**EXERCISES FROM HARTSHORNE**

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Contents



# 1. Chapter 1

## 1.1. **Affine Varieties.** Exercises

**Exercise 1.1.1** (1.1)**.**
(1) Let $Y$ be the plane curve $y = x^2$ (i.e., $Y$ is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$.

*Proof.* We have that $I(y - x^2) = (y - x^2)$, so $A(Y) = k[x, y]/(y - x^2)$, which is obviously isomorphic to $k[x]$. □

(2) Let $Z$ be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over $k$.

*Proof.* We want to show that $A(Z) = k[x, y]/(xy - 1) \ncong k[x]$. Suppose FTSOC that there was an isomorphism $f : A(Z) \to k[x]$. Then $f(xy) = f(1) = 1 = f(x)f(y)$ gives us that $f(x)$ is a unit. Similarly, we can see that $f(kx)$ are also distinct units.

As $k$ is a field, $f(k)$ is a field. The only subfield of $k[x]$ is $k$, so $f(k) = k$. But $f(x)$ is in $k$, so we don't have bijectivity. □

(3) Let $f$ be any irreducible quadratic polynomial in $k[x, y]$, and let $W$ be the conic defined by $f$. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

*Proof.* By Proposition 1.7, $\dim W = \operatorname{height} I(W) \implies$ by Proposition 1.8A $\dim W + \dim k[x, y]/I(W) = 2$. As dimension is non-negative, there are three possibilities for $\dim W$.

It can't be 2 because these correspond to maximal ideals, and maximal ideals here are of the form $(x - a, y - b)$, which isn't principal.

If it is 1, then $\dim k[x, y]/I(W) = 1$ and thus $A(W) = k[x, y]/I(W) \cong k[f]$ by definition of transcendence degree. This corresponds to the $Y$ case.

If it is 0, then $\dim k[x, y]/I(W) = 2$ and thus $A(W) = k[x, y]/I(W) \cong k[f, g]$ by definition of transcendence degree. This corresponds to the $Z$ case, as $k(Z) \cong k[x, x^{-1}]$. □

**Exercise 1.1.2** (1.2)**.** `The Twisted Cubic Curve`. Let $Y \subseteq A^3$ be the set $Y = \{(t, t^2, t^3) | t \in k\}$. Show that $Y$ is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. SHow that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$. We say that $Y$ is given by the parametric representation $x = t, y = t^2, z = t^3$.

*Proof.* $Y$ is an affine variety because it is closed due to being $Z(y - x^2, z - x^3)$ and irreducible by quotienting out $(k[x, y, z]/I(Y) = k[x])$. It is dimension 1 because $\operatorname{height} I(Y) + \dim k[x, y, z]/\mathfrak{p} = \dim k[x, y, z]$ and $1 + 2 = 3$ by Proposition 1.8A.

The generators are $y - x^2, z - x^3$ and $A(Y) \cong k[x]$. □

**Exercise 1.1.3** (1.3)**.** Let $Y$ be the algebraic set in $\boldsymbol{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that $Y$ is a union of three irreducible components. Describe them and find their prime ideals.

*Proof.* We can see that $Z(x^2 - yz, xz - x) = Z(x^2 - yz, x) \cup Z(x^2 - yz, z - 1)$ because $k$ is an integral domain. Then $Z(x^2 - yz, x) = Z(yz, x) = Z(y, x) \cup Z(z, x)$, both of which are irreducible by quotienting out.

Finally, we can see that $Z(x^2 - yz, z - 1) = Z(x^2 - y, z - 1)$, which is irreducible by quotienting out. Thus $Z(x^2 - yz, xz - x) = Z(x, y) \cup Z(x, z) \cup Z(x^2 - y, z - 1)$. $\square$

**Exercise 1.1.4** (1.4)**.** Let $Y$ be the algebraic set in $\boldsymbol{A}^3$ with $\boldsymbol{A} \times \boldsymbol{A}$ in the natural way, show that the Zariski topology on $\boldsymbol{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\boldsymbol{A}^1$.

*Proof.* In the Zariski topology, we can see that $Z(xy)$ is $\square$

**Exercise 1.1.5** (1.5)**.** Show that a $k$-algebra $B$ is isomorphic to the affine coordinate ring of some algebraic set in $\boldsymbol{A}^n$, for some $n$, if and only if $B$ is a finitely generated $k$-algebra with no nilpotent elements.

**Exercise 1.1.6** (1.6)**.** Any nonempty open subset of an irreducible topological space is dense and irreducible. If $Y$ is a subsut of a topological space $X$, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.