# TWO RESULTS ON CARDINAL INVARIANTS AT UNCOUNTABLE CARDINALS

DILIP RAGHAVAN AND SAHARON SHELAH

Abstract. We prove two ZFC theorems about cardinal invariants above the continuum which are in sharp contrast to well-known facts about these same invariants at the continuum. It is shown that for an uncountable regular cardinal $\kappa$, $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$. This improves an earlier result of Blass, Hyttinen, and Zhang [3]. It is also shown that if $\kappa \geq \beth_\omega$ is an uncountable regular cardinal, then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. This result partially dualizes an earlier theorem of the authors [7].

## 1. Introduction

The theory of cardinal invariants at uncountable regular cardinals remains less developed than the theory at $\omega$. One of the first papers to explore the situation above $\omega$ was by Cummings and Shelah [4]. In that paper, they considered the direct analogues of the bounding and dominating numbers. They also considered bounding and domination modulo the club filter, a notion which has no counterpart at $\omega$ but which becomes very natural at uncountable regular cardinals. Recall the following definitions.

**Definition 1.** Let $\kappa > \omega$ be a regular cardinal. Let $f, g \in \kappa^\kappa$. $f \leq^* g$ means that $|\{\alpha < \kappa : g(\alpha) < f(\alpha)\}| < \kappa$ and $f \leq_{\mathrm{cl}} g$ means that $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ is non-stationary. We say that $F \subset \kappa^\kappa$ is $*$-*unbounded* if $\neg\exists g \in \kappa^\kappa \forall f \in F\, [f \leq^* g]$ and we say that $F$ is cl-*unbounded* if $\neg\exists g \in \kappa^\kappa \forall f \in F\, [f \leq_{\mathrm{cl}} g]$. Define

$$\mathfrak{b}(\kappa) = \min\{|F| : F \subset \kappa^\kappa \wedge F \text{ is } *\text{-unbounded}\},$$
$$\mathfrak{b}_{\mathrm{cl}}(\kappa) = \min\{|F| : F \subset \kappa^\kappa \wedge F \text{ is cl-unbounded}\}.$$

We say that $F \subset \kappa^\kappa$ is $*$-*dominating* if $\forall g \in \kappa^\kappa \exists f \in F\, [g \leq^* f]$ and we say that $F$ is cl-*dominating* if $\forall g \in \kappa^\kappa \exists f \in F\, [g \leq_{\mathrm{cl}} f]$. Define

$$\mathfrak{d}(\kappa) = \min\{|F| : F \subset \kappa^\kappa \text{ and } F \text{ is } *\text{-dominating}\}.$$
$$\mathfrak{d}_{\mathrm{cl}}(\kappa) = \min\{|F| : F \subset \kappa^\kappa \text{ and } F \text{ is cl-dominating}\}.$$

Cummings and Shelah [4] proved that for any regular $\kappa$, $\kappa^+ \leq \mathrm{cf}(\mathfrak{b}(\kappa)) = \mathfrak{b}(\kappa) \leq \mathrm{cf}(\mathfrak{d}(\kappa)) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$, and that these are the only relations between $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ that are provable in ZFC, thereby generalizing a classical result of Hechler from the case $\kappa = \omega$. Quite remarkably, they also showed that for every regular $\kappa > \omega$, $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$, and that if $\kappa \geq \beth_\omega$ is regular, then $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$. The question of whether $\mathfrak{d}_{\mathrm{cl}}(\kappa) < \mathfrak{d}(\kappa)$ is consistent for any $\kappa$ was left open; as far as we are aware, it remains open.

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Other early papers which studied the splitting number at uncountable cardinals revealed interesting differences with the situation at $\omega$. Recall the following definitions.

**Definition 2.** Let $\kappa > \omega$ be a regular cardinal. For $A, B \in \mathcal{P}(\kappa)$, $A \subset^* B$ means $|A \setminus B| < \kappa$. For a family $F \subset [\kappa]^\kappa$ and a set $B \in \mathcal{P}(\kappa)$, $B$ is said to *reap* $F$ if for every $A \in F$, $|A \cap B| = |A \cap (\kappa \setminus B)| = \kappa$. We say that $F \subset [\kappa]^\kappa$ is *unreaped* if there is no $B \in \mathcal{P}(\kappa)$ that reaps $F$.

$$\mathfrak{r}(\kappa) = \min\left\{|F| : F \subset [\kappa]^\kappa \text{ and } F \text{ is unreaped}\right\}.$$

A family $F \subset \mathcal{P}(\kappa)$ is called a *splitting family* if

$$\forall B \in [\kappa]^\kappa \exists A \in F\left[|B \cap A| = |B \cap (\kappa \setminus A)| = \kappa\right].$$

$$\mathfrak{s}(\kappa) = \min\left\{|F| : F \subset \mathcal{P}(\kappa) \text{ and } F \text{ is a splitting family}\right\}.$$

For instance, Suzuki [11] showed that for a regular cardinal $\kappa > \omega$, $\mathfrak{s}(\kappa) \geq \kappa$ iff $\kappa$ is strongly inaccessible and $\mathfrak{s}(\kappa) \geq \kappa^+$ iff $\kappa$ is weakly compact. Zapletal [12] additionally showed that the statement that there exists some regular uncountable cardinal $\kappa$ for which $\mathfrak{s}(\kappa) \geq \kappa^{++}$ has large consistency strength, significantly more than a measurable cardinal. More recently, the authors proved in [7] that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ for all regular $\kappa > \omega$. This is in marked contrast to the situation at $\omega$, where it is known that $\mathfrak{s}(\omega)$ and $\mathfrak{b}(\omega)$ are independent. More information about cardinal invariants at $\omega$ can be found in [2].

Blass, Hyttinen, and Zhang [3] is a work about the almost disjointness number at regular uncountable cardinals. Let us recall the definition of maximal almost disjoint families.

**Definition 3.** Let $\kappa > \omega$ be a regular cardinal. $A, B \in [\kappa]^\kappa$ are said to be *almost disjoint* or *a.d.* if $|A \cap B| < \kappa$. A family $\mathscr{A} \subset [\kappa]^\kappa$ is said to be *almost disjoint* or *a.d.* if the members of $\mathscr{A}$ are pairwise a.d. Finally $\mathscr{A} \subset [\kappa]^\kappa$ is called *maximal almost disjoint* or *m.a.d.* if $\mathscr{A}$ is an a.d. family, $|\mathscr{A}| \geq \kappa$, and $\mathscr{A}$ cannot be extended to a larger a.d. family in $[\kappa]^\kappa$.

$$\mathfrak{a}(\kappa) = \min\left\{|\mathscr{A}| : \mathscr{A} \subset [\kappa]^\kappa \text{ and } \mathscr{A} \text{ is m.a.d.}\right\}.$$

Blass, Hyttinen, and Zhang [3] proved that if $\kappa > \omega$ is regular, then $\mathfrak{d}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$. This is potentially different from the situation at $\omega$: it remains an open problem whether $\mathfrak{d}(\omega) = \aleph_1$ implies $\mathfrak{a}(\omega) = \aleph_1$, while Shelah [9] showed the consistency of $\mathfrak{d}(\omega) = \aleph_2 < \aleph_3 = \mathfrak{a}(\omega)$ (see also Question 15).

There is also a well-developed theory of duality for cardinal invariants at $\omega$. Thus, for example, $\mathfrak{b}(\omega)$ and $\mathfrak{d}(\omega)$ are dual to each other, while $\mathfrak{s}(\omega)$ and $\mathfrak{r}(\omega)$ are duals. The ZFC inequality $\mathfrak{s}(\omega) \leq \mathfrak{d}(\omega)$ dualizes to the inequality $\mathfrak{b}(\omega) \leq \mathfrak{r}(\omega)$, and indeed even the proof of $\mathfrak{s}(\omega) \leq \mathfrak{d}(\omega)$ dualizes to the proof of $\mathfrak{b}(\omega) \leq \mathfrak{r}(\omega)$. It is possible to make this notion of duality precise using Galois-Tukey connections. We refer the reader to [2] for further details about duality of cardinal invariants at $\omega$. It is unclear at present if there can be a smooth theory of duality for cardinal invariants at uncountable cardinals too. For example, if we try to naïvely dualize Suzuki's result mentioned above that $\mathfrak{s}(\kappa)$ is small for most $\kappa$, then we would be trying to show that $\mathfrak{r}(\kappa)$ is large for most $\kappa$. In other words, we might expect to show that if $\kappa$ is not weakly compact, then $\mathfrak{r}(\kappa) = 2^\kappa$. However it is still an open problem whether the inequality $\mathfrak{r}(\aleph_1) < 2^{\aleph_1}$ is consistent (see Question 17). Nevertheless, it is of interest to ask whether for all regular $\kappa > \omega$ the result from [7] that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ can be dualized to the result that $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

We present two further ZFC theorems on cardinal invariants at uncountable regular cardinals in the paper. Our first result, Theorem 5, says that if $\kappa > \omega$ is regular, then $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$. This improves the above mentioned result of Blass, Hyttinen, and Zhang [3]. It also shows that $\omega$ is unique among regular cardinals in that it is the only such $\kappa$ where $\mathfrak{b}(\kappa) = \kappa^+ < \kappa^{++} = \mathfrak{a}(\kappa)$ is consistent. Our next result, Theorem 13, is a partial dual to our earlier result from [7]. It says that for all regular cardinals $\kappa \geq \beth_\omega$, $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. Thus for sufficiently large $\kappa$, the invariants $\mathfrak{s}(\kappa), \mathfrak{b}(\kappa), \mathfrak{d}(\kappa)$, and $\mathfrak{r}(\kappa)$ are provably comparable and ordered as $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. The proof of our first theorem makes use of the equality $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$ of Cummings and Shelah [4] discussed before. Their theorem that $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ for all regular $\kappa \geq \beth_\omega$ is not directly used. However the main idea of the proof of our Theorem 13 is similar to the main idea in the proof of $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ – both results use the revised GCH of Shelah, which is a striking application of PCF theory exposed in [8].

Finally one word about our notation, which is standard. $X \subset Y$ means that $\forall x \left[x \in X \implies x \in Y\right]$. So the symbol "$\subset$" does not mean "proper subset". If $f$ is a function and $X \subset \mathrm{dom}(f)$, then $f''X$ is the image of $X$ under $f$, that is $f''X = \{f(x) : x \in X\}$.

## 2. The bounding and almost disjointness numbers: A ZFC result

We will quote the following well-known result of Cummings and Shelah [4].

**Theorem 4** (see Theorem 6 of [4])**.** *For every regular cardinal $\kappa > \omega$, $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$.*

**Theorem 5.** *Let $\kappa > \omega$ be a regular cardinal. If $\mathfrak{b}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$.*

*Proof.* The hypothesis and Theorem 4 imply that there exists a sequence $\langle f_\delta : \delta < \kappa^+ \rangle$ of functions in $\kappa^\kappa$ with the property that for any $g \in \kappa^\kappa$, there is a $\delta < \kappa^+$ such that $\{\alpha < \kappa : g(\alpha) < f_\delta(\alpha)\}$ is stationary in $\kappa$. For any $E \subset \kappa$, if $\mathrm{otp}(E) = \kappa$, then let $\langle \mu_{E,\xi} : \xi < \kappa \rangle$ be the increasing enumeration of $E$. For each $\delta < \kappa^+$, let $C_\delta = \{\alpha < \kappa : \alpha \text{ is closed under } f_\delta\}$. Recall that $C_\delta$ is a club in $\kappa$. Also, fix a sequence $\langle e_\delta : \kappa \leq \delta < \kappa^+ \rangle$ of bijections $e_\delta : \kappa \to \delta$. We will construct a sequence $\langle \langle A_\delta, E_\delta \rangle : \delta < \kappa^+ \rangle$ satisfying the following conditions for each $\delta < \kappa^+$:

(1) $A_\delta \in [\kappa]^\kappa$ and $E_\delta \subset C_\delta$ is a club in $\kappa$;
(2) $\forall \gamma < \delta \left[|A_\gamma \cap A_\delta| < \kappa\right]$;
(3) if $\kappa \leq \delta$, then $A_\delta = \bigcup_{\xi < \kappa} B_{\delta,\xi}$, where for each $\xi < \kappa$, $B_{\delta,\xi}$ is defined to be

$$\left\{\mu_{E_\delta,\xi} \leq \alpha < \mu_{E_\delta,\xi+1} : \forall \nu < \mu_{E_\delta,\xi} \left[\alpha \notin A_{e_\delta(\nu)}\right]\right\}.$$

Suppose for a moment that such a sequence can be constructed. Let $\mathscr{A} = \{A_\delta : \delta < \kappa^+\}$. By (1) and (2), $\mathscr{A}$ is an a.d. family in $[\kappa]^\kappa$ of size $\kappa^+$. We claim that it is maximal. To see this, fix $B \in [\kappa]^\kappa$. Define a function $g : \kappa \to \kappa$ by stipulating that for each $\mu \in \kappa$, $g(\mu) = \sup\left(\{\min(B \setminus (\mu+1))\} \cup \{f_\nu(\mu) : \nu \leq \mu\}\right)$. Find $\delta < \kappa^+$ such that $S = \{\mu \in \kappa : g(\mu) < f_\delta(\mu)\}$ is stationary in $\kappa$. Note that $\kappa \leq \delta$. Therefore the consequent of (3) applies to $\delta$. Let $I = \{\xi < \kappa : B_{\delta,\xi} \cap B \neq 0\}$. If $|I| = \kappa$, then $|A_\delta \cap B| = \kappa$, and we are done. So assume that $|I| < \kappa$. Then $\{\mu_{E_\delta,\xi} : \xi \in I\} \subset E_\delta \subset \kappa$ and $|\{\mu_{E_\delta,\xi} : \xi \in I\}| \leq |I| < \kappa$. Therefore $\sup\left(\{\mu_{E_\delta,\xi} : \xi \in I\}\right) = \nu_0 < \kappa$. Now $\{\mu \in E_\delta : \mu > \nu_0\}$ is a club in $\kappa$ and $T = S \cap \{\mu \in E_\delta : \mu > \nu_0\}$ is stationary in $\kappa$. Consider any $\mu \in T$. There exists $\xi \in \kappa \setminus I$ with $\mu = \mu_{E_\delta,\xi}$. Note that $B_{\delta,\xi} \cap B = 0$ because $\xi \notin I$. On the other hand, $\mu_{E_\delta,\xi} = \mu < \min(B \setminus (\mu+1)) \leq g(\mu) < f_\delta(\mu) < \mu_{E_\delta,\xi+1}$ because $\mu \in S$ and because $\mu_{E_\delta,\xi+1} \in C_\delta$. Since $\min(B \setminus (\mu+1)) \notin B_{\delta,\xi}$, it follows from the definition of $B_{\delta,\xi}$ that $\exists \nu < \mu \left[\min(B \setminus (\mu+1)) \in A_{e_\delta(\nu)}\right]$. Thus we have proved that for each $\mu \in T$, $\exists \nu < \mu \exists \beta \in B \left[\mu < \beta \wedge \beta \in A_{e_\delta(\nu)}\right]$. Since $T$ is stationary in $\kappa$, there exist $T^* \subset T$ and $\nu$ such that $T^*$ is stationary in $\kappa$ and for each $\mu \in T^*$, $\nu < \mu$ and $\exists \beta \in B \left[\mu < \beta \wedge \beta \in A_{e_\delta(\nu)}\right]$. It now easily follows that $\left|A_{e_\delta(\nu)} \cap B\right| = \kappa$. This proves the maximality of $\mathscr{A}$. Since $|\mathscr{A}| = \kappa^+$, we have $\mathfrak{a}(\kappa) \leq \kappa^+$, while standard arguments (see Theorem 1.2 of [5]) show that $\kappa^+ \leq \mathfrak{a}(\kappa)$. Hence we have $\mathfrak{a}(\kappa) = \kappa^+$.

Thus it suffices to construct a sequence satisfying (1)–(3) above. Let $\langle A_\gamma : \gamma \in \kappa \rangle$ be any partition of $\kappa$ into $\kappa$ many pairwise disjoint pieces of size $\kappa$. For each $\gamma < \kappa$, let $E_\gamma = C_\gamma$. It is clear that the sequence $\langle \langle A_\gamma, E_\gamma \rangle : \gamma < \kappa \rangle$ satisfies (1)–(3). Now fix $\kappa^+ > \delta \geq \kappa$ and assume that $\langle \langle A_\gamma, E_\gamma \rangle : \gamma < \delta \rangle$ satisfying (1)–(3) is given. We construct $A_\delta$ and $E_\delta$ as follows. Let $\theta$ be a sufficiently large regular cardinal. Let $x = \{\kappa, \langle f_\delta : \delta < \kappa^+ \rangle, \langle C_\delta : \delta < \kappa^+ \rangle, \langle e_\delta : \kappa \leq \delta < \kappa^+ \rangle, \delta, \langle \langle A_\gamma, E_\gamma \rangle : \gamma < \delta \rangle\}$. Let $\langle N_\xi : \xi < \kappa \rangle$ be such that

(4) $\forall \xi < \kappa \, [N_\xi \prec H(\theta) \wedge x \in N_\xi]$;
(5) $\forall \xi < \kappa \, [|N_\xi| < \kappa \wedge \mu_\xi = N_\xi \cap \kappa \in \kappa]$;
(6) $\forall \xi < \xi + 1 < \kappa \, [\langle N_\zeta : \zeta \leq \xi \rangle \in N_{\xi+1}]$;
(7) $\forall \xi < \kappa \left[\xi \text{ is a limit ordinal } \implies N_\xi = \bigcup_{\zeta < \xi} N_\zeta\right]$.

Observe that these conditions imply that $\forall \zeta < \xi < \kappa \, [N_\zeta \in N_\xi \wedge N_\zeta \subset N_\xi]$. Observe also that $E_\delta = \{\mu_\xi : \xi < \kappa\}$ is a club in $\kappa$ and that $\mu_{E_\delta, \xi} = \mu_\xi$, for all $\xi < \kappa$. Next for each $\xi < \kappa$, $C_\delta \in N_\xi$. It follows that $\mu_\xi \in C_\delta$ because $C_\delta$ is a club in $\kappa$. So $E_\delta \subset C_\delta$. Now define $A_\delta = \bigcup_{\xi < \kappa} B_{\delta, \xi}$, where for each $\xi < \kappa$, $B_{\delta,\xi}$ is

$$\left\{\mu_\xi \leq \alpha < \mu_{\xi+1} : \forall \nu < \mu_\xi \left[\alpha \notin A_{e_\delta(\nu)}\right]\right\}.$$

It is clear that (3) is satisfied by definition and that $A_\delta \subset \kappa$. So to complete the proof, it suffices to check that $|A_\delta| = \kappa$ and that $\forall \gamma < \delta \, [|A_\gamma \cap A_\delta| < \kappa]$. To see the second statement, fix any $\gamma < \delta$. Since $e_\delta : \kappa \to \delta$ is a bijection, we can find $\nu \in \kappa$ with $e_\delta(\nu) = \gamma$. Find $\zeta < \kappa$ with $\nu < \mu_\zeta$. Consider any $\xi < \kappa$ so that $\zeta \leq \xi$. Then $\nu < \mu_\zeta \leq \mu_\xi$. It follows that $A_\gamma \cap B_{\delta,\xi} = A_{e_\delta(\nu)} \cap B_{\delta,\xi} = 0$. Therefore, $A_\gamma \cap A_\delta = \bigcup_{\xi < \kappa} (A_\gamma \cap B_{\delta,\xi}) = \bigcup_{\xi < \zeta} (A_\gamma \cap B_{\delta,\xi}) \subset \bigcup_{\xi < \zeta} B_{\delta,\xi}$. For each $\xi < \zeta$, $|B_{\delta,\xi}| < \kappa$. So $\bigcup_{\xi < \zeta} B_{\delta,\xi}$ is the union of $\leq |\zeta| \leq \zeta < \kappa$ many sets each of size $< \kappa$. Since $\kappa$ is regular, we conclude that $\left|\bigcup_{\xi < \zeta} B_{\delta,\xi}\right| < \kappa$. So $|A_\gamma \cap A_\delta| < \kappa$, as needed.

Finally we check that for each $\xi < \kappa$, $B_{\delta,\xi} \neq 0$. This will imply that $|A_\delta| = \kappa$. Fix any $\xi < \kappa$. Note that for each $\nu < \mu_\xi$, $\left|A_{e_\delta(\mu_\xi)} \cap A_{e_\delta(\nu)}\right| < \kappa$. Therefore $R_\xi = \bigcup_{\nu < \mu_\xi} \left(A_{e_\delta(\mu_\xi)} \cap A_{e_\delta(\nu)}\right)$ is the union of at most $|\mu_\xi| \leq \mu_\xi < \kappa$ many sets each having size $< \kappa$. Since $\kappa$ is regular, it follows that $|R_\xi| < \kappa$. Hence there is an $\alpha \in A_{e_\delta(\mu_\xi)} \setminus R_\xi$ with $\mu_\xi \leq \alpha$ because $\left|A_{e_\delta(\mu_\xi)}\right| = \kappa$. Since $N_{\xi+1} \prec H(\theta)$ and since all the relevant parameters belong to $N_{\xi+1}$, we conclude that there exists $\alpha \in N_{\xi+1}$ such that $\alpha \in \kappa$, $\mu_\xi \leq \alpha$, and $\forall \nu \in \mu_\xi \left[\alpha \notin A_{e_\delta(\nu)}\right]$. Now we have that $\mu_\xi \leq \alpha < \mu_{\xi+1}$ and so $\alpha \in B_{\delta,\xi}$. This shows that $B_{\delta,\xi} \neq 0$ and concludes the proof. $\dashv$

## 3. The reaping and dominating numbers: an application of PCF theory

We begin with a well-known fact, whose proof we include for completeness.

**Definition 6.** Let $\kappa > \omega$ be a regular cardinal. If $A \in [\kappa]^\kappa$, then we let $e_A : \kappa \to A$ be the order isomorphism from $\langle \kappa, \in \rangle$ to $\langle A, \in \rangle$. We also define a function $s_A : \kappa \to A$ by setting $s_A(\alpha) = \min(A \setminus (\alpha + 1))$, for each $\alpha \in \kappa$. We also write $\lim(\kappa) = \{\alpha < \kappa : \alpha \text{ is a limit ordinal}\}$ and $\operatorname{succ}(\kappa) = \{\alpha < \kappa : \alpha \text{ is a successor ordinal}\}$.

**Lemma 7** (Folklore)**.** *If $\kappa > \omega$ is a regular cardinal, then $\mathfrak{r}(\kappa) \geq \kappa^+$.*

*Proof.* Let $F \subset [\kappa]^\kappa$ be a family with $|F| \leq \kappa$. We must find a $B \in \mathcal{P}(\kappa)$ which reaps $F$. If $F$ is empty, then $B = \kappa$ will work. So assume $F$ is non-empty. Let $\{A_\alpha : \alpha < \kappa\}$ enumerate $F$, possibly with repetitions. For each $\alpha < \kappa$, let $C_\alpha = \{\delta < \kappa : \delta \text{ is closed under } s_{A_\alpha}\}$. Then $C = \{\delta < \kappa : \forall \alpha < \delta \, [\delta \in C_\alpha]\}$ is a club in $\kappa$. For each $\xi \in \kappa$, let $B_\xi = \{\zeta < e_C(\xi+1) : e_C(\xi) \leq \zeta\}$. Note that for all $\alpha < e_C(\xi+1)$, $A_\alpha \cap B_\xi \neq 0$. Also for any distinct $\xi, \xi' \in \kappa$, $B_\xi \cap B_{\xi'} = 0$. Put $B = \bigcup\{B_\xi : \xi \in \lim(\kappa)\}$. Then $B \in \mathcal{P}(\kappa)$ and since for each $\alpha < \kappa$ and each $\xi \in \lim(\kappa) \setminus \alpha$, $A_\alpha \cap B_\xi \neq 0$, $|A_\alpha \cap B| = \kappa$, for all $\alpha < \kappa$. Furthermore, $\bigcup\{B_{\xi'} : \xi' \in \mathrm{succ}(\kappa)\} \subset \kappa \setminus B$, and since for each $\alpha < \kappa$ and for each $\xi' \in \mathrm{succ}(\kappa) \setminus \alpha$, $A_\alpha \cap B_{\xi'} \neq 0$, $|A_\alpha \cap (\kappa \setminus B)| = \kappa$, for all $\alpha < \kappa$. Thus $B$ reaps $F$. ⊣

The above proof really shows that $\mathfrak{r}(\kappa) \geq \mathfrak{b}(\kappa)$. However we will not need this in what follows. The proof of the main theorem is broken into two cases. For the remainder of this section, let $\kappa > \omega$ be a fixed regular cardinal. The crucial definition is the following.

**Definition 8.** Let $E_2 \subset E_1$ both be clubs in $\kappa$. For each $\xi \in \kappa$, define $\mathrm{set}(E_1, \xi) = \{\zeta < s_{E_1}(\xi) : \xi \leq \zeta\}$. Define $\mathrm{set}(E_2, E_1) = \bigcup\{\mathrm{set}(E_1, \xi) : \xi \in E_2\}$.

**Lemma 9.** *Suppose that $F \subset [\kappa]^\kappa$ is an unreaped family with $|F| = \mathfrak{r}(\kappa)$. Assume there is a club $E_1 \subset \kappa$ such that for each club $E \subset E_1$, there exists $A \in F$ with $A \subset^* \mathrm{set}(E, E_1)$. Then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.*

*Proof.* For each $A \in F$ define a function $g_A : \kappa \to \kappa$ as follows. Given $\beta \in \kappa$, $g_A(\beta) = s_A(s_{E_1}(\beta))$. Then $|\{g_A : A \in F\}| \leq |F| = \mathfrak{r}(\kappa)$, and we will check that this is a dominating family of functions. To this end, fix any $f \in \kappa^\kappa$. Put

$$E_f = \{\xi \in E_1 : \xi \text{ is closed under } f\}.$$

Then $E_f \subset E_1$ and it is a club in $\kappa$. By hypothesis there exist $A \in F$ and $\delta \in \kappa$ with $A \setminus \delta \subset \mathrm{set}(E_f, E_1)$. We claim that for any $\zeta \in \kappa$, if $\zeta \geq \delta$, then $f(\zeta) < g_A(\zeta)$. Indeed suppose $\delta \leq \zeta < \kappa$ is given. Let $\gamma = s_{E_1}(\zeta) > \zeta$ and let $g_A(\zeta) = \beta = s_A(s_{E_1}(\zeta))$. Then $\beta \in A$ and $\delta \leq \zeta < s_{E_1}(\zeta) < \beta$. Thus $\beta \in \mathrm{set}(E_f, E_1)$. Let $\zeta' \in E_f$ be such that $\zeta' \leq \beta < s_{E_1}(\zeta')$. It could not be the case that $\zeta' < \gamma$, for if that were the case, then the inequality $\beta < s_{E_1}(\zeta') \leq \gamma = s_{E_1}(\zeta) < \beta$ would be true, which is impossible. Therefore $\gamma \leq \zeta'$ and since $\zeta < \gamma \leq \zeta'$ and $\zeta'$ is closed under $f$, we have $f(\zeta) < \zeta' \leq \beta = g_A(\zeta)$, as claimed. Hence $f \leq^* g_A$. As $f \in \kappa^\kappa$ was arbitrary, this proves that $\{g_A : A \in F\}$ is dominating, and so $\mathfrak{d}(\kappa) \leq |\{g_A : A \in F\}| \leq \mathfrak{r}(\kappa)$. ⊣

The proof in the case when the hypothesis of Lemma 9 fails will make use of Shelah's Revised GCH, which is a theorem of ZFC. Let us recall the definition of various notions that are relevant to the revised GCH.

**Definition 10.** Let $\kappa$ and $\lambda$ be cardinals. Define $\lambda^{[\kappa]}$ to be

$$\min\left\{|\mathcal{P}| : \mathcal{P} \subset [\lambda]^{\leq\kappa} \text{ and } \forall u \in [\lambda]^\kappa \exists \mathcal{P}_0 \subset \mathcal{P}\left[|\mathcal{P}_0| < \kappa \text{ and } u = \bigcup \mathcal{P}_0\right]\right\}.$$

The operation $\lambda^{[\kappa]}$ is sometimes referred to as the *weak power*.

The following remarkable ZFC result was obtained by Shelah in [8] as one of the many fruits of his PCF theory. A nice exposition of its proof may also be found in Abraham and Magidor [1]. Another relevant reference is Shelah [10].

**Theorem 11** (The Revised GCH)**.** *If $\theta$ is a strong limit uncountable cardinal, then for every $\lambda \geq \theta$, there exists $\sigma < \theta$ such that for every $\sigma \leq \kappa < \theta$, $\lambda^{[\kappa]} = \lambda$.*

**Corollary 12.** *Let $\mu \geq \beth_\omega$ be any cardinal. There exists an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subset [\mu]^{\leq\theta}$ such that $|\mathcal{P}| \leq \mu$ and for each $u \in [\mu]^\theta$, there exists $v \in \mathcal{P}$ with the property that $v \subset u$ and $|v| \geq \aleph_0$.*

*Proof.* $\beth_\omega$ is a strong limit uncountable cardinal. Therefore Theorem 11 applies and implies that there exists $\sigma < \beth_\omega$ such that for every $\sigma \leq \theta < \beth_\omega$, $\mu^{[\theta]} = \mu$. It is possible to choose an uncountable regular cardinal $\theta$ satisfying $\sigma \leq \theta < \beth_\omega$. Since $\mu^{[\theta]} = \mu$, there exists $\mathcal{P} \subset [\mu]^{\leq\theta}$ such that $|\mathcal{P}| = \mu$ and for each $u \in [\mu]^\theta$, there exists $\mathcal{P}_0 \subset \mathcal{P}$ with the property that $|\mathcal{P}_0| < \theta$ and $u = \bigcup \mathcal{P}_0$. Now suppose that $u \in [\mu]^\theta$ is given. Let $\mathcal{P}_0 \subset \mathcal{P}$ be such that $|\mathcal{P}_0| < \theta$ and $u = \bigcup \mathcal{P}_0$. Since $\theta$ is a regular cardinal and $|u| = \theta$, it follows that $|v| = \theta \geq \aleph_0$, for some $v \in \mathcal{P}_0$. This is as required because $v \in \mathcal{P}$ and $v \subset u$. $\dashv$

The proof of the following theorem is similar to the proof of Cummings and Shelah's theorem from [4] that if $\kappa \geq \beth_\omega$, then $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$.

**Theorem 13.** *If $\kappa \geq \beth_\omega$, then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.*

*Proof.* Write $\mu = \mathfrak{r}(\kappa)$. Let $F \subset [\kappa]^\kappa$ be such that $F$ is unreaped and $|F| = \mu$. Then $\beth_\omega \leq \kappa < \kappa^+ \leq \mathfrak{r}(\kappa) = \mu$. So applying Corollary 12, fix an uncountable regular cardinal $\theta < \beth_\omega$ satisfying the conclusion of Corollary 12. Note that $|\theta \times \mu| = \mu$ because $\theta < \beth_\omega < \mu$. So $|\theta \times F| = \mu$. Therefore applying Corollary 12, find a family $\mathcal{P} \subset [\theta \times F]^{\leq \theta}$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^\theta$, there exists $v \in \mathcal{P}$ satisfying $v \subset u$ and $|v| \geq \aleph_0$. Put $X = F \cup \mu \cup \mathcal{P} \cup \{\theta, \mu, \kappa, \kappa^\kappa, \mathcal{P}(\kappa)\}$. Then $|X| = \mu$, and so if $\chi$ is a sufficiently large regular cardinal, then there exists $M \prec H(\chi)$ with $|M| = \mu$ and $X \subset M$. We will aim to prove that $M \cap \kappa^\kappa$ is a dominating family.

In view of Lemma 9 it may be assumed that for any club $E_1 \subset \kappa$, there exists a club $E_2 \subset E_1$ such that for all $B \in F$, $B \not\subset^* \mathrm{set}(E_2, E_1)$. Since $F$ is an unreaped family and since $\mathrm{set}(E_2, E_1) \in \mathcal{P}(\kappa)$ whenever $E_2 \subset E_1$ are both clubs in $\kappa$, it follows that for each club $E_1 \subset \kappa$, there exist a club $E_2 \subset E_1$ and a $B \in F$ such that $B \subset^* \kappa \setminus \mathrm{set}(E_2, E_1)$. Let $f \in \kappa^\kappa$ be a fixed function. Construct a sequence $\langle \langle E_i, E_i^1, B_i \rangle : i < \theta \rangle$ by induction on $i < \theta$ so that the following conditions are satisfied at each $i < \theta$:

(1) $E_i$ and $E_i^1$ are both clubs in $\kappa$, $E_i^1 \subset E_i$, and $\forall j < i \left[E_i \subset E_j^1\right]$;
(2) $B_i \in F$ and $B_i \subset^* \kappa \setminus \mathrm{set}(E_i^1, E_i)$;
(3) if $i = 0$, then $E_i = \{\alpha < \kappa : \alpha \text{ is closed under } f\}$.

We first show how to construct such a sequence. When $i = 0$, put $E_i = \{\alpha < \kappa : \alpha \text{ is closed under } f\}$. Then $E_i$ is a club in $\kappa$, and so there exist a club $E_i^1 \subset E_i$ and a $B_i \in F$ with $B_i \subset^* \kappa \setminus \mathrm{set}(E_i^1, E_i)$. Next suppose that $\theta > i > 0$ and that $\langle \langle E_j, E_j^1, B_j \rangle : j < i \rangle$ satisfying (1)–(3) is given. Then $\{E_j^1 : j < i\}$ is a collection of $\leq |i| \leq i < \theta < \beth_\omega \leq \kappa$ many clubs in $\kappa$. Therefore $E_i = \bigcap_{j<i} E_j^1$ is a club in $\kappa$. We have $\forall j < i \left[E_i \subset E_j^1\right]$ and moreover there exist a club $E_i^1 \subset E_i$ and a $B_i \in F$ such that $B_i \subset^* \kappa \setminus \mathrm{set}(E_i^1, E_i)$. It is clear that $E_i, E_i^1$, and $B_i$ are as required. This completes the construction of the sequence $\langle \langle E_i, E_i^1, B_i \rangle : i < \theta \rangle$.

Now define a function $u : \theta \to F$ by setting $u(i) = B_i$ for all $i \in \theta$. Then $u \subset \theta \times F$ and $|u| = |\mathrm{dom}(u)| = \theta$. Hence by the choice of $\mathcal{P}$ and $M$, there exists $v \in \mathcal{P} \subset X \subset M$ such that $v \subset u$ and $|v| \geq \aleph_0$. $v$ is a function and $c = \mathrm{dom}(v) \subset \mathrm{dom}(u) = \theta$. Moreover, $\aleph_0 \leq |v| = |c|$ and $c \in M$. Hence we can find $d \in M$ so that $d \subset c$ and $\mathrm{otp}(d) = \omega$. Let $w = v \restriction d \in M$. Since $\kappa > \omega$ is regular, there exists a function $g \in \kappa^\kappa$ with the property that for each $\alpha \in \kappa$, $\forall i \in d \exists \beta \in w(i) = B_i \left[\alpha < \beta < g(\alpha)\right]$. We may further assume that $g \in M$ because all of the relevant parameters belong to $M$. Let $\langle i_n : n \in \omega \rangle$ be the strictly increasing enumeration of $d$. Recall that for each $n \in \omega$, $E_{i_n}^1 \subset E_{i_n} \subset \kappa$ are both clubs in $\kappa$ and that $B_{i_n} \subset^* \kappa \setminus \mathrm{set}(E_{i_n}^1, E_{i_n})$. In particular, for each $n \in \omega$, there exists $\delta_n \in \kappa$ so that $B_{i_n} \setminus \delta_n \subset \kappa \setminus \mathrm{set}(E_{i_n}^1, E_{i_n})$, and also $\min(E_{i_n}) \in \kappa$. Hence $\{\delta_n : n \in \omega\} \cup \{\min(E_{i_n}) : n \in \omega\}$ is a countable subset of $\kappa$, whence $\{\delta_n : n \in \omega\} \cup \{\min(E_{i_n}) : n \in \omega\} \subset \delta$, for some $\delta \in \kappa$. We will argue that for each $\alpha \in \kappa$, if $\alpha \geq \delta$, then $f(\alpha) < g(\alpha)$. To this end, let $\alpha \in \kappa$ be fixed, and assume that $\delta \leq \alpha$. For each $n \in \omega$, since $E_{i_n} \subset \kappa$ is a club in $\kappa$ and since $\min(E_{i_n}) < \delta \leq \alpha < \alpha + 1 < \kappa$, it follows that $\xi_n = \sup(E_{i_n} \cap (\alpha + 1)) \in E_{i_n}$. Also $\forall n \in \omega \left[\xi_{n+1} \leq \xi_n\right]$

because $\forall n \in \omega \left[E_{i_{n+1}} \subset E_{i_n}\right]$. It follows that there exist $\xi$ and $N \in \omega$ such that $\forall n \geq N\,[\xi_n = \xi]$. Note that $\xi \in E_{i_{N+1}} \subset E^1_{i_N}$. Consider $s_{E_{i_N}}(\xi)$. $s_{E_{i_N}}(\xi) \in E_{i_N}$ and $s_{E_{i_N}}(\xi) > \xi = \xi_N = \sup(E_{i_N} \cap (\alpha+1))$. Therefore $s_{E_{i_N}}(\xi) \geq \alpha + 1 > \alpha$. Since $s_{E_{i_N}}(\xi) \in E_{i_N} \subset E_0$, $s_{E_{i_N}}(\xi)$ is closed under $f$. Therefore $f(\alpha) < s_{E_{i_N}}(\xi)$. Next by the choice of $g$, there exists $\beta \in B_{i_N}$ with $\alpha < \beta < g(\alpha)$. Note that $\delta_N < \delta \leq \alpha < \beta$. Hence $\beta \in B_{i_N} \setminus \delta_N \subset \kappa \setminus \operatorname{set}(E^1_{i_N}, E_{i_N})$, in other words, $\beta \notin \operatorname{set}(E^1_{i_N}, E_{i_N})$. Note that $\xi = \sup(E_{i_N} \cap (\alpha+1)) \leq \alpha < \beta$. Since $\xi \in E^1_{i_N}$, $\beta \geq s_{E_{i_N}}(\xi)$. Putting all this information together, we have $f(\alpha) < s_{E_{i_N}}(\xi) \leq \beta < g(\alpha)$, as required.

Thus we have proved that $f \leq^* g$. Since $f \in \kappa^\kappa$ was arbitrary and since $g \in M \cap \kappa^\kappa$, we have proved that $M \cap \kappa^\kappa$ is a dominating family. Therefore $\mathfrak{d}(\kappa) \leq |M \cap \kappa^\kappa| \leq |M| = \mu = \mathfrak{r}(\kappa)$. ⊣

## 4. Questions

Raghavan and Shelah [6] introduced the method of forcing with a carefully chosen Boolean ultrapower of a forcing iteration to obtain the following result.

**Theorem 14** ([6])**.** *Let $\kappa \geq \omega$ be any regular cardinal. If there is a supercompact cardinal $\theta > \kappa$, then there is a cardinal preserving forcing extension in which $\theta < \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$. There is also a cardinal preserving forcing extension in which $\theta < \mathfrak{b}(\kappa) < \mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$.*

In the models of $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$ obtained in [6], the value of $\mathfrak{b}(\kappa)$ is much larger than $\kappa$. It is unknown how large $\mathfrak{b}(\kappa)$ needs to be for the configuration $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$ to be consistent. So we ask

**Question 15.** *Does $\mathfrak{b}(\kappa) = \kappa^{++}$ imply that $\mathfrak{a}(\kappa) = \kappa^{++}$, for every regular cardinal $\kappa > \omega$?*

It is not possible to step-up the proof of Theorem 5 in any straightforward way. If Question 15 has a positive answer, then the proof is likely to involve quite a different argument.

Theorem 13 of course gives no information about the relationship between $\mathfrak{d}(\kappa)$ and $\mathfrak{r}(\kappa)$ when $\kappa < \beth_\omega$.

**Question 16.** *If $\omega < \kappa < \beth_\omega$ is a regular cardinal, then does $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ hold? In particular, is $\mathfrak{d}(\aleph_n) \leq \mathfrak{r}(\aleph_n)$, for all $1 \leq n < \omega$?*

In trying to tackle this problem, it may seem reasonable to first try to produce a model where $\mathfrak{r}(\aleph_n) < 2^{\aleph_n}$, for if $\mathfrak{r}(\aleph_n)$ is provably equal to $2^{\aleph_n}$, then of course $\mathfrak{d}(\aleph_n) \leq \mathfrak{r}(\aleph_n)$. This is closely related to a well-known question of Kunen about the minimal size of a base for a uniform ultrafilter on $\aleph_1$.

**Question 17.** *Is $\mathfrak{r}(\aleph_1) < 2^{\aleph_1}$ consistent? Is $\mathfrak{u}(\aleph_1) < 2^{\aleph_1}$ consistent?*

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Department of Mathematics, National University of Singapore, Singapore 119076
*Email address*: `raghavan@math.nus.edu.sg`
*URL*: `http://www.math.nus.edu.sg/~raghavan`