EXERCISES FROM HARTSHORNE

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Contents

1. Schemes

1.1. Sheaves.

Exercise 1.1.1. Let A be an abelian group, and define the *constant presheaf* assicated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.

Proof. Let the constant presheaf be $\mathcal{C}$. We want to show that $\forall$ open $U \subseteq X$, $\mathcal{C}^+(U) = A$. Clearly $\mathcal{C}_P = A$ since $\mathcal{C}(U) = A$ for all subsets U with the map from $\mathcal{C}(U) \to \mathcal{C}_P$ the identity.

Fix a connected, open U. Then for a fixed but arbitrary $P \in U$, we have A choices for $s(P)$. We can then see that by the second condition of $\mathcal{C}^+$ and making V small enough that it falls in the connected open set, there exists t in $\mathcal{C}(V) = A$ such that for all $Q \in V, t_Q = t = s(Q)$. Thus s is constant on V as t is in A.

Finally, take the collection of these neighborhoods. Pick one. Because U is connected, there must be non-empty intersection between this neighborhood and the union of the other neighborhoods. By doing the above argument for a point in their intersection, s is constant on all of U.

By construction, s satisfies conditions (1) and (2) on U, putting $s \in \mathcal{C}^+(U)$. Thus $\mathcal{C}^+(U) = A$. By picking constants on connected components, we can see that such a function satisfies (1) and (2), making $\mathcal{C}^+(U) = \mathcal{A}(U)$. □

Exercise 1.1.2.

(1) For any morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, show that for any point P, $(\ker \phi)_P = \ker(\phi_P)$ and $(im\phi)_P = \mathrm{im}(\phi_P)$.
(2) Show that ϕ is injective (respectively, surjective) if and only if the induced map on the stalks ϕ_P is injective (respectively, surjective) for all P.
(3) Show that a sequence

$$\cdots \to \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \to \cdots$$

of sheaves and morphisms is exact if and only if for each $P \in X$, the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof. a) We have the following commutative diagram by definition:

$$\begin{tikzcd}
\cdots \arrow[r] & \ker\phi(U) \arrow[rr, "\iota"] \arrow[dr] \arrow[dd] & & \ker\phi(V) \arrow[r] \arrow[dl] \arrow[dd] & \cdots \\
& & \lim\ker\phi \arrow[dd, dotted] & & \\
\cdots \arrow[r] & \mathcal{F}(U) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{F}(V) \arrow[r] \arrow[dl] \arrow[dd, "\phi(V)"] & \cdots \\
& & \mathcal{F}_P \arrow[dd, dotted, "\phi_P"] & & \\
\cdots \arrow[r] & \mathcal{G}(U) \arrow[rr] \arrow[dr] & & \mathcal{G}(V) \arrow[r] \arrow[dl, "\rho_{G,V}"] & \cdots \\
& & \mathcal{G}_P & &
\end{tikzcd}$$

We have a map $\lim\ker\phi \to \ker\phi_P$ because any element $u \in \lim\ker\phi$ can be represented by $(v, V), v \in \ker\phi(V)$, and v maps to 0 in $\mathcal{G}_P$ (so map u to v_P). Further this is injective, because each map $\ker\phi(U) \to \mathcal{F}(U)$ is injective, making the induced map injective (Proposition 1.1). Hence if we show it is surjective, we have a bijective homomorphism and thus are the same.

Then for any element $\ell \in \ker\phi_P \subseteq \mathcal{F}_P$, let (v_1, V_1) represent it. Let $(0, V_2)$ represent $\phi_P(\ell)$. Then $\rho_{G,V}(\phi(V)(v)) = 0$ by commutativity. Next restrict 0 to $V_1 \cap V_2$ to get $0 \in \mathcal{G}(V_1 \cap V_2)$, implying that via commuting that $v_1|_{V_1\cap V_2}$ is mapped to 0 under $\phi(V_1 \cap V_2)$.

So $v_1|_{V_1\cap V_2} \in \ker\phi(V_1 \cap V_2)$. We have thus found an element $\iota(v_1|_{V_1\cap V_2})$ that gets mapped to ℓ as $\iota(v_1|_{V_1\cap V_2})$ can be represented by $(v_1|_{V_1\cap V_2}, V_1 \cap V_2)$ and $(v_1|_{V_1\cap V_2})_P \in \ker\phi_P$.

I don't really want to make another large diagram to trace it for the image, so I'll just trust that it is very similar.

b) Follows from a).

c) If $\ker\phi^i = \operatorname{im}\phi^{i-1}$, then $(\ker\phi^i)_P = (\operatorname{im}\phi^{i-1})_P$. By a), we then have $\ker\phi^i_P = \operatorname{im}\phi^{i-1}_P$, showing exactness of stalks. $\square$

Exercise 1.1.3 (Extending a Sheaf by Zero)**.** Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X \setminus Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

(1) Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathcal{F})_P$ of the direct image sheaf on X is $\mathcal{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z. By abuse of notation, we will sometimes write $\mathcal{F}$ instead of $i_*\mathcal{F}$, and say "consider $\mathcal{F}$ as a sheaf on X," when we mean "consider $i_*\mathcal{F}$".

(2) Now let $\mathcal{F}$ be a sheaf on U. Let $j_!(\mathcal{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathcal{F}))_P$ is equal to $\mathcal{F}_P$ is $P \in U$, 0 if $P \notin U$, and show that $j_!\mathcal{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathcal{F}$. We call $j_!\mathcal{F}$ the sheaf obtained by *extending* $\mathcal{F}$ *by zero* outside U.

(3) Now let $\mathcal{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

Proof. (a): First we have by definition that

$$(i_*\mathcal{F})_P = \varinjlim_{V,\text{open in } X \ni P} \mathcal{F}(i^{-1}(V)).$$

If $P \notin Z$, then because U is open and contains P, this is a set the limit goes over. Then because $U \cap Z = \emptyset$, $i^{-1}(U) = \emptyset$, so $\mathcal{F}(i^{-1}(U)) = 0 \implies (i_*\mathcal{F})_P = 0$.

If $P \in Z$, then because i is the inclusion map and is continuous, we can reparameterize what the limit goes over to be U open in Z that contain P. This is just the stalk of $\mathcal{F}$.

(b): First we have by definition that

$$(j_!(\mathcal{F}))_P = \varinjlim_{V,\text{open in } X \ni P} j_!(\mathcal{F})(V).$$

If $P \notin V$, then by clearly $V \not\subseteq U$, so $j_!(\mathcal{F})(V) = 0$, making the limit 0.

If $P \in V$, then because the stalks of sheafication and the presheaf are the same, the limit is $\mathcal{F}_P$ as desired.

TODO: uniqueness ∎

1.2. **Schemes.**

Exercise 1.2.1. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of $\operatorname{Spec} K$ to X, it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof. A morphism $f : \operatorname{Spec} K \to X$ gives us a local homomorphism $f_x^\# : \mathcal{O}_x \to K$ where x is the image of the unique point of $\operatorname{Spec} K$. As this is a local homomorphism, this implies that the maximal ideal of $\mathcal{O}_x$ is $\ker f_x^\#$. By the first iso theorem, we then have that $\mathcal{O}_x / \ker f_x^\# \cong \operatorname{im}(f_x^\#) \subseteq K$, giving an inclusion $k(x) \to K$.

If we have a point $x \in X$ and an inclusion map $k(x) \to K$, then we have the desired morphism by sending $(0) \to x$ and we have two cases for defining $f^\#$. For U open, $x \in U$, then $f_*(\mathcal{O}_K(U)) = K$, and we can define the map $\mathcal{O}_X(U)$ to K via $\mathcal{O}_X(U) \to \mathcal{O}_x \to k(x) \to K$.

If $x \notin U$, then $f_*(\mathcal{O}_K(U)) = 0$, so the map is just given by 0.

Then we have that the needed diagram commute with three cases:

$x \in U \subseteq V$:

$$\begin{tikzcd} \mathcal{O}_X(V) \arrow[r] \arrow[d, "\rho_{UV}"] & K \arrow[d, "\rho_{UV}"] \\ \mathcal{O}_X(U) \arrow[r] & K \end{tikzcd}$$

The top and bottom row commute by commutativity of ρ in the direct system that is part of $\mathcal{O}_x$.

$x \notin U, x \in V$:

$$\begin{tikzcd} \mathcal{O}_X(V) \arrow[r] \arrow[d, "\rho_{UV}"] & K \arrow[d, "\rho_{UV}"] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

$x \notin U, x \notin V$:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\rho_{UV}"] & 0 \arrow[d, "\rho_{UV}"] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

∎

Exercise 1.2.2. Let X be a scheme. For any point $x \in X$, we define the *Zariski tangent space* T_x to X to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\epsilon]/\epsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of $\operatorname{Spec} k[\epsilon]/\epsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e. such that $k(x) = k$) and an element of T_x.

Proof. TODO ∎

Exercise 1.2.3. Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.

(a) If $U = \operatorname{Spec} B$ is an open *affine* subscheme of X, and if $\overline{f} \in B = \Gamma(Y, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$. Conclude that X_f is an open subset of X.

Proof. $U \cap X_f \subseteq D(\overline{f})$: Take a point $x \in U \cap X_f$. Then because $\mathcal{O}_x = \mathcal{O}_{U,x} = B_x$, f_x not being in the maximal ideal of $\mathcal{O}_x$ implies that $\overline{f}_x$ isn't in the maximal ideal of B_x. So $\overline{f}_x$ is invertible, implying that $\overline{f}_x = \frac{\overline{f}}{1} \notin x$. Thus $x \in D(\overline{f})$.

$U \cap X_f \supseteq D(\overline{f})$: Take some point $x \in D(\overline{f})$, so $\overline{f} \notin x$. Thus $\frac{\overline{f}}{1}$ is invertible in B_x. Hence $\frac{\overline{f}}{1} = \overline{f}_x = f_x$ is not in the maximal ideal of $\mathcal{O}_x = \mathcal{O}_{U,x} = B_x$, putting $x \in X_f$.

To see that X_f is open, for every point of X_f, we can find an open affine neighborhood of it, whose intersection with X_f is an open set. A union of open sets is open. □

(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$. [Hint: Use an open affine cover of X.]

Proof. Take an open affine cover of X. Because X is quasi-compact, we can take a finite number, say $U_1, \ldots, U_n$ and say they equal $\operatorname{Spec} A_i$.

Let $\overline{f}_i$ be the image of f in U_i. This then covers X_f, and because of (a), $U_i \cap X_f = D(\overline{f}_i)$.

As $D(\overline{f}_i) \cong \operatorname{Spec}(A_i)_{\overline{f}_i}$, $a|_{X_f} = 0 \implies a|_{U_i} = 0 \in (A_i)_{\overline{f}_i}$. Thus $\exists n_i$ s.t. $\overline{f}_i^{n_i} a|_{U_i} = (f^{n_i} a)|_{U_i} = 0$ in A_i by definition of localizing. Because there are finitely many, we can take a common n for all $\overline{f}_i$. As $\mathcal{O}_X$ is a sheaf, $f^n a$ being 0 on the restictions to an open cover implies that $f^n a$ is globally 0. □

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied for example, if $\operatorname{sp}(X)$ is Noetherian.) Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

Proof. Let $U_i = \operatorname{Spec} A_i$. Let $b|_{X_f \cap U_i} = \frac{b_i}{f^{n_i}}$ for $b_i \in A_i$. Because there are finitely many U_i, we can pick a sufficiently large common n for all of them. Then $f^n b|_{X_f \cap U_i} = b_i$.

From this is follows that restricting b_i, b_j to $U_{ij} := U_i \cap U_j$ equals $f^n b|_{X_f \cap U_{ij}}$, so $b_i - b_j = 0$ in $\Gamma(U_{ij} \cap X_f, \mathcal{O}_X)$. As U_{ij} is quasi-compact by hypothesis, from (b) we can conclude that there is a m_{ij} and s.t. $f^{m_{ij}}(b_i - b_j) = 0$. Because there are finitely many, we can pick a universal m that works for all of i, j. Then by $\mathcal{O}_X$ being a sheaf (and they agree on intersections), the $f^m b_i \in \Gamma(U_i, \mathcal{O}_X)$ glue together to get a global section $s \in A$. Finally, $s|_{X_f} = f^{n+m} b$ because $s|_{U_i \cap X_f} - f^{n+m} b|_{U_i \cap X_f} = f^m b_i - f^m b_i = 0$ on a cover of X_f, so by sheaf property (i) $s|_{X_f} - f^m b = 0$. □

(d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Proof. Technically, there should be some $\overline{f}$ to indicate the image in $\Gamma(X_f, \mathcal{O}_{X_f})$, but it doesn't really add much. We have a map $\Gamma(X_f, \mathcal{O}_{X_f}) \to A_f$ by sending $b \to \frac{a}{f^n}$ where a is the element of A and n is the associated n via (c). This is a homomorphism because obviously 0 and 1 are sent to the right places. Then with $f^n b_1 = a_1|_{X_f}, f^m b_2 = a_2|_{X_f}$, we have that $f^{n+m}(b_1 + b_2)$ is the restriction of $f^m a_1 + f^n a_2$. Hence $b_1 + b_2$ gets mapped to $\frac{a_1 f^m + a_2 f^n}{f^{n+m}} = \frac{a_1}{f^n} + \frac{a_2}{f^m}$, the sum of the images of b_1, b_2.

This is an isomorphism because it is surjective with $\frac{a|_{X_f}}{f^n}$ mapping to $\frac{a}{f^n} \in A_f \forall a \in A$. This is in $\Gamma(X_f, \mathcal{O}_{X_f})$ because we can find for any point $\mathfrak{p}$ an open neighborhood of X_f s.t. $f \notin \mathfrak{q}$ for all points $\mathfrak{q}$ in that neighborhood. Take any open affine neighborhood of $\mathfrak{p}$. By definition, $f \notin \mathfrak{m}_{\mathfrak{q}} = \mathfrak{q} \forall \mathfrak{q} \in X_f$ (note that $\mathfrak{m}_{\mathfrak{q}}$ is the maximal ideal of the stalk at $\mathfrak{q}$ of the open affine neighborhood, which equals $\mathcal{O}_{\mathfrak{q}}$). □

Exercise 1.2.4. (a) Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

Proof. TODO □

(b) A scheme is affine if and only if there is a finite set of elements $f_1, \ldots, f_r \in A = \Gamma(X, \mathcal{O}_X)$ such that the open subsets X_{f_i} are affine and $f_1, \ldots, f_r$ generate the unit ideal in A. [Hint: Use (Ex. 2.4) and (Ex. 2.16d) above.]

Proof. The if direction is easy.

For the only if direction, we will show that $X \cong \operatorname{Spec} A$. We can cover $\operatorname{Spec} A$ with a finite number of $U_i := \operatorname{Spec} A_{g_i}$. We have the map $f : X \to \operatorname{Spec} A$ via Exercise 2.4 and the map $\iota : A \to \Gamma(X, \mathcal{O}_X)$. So by (a), if we show that $f^{-1}(U_i) \to U_i$ is an isomorphism, we are done. TODO □

1.3. **First Properties of Schemes.**

Exercise 1.3.1. Show that a morphism $f : X \to Y$ is locally of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of $Y, f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

Proof. The if direction is easy.

We can use the Affine Communication Lemma found in Vakil. Let P be the property of $V = \operatorname{Spec} B$ that $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$ with A_j a finitely generated B-algebra. The first condition is met because $\operatorname{Spec} B_{f_i} \subseteq \operatorname{Spec} B$, so by covering $f^{-1}(\operatorname{Spec} B_{f_i})$ with the same U_j from above and using the same generators of A_j as a B-algebra, A_j is a finitely generated B_{f_i}-algebra.

Then for condition two, I claim that by embedding the open cover of $\operatorname{Spec} B_{f_i}$ into $\operatorname{Spec} B$, we have the conditions met for P on $\operatorname{Spec} B$. First, because $(f_1, \ldots, f_n) = (1)$, for any point in $\mathfrak{p} \in \operatorname{Spec} B$ there is some $f_i \notin \mathfrak{p}$ (otherwise $\mathfrak{p}$ would contain 1). So the open covers of $f^{-1}(\operatorname{Spec} B_{f_i})$ cover $f^{-1}(\operatorname{Spec} B)$. Then each A_{ij} whose spectra cover $f^{-1}(\operatorname{Spec} B_{f_i})$ are finitely generated B-algebras because they are finitely generated B_{f_i}-algebras, and B_{f_i} is a finitely generated B-algebra by using $1, \frac{1}{f_i}$. Thus this open cover of $f^{-1}(B)$ give us the condition for P. □

Exercise 1.3.2. A morphism $f : X \to Y$ of schemes is *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for *every* open affine subset $V \subseteq Y, f^{-1}(V)$ is quasi-compact.

Proof. The if direction is easy.

For the other direction, cover V by the V_i s.t. $f^{-1}(V_i)$ is quasi-compact. Because V is open affine, it is quasi-compact, so we can pick a finite number of these. Then any open cover of $f^{-1}(V)$ has a finite subcover formed by putting together the finite subcovers of $f^{-1}(V) \cap f^{-1}(V_i)$, of which there are a finite number of V_i. □

Exercise 1.3.3.

(a) Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

(b) Conclude from this that f is of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $Y_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

(c) Show also if f is of finite type, then for *every* open affine subset $V = \operatorname{Spec} B \subseteq Y$ and for *every* open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

Proof. a) The if direction is trivially. Finite type implies locally finite type by definition. Now for quasi-compact, take the cover of Y from locally finite type, let it be V_i. Because it is finite type, there is a finite open affine cover of $f^{-1}(V_i)$, say U_{ij}. Then any open cover of $f^{-1}(V_i)$ then covers U_{ij}, which is quasi-compact because it is affine, so we can take a finite subcover. We then have a finite subcover of $f^{-1}(V_i)$ by putting together all the finite subcovers from U_{ij}.

b) Easy by using 3.1 and 3.2.

c) WLOG, we can have $Y = \operatorname{Spec} B$ be affine. Let P be the property of $\operatorname{Spec} A \subseteq X$ that A is a finitely generated B-algebra. We want to use the affine communication lemma. Clearly $\operatorname{Spec} A_g$ satisfies P, giving us condition 1 (add $\frac{1}{g}$ as a generator). Then for condition 2, by the proof of the second condition in 3.1, having a cover of $\operatorname{Spec} A \subseteq \operatorname{Spec} B$ of $\operatorname{Spec} A_f$ with A_f f.g. B algebras gives us that A is a f.g. B algebra. □

Exercise 1.3.4. Show that a morphism $f : X \to Y$ is finite if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.

Proof. The if direction is trivial.

Let $\operatorname{Spec} B_{f_i}$ be a cover of Y s.t. $f^{-1}(\operatorname{Spec} B_{f_i}) = \operatorname{Spec} A_i$ with A_i finite B-modules. Because V is quasi-compact, we can pick a finite subcover of this. Then in V, $(f_{i_1}, \cdots, f_{i_n}) = (1)$. Let $X = f^{-1}(\operatorname{Spec} B)$ and $\overline{f_i}$ be the image of f_i in B. Because $X_{\overline{f_i}} = X \setminus V(\overline{f_i})$ and $f^{-1}(\operatorname{Spec} B_{f_i}) = f^{-1}(\operatorname{Spec} B) \setminus f^{-1}(V(\overline{f_i})) = f^{-1}(\operatorname{Spec} B) \setminus V(\overline{f_i}) = \operatorname{Spec} A_i$, we can apply the affine criterion to f_{i_n} from 2.17 to conclude that $X = f^{-1}(\operatorname{Spec} B)$ is affine.

As $f^{-1}(\operatorname{Spec} B_{f_i}) = \operatorname{Spec} A_{f^\#(\overline{f_i})}$, we have each A_i a localization of A. Because $(f_{i_1}, \ldots, f_{i_n}) = (1)$, the ideal generated by $f^\#(\overline{f_i})$ also generates (1) in A. Let $g_i = f^\#(\overline{f_i})$. Say A_i is generated by $\frac{a_{i_1}}{f_i^n}, \ldots, \frac{a_{i_j}}{f_j^n}, \ldots$ where we pick n sufficiently large s.t. it is the same for all i, j. Because $(f_{i_1}, \ldots, f_{i_n}) = (1)$, $\exists d_i$ s.t. $\sum d_i f_{i_1}^n = 1$. Then we can see that A is generated by $d_i, f_{i_j}^n, a_{i_j}$ because any element of B can be written using the generators of A_{f_1}, which upon multiplication by 1 and substituting kills the denominators and leaves these terms to generate it, which is a finite number. $\square$

Exercise 1.3.5. A morphism $f : X \to Y$ is *quasi-finite* if for every point $y \in Y, f^{-1}(y)$ is a finite set.

(a) Show that a finite morphism is quasi-finite.
(b) Show that a finite morphism is *closed*, i.e. the image of any closed subset is closed.
(c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

Proof. a) By 3.4 we can reduce to the case of affine X, Y, say $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$ with B a finite A-module. Because the topological space of the fibre of f over y is homeomorphic to $f^{-1}(y)$ (3.10), if the fibre is finite we are done. Then we have that $X_y = X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} B \times_Y \operatorname{Spec} k(y) = \operatorname{Spec}(B \otimes k(y))$. Because $k(y)$ is a field and B is a finite A-module, $B \otimes k(y)$ is a finite dimensional vector space.

This is then a finite field extension, and by commutative algebra thus an integral extension of a field. By the lying down lemma, every prime of $B \otimes k(y)$ lies over a prime of $k(y)$, and by commutative algebra there are finitely many. Because $k(y)$ has only one prime, $\operatorname{Spec}(B \otimes k(y))$ is finite.

b) Fix a closed set C and suppose FSTOC that $f(C)$ wasn't closed. Then there is a limit point p that isn't in $f(C)$. As this is a limit point, for all open sets U in Y containing p, $U \cap f(C) \neq \emptyset$.

Because f is continuous, $f^{-1}(U \cap f(C))$ is open, and, from the above, it is non-empty.

c) Take $\operatorname{Spec} \mathbb{F}_p[x]/(x^2+1) \to \operatorname{Spec} \mathbb{F}_p$. $\square$

Exercise 1.3.6. Let X be an integral scheme. Show that the local ring $\mathcal{O}_\zeta$ of the generic point ζ of X is a field. It is called the *function field* of X, and is denoted by $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

Proof. We can find an open affine neighborhood of ζ $U = \operatorname{Spec} A$ by definition of a scheme. Because X is integral, and the generic point is unique, the generic point is $(0) \in U$. Further, $\mathcal{O}_{(0),U}$ is a field because it is an integral domain. Because the local ring of a restricted scheme and the scheme are the same, $\mathcal{O}_\zeta$ is a field.

The quotient field of A is $A_{(0)} \cong \mathcal{O}(U)_\zeta$. Because the local ring of a restricted scheme and the scheme are the same, $K(X) = \mathcal{O}_\zeta \cong A_{(0)}$. $\square$

Exercise 1.3.7. A morphism $f : X \to Y$, with Y irreducible is *generically finite* if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f : X \to Y$ is *dominant* is $f(X)$ is dense in Y. Now let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $Y \subseteq Y$ scuh that the induced morphism $f^{-1}(U) \to U$ is finite. [Hint: First show that the function field of X is a finite field extension of the function field of Y.]

Proof. We show the hint first. Let η_X be the generic point of X. Then because $f(\overline{\eta_X}) \subseteq \overline{f(\eta_X)}$ and η_X is the generic point of X, we have that $f(X) \subseteq \overline{f(\eta_X)}$. Because f is dominant, $f(X)$ is dense, so by definition of closure we have $\overline{f(X)} = Y \subseteq \overline{f(\eta_X)} \implies f(\eta_X) = \eta$. So f maps the generic point of X to the generic point of Y.

Because f is generically finite and $f^{-1}(\eta)$ is homeomorphic to $X \times_Y \operatorname{Spec}(K(Y))$. $\square$

Exercise 1.3.8. *The Topological Space of a Product.* Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology (I, Ex. 1.4). Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.

(a) Let k be a field, and let $A_k^1 = \operatorname{Spec} k[x]$ be the affine line over k. Show that $A_k^1 \times A_k^1 \cong A_k^2$, and show that the udnerlying point set of the product is the not product of the underlying point sets of the factors (even if k is algebraically closed).