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Splitting Families and Complete Separability

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Abstract. We answer a question from Raghavan and Steprāns by showing that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. Then we use this to construct a completely separable maximal almost disjoint family under $\mathfrak{s} \leq \mathfrak{a}$, partially answering a question of Shelah.

1 Introduction

The purpose of this short note is to answer a question posed by the second and third authors in [5] and to use this to solve a problem of Shelah [6]. We say that two infinite subsets a and b of ω are *almost disjoint* or *a.d.* if $a \cap b$ is finite. We say that a family $\mathscr{A}$ of infinite subsets of ω is *almost disjoint* or *a.d.* if its members are pairwise almost disjoint. A *Maximal Almost Disjoint family* or *MAD family* is an infinite a.d. family that is not properly contained in a larger a.d. family.

For an a.d. family $\mathscr{A}$, let $\mathcal{I}(\mathscr{A})$ denote *the ideal on ω generated by $\mathscr{A}$*—that is, $a \in \mathcal{I}(\mathscr{A})$ if and only if $\exists a_0, \dots, a_k \in \mathscr{A} [\, a \subset^* a_0 \cup \cdots \cup a_k \,]$. For any ideal $\mathcal{I}$ on ω, $\mathcal{I}^+$ denotes $\mathcal{P}(\omega) \setminus \mathcal{I}$. An a.d. family $\mathscr{A} \subset [\omega]^\omega$ is said to be *completely separable* if for any $b \in \mathcal{I}^+(\mathscr{A})$, there is an $a \in \mathscr{A}$ with $a \subset b$. Notice that an infinite completely separable a.d. $\mathscr{A}$ must be MAD. Though the following is one of the most well-studied problems in set theory, it continues to remains open.

Question 1 (Erdős and Shelah [3]) *Does there exist a completely separable MAD family $\mathscr{A} \subset [\omega]^\omega$?*

Progress on Question 1 was made by Balcar, Dočkálková, and Simon who showed in a series of papers that completely separable MAD families can be constructed from any of the assumptions $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{s} = \omega_1$, or $\mathfrak{d} \leq \mathfrak{a}$. See [1], [2], and [7] for this work. Then Shelah [6] recently showed that the existence of completely separable MAD families is *almost* a theorem of ZFC. His construction is divided into three cases. The first case is when $\mathfrak{s} < \mathfrak{a}$, and he shows on the basis of ZFC alone that a completely separable MAD family can be constructed in this case. The second and third cases are when $\mathfrak{s} = \mathfrak{a}$ and $\mathfrak{a} < \mathfrak{s}$ respectively, and Shelah shows that a completely separable MAD family can be constructed in these cases *provided* that certain PCF-type hypotheses are satisfied. More precisely, he shows that there is a completely separable MAD family when $\mathfrak{s} = \mathfrak{a}$ and $U(\mathfrak{s})$ holds, or when $\mathfrak{a} < \mathfrak{s}$ and $P(\mathfrak{s}, \mathfrak{a})$ holds.

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Definition 2 For a cardinal $\kappa > \omega$, $U(\kappa)$ is the following principle. There is a sequence $\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ such that

(1) $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$,
(2) $\forall X \in [\kappa]^\kappa \exists \omega \leq \alpha < \kappa [\, |u_\alpha \cap X| = \omega\,]$.

For cardinals $\kappa > \lambda > \omega$, $P(\kappa, \lambda)$ says that there is a sequence $\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ such that

(3) $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$,
(4) for each $X \subset \kappa$, if X is bounded in κ and $\mathrm{otp}\,(X) = \lambda$, then $\exists \omega \leq \alpha < \sup\,(X) [\, |u_\alpha \cap X| = \omega\,]$.

It is easy to see that both $U(\mathfrak{s})$ and $P(\mathfrak{s}, \mathfrak{a})$ are satisfied when $\mathfrak{s} < \aleph_\omega$, so in particular, the existence of a completely separable MAD family is a theorem of ZFC when $\mathfrak{c} < \aleph_\omega$. Shelah [6] asked whether all uses of PCF-type hypotheses can be eliminated from the second and third cases.

The second and third authors modified the techniques of Shelah [6] in order to treat MAD families with few partitioners in [5] (see the introduction there). In that paper they introduced a cardinal invariant $\mathfrak{s}_{\omega,\omega}$, which is a variation of the splitting number $\mathfrak{s}$. They showed that if $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{b}$, then there is a weakly tight family. Recall that an a.d. family $\mathscr{A} \subset [\omega]^\omega$ is called *weakly tight* if for every countable collection $\{b_n : n \in \omega\} \subset \mathcal{I}^+(\mathscr{A})$, there is $a \in \mathscr{A}$ such that $\exists^\infty n \in \omega [\, |b_n \cap a| = \omega\,]$. The question of whether $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$ was raised in [5], and the authors pointed out that an affirmative answer to this question could help eliminate the use of PCF-type hypotheses from the second case of Shelah's construction.

In this paper we answer this question from [5] by proving that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. We then use this information to partially answer the question from Shelah [6]. We show that the second case can be done without any additional hypothesis. So it is a theorem of ZFC alone that a completely separable MAD family exists when $\mathfrak{s} \leq \mathfrak{a}$. We give a single construction from this assumption, so Shelah's first and second cases are unified into a single case.

The question of whether the hypothesis $P(\mathfrak{s}, \mathfrak{a})$ can be eliminated from the case when $\mathfrak{a} < \mathfrak{s}$ remains open.

2 $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$

In this section we answer Question 21 from [5] by showing that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. For a set $x \subset \omega$, x^0 is used to denote x and x^1 is used to denote $\omega \setminus x$. This notation will be used in the next section also. Recall the following definitions.

Definition 3 For $x, a \in \mathcal{P}(\omega)$, x *splits* a if $|x^0 \cap a| = |x^1 \cap a| = \omega$. $\mathcal{F} \subset \mathcal{P}(\omega)$ is called a *splitting family* if $\forall a \in [\omega]^\omega \exists x \in \mathcal{F} [\, x \text{ splits } a\,]$. $\mathcal{F} \subset \mathcal{P}(\omega)$ is said to be (ω,ω)-*splitting* if for each countable collection $\{a_n : n \in \omega\} \subset [\omega]^\omega$, there exists $x \in \mathcal{F}$ such that $\exists^\infty n \in \omega [\, |x^0 \cap a_n| = \omega\,]$ and $\exists^\infty n \in \omega [\, |x^1 \cap a_n| = \omega\,]$. Define

$$\mathfrak{s} = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{P}(\omega) \wedge \mathcal{F} \text{ is a splitting family}\}$$
$$\mathfrak{s}_{\omega,\omega} = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{P}(\omega) \wedge \mathcal{F} \text{ is } (\omega,\omega)\text{-splitting}\}.$$

Obviously every (ω, ω)-splitting family is a splitting family. So $\mathfrak{s} \leq \mathfrak{s}_{\omega,\omega}$. It was shown in Theorem 13 of [5] that if $\mathfrak{s} < \mathfrak{b}$, then $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. We reproduce that result here for the reader's convenience.

Lemma 4 (Theorem 13 of [5]) *If* $\mathfrak{s} < \mathfrak{b}$, *then* $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.

Proof Let $\langle e_\alpha : \alpha < \kappa \rangle$ witness that $\kappa = \mathfrak{s}$. Suppose $\{b_n : n \in \omega\} \subset [\omega]^\omega$ is a countable collection such that $\forall \alpha < \kappa \exists i \in 2 \forall^\infty n \in \omega [\, b_n \subset^* e^i_\alpha \,]$. By shrinking them if necessary we may assume that $b_n \cap b_m = 0$ whenever $n \neq m$. Now, for each $\alpha < \kappa$ define $f_\alpha \in \omega^\omega$ as follows. We know that there is a unique $i_\alpha \in 2$ such that there is a $k_\alpha \in \omega$ such that $\forall n \geq k_\alpha [\, |b_n \cap e^{i_\alpha}_\alpha| < \omega \,]$. We define $f_\alpha(n) = \max(b_n \cap e^{i_\alpha}_\alpha)$ if $n \geq k_\alpha$, and $f_\alpha(n) = 0$ if $n < k_\alpha$. As $\kappa < \mathfrak{b}$, there is an $f \in \omega^\omega$ with $f >^* f_\alpha$ for each $\alpha < \kappa$. Now, for each $n \in \omega$, choose $l_n \in b_n$ with $l_n \geq f(n)$. Since the b_n are pairwise disjoint, $c = \{l_n : n \in \omega\} \in [\omega]^\omega$. So by definition of $\mathfrak{s}$, there is $\alpha < \kappa$ such that $|c \cap e^0_\alpha| = |c \cap e^1_\alpha| = \omega$. In particular, $c \cap e^{i_\alpha}_\alpha$ is infinite. However we know that there is an $m_\alpha \in \omega$ such that $\forall n \geq m_\alpha [\, f_\alpha(n) < f(n) \,]$. So there exists $n \geq \max\{m_\alpha, k_\alpha\}$ with $l_n \in b_n \cap e^{i_\alpha}_\alpha$. But this is a contradiction because $l_n \leq f_\alpha(n) < f(n)$. ■

In the case when $\mathfrak{b} \leq \mathfrak{s}$ it turns out that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$ can still be proved by considering the following notion appearing in [4].

Definition 5 $\mathcal{F}$ is called *block-splitting* if given any partition $\langle a_n : n \in \omega \rangle$ of ω into finite sets there is a set $x \in \mathcal{F}$ such that there are infinitely many n with $a_n \subset x$ and there are infinitely many n with $a_n \cap x = 0$.

It was proved by Kamburelis and Węglorz [4] that the least size of a block-splitting family is $\max\{\mathfrak{b}, \mathfrak{s}\}$. Therefore, when $\mathfrak{b} \leq \mathfrak{s}$, there is a block-splitting family of size $\mathfrak{s}$.

Theorem 6 $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.

Proof In view of Lemma 4, we may assume that $\mathfrak{b} \leq \mathfrak{s}$. By results of Kamburelis and Węglorz [4] fix $\langle x_\alpha : \alpha < \mathfrak{s} \rangle \subset \mathcal{P}(\omega)$, a block-splitting family. We show that $\langle x_\alpha : \alpha < \mathfrak{s} \rangle$ is an (ω, ω)-splitting family. Let $\{a_n : n \in \omega\} \subset [\omega]^\omega$ be given. For $n \in \omega$, define $s_n \in [\omega]^{<\omega}$ as follows. Suppose $\langle s_i : i < n \rangle$ have been defined. Put $s = \bigcup_{i<n} s_i$. Put $s_n = \{\min(\omega \setminus s)\} \cup \{\min(a_i \setminus s) : i \leq n\}$. Note that $\langle s_n : n \in \omega \rangle$ is a partition of ω into finite sets and that $\forall i \in \omega \forall^\infty n \in \omega [\, s_n \cap a_i \neq 0 \,]$. Now choose $\alpha < \mathfrak{s}$ such that $\exists^\infty n \in \omega [\, s_n \subset x^0_\alpha \,]$ and $\exists^\infty n \in \omega [\, s_n \subset x^1_\alpha \,]$. So for each $i \in \omega$, $\exists^\infty n \in \omega [\, s_n \cap a_i \cap x^0_\alpha \neq 0 \,]$ and $\exists^\infty n \in \omega [\, s_n \cap a_i \cap x^1_\alpha \neq 0 \,]$. Since the s_n are pairwise disjoint, it follows that $|a_i \cap x^0_\alpha| = |a_i \cap x^1_\alpha| = \omega$, for each $i \in \omega$. ■

3 Constructing a Completely Separable MAD Family from $\mathfrak{s} \leq \mathfrak{a}$

As $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$ and as every (ω, ω)-splitting family is also a splitting family, fix once and for all a sequence $\langle x_\alpha : \alpha < \kappa \rangle$ witnessing that $\kappa = \mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. We will construct a completely separable MAD family assuming that $\kappa \leq \mathfrak{a}$. The construction closely follows the proof of Lemma 8 in [5], which in turn is based on Shelah [6]. An important point of the construction is that if $\mathscr{A}$ is an arbitrary a.d. family and $b \in \mathcal{I}^+(\mathscr{A})$, then every (ω, ω)-splitting family contains an element which splits b into two *positive* pieces.

Lemma 7 *Let $\mathscr{A} \subset [\omega]^\omega$ be any a.d. family. Suppose $b \in \mathcal{I}^+(\mathscr{A})$. Then there is $\alpha < \kappa$ such that $b \cap x^0_\alpha \in \mathcal{I}^+(\mathscr{A})$ and $b \cap x^1_\alpha \in \mathcal{I}^+(\mathscr{A})$.*

Proof See proof of Lemma 7 of [5]. ■

At a stage $\delta < \mathfrak{c}$, an a.d. family $\mathscr{A}_\delta = \langle a_\alpha : \alpha < \delta \rangle \subset [\omega]^\omega$ is given. Moreover we assume that there is also a family $\langle \sigma_\alpha : \alpha < \delta \rangle \subset 2^{<\kappa}$ such that for each $\alpha < \delta$, $\forall \xi < \mathrm{dom}(\sigma_\alpha)[\, a_\alpha \subset^* x^{\sigma_\alpha(\xi)}_\xi \,]$. We say that σ_α is *the node associated with a_α*. The next lemma says that under the assumption $\kappa \leq \mathfrak{a}$, such an a.d. family must be "nowhere maximal", which is of course a property that we need to maintain in order to end up with a completely separable MAD family.

Definition 8 Let $\eta \in 2^{<\kappa}$. Define $\mathcal{I}_\eta = \{a \in \mathcal{P}(\omega) : \forall \xi < \mathrm{dom}(\eta)[\, a \subset^* x^{\eta(\xi)}_\xi \,]\}$.

Lemma 9 (Main Lemma) *Let $\kappa \leq \mathfrak{a}$ and $\delta < \mathfrak{c}$. Suppose that $\mathscr{A}_\delta = \langle a_\alpha : \alpha < \delta \rangle$ and $\langle \sigma_\alpha : \alpha < \delta \rangle$ are as above. Assume also that $\forall \alpha, \beta < \delta[\, \alpha \neq \beta \implies \sigma_\alpha \neq \sigma_\beta \,]$. Let $b \in \mathcal{I}^+(\mathscr{A}_\delta)$. Then there exist $a \in [b]^\omega$ and $\sigma \in 2^{<\kappa}$ such that*

(1) $\forall \alpha < \delta[\, |a \cap a_\alpha| < \omega \,]$,
(2) *for each $\alpha < \delta$, $\sigma \not\subset \sigma_\alpha$ and $a \in I_\sigma$.*

Proof Applying Lemma 7, let $\alpha_0 < \kappa$ be least such that $b \cap x^0_{\alpha_0} \in \mathcal{I}^+(\mathscr{A}_\delta)$ and $b \cap x^1_{\alpha_0} \in \mathcal{I}^+(\mathscr{A}_\delta)$. Define $\tau_0 \in 2^{\alpha_0}$ by stipulating that

$$\forall \xi < \alpha_0 \forall i \in 2[\, \tau_0(\xi) = i \leftrightarrow b \cap x^i_\xi \in \mathcal{I}^+(\mathscr{A}_\delta) \,].$$

By choice of α_0 and by the hypothesis that $b \in \mathcal{I}^+(\mathscr{A}_\delta)$, τ_0 is well defined. Now construct two sequences $\langle \alpha_s : s \in 2^{<\omega} \rangle \subset \kappa$ and $\langle \tau_s : s \in 2^{<\omega} \rangle \subset 2^{<\kappa}$ such that the following hold:

(3) $\forall s \in 2^{<\omega} \forall i \in 2[\, \alpha_s = \mathrm{dom}(\tau_s) \wedge \alpha_{s^\frown\langle i \rangle} > \alpha_s \wedge \tau_{s^\frown\langle i \rangle} \supset \tau_s{}^\frown\langle i \rangle \,]$.
(4) For each $s \in 2^{<\omega}$ and for each $\xi < \alpha_s$, $x^{1-\tau_s(\xi)}_\xi \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \in \mathcal{I}(\mathscr{A}_\delta)$. Here, $\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}$ is taken to be ω when $s = 0$.
(5) For each $s \in 2^{<\omega}$, both

$$x^0_{\alpha_s} \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \in \mathcal{I}^+(\mathscr{A}_\delta) \quad \text{and} \quad x^1_{\alpha_s} \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \in \mathcal{I}^+(\mathscr{A}_\delta).$$

α_0 and τ_0 are already defined. Suppose that α_s and τ_s are given. By (5), for each $i \in 2$, $x^i_{\alpha_s} \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \in \mathcal{I}^+(\mathscr{A}_\delta)$. Apply Lemma 7 to let $\alpha_{s^\frown\langle i \rangle}$ be the least $\alpha < \kappa$ such that both $x^i_{\alpha_s} \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \cap x^0_\alpha$ and $x^i_{\alpha_s} \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \cap x^1_\alpha$ are in $\mathcal{I}^+(\mathscr{A}_\delta)$. Again define $\tau_{s^\frown\langle i \rangle} \in 2^{\alpha_{s^\frown\langle i \rangle}}$ by stipulating that

$$\forall \xi < \alpha_{s^\frown\langle i \rangle} \forall j \in 2[\, \tau_{s^\frown\langle i \rangle}(\xi) = j \leftrightarrow x^i_{\alpha_s} \cap b \cap \Big(\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}\Big) \cap x^j_\xi \in \mathcal{I}^+(\mathscr{A}_\delta) \,]$$

$\tau_{s^\frown\langle i \rangle}$ is well defined because $x^i_{\alpha_s} \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \in \mathcal{I}^+(\mathscr{A}_\delta)$ and because of the choice of $\alpha_{s^\frown\langle i \rangle}$. Now, for each $\xi < \alpha_s$, $x^i_{\alpha_s} \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \subset b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t})$

and, by (4), $b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \cap x^{1-\tau_s(\xi)}_{\xi} \in \mathcal{I}(\mathscr{A}_\delta)$. It follows that $\alpha_{s^\frown\langle i\rangle} \geq \alpha_s$ and that for each $\xi < \alpha_s$, $\tau_s(\xi) = \tau_{s^\frown\langle i\rangle}(\xi)$. Next, since $x^i_{\alpha_s} \cap b \cap (\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}) \cap x^{1-i}_{\alpha_s} = 0$, $\alpha_{s^\frown\langle i\rangle} > \alpha_s$, and $\tau_{s^\frown\langle i\rangle} \supset \tau_s{}^\frown\langle i\rangle$. Now, it is clear that (4) and (5) hold for $s^\frown\langle i\rangle$. This completes the construction of $\langle \alpha_s : s \in 2^{<\omega}\rangle$ and $\langle \tau_s : s \in 2^{<\omega}\rangle$.

For each $f \in 2^\omega$, put $\alpha_f = \sup\{\alpha_{f\restriction n} : n \in \omega\}$ and $\tau_f = \bigcup_{n\in\omega} \tau_{f\restriction n}$. As $\kappa = \mathfrak{s}$, $\mathrm{cf}(\kappa) > \omega$. Therefore, $\alpha_f < \kappa$. Note that $\tau_f \in 2^{\alpha_f}$. Also, if $f, g \in 2^\omega$, $f \neq g$, and $n \in \omega$ is least such that $f(n) \neq g(n)$, then $\tau_f \supset \tau_s{}^\frown\langle i\rangle$ and $\tau_g \supset \tau_s{}^\frown\langle 1-i\rangle$, where $s = f\restriction n = g\restriction n$ and $i \in 2$. So there cannot be $\alpha < \delta$ such that both $\tau_f \subset \sigma_\alpha$ and $\tau_g \subset \sigma_\alpha$ hold. Therefore, it is possible to find $f \in 2^\omega$ such that $\tau_f \notin \{\sigma \in 2^{<\kappa} : \exists \alpha < \delta[\, \sigma \subset \sigma_\alpha\,]\}$. Fix such f and for each $n \in \omega$, define e_n to be $b \cap (\bigcap_{m<n} x^{\tau_f(\alpha_{f\restriction m})}_{\alpha_{f\restriction m}})$. By (5) each $e_n \in \mathcal{I}^+(\mathscr{A}_\delta)$. Moreover, $e_{n+1} \subset e_n \subset b$. Therefore, by a standard argument, there is $e \in [b]^\omega \cap \mathcal{I}^+(\mathscr{A}_\delta)$ such that $\forall n \in \omega[\, e \subset^* e_n\,]$.

Now suppose $\xi < \alpha_f$. Since $\alpha_{f\restriction n+1} > \alpha_{f\restriction n}$ for all $n \in \omega$, it follows that $\xi < \alpha_{f\restriction n}$ for some n. By (4) applied to $s = f\restriction n$, we have $x^{1-\tau_f(\xi)}_\xi \cap e_n \in \mathcal{I}(\mathscr{A}_\delta)$. Since $e \subset^* e_n$, $x^{1-\tau_f(\xi)}_\xi \cap e \in \mathcal{I}(\mathscr{A}_\delta)$. Thus we conclude that $\forall \xi < \alpha_f[\, x^{1-\tau_f(\xi)}_\xi \cap e \in \mathcal{I}(\mathscr{A}_\delta)\,]$. So for each $\xi < \alpha_f$, fix $F_\xi \in [\delta]^{<\omega}$ such that

$$(x^{1-\tau_f(\xi)}_\xi \cap e) \subset^* \Big(\bigcup_{\alpha\in F_\xi} a_\alpha\Big).$$

Now put $\mathcal{F} = \bigcup_{\xi<\alpha_f} F_\xi$ and $\mathcal{G} = \{\alpha < \delta : \sigma_\alpha \subset \tau_f\}$. Note that $|\mathcal{F} \cup \mathcal{G}| < \kappa \leq \mathfrak{a}$ because of the assumption that $\forall \alpha, \beta < \delta[\, \alpha \neq \beta \implies \sigma_\alpha \neq \sigma_\beta\,]$. Since $e \in \mathcal{I}^+(\mathscr{A}_\delta)$, there is $a \in [e]^\omega$ such that $\forall \alpha \in \mathcal{F} \cup \mathcal{G}[\, |a \cap a_\alpha| < \omega\,]$. Note that for each $\xi < \alpha_f$, $x^{1-\tau_f(\xi)}_\xi \cap a$ is finite. Thus, putting $\sigma = \tau_f$, we have that $\forall \alpha < \delta[\, \sigma \not\subset \sigma_\alpha\,]$ and $a \in I_\sigma$. In order to finish the proof, it is enough to check that $\forall \alpha < \delta[\, |a_\alpha \cap a| < \omega\,]$.

Fix $\alpha < \delta$. If $\alpha \in \mathcal{G}$, then $|a \cap a_\alpha| < \omega$ simply by choice of a. Suppose $\alpha \notin \mathcal{G}$. Then there must be $\xi \in \mathrm{dom}(\sigma_\alpha) \cap \alpha_f$ such that $\sigma_\alpha(\xi) = 1 - \tau_f(\xi)$. However, since $a_\alpha \subset^* x^{\sigma_\alpha(\xi)}_\xi$ and $a \cap x^{1-\tau_f(\xi)}_\xi$ is finite, it follows that $|a \cap a_\alpha| < \omega$. ∎

Theorem 10 *If $\mathfrak{s} \leq \mathfrak{a}$, then there is a completely separable MAD family.*

Proof Fix an enumeration $\langle b_\alpha : \alpha < \mathfrak{c}\rangle$ of $[\omega]^\omega$. Let $\langle x_\alpha : \alpha < \kappa\rangle$ witness $\kappa = \mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. Build two sequences $\langle a_\delta : \delta < \mathfrak{c}\rangle$ and $\langle \sigma_\delta : \delta < \mathfrak{c}\rangle$ such that the following hold:

(1) For each $\delta < \mathfrak{c}$, $a_\delta \in [\omega]^\omega$, $\sigma_\delta \in 2^{<\kappa}$, and $a_\delta \in I_{\sigma_\delta}$.
(2) $\forall \gamma, \delta < \mathfrak{c}[\, \gamma \neq \delta \implies (|a_\gamma \cap a_\delta| < \omega \wedge \sigma_\gamma \neq \sigma_\delta)\,]$.
(3) For each $\delta < \mathfrak{c}$, if $b_\delta \in \mathcal{I}^+(\mathscr{A}_\delta)$, then $a_\delta \subset b_\delta$, where $\mathscr{A}_\delta = \{a_\alpha : \alpha < \delta\}$.

Note that if we succeed in this, then $\mathscr{A}_\mathfrak{c} = \{a_\delta : \delta < \mathfrak{c}\}$ will be completely separable. For given any $b \in \mathcal{I}^+(\mathscr{A}_\mathfrak{c})$, b is in $\mathcal{I}^+(\mathscr{A}_\delta)$ for every $\delta < \mathfrak{c}$ and so there is a $\delta < \mathfrak{c}$, where $b_\delta = b$ and $b_\delta \in \mathcal{I}^+(\mathscr{A}_\delta)$, whence by (3), $a_\delta \subset b$.

At a stage $\delta < \mathfrak{c}$ suppose $\langle a_\alpha : \alpha < \delta\rangle$ and $\langle \sigma_\alpha : \alpha < \delta\rangle$ are given. If $b_\delta \in \mathcal{I}^+(\mathscr{A}_\delta)$, then let $b = b_\delta$, else let $b = \omega$. In either case, the hypotheses of Lemma 9 are satisfied. So find $a_\delta \in [b]^\omega$ and $\sigma_\delta \in 2^{<\kappa}$ such that

(4) $\forall \alpha < \delta[\, |a_\delta \cap a_\alpha| < \omega\,]$,

(5) for each $\alpha < \delta$, $\sigma_\delta \not\subset \sigma_\alpha$ and $a_\delta \in I_{\sigma_\delta}$.

It is clear that a_δ and σ_δ are as needed. ∎

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