{F})$ into a topological space by giving it the strongest topology such that all the maps $\bar{s} : U \rightarrow \text{Spé}(\mathcal{F})$ for all U , and all $s \in \mathcal{F}(U)$, are continuous. Now show that the sheaf \mathcal{F}^+ associated to \mathcal{F} can be described as follows: for any open set $U \subseteq X$, $\mathcal{F}^+(U)$ is the set of *continuous sections* of $\text{Spé}(\mathcal{F})$ over U . In particular, the original presheaf \mathcal{F} was a sheaf if and only if for each U , $\mathcal{F}(U)$ is equal to the set of all continuous sections of $\text{Spé}(\mathcal{F})$ over U .

Proof. We show that a continuous section has the properties of a section of the sheafification: Condition 1 is obviously met. Condition 2 is met by letting $V = U$ and $t = s$.

Finally we can see that a section u of the sheafification (say over U) is a continuous section of $\text{Spé}(\mathcal{F})$ over U : It suffices to check that the preimage of the open set $\bar{s}(U)$ for all open sets U and sections $s \in \mathcal{F}(U)$. Fix a $\bar{s}(U)$.

For every point P in the preimage, by condition 2 of being a section of the sheafification, there is a neighborhood V of P and an element $t \in \mathcal{F}(V)$ such that $t_Q = u(Q) \forall Q \in V$. But $u(P) = t_P = s_P$, so $t = s$ (the latter equality is due to P being in the preimage of $\bar{s}(U)$ and $u(P) \in \mathcal{F}_P$). Then V is in the preimage as well, so the preimage is a union of open V 's, implying that the preimage is open, giving us continuity of s . \square

Exercise 1.1.14 (Support). Let \mathcal{F} be a sheaf on X , and let $s \in \mathcal{F}(U)$ be a section over an open set U . The *support* of s , denoted $\text{Supp } s$, is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk \mathcal{F}_P . Show that $\text{Supp } s$ is a closed subset of U . We define the *support* of \mathcal{F} , $\text{Supp } \mathcal{F}$, to be $\{P \in X \mid \mathcal{F}_P \neq 0\}$. It need not be a closed subset.

Proof. By the above exercise, $\mathcal{F}(U)$ is the set of all continuous sections of $\text{Spé}(\mathcal{F})$ over U . Then $\text{Supp } s$ equals $\bar{s}^{-1}(\{\bigsqcup_{P \in X} 0_P\}^C)$. As \mathcal{F}_P has the discrete topology, $\{0\}$ is open, hence $\{\bigsqcup 0_P\}^C$ is closed, which by continuity of \bar{s} gives us that $\text{Supp } s$ is closed. \square

Exercise 1.1.15 (Sheaf Hom). Let \mathcal{F}, \mathcal{G} be sheaves of abelian groups on X . For any open set $U \subseteq X$, show that the set $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of an abelian group. Show that the presheaf $U \mapsto \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of \mathcal{F} into \mathcal{G} , “sheaf hom” for short, and is denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Proof. The abelian group operation is simply by defining $(f+g)(V)$, $f, g \in \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ to be $f + g$ with $+$ the group operation of $\mathcal{G}|_U(V)$. This satisfies the group properties because $+$ has this property. It is in $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ because the below diagram commutes:

$$\begin{array}{ccc} \mathcal{F}|_U(V) & \xrightarrow{(f+g)(V)} & \mathcal{G}|_U(V) \\ \downarrow \rho(VW) & & \downarrow \rho(VW) \\ \mathcal{F}|_U(W) & \xrightarrow{(f+g)(W)} & \mathcal{G}|_U(W) \end{array}$$

by the commuting properties in the definition of the morphisms f, g .

We show that it is a sheaf:

i) If we have a morphism $f \in \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ that restricts to 0 morphisms on an open cover of U , call it $\{U_i\}$, then for any open set $V \subseteq U$,

$$\begin{array}{ccc} \mathcal{F}|_U(V) & \xrightarrow{f(V)} & \mathcal{G}|_U(V) \\ \downarrow & & \downarrow \rho_{\mathcal{G}}(V \cap U_i) \\ \mathcal{F}|_U(V \cap U_i) & \xrightarrow{f(V \cap U_i)} & \mathcal{G}|_U(V \cap U_i) \end{array}$$

commutes for all i . As $f(V \cap U_i) = f|_{U_i}(V \cap U_i)$, $f(V \cap U_i) = 0$. Finally, if there is an element x that isn't in the kernel of $f(V)$, then this element is in the kernel of $\rho_{\mathcal{G}}(V, V \cap U_i) \forall i$. But then by \mathcal{G} being a sheaf, this implies that $f(V)(x) = 0$, a contradiction. Thus $f(V) = 0$, showing that $f = 0$.

ii) Say we have $f_i \in \text{Hom}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i})$ (sections of the $\mathcal{H}om$ sheaf) for an open cover U_i of some open subset V that agree on intersections. Then we can define a function $f \in \text{Hom}(\mathcal{F}|_U(V), \mathcal{G}|_U(V))$ by defining it for open $V' \subseteq V$ by taking $x \in \mathcal{F}|_U(V')$ to the lift of $f_i(x|_{U_i})$. Because the f_i agree on intersections, we can lift $f_i(x|_{U_i})$.

Finally, f satisfies the commuting diagram

$$\begin{array}{ccc} \mathcal{F}|_U(V') & \xrightarrow{f(V')} & \mathcal{G}|_U(V') \\ \downarrow & & \downarrow \\ \mathcal{F}|_U(V'') & \xrightarrow{f(V'')} & \mathcal{G}|_U(V'') \end{array}$$

for all open $V'' \subseteq V' \subseteq V$ because the f_i are sheaf morphisms, so they satisfy the commuting diagrams

$$\begin{array}{ccc} \mathcal{F}|_U(V' \cap U_i) & \longrightarrow & \mathcal{G}|_U(V' \cap U_i) \\ \downarrow & & \downarrow \\ \mathcal{F}|_U(V'' \cap U_i) & \longrightarrow & \mathcal{G}|_U(V'' \cap U_i) \end{array}$$

and the lift is unique and restricts properly (i.e. going along the top gives an element that should be the lift, and uniqueness tells us it is the lift given from going along the bottom). \square

Exercise 1.1.16 (Flasque Sheaves.). A sheaf \mathcal{F} on a topological space X is *flasque* if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective.

- Show that a constant sheaf on an irreducible topological space is flasque. See (I, Section 1) for irreducible topological spaces.
- If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of sheaves, and if \mathcal{F}' is flasque, then for any open set U , the sequence $0 \rightarrow \mathcal{F}'(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}''(U) \rightarrow 0$ of abelian groups is also exact.
- If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of sheaves, and if \mathcal{F}' and \mathcal{F} are flasque, then \mathcal{F}'' is flasque.
- If $f : X \rightarrow Y$ is a continuous map, and if \mathcal{F} is a flasque sheaf on X , then $f_*\mathcal{F}$ is a flasque sheaf on Y .
- Let \mathcal{F} be any sheaf on X . We define a new sheaf \mathcal{G} , called the sheaf of *discontinuous sections* of \mathcal{F} as follows. For each open set $U \subseteq X$, $\mathcal{G}(U)$ is the set of maps $s : U \rightarrow \bigcup_{P \in U} \mathcal{F}_P$ such that for each $P \in U$, $s(P) \in \mathcal{F}_P$. Show that \mathcal{G} is a flasque sheaf, and that there is a natural injective morphism of \mathcal{F} to \mathcal{G} .

Proof. a) First we can observe that all open subsets of X are connected because if we have open $U = C_1 \sqcup C_2$ and closed C_1, C_2 , then C_1, C_2 come from closed sets in X , call them C'_1, C'_2 . Then $X = (C'_1 \cup C'_2) \cup U^c$, contradicting X 's irreducibility.

As $\mathcal{A}(U) \cong A$ for connected open sets, it suffices to show that the restriction map is injective ($A/\ker \rho \cong \text{im } \rho \subseteq A$). Suppose that we have $V \subseteq U$ and $s \in \mathcal{A}(U)$ such that $s|_V = 0$. Then $s^{-1}(\{0\}^c)$ is open and closed by the discrete topology on A and continuity of s . But this contradicts the connectedness of U , unless $s^{-1}(\{0\}^c) = \emptyset$. Thus $s(U) = 0$.

b) Suppose we have an open subset U of X , a section $s \in \mathcal{F}''(U)$, and call the maps $f : \mathcal{F}' \rightarrow \mathcal{F}$ and $g : \mathcal{F} \rightarrow \mathcal{F}''$. By Exercise 1.3a and hypothesis, we have an open cover of U , call it $\{U_i\}$, and sections $t_i \in \mathcal{F}(U_i)$ such that $g_{U_i}(t_i) = s|_{U_i}$.

Now we show an intermediary result. Consider the set $\mathcal{C} = \{(P, t)\}$ with P an arbitrary union of the U_i 's and $t \in \mathcal{F}(P)$ such that $g_P(t) = s|_P$. Give it a partial

order via $(U_1, t_1) \leq (U_2, t_2)$ if $U_1 \subseteq U_2$ and $t_2|_{U_1} = t_1$. Then this set has a maximal element by Zorn's lemma: if we have a chain

$$(V_1, t_1) \subseteq (V_2, t_2) \subseteq (V_3, t_3) \subseteq \cdots,$$

then the t_i 's lift to a section t on $V = \cup V_i$ by \mathcal{F} being a sheaf and agreeing on intersections by definition of the partial order. Because the V_i 's are unions of the U_i 's, V is too. Then because g is a sheaf morphism, $g_V(t)|_{V_i} = s|_{V_i}$, and as the V_i are an open cover of V , by the uniqueness in sheaf property 2, $g_V(t) = s|_V$.

Next take a maximal element of \mathcal{C} with the open sets contained in U , call it (A, t) . If $A = U$, then we have a section of $\mathcal{F}(U)$ that maps to $s|_U = s$. So suppose that $A \subsetneq U$. Then $A \subsetneq A \cup U_k$ for some k .

We can see that $g_{U_k \cap A}(t_k) = (g_{U_k}(t_k))|_{U_k \cap A} = (s|_{U_k})|_{U_k \cap A} = s|_{U_k \cap A}$. Similarly, $g_{U_k \cap A}(t) = (g_A(t))|_{U_k \cap A} = (s|_A)|_{U_k \cap A} = s|_{U_k \cap A}$. Thus $t|_{U_k \cap A} - (t_k)|_{U_k \cap A} \in \ker g_{U_k \cap A}$. Hence by left exactness from Exercise 1.8, there is $u \in \mathcal{F}'(U_k \cap A)$.

By \mathcal{F}' being flasque, u lifts to an element $u' \in \mathcal{F}'(A)$. Now consider $t - f_A(u') \in \mathcal{F}'(A)$. Then $(t - f_A(u'))|_{U_k \cap A} - (t_k)|_{U_k \cap A} = t|_{U_k \cap A} - (t_k)|_{U_k \cap A} - (t|_{U_k \cap A} - (t_k)|_{U_k \cap A}) = 0$ because of the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}'(A) & \xrightarrow{f_A} & \mathcal{F}(A) & \xrightarrow{g_A} & \mathcal{F}''(A) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}'(U_k \cap A) & \xrightarrow{f_{U_k \cap A}} & \mathcal{F}(U_k \cap A) & \xrightarrow{g_{U_k \cap A}} & \mathcal{F}''(U_k \cap A) \end{array}$$

As they agree on intersections, by sheaf property 2 we can lift $t - f_A(u')$ to a $t' \in \mathcal{F}(A \cap U_k)$, which then contradicts (A, t) 's maximality

c) Kind of a ridiculous proof but here goes: By b), we have this diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker a & \longrightarrow & \ker b & \longrightarrow & \ker c \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) \longrightarrow 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ 0 & \longrightarrow & \mathcal{F}'(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}''(V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{coker } a & \longrightarrow & \text{coker } b & \longrightarrow & \text{coker } c \end{array}$$

Because a, b are surjective, $\text{coker } a = \text{coker } b = 0$. Then by the snake lemma, we have exactness of a map $\ker c \rightarrow \text{coker } a$. As $\text{coker } a = 0$, the top row is then exact. Finally, by abusing first isomorphism, we have

$$\begin{aligned} \mathcal{F}(V) &\cong \mathcal{F}(U)/\ker b & \mathcal{F}'(V) &\cong \mathcal{F}'(U)/\ker a \\ \ker c &\cong \ker b/\ker a & \text{im } c &\cong \mathcal{F}''(U)/\ker c \\ & & \mathcal{F}''(V) &\cong \mathcal{F}(V)/\mathcal{F}'(V). \end{aligned}$$

Finally,

$$\begin{aligned}
 \mathcal{F}''(V) &\cong \mathcal{F}(V)/\mathcal{F}'(V) \\
 &\cong (\mathcal{F}(U)/\ker b)/(\mathcal{F}'(U)/\ker a) \\
 &\cong (\mathcal{F}(U)/\mathcal{F}'(U))/(\ker b/\ker a) \\
 &\cong \mathcal{F}''(U)/\ker c \\
 &\cong \operatorname{im} c.
 \end{aligned}$$

d) Take open $V \subseteq U \subseteq Y$. Then $f_*\mathcal{F}(U) \rightarrow f_*\mathcal{F}(V)$ is surjective because this morphism is, by definition, $\mathcal{F}(f^{-1}(U)) \rightarrow \mathcal{F}(f^{-1}(V))$, and $f^{-1}(U), f^{-1}(V)$ are open by continuity of f . This morphism is surjective by definition of \mathcal{F} being flasque.

e) Suppose we have open $V \subseteq U \subseteq X$. Then \square

Exercise 1.1.17 (Skyscraper Sheaves.). Let X be a topological space, let P be point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$ denotes the closure of the set consisting of the point P . Hence the name “skyscraper sheaf”. Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i : \{P\}^- \rightarrow X$ is the inclusion.

Proof. For every point $Q \in \{P\}^-$, all neighborhoods of Q contain P , so all the terms in the direct limit are A , making the stalk at Q equal A . Then elsewhere, we can find a neighborhood of Q that doesn't contain P by definition of not being in the closure. Then the section of i_P at that neighborhood is 0 by definition, making the direct limit equal 0.

It can also be described as $i_*(A)$ because for any open set U , if $i^{-1}(U) \neq \emptyset$, then it contains a point of $\{P\}^-$, and as it is an open set containing a point in the closure of P , the open set contains P and thus $\mathcal{A}(i^{-1}(U))$ equals A (the closure of a point is an irreducible space, so all open subsets of $\{P\}^-$ are connected). Then if $i^{-1}(U) = \emptyset$, note that U doesn't contain P and $i_*(U) = \mathcal{A}(\emptyset) = 0$. \square

Exercise 1.1.18 (Adjoint Property of f^{-1}). Let $f : X \rightarrow Y$ be a continuous map of topological spaces. Show that for any sheaf \mathcal{F} on X there is a natural map $f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$, and for any sheaf \mathcal{G} on Y there is a natural map $\mathcal{G} \rightarrow f_*f^{-1}\mathcal{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves \mathcal{F} on X and \mathcal{G} on Y ,

$$\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Hence we say that f^{-1} is a *left adjoint* of f_* , and that f_* is a *right adjoint* of f^{-1} .

Proof. We have a natural map $f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$ because $f^{-1}f_*\mathcal{F}(U) = \lim_{\text{open } V \supseteq f(U)} \mathcal{F}(f^{-1}(V))$ and each $f^{-1}(V)$ contains U so that we have the restriction maps $\mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(U)$ that commute with the system. By the universal property of the limit, this induces a map $\lim_{\text{open } V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(U)$. This then gives us a morphism

of sheaves because if we have $X \supseteq Y$ and arbitrary open $V \supseteq f(X)$ this commutes:

$$\begin{array}{ccccc} \mathcal{F}(f^{-1}(V)) & \longrightarrow & \lim_{V \supseteq f(X)} \mathcal{F}(f^{-1}(V)) & \longrightarrow & \mathcal{F}(X) \\ & & \downarrow & & \downarrow \\ & & \lim_{U \supseteq f(Y)} \mathcal{F}(f^{-1}(U)) & \longrightarrow & \mathcal{F}(Y) \end{array}$$

Hence by definition of the limit, we induce a unique map from the top middle to the bottom right:

$$\begin{array}{ccccc} \mathcal{F}(f^{-1}(V)) & \longrightarrow & \lim_{V \supseteq f(X)} \mathcal{F}(f^{-1}(V)) & \longrightarrow & \mathcal{F}(X) \\ & & \downarrow & \nearrow \exists! & \downarrow \\ & & \lim_{U \supseteq f(Y)} \mathcal{F}(f^{-1}(U)) & \longrightarrow & \mathcal{F}(Y) \end{array}$$

This shows the required commutativity.

We have a map $\mathcal{G} \rightarrow f_* f^{-1} \mathcal{G}$ because

$$f_* f^{-1} \mathcal{G}(U) = \lim_{V \supseteq f^{-1}(U)} \mathcal{G}(V)$$

has a morphism $\mathcal{G}(U) \rightarrow \lim_{V \supseteq f^{-1}(U)} \mathcal{G}(V)$ by the definition of a limit. It exists because by picking an arbitrary V in the index, there is a morphism $\mathcal{G}(U) \rightarrow \mathcal{G}(V)$ the restriction (the proper commuting relations allows us to pick arbitrarily). Finally, this is a sheaf morphism because given $A \subseteq B$,

$$\begin{array}{ccc} \mathcal{G}(B) & \xrightarrow{y(B)} & \lim_{V \supseteq f^{-1}(B)} \mathcal{G}(V) \\ \downarrow & & \downarrow \\ \mathcal{G}(A) & \xrightarrow{y(A)} & \lim_{U \supseteq f^{-1}(A)} \mathcal{G}(U) \end{array}$$

commutes. It commutes because the top left down route to the bottom right induces a unique map from the top right down, forcing the whole thing to commute.

Call the natural maps x and y (after their topological space). Now that we have those two maps, if we have $a \in \text{Hom}_X(f^{-1} \mathcal{G}, \mathcal{F})$, then we have $f_* a : f_* f^{-1} \mathcal{G} \rightarrow f_* \mathcal{F}$. Precomposing with y gives us $f_* a y : \mathcal{G} \rightarrow f_* \mathcal{F} \in \text{Hom}_Y(\mathcal{G}, f_* \mathcal{F})$. Similarly, given $b \in \text{Hom}_Y(\mathcal{G}, f_* \mathcal{F})$, $f^{-1} b : f^{-1} \mathcal{G} \rightarrow f^{-1} f_* \mathcal{F}$, which composing with x gives us $x f^{-1} b \in \text{Hom}_X(f^{-1} \mathcal{G}, \mathcal{F})$. This is a bijection because these are inverses: given $a \in \text{Hom}_X(f^{-1} \mathcal{G}, \mathcal{F})$, we want to show that this diagram commutes

$$\begin{array}{ccccc} \mathcal{G}(V) & \xrightarrow{y(V)} & f_* f^{-1} \mathcal{G}(V) = \lim_{V' \supseteq f(f^{-1}(V))} \mathcal{G}(f(V')) & \xrightarrow{a(f^{-1}(V))} & f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \\ \downarrow & & & & \downarrow \\ f^{-1} \mathcal{G}(U) = \lim_{V \supseteq f(U)} \mathcal{G}(V) & \xrightarrow{\lim_{V \supseteq f(U)} a(f^{-1}(V)) y(V)} & f^{-1} f_* \mathcal{F}(U) = \lim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) & & \\ & \searrow a(U) & & \downarrow x(U) & \\ & & & & \mathcal{F}(U) \end{array}$$

But this commutativity is guaranteed because the morphisms $\mathcal{G}(V) \rightarrow \mathcal{F}(U)$ commute so that we induce a unique map from $f^{-1} \mathcal{G}(U) \rightarrow \mathcal{F}(U)$ which must be a and the top rectangle simultaneously. \square

Exercise 1.1.19 (Extending a Sheaf by Zero). Let X be a topological space, let Z be a closed subset, let $i : Z \rightarrow X$ be the inclusion, let $U = X \setminus Z$ be the complementary open subset, and let $j : U \rightarrow X$ be its inclusion.

- (a) Let \mathcal{F} be a sheaf on Z . Show that the stalk $(i_*\mathcal{F})_P$ of the direct image sheaf on X is \mathcal{F}_P if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathcal{F}$ the sheaf obtained by extending \mathcal{F} by zero outside Z . By abuse of notation, we will sometimes write \mathcal{F} instead of $i_*\mathcal{F}$, and say “consider \mathcal{F} as a sheaf on X ,” when we mean “consider $i_*\mathcal{F}$ ”.
- (b) Now let \mathcal{F} be a sheaf on U . Let $j_!(\mathcal{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathcal{F}))_P$ is equal to \mathcal{F}_P if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathcal{F}$ is the only sheaf on X which has this property, and whose restriction to U is \mathcal{F} . We call $j_!\mathcal{F}$ the sheaf obtained by *extending \mathcal{F} by zero outside U* .
- (c) Now let \mathcal{F} be a sheaf on X . Show that there is an exact sequence of sheaves on X ,

$$0 \rightarrow j_!(\mathcal{F}|_U) \rightarrow \mathcal{F} \rightarrow i_*(\mathcal{F}|_Z) \rightarrow 0.$$

Proof. (a): First we have by definition that

$$(i_*\mathcal{F})_P = \varinjlim_{V, \text{open in } X \ni P} \mathcal{F}(i^{-1}(V)).$$

If $P \notin Z$, then because U is open and contains P , this is a set the limit goes over. Then because $U \cap Z = \emptyset$, $i^{-1}(U) = \emptyset$, so $\mathcal{F}(i^{-1}(U)) = 0 \implies (i_*\mathcal{F})_P = 0$.

If $P \in Z$, then because i is the inclusion map and is continuous, we can reparameterize what the limit goes over to be U open in Z that contain P . This is just the stalk of \mathcal{F} .

(b): First we have by definition that

$$(j_!(\mathcal{F}))_P = \varinjlim_{V, \text{open in } X \ni P} j_!(\mathcal{F})(V).$$

If $P \notin U$, then by clearly $V \not\subseteq U$, so $j_!(\mathcal{F})(V) = 0$, making the limit 0.

If $P \in U$, then because the stalks of sheafification and the presheaf are the same, the limit is \mathcal{F}_P as desired.

For uniqueness, suppose we have \mathcal{F}' with these properties. Call the presheaf defined in the problem j . Then we have a presheaf morphism $j \rightarrow \mathcal{F}'$ via $\mathcal{F}(V) \rightarrow \mathcal{F}'(V)$ when $V \subseteq U$ and $0 \rightarrow \mathcal{F}'(V)$ otherwise. Thus we have a sheaf morphism $j_!(\mathcal{F}) \rightarrow \mathcal{F}'$. As they are isomorphic on stalks, we are done.

(c): First we define the morphisms: the first is just inclusion and the second is via the natural morphism from $\mathcal{F}(U) \rightarrow \varinjlim_{V \supseteq U \cap Z} \mathcal{F}(V)$ as U is in the indexing set. This is a sheaf morphism because the limit commutes with restrictions as these are the maps in the direct system.

We can then check exactness on the stalks. If $P \in U$, then the stalk sequence is

$$0 \rightarrow j_!(\mathcal{F}|_U)_P = \mathcal{F}_P \rightarrow \mathcal{F}_P \rightarrow 0 \rightarrow 0$$

with the last term due to $P \notin Z$. This is obviously exact. Then if $P \notin U$, the stalk sequence is

$$0 \rightarrow 0 \rightarrow \mathcal{F}_P \rightarrow \mathcal{F}_P \rightarrow 0,$$

which is obviously exact. Thus the sequence is exact and we are done. \square

Exercise 1.1.20 (Subsheaf with Supports.). Let Z be a closed subset of X , and let \mathcal{F} be a sheaf on X . We define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z .

- (a) Show that the presheaf $V \mapsto \Gamma_{Z \cap V}(V, \mathcal{F}|_V)$ is a sheaf. It is called the subsheaf of \mathcal{F} with supports in Z , and is denoted by $\mathcal{H}_Z^0(\mathcal{F})$.
- (b) Let $U = X \setminus Z$, and let $j : U \rightarrow X$ be the inclusion. Show that there is an exact sequence of sheaves on X

$$0 \rightarrow \mathcal{H}_Z^0(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow j_*(\mathcal{F}|_U).$$

Furthermore, if \mathcal{F} is flasque, the map $\mathcal{F} \rightarrow j_*(\mathcal{F}|_U)$ is surjective.

Proof. (a) Suppose we have $s \in \mathcal{H}_Z^0(\mathcal{F})(U)$ such that it restricts to 0 on an open cover $\{U_i\}$ of U and respects intersections. Then by property 1 of sheaves, $s = 0$.

If we have sections $\{s_i\}$ of an open cover $\{U_i\}$, then it lifts to a section $s \in \mathcal{F}(U)$. Then if we have $P \in U \cap Z$ such that $s_P = 0$ (i.e. $s \notin \mathcal{H}_Z^0(\mathcal{F})(U)$), P is in some U_i . But because $(s|_{U_i})_P = s_P = 0$, this implies that $\text{Supp } s$ doesn't contain P , a contradiction as $s|_{U_i} \in \mathcal{H}_Z^0(\mathcal{F})(U_i) \implies \text{Supp } s|_{U_i} \subseteq U_i \cap Z$ and $P \in U_i \cap Z$ because $P \in U \cap Z$. Thus $s \in \mathcal{H}_Z^0(\mathcal{F})(U)$.

(b) We obviously have the existence and injectivity of the first morphism. The second morphism is defined via restriction (and thus obviously is a sheaf morphism) as $j_*(\mathcal{F}|_U)(V) = \mathcal{F}|_U(U \cap V)$.

Call the first morphism a and the second b . Since a is obviously injective, the image presheaf is a sheaf, so we just need to show that $\text{im } a = \ker b$.

We can see that $\text{im } a(V) \subseteq \ker b(V)$ because restricting a section $s \in \mathcal{H}_Z^0(V)$ with support contained in $Z \cap V$ to $U \cap V$ means that $s_Q = 0 \forall Q \in U \cap V$, making $s = 0$ in $\mathcal{F}|_U(U \cap V)$. Then if we have $s \in \ker b(V)$, then because $s|_{U \cap V} = 0$, $\text{Supp } s \subseteq V \setminus (U \cap V) = Z \cap V$, so $s \in \mathcal{H}_Z^0(\mathcal{F})(V)$. \square

Exercise 1.1.21 (Some Examples of Sheaves on Varieties). Let X be a variety over an algebraically closed field k , as in Ch I. Let \mathcal{O}_X be the sheaf of regular functions on X (1.0.1).

- (a) Let Y be a closed subset of X . For each open set $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is called the sheaf of ideals \mathcal{I}_Y of Y , and it is a subsheaf of the sheaf of rings \mathcal{O}_X .
- (b) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \rightarrow X$ is the inclusion, and \mathcal{O}_Y is the sheaf of regular functions on Y .
- (c) Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X , where $\mathcal{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0.$$

Show however that the induced map on global sections $\Gamma(X, \mathcal{O}) \rightarrow \Gamma(X, \mathcal{F})$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

- (d) Again let $X = \mathbb{P}^1$, and let \mathcal{O} be the sheaf of regular functions. Let \mathcal{H} be the constant sheaf on X associated to the function field K of X . Show that

there is a natural injection $\mathcal{O} \rightarrow \mathcal{H}$. Show that the quotient sheaf, \mathcal{H}/\mathcal{O} is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$, where I_P is the group K/\mathcal{O}_P , and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P ,

(e) Finally show that in the case of (d) the sequence

$$0 \rightarrow \Gamma(X, \mathcal{O}) \rightarrow \Gamma(X, \mathcal{H}) \rightarrow \Gamma(X, \mathcal{H}/\mathcal{O}) \rightarrow 0$$

is exact. (This is an analogue of what is called the “first Cousin problem” in several complex variables, See Gunning and Rossi [1, p. 248]).

Proof. (a) Property 1: If we have $s \in \mathcal{I}(U)$ such that $s|_{U_i} = 0$ for an open cover $\{U_i\}$ of U , then by sheaf property 1 $s = 0$. Because s vanishes every, it vanishes on $Y \cap U$ and thus $s \in \mathcal{I}(U)$.

Property 2: If we have $s_i \in \mathcal{I}(U_i)$ that agree on intersections, then by sheaf property 2 we can lift it to a section $s \in \mathcal{O}(U)$ that restricts to s_i . As restriction in the sheaf of regular functions is just restriction as a function, $s|_{U_i}$ vanishing on $Y \cap U_i$ implies that s vanishes on $Y \cap U_i$. As it vanishes for all i , s vanishes on $Y \cap U$.

(b) We have a morphism $\mathcal{O}_X/\mathcal{I}_Y \rightarrow i_*(\mathcal{O}_Y)$ by sending $[f] \in (\mathcal{O}_X/\mathcal{I}_Y)(U)$ (because of Lemma 1.1 and Exercise 1.6b) to $f|_{Y \cap U} \in \mathcal{O}_Y(Y \cap U) = i_*(\mathcal{O}_Y)(U)$, which is well-defined because $\mathcal{I}_Y(U)$ vanishes on $Y \cap U$.

(c) This is exact because $\mathcal{F} \cong i_*(\mathcal{O}_Y)$ because $\mathcal{O}_Y = \mathcal{O}_P \oplus \mathcal{O}_Q$ and $i_*(\mathcal{O}_P \oplus \mathcal{O}_Q) = i_*(\mathcal{O}_P) \oplus i_*(\mathcal{O}_Q)$. Then by (b), $i_*(\mathcal{O}_Y) \cong \mathcal{O}_X/\mathcal{I}_Y$. Thus the sequence is exact.

Clearly $\mathcal{F}(X) = k \oplus k$. But $\mathcal{O}_X(X) = k$ by Theorem 3.4 so that this map is not surjective.

(d) We have that $K = k[x_0, x_1]_{((0))}$ by Theorem 3.4. As we have a map $\mathcal{O}(U) \rightarrow \mathcal{O}_P$ for some $P \in U$ and we have a natural map $\mathcal{O}_P = S(X)_{\mathfrak{m}_P} \rightarrow K(X) = K$ by Theorem 3.4, we have a natural injection $\mathcal{O} \rightarrow \mathcal{H}$. Namely this is via $\mathcal{O}(U) \rightarrow \mathcal{O}_P \rightarrow K \xrightarrow{\oplus n} K^n = \mathcal{H}(U)$ where n is the number of connected components of U by Example 1.0.3. Also note that \mathbb{P}^1 is locally connected because every point is in an affine neighborhood.

Let $\mathcal{F} = \sum_{P \in X} i_P(I_P)$. We can first realize that the stalks of \mathcal{H}/\mathcal{O} are K/\mathcal{O}_P because sheafification preserves stalks. Then by definition, an element s of the section $\mathcal{H}/\mathcal{O}(U)$ is a function $U \rightarrow \sqcup_{P \in U} (K/\mathcal{O}_P)$ with certain properties. By letting π_P be the projection to the P -th coordinate, we can see that each $\pi_P \circ s(P)$ is in K/\mathcal{O}_P . Thus we have a morphism that sends s to $(\pi_P s(P))_{P \in U} \in \mathcal{F}(U)$. This commutes properly because

$$\begin{array}{ccc} \mathcal{H}/\mathcal{O}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{H}/\mathcal{O}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

sends s to $(\pi_P s(P))_{P \in U}$ to $(\pi_P s(P))_{P \in V}$ along the top path and sends s to $s|_V$ to $(\pi_P s|_V(P))_{P \in V} = (\pi_P s(P))_{P \in V}$ by construction of the sections of the sheafification. Finally, to see this isomorphism, we can check it on stalks by Proposition 1.1. At P , $(\mathcal{H}/\mathcal{O})_P = K/\mathcal{O}_P$ and $\mathcal{F}_P = \sum (i_Q(I_Q))_P = (i_P(I_P))_P = I_P = K/\mathcal{O}_P$.

(e) By Exercise 1.6a we have the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{H} \rightarrow \mathcal{H}/\mathcal{O} \rightarrow 0.$$

Then by Exercise 1.8 we have left exactness. By part (d), $\Gamma(X, \mathcal{H}/\mathcal{O}) \cong \sum_{P \in X} (K/\mathcal{O}_P)$. Then because \mathbb{P}^1 is connected, $\mathcal{H}(X) = K$. By Theorem 3.4, $\mathcal{O}_P = S(X)/(\mathfrak{m}_P)$ where $S = k[x_0, x_1]$. Thus we are trying to show surjectivity of this:

$$K \rightarrow \sum_{P \in X} (K/(k[x_0, x_1]_{(\mathfrak{m}_P)})).$$

Suppose we have $f \in \sum_{P \in X} (K/(S_{(\mathfrak{m}_P)}))$. Then $s = ([k_P])_{P \in X}$ where k_P is the projection of s into the P component and $[\cdot]$ is the equivalent class. Because this is a direct sum, s has all but finitely many zeros, say $\{[k_{P_i}]\}$ are non-zero.

We can find an element s' that maps to s by letting s' be $\sum k_{P_i}$ for representatives k_{P_i} of $[k_{P_i}]$. Then $s' \mapsto s$ because for $Q \notin \{P_i\}$, $\sum k_{P_i} \equiv 0 \pmod{S_{(\mathfrak{m}_Q)}}$ as the denominator of each k_{P_i} is not in \mathfrak{m}_P , so k_{P_i} is an element of $S_{(\mathfrak{m}_Q)}$. Thus this last map is surjective, showing exactness. \square

Exercise 1.1.22 (Gluing Sheaves.). Let X be a topological space, let $\mathcal{U} = \{U_i\}$ be an open cover of X , and suppose we are given for each i a sheaf \mathcal{F}_i on U_i and for each i, j an isomorphism $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \rightarrow \mathcal{F}_j|_{U_i \cap U_j}$ such that (1) for each i , $\phi_{ii} = \text{id}$, and (2) for each i, j, k , $\phi_{ik} = \phi_{jk}\phi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf \mathcal{F} on X , together with isomorphisms $\psi : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that for each i, j , $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that \mathcal{F} is obtained by *glueing* the sheaves \mathcal{F}_i via the isomorphisms ϕ_{ij} .

Proof. Consider $U \subseteq X$ be an open set and $\mathcal{U}_i = U \cap U_i$. Define $\Gamma(U, \mathcal{F})$ to be the pullback of the diagram $\mathcal{F}_i(V)$ with $V \subseteq \mathcal{U}_i$ for all i and the restriction morphisms, i.e. the diagrams $\mathcal{F}_i|_{\mathcal{U}_i}$ and the morphisms ϕ_{ij} . The restriction map into open subsets of the U_i are well-defined by having the isomorphism ϕ_{ii} . The restriction maps into these open subsets commute properly because $\phi_{ik} = \phi_{jk}\phi_{ij}$.

Finally, the restriction maps into any open subset of U , call it V , commutes properly because $V \subseteq U \implies V \cap U_i \subseteq \mathcal{U}_i$, giving us maps from $\Gamma(V, \mathcal{F})$ to $\mathcal{F}_i(\mathcal{U}_i)$ for all i . This then induces a map $\Gamma(V, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F})$ by the universal property of the pullback. \square

1.2. Schemes.

Exercise 1.2.1. Let A be a ring, let $X = \text{Spec } A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\text{Spec } A_f$.

Proof. Because we have the homomorphism $\iota : A \rightarrow A_f$, can see that $\text{Spec } A_f$ pullback to prime ideals in A that don't contain f . The prime ideals of A that don't contain f also correspond to prime ideals in A_f because if $A_f/\iota(\mathfrak{p})$ had a zero divisor, say $ab = 0$, then by cancelling the denominators and pulling back into A , we get a zero divisor in A/\mathfrak{p} , a contradiction. Thus we have a homeomorphism $D(f) \rightarrow \text{Spec } A_f$.

Finally, it suffices to show that we have a sheaf isomorphism $\mathcal{O}_X|_{D(f)} \rightarrow \text{Spec } A_f$. Obviously both are isomorphic on stalks: all points (i.e. primes) in $\mathcal{O}_X|_{D(f)}$ don't contain f , so $A_{\mathfrak{p}} = (A_f)_{\mathfrak{p}}$. Thus all we need is a sheaf morphism. We have one via composing the isomorphism of the stalks with elements of the section, which commutes because the restriction homomorphisms are just restriction of domains. \square

Exercise 1.2.2. Let (X, \mathcal{O}_X) be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the *induced scheme structure* on the open set U , and we refer to $(U, \mathcal{O}_X|_U)$ as an *open subscheme* of X .

Proof. Clearly this is a locally ringed space. To check that this is locally affine, take some point $p \in U$. Then we have a local open affine neighborhood $\text{Spec } A$ around p . By intersecting this with U , we get an open subset of $\text{Spec } A$ around p . The complement of this open subset in $\text{Spec } A$ is then $V(\mathfrak{a})$ for some ideal \mathfrak{a} . By picking an element in \mathfrak{a} , we get a open affine neighborhood $\text{Spec } A_f$ around p in U . \square

Exercise 1.2.3. A scheme (X, \mathcal{O}_X) is *reduced* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

- (a) Show that (X, \mathcal{O}_X) is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.
 - (b) Let (X, \mathcal{O}_X) be a scheme. Let $(\mathcal{O}_X)_{red}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{red}$, where for any ring A , we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{red})$ is a scheme. We it the *reduced scheme* associated to X , and denote it by X_{red} . Show that there is a morphism of schemes $X_{red} \rightarrow X$, which is a homeomorphism on the underlying topological spaces.
 - (c) Let $f : X \rightarrow Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \rightarrow Y_{red}$ such that f is obtained by composing g with the natural map $Y_{red} \rightarrow Y$.
-

Proof. (a) Clearly if all the local rings are reduced, the sections are reduced, for if we had a nilpotent element, it would become a nilpotent element in the local ring through the ring morphism. Conversely, if all sections are reduced and we had a nilpotent in some local ring, then it, by definition of limit, has a power that is 0 on some open set. But then on the section of that open set, that element is a nilpotent, contradiction our hypothesis.

(b)

Lemma 1.2. *Limits commute with quotients in the category of groups.*

Proof. Suppose we have directed systems (A_i) and (I_i) with $I_i \triangleleft A_i$ with $\rho_{ij}(I_i) \subseteq I_j$ with limits A, I . Then if we have a collection of morphisms that commute with the system (A_i/I_i) to a group P , induce a map $\varinjlim A_i \rightarrow P$ because of the maps $A_i \rightarrow A_i/I_i$ (which commute because $\rho_{ij}(I_i) \subseteq I_j$). By a similar proof from Exercise 1.2a, we see that $\ker(\varinjlim A_i \rightarrow P) = \varinjlim \ker(A_i \rightarrow P) = I$. Hence $A \rightarrow P$ factors uniquely through $\varinjlim (A_i/I_i) \rightarrow P$ by the universal property of quotients. As this happens uniquely for all P , $\varinjlim (A_i/I_i) \cong A/I$. \square

By the above lemma (which applies because nilpotents restrict to nilpotents) and the fact that quotients of a local ring are local, $(\mathcal{O}_X)_{red}$ is a locally ringed space. Clearly it is also affine because a point $p \in X$ has an open affine $\text{Spec } A = \Gamma(U, \mathcal{O}_X|_U)$ which produces an open affine of $(\mathcal{O}_X)_{red}$ being $\text{Spec } A_{red}$.

Let f be the identity map and let $f^\#$ be the quotient map $f^\#(U) : \mathcal{O}_X(U) \rightarrow (\mathcal{O}_X)_{red}(U)$. By Lemma 1.2, the map $(\mathcal{O}_X)_P \rightarrow ((\mathcal{O}_X)_{red})_P$ is the quotient map.

Because the maximal ideal of the RHS is the image of the maximal ideal of the LHS and the quotient map is surjective, the preimage of the maximal ideal on the right is the maximal ideal on the left. Hence $f_P^\#$ is a local homomorphism.

(c)

Lemma 1.3. *To define a locally ringed morphism $X \rightarrow Y$, it is enough to define locally ringed morphisms $f_i : U_i \rightarrow Y$ on an open cover $\{U_i\}$ of X such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. It is also unique.*

Proof. Denote $f_i^\# : \mathcal{O}_Y \rightarrow f_{i*}\mathcal{O}_{U_i} = f_{i*}\mathcal{O}_X|_{U_i}$. Defining the topological map is trivial: just let $f(x) = f_i(x)$ for some i such that $x \in U_i$. This is well-defined by hypothesis. In this way $f^{-1}(U) = \cup f_i^{-1}(U)$ for any open set $U \subseteq Y$.

Now take an arbitrary open subset of X . We can define a map $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ by letting the map $f^\#(U)$ take $s \in \mathcal{O}_Y(U)$ to the section lifted from the sections $f_i^\#(s) \in \mathcal{O}_X(f_i^{-1}(U))$ to $\mathcal{O}_X(\cup f_i^{-1}(U))$. Let $V_i = f_i^{-1}(U)$. We can lift like this because $f_i^\#(s)|_{V_i \cap V_j} = f_j^\#(s)|_{V_i \cap V_j}$ by hypothesis. Then $f^\#$ is a morphism because the image of $f^\#(U)$ agree on restrictions by property 2 of the sheaf giving us a section that agrees on restrictions. This is unique because of property 1 of sheaves. \square

Because X is reduced, for every open affine subset $U = \text{Spec } A$ of Y , the map $f^\#(U) : A \rightarrow f_*\mathcal{O}_X(U)$ has kernel containing the ideal of nilpotents of $\mathcal{O}_X(f^{-1}(U))$. Thus it factors through a map $\iota(U) : \mathcal{O}_Y(U)_{\text{red}} \rightarrow f_*\mathcal{O}_X(U)$, allowing us to define $g_U(U)$. The restriction morphisms are just via composition with restrictions. The diagram later will illuminate. We can then check that this map agrees on intersections: say we have $U = \text{Spec } A, V = \text{Spec } B$. Then $g(U)|_{U \cap V} = g(U \cap V)$ because $f^\#$ commutes with restrictions, i.e.

$$\begin{array}{ccccc}
 & & \Gamma(U, \mathcal{O}_Y)_{\text{red}} & & \\
 & \nearrow & \downarrow & \searrow & \\
 \Gamma(U, \mathcal{O}_Y) & \longrightarrow & \Gamma(U, f_*\mathcal{O}_X) & \xrightarrow{g_U(U \cap V)} & \\
 \downarrow & & \downarrow & & \\
 \Gamma(U \cap V, \mathcal{O}_Y) & \longrightarrow & \Gamma(U \cap V, f_*\mathcal{O}_X) & & \\
 & \searrow & \uparrow & \nwarrow & \\
 & & \Gamma(U \cap V, \mathcal{O}_Y)_{\text{red}} & &
 \end{array}$$

By similar logic, $g(U \cap V) = g(V)|_{U \cap V}$, showing that they agree on intersections. By Lemma 1.3 we then have a unique map $Y_{\text{red}} \rightarrow Y$. \square

Exercise 1.2.4. Let A be a ring and let (X, \mathcal{O}_X) be a scheme. Given a morphism $f : X \rightarrow \text{Spec } A$, we have an associated map on sheaves $f^\# : \mathcal{O}_{\text{Spec } A} \rightarrow f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \rightarrow \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map

$$\alpha : \text{Hom}_{\text{Sch}}(X, \text{Spec } A) \rightarrow \text{Hom}_{\text{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

Proof. First we show that it is surjective. Cover X with open affines $U_i = \text{Spec } A_i$. Then for some map $f : A \rightarrow \Gamma(X, \mathcal{O}_X)$, we have a map $f_i^\#(X) : A \rightarrow \Gamma(U_i, \mathcal{O}_X) = A_i$ defined by composing with $\rho_{XU_i}^X$. Hence we have a morphism of schemes induced by the ring homomorphism $f_i : \text{Spec } A_i \rightarrow \text{Spec } A$.

These maps agree on intersections because given $f_i^\#, f_j^\#, f_i^\#|_{U_i \cap U_j} = (\rho_{XU_i}^X f)|_{U_i \cap U_j} = \rho_{X(U_i \cap U_j)}^X f = f_j^\#|_{U_i \cap U_j}$. Hence by Lemma 1.3, we have a map $X \rightarrow \text{Spec } A$. By uniqueness given in the lemma, we have injectivity, giving us a bijection. \square

Exercise 1.2.5. Describe $\text{Spec } \mathbf{Z}$, and show that it is a final object for the category of schemes, i.e., each scheme X admits a unique morphism to $\text{Spec } \mathbf{Z}$.

Proof. We have the $\text{Spec } \mathbf{Z}$ bijects to the set of prime numbers in \mathbf{Z} and 0. It is a final object in the category of schemes because we have a morphism $\mathbf{Z} \rightarrow \Gamma(X, \mathcal{O}_X)$ as \mathbf{Z} is the initial object in **Ring**, giving us a scheme morphism by Exercise 1.2.4. \square

Exercise 1.2.6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R).

Proof. The spectrum of the zero ring is \emptyset , as there are no proper ideals. Then the map $\emptyset \rightarrow X$ is just no map on points and the trivial map on sections. \square

Exercise 1.2.7. Let X be a scheme. For any $x \in X$, let \mathcal{O}_x be the local ring at x , and \mathfrak{m}_x its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x / \mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of $\text{Spec } K$ to X , it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \rightarrow K$.

Proof. A morphism $f : \text{Spec } K \rightarrow X$ gives us a local homomorphism $f_x^\# : \mathcal{O}_x \rightarrow K$ where x is the image of the unique point of $\text{Spec } K$. As this is a local homomorphism, this implies that the maximal ideal of \mathcal{O}_x is $\ker f_x^\#$. By the first iso theorem, we then have that $\mathcal{O}_x / \ker f_x^\# \cong \text{im}(f_x^\#) \subseteq K$, giving an inclusion $k(x) \rightarrow K$.

If we have a point $x \in X$ and an inclusion map $k(x) \rightarrow K$, then we have the desired morphism by sending $(0) \rightarrow x$ and we have two cases for defining $f^\#$. For U open, $x \in U$, then $f_*(\mathcal{O}_K(U)) = K$, and we can define the map $\mathcal{O}_X(U) \rightarrow K$ via $\mathcal{O}_X(U) \rightarrow \mathcal{O}_x \rightarrow k(x) \rightarrow K$.

If $x \notin U$, then $f_*(\mathcal{O}_K(U)) = 0$, so the map is just given by 0.

Then we have that the needed diagram commute with three cases:

$x \in U \subseteq V$:

$$\begin{array}{ccc} \mathcal{O}_X(V) & \longrightarrow & K \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ \mathcal{O}_X(U) & \longrightarrow & K \end{array}$$

The top and bottom row commute by commutativity of ρ in the direct system that is part of \mathcal{O}_x .

$x \notin U, x \in V$:

$$\begin{array}{ccc} \mathcal{O}_X(V) & \longrightarrow & K \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ 0 & \longrightarrow & 0 \end{array}$$

$x \notin U, x \notin V$:

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ 0 & \longrightarrow & 0 \end{array}$$

□

Exercise 1.2.8. Let X be a scheme. For any point $x \in X$, we define the *Zariski tangent space* T_x to X to be the dual of the $k(x)$ -vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k , and let $k[\epsilon]/\epsilon^2$ be the *ring of dual numbers* over k . Show that to give a k -morphism of $\text{Spec } k[\epsilon]/\epsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e. such that $k(x) = k$) and an element of T_x .

Proof. \implies) Let x be the image of (ϵ) in X . Then because we have this diagram

$$\begin{array}{ccc} \text{Spec } k[\epsilon]/(\epsilon^2) & \longrightarrow & X \\ & \searrow & \swarrow \\ & \text{Spec } k & \end{array},$$

we have this diagram

$$\begin{array}{ccc} k[\epsilon]/(\epsilon^2) & \longleftarrow & \mathcal{O}_x \\ & \nwarrow \quad \nearrow & \\ & k & \end{array}.$$

First we can see that $k(x) = k$ because the image of k in \mathcal{O}_x is a field, and $\mathcal{O}_x/\mathfrak{m}_x$ is a field.

Because $f_x^\#$ is a local homomorphism, $f_x^\#(\mathfrak{m}_x) \subseteq (\epsilon)$. Thus $f_x^\#(\mathfrak{m}_x^2) \subseteq (\epsilon^2) = 0 \implies \mathfrak{m}_x^2 \subseteq \ker f_x^\#$. As such it induces a ring map $\mathcal{O}_x/\mathfrak{m}_x^2 \rightarrow k[\epsilon]/(\epsilon^2)$. By restricting to $\mathfrak{m}_x/\mathfrak{m}_x^2$ and composing with just taking the coefficient of ϵ in the image, we get a k -linear vector space map $\mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow k$. (Note that (ϵ) has no elements of the form $\epsilon + a$).

\Leftarrow) Suppose we have $x \in X$ such that $k(x) = k$ and $e \in T_x$. Define the map $(f, f^\#)$ by letting f map (ϵ) to x . Then define $f^\#(U) : \mathcal{O}_X(U) \rightarrow 0$ for U with $x \notin U$ to just be 0.

Finally define $f^\#(U) : \mathcal{O}_X(U) \rightarrow k[\epsilon]/(\epsilon^2)$ for $x \in U$ to be the composition of $\mathcal{O}_X(U) \rightarrow \mathcal{O}_x \xrightarrow{e'} k[\epsilon]/(\epsilon^2)$ where e' is defined as follows: write $h \in \mathcal{O}_x$ as $k' + m$ with $k' \in k$ and $m \in \mathfrak{m}_x$, which exists because $\mathcal{O}_x/\mathfrak{m}_x = k$. Then send h to $k' + e([m])\epsilon$. This is a ring morphism because $(k_1 + m_1)(k_2 + m_2) \mapsto k_1k_2 + e([k_1m_2 + k_2m_1])\epsilon$, which is the result expected. □

Exercise 1.2.9. If X is a topological space, and Z an irreducible closed subset of X , a *generic point* for Z is a point ζ such that $Z = \{\zeta\}^-$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Proof. If we have an open subset of Z s.t. $\{\zeta\}^- \cap U = U$, then $(U^C \cap Z) \cup ((\{\zeta\}^- \cap U) \cap Z) = Z$. Both of these are closed, so by irreducibility of Z , $\{\zeta\}^- \cap U \cap Z = Z$. As the closure is the smallest closed set containing ζ , and Z is closed, we have that $\overline{\{\zeta\}^- \cap U} = Z$ (as $\{\zeta\}^- \subseteq Z$). Because $\{\zeta\}^- \cap U \subseteq \{\zeta\}^-$, $Z \subseteq \{\zeta\}^-$, showing that $\{\zeta\}^- = Z$.

Hence we can reduce it to showing that there is an open affine space $\text{Spec } A$, there is a point ζ s.t. $\{\zeta\}^- = \text{Spec } A$. Further, we have that $\text{Spec } A$ is irreducible, since if we had a closed partition of $\text{Spec } A$, the complement of $\text{Spec } A$ is closed, giving rise to a closed partition of Z . Hence there is only one minimal prime ideal of $\text{Spec } A$, with which the closure of it is $\text{Spec } A$ (there is only one minimal prime ideal because otherwise, the closures of them would be a closed partition of $\text{Spec } A$).

The uniqueness follows from the uniqueness of the minimal prime ideal. \square

Exercise 1.2.10. Describe $\text{Spec } \mathbf{R}[x]$. How does its topological space compare to the set \mathbf{R} ? To \mathbf{C} ?

Proof. $\text{Spec } \mathbf{R}[x] = \{(0), (x-a), (x^2+ax+b) | a, b \in \mathbf{R}, a^2-4b < 0\}$ as $\mathbf{R}[x]$ is a PID and these are the irreducible elements. Its set of closed points has a copy of \mathbf{R} in it, through $\{x-a\}$ and a copy of $\mathbf{C} \setminus \mathbf{R}$ glued along the real line, as each (x^2+ax+b) corresponds to its two roots glued, and each complex, non-real number corresponds to such a x^2+ax+b . \square

Exercise 1.2.11. Let $k = \mathbf{F}_p$ be the finite field with p elements. Describe $\text{Spec } k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Proof. Because $k[x]$ is a PID, the prime ideals correspond to irreducible elements. Thus the residue field is just a finite field, as $(k[x]/\mathfrak{p})_{\mathfrak{p}} \cong k[x]/\mathfrak{p}$ due to $k[x]/\mathfrak{p}$ being a field already. The fact that $k[x]/\mathfrak{p}$ is a finite field is due to \mathfrak{p} being generated by an irreducible.

The number of points with a given residue field, say \mathbf{F}_{p^n} equals

$$\frac{1}{n} \sum_{d|n} \mu(d) p^{n/d}$$

by Möbius inverting the formula $\sum_{d|n} d\tau(d) = p^n$. We get this formula from counting degrees in the fact that $x^{p^n} - x$ factors into all the irreducibles of degree d for all $d|n$.

The topology here is the Zariski topology, i.e. cofinite sets because $k[x]$ is a PID, so $V(\mathfrak{a}) = V((f))$ for some $f \in k[x]$, and because $k[x]$ has unique factorization, $V(\mathfrak{a})$ is finite. This proof works for the previous question too.

The set of closed points of degree n “look” like \mathbf{F}_{p^n} with each point glued to n other points glued along the Galois automorphisms. \square

Exercise 1.2.12 (Gluing Lemma). Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$ and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\phi_{ij} : U_{ij} \rightarrow U_{ji}$ such that (1) for each i, j , $\phi_{ji} = \phi_{ij}^{-1}$, and (2) for each

$i, j, k, \phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X , together with morphisms $\psi_i : X_i \rightarrow X$ for each i , such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X , (2) the $\psi_i(X_i)$ cover X , (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij} . We say that X is obtained by *glueing* the schemes X_i along the isomorphisms ϕ_{ij} . An interesting special case is when the family X_i is arbitrary, but the U_{ij} and ϕ_{ij} are all empty. Then the scheme X is called the *disjoint union* of the X_i , and is denoted $\sqcup X_i$.

Proof. Let $sp(X)$ be the pushout in the category of topological spaces of the ϕ_{ij} of the diagrams

$$\begin{array}{ccc} U_{ij} = U_{ji} & \longrightarrow & X_j \\ \downarrow & & \\ X_i & & \end{array}$$

(the extra properties are needed to ensure that the pushout exists, namely for ensuring that $x \sim x' \iff x \in U_{ij}, x' \in U_{ji}, \phi_{ij}(x) = x'$). We want to let the sheaf structure be the one obtained from gluing the sheaves together, a la Exercise 1.22. To glue them together, let the topological space be $sp(X)$. Because this is the pushforward, $\{sp(X_i)\}$ form an open cover of $sp(X)$. Then each $sp(X_i)$ has a sheaf on it, X_i . Then because $sp(X_i) \cap sp(X_j)$ in X is just U_{ij} by gluing. Thus by letting the gluing maps be ϕ_{ij} and id when $j = i$, we have the maps that satisfy conditions 1 and 2 needed. \square

Exercise 1.2.13. A topological space is *quasi-compact* if every open cover has a finite subcover.

- (a) Show that a topological space is noetherian (I, §1) if and only if every open subset is quasi-compact.
 - (b) If X is an affine scheme, show that $sp(X)$ is quasi-compact, but not in general noetherian. We say a scheme is *quasi-compact* if $sp(X)$ is.
 - (c) If A is a noetherian ring, show that $sp(\text{Spec } A)$ is a noetherian topological space.
 - (d) Give an example to show that $sp(\text{Spec } A)$ can be noetherian even when A is not.
-

Proof. (a) \implies) Suppose X is noetherian and we had an open cover with no finite subcover $\{U_i\}$. Then we can get a non-stabilizing descending chain of closed sets by starting with some $V_1 \in \{U_i\}$, picking V_{i+1} not contained in V_i . So we have

$$V_1^C \supseteq (V_1 \cup V_2)^C \supseteq \cdots \supseteq (\cup_{n=1}^i V_n)^C \supseteq \cdots$$

It is obviously descending and consists of closed sets, and it is non-stabilizing because if it stabilizes, then by how we pick V_{i+1} , it must terminate in \emptyset . But then we have a finite subcover.

\Leftarrow) Suppose every open subset is quasi-compact. Now FTSOC suppose we have a descending chain of closed sets:

$$C_1 \supseteq C_2 \supseteq \cdots$$

Then consider the open set $\bigcup C_i^C$ and the open cover $\{\bigcup_{i=1}^n C_i^C\}_{n \in \mathbb{N}}$. As $\bigcup C_i^C$ is quasi-compact, we have a finite subcover $\bigcup_{i=1}^{n_1} C_i^C, \dots, \bigcup_{i=1}^{n_m} C_i^C$. Let $N = \max n_i$.

Then the (finite) union of the subcover is X and also $\cup_{i=1}^N C_i^C$. Taking complements, we get that $\emptyset = \cap_{i=1}^N C_i$. Thus this sequence terminates.

(b) Let $X = \operatorname{Spec} A$ and suppose we have an open cover $\{U_i\}$. Then $X = \bigcup U_i \implies \emptyset = \bigcap V(\mathfrak{a}_i) = V(\sum \mathfrak{a}_i)$. Because $V(\sum \mathfrak{a}_i) = \emptyset \iff \sum \mathfrak{a}_i = (1)$, this implies that $1 = \sum_{j=1}^m f_{n_j}$ for $f_{n_j} \in \mathfrak{a}_{n_j}$. Now take as subcover the \mathfrak{a}_{n_j} . This forms a cover because $(\cup_{j=1}^m U_{n_j})^C = \cap_{j=1}^m V(\mathfrak{a}_{n_j}) = V(\sum_{j=1}^m \mathfrak{a}_{n_j}) = \emptyset$ because $1 \in \sum_{j=1}^m \mathfrak{a}_{n_j}$.

For an example of a quasi-compact non-Noetherian affine scheme, consider $\operatorname{Spec}(\mathbf{C}[x_1, x_2, \dots])_{(x_1, x_2, \dots)}$. As we have localized all ideals not contained in (x_1, x_2, \dots) , the prime spectrum falls into this ascending sequence:

$$(x_1) \subseteq (x_1, x_2) \subseteq \dots$$

Then we have an non-stabilizing chain of closed subsets:

$$V(x_1) \supseteq V(x_1, x_2) \supseteq \dots$$

Furthermore, this is quasi-compact, because the topology is cofinite (see Exercise 2.11).

(c) Suppose we have a descending non-stabilizing chain of closed sets

$$V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \dots$$

Then we can WLOG suppose that \mathfrak{a}_i are radical, because $V(\mathfrak{a}_i) = V(\sqrt{\mathfrak{a}_i})$ —clearly $V(\sqrt{\mathfrak{a}}) \supseteq V(\mathfrak{a})$ and $V(\sqrt{\mathfrak{a}}) \subseteq V(\mathfrak{a})$ because any prime that contains \mathfrak{a} also contains $\sqrt{\mathfrak{a}}$, being the intersection of all prime ideals containing \mathfrak{a} . By 2.1c,

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$$

As this is an ascending chain of ideals, it stabilizes, at which point the chain of closed sets also stabilize.

(d) Consider $\operatorname{Spec} \mathbf{C}[x_1, x_2, \dots]$. The topology is the cofinite topology (see Exercise 2.11), so every open subset is quasi-compact, allowing us to use a). But $\mathbf{C}[x_1, \dots]$ is not Noetherian as we have $(x_1) \subseteq (x_1, x_2) \subseteq \dots$, each of which is prime. \square

Exercise 1.2.14.

- Let S be a graded ring. Show that $\operatorname{Proj} S = \emptyset$ if and only if every element of S_+ is nilpotent.
- Let $\phi : S \rightarrow T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supseteq \phi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj} T$, and show that ϕ determines a natural morphism $f : U \rightarrow \operatorname{Proj} S$.
- The morphism f can be an isomorphism even when ϕ is not. For example, suppose that $\phi_d : S_d \rightarrow T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \rightarrow \operatorname{Proj} S$ is an isomorphism.
- Let V be a projective variety with homogenous coordinate ring S (I, §2). Show that $t(V) \cong \operatorname{Proj} S$.

Proof. (a) If $\operatorname{Proj} S = \emptyset$, and there is a non-nilpotent element of S_+ , then because this is contained in a prime (non-necessarily homogenous) ideal of S \mathfrak{p} , by taking the ideal generated by the homogenous elements of \mathfrak{p} we get a homogenous prime ideal containing the element. This homogenous ideal is prime because it is a subset

of \mathfrak{p} . But then we get a contradiction with our starting assumption, and hence every element of S_+ is nilpotent.

Conversely, if every element is nilpotent, then if we had $\mathfrak{p} \in \text{Proj } S$, then because $0 \in \mathfrak{p}$ and \mathfrak{p} is a homogenous prime ideal, $\forall a \in S_i, a^n = 0 \implies a \in \mathfrak{p}$ as a is homogenous. But these $a \in S_i$ for all i generate S_+ , so $S_+ \subseteq \mathfrak{p}$, a contradiction.

Hence we get a morphism f by having a topological map via taking preimages: this is well-defined because the preimage is prime, homogenous, and doesn't contain S_+ . The map on sheaves is the one induced by gluing together the open covers $\text{Spec } T_{(\phi(f))} \rightarrow \text{Spec } S_{(f)}$ induced by $S_{(f)} \rightarrow T_{\phi(f)}$ via [Lemma 1.3](#). The conditions for gluing are met because on intersections, it is just $\text{Spec } T_{(\phi(f_i f_j))}$, which is well-defined up to order. It is a local homomorphism because the map $\mathcal{O}_{T, \mathfrak{p}} \rightarrow \mathcal{O}_{S, \phi^{-1}(\mathfrak{p})}$ is clearly a local homomorphism.

(b) U is an open subset because $U^C = \{\mathfrak{p} \in \text{Proj } T \mid \mathfrak{p} \supseteq \phi(S_+) = V(\langle \phi(S_+) \rangle)\}$ where $\langle \phi(S_+) \rangle$ indicates the homogenous ideal generated by it. Such an ideal exists because ϕ is degree preserving (namely, the homogenous generators of S_+ will be sent to homogenous generators of $\langle \phi(S_+) \rangle$). The equality is obvious: a homogenous ideal containing the homogenous ideal generated by $\phi(S_+)$ must contain $\phi(S_+)$ and a homogenous ideal containing $\phi(S_+)$ must contain the ideal generated by $\phi(S_+)$.

(c) First we show that $U = \text{Proj } T$: we want to show that any $\mathfrak{p} \in \text{Proj } T$ doesn't contain $\phi(S_+)$. We do so by using the observation that homogenous prime $\mathfrak{p} \subseteq T$ contains $\phi_{\geq d_0}(S_{\geq d_0})$ if and only if \mathfrak{p} contains T_+ . One direction is obvious because $\phi_{\geq d_0}$ is an isomorphism.

Now if we had a homogenous element a of T_+ , say it is of degree i , there n such that $a^n \in T_{\geq d_0}$ by simply picking n such that $in \geq d_0$. Thus $a^n \in \mathfrak{p}$, but because \mathfrak{p} is prime, $a \in \mathfrak{p}$. Because homogenous elements span T_+ , $T_+ \subseteq \mathfrak{p}$.

Because all $\mathfrak{p} \in \text{Proj } T$ don't contain $\phi(S_+)$, $U = \text{Proj } T$. Next we show that f is a homeomorphism. Continuity of f is obvious.

injective: suppose we had $\mathfrak{p}, \mathfrak{q} \in \text{Proj } T$ such that their images were equal. Then $f(\mathfrak{p})_{\geq d_0} = f(\mathfrak{q})_{\geq d_0}$. Now take some homogenous element a of $\mathfrak{p}_{< d_0}$. Then for some n , $a^n \in \mathfrak{p}_{\geq d_0}$. Thus $\phi^{-1}(a^n) \in f(\mathfrak{p})_{\geq d_0} = f(\mathfrak{q})_{\geq d_0}$ so that $a^n \in \mathfrak{q}$. Because \mathfrak{q} is a homogenous prime, $a \in \mathfrak{q}$, proving their equality.

surjective: Define the map $f^{-1} : \text{Proj } S \rightarrow \text{Proj } T$ via $\mathfrak{p} \mapsto \sqrt{\langle \phi(\mathfrak{p}) \rangle}$ where $\langle \cdot \rangle$ indicates the homogenous ideal generated by it. Because radicals of homogenous are homogenous, $\sqrt{\langle \phi(\mathfrak{p}) \rangle}$ is homogenous. It is also prime because given homogenous $ab \in \sqrt{\langle \phi(\mathfrak{p}) \rangle}$, $\exists m$ such that $(ab)^m \in \langle \phi(\mathfrak{p}) \rangle$. We can find n such that $(ab)^{mn} \in \phi_{\geq d_0}(\mathfrak{p})$ and large enough that $a^{mn}, b^{mn} \in S_{\geq d_0}$. Pulling back with the isomorphism $\phi_{\geq d_0}$ and using the fact that \mathfrak{p} is a homogenous prime, we get that $\phi^{-1}(a^{mn})$ or $\phi^{-1}(b^{mn})$ is in \mathfrak{p} . Thus either a^{mn} or b^{mn} are in $f(\mathfrak{p})$. Hence a or b are in $\sqrt{\langle \phi(\mathfrak{p}) \rangle}$.

Let $f^{-1}(\mathfrak{p}) = \mathfrak{q}$. Finally, we can show that $f(\mathfrak{q}) = \mathfrak{p}$. One direction is obvious. Now we want to show that $\phi^{-1}(\sqrt{\langle \phi(\mathfrak{p}) \rangle}) \subseteq \mathfrak{p}$.

Suppose that we have $a \in \phi^{-1}(\sqrt{\langle \phi(\mathfrak{p}) \rangle})$. Consider it's homogenous components a_i . Then $\phi(a_i) \in \sqrt{\langle \phi(\mathfrak{p}) \rangle}$. By definition, we have m such that $\phi(a_i)^m = \phi(a_i^m) \in \langle \phi(\mathfrak{p}) \rangle$. Thus $\phi(a_i^m) = \sum t_i \phi(p_i)$ with $t_i \in T$ and $p_i \in \mathfrak{p}$. By finding n large enough that each $t_i \phi(p_i)$ has degree larger than d_0 (possible by finding the least degree $t_i \phi(p_i)$ and multiplying it enough times to be larger than d_0), we find that $\phi(a_i^{mn}) \in \langle \phi(\mathfrak{p}) \rangle$ via the isomorphism.

Thus $a_i^{mn} \in \mathfrak{p}$, so $a_i \in \mathfrak{p}$ by \mathfrak{p} being homogenous prime and a_i being homogenous. As all the components of a are in \mathfrak{p} , $a \in \mathfrak{p}$.

Because we have a morphism $\text{Proj } T \rightarrow \text{Proj } S$, it suffices to show that they are isomorphic on stalks. The stalks are $T_{\mathfrak{p}}$ and $S_{f(\mathfrak{p})}$ respectively. Because every element of $T_{\mathfrak{p}}$ is of the form $\frac{t}{a}$ for $t \in T$ and $a \in T \setminus \mathfrak{p}$ such that t and a have the same degree. Then we can find n such that $\frac{t}{a} = \frac{ta^{n-1}}{a^n}$ with $ta^{n-1} \in T_{\geq d_0}$. Hence $\phi_{\geq d_0}$ is an isomorphism of $T_{\mathfrak{p}}$ to $S_{f(\mathfrak{p})}$, showing that f is an isomorphism of schemes.

(d) Using Chapter 2.4.10, we get a morphism $t(V) \rightarrow \text{Proj } S$. By checking the stalks, we can see that this is an isomorphism. \square

Exercise 1.2.15.

- (a) Let V be a variety over the algebraically closed field k . Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k .
- (b) If $f : X \rightarrow Y$ is a morphism of schemes over k , and if $P \in X$ is a point with residue field k , then $f(P) \in Y$ also has residue field k .
- (c) Now show that if V, W are any two varieties over k , then the natural map

$$\text{Hom}_{\mathbf{Var}}(V, W) \rightarrow \text{Hom}_{\mathbf{Sch}/k}(t(V), t(W))$$

is bijective. (Injectivity is easy. The hard part is to show it is surjective).

Proof. (a) Suppose P is closed. Because α is a homeomorphism of V to closed points of $t(V)$, $P = \{P'\}^- = \{\eta\}$, i.e. P' is a closed point.

Now consider

$$\lim_{\substack{\longrightarrow \\ P \in \text{open } U \subseteq t(V)}} \mathcal{O}_V(\alpha^{-1}(U)).$$

Because P is in all the U , $\alpha^{-1}(U)$ contains P' for all U . As α produces a bijection between open subsets of $t(V)$ and V , this limit is then just $\mathcal{O}_{V, P'}$.

Next note that because V is a variety over k , there is a map $\mathcal{O}_V \rightarrow \text{Spec } k$. Now let $\text{Spec } A$ be an open affine neighborhood of P' . Because of the map $\mathcal{O}_V \rightarrow \text{Spec } k$, we have a map $k \rightarrow A$, making A a k -algebra. Finally, because $\mathcal{O}_{V, P'} = \mathcal{O}_{\text{Spec } A, P'}$ and P' is a closed point, P' be maximal. Thus the residue field is k because k is algebraically closed ($k(P')$ is a finite field extension of k because V being a variety implies that $k(P')$ is finitely generated over k , but k is algebraically closed).

If $k(P) = k$: Take an open affine cover of $t(V)$, $\{\text{Spec } A_i\}$. Then for all $\text{Spec } A_i$ with P in them, $\mathcal{O}_{t(V), P} = \mathcal{O}_{\text{Spec } A_i, P}$ has residue field k . But this only happens if P is maximal in A_i , i.e. P is closed in $\text{Spec } A_i$ (Nullstellensatz). A point closed in an open cover is closed.

(b) By Exercise 2.7, we have a map $\text{Spec } k \rightarrow X$ because we have a point $P \in X$ and an inclusion $k(P) = k \rightarrow k$. Then just composing, we get a map $\text{Spec } k \rightarrow Y$. By using Exercise 2.7 again, we get an inclusion $k(f(P)) \rightarrow k$. Because Y is a scheme over k , we get a map $k \rightarrow k(f(P))$, making $k(f(P))$ equal to k .

(c) The natural map is by taking $f : V \rightarrow W$ to a map of k schemes with the topological map being $t(f)$ and $f^\#(U) : \alpha_*(\mathcal{O}_W)(U) \rightarrow t(f)_*\alpha_*(\mathcal{O}_V)(U)$ for $U \subseteq t(W)$ as follows:

First take a section $s \in \mathcal{O}_W(\alpha^{-1}(U))$. Next, notice that $t(f)^{-1}(\alpha^{-1}(U)) = f^{-1}(\alpha^{-1}(U))$ because $t(f)(\alpha(P)) = \overline{f(P)} \supseteq \overline{f(P)}$ as $f(P) \in \overline{f(P)}$. We also have $\overline{f(P)} \subseteq \overline{f(P)}$ because $\overline{f(P)}$ is a closed set containing $f(P)$ as $f^{-1}\overline{f(P)}$ is closed and contains P , so $\overline{P} \subseteq f^{-1}\overline{f(P)}$.

Finally, by precomposing it with f , we get a regular map on $f^{-1}(\alpha^{-1}(U)) = \alpha^{-1}(t(f)^{-1}(U))$, i.e. $s \circ f : f^{-1}(\alpha^{-1}(U)) \rightarrow k$ by definition of a morphism of varieties.

This map is injective because the codomain is not correct ($\mathcal{O}(f^{-1}\alpha^{-1}(U))$ isn't always $\mathcal{O}(g^{-1}(\alpha^{-1}(U)))$). For surjectivity, suppose we have a morphism $f : t(V) \rightarrow t(W)$. Then we can see that f takes closed points to closed points because it is a local homomorphism and the stalks of $t(V)$ and $t(W)$ at closed points is just the stalk of the sheaf of regular functions. Thus define $f' : V \rightarrow W$ by $P \mapsto \alpha^{-1}(f(\{P\}))$ (all points of a variety are closed). This is continuous because it satisfies the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{f'} & W \\ \downarrow \alpha & & \downarrow \alpha \\ t(V) & \xrightarrow{f} & t(W) \end{array}$$

and an open set in W bijects via α to an open in $t(W)$, preimages to an open in $t(V)$, and bijects to an open in V via α .

Finally, to see that this is a morphism of varieties, suppose we have a regular function $s \in \mathcal{O}_W(U)$. To show that $s \circ f'$ is regular on $(f')^{-1}(U)$, take a point $v \in (f')^{-1}(U)$. We can pick an open affine subneighborhood, $U'' = \text{Spec } A$ of U' . Then pick an open affine neighborhood $N = \text{Spec } B$ of v contained in $f^{-1}(U'')$. By making these choices, we have a map $\text{Spec } B \rightarrow \text{Spec } A$, inducing a map $\mathcal{O}_W(U'') = A \rightarrow B = f'_*\mathcal{O}_V(N)$, implying that $s|_{\text{Spec } B} = \frac{P}{Q}$ into a rational polynomial in k with a denominator that doesn't vanish on N (as otherwise the denominator would become a polynomial th \square

Exercise 1.2.16. Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal \mathfrak{m}_x of the local ring \mathcal{O}_x .

- (a) If $U = \text{Spec } B$ is an open *affine* subscheme of X , and if $\bar{f} \in B = \Gamma(Y, \mathcal{O}_X|_U)$ is the restriction of f , show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X .

Proof. $U \cap X_f \subseteq D(\bar{f})$: Take a point $x \in U \cap X_f$. Then because $\mathcal{O}_x = \mathcal{O}_{U,x} = B_x$, f_x not being in the maximal ideal of \mathcal{O}_x implies that \bar{f}_x isn't in the maximal ideal of B_x . So \bar{f}_x is invertible, implying that $\bar{f}_x = \frac{\bar{f}}{1} \notin \mathfrak{m}_x$. Thus $x \in D(\bar{f})$.

$U \cap X_f \supseteq D(\bar{f})$: Take some point $x \in D(\bar{f})$, so $\bar{f} \notin \mathfrak{m}_x$. Thus $\frac{\bar{f}}{1}$ is invertible in B_x . Hence $\frac{\bar{f}}{1} = \bar{f}_x = f_x$ is not in the maximal ideal of $\mathcal{O}_x = \mathcal{O}_{U,x} = B_x$, putting $x \in X_f$.

To see that X_f is open, for every point of X_f , we can find an open affine neighborhood of it, whose intersection with X_f is an open set. A union of open sets is open. \square

- (b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$. [Hint: Use an open affine cover of X .]

Proof. Take an open affine cover of X . Because X is quasi-compact, we can take a finite number, say U_1, \dots, U_n and say they equal $\text{Spec } A_i$.

Let \bar{f}_i be the image of f in U_i . This then covers X_f , and because of (a), $U_i \cap X_f = D(\bar{f}_i)$.

As $D(\bar{f}_i) \cong \text{Spec}(A_i)_{\bar{f}_i}$, $a|_{X_f} = 0 \implies a|_{U_i} = 0 \in (A_i)_{\bar{f}_i}$. Thus $\exists n_i$ s.t. $\bar{f}_i^{n_i} a|_{U_i} = (f^{n_i} a)|_{U_i} = 0$ in A_i by definition of localizing. Because there are finitely many, we can take a common n for all \bar{f}_i . As \mathcal{O}_X is a sheaf, $f^n a$ being 0 on the restrictions to an open cover implies that $f^n a$ is globally 0. \square

- (c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied for example, if $\text{sp}(X)$ is Noetherian.) Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A .

Proof. Let $U_i = \text{Spec } A_i$. Let $b|_{X_f \cap U_i} = \frac{b_i}{\bar{f}_i^{n_i}}$ for $b_i \in A_i$. Because there are finitely many U_i , we can pick a sufficiently large common n for all of them. Then $f^n b|_{X_f \cap U_i} = b_i$.

From this it follows that restricting b_i, b_j to $U_{ij} := U_i \cap U_j$ equals $f^n b|_{X_f \cap U_{ij}}$, so $b_i - b_j = 0$ in $\Gamma(U_{ij} \cap X_f, \mathcal{O}_X)$. As U_{ij} is quasi-compact by hypothesis, from (b) we can conclude that there is a m_{ij} and s.t. $f^{m_{ij}}(b_i - b_j) = 0$. Because there are finitely many, we can pick a universal m that works for all of i, j . Then by \mathcal{O}_X being a sheaf (and they agree on intersections), the $f^m b_i \in \Gamma(U_i, \mathcal{O}_X)$ glue together to get a global section $s \in A$. Finally, $s|_{X_f} = f^{n+m} b$ because $s|_{U_i \cap X_f} - f^{n+m} b|_{U_i \cap X_f} = f^m b_i - f^m b_i = 0$ on a cover of X_f , so by sheaf property (i) $s|_{X_f} - f^{n+m} b = 0$. \square

- (d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Proof. Technically, there should be some \bar{f} to indicate the image in $\Gamma(X_f, \mathcal{O}_{X_f})$, but it doesn't really add much. We have a map $\Gamma(X_f, \mathcal{O}_{X_f}) \rightarrow A_f$ by sending $b \rightarrow \frac{a}{\bar{f}^n}$ where a is the element of A and n is the associated n via (c). This is a homomorphism because obviously 0 and 1 are sent to the right places. Then with $f^n b_1 = a_1|_{X_f}$, $f^m b_2 = a_2|_{X_f}$, we have that $f^{n+m}(b_1 + b_2)$ is the restriction of $f^m a_1 + f^n a_2$. Hence $b_1 + b_2$ gets mapped to $\frac{a_1 f^m + a_2 f^n}{f^{n+m}} = \frac{a_1}{f^n} + \frac{a_2}{f^m}$, the sum of the images of b_1, b_2 .

This is an isomorphism because it is surjective with $\frac{a|_{X_f}}{f^n}$ mapping to $\frac{a}{f^n} \in A_f \forall a \in A$. This is in $\Gamma(X_f, \mathcal{O}_{X_f})$ because we can find for any point \mathfrak{p} an open neighborhood of X_f s.t. $f \notin \mathfrak{q}$ for all points \mathfrak{q} in that neighborhood. Take any open affine neighborhood of \mathfrak{p} . By definition, $f \notin \mathfrak{m}_{\mathfrak{q}} = \mathfrak{q} \forall \mathfrak{q} \in X_f$ (note that $\mathfrak{m}_{\mathfrak{q}}$ is the maximal ideal of the stalk at \mathfrak{q} of the open affine neighborhood, which equals $\mathcal{O}_{\mathfrak{q}}$). \square

Exercise 1.2.17. (a) Let $f : X \rightarrow Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i , such that for each i , the induced map $f^{-1}(U_i) \rightarrow U_i$ is an isomorphism. Then f is an isomorphism.

Proof. First we can see that f is a homeomorphism as it is:

bijjective: surjectivity comes from every point in Y being the output of some $x \in f^{-1}(U_i)$ and injectivity comes from the fact that if two points map to the same point, they are in the same preimage, which implies they are the same point by bijectivity of $f^{-1}(U_i) \rightarrow U_i$.

bicontinuous: follows from it being a local homeomorphism and a bijection.

Thus it suffices to check it is sheaf isomorphism on stalks. Take some $x \in X$ and some j such that $x \in f^{-1}(U_i)$. Then $\mathcal{O}_{Y,x} = (\mathcal{O}_Y|_{U_i})_x = \mathcal{O}_{U_i,x}$. As $\mathcal{O}_{U_i} \rightarrow \mathcal{O}_{f^{-1}(U_i)}$ is an isomorphism, we have an isomorphism $\mathcal{O}_{U_i,x} \rightarrow \mathcal{O}_{f^{-1}(U_i),x}$. Hence $\mathcal{O}_{Y,x} \cong (\mathcal{O}_Y|_{U_i})_x \cong \mathcal{O}_{U_i,x} \cong \mathcal{O}_{f^{-1}(U_i),x} \cong (\mathcal{O}_X|_{f^{-1}(U_i)})_x$. Thus the stalks are isomorphic. \square

- (b) A scheme is affine if and only if there is a finite set of elements $f_1, \dots, f_r \in A = \Gamma(X, \mathcal{O}_X)$ such that the open subsets X_{f_i} are affine and f_1, \dots, f_r generate the unit ideal in A . [Hint: Use (Ex. 2.4) and (Ex. 2.16d) above.]

Proof. The if direction is easy because $X_{f_i} \cong D(f_i)$.

For the only if direction, we will show that $X \cong \text{Spec } A$. We want to use 2.16c, so we first show that with the finite open cover of affines X_{f_i} , the intersections are quasi-compact. By 2.16a, $X_{f_i} \cap X_{f_j} = D(f_j|_{X_{f_i}}) \subseteq \text{Spec } \Gamma(X_{f_i}, \mathcal{O}_X|_{X_{f_i}})$. As a distinguished open subset of an affine space, this intersection is then affine. Thus by 2.16d we have that $\Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) \cong A_{f_i}$. As X_{f_i} is affine, $f_i : X_{f_i} \xrightarrow{\sim} \text{Spec } A_{f_i}$. By looking at the map $A_{f_i} \rightarrow \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$, we see that

$$\begin{array}{ccc} A & \longrightarrow & \Gamma(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \sim \\ A_{f_i} & \longrightarrow & \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) \end{array}$$

commutes. Thus the morphism $f : X \rightarrow \text{Spec } A$ induced by identity from 2.4a gives us isomorphisms $X_{f_i} \rightarrow \text{Spec } A_{f_i}$. By part (a), this makes f an isomorphism (because $\text{Spec } A_{f_i}$ cover $\text{Spec } A$ as the ideal generated by all the f_i 's contains 1), making X affine. \square

Exercise 1.2.18. In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

- (a) Let A be a ring, $X = \text{Spec } A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.
- (b) Let $\phi : A \rightarrow B$ be a homomorphism of rings, and let $f : Y = \text{Spec } B \rightarrow X = \text{Spec } A$ be the induced morphism of affine schemes. Show that ϕ is injective if and only if the map of sheaves $f^\# : \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is *dominant*, i.e. $f(Y)$ is dense in X .
- (c) With the same notation, show that if ϕ is surjective, then f is a homeomorphism of Y onto a closed subset of X , and $f^\# : \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is surjective.
- (d) Prove the converse to (c), namely, if $f : Y \rightarrow X$ is a homeomorphism onto a closed subset, and $f^\# : \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is surjective, then ϕ is surjective. [Hint: Consider $X' = \text{Spec}(A/\ker \phi)$ and use (b) and (c).]

Proof. (a) Begin with the fact that $D(f) = \text{Spec } A_f$. Next, notice that the elements of $\text{Spec } A_f$ don't meet f . If f is nilpotent, every prime ideal must contain f (as they contain 0), so $\text{Spec } A_f = \emptyset$.

Conversely, if $\text{Spec } A_f = \emptyset$, then $A_f = 0$ because every non-zero ring has a maximal ideal. Thus there is some n such that $1 \cdot f^n = 0$ in A .

(b) The map of sheaves is injective iff it is injective on the stalks. This happens iff $f_y^\# : \mathcal{O}_{X,f(y)} \rightarrow (f_* \mathcal{O}_Y)_{f(y)}$ is injective. Note that the domain and codomain are $A_{f(y)}$ and B . To see this codomain equality, we can first see that $(f_* \mathcal{O}_Y)_{f(y)} \cong B \otimes A_{f(y)}$. But this happens iff the map $A \rightarrow B$ is injective as these localizations range over all prime ideals (by commutative algebra, can see Atiyah-Macdonald Chapter 5).

Finally, $f(Y)$ is dense because we can show that every open set in X intersects $f(Y)$. It suffices to show this for distinguished open sets $D(g) \subseteq X$ because every open set contains a non-empty distinguished open. Because $D(g) \cong \text{Spec } A_g$, if we show that there is an element of $\text{Spec } A_g$ that is the image of an element of $\text{Spec } B$, we are done. Now consider $\text{Spec } B_{\phi(g)}$. This is non-empty because it is empty iff $\phi(g)$ is nilpotent, which would imply that g is nilpotent (using injectivity of ϕ), contradicting the non-emptiness of $\text{Spec } A_g$. Then the image of an element of $\text{Spec } B_{\phi(g)}$ is in $\text{Spec } A_g$ because primes in $B_{\phi(g)}$ don't contain $\phi(g)$, so their pullbacks don't either.

(c) To see that f is a homeomorphism onto a closed subset of X , realize that because ϕ is surjective, $A/\ker \phi \cong B$. Thus $\text{Spec } B \cong \text{Spec } A/\ker \phi \cong V(\ker \phi) \subseteq \text{Spec } A$.

Finally, to see that $f^\#$ is surjective, \square

Exercise 1.2.19. Let A be a ring. Show that the following conditions are equivalent:

- (a) $\text{Spec } A$ is disconnected;
- (b) there exist nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0, e_1^2 = e_1, e_2^2 = e_2, e_1 + e_2 = 1$ (these elements are called *orthogonal idempotents*);
- (c) A is isomorphism to a direct product $A_1 \times A_2$ of two nonzero rings.

Proof. $ii) \implies iii)$ We have that $A \cong Ae_1 \times Ae_2$. First note that Ae_1 is a ring because we have addition and multiplication from A , and it is closed: $ae_1 + be_1 = (a+b)e_1$ and $(ae_1)(be_1) = abe_1^2 = abe_1$. The isomorphism is via the forward direct mapping $a \in A$ to (ae_1, ae_2) and the backwards mapping (a, b) to $ae_1 + be_2$. These are inverses because $e_1 e_2 = 0$, so $ae_1 + be_2 \mapsto (a, b)$.

Finally, the forward map is injective because if $(ae_1, ae_2) = (be_1, be_2)$, then $((a-b)e_1, (a-b)e_2) = (0, 0)$, so $a-b = (a-b)(e_1 + e_2) = 0$.

$iii) \implies ii)$ Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

$i) \implies iii)$ By definition, $\text{Spec } A = V(\mathfrak{a}) \sqcup V(\mathfrak{b})$ for ideals $\mathfrak{a}, \mathfrak{b}$. Then $\text{Spec } A \cong \text{Spec } A/\mathfrak{a} \sqcup \text{Spec } A/\mathfrak{b}$. Because Spec is a contravariant functor, this implies that $A = A/\mathfrak{a} \times A/\mathfrak{b}$ via universal properties. Proof of this: We have this diagram:

$$\begin{array}{ccc}
 C & & \\
 \swarrow \exists! & \searrow & \\
 & A/\mathfrak{a} \times A/\mathfrak{b} & \longrightarrow A/\mathfrak{b} \\
 \searrow & \downarrow & \\
 & A/\mathfrak{a} &
 \end{array}$$

Applying Spec we get

$$\begin{array}{ccc}
 & \text{Spec } A/\mathfrak{b} & \\
 & \downarrow & \\
 \text{Spec } A/\mathfrak{a} & \longrightarrow & \text{Spec}(A/\mathfrak{a} \times A/\mathfrak{b}) \\
 & \searrow & \nearrow \exists! \\
 & & \text{Spec } C
 \end{array}$$

This implies by universality of disjoint union that

$$\text{Spec}(A/\mathfrak{a}) \sqcup \text{Spec}(B/\mathfrak{b}) \cong \text{Spec}(A/\mathfrak{a} \times A/\mathfrak{b}).$$

(Note that this is only in the category of affine schemes, but that is enough here. By Lemma 26.6.7 of the Stacks, finite disjoint unions and products in the category of affine schemes are also disjoint unions and products in the category of locally ringed spaces.).

iii) \implies i) Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then $\text{Spec } A = V((e_1)) \sqcup V((e_2))$ because $V((e_1)) \sqcup V((e_2)) = V((e_1 e_2)) = V((0)) = \text{Spec } A$ and $V((e_1)) \cap V((e_2)) = V((e_1, e_2)) = V((1)) = \emptyset$. \square

1.3. First Properties of Schemes.

Exercise 1.3.1. Show that a morphism $f : X \rightarrow Y$ is locally of finite type if and only if for *every* open affine subset $V = \text{Spec } B$ of Y , $f^{-1}(V)$ can be covered by open affine subsets $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B -algebra.

Proof. The if direction is easy.

We can use the Affine Communication Lemma found in Vakil. Let P be the property of $V = \text{Spec } B$ that $f^{-1}(V)$ can be covered by open affine subsets $U_j = \text{Spec } A_j$ with A_j a finitely generated B -algebra. The first condition is met because $\text{Spec } B_{f_i} \subseteq \text{Spec } B$, so by covering $f^{-1}(\text{Spec } B_{f_i})$ with the same U_j from above and using the same generators of A_j as a B -algebra, A_j is a finitely generated B_{f_i} -algebra.

Then for condition two, I claim that by embedding the open cover of $\text{Spec } B_{f_i}$ into $\text{Spec } B$, we have the conditions met for P on $\text{Spec } B$. First, because $(f_1, \dots, f_n) = (1)$, for any point in $\mathfrak{p} \in \text{Spec } B$ there is some $f_i \notin \mathfrak{p}$ (otherwise \mathfrak{p} would contain 1). So the open covers of $f^{-1}(\text{Spec } B_{f_i})$ cover $f^{-1}(\text{Spec } B)$. Then each A_{ij} whose spectra cover $f^{-1}(\text{Spec } B_{f_i})$ are finitely generated B -algebras because they are finitely generated B_{f_i} -algebras, and B_{f_i} is a finitely generated B -algebra by using $1, \frac{1}{f_i}$. Thus this open cover of $f^{-1}(B)$ give us the condition for P . \square

Exercise 1.3.2. A morphism $f : X \rightarrow Y$ of schemes is *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i . Show that f is quasi-compact if and only if for *every* open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

Proof. The if direction is easy.

For the other direction, cover V by the V_i s.t. $f^{-1}(V_i)$ is quasi-compact. Because V is open affine, it is quasi-compact, so we can pick a finite number of these. Then

any open cover of $f^{-1}(V)$ has a finite subcover formed by putting together the finite subcovers of $f^{-1}(V) \cap f^{-1}(V_i)$, of which there are a finite number of V_i . \square

Exercise 1.3.3.

- (a) Show that a morphism $f : X \rightarrow Y$ is of finite type if and only if it is locally of finite type and quasi-compact.
- (b) Conclude from this that f is of finite type if and only if for *every* open affine subset $V = \text{Spec } B$ of Y , $f^{-1}(V)$ can be covered by a finite number of open affines $Y_j = \text{Spec } A_j$, where each A_j is a finitely generated B -algebra.
- (c) Show also if f is of finite type, then for *every* open affine subset $V = \text{Spec } B \subseteq Y$ and for *every* open affine subset $U = \text{Spec } A \subseteq f^{-1}(V)$, A is a finitely generated B -algebra.

Proof. a) The if direction is trivial. Finite type implies locally finite type by definition. Now for quasi-compact, take the cover of Y from locally finite type, let it be V_i . Because it is finite type, there is a finite open affine cover of $f^{-1}(V_i)$, say U_{ij} . Then any open cover of $f^{-1}(V_i)$ then covers U_{ij} , which is quasi-compact because it is affine, so we can take a finite subcover. We then have a finite subcover of $f^{-1}(V_i)$ by putting together all the finite subcovers from U_{ij} .

b) Easy by using Exercises 3.1 and 3.2.

c) WLOG, we can have $Y = \text{Spec } B$ be affine. Let P be the property of $\text{Spec } A \subseteq X$ that A is a finitely generated B -algebra. We want to use the affine communication lemma. Clearly $\text{Spec } A_g$ satisfies P , giving us condition 1 (add $\frac{1}{g}$ as a generator). Then for condition 2, by the proof of the second condition in 3.1, having a cover of $\text{Spec } A \subseteq \text{Spec } B$ of $\text{Spec } A_f$ with A_f f.g. B algebras gives us that A is a f.g. B algebra. \square

Exercise 1.3.4. Show that a morphism $f : X \rightarrow Y$ is finite if and only if for *every* open affine subset $V = \text{Spec } B$ of Y , $f^{-1}(V)$ is affine, equal to $\text{Spec } A$, where A is a finite B -module.

Proof. The if direction is trivial.

Let $\text{Spec } B_{f_i}$ be a cover of Y s.t. $f^{-1}(\text{Spec } B_{f_i}) = \text{Spec } A_i$ with A_i finite B -modules. Because V is quasi-compact, we can pick a finite subcover of this. Then in V , $(f_{i_1}, \dots, f_{i_n}) = (1)$. Let $X = f^{-1}(\text{Spec } B)$ and $\overline{f_i}$ be the image of f_i in B . Because $X_{\overline{f_i}} = X \setminus V(\overline{f_i})$ and $f^{-1}(\text{Spec } B_{f_i}) = f^{-1}(\text{Spec } B) \setminus f^{-1}(V(\overline{f_i})) = f^{-1}(\text{Spec } B) \setminus V(\overline{f_i}) = \text{Spec } A_i$, we can apply the affine criterion to f_{i_n} from 2.17 to conclude that $X = f^{-1}(\text{Spec } B)$ is affine.

As $f^{-1}(\text{Spec } B_{f_i}) = \text{Spec } A_{f_i}$, we have each A_i a localization of A . Because $(f_{i_1}, \dots, f_{i_n}) = (1)$, the ideal generated by $f_i^{\#}(\overline{f_i})$ also generates (1) in A . Let $g_i = f_i^{\#}(\overline{f_i})$. Say A_i is generated by $\frac{a_{i_1}}{f_i^n}, \dots, \frac{a_{i_j}}{f_i^n}, \dots$ where we pick n sufficiently large s.t. it is the same for all i, j . Because $(f_{i_1}, \dots, f_{i_n}) = (1)$, $\exists d_i$ s.t. $\sum d_i f_{i_1}^n = 1$. Then we can see that A is generated by $d_i, f_{i_j}^n, a_{i_j}$ because any element of B can be written using the generators of A_{f_i} , which upon multiplication by 1 and substituting kills the denominators and leaves these terms to generate it, which is a finite number. \square

Exercise 1.3.5. A morphism $f : X \rightarrow Y$ is *quasi-finite* if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

- (a) Show that a finite morphism is quasi-finite.
- (b) Show that a finite morphism is *closed*, i.e. the image of any closed subset is closed.
- (c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

Proof. a) By 3.4 we can reduce to the case of affine X, Y , say $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$ with B a finite A -module. Because the topological space of the fibre of f over y is homeomorphic to $f^{-1}(y)$ (3.10), if the fibre is finite we are done. Then we have that $X_y = X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} B \times_Y \operatorname{Spec} k(y) = \operatorname{Spec}(B \otimes k(y))$. Because $k(y)$ is a field and B is a finite A -module, $B \otimes k(y)$ is a finite dimensional vector space.

This is then a finite field extension, and by commutative algebra thus an integral extension of a field. By the lying down lemma, every prime of $B \otimes k(y)$ lies over a prime of $k(y)$, and by commutative algebra there are finitely many. Because $k(y)$ has only one prime, $\operatorname{Spec}(B \otimes k(y))$ is finite.

b) Fix a closed set C and suppose FSTOC that $f(C)$ wasn't closed. Then there is a limit point p that isn't in $f(C)$. As this is a limit point, for all open sets U in Y containing p , $U \cap f(C) \neq \emptyset$.

Because f is continuous, $f^{-1}(U \cap f(C))$ is open, and, from the above, it is non-empty.

c) Take $\operatorname{Spec} \mathbb{F}_p[x]/(x^2 + 1) \rightarrow \operatorname{Spec} \mathbb{F}_p$. □

Exercise 1.3.6. Let X be an integral scheme. Show that the local ring \mathcal{O}_ζ of the generic point ζ of X is a field. It is called the *function field* of X , and is denoted by $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X , then $K(X)$ is isomorphic to the quotient field of A .

Proof. We can find an open affine neighborhood of ζ $U = \operatorname{Spec} A$ by definition of a scheme. Because X is integral, and the generic point is unique, the generic point is $(0) \in U$. Further, $\mathcal{O}_{(0),U}$ is a field because it is an integral domain. Because the local ring of a restricted scheme and the scheme are the same, \mathcal{O}_ζ is a field.

The quotient field of A is $A_{(0)} \cong \mathcal{O}(U)_\zeta$. Because the local ring of a restricted scheme and the scheme are the same, $K(X) = \mathcal{O}_\zeta \cong A_{(0)}$. □

Exercise 1.3.7. A morphism $f : X \rightarrow Y$, with Y irreducible is *generically finite* if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y . A morphism $f : X \rightarrow Y$ is *dominant* if $f(X)$ is dense in Y . Now let $f : X \rightarrow Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $Y \subseteq Y$ such that the induced morphism $f^{-1}(U) \rightarrow U$ is finite.

Proof. We show the hint first, namely that the function field K_X of X is a finite field extension of K_Y . Let η_X be the generic point of X , which exists because X is integral, so irreducible, so Exercise 2.9 applies. Then because $f(\overline{\eta_X}) \subseteq \overline{f(\eta_X)}$ and η_X is the generic point of X , we have that $f(X) \subseteq \overline{f(\eta_X)}$. Because f is dominant,

$f(X)$ is dense, so by definition of closure we have $\overline{f(X)} = Y \subseteq \overline{f(\eta_X)} \implies f(\eta_X) = \eta$. So f maps the generic point of X to the generic point of Y .

Next notice that $f^{-1}(\eta_X)$ is homeomorphic to $X \times_Y \text{Spec}(k(\eta_Y))$. As η_Y is the generic point of Y , $k(\eta_Y) = K_Y$. \square

Exercise 1.3.8 (Normalization.). A scheme is *normal* if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \text{Spec } A$ of X , let \tilde{A} be the integral closure of A in its quotient field, and let $\tilde{U} = \text{Spec } \tilde{A}$. Show that one can glue the schemes \tilde{U} to obtain a normal integral scheme \tilde{X} , called the *normalization* of X . Show also that there is a morphism $\tilde{X} \rightarrow X$, having the following universal property: for every normal integral scheme Z , and for every dominant morphism $f : Z \rightarrow X$, f factors uniquely through \tilde{X} . If X is of finite type over a field k , then the morphism $\tilde{X} \rightarrow X$ is a finite morphism. This generalizes (I, Ex. 3.17).

Proof. We wish to glue together the schemes via Exercise 2.12, so we have to check the conditions. \square

Exercise 1.3.9. *The Topological Space of a Product.* Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology (I, Ex. 1.4). Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.

- (a) Let k be a field, and let $A_k^1 = \text{Spec } k[x]$ be the affine line over k . Show that $A_k^1 \times A_k^1 \cong A_k^2$, and show that the underlying point set of the product is the not product of the underlying point sets of the factors (even if k is algebraically closed).

Exercise 1.3.10 (3.11 Closed Subschemes). (a) Closed immersions are stable under base extension: if $f : Y \rightarrow X$ is a closed immersion, and if $X' \rightarrow X$ is any morphism, then $f' : Y \times_X X' \rightarrow X'$ is also a closed immersion.

Proof. \square

Exercise 1.3.11 (3.18, Constructible Sets.). Let X be a Zariski topological. A *constructible subset* of X is a subset which belongs to the smallest family \mathcal{F} of subsets such that (1) every open subset is in \mathcal{F} , (2) a finite intersection of elements of \mathcal{F} is in \mathcal{F} , and (3) the complement of an element of \mathcal{F} is in \mathcal{F} .

- (a) A subset of X is *locally closed* if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.

Proof. First we can see that every disjoint union of locally closed sets is constructible. Suppose we had $X = \sqcup_{i=1}^n U_i \cap C_i$ with U_i open and C_i closed.

Because all open subsets are in the family, and closed sets are the complement of an open set, closed subsets are also in the family by (3). Further, finite unions of constructible sets are constructible. Say we had $X = \cup_{i=1}^n C_i$

with c_i constructible. Then c_i^C is constructible by (3), so $\cap c_i^C$ is constructible by (2). Finally, by (3) $(\cap c_i^C)^C = \cup c_i$ is constructible.

Because U_i, C_i are constructible, $U_i \cap C_i$ is constructible. As X is a finite union, we are done.

To show the reverse direction, we show that $S = \{ \text{finite disjoint union of locally closed subsets} \}$ satisfies properties (1),(2),(3).

(1): Obviously met since open sets are itself intersected with the whole space, a closed subset, making open sets locally closed. Further, (as a fact needed in (3)), closed sets are in S because a closed set is itself intersected with the whole space, an open set, making closed sets locally closed.

(2): Say we have $X = \sqcup_{i=1}^m U_i \cap C_i$ and $Y = \sqcup_{i=1}^n U'_i \cap C'_i$. Then $X \cap Y = \sqcup_{i=1}^m \sqcup_{j=1}^n (U_i \cap C_i) \cap (U'_j \cap C'_j)$. This is because of how intersections distribute across unions. The disjoint union is due to each $U_i \cap C_i, U'_j \cap C'_j$ being disjoint as i varies and as j varies. Finally, $(U_i \cap C_i) \cap (U'_j \cap C'_j) = (U_i \cap U'_j) \cap (C_i \cap C'_j)$, which is constructible as the two sides are open and closed sets respectively, making their intersection constructible.

(3): Suppose we have $X = \sqcup_{i=1}^n U_i \cap C_i$. Then $X^C = \cap_{i=1}^n (U_i \cap C_i)^C = \cap_{i=1}^n U_i^C \cup C_i^C$. As U_i^C, C_i^C are locally closed, $U_i^C \cup C_i^C \in S$ by definition. By (2), this intersection is in S . So S is closed under complements. \square

- (b) Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a non-empty open subset.

Proof. If it contains the generic point, it is obviously dense.

Suppose we have a dense constructible set S . Then by the above, it is of the form $S = \sqcup_{i=1}^n U_i \cap C_i$. Because it is dense, $\overline{S} = X$. But $\overline{S} \supseteq \cup \overline{U_i \cap C_i}$.

As X is irreducible, if the right hand side equalled X , then we would get a closed decomposition of X . So in this case, $\overline{U_i \cap C_i} = X$, and because $C_i \supseteq U_i \cap C_i$ and C_i is closed, $\overline{U_i \cap C_i} \subseteq C_i \implies C_i = X$. So X is a union of open sets in X , so X contains the generic point as all open sets contain the generic point.

So $\cup \overline{U_i \cap C_i} \subsetneq X$. As $\overline{U_i \cap C_i} \supseteq \overline{U_i} \cap \overline{C_i}$, \square

1.4. Sheaves of Modules.

Exercise 1.4.1. Show that a sheaf of \mathcal{O}_X -modules \mathcal{F} on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U , such that $\mathcal{F}|_U$ is isomorphic to a cokernel of free sheaves on U . If X is noetherian, then \mathcal{F} is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank. (These properties were originally the definition of quasi-coherent and coherent sheaves).

Proof. Fix a point $x \in X$. Let the neighborhood be an open affine set $U = \text{Spec } A$ containing x . Then $\mathcal{F}|_U \cong \tilde{M}$ by Proposition 5.4. For all open U , we have the exact sequence

$$0 \rightarrow A^n \rightarrow A^m \rightarrow M \rightarrow 0$$

because all modules are the quotient of a free module.

Because $\tilde{\cdot}$ is exact, we have that

$$0 \rightarrow \tilde{A}^n \rightarrow \tilde{A}^m \rightarrow \tilde{M} \rightarrow 0$$

is exact. By Proposition 5.2c, the first two sheaves are free sheaves, and by hypothesis the last sheaf is $\mathcal{F}|_U$. Finally, $\tilde{M} \cong \tilde{A}^m / \tilde{A}^n$ by Exercise 1.6.

Similarly, if X is Noetherian, then use the same notation as above. M is finitely generated by coherence of \mathcal{F} . Hence m is finite, and n is finite because a submodule of a finite module over a Noetherian ring is finite. \square

Exercise 1.4.2 (5.5). Let $f : X \rightarrow Y$ be a morphism of schemes.

- (a) Show by example that if \mathcal{F} is coherent on X , then $f_*\mathcal{F}$ need not be coherent on Y , even if X and Y are varieties over a field k .
- (b) Show that a closed immersion is a finite morphism (§ 3).
- (c) If f is a finite morphism of noetherian schemes, and if \mathcal{F} is coherent on X , then $f_*\mathcal{F}$ is coherent on Y .

Proof. a) Let $X = \operatorname{Spec} \mathbf{Q}$ and $Y = \operatorname{Spec} \mathbf{Z}$. Let \mathcal{F} be the coherent sheaf assigning to $(0) \in X$ the finite \mathbf{Q} -module \mathbf{Q} . Then $f_*\mathcal{F}(Y) = \mathbf{Q}$, which isn't finite over \mathbf{Z} .

b) By exercise 3.4, it suffices to show that for all affines $V = \operatorname{Spec} B$, $f^{-1}(V) = \operatorname{Spec} A$ with A a finite B module. By exercise 3.11, $f^{-1}(V) = \operatorname{Spec} B/I$ for some ideal $I \subseteq B$. As B/I is clearly finite over B , we are done.

c) Fix an arbitrary open affine $U = \operatorname{Spec} A$ of Y . Because f is finite, $f^{-1}(U) = \operatorname{Spec} B$ for B a finite A module. Because \mathcal{F} is coherent on X , for all open affines $\operatorname{Spec} A$ of Y , $\mathcal{F}(f^{-1}(\operatorname{Spec} A)) = \mathcal{F}(\operatorname{Spec} B) = \tilde{M}$ with M a finite B module. As M is finite over B and B is finite over A , M is finite over A . Thus $f_*\mathcal{F}$ is coherent on Y . \square

Exercise 1.4.3 (Support.). Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from (Ex. 1.14) and (Ex. 1.20).

- (a) Let A be a ring, let M be an A -module, let $X = \operatorname{Spec} A$, and let $\mathcal{F} = \tilde{M}$. For any $m \in M = \Gamma(X, \mathcal{F})$, show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m$ is the annihilator of $m = \{a \in A \mid am = 0\}$.
- (b) Now suppose that A is noetherian, and M finitely generated. Show that $\operatorname{Supp} \mathcal{F} = V(\operatorname{Ann} M)$.
- (c) The support of a coherent sheaf on a noetherian scheme is closed.
- (d) For any ideal $\mathfrak{a} \subseteq A$, we define a submodule $\Gamma_{\mathfrak{a}}(M)$ of M by $\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A -module. Show that $\Gamma_{\mathfrak{a}}(\tilde{M}) \cong \mathcal{H}_Z^0(\mathcal{F})$, where $Z = V(\mathfrak{a})$ and $\mathcal{F} = \tilde{M}$.
[Hint: Use (Ex. 1.20) and (5.8) to show a priori that $\mathcal{H}_Z^0(\mathcal{F})$ is quasi-coherent. Then show that $\Gamma_{\mathfrak{a}}(M) \cong \Gamma_Z(\mathcal{F})$.]
- (e) Let X be a noetherian scheme, and let Z be a closed subset. If \mathcal{F} is a quasi-coherent (respectively, coherent) \mathcal{O}_X -module, then $\mathcal{H}_Z^0(\mathcal{F})$ is also quasi-coherent (respectively, coherent).

Proof. a) Clearly $V(\operatorname{Ann} m) \subseteq \operatorname{Supp} m$, as $\forall \mathfrak{p} \in V(\operatorname{Ann} m)$, $m_{\mathfrak{p}} \neq 0$ because otherwise, $\exists a \in A \setminus \mathfrak{p}$ such that $am = 0$. But then $a \in \operatorname{Ann} m$, a contradiction. Thus $\mathfrak{p} \in \operatorname{Supp} m$.

Finally, take some $\mathfrak{p} \in \text{Supp } m$. Then $m_{\mathfrak{p}} \neq 0$, which by definition means that $\forall a \in A \setminus \mathfrak{p}, am \neq 0$. Thus $a \in \text{Ann } m \implies am = 0 \implies a \notin A \setminus \mathfrak{p}$, so $a \in \mathfrak{p}$. Hence $\text{Ann } m \subseteq \mathfrak{p}$.

b) We have that $\text{Supp } \mathcal{F} = \bigcap_{m \in \mathcal{F}(X)} \text{Supp } m$ by definition. By part a, this equals $\text{Supp } \mathcal{F} = \bigcap V(\text{Ann } m)$. Because M is finitely generated, it suffices to consider $\bigcap V(\text{Ann } m_i)$ with m_i a finite set of generators of M . Finally, $\bigcap V(\text{Ann } m_i) = V(\bigcup \text{Ann } m_i)$. But $\bigcup \text{Ann } m_i = \text{Ann } M$. Thus $\text{Supp } \mathcal{F} = V(\text{Ann } M)$.

c) By b, locally $\text{Supp } \mathcal{F}|_{\text{Spec } A} = V(\text{Ann } M)$ with M finitely generated and A noetherian.

d) By Exercise 1.20, we have

$$0 \rightarrow \mathcal{H}_Z^0(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow j_*(\mathcal{F}|_U).$$

By 5.8, $j_*(\mathcal{F}|_U)$. Because $\mathcal{H}_Z^0(\mathcal{F}) \cong \ker(\mathcal{F} \rightarrow j_*(\mathcal{F}|_U))$. But the kernel of a map between quasi-coherent sheaves is quasi-coherent, so $\mathcal{H}_Z^0(\mathcal{F})$.

e) □

Exercise 1.4.4. Let X be noetherian scheme, and let \mathcal{F} be a coherent sheaf.

- (a) If the stalk \mathcal{F}_x is a free \mathcal{O}_X -module for some point $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.
- (b) \mathcal{F} is locally free if and only if its stalks \mathcal{F}_x are free \mathcal{O}_X -module for all $x \in X$.
- (c) \mathcal{F} is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf \mathcal{G} such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. (This justifies the terminology invertible: it means that \mathcal{F} is an invertible element of the monoid of coherent sheaves under the operation \otimes).

Proof. a)

b) The if direction follows from a).

c) □

Exercise 1.4.5 (Vector Bundles). Let Y be a scheme. A (*geometric*) *vector bundle* of rank n over Y is a scheme X and a morphism $f : X \rightarrow Y$, together with additional data consisting of an open covering $\{U_i\}$ of Y , and isomorphisms $\psi_i : f^{-1}(U_i) \rightarrow \mathbb{A}_{U_i}^n$, such that for any i, j , and for any open affine subset $V = \text{Spec } A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{A}_V^n = \text{Spec } A[x_1, \dots, x_n]$, i.e. $\theta(a) = a$ for any $a \in A$, and $\theta(x_i) = \sum a_{ij} x_j$ for suitable $a_{ij} \in A$.

An *isomorphism* $g : (X, f, \{U_i\}, \{\psi_i\}) \rightarrow (X', f', \{U'_i\}, \{\psi'_i\})$ of one vector bundle of rank n to another one is an isomorphism $g : X \rightarrow X'$ of the underlying schemes, such that $f = f' \circ g$, and such that X, f , together with the covering of Y consisting of all the U_i and U'_i , and the isomorphisms ψ_i and $\psi'_i \circ g$, is also a vector bundle structure on X .

- (a) Let \mathcal{E} be a locally free sheaf of rank n on a scheme Y . Let $S(\mathcal{E})$ be the symmetric algebra on \mathcal{E} , and let $X = \text{Spec } S(\mathcal{E})$, with projection morphism $f : X \rightarrow Y$. For each open affine subset $U \subseteq Y$ for which $\mathcal{E}|_U$ is free, choose a basis of \mathcal{E} , and let $\psi : f^{-1}(U) \rightarrow \mathbb{A}_U^n$ be the isomorphism resulting from the identification of $S(\mathcal{E}(U))$ with $\mathcal{O}(U)[x_1, \dots, x_n]$. Then (X, f, ψ) is a vector bundle of rank n over Y , which (up to isomorphism) does not depend on the bases of \mathcal{E}_U chosen. We call it the geometric vector bundle associated to \mathcal{E} , and denote it by $V(\mathcal{E})$.

