# SEPARATING FAMILIES AND ORDER DIMENSION OF TURING DEGREES

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Abstract. We study families of functions and linear orders which separate countable subsets of the continuum from points. As an application, we show that the order dimension of the Turing degrees, denoted $\mathfrak{dim}_T$, cannot be decided in ZFC. We also provide a combinatorial description of $\mathfrak{dim}_T$ and show that the Turing degrees have the largest order dimension among all locally countable partial orders of size continuum. Finally, we prove that it is consistent that the number of linear orders needed to separate countable subsets of the continuum from points is strictly smaller than the number of functions necessary for separating them.

## 1. Introduction

The order dimension of a partial order was defined by Dushnik and Miller [3].

**Definition 1.1** ([3]). *Suppose $\langle P, \prec\rangle$ is a partial order. The* order dimension *of $P$, denoted $\mathsf{odim}(P)$, is the least $\kappa$ such that there exists $\langle \prec_i : i < \kappa\rangle$ such that each $\prec_i$ is a linear order on $P$ that extends $\prec$ and for every $a, b \in P$, if $a \not\preceq b$, then for some $i < \kappa$, $b \prec_i a$.*

Equivalently, the order dimension of $\langle P, \prec\rangle$ is the minimal $\kappa$ such that there is a sequence $\langle \prec_i : i \in \kappa\rangle$ of partial orders on $P$ extending $\prec$ such that for any $x, y \in P$, if $x \not\preceq y$, then $y \prec_i x$, for some $i \in \kappa$. Dushnik and Miller [3] showed that the order dimension of $\langle P, \prec\rangle$ is also the minimal $\kappa$ such that $\langle P, \prec\rangle$ can be embedded into a product of $\kappa$ many linear orders (with the coordinate-wise ordering on the product). In this paper, we will mostly deal with partial orders of size continuum which have the property that every element has only countably many predecessors.

**Definition 1.2.** *A partial order $\langle P, \prec\rangle$ is said to be* locally countable *if for every $x \in P$, $\{y \in P : y \preceq x\}$ is countable.*

The reason for this restriction is that this work arose out of our attempt to resolve some of the questions left open in a recent paper of Higuchi, Lempp, Raghavan, and Stephan [7], who investigated the order dimension of the Turing degrees. Let $\mathcal{D}$ denote the collection of Turing degrees and let $<_T$ be the partial ordering of Turing reducibility. Note that $\langle \mathcal{D}, <_T\rangle$ is a locally countable partial order of size $\mathfrak{c}$, where $\mathfrak{c}$ denotes $2^{\aleph_0}$. The cardinal invariant $\mathfrak{dim}_T = \mathsf{odim}(\langle \mathcal{D}, <_T\rangle)$ was introduced in [7] where it was observed that $\aleph_1 \leq \mathfrak{dim}_T \leq \mathfrak{c}$. It was also proved in [7] that if $\mathfrak{c} = \kappa^+$ where $\kappa$ is a cardinal of uncountable cofinality, then the order dimension of every locally countable partial order of size $\mathfrak{c}$ is $\leq \kappa$. This implies that if $\mathfrak{dim}_T = \mathfrak{c} > \aleph_1$,

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then either $\mathfrak{c}$ is a limit cardinal or it is the successor of a cardinal of countable cofinality. In particular, if $\mathfrak{c} \leq \aleph_2$, then $\mathfrak{dim}_T = \aleph_1$. They also left open the possibility that $\mathfrak{dim}_T = \aleph_1$ in ZFC.

**Question 1.3** ([7])**.** *Does* ZFC *decide the order dimension of the Turing degrees* $\langle \mathcal{D}, <_T \rangle$*?*

In this paper, we provide a strong negative answer to Question 1.3 by showing, among other things, that each of the following is consistent:

(1) $\aleph_1 < \mathfrak{dim}_T < \mathfrak{c}$;
(2) $\mathfrak{dim}_T = \mathfrak{c} = \aleph_{\omega_1}$;
(3) $\mathfrak{dim}_T = \mathfrak{c} = \aleph_{\omega+1}$.

One of the original motivations behind investigating the order dimension of Turing degrees was the following question of Sacks [11]: Must every locally countable partial order of size continuum embed into the Turing degrees? Sacks showed that the answer is yes under the continuum hypothesis. In [6], Groszek and Slaman showed that a positive answer to Sacks' question cannot be established via "extensions of embedding" by constructing a model of $\mathfrak{c} > \aleph_1$ where there is a maximal Turing independent set of size $\aleph_1$. A plausible way of showing a consistent negative answer to Sacks' question could have been to find a model of set theory where the order dimension of the Turing degrees is strictly smaller than the order dimension of some locally countable partial order of size continuum. We show here that this approach fails by proving that among all locally countable partial orders of size continuum, the Turing degrees have the largest order dimension.

The cardinal $\mathfrak{dim}_T$ turns out to have a simple combinatorial description. In fact, the order dimension of $\langle \mathcal{D}, <_T \rangle$ is equal to the order dimension of one of its suborders which lacks chains of length 3. This two-layered partial order, which we denote $\mathbb{H}_\mathfrak{c}$, was first defined by Higuchi in [7]. Higuchi, Lempp, Raghavan, and Stephan essentially established (see Lemma 3.7 in [7] and Lemmas 2.8 and 2.9 below) that $\mathbb{H}_\mathfrak{c}$ has the largest order dimension among all locally countable partial orders of size $\mathfrak{c}$. In Section 2, we show that $\mathbb{H}_\mathfrak{c}$ embeds into the Turing degrees, proving that $\mathfrak{dim}_T$ is equal to the order dimension of $\mathbb{H}_\mathfrak{c}$, and hence that the Turing degrees also have maximal order dimension among all locally countable partial orders of size $\mathfrak{c}$.

The cardinal $\mathfrak{dim}_T$ is closely connected to various types of separating families. The notion of a family of objects of a certain type which separates sets from points or from other sets is familiar from topology where one asks, for instance, whether any two disjoint closed sets can be separated by a continuous function. In the combinatorial context of order dimension, the relevant notion is that of separating points from countable sets using linear orders. A family $\mathcal{F}$ of functions from a set $X$ into $\{0,1\}$ *separates* all members of a family $\mathcal{C}$ of subsets of $X$ from points if for each $B \in \mathcal{C}$ and $x \in X \setminus B$, there is a member of $\mathcal{F}$ that is constantly 0 on $B$, while sending $x$ to 1. It is well-known that there is always a countable family of functions from $\mathbb{R}$ into $\{0,1\}$ separating finite sets from points. It is more intricate to separate countable subsets of $\mathbb{R}$ from points using functions. It was proved in [7] that if $\mathsf{cf}(\kappa) > \omega$ and $\mathfrak{c} = \kappa^+$, then there is a family of at most $\kappa$ many functions from $\mathbb{R}$ into $\{0,1\}$ which separates countable sets from points. Observe that every $f : \mathbb{R} \to \{0,1\}$ induces a linear order on $\mathbb{R}$ that puts all members of $f^{-1}(\{0\})$ before any member of $f^{-1}(\{1\})$. If $\mathcal{F}$ is a family of functions from $\mathbb{R}$ into $\{0,1\}$ which separates countable sets from points, then the family of linear orders on $\mathbb{R}$ induced in this manner by the members of $\mathcal{F}$ also separates countable sets from points in the sense that for every countable $B \subseteq \mathbb{R}$ and $x \in \mathbb{R} \setminus B$, there is a linear order in the family which puts $x$ above every member of $B$. It was essentially noted in [7] that if there is a family of $\theta$ many linear orders on $\mathbb{R}$ which separates countable sets from points in this sense, then the order dimension of any locally countable partial order of size $\mathfrak{c}$ is at most $\theta$. The results in Section 2 below show that $\mathfrak{dim}_T$ is precisely the minimal number of linear orders on $\mathbb{R}$ required to separate countable subsets of $\mathbb{R}$ from points in this sense, which is a purely combinatorial property of the cardinal $\mathfrak{c}$. A noteworthy result from Section 5 is that it is fundamentally easier to separate countable subsets of $\mathbb{R}$ from points using linear orders than it is to separate them via functions. For instance, after adding $\aleph_3$ Cohen reals to a model of GCH, there is a family of $\aleph_1$ many linear orders on $\mathbb{R}$ separating countable sets from points, but there is no such family of functions from $\mathbb{R}$ into $\{0, 1\}$ having size $\aleph_1$. We also show in Section 5 that after adding $\aleph_4$ Cohen reals to a model of GCH, there is no family of $\aleph_1$ many linear orders on $\mathbb{R}$ separating countable sets from points.

Note that $\langle \mathcal{D}, <_T \rangle$ is *$\sigma$-directed*, meaning that every countable subset of $\mathcal{D}$ has a $\leq_T$-upper bound in $\mathcal{D}$. The following question about $\sigma$-directed locally countable partial orders was raised in [7].

**Question 1.4.** *In* ZFC*, is there any $\sigma$-directed locally countable partial order $\mathbb{H}$ such that $\mathsf{odim}(\mathbb{H}) = \aleph_1$ and $|\mathbb{H}| = \mathfrak{c}$?*

We show in Section 6 that the answer is no: Consistently, there is no locally countable partial order of size continuum and order dimension $\aleph_1$ in which every countable set has an upper bound.

**Notation**: $\mathsf{Cohen}_\kappa$ denotes the forcing for adding $\kappa$ Cohen reals. For a set $X$, $\mathsf{Random}_X$ is the measure algebra on $2^X$ equipped with the usual product measure which we denote by $\mu_X$. If $X = \emptyset$, we interpret $(2^X, \mu_X)$ as the measure space with one point of measure one.

## 2. A partial order inside the Turing degrees

This section provides a combinatorial description of $\mathfrak{dim}_T$ (Corollary 2.11). The following poset was introduced by Higuchi.

**Definition 2.1.** *Let $\mathbb{H}_\kappa$ be the partial order consisting of $\kappa \sqcup [\kappa]^{\leq\aleph_0}$ with the ordering $a < B$ iff $a \in \kappa$, $B \in [\kappa]^{\leq\aleph_0}$ and $a \in B$.*

**Theorem 2.2.** *$\mathbb{H}_\mathfrak{c}$ embeds into the Turing degrees.*

It follows that the order dimension of the Turing degrees is at least that of $\mathbb{H}_\mathfrak{c}$. Lemma 2.10 will show that they are in fact equal. For the proof of Theorem 2.2, we'll make use of the following perfect set version of the exact pair theorem of Spector. Since we couldn't find a reference in the literature, we include a proof.

**Lemma 2.3.** *Suppose $\langle a_n : n < \omega \rangle$ is $\leq_T$-increasing. Put $\mathcal{I} = \{z : (\exists n)(z \leq_T a_n)\}$. Then there is a perfect set $P \subseteq 2^\omega$ such that for any $x \neq y$ in $P$, $x$ and $y$ form an exact pair for $\mathcal{I}$, i.e., $\{z : z \leq_T x \wedge z \leq_T y\} = \mathcal{I}$.*

*Proof.* Let $A = \oplus_{n<\omega} a_n = \{(n, m) : m \in a_n\}$. Let $\mathbb{P}$ be the forcing whose conditions are $p = \langle p_\sigma : \sigma \in {}^K 2 \rangle$ where

- $K = K_p < \omega$.
- For every $\sigma \in {}^{K}2$, $p_\sigma : K \times \omega \to 2$ and $\sigma \neq \tau$ implies $p_\sigma \neq p_\tau$.
- $\{(n,m) \in \mathsf{dom}(p_\sigma) : p_\sigma(n,m) \neq 1_A(n,m)\}$ is finite.

For $p, q \in \mathbb{P}$, $p \leq q$ iff $K_p \geq K_q$ and for every $\sigma \in {}^{K_p}2$, $p_\sigma \restriction (K_q \times \omega) = q_\tau$ where $\tau = \sigma \restriction K_q$.

**Claim 2.4.** *There is a countable family $\mathcal{F}$ of dense subsets of $\mathbb{P}$ such that if $G$ is a $\mathbb{P}$-generic filter over $\mathcal{F}$, then $P = \{x \in 2^{\omega\times\omega} : (\exists y \in 2^\omega)(\forall K < \omega)(\exists p \in G)(K_p = K \wedge p_{y\restriction K} \subseteq x)\}$ is as required.*

*Proof.* Let $\mathcal{F}$ consist of the following.

(a) For $k < \omega$, $D_k = \{p \in \mathbb{P} : K_p > k\}$

(b) (Splitting) For every pair of Turing functionals $\Phi, \Psi$ and $K < \omega$, $D_{\Phi,\Psi,K}$ is the set of all $p \in \mathbb{P}$ such that $K_p \geq K$ and for every $\sigma, \tau$ in ${}^{K_p}2$, for every $x, y \in 2^{\omega\times\omega}$, if $\sigma \restriction K \neq \tau \restriction K$, $p_\sigma \subseteq x$ and $p_\tau \subseteq y$, and if $\Phi(x)$, $\Psi(y)$ both converge and are equal to say $z$, then $z \leq_T a_{K_p}$.

It is easy to see that each $D_k$ is dense in $\mathbb{P}$. Next suppose $\Phi, \Psi$ are Turing functionals, $K < \omega$ and $q \in \mathbb{P}$. We'll construct $p \leq q$ such that $p \in D_{\Phi,\Psi,K}$. By extending $q$, we can assume that $K_q > K$.

**Subclaim 2.5.** *There exists $\{r_\sigma : \sigma \in {}^{K_q}2\}$ such that the following hold.*

(i) *Each $r_\sigma$ is a partial function from $\omega \times \omega$ to 2 such that $r_\sigma$ extends $q_\sigma$ and $r_\sigma \setminus q_\sigma$ is finite.*

(ii) *For every $\sigma \neq \tau$ in ${}^{K_q}2$, $\star(r_\sigma, r_\tau)$ holds where*
*$\star(r,s)$: For every $x, y \in {}^{\omega\times\omega}2$, if $r \subseteq x$, $s \subseteq y$, $\Phi(x)$ and $\Psi(y)$ both converge and are equal to $z$, then $z \leq_T a_{K_q}$.*

*Proof.* First note that if $\star(r,s)$ holds and $r', s'$ extend $r, s$ respectively, then $\star(r', s')$ holds as well, so we can construct $\{r_\sigma : \sigma \in {}^{K_q}2\}$ one step at a time. Fix $\sigma \neq \tau$ in ${}^{K_q}2$ and suppose $r, s$ are partial functions from $\omega \times \omega$ to 2 such that $p_\sigma \subseteq r$, $p_\tau \subseteq s$ and $r \setminus p_\sigma$, $s \setminus p_\tau$ are finite. If there are $x, y \in {}^{\omega\times\omega}2$ such that $r \subseteq x$, $s \subseteq y$ and on some input $n < \omega$, both $\Phi(x)(n)$ and $\Psi(y)(n)$ converge and give different outputs, then we extend $r$ to $r_1 \subseteq x$ and $s$ to $s_1 \subseteq y$ by finite amounts such that they contain the use of these disagreeing computations of $\Phi(x)(n)$ and $\Psi(y)(n)$. Let us say we forced a disagreement if we did manage to find such $x, y$. If there are no such $x, y$, we set $r_1 = r$ and $s_1 = s$. We claim that $\star(r_1, s_1)$ holds. Note that this is clear if we did force a disagreement. So assume no disagreement could be forced over $r, s$. Observe that, in this case, for every $x, x', y, y'$ and $n < \omega$ where $x, x'$ extend $r$ and $y, y'$ extend $s$, if $\Phi(x)(n), \Phi(x')(n), \Psi(y)(n), \Psi(y')(n)$ all converge, then they must all be equal. Hence if $x, y$ extend $r, s$ respectively and $\Phi(x), \Psi(y)$ both converge to $z$, then $z$ can be computed from $r$ – On input $n$, look for some $r_1$ that extends $r$ by finitely many points such that $\Phi(r_1)(n)$ converges and output this value. □

Let $\{r_\sigma : \sigma \in {}^{K_q}2\}$ be as in Subclaim 2.5. Choose $p \leq q$ such that $K_p > K_q$ is large enough so that $K_p \times \omega$ contains the domain of every member of $\{r_\sigma : \sigma \in {}^{K_q}2\}$. For each $\sigma \in {}^{K_p}2$, choose $p_\sigma$ extending $r_\sigma$ such that $p_\sigma$'s are pairwise distinct and each $p_\sigma$ differs from $1_A \restriction K_p \times \omega$ on a finite set. It is clear that $p \in D_{\Phi,\Psi,K}$. □

Let $G$ be a filter on $\mathbb{P}$ that meets every set in $\mathcal{F}$. Then it is easy to check that $G$ is as required. This completes the proof of Lemma 2.3. □

*Proof of Theorem 2.2.* Let $\{c_\xi : \xi < \mathfrak{c}\}$ be a Turing independent set of reals. This means that for every finite $F \subseteq \mathfrak{c}$ and $\eta \in \mathfrak{c} \setminus F$, the join of $\{c_\xi : \xi \in F\}$ does not compute $c_\eta$. Let $\langle X_i : i < \mathfrak{c}\rangle$ list $[\mathfrak{c}]^{\aleph_0}$. For each $i < \mathfrak{c}$, fix a perfect set $P_i$ of exact pairs for the Turing ideal generated by $\{c_\xi : \xi \in X_i\}$. Inductively construct $\langle (W_i, Y_j, y_j) : i, j < \mathfrak{c}, j \text{ limit}\rangle$ such that the following hold.

(i) $W_i$ is a subset of $\mathfrak{c}$ of order type $i$.
(ii) $i < j < \mathfrak{c}$ implies $W_i \subseteq W_j$ and $\sup(W_i) < \min(W_j \setminus W_i)$.
(iii) $j < \mathfrak{c}$ limit implies $W_j = \bigcup_{i<j} W_i$.
(iv) $j < \mathfrak{c}$ limit implies $Y_j \in [W_j]^{\aleph_0}$.
(v) For every $i < \mathfrak{c}$, for every $X \in [W_i]^{\aleph_0}$, there exists $j < \mathfrak{c}$ such that $Y_j = X$.
(vi) $y_j \in P_{i(j)}$ where $X_{i(j)} = Y_j$.
(vii) $j < j' < \mathfrak{c}$ limit implies $y_j, y_{j'}$ are Turing incomparable.
(viii) For every $j < \mathfrak{c}$ limit and $\xi \in \bigcup_{i<\mathfrak{c}} W_i$, if $c_\xi \leq_T y_j$, then $\xi \in Y_j$.

At a successor stage $i+1$, we just add some $\xi < \mathfrak{c}$ to $W_{i+1}$ such that $\xi$ is above $\sup(W_i)$ and no $y_j$ computes $c_\xi$ for $j \leq i$ limit. This can be done because each $y_j$ computes only countably many $c_\xi$'s. At a limit stage $j < \mathfrak{c}$, we define $Y_j, y_j$ as follows. Let $j_1 < \mathfrak{c}$ be least such that $X_{j_1} \subseteq W_j$ and there is no $j' < j$ limit such that $Y_{j'} = X_{j_1}$. Put $Y_j = X_{j_1}$ and choose $y_j \in P_{j_1}$ such that $y_j$ is Turing incomparable with every member of $\{y_k : k < j \text{ limit}\}$ and $y_j$ does not compute any member of $\{c_\xi : \xi \in W_j \setminus Y_j\}$. This can be done because no two members of $P_{j_1}$ can compute the same member of $\{c_\xi : \xi \in W_j \setminus Y_j\} \cup \{y_k : k < j \text{ limit}\}$. Note that this also takes care of clause (v).

Put $W = \bigcup_{i<\mathfrak{c}} W_i$, $C = \{c_\xi : \xi \in W\}$. For each $X \in [W]^{\aleph_0}$, choose $j < \mathfrak{c}$ such that $Y_j = X$ and put $y_X = y_j$. Fix $T \in [W]^{\aleph_0}$ and put $S = W \setminus T$. Define $f : S \sqcup [S]^{\leq\aleph_0} \to 2^\omega$ as follows.

(a) If $\xi \in S$, then $f(\xi) = c_\xi$.
(b) If $X \in [S]^{\leq\aleph_0}$, then $f(X) = y_{(X \cup T)}$.

It is easy to see that $f$ witnesses that $\mathbb{H}_\mathfrak{c}$ embeds into the Turing degrees. □

The proof of Theorem 2.2 made essential use of of the fact that $\{c_i : i < \mathfrak{c}\}$ was Turing independent. The embedding of $\mathbb{H}_\mathfrak{c}$ given there doesn't necessarily send $[\mathfrak{c}]^{\leq\aleph_0}$ to a Turing independent set. So we can ask the following.

**Question 2.6.** *Does $\mathbb{H}^2_\mathfrak{c}$ embed into the Turing degrees? Here, $\mathbb{H}^2_\mathfrak{c}$ is the partial order whose domain is $\mathfrak{c} \sqcup [\mathfrak{c}]^{\leq\aleph_0} \sqcup [[\mathfrak{c}]^{\leq\aleph_0}]^{\leq\aleph_0}$, with the ordering $<$ defined by $x < y$ iff one of the following holds.*

(a) *$x \in \mathfrak{c}$, $y \in [\mathfrak{c}]^{\leq\aleph_0}$ and $x \in y$.*
(b) *$x \in [\mathfrak{c}]^{\leq\aleph_0}$, $y \in [[\mathfrak{c}]^{\leq\aleph_0}]^{\leq\aleph_0}$ and $x \in y$.*
(c) *$x \in \mathfrak{c}$, $y \in [[\mathfrak{c}]^{\leq\aleph_0}]^{\leq\aleph_0}$ and for some $z \in [\mathfrak{c}]^{\leq\aleph_0}$, $x \in z$ and $z \in y$.*

**Definition 2.7.** *For infinite cardinals $\theta$ and $\kappa$, $\star(\theta, \kappa)$ is the following statement: For every sequence $\langle <_i : i < \theta\rangle$ of linear orders on $\kappa$, there exist $X, \alpha$ such that $X \in [\kappa]^{\aleph_0}$, $\alpha \in \kappa \setminus X$ and for every $i < \theta$, there exists $\beta \in X$ such that $\alpha <_i \beta$.*

Note that the failure of $\star(\theta, \kappa)$ says that there is a family of at most $\theta$ many linear orders on $\kappa$ separating points from countable sets. It was essentially shown by Higuchi, Lempp, Raghavan, and Stephan in [7] that the failure of $\star(\theta, \kappa)$ implies that the dimension of every locally countable partial order of size $\kappa$ is at most $\theta$. More precisely, the next lemma follows from the proof of Lemma 3.7 of [7].

**Lemma 2.8** (Higuchi, Lempp, Raghavan, and Stephan)**.** *Let $\kappa$ and $\theta$ be infinite cardinals. Let $\langle P, \prec \rangle$ be any locally countable partial order with $|P| = \kappa$. If $\star(\theta, \kappa)$ fails, then the dimension of $\langle P, \prec \rangle$ is at most $\theta$.*

*Proof.* Suppose $\star(\theta, \kappa)$ fails. Fix a locally countable partial order $\langle P, \prec \rangle$ with $|P| = \kappa$. By hypothesis, we can find a sequence $\langle <_i : i < \theta \rangle$ of linear orders on $P$ with the property that for any $B \in [P]^{\aleph_0}$ and any $x \in P \setminus B$, there is $i < \theta$ such that $y <_i x$, for all $y \in B$. For any $i < \theta$, let $R_i$ be defined as follows: for $x, y \in P$, $y \, R_i \, x$ if and only if $\exists u \preceq x \forall v \preceq y \, [v <_i u]$. Define $\prec_i$ as follows: for any $x, y \in P$, $x \prec_i y$ if and only if either $x \prec y$ or $x \, R_i \, y$. It is not difficult to verify that each $\prec_i$ is a partial order on $P$ extending $\prec$. Thus $\{\prec_i : i < \theta\}$ is a collection of at most $\theta$ many partial orders on $P$ extending $\prec$. Now consider any $x, y \in P$ and suppose $x \not\preceq y$. Let $A = \{v \in P : v \preceq y\}$. Since $\langle P, \prec \rangle$ is locally countable, $A$ is countable, though it need not be an infinite set. However since $|P| = \kappa$ is an infinite cardinal and since $A \subseteq P \setminus \{x\}$, it is possible to find $B \in [P \setminus \{x\}]^{\aleph_0}$ with $A \subseteq B$. There is $i < \theta$ so that $v <_i x$, for all $v \in B$. By definition of $R_i$, $y \, R_i \, x$ and so $y \prec_i x$. It now follows that the order dimension of $\langle P, \prec \rangle$ is at most $\theta$. □

The following lemma provides a combinatorial description of $\mathsf{odim}(\mathbb{H}_\kappa)$.

**Lemma 2.9** (Higuchi, Lempp, Raghavan, and Stephan)**.** *Let $\kappa$ and $\theta$ be infinite cardinals. Then the following are equivalent:*

1. *The order dimension of $\mathbb{H}_\kappa$ is at most $\theta$;*
2. *$\star(\theta, \kappa)$ fails.*

*Proof.* The proof of (2) $\implies$ (1) is very similar to the argument for Lemma 2.8. To prove (1) $\implies$ (2), fix a sequence of linear orders $\langle \prec_i : i < \theta \rangle$ witnessing that the order dimension of $\mathbb{H}_\kappa$ is at most $\theta$. For each $i < \theta$, define $<_i = \prec_i \cap (\kappa \times \kappa)$. Then $<_i$ is a linear order on $\kappa$. If $B \in [\kappa]^{\aleph_0}$ and $\alpha \in \kappa \setminus B$, then $\alpha$ is not below $B$ in $\mathbb{H}_\kappa$ and so $B \prec_i \alpha$ for some $i < \theta$. For any $\beta \in B$, $\beta \prec_i B \prec_i \alpha$, whence $\beta <_i \alpha$. Therefore $\langle <_i : i < \theta \rangle$ is a witness to the failure of $\star(\theta, \kappa)$. □

An easy corollary is that ccc forcings do not increase the order dimension of $\mathbb{H}_\kappa$. Also the order dimension of any locally countable partial order of size $\kappa$ is bounded above by that of $\mathbb{H}_\kappa$. If $\kappa^{\aleph_0} = \kappa$, then $\mathbb{H}_\kappa$ itself has size $\kappa$.

**Lemma 2.10.** *Let $\kappa$ be any infinite cardinal. If $\langle P, \prec \rangle$ is any locally countable partial order with $|P| = \kappa$, then the order dimension of $\langle P, \prec \rangle$ is at most that of $\mathbb{H}_\kappa$. If $\kappa^{\aleph_0} = \kappa$, then $\mathbb{H}_\kappa$ has the largest order dimension among all locally countable partial orders of size $\kappa$.*

*Proof.* Let $\langle P, \prec \rangle$ be any locally countable partial order with $|P| = \kappa$. Let $\theta$ be the order dimension of $\mathbb{H}_\kappa$. It is easy to see that $\theta$ must be an infinite cardinal because $\kappa$ is infinite. By Lemma 2.9, $\star(\theta, \kappa)$ fails, and so by Lemma 2.8, the order dimension of $\langle P, \prec \rangle$ is at most $\theta$. The second statement follows from the first and the fact that if $\kappa^{\aleph_0} = \kappa$, then $\mathbb{H}_\kappa$ is a locally countable partial order of size $\kappa$. □

**Corollary 2.11.** *Among all locally countable partial orders of size $\mathfrak{c}$, the Turing degrees have the largest order dimension. Furthermore, $\mathfrak{dim}_T$ is equal to the order dimension of $\mathbb{H}_\mathfrak{c}$, which is the least cardinal $\theta$ for which $\star(\theta, \mathfrak{c})$ fails.*

*Proof.* By Theorem 2.2 and Lemma 2.10. □

## 3. Small order dimension and MA

**Definition 3.1.** *Call a family $\mathcal{F}$ of subsets of $\kappa$ $(\aleph_1, 2)$-separating, if for every $X \in [\kappa]^{\aleph_0}$ and $\alpha \in \kappa \setminus X$, there exists $A \in \mathcal{F}$ such that $X \cap A = \emptyset$ and $\alpha \in A$.*

The following fact was shown in [7]. Indeed it is a simple consequence of Lemma 2.8. It was also proved in [7] that if $\kappa$ and $\lambda$ are cardinals with $\omega < \mathsf{cf}(\kappa) \leq \kappa \leq \lambda$, and if there is an almost disjoint family of subsets of $\kappa$ having size $\lambda$, then there is an $(\aleph_1, 2)$-separating family of subsets of $\lambda$ of size $\kappa$. It is clear that separating countable sets from points using subsets is harder to do than separating countable sets from points using linear orders. In other words, if there is an $(\aleph_1, 2)$-separating family of subsets of $\lambda$ of size $\kappa$, then $\star(\kappa, \lambda)$ fails.

**Fact 3.2.** *Suppose $\langle P, \prec\rangle$ is any locally countable partial order. Let $\kappa$ and $\lambda$ be infinite cardinals. If $|P| = \kappa$ and if there is an $(\aleph_1, 2)$-separating family $\mathcal{F}$ of subsets of $\kappa$ with $|\mathcal{F}| = \lambda$, then the order dimension of $\langle P, \prec\rangle$ is at most $\lambda$.*

The following claim shows that consistently, continuum can be arbitrarily large while $\mathfrak{dim}_T = \aleph_1$.

**Claim 3.3.** *Suppose $V \models GCH$ and $\kappa$ is regular uncountable. Let $\mathbb{P} = \mathsf{Add}(\omega_1, \kappa)$ be the forcing for adding $\omega_1$ subsets of $\kappa$ with countable conditions. Let $\mathbb{Q} = \mathsf{Cohen}_\kappa$. Then after forcing with $\mathbb{P} \times \mathbb{Q}$, all cofinalities are preserved, $\mathfrak{c} = \kappa$ and $\mathfrak{dim}_T = \aleph_1$.*

*Proof of Claim 3.3.* Note that $\mathbb{P}$ is countably closed and satisfies the $\aleph_2$-cc. Also $\mathbb{Q}$ is ccc (in $V^{\mathbb{P}}$), so all cofinalities are preserved. For $i < \omega_1$, let $A_i = \{\alpha < \kappa : (\exists p \in G_{\mathbb{P}})(p(i, \alpha) = 1)\}$. Then, by an easy density argument, the family $\mathcal{A} = \{A_i : i < \omega_1\}$ is an $(\aleph_1, 2)$-separating family of subsets of $\kappa$ in $V^{\mathbb{P}}$. Since $\mathbb{Q}$ is ccc, $\mathcal{A}$ remains $(\aleph_1, 2)$-separating in $V^{\mathbb{P}\times\mathbb{Q}}$. Since $V^{\mathbb{P}\times\mathbb{Q}} \models \mathfrak{c} = \kappa$, by Fact 3.2, it follows that the order dimension of Turing degrees is $\aleph_1$ in this model. □

The next claim says that Martin's axiom (MA) does not increase $\mathfrak{dim}_T$.

**Claim 3.4.** *Suppose $V \models GCH$ and $\kappa \geq \aleph_2$ is regular. Let $\mathbb{P} = \mathsf{Add}(\omega_1, \kappa)$. Then, in $V^{\mathbb{P}}$, there is a ccc forcing $\mathbb{Q}$ such that $V^{\mathbb{P}\star\mathbb{Q}}$ satisfies MA, $\mathfrak{c} = \kappa$ and $\mathfrak{dim}_T = \aleph_1$.*

*Proof.* Note that $V^{\mathbb{P}} \models \kappa^{<\kappa} = \kappa$ and $\mathsf{odim}(\mathbb{H}_\kappa) = \aleph_1$. So, in $V^{\mathbb{P}}$, let $\mathbb{Q}$ be the standard ccc poset for forcing MA and $\mathfrak{c} = \kappa$. □

## 4. Saturated ideals and order dimension

This section shows that the existence of certain types of highly saturated ideals on the continuum implies that $\mathfrak{dim}_T = \mathfrak{c}$. Furthermore, starting with a measurable cardinal, we construct some models where such ideals exist.

**Definition 4.1.** *Suppose $\kappa$ is regular uncountable and $\mathcal{I}$ is a normal ideal on $\kappa$. We say that $\mathcal{I}$ is supersaturated if for every $\mathcal{A} \subseteq \mathcal{I}^+$, if $|\mathcal{A}| < \kappa$, then there exists $X \in [\kappa]^{\aleph_0}$ such that for every $A \in \mathcal{A}$, $X \cap A \neq \emptyset$.*

Note that if $\mathcal{I}$ is supersaturated, then it is also $\aleph_1$-saturated which means that every disjoint subfamily of $\mathcal{I}^+$ is countable. It follows that if there is a supersaturated ideal on $\kappa$, then either $\kappa \leq \mathfrak{c}$ or $\kappa$ is a measurable cardinal.

**Lemma 4.2.** *Suppose $\kappa$ is regular uncountable and there is a supersaturated ideal on $\kappa$. Then for every $\theta < \kappa$, $\star(\theta, \kappa)$ holds.*

*Proof.* Let $\mathcal{I}$ be a supersaturated ideal on $\kappa$ and $\theta < \kappa$. Let $\{<_i : i < \theta\}$ be a family of linear orders on $\kappa$. We'll construct $A \in [\kappa]^{\aleph_0}$ and $\alpha \in \kappa \setminus A$ such that for every $i < \theta$, some member of $A$ is $<_i$-above $\alpha$. For each $i < \theta$, let $R_i = \{\alpha < \kappa : \{\beta < \kappa : \beta \geq_i \alpha\} \in \mathcal{I}\}$. Let $\Gamma = \{i < \theta : R_i \in \mathcal{I}^+\}$. Choose $A_0 \in [\kappa]^{\aleph_0}$ such that $A_0 \cap R_i$ is infinite for each $i \in \Gamma$. Let $W = \kappa \setminus \bigcup_{i\in\Gamma}\{\beta < \kappa : (\exists \alpha \in A_0 \cap R_i)(\beta \geq_i \alpha)\}$. Note that $\kappa \setminus W \in \mathcal{I}$. Choose $\alpha \in W \setminus \bigcup_{i\in\theta\setminus\Gamma} R_i$. Choose $A_1 \in [\kappa]^{\aleph_0}$ such that it meets $\{\beta < \kappa : \beta \geq_i \alpha\}$ at an infinite set for every $i \in \theta \setminus \Gamma$. Then $A = (A_0 \cup A_1) \setminus \{\alpha\}$, $\alpha$ are as required. □

**Corollary 4.3.** *Suppose that there is supersaturated ideal on $\mathfrak{c}$. Then $\mathfrak{dim}_T = \mathfrak{c}$.*

*Proof.* By Lemma 4.2, the order dimension of $\mathbb{H}_\mathfrak{c}$ is $\mathfrak{c}$. By Theorem 2.2, $\mathbb{H}_\mathfrak{c}$ embeds into the Turing degrees. Hence the order dimension of Turing degrees is also $\mathfrak{c}$. □

In the remainder of this section, we produce some models in which there is a supersaturated ideal on $\mathfrak{c}$.

**Theorem 4.4.** *Suppose $\kappa$ is measurable and let $\mathcal{I}$ be a normal ideal on $\kappa$. Let $\mathbb{P} \in \{\mathsf{Cohen}_\kappa, \mathsf{Random}_\kappa\}$. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}}$. Then the following hold in $V^{\mathbb{P}}$*

1. $\mathfrak{c} = \kappa$.
2. *$\mathcal{J}$ is a supersaturated ideal on $\kappa$.*

*Proof.* Fix a witnessing normal measure $m : \mathcal{P}(\kappa) \to 2$ such that $\mathcal{I} = \{X \subseteq \kappa : m(X) = 0\}$. First suppose $\mathbb{P} = \mathsf{Cohen}_\kappa = \mathbb{C}$. We think of $\mathbb{C}$ as the complete Boolean algebra of Baire subsets of $2^\kappa$ modulo the meager ideal on $2^\kappa$. In $V[G_\mathbb{C}]$, let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$. It is easy to check that $\mathcal{J}$ is a normal ideal on $\kappa$. To show that $\mathcal{J}$ is supersaturated, it suffices to prove the following.

**Claim 4.5.** *Suppose $p \in \mathbb{C}$, $\theta < \kappa$ and $\mathring{A}_i \in V^{\mathbb{C}} \cap \mathcal{P}(\kappa)$ for $i < \theta$ are such that $p \Vdash \mathring{A}_i \in \mathcal{J}^+$ for every $i < \theta$. Then there exists $X \in [\kappa]^{\aleph_0}$ such that $p \Vdash (\forall i < \theta)(\mathring{A}_i \cap X \neq \emptyset)$.*

*Proof.* Without loss of generality, $p = 1_\mathbb{C}$. Let $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]]_\mathbb{C}$. Note that for every $X \subseteq \kappa$, if $m(X) = 1$, then $\bigcup_{\alpha\in X} p_{i,\alpha} = 1_\mathbb{C}$ since otherwise, the complement of $\bigcup_{\alpha\in X} p_{i,\alpha}$ will force that $\mathring{A}_i \in \mathcal{J}$. Choose $X \subseteq \kappa$ such that $m(X) = 1$ and for every $\alpha \in X$ and $i < \theta$, $p_{i,\alpha} \neq 0_\mathbb{C}$. Let $\mathsf{supp}(p_{i,\alpha}) = S_{i,\alpha} \in [\kappa]^{\aleph_0}$. Using the normality of $\mathcal{I}$, for each $i < \theta$, choose $Y_i \subseteq X$ such that $m(Y_i) = 1$ and for every $\alpha, \beta \in Y_i$,

- $S_{i,\alpha}$ and $S_{i,\beta}$ have the same order type,
- $S_{i,\alpha} \cap \alpha = W_i$ does not depend on $\alpha$ and
- $(2^{S_{i,\alpha}}, p_{i,\alpha}) \cong (2^{S_{i,\beta}}, p_{i,\beta})$ which means that the unique order preserving bijection from $S_{i,\alpha}$ to $S_{i,\beta}$ sends $p_{i,\alpha}$ to $p_{i,\beta}$.

For $i < \theta$ and $\alpha \in Y_i$, put

$$B_{i,\alpha} = \{x \in 2^{W_i} : \{y \restriction (\kappa \setminus W_i) : y \in p_{i,\alpha} \wedge y \restriction W_i = x\} \text{ is meager}\}$$

Note that $B_{i,\alpha} = B_i$ does not depend on $\alpha \in Y_i$. Also, $B_i$ is meager in $2^{W_i}$, otherwise $\bigcup_{\alpha\in Y_i} p_{i,\alpha}$ would not be open dense in $2^\kappa$.

Let $Y = \bigcap_{i<\theta} Y_i$. Choose $X \in [Y]^{\aleph_0}$ such that for every $i < \theta$, $\{S_{i,\alpha} \setminus W_i : \alpha \in X\}$ has pairwise disjoint members. It follows that for every $i < \theta$, $\bigcup_{\alpha\in X} p_{i,\alpha}$ is open dense in $2^\kappa$. Hence $\Vdash_\mathbb{C} (\forall i < \theta)(X \cap \mathring{A}_i \neq \emptyset)$. □

Next suppose $\mathbb{P} = \mathsf{Random}_\kappa = \mathbb{B}$. $\mathbb{B}$ is the measure algebra on $2^\kappa$ and $\mu$ is the corresponding product measure. In $V[G_{\mathbb{B}}]$, define $m_1 : \mathcal{P}(\kappa) \to [0,1]$ by

$$m_1(A) = \sup_{p \in G_{\mathbb{B}}} \inf_{q \leq p} \int_\kappa \frac{\mu([[\alpha \in \mathring{A}]]_{\mathbb{B}} \cap q)}{\mu(q)} dm$$

It is well known that $m_1$ is a $\kappa$-additive atomless probability measure in $V[G_{\mathbb{B}}]$ whose null ideal $\mathcal{J}$ is the ideal generated by $\mathcal{I}$ (See [12]). To show that $\mathcal{J}$ is supersaturated, it suffices to prove the following.

**Claim 4.6.** *Suppose $\varepsilon > 0$, $\theta < \kappa$, $p \in \mathbb{B}$ and $\mathring{A}_i \in V^{\mathbb{B}} \cap \mathcal{P}(\kappa)$ for $i < \theta$ are such that $p \Vdash m_1(\mathring{A}_i) > \varepsilon$ for every $i < \theta$. Then there exists $A \in [\kappa]^{\aleph_0}$ such that $p \Vdash (\forall i < \theta)(\mathring{A}_i \cap A \neq \emptyset)$.*

Without loss of generality, $p = 1_{\mathbb{B}}$. Let $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]]_{\mathbb{B}}$.

**Subclaim 4.7.** *For every $i < \theta$ there exists $X_i \subseteq \kappa$ such that $m(X_i) = 1$ and for every $q \in \mathbb{B}$, $\alpha \in X_i$,*

$$(\star) \qquad \frac{\mu(p_{i,\alpha} \cap q)}{\mu(q)} \geq \varepsilon$$

*Proof.* Suppose not and fix $i < \theta$, $q \in \mathbb{B}$ and $X \subseteq \kappa$ such that $m(X) = 1$ and for every $\alpha \in X$,

$$\frac{\mu(p_{i,\alpha} \cap q)}{\mu(q)} < \varepsilon$$

Since $\Vdash m_1(\mathring{A}_i) > \varepsilon$, using the definition of $m_1$, we can find a maximal antichain $\{q_n : n < \omega\}$ below $q$ such that for every $n < \omega$,

$$\int_\kappa \frac{\mu(p_{i,\alpha} \cap q_n)}{\mu(q_n)} dm > \varepsilon$$

Since $\{q_n : n < \omega\}$ is a partition of $q$, for each $\alpha \in X$, we can find $n_\alpha < \omega$ such that $\mu(p_{i,\alpha} \cap q_{n_\alpha}) < \varepsilon\mu(q_{n_\alpha})$. Choose $Y \subseteq X$ such that $m(Y) = 1$ and $n_\alpha = n_\star$ does not depend on $\alpha \in Y$. But this implies that

$$\int_\kappa \frac{\mu(p_{i,\alpha} \cap q_{n_\star})}{\mu(q_{n_\star})} dm < \varepsilon$$

which is impossible. $\square$

Fix $X_i$ as in Subclaim 4.7. Let $\mathsf{supp}(p_{i,\alpha}) = S_{i,\alpha} \in [\kappa]^{\aleph_0}$. Since $m$ is normal, for each $i < \theta$, we can choose $Y_i \subseteq X_i$ such that $m(Y_i) = 1$ and for every $\alpha, \beta \in Y_i$,

- $S_{i,\alpha}$ and $S_{i,\beta}$ are order isomorphic,
- $S_{i,\alpha} \cap \alpha = W_i$ does not depend on $\alpha$ and
- $(2^{S_{i,\alpha}}, p_{i,\alpha}) \cong (2^{S_{i,\beta}}, p_{i,\beta})$ which means that the unique order preserving bijection from $S_{i,\alpha}$ to $S_{i,\beta}$ sends $p_{i,\alpha}$ to $p_{i,\beta}$.

For $i < \theta$ and $\alpha \in Y_i$, put

$$B_{i,\alpha} = \{x \in 2^{W_i} : \mu_{\kappa \setminus W_i}(\{y \restriction (\kappa \setminus W_i) : y \in p_{i,\alpha} \wedge y \restriction W_i = x\}) \leq \varepsilon/2\}$$

Note that $B_{i,\alpha} = B_i$ does not depend on $\alpha \in Y_i$. Also $\mu_{W_i}(B_i) = 0$ since otherwise, by Fubini's theorem, $q = \{x \in 2^\kappa : x \restriction W_i \in B_i\}$ would violate ($\star$).

Let $Y = \bigcap_{i<\theta} Y_i$. Choose $X \in [Y]^{\aleph_0}$ such that for every $i < \theta$, $\langle S_{i,\alpha} : \alpha \in X\rangle$ forms a $\Delta$-system with root $W_i$. To complete the proof of Claim 4.6, it suffices to show the following.

**Claim 4.8.** *For every $i < \theta$, $\mu(\bigcup_{\alpha \in X} p_{i,\alpha}) = 1$.*

*Proof.* Since $\mu_{W_i}(B_i) = 0$ and $\langle S_{i,\alpha} \setminus W_i : \alpha \in X\rangle$ is a sequence of pairwise disjoint sets, it follows that for every $x \in 2^{W_i} \setminus B_i$,

$$\{y \restriction (\kappa \setminus W_i) : y \in 2^\kappa \wedge y \restriction W_i = x \wedge (\forall \alpha \in X)(y \notin p_{i,\alpha})\}$$

is null in $2^{\kappa \setminus W_i}$. So by Fubini's theorem, $\mu(\{y \in 2^\kappa : (\exists \alpha \in X)(y \in p_{i,\alpha})\}) = 1$. □

This completes the proof of Claim 4.6 and Theorem 4.4. □

We can generalize the proof of the $\mathbb{P} = \mathsf{Cohen}_\kappa$ case in Theorem 4.4 to a finite support iteration of small ccc forcings as follows.

**Theorem 4.9.** *Suppose $\mathcal{I}$ is a normal supersaturated ideal on $\kappa$. Let $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \kappa\rangle$ be a finite support iteration with limit $\mathbb{P}$ where $\mathbb{Q}_i$ is a ccc poset in $V^{\mathbb{P}_i}$ and $|\mathbb{P}_i| < \kappa$ for every $i < \kappa$. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}}$. Then $\mathcal{J}$ is a supersaturated ideal on $\kappa$.*

*Proof.* Since $\mathbb{P}$ is ccc and $\mathcal{I}$ is normal, $\mathcal{J}$ is a normal ideal on $\kappa$. Suppose $\theta < \kappa$ and $p \Vdash_{\mathbb{P}} \mathring{A}_i \in \mathcal{J}^+$ for every $i < \theta$. Without loss of generality, $p$ is the empty condition. Let $P_{i,\alpha}$ be a (possibly empty) maximal family of pairwise incompatible conditions in $\mathbb{P}$ each of which forces that $\alpha \in \mathring{A}_i$. Let $T_i = \{\alpha < \kappa : P_{i,\alpha} \neq \emptyset\}$. Then $T_i \in \mathcal{I}^+$. For each $\alpha \in T_i$, choose $S_{i,\alpha} \in [\kappa]^{\aleph_0}$ such that for every $p \in \mathbb{P}_{i,\alpha}$, $\mathsf{dom}(p) \subseteq S_{i,\alpha}$.

**Claim 4.10.** *For each $i < \theta$, we can find $\mathcal{F}_i \subseteq \mathcal{I}^+$ and $\langle (\delta_{i,Y}, W_{i,Y}) : Y \in \mathcal{F}_i\rangle$ such that the following hold.*

(1) *$\mathcal{F}_i$ is a countable family of pairwise disjoint sets and $T_i \setminus \bigcup \mathcal{F}_i \in \mathcal{I}$.*
(2) *For each $Y \in \mathcal{F}_i$,*
   *(a) $\delta_{i,Y} < \kappa$ and $W_{i,Y} \in [\mathbb{P}_{\delta_{i,Y}}]^{\aleph_0}$,*
   *(b) for every $\alpha \in Y$, $S_{i,\alpha} \cap \alpha \subseteq \delta_{i,Y}$ and*
   *(c) for every $\alpha \in Y$, $\{p \restriction \delta_{i,Y} : p \in P_{i,\alpha}\} \subseteq W_{i,Y}$.*

*Proof.* It is enough to show that for every $T \subseteq T_i$, $T \in \mathcal{I}^+$, there exists $Y \subseteq T$, $Y \in \mathcal{I}^+$ such that there are $\delta_{i,Y}$ and $W_{i,Y} \in [\mathbb{P}_{\delta_{i,Y}}]^{\aleph_0}$ such that Clauses 2(a)-(c) above hold. For then we can take $\mathcal{F}_i$ to be a maximal disjoint family of such $Y$'s. Suppose $T \subseteq T_i$ and $T \in \mathcal{I}^+$. Since $\mathcal{I}$ is an $\aleph_1$-saturated normal ideal on $\kappa$, it concentrates on the set of weakly inaccessible cardinals below $\kappa$. So we can assume that every $\alpha \in T$ is weakly inaccessible. As the map $\alpha \mapsto \sup(S_{i,\alpha} \cap \alpha)$ is regressive on $T$, we can choose $Y_1 \subseteq T$ on which it is constant say $\delta < \kappa$. We need the following.

**Subclaim 4.11.** *Suppose $\gamma < \kappa$ and $f : \kappa \to [\gamma]^{\leq \aleph_0}$. Then, there exists $W \in [\kappa]^{\aleph_0}$ such that $\{\alpha < \kappa : f(\alpha) \nsubseteq W\} \in \mathcal{I}$.*

*Proof.* Towards a contradiction, fix $\gamma < \kappa$ least such that there exists $f : \kappa \to [\gamma]^{\leq \aleph_0}$ such that for every $W \in [\kappa]^{\aleph_0}$, $\{\alpha < \kappa : f(\alpha) \nsubseteq W\} \in \mathcal{I}^+$.

We claim that $\mathsf{cf}(\gamma) \geq \aleph_1$. Suppose not and let $\{\gamma_n : n < \omega\}$ be increasing cofinal in $\gamma$. Let $f_n : \kappa \to [\gamma_n]^{\leq \aleph_0}$ be defined by $f_n(\alpha) = f(\alpha) \cap \gamma_n$. Note that $f(\alpha) = \bigcup_{n<\omega} f_n(\alpha)$. Since $\gamma_n < \gamma$, we can find $W_n \in [\gamma_n]^{\aleph_0}$ such that $\{\alpha < \kappa : f_n(\alpha) \nsubseteq W_n\} \in \mathcal{I}$. Let $W = \bigcup_{n<\omega} W_n$. Then $\{\alpha < \kappa : f(\alpha) \nsubseteq W\} \in \mathcal{I}$: Contradiction. So $\mathsf{cf}(\gamma) \geq \aleph_1$.

Since $\mathcal{I}$ is $\aleph_1$-saturated, we can choose a countable family $\mathcal{F} \subseteq \mathcal{I}^+$ of pairwise disjoint sets such that $\kappa \setminus \bigcup \mathcal{F} \in \mathcal{I}$ and for every $Z \in \mathcal{F}$, there exists $\gamma_Z < \gamma$ such that for every $\alpha \in Z$, $\sup(f(\alpha)) < \gamma_Z$. For each $Z \in \mathcal{F}$, define $g_Z : \kappa \to [\gamma_Z]^{\leq \aleph_0}$ by $g_Z(\alpha) = f(\alpha) \cap \gamma_Z$. As $\gamma_Z < \gamma$, we can find $W_Z \in [\kappa]^{\aleph_0}$ such that $\{\alpha < \kappa : g_Z(\alpha) \nsubseteq W_Z\} \in \mathcal{I}$. Put $W = \bigcup_{Z \in \mathcal{F}} W_Z$. Then $\{\alpha < \kappa : f(\alpha) \nsubseteq W\} \in \mathcal{I}$: Contradiction. $\square$

Since $|\mathbb{P}_\delta| < \kappa$, using Subclaim 4.11, we can find $Y \subseteq Y_1$, $Y \in \mathcal{I}^+$ and $W \in [\mathbb{P}_\delta]^{\aleph_0}$ such that for every $\alpha \in Y$, $\{p \restriction \delta : p \in P_{i,\alpha}\} \subseteq W$. Put $\delta_{i,Y} = \delta$ and $W = W_{i,Y}$. This completes the proof of Claim 4.10. $\square$

Fix $\mathcal{F}_i$ and $\{(\delta_{i,Y}, W_{i,Y}) : Y \in \mathcal{F}_i\}$ as in Claim 4.10. Fix $Y \in \mathcal{F}_i$. For $\alpha \in Y$, let $B_{i,\alpha} = \{p \restriction \delta_{i,Y} : p \in P_{i,\alpha}\}$. For each $q \in W_{i,Y}$, let $X_{i,Y,q} = \{\alpha \in Y : q \in B_{i,\alpha}\}$. Let $W'_{i,Y} = \{q \in W_{i,Y} : X_{i,Y,q} \in \mathcal{I}^+\}$ and $Y' = \{\alpha \in Y : B_{i,\alpha} \subseteq W'_{i,Y}\}$. Note that $Y \setminus Y' \in \mathcal{I}$.

**Claim 4.12.** *$\mathcal{W} = \bigcup\{W'_{i,Y} : Y \in \mathcal{F}_i\}$ is predense in $\mathbb{P}$.*

*Proof.* Suppose not and fix $q \in \mathbb{P}$ incompatible with every member of $\mathcal{W}$. We claim that for every $Y \in \mathcal{F}_i$ and $\alpha \in Y'$, $q \Vdash \alpha \notin \mathring{A}_i$. To see this, note that $B_{i,\alpha} \subseteq W'_{i,Y}$ and $q$ is incompatible with every member of $W'_{i,Y}$ so that it is also incompatible with every member of $P_{i,\alpha}$. But this means that $q$ forces that $\mathring{A}_i \cap \bigcup\{Y' : Y \in \mathcal{F}_i\}$ is empty and hence $q \Vdash \mathring{A}_i \in \mathcal{J}$ which is impossible. $\square$

**Claim 4.13.** *There exists $X \in [\kappa]^{\aleph_0}$ such that for every $i < \theta$, $Y \in \mathcal{F}_i$ and $q \in W'_{i,Y}$, there exists $X_q \in [X]^{\aleph_0}$ such that $X_q \subseteq X_{i,Y,q}$ and $\{S_{i,\alpha} \setminus \delta_{i,Y} : \alpha \in X_q\}$ consists of pairwise disjoint sets.*

*Proof.* Note that $\mathcal{A} = \{X_{i,Y,q} : i < \theta, Y \in \mathcal{F}_i, q \in W'_{i,Y}\} \subseteq \mathcal{I}^+$ has size $\leq |\theta|$. Construct an increasing sequence $\langle X_k : k < \omega \rangle$ of members of $[\kappa]^{\aleph_0}$ as follows. Since $\mathcal{I}$ is supersaturated, we can find $X_0 \in [\kappa]^{\aleph_0}$ that meets every member of $\mathcal{A}$. Suppose $X_k$ for $k \leq n$ has been defined. For each $i < \theta$, $Y \in \mathcal{F}_i$ and $q \in W'_{i,Y}$, put $X^n_{i,Y,q} = \{\alpha \in X_{i,Y,q} : (\forall \beta \in X_n)(S_{i,\alpha} \cap S_{i,\beta} \subseteq \delta_{i,Y})\}$. Note that $|X_{i,Y,q} \setminus X^n_{i,Y,q}| < \kappa$ so that each $X^n_{i,Y,q} \in \mathcal{I}^+$. Put $\mathcal{A}_n = \{X^n_{i,Y,q} : i < \theta, Y \in \mathcal{F}_i, q \in W'_{i,Y}\}$. Choose $X_{n+1} \in [\kappa]^{\aleph_0}$ such that $X_n \subseteq X_{n+1}$ and $X_{n+1}$ meets every member of $\mathcal{A}_n$. It is clear that $X = \bigcup_{k<\omega} X_k$ is as required. $\square$

We claim that for every $i < \theta$, $\bigcup_{\alpha \in X} P_{i,\alpha}$ is predense in $\mathbb{P}$. Fix $i < \theta$ and $p \in \mathbb{P}$. By Claim 4.12, we can find $Y \in \mathcal{F}_i$ and $q \in W'_{i,Y}$ such that $p, q$ are compatible. Choose $X_q \in [X]^{\aleph_0}$ as in Claim 4.13. Since $\{S_{i,\alpha} \setminus \delta_{i,Y} : \alpha \in X_q\}$ has pairwise disjoint members, we can find $\alpha \in X_q$ such that $\mathsf{dom}(p) \cap S_{i,\alpha} \subseteq \delta_{i,Y}$. Since $X_q \subseteq X_{i,Y,q}$, we can find $r \in P_{i,\alpha}$ such that $r \restriction \delta_{i,Y} = q$. It follows that $q, r$ are compatible. Hence the empty condition forces that $X$ meets $\mathring{A}_i$ for every $i < \theta$. This completes the proof of Theorem 4.9. $\square$

**Corollary 4.14.** *Suppose $\kappa$ is measurable. There exists a ccc forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, Martin's axiom holds and $\mathfrak{dim}_T = \mathfrak{c} = \kappa$.*

*Proof.* Take $\mathbb{P}$ to be the usual finite support iteration of ccc forcings for forcing MA and $\mathfrak{c} = \kappa$ and apply Theorem 4.9 and Corollary 4.3. $\square$

It is natural to ask if every $\aleph_1$-saturated normal ideal $\mathcal{I}$ on $\kappa$ must necessarily be supersaturated. In the case where $\kappa$ is real valued measurable and $\mathcal{I}$ is the null ideal of a witnessing normal measure on $\kappa$, this is Problem EG(h) in Fremlin's list [5].

**Question 4.15.** [1] *Suppose $\mathcal{I}$ is an $\aleph_1$-saturated normal ideal on $\kappa$. Must $\mathcal{I}$ be supersaturated?*

## 5. A closer look

One of the goals of this section is to show, for example, that each of the following is consistent.

(1) $\aleph_1 < \mathfrak{dim}_T < \mathfrak{c}$.
(2) $\mathfrak{dim}_T = \mathfrak{c} = \aleph_{\omega_1}$.
(3) $\mathfrak{dim}_T = \mathfrak{c} = \aleph_{\omega+1}$.

Another goal is to show that separating countable sets from points using linear orders really is easier than separating countable sets from points using functions. We begin by showing that the least size of an $(\aleph_1, 2)$-separating family of subsets of $\mathfrak{c}$ could be strictly larger than $\mathsf{odim}(\mathbb{H}_{\mathfrak{c}})$.

A family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is called a *weak P-family* if for every countable $\mathcal{B} \subseteq \mathcal{F}$ and $E' \in \mathcal{F} \setminus \mathcal{B}$, there exists a finite set $F \subseteq \kappa$ so that

$$\forall E \in \mathcal{B}\, [E \cap F \neq E' \cap F]\,.$$

When a weak P-family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is viewed as a subspace of $2^\kappa$, $\mathcal{F}$ has the property that all of its countable subsets are relatively closed in $\mathcal{F}$. Thus, every member of $\mathcal{F}$ is a weak P-point in $\mathcal{F}$. This notion has been extensively studied by topologists (see, for example, [10]). In [7], it was proved that for any uncountable cardinal $\theta$, the minimal size of an $(\aleph_1, 2)$-separating family of subsets of $\theta$ is equal to the minimal $\lambda$ for which there is a weak P-family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$ with $|\mathcal{F}| \geq \theta$. It was also shown in [7] that when $\mathfrak{c} \leq \aleph_2$, there is a weak P-family $\mathcal{F} \subseteq \mathcal{P}(\aleph_1)$ with $|\mathcal{F}| = \mathfrak{c}$. In May 2018, Kunen suggested to the second author in a conversation that there ought to be no weak P-families of size $\mathfrak{c}$ in $\mathcal{P}(\aleph_1)$ after adding $\aleph_3$ Cohen reals to a model of GCH. Our next lemma verifies Kunen's remark.

**Lemma 5.1.** *Suppose $\aleph_0 \leq \theta$, $\kappa = (2^\theta)^+$ and $\mathbb{P} = \mathsf{Cohen}_\kappa$. Then, in $V^{\mathbb{P}}$, there is no $(\aleph_1, 2)$-separating family of subsets of $\kappa$ of size $\theta$.*

*Proof.* Suppose not. In $V^{\mathbb{P}}$, fix an $(\aleph_1, 2)$-separating family $\langle \mathring{A}_i : i < \theta \rangle$ of subsets of $\kappa$. Without loss of generality, assume that the empty condition forces this. We'll find $X \in [\kappa]^{\aleph_0}$ and $\alpha \in (\kappa \setminus X)$ such that $\Vdash_{\mathbb{P}} (\forall i < \theta)(\alpha \notin \mathring{A}_i \vee X \cap \mathring{A}_i \neq \emptyset)$. Recall that the non-stationary ideal on $\kappa$ is $\kappa$-complete as $\kappa$ is regular uncountable. Let $T = \{\alpha < \kappa : \mathsf{cf}(\alpha) \geq \theta^+\}$. $T$ is stationary, and for each $\alpha \in T$, let $G_\alpha = \left\{ i < \theta : \exists p \in \mathbb{P} \left[ p \Vdash_{\mathbb{P}} \alpha \in \mathring{A}_i \right] \right\}$. Since $2^\theta < \kappa$, there is a stationary subset $T' \subseteq T$ and a set $G \subseteq \theta$ such that $G = G_\alpha$, for every $\alpha \in T'$. For each $\alpha \in T'$ and

[1] The answer is negative.

$i \in G$, choose a sequence $\langle p_{\alpha,i,m} : m < \omega \rangle$ with the property that for each $m < \omega$, $p_{\alpha,i,m} \Vdash_{\mathbb{P}} \alpha \in \mathring{A}_i$, and for any $p \in \mathbb{P}$, if $p \Vdash_{\mathbb{P}} \alpha \in \mathring{A}_i$, then for some $m < \omega$, $p \not\perp_{\mathbb{P}} p_{\alpha,i,m}$. For $\alpha \in T'$, define $S_\alpha = \bigcup\{\mathsf{dom}(p_{\alpha,i,m}) : i \in G \text{ and } m < \omega\}$. Note that $|S_\alpha| \leq \theta$. Since $T'$ is a stationary set consisting of ordinals of cofinality greater than $\theta$, and since $2^\theta < \kappa$, there is a stationary set $E \subseteq T'$ satisfying the following:

- there is an $S \subseteq \kappa$ such that for each $\alpha \in E$, $S = S_\alpha \cap \alpha$;
- for any $\alpha, \beta \in E$, if $\alpha < \beta$, then $S_\alpha \subseteq \beta$, which implies that $(S_\alpha \setminus S) \cap (S_\beta \setminus S) = 0$;
- there exists a sequence $\langle p_{i,m} : i \in G \wedge m < \omega \rangle$ of conditions in $\mathrm{Fn}(S, 2)$ so that

$$\forall \alpha \in E \, \forall i \in G \, \forall m < \omega \, [p_{\alpha,i,m} \restriction S = p_{i,m}] \, .$$

Consider any $X \in [E]^{\aleph_0}$ and $\alpha \in (E \setminus X)$. Suppose for a contradiction that there is $p \in \mathbb{P}$ and $i < \theta$ so that $p \Vdash_{\mathbb{P}} \alpha \in \mathring{A}_i \wedge X \cap \mathring{A}_i = 0$. By definition, $i \in G$. Fix $m < \omega$ and $q \in \mathbb{P}$, so that $q \leq p, p_{\alpha,i,m}$. Choose any $\beta \in X$ with $\mathsf{dom}(q) \cap (S_\beta \setminus S) = 0$. Note that $q \restriction S \leq p_{\alpha,i,m} \restriction S = p_{i,m} = p_{\beta,i,m} \restriction S$. Since $\mathsf{dom}(q) \cap \mathsf{dom}(p_{\beta,i,m}) \subseteq S$, there is $r \leq q, p_{\beta,i,m}$. However this means that $r \Vdash_{\mathbb{P}} \beta \in \mathring{A}_i$, contradicting $p \Vdash_{\mathbb{P}} X \cap \mathring{A}_i = 0$, and finishing the proof. □

Thus if we start with a model of GCH and add $\aleph_3$ Cohen reals, then there are no $(\aleph_1, 2)$-separating families of subsets of $\aleph_3$ having size $\aleph_1$. As a consequence, there are no weak P-families $\mathcal{F} \subseteq \mathcal{P}(\aleph_1)$ of size $\mathfrak{c}$. A result from [7] says that if $\kappa$ is any cardinal with $\mathsf{cf}(\kappa) > \omega$, then there is an $(\aleph_1, 2)$-separating family of subsets of $\kappa^+$ of size $\kappa$. In particular, there is always an $(\aleph_1, 2)$-separating family of subsets of $\aleph_3$ of size $\aleph_2$. On the other hand, it is a consequence of a theorem of Kierstead and Milner that there is a family of $\aleph_1$ many linear orders on $\aleph_3$ separating countable sets from points in the model obtained by adding $\aleph_3$ Cohen reals to a model of GCH. The following lemma follows from Theorem 1.2 in [8].

**Lemma 5.2.** *Suppose $\theta^{\aleph_0} = \theta$ and $\kappa = \beth_2(\theta)$. Then $\mathsf{odim}(\mathbb{H}_\kappa) \leq \theta$. If $2^{<\theta} = \theta$, then $\mathsf{odim}(\mathbb{H}_\kappa) = \theta$.*

Assuming GCH in $V$, Lemma 5.2 implies that $\mathsf{odim}(\mathbb{H}_{\aleph_3}) = \aleph_1$ in $V$. So there is a family $\langle <_i : i \in \aleph_1 \rangle$ of linear orders on $\aleph_3$ separating points from countable sets in $V$. If $\mathbb{P} = \mathsf{Cohen}_{\aleph_3}$ and if $B$ is any countable subset of $\aleph_3$ in $V^{\mathbb{P}}$ and $\alpha \in \aleph_3 \setminus B$, then there is a countable set $A$ in $V$ with $B \subseteq A \subseteq \aleph_3 \setminus \{\alpha\}$ because $\mathbb{P}$ is proper. Now there is an $i \in \theta$ so that $<_i$ puts $\alpha$ above every element of $A$, and hence above every element of $B$. Thus $\langle <_i : i \in \aleph_1 \rangle$ still separates points from countable subsets of $\aleph_3$ in $V^{\mathbb{P}}$. Therefore after adding $\aleph_3$ Cohen reals to a model of GCH, we find that separating points from countable sets using linear orders on $\aleph_3$ becomes provably easier than separating them using subsets of $\aleph_3$. Moreover by Lemma 2.8, $\mathsf{odim}(\mathbb{H}_{\aleph_3}) = \aleph_1$, even though there are no $(\aleph_1, 2)$-separating families on $\aleph_3$ of size $\aleph_1$.

Lemma 5.1 should be compared to the classical result of Baumgartner [2] (see also Exercise B4 in Chapter VIII of Kunen [9]) saying that after adding $\aleph_3$ Cohen reals to a model of GCH, there are no almost disjoint families of subsets of $\aleph_1$ having size $\aleph_3$. By Lemma 3.7 of [7], if there is an almost disjoint family of subsets of $\aleph_1$ of size $\aleph_3$, then there is an $(\aleph_1, 2)$-separating family of subsets of $\aleph_3$ of size $\aleph_1$. Therefore Lemma 5.1 implies Baumgartner's Theorem about almost disjoint families. Baumgartner used a partition relation of Erdős and Rado to prove his result. It is noteworthy that partition calculus is not directly involved in the proof of Lemma 5.1. Nevertheless, we do not know if Baumgartner's result is strictly weaker than Lemma 5.1. So we propose the following question.

**Question 5.3.** *Is it consistent with* ZFC *that* $\mathfrak{c} = \aleph_3$ *and there is no almost disjoint family of subsets of* $\aleph_1$ *of size* $\aleph_3$*, but there is an* $(\aleph_1, 2)$*-separating family of subsets of* $\aleph_3$ *of size* $\aleph_1$*?*

The following theorem shows that certain lower bounds on $\mathsf{odim}(\mathbb{H}_\kappa)$ are preserved after adding $\kappa$ random/Cohen reals. Note that Lemma 5.2 and the remarks following it imply that $(\beth_2(\theta))^+$ cannot be replaced with $\beth_2(\theta)$ in the statement of Theorem 5.4. The proof of Theorem 5.4 makes use of a standard "double-delta system" argument. Such double-delta system arguments were used extensively by Todorcevic in [14] to obtain results on partition calculus.

**Theorem 5.4.** *Suppose* $\aleph_0 \leq \theta$*,* $\kappa = (\beth_2(\theta))^+$ *and* $\mathbb{P} \in \{\mathsf{Cohen}_\kappa, \mathsf{Random}_\kappa\}$*. Then* $V^{\mathbb{P}}$ *satisfies* $\star(\theta, \kappa)$*. Hence* $V^{\mathbb{P}} \models \mathsf{odim}(\mathbb{H}_\kappa) \geq \theta^+$*.*

*Proof.* We first give the proof for $\mathbb{P} = \mathsf{Random}_\kappa$ and then comment on how to modify it to deal with the Cohen case. Let $\{<_i : i < \theta\}$ be a family of linear orders on $\kappa$ in $V^{\mathbb{P}}$. For $i < \theta$ and $\alpha < \beta < \kappa$, let $p_{i,\alpha,\beta} = [[\alpha <_i \beta]]_{\mathbb{P}}$ and $\bar{p}_{i,\alpha,\beta} = [[\beta <_i \alpha]]_{\mathbb{P}}$. So $p_{i,\alpha,\beta}$ and $\bar{p}_{i,\alpha,\beta}$ are disjoint Baire subsets of $2^\kappa$ and their union has measure one. Let $S_{i,\alpha,\beta} \in [\kappa]^{\aleph_0}$ contain their common support.

Let $\gamma_{i,\alpha,\beta} = \mathsf{otp}(S_{i,\alpha,\beta})$ and $h_{i,\alpha,\beta} : \gamma_{i,\alpha,\beta} \to S_{i,\alpha,\beta}$ be the unique order isomorphism. For $r \in \mathsf{Random}_{\gamma_{i,\alpha,\beta}}$, let $r_{i,\alpha,\beta} \in \mathsf{Random}_\kappa$ be obtained by applying the isomorphism induced by $h_{i,\alpha,\beta}$ – So $r_{i,\alpha,\beta} = \{x \in 2^\kappa : (\exists y \in r)(y \circ h^{-1}_{i,\alpha,\beta} = x \restriction S_{i,\alpha,\beta})\}$.

Let $\mathcal{F} = \{g_{i,0}, g_{i,1}, f_{i,\gamma} : i < \theta, \gamma < \omega_1\}$ where $g_{i,0}$, $g_{i,1}$ and $f_{i,\gamma}$ are defined as follows.

(i) $g_{i,0} : [\kappa]^2 \to \omega_1$ and for every $\alpha < \beta < \kappa$, $g_{i,0}(\{\alpha, \beta\}) = \gamma_{i,\alpha,\beta}$.
(ii) $g_{i,1} : [\kappa]^2 \to \mathsf{Random}_{\omega_1}$ and for every $\alpha < \beta < \kappa$, $g_{i,0}(\{\alpha, \beta\}) = r$ where $r_{i,\alpha,\beta} = p_{i,\alpha,\beta}$.
(iii) $f_{i,\gamma} : [\kappa]^2 \to \kappa$ and for every $\alpha < \beta < \kappa$, $f_{i,\gamma}(\{\alpha, \beta\}) = h_{i,\alpha,\beta}(\gamma)$ if $\gamma < \gamma_{i,\alpha,\beta}$ and 0 otherwise.

Suppose $f : [\kappa]^2 \to \kappa$ and $Y \subseteq \kappa$. We say that $f$ is *canonical* on $Y$ if for some $\Delta_f \subseteq 2$, for every $\alpha_0 < \alpha_1$ and $\beta_0 < \beta_1$ in $Y$, $f(\{\alpha_0, \alpha_1\}) = f(\{\beta_0, \beta_1\})$ iff $(\forall k \in \Delta_f)(\alpha_k = \beta_k)$. Erdős and Rado [4] showed that functions from $[\omega]^n$ to $\omega$ can be canonized on some infinite subset of $\omega$, for all $n < \omega$. We would like to canonize all of the functions in the family $\mathcal{F}$ defined above on a sufficiently large subset of $\kappa$. The fact that this can be done follows from Theorem 1 in Baumgartner [1], and the reason for taking $\kappa = (\beth_2(\theta))^+$ is precisely so that this theorem of Baumgartner is applicable. Much more general canonization theorems are derivable from the partition relations for partial orders established in Todorcevic [13].

**Fact 5.5** (Baumgartner [1])**.** *Suppose* $\lambda < \mu$ *and* $\rho = (2^{<\mu})^+$*. Suppose* $\mathcal{F}$ *is a family of functions from* $[\rho]^2$ *to* $\rho$ *such that* $|\mathcal{F}| \leq \lambda$*. Then there exists* $Y \in [\rho]^\mu$ *such that for every* $f \in \mathcal{F}$*,* $f$ *is canonical on* $Y$*.*

Using Fact 5.5 with $\mu = (2^\theta)^+$ and $\lambda = \theta$, choose $Y \in [\kappa]^{(2^\theta)^+}$ such that for every $f \in \mathcal{F}$, $f$ is canonical on $Y$. For $i < \theta$ and $k < 2$, let $\Delta_{i,k} \subseteq 2$ witness that $g_{i,k}$ is canonical on $Y$. Since $|\mathsf{range}(g_{i,k})| < 2^\theta$, it follows that $\Delta_{i,k} = \emptyset$ and $g_{i,k} \restriction [Y]^2$ is constant. It follows that for each $i < \theta$, we can find $\gamma^i_\star < \omega_1$ and $r^i \in \mathsf{Random}_{\gamma^i_\star}$ such that for every $\alpha < \beta$ in $Y$, $\mathsf{otp}(S_{i,\alpha,\beta}) = \gamma^i_\star$ and $r^i_{i,\alpha,\beta} = p_{i,\alpha,\beta}$.

For $i < \theta$, let $R^i_{\mathsf{up}} = \{\gamma < \gamma^i_\star : 1 \notin \Delta_{i,\gamma}\}$ and $R^i_{\mathsf{down}} = \{\gamma < \gamma^i_\star : 0 \notin \Delta_{i,\gamma}\}$ where $\Delta_{i,\gamma} \subseteq 2$ witnesses that $f_{i,\gamma}$ is canonical on $Y$.

**Lemma 5.6.** *For every $\alpha < \beta_1 < \beta_2$ in $Y$ and $\gamma < \gamma^i_\star$, $\gamma \in R^i_{\mathsf{up}}$ iff $h_{i,\alpha,\beta_1}(\gamma) = h_{i,\alpha,\beta_2}(\gamma)$.*

*Proof.* Should be clear using $1 \notin \Delta_{i,\gamma}$. □

Similarly, using $0 \notin \Delta_{i,\gamma}$ we get the following.

**Lemma 5.7.** *For every $\beta_1 < \beta_2 < \alpha$ in $Y$ and $\gamma < \gamma^i_\star$, $\gamma \in R^i_{\mathsf{down}}$ iff $h_{i,\beta_1,\alpha}(\gamma) = h_{i,\beta_2,\alpha}(\gamma)$.*

Together, they imply that for every $i < \theta$ the following hold.

(a) $h_{i,\alpha,\beta} \restriction R^i_{\mathsf{up}} = h_{i,\alpha,\mathsf{up}}$ does not depend on $\beta$ in $Y$ above $\alpha$.
(b) $h_{i,\beta,\alpha} \restriction R^i_{\mathsf{down}} = h_{i,\alpha,\mathsf{down}}$ does not depend on $\beta$ in $Y$ below $\alpha$.
(c) For every $\gamma \in \gamma^i_\star \setminus R^i_{\mathsf{up}}$, $\beta \mapsto h_{i,\alpha,\beta}(\gamma)$ is injective w.r.t. $\beta$ in $Y$ above $\alpha$.
(d) For every $\gamma \in \gamma^i_\star \setminus R^i_{\mathsf{down}}$, $\beta \mapsto h_{i,\beta,\alpha}(\gamma)$ is injective w.r.t. $\beta$ in $Y$ below $\alpha$.

**Lemma 5.8.** *For every $\gamma \in R^i_{\mathsf{up}} \cap R^i_{\mathsf{down}}$ and $\beta < \alpha < \delta$ in $Z$, $h_{i,\alpha,\delta}(\gamma) = h_{i,\beta,\alpha}(\gamma)$. Hence $h_{i,\alpha,\mathsf{up}}$ and $h_{i,\alpha,\mathsf{down}}$ agree on $R^i_{\mathsf{up}} \cap R^i_{\mathsf{down}}$ for every $\alpha$ in $Z$. Hence for every $T \in \mathsf{Random}_{\gamma^i_\star}$, if $T$ is supported in $R^i_{\mathsf{up}} \cap R^i_{\mathsf{down}}$, then $T_{i,\delta,\alpha} = T_{i,\alpha,\beta}$.*

*Proof.* Since $\gamma \in R^i_{\mathsf{up}}$, $h_{i,\beta,\alpha}(\gamma) = h_{i,\beta,\delta}(\gamma)$. Since $\gamma \in R^i_{\mathsf{down}}$, $h_{i,\beta,\delta}(\gamma) = h_{i,\alpha,\delta}(\gamma)$. Hence $h_{i,\alpha,\delta}(\gamma) = h_{i,\beta,\alpha}(\gamma)$. □

**Lemma 5.9.** *There exist $A, B, \alpha$ such that $A, B \in [Y]^{\aleph_0}$, $\alpha \in Y$, $A < \alpha < B$ such that for every $i < \theta$*

(i) *$\{S_{i,\delta,\alpha} : \delta \in A\}$ is a $\Delta$-system with root $h_{i,\alpha,\mathsf{down}}[R^i_{\mathsf{down}}]$ and*
(ii) *$\{S_{i,\alpha,\beta} : \beta \in B\}$ is a $\Delta$-system with root $h_{i,\alpha,\mathsf{up}}[R^i_{\mathsf{up}}]$.*

*Proof.* Choose $\alpha \in Y$ such that $|Y \cap \alpha|, |Y \setminus \alpha| \geq 2^\theta$. Let $A_1 \in [Y \cap \alpha]^{\theta^+}$ and $B_1 \in [Y \setminus (\alpha+1)]^{\theta^+}$. We'll find $A \in [A_1]^{\aleph_0}$ and $B \in [B_1]^{\aleph_0}$ such that (i) and (ii) hold.

Let $\Gamma = \{i < \theta : \gamma^i_\star \setminus R^i_{\mathsf{down}} \neq \emptyset\}$. If $\Gamma = \emptyset$, then (i) is trivial since any $A \in [A_1]^{\aleph_0}$ satisfies it. So assume $\Gamma \neq \emptyset$. Let $S = \bigcup\{S_{i,\delta,\alpha} : i < \theta, \delta \in A_1\}$. Note that $|S| = \theta^+$ because $\Gamma \neq \emptyset$. Let $\{\alpha_\xi : \xi < \theta^+\}$ be a one-one listing of $S$. Inductively choose $\{\delta_n : n < \omega\} \subseteq A_1$ as follows. $\delta_0 \in A_1$ is arbitrary. Having chosen $\delta_n$, let $\xi_n < \theta^+$ be large enough so that for every $\xi' > \xi_n$, $\alpha_{\xi'} \notin \bigcup_{i<\theta} h_{i,\delta_n,\alpha}[\gamma^i_\star]$. We claim that there is some $\delta_{n+1} \in A_1$ such that for every $\xi < \theta^+$, if $\alpha_\xi \in \bigcup_{i\in\Gamma} h_{i,\delta_{n+1},\alpha}[\gamma^i_\star \setminus R^i_{\mathsf{down}}]$, then $\xi > \xi_n$. Suppose there is no such $\delta_{n+1} \in A_1$. Then since $|A_1| = \theta^+$, we can find $i \in \Gamma$ and $\gamma \in \gamma^i_\star \setminus R^i_{\mathsf{down}}$ such that $\{\delta \in A_1 : (\exists \xi' < \xi_n)(h_{i,\delta,\alpha}(\gamma) = \alpha_{\xi'})\}$ has size $\theta^+$. But this is impossible because the map $\delta \mapsto h_{i,\delta,\alpha}(\gamma)$ is injective on $A_1$. It is easy to see that $A = \{\beta_n : n < \omega\}$ satisfies (i). The construction of $B$ is similar. □

Let $A, B, \alpha$ be as in Lemma 5.9. Towards a contradiction, fix $p \in \mathbb{P}$ and $i < \theta$ such that $p \Vdash (\forall \beta \in A \cup B)(\beta <_i \alpha)$. Recall that $r^i \in \mathsf{Random}_{\gamma^i_\star}$ satisfies: For every $\alpha_1 < \alpha_2$ in $Y$, $r^i_{i,\alpha_1,\alpha_2} = p_{i,\alpha_1,\alpha_2} = [[\alpha_1 <_i \alpha_2]]_{\mathbb{P}}$.

**Claim 5.10.** $0 < \mu(r^i) < 1$.

*Proof.* If not, then then the empty condition forces that $\alpha$ is $<_i$-between $\gamma \in A$ and $\delta \in B$: Contradiction. $\square$

In the next lemma, for the definitions of $W^1$ and $W^2$, we follow the convention that $(2^\emptyset, \mu_\emptyset)$ is the measure space with one point of measure one. For $r \subseteq 2^{\gamma^i_\star}$, $R \subseteq \gamma^i_\star$ and $z \in 2^R$, let

$$\pi_z(r) = \{y \upharpoonright \gamma^i_\star \setminus R : y \in r \wedge y \upharpoonright R = z\} \subseteq 2^{\gamma^i_\star \setminus R}.$$

**Lemma 5.11.** *Let $W^1 = \{y \in 2^{\gamma^i_\star} : \mu_{\gamma^i_\star \setminus R^i_{down}}(\pi_{y \upharpoonright R^i_{down}}(r^i)) = 1\}$ and $W^2 = \{y \in 2^{\gamma^i_\star} : \mu_{\gamma^i_\star \setminus R^i_{up}}(\pi_{y \upharpoonright R^i_{up}}(r^i)) = 0\}$. Then the following hold.*

(a) *For every $\delta \in A$, $p \leq W^1_{i,\delta,\alpha}$.*
(b) *For every $\beta \in B$, $p \leq W^2_{i,\alpha,\beta}$.*

*Proof.* Note that $W^1, W^2$ are Baire by Fubini's theorem.

(a) Suppose $y \in 2^{\gamma^i_\star} \setminus W^1$. Put $z = y \upharpoonright R^i_{\mathsf{down}}$ and $C = \pi_z(r^i)$. So $\mu_{\gamma^i_\star \setminus R^i_{\mathsf{down}}}(C) < 1 - a$ for some $a > 0$. Let $z_\star = z \circ h^{-1}_{i,\alpha,\mathsf{down}}$. Then $z_\star \in 2^{S_{\alpha,\mathsf{down}}}$ where $S_{\alpha,\mathsf{down}} = h_{i,\alpha,\mathsf{down}}[R^i_{\mathsf{down}}]$. Define

$$p_{z_\star} = \{x \in 2^\kappa : (\exists x' \in p)(x' \upharpoonright S_{\alpha,\mathsf{down}} = z_\star \wedge x \upharpoonright (\kappa \setminus S_{\alpha,\mathsf{down}}) = x' \upharpoonright (\kappa \setminus S_{\alpha,\mathsf{down}}))\}$$

We claim that $p_{z_\star}$ is null. Since $C$ is supported in $\gamma^i_\star \setminus R^i_{\mathsf{down}}$, the family $\{C_{i,\delta,\alpha} : \delta \in A\}$ consists of sets with pairwise disjoint supports and each $C_{i,\delta,\alpha}$ has measure $< 1 - a$. It follows that this family has null intersection. Since $p \leq r^i_{i,\delta,\alpha}$ for every $\delta \in A$, $p_{z_\star} \leq C_{i,\delta,\alpha}$ for every $\delta \in A$. Hence $p_{z_\star}$ is null. It follows that $p \leq W^1_{i,\delta,\alpha}$ for every $\delta \in A$.

(b) Let $y \in 2^{\gamma^i_\star} \setminus W^2$. Put $z = y \upharpoonright R^i_{\mathsf{up}}$ and $C = \pi_z(r^i)$. Then $\mu_{\gamma^i_\star \setminus R^i_{\mathsf{up}}}(C) > a$ for some $a > 0$. Let $z_\star = z \circ h^{-1}_{i,\alpha,\mathsf{up}}$. Then $z_\star \in 2^{S_{\alpha,\mathsf{up}}}$ where $S_{\alpha,\mathsf{up}} = h_{i,\alpha,\mathsf{up}}[R^i_{\mathsf{up}}]$. Define

$$p_{z_\star} = \{x \in 2^\kappa : (\exists x' \in p)(x' \upharpoonright S_{\alpha,\mathsf{up}} = z_\star \wedge x \upharpoonright (\kappa \setminus S_{\alpha,\mathsf{up}}) = x' \upharpoonright (\kappa \setminus S_{\alpha,\mathsf{up}}))\}$$

We claim that $p_{z_\star}$ is null. Since $C$ is supported in $\gamma^i_\star \setminus R^i_{\mathsf{up}}$, the family $\{C_{i,\alpha,\beta} : \beta \in B\}$ consists of sets with pairwise disjoint supports and each $C_{i,\alpha,\beta}$ has measure $> a$. It follows that the union of this family has full measure. Since $p \perp r^i_{i,\alpha,\beta}$ for every $\beta \in B$, $p_{z_\star} \perp C_{i,\alpha,\beta}$ for every $\beta \in B$. Hence $p_{z_\star}$ is null. It follows that $p \leq W^2_{i,\alpha,\beta}$ for every $\beta \in B$. $\square$

Let $W = W^1 \cap W^2$. $W$ is null because $W^1 \leq r^i$ while $W^2 \perp r^i$. Put $\Delta = R^i_{\mathsf{up}} \cap R^i_{\mathsf{down}}$. Let $T^1 = \{x \in 2^{\gamma^i_\star} : (\exists y \in W^1)(x \upharpoonright \Delta = y \upharpoonright \Delta)\}$ and $T^2 = \{x \in 2^{\gamma^i_\star} : (\exists y \in W^2)(x \upharpoonright \Delta = y \upharpoonright \Delta)\}$. So $T^j$ is the projection of $W^j$ on $\Delta$. Since the supports of $W^1, W^2$ are disjoint outside $\Delta$, $T^1 \cap T^2$ must also be null. Since $T_1, T_2$ are supported in $\Delta$, by Lemma 5.8, for every $\beta \in B$ and $\delta \in A$, $T^1_{i,\delta,\alpha} = T^1_{i,\alpha,\beta}$ and $T^2_{i,\delta,\alpha} = T^2_{i,\alpha,\beta}$. But both of these contain the projection of $p$ on $h_{i,\alpha,\mathsf{up}}[\Delta] = h_{i,\alpha,\mathsf{down}}[\Delta]$ which is non null: Contradiction.

The proof for the Cohen case is similar. We just replace "null" by "meager" and "measure one" by "comeager" everywhere. For example, in the proof of Lemma

5.11, the family $\{C_{i,\delta,\alpha} : \delta \in A\}$ has meager intersection because the sets in this family have pairwise disjoint supports and each $C_{i,\delta,\alpha}$ is not comeager. This completes the proof of Theorem 5.4. $\square$

**Corollary 5.12.** *Suppose $V \models GCH$. Then after adding $\aleph_{\omega_1}$ Cohen/random reals, $\mathfrak{dim}_T = \mathfrak{c} = \aleph_{\omega_1}$.*

**Corollary 5.13.** *Suppose $V \models GCH$. Then after adding $\aleph_9$ Cohen/random reals, $\aleph_7 = \mathfrak{dim}_T < \mathfrak{c} = \aleph_9$.*

*Proof.* By Theorem 5.4, $\mathsf{odim}(\mathbb{H}_\mathfrak{c}) \geq \aleph_7$ in this model. By Lemma 5.2 and the fact that this forcing is ccc, $\mathsf{odim}(\mathbb{H}_\mathfrak{c}) \leq \aleph_7$. Finally, by Corollary 2.11, $\mathfrak{dim}_T = \mathsf{odim}(\mathbb{H}_\mathfrak{c}) = \aleph_7$. $\square$

The next corollary illustrates that using a preparatory forcing, we can get large gaps between $\mathfrak{dim}_T$ and $\mathfrak{c}$.

**Corollary 5.14.** *Suppose $V \models GCH$. Let $\mathbb{P} = \mathsf{Add}(\omega_5, \omega_9)$ be the forcing for adding a subset of $\omega_9$ with conditions of size $< \aleph_5$. In $V^\mathbb{P}$, let $\mathbb{Q}$ be the forcing for adding $\aleph_9$ Cohen/random reals. Then $\mathbb{V}^{\mathbb{P}\star\mathbb{Q}} \models \mathfrak{dim}_T = \aleph_4 < \mathfrak{c} = \aleph_9$.*

*Proof.* Note that forcing with $\mathbb{P} \star \mathbb{Q}$ preserves all cofinalities and $V^{\mathbb{P}\star\mathbb{Q}} \models \mathfrak{c} = \aleph_9$. Since in $V^\mathbb{P}$, $(\aleph_4)^{\aleph_0} = \aleph_4$ and $\aleph_9 = 2^{\aleph_5} = \beth_2(\aleph_4)$, by Lemma 5.2, we get $V^\mathbb{P} \models \mathsf{odim}(\mathbb{H}_{\omega_9}) \leq \aleph_4$. Since $V^\mathbb{P} \models \beth_2(\aleph_3) = \aleph_5 < \aleph_9$, by Theorem 5.4, $V^{\mathbb{P}\star\mathbb{Q}} \models (\aleph_3)^+ = \aleph_4 \leq \mathsf{odim}(\mathbb{H}_{\omega_9})$. The result follows. $\square$

**Lemma 5.15.** *Suppose $\kappa$ is a cardinal of uncountable cofinality. Then the order dimension of $\mathbb{H}_\kappa$ cannot have cofinality $\aleph_0$.*

*Proof.* Suppose not. Let $\theta < \kappa$ be a cardinal of cofinality $\aleph_0$ such that $\mathsf{odim}(\mathbb{H}_\kappa) = \theta$. Using Lemma 2.9, fix a family $\{<_i : i < \theta\}$ of linear orders on $\kappa$ such that for every countable $X \in [\kappa]^{\aleph_0}$ and $\alpha \in \kappa \setminus X$, there exists $i < \theta$ such that $\alpha$ is $<_i$-above every member of $X$.

Choose $\{\theta_n : n < \omega\}$ increasing cofinal in $\theta$. Call $\alpha < \kappa$ $n$-good if there exists $X \in [\kappa]^{\aleph_0}$ such that for every $i < \theta_n$, some member of $X$ is $<_i$-above $\alpha$.

**Claim 5.16.** *For $n < \omega$, put $B_n = \{\alpha < \kappa : \alpha$ is not $n$-good$\}$. Then $|B_n| < \kappa$.*

*Proof.* Suppose not and fix $n < \omega$ such that $|B_n| = \kappa$. Then, by Lemma 2.9, $\{<_i\restriction B_n : i < \theta_n\}$ witnesses that the order dimension of $\mathbb{H}_{|B_n|} = \mathbb{H}_\kappa$ is at most $\theta_n$. $\square$

Since $\mathsf{cf}(\kappa) > \aleph_0$, we can find $\alpha \in \kappa \setminus \bigcup_{n<\omega} B_n$. Let $X_n \in [\kappa]^{\aleph_0}$ witness the $n$-goodness of $\alpha$. Put $X = \bigcup_{n<\omega} X_n$. Then for every $i < \theta$, some member of $X$ is $<_i$-above $\alpha$: Contradiction. $\square$

**Corollary 5.17.** *Suppose $V \models GCH$. Then after adding $\aleph_\omega$ Cohen/random reals, $\mathfrak{dim}_T = \mathfrak{c} = \aleph_{\omega+1}$.*

*Proof.* Use Theorem 5.4 and Lemma 5.15. $\square$

We can also have $\aleph_1 < \mathfrak{dim}_T$ and $\mathfrak{dim}_T^+ = \mathfrak{c}$.

**Corollary 5.18.** *Suppose $V \models GCH$. Then after adding $\aleph_{\omega+2}$ Cohen/random reals, $\mathfrak{dim}_T = \aleph_{\omega+1} < \mathfrak{c} = \aleph_{\omega+2}$.*

*Proof.* Since $\mathfrak{c} = \aleph_{\omega+1}$ is a successor of a cardinal of uncountable cofinality, by [7], $\mathfrak{dim}_T \leq \aleph_{\omega+1}$. By Theorem 5.4 and Lemma 5.15, $\mathfrak{dim}_T \geq \aleph_{\omega+1}$. $\square$

We don't know if we can have $\mathfrak{dim}^+_T = \mathfrak{c}$ with $\mathfrak{c} < \aleph_\omega$. For example, we can ask the following.

**Question 5.19.** *Can we have $\mathfrak{dim}_T = \aleph_2 < \mathfrak{c} = \aleph_3$?*

## 6. On a question of Higuchi, Lempp, Raghavan, and Stephan

**Definition 6.1.** *Suppose $\mathcal{F} \subseteq [\kappa]^{\aleph_0}$. Define $\mathbb{H}_\mathcal{F}$ to be the partial order whose universe is $\kappa \sqcup \mathcal{F}$ with the order $a < b$ iff either $a \in \kappa$, $b \in \mathcal{F}$ and $a \in b$ or $a, b \in \mathcal{F}$ and $a \subsetneq b$.*

Recall that a poset is $\sigma$-directed if every countable subset of the poset has an upper bound. Note that if $\mathcal{F} \subseteq [\kappa]^{\aleph_0}$ is $\subseteq$-cofinal, then $\mathbb{H}_\mathcal{F}$ is $\sigma$-directed.

**Definition 6.2.** *For $\theta \leq \kappa$ and $\mathcal{F} \subseteq [\kappa]^{\aleph_0}$, define $\star(\theta, \kappa, \mathcal{F})$ to be the following statement: For every family $\{<_i : i < \theta\}$ of linear orders on $\kappa$, there exist $A \in \mathcal{F}$ and $\alpha \in \kappa \setminus A$ such that for every $i < \theta$, some member of $A$ is $<_i$ above $\alpha$.*

The following analogue of Lemma 2.9 characterizes $\mathsf{odim}(\mathbb{H}_\mathcal{F})$.

**Lemma 6.3.** *Let $\prec$ be a partial order on $\mathbb{H}_\mathcal{F}$ such that for every $a \in \kappa$, $B \in \mathcal{F}$, $a \in B$ iff $a \prec B$ and $\prec_{\mathbb{H}_\mathcal{F}}$ extends $\prec$. Then the order dimension of $\prec$ is equal to the least $\theta$ for which $\star(\theta, \kappa, \mathcal{F})$ fails.*

*Proof.* It is easy to check that if $\mathsf{odim}(\mathbb{H}_\mathcal{F}) \leq \theta$, then $\star(\theta, \kappa, \mathcal{F})$ fails. Next, assume $\star(\theta, \kappa, \mathcal{F})$ fails and fix a family $\{<_i : i < \theta\}$ of linear orders on $\kappa$ witnessing this. We can assume that the well ordering and the reverse well ordering of $\kappa$ belong to this family. For each $i < \theta$, define a partial order $\prec_i$ on $\kappa \sqcup \mathcal{F}$ as follows.

(a) For every $\alpha, \beta$ in $\kappa$, $\alpha \prec_i \beta$ iff $\alpha <_i \beta$.
(b) For every $X \in \mathcal{F}$ and $\alpha \in \kappa \setminus X$, $\alpha \prec_i X$ iff $(\exists \beta \in X)(\alpha \leq_i \beta)$.
(c) For every $X \in \mathcal{F}$ and $\alpha \in \kappa \setminus X$, $X \prec_i \alpha$ iff $(\forall \beta \in X)(\beta <_i \alpha)$.
(d) For every $X, Y \in \mathcal{F}$, $X \prec_i Y$ iff $X <_{\mathbb{H}_\mathcal{F}} Y$ or $(\exists \beta \in Y)(\forall \alpha \in X)(\alpha <_i \beta)$.

It is easy to check that $\{\prec_i : i < \theta\}$ is a family of partial orders on $\kappa \sqcup \mathcal{F}$ such that each $\prec_i$ extends $<_{\mathbb{H}_\mathcal{F}}$ and their intersection is $<_{\mathbb{H}_\mathcal{F}}$. Hence $\mathsf{odim}(\mathbb{H}_\mathcal{F}) \leq \theta$. $\square$

Assume $\mathfrak{c} > \aleph_1$. Suppose $\mathbb{H}$ is a $\sigma$-directed locally countable partial order of size continuum. Since $\mathfrak{c} > \aleph_1$, by Fodor's lemma, we can choose $W = \{p_i : i < \mathfrak{c}\} \subseteq \mathbb{H}$ such that no two members of $W$ are $<_\mathbb{H}$-comparable. Let $\mathcal{F} = \{A \in [\mathfrak{c}]^{\aleph_0} : (\exists p \in \mathbb{H})(A = \{i : p_i \leq_\mathbb{H} p\})\}$. Then $\mathcal{F} \subseteq [\mathfrak{c}]^{\aleph_0}$ is $\subseteq$-cofinal. For each $A \in \mathcal{F}$, fix $p_A \in \mathbb{H}$ such that $\{i < \mathfrak{c} : p_i \leq_\mathbb{H} p_A\} = A$. Consider the map $f : \mathbb{H}_\mathcal{F} \to \mathbb{H}$ defined by $f(i) = p_i$ for $i < \mathfrak{c}$ and $f(A) = p_A$ for $A \in \mathcal{F}$. Note that $f$ satisfies the following.

(i) $f(i)$ is $<_\mathbb{H}$-incomparable with $f(j)$ for every $i < j < \mathfrak{c}$.
(ii) $i \in A$ implies $f(i) <_\mathbb{H} f(A)$.
(iii) If $f(A) \leq_\mathbb{H} f(B)$, then $A \subseteq B$.

It follows that there is a partial order $\prec$ on $\mathbb{H}_\mathcal{F}$ such that $<_{\mathbb{H}_\mathcal{F}}$ extends $\prec$, for every $i < \mathfrak{c}$ and $A \in \mathcal{F}$, $i \in A$ iff $i \prec A$ and $f$ is an embedding from $(\mathbb{H}_\mathcal{F}, \prec)$ to $\mathbb{H}$. Hence $\mathsf{odim}(\mathbb{H}_\mathcal{F}) \leq \mathsf{odim}(\mathbb{H})$. One way of getting a negative answer to Question 1.4 would be to show that for every $\subseteq$-cofinal $\mathcal{F} \subseteq [\mathfrak{c}]^{\aleph_0}$, $\mathsf{odim}(\mathbb{H}_\mathcal{F}) = \mathsf{odim}(\mathbb{H}_\mathfrak{c})$. We don't know if this is possible, so we ask the following.

**Question 6.4.** *Is it consistent to have a $\subseteq$-cofinal $\mathcal{F} \subseteq [\mathfrak{c}]^{\aleph_0}$ for which $\aleph_1 \leq \mathsf{odim}(\mathbb{H}_{\mathcal{F}}) < \mathsf{odim}(\mathbb{H}_{\mathfrak{c}})$? Equivalently, is it consistent that for some $\subseteq$-cofinal $\mathcal{F} \subseteq [\mathfrak{c}]^{\aleph_0}$ and $\theta < \mathfrak{c}$, $\star(\theta, \mathfrak{c})$ holds and $\star(\theta, \mathfrak{c}, \mathcal{F})$ fails?*

**Remark**: If $\mathcal{F} \subseteq [\kappa]^{\aleph_0}$ is $\subseteq$-cofinal and for every $X \in [\kappa]^{\aleph_0}$, $\{A \cap X : A \in \mathcal{F}\}$ is countable, then $\mathcal{F}$ is called a *cofinal Kurepa family* in $[\kappa]^{\aleph_0}$. If $\mathfrak{c} < \aleph_\omega$, then there is a cofinal Kurepa family $\mathcal{F} \subseteq [\mathfrak{c}]^{\aleph_0}$ (see, for example, Todorcevic [15]). It follows that not even $\mathbb{H}_{[\omega]^{\aleph_0}}$ embeds into $\mathbb{H}_{\mathcal{F}}$.

Nevertheless, the following results show that the answer to Question 1.4 is no.

**Definition 6.5.** *For $\theta \leq \kappa$, $\dagger(\theta, \kappa)$ is the following statement: For every family $\{<_i : i < \theta\}$ of linear orders on $\kappa$, there exist $X \in [\kappa]^{\aleph_1}$ and $A \in [\kappa]^{\aleph_0}$ such that for every $i < \theta$ and $\alpha \in X$, there exists $\beta \in A$ such that $\alpha <_i \beta$.*

**Lemma 6.6.** *Assume $\dagger(\theta, \kappa)$. Let $\mathbb{H}$ be a $\sigma$-directed locally countable partial order of size $\kappa$. Then $\mathsf{odim}(\mathbb{H}) > \theta$.*

*Proof.* Suppose not and let $\{<_i : i < \theta\}$ be a family of linear orders on $\mathbb{H}$ witnessing $\mathsf{odim}(\mathbb{H}) \leq \theta$. We can assume that $\mathbb{H} = (\kappa, \prec)$. Choose $X \in [\kappa]^{\aleph_1}$ and $A \in [\kappa]^{\aleph_0}$ such that for every $i < \theta$ and $\alpha \in X$, there exists $\beta \in A$ such that $\alpha <_i \beta$. Since every countable subset of $\mathbb{H}$ has a $\prec$-upper bound, we can find $\gamma < \kappa$ such that for every $\beta \in A$, $\beta \prec \gamma$. Since $\mathbb{H}$ is locally countable, we can find $\alpha \in X$ such that $\alpha \not\prec \gamma$. Then for some $i < \theta$, $\gamma <_i \alpha$. But this this implies that every member of $A$ is $<_i$-below $\alpha$: Contradiction. ☐

**Lemma 6.7.** *Suppose there is a supersaturated ideal on $\kappa$. Then for every $\theta < \kappa$, $\dagger(\theta, \kappa)$ holds.*

*Proof.* Let $\mathcal{I}$ be a supersaturated ideal on $\kappa$. Suppose $\theta < \kappa$ and $\{<_i : i < \theta\}$ is a family of linear orders on $\kappa$. For each $\alpha < \kappa$, let $R_{i,\alpha} = \{\beta < \kappa : \alpha \leq_i \beta\}$ and $W_i = \{\alpha < \kappa : R_{i,\alpha} \in \mathcal{I}\}$. Let $\Gamma = \{i < \theta : W_i \in \mathcal{I}^+\}$. Choose $A_0 \in [\kappa]^{\aleph_0}$ such that for every $i \in \Gamma$, $W_i \cap A_0$ is infinite. Let $Y = \bigcup_{i \in \Gamma} \{\alpha < \kappa : (\exists \beta \in W_i \cap A_0)(\beta \leq_i \alpha)\}$ and note that $Y \in \mathcal{I}$. Choose $X \subseteq \kappa \setminus (Y \cup \bigcup_{i \in \theta \setminus \Gamma} W_i)$ of size $\aleph_1$. Choose $A_1 \in [\kappa]^{\aleph_0}$ such that for every $i \in \theta \setminus \Gamma$ and $\alpha \in X$, $R_{i,\alpha} \cap A_1$ is infinite. Put $A = A_0 \cup A_1$ and note that for every $\alpha \in X$ and $i < \theta$, $R_{i,\alpha} \cap A \neq \emptyset$. ☐

**Corollary 6.8.** *Suppose there is a supersaturated ideal on $\mathfrak{c}$. Let $\mathbb{H}$ be a $\sigma$-directed locally countable partial order of size $\mathfrak{c}$. Then $\mathsf{odim}(\mathbb{H}) = \mathfrak{c}$.*

**Lemma 6.9.** *Suppose $\aleph_1 \leq \theta$ and $\kappa = (\beth_2(\theta))^+$. Let $\mathbb{P} \in \{\mathsf{Cohen}_\kappa, \mathsf{Random}_\kappa\}$. Then $\dagger(\theta, \kappa)$ holds in $V^{\mathbb{P}}$.*

*Proof.* Suppose $\{<_i : i < \theta\}$ is a family of linear orders on $\kappa$ in $V^{\mathbb{P}}$. We start repeating the proof of Theorem 5.4. Note that, as $\theta^+ \geq \aleph_2$, in Lemma 5.9, we can choose $A \sqcup X \sqcup B \subseteq Y$ such that $A < X < B$, $A, B \in [Y]^{\aleph_0}$, $X \in [Y]^{\aleph_1}$ and for every $\alpha \in X$, clauses (i), (ii) hold for $A, B, \alpha$. The rest of the argument given there implies that $C = A \cup B$ and $X$ are as required. ☐

**Corollary 6.10.** *Suppose $V \models GCH$. Let $\mathbb{P} \in \{\mathsf{Cohen}_\kappa, \mathsf{Random}_\kappa\}$ and let $\mathbb{H} \in V^{\mathbb{P}}$ be a locally countable $\sigma$-directed partial order of size $\mathfrak{c}$.*

(i) *If $\kappa = \aleph_{\omega_1}$, then $V^{\mathbb{P}} \models \mathsf{odim}(\mathbb{H}) = \mathfrak{c} = \aleph_{\omega_1}$.*
(ii) *If $\kappa = \aleph_\omega$, then $V^{\mathbb{P}} \models \aleph_\omega \leq \mathsf{odim}(\mathbb{H}) \leq \mathfrak{c} = \aleph_{\omega+1}$.*
(iii) *If $\kappa = \aleph_{10}$, then $V^{\mathbb{P}} \models \mathsf{odim}(\mathbb{H}) = \aleph_8 < \mathfrak{c} = \aleph_{10}$.*

*Proof.* (i) and (ii) follow from Lemmas 6.6 and 6.9. For (iii), note that Lemmas 6.6 and 6.9 imply that $\mathsf{odim}(\mathbb{H}) \geq \aleph_8$. For the reverse inequality, use Lemma 2.10 and the fact that $\mathsf{odim}(\mathbb{H}_\kappa) = \aleph_8$ in this model. $\square$

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