THE DENSITY ZERO IDEAL AND THE SPLITTING NUMBER

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Abstract. The main result of this paper is an improvement of the upper bound on the cardinal invariant $\mathrm{cov}^*(\mathcal{Z}_0)$ that was discovered in [11]. Here $\mathcal{Z}_0$ is the ideal of subsets of the set of natural numbers that have asymptotic density zero. This improved upper bound is also dualized to get a better lower bound on the cardinal $\mathrm{non}^*(\mathcal{Z}_0)$. En route some variations on the splitting number are introduced and several relationships between these variants are proved.

1. Introduction

We use ω to denote the set of natural number in keeping with usual set-theoretic convention. Recall that a set $A \subset \omega$ is said to have *asymptotic density 0* if $\lim\limits_{n\to\infty} \frac{|A \cap n|}{n} = 0$. By $\mathcal{Z}_0$ we denote the set $\{A \subset \omega : A \text{ has asymptotic density } 0\}$. Recall that given a set a, $\mathcal{I}$ is said to be an *ideal* on a if $\mathcal{I}$ is a subset of $\mathcal{P}(a)$ such that the following conditions hold: if $b \subseteq a$ is finite, then $b \in \mathcal{I}$; if $b \in \mathcal{I}$ and $c \subseteq b$, then $c \in \mathcal{I}$; if $b \in \mathcal{I}$ and $c \in \mathcal{I}$, then $b \cup c \in \mathcal{I}$; and $a \notin \mathcal{I}$. It is easily seen that $\mathcal{Z}_0$ is an ideal on ω. It is moreover a *P-ideal*, which means that for every collection $\{a_n : n \in \omega\} \subset \mathcal{Z}_0$, there exists $a \in \mathcal{Z}_0$ such that $\forall n \in \omega\, [a_n \subset^* a]$, where $X \subset^* Y$ if and only if $X \setminus Y$ is finite. $\mathcal{Z}_0$ is also a *tall ideal on* ω, which means that $\forall a \in [\omega]^\omega \exists b \in [a]^\omega\, [b \in \mathcal{Z}_0]$. In terms of the Borel hierarchy of $\mathcal{P}(\omega)$, $\mathcal{Z}_0$ is $F_{\sigma\delta}$ but not $G_{\delta\sigma}$.

Cardinal invariants associated with such tall analytic P-ideals have been studied in several works, principally by Hernández-Hernández and Hrušák [7]. Among the various invariants that have been considered, $\mathrm{cov}^*(\mathcal{Z}_0)$ and $\mathrm{non}^*(\mathcal{Z}_0)$ are of particular interest.

Definition 1.

$$\mathrm{cov}^*(\mathcal{Z}_0) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{Z}_0 \wedge \forall a \in [\omega]^\omega \exists b \in \mathcal{F}\, [|a \cap b| = \aleph_0]\},$$
$$\mathrm{non}^*(\mathcal{Z}_0) = \min\{|\mathcal{F}| : \mathcal{F} \subset [\omega]^\omega \wedge \forall b \in \mathcal{Z}_0 \exists a \in \mathcal{F}\, [|a \cap b| < \aleph_0]\}.$$

Of course there is nothing special about $\mathcal{Z}_0$ here and these invariants can be defined for any tall P-ideal $\mathcal{I}$ on ω. In fact, these invariants are special cases of the invariants $\mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I})$, which make sense for any ideal $\mathcal{I}$ on any set X. To see how, for each $a \subset \omega$, let $\hat{a} = \{b \subset \omega : |b \cap a| = \aleph_0\}$. This is a G_δ subset of $\mathcal{P}(\omega)$. Let $\hat{\mathcal{Z}}_0 = \{X \subset \mathcal{P}(\omega) : \exists a \in \mathcal{Z}_0\, [X \subset \hat{a}]\}$. Now $\hat{\mathcal{Z}}_0$ is a σ-ideal on $\mathcal{P}(\omega)$ generated by Borel sets, and it is not hard to show (see Proposition 1.2 of [7]) that $\mathrm{cov}(\hat{\mathcal{Z}}_0) = \mathrm{cov}^*(\mathcal{Z}_0)$ and that $\mathrm{non}(\hat{\mathcal{Z}}_0) = \mathrm{non}^*(\mathcal{Z}_0)$.

$\mathcal{Z}_0$ turned out to be a critical object of study in [7], where the invariants associated to $\mathcal{Z}_0$ were shown to be closely connected to many others, including

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$\mathrm{add}(\mathcal{N}), \mathrm{cov}(\mathcal{N})$, and $\mathrm{non}(\mathcal{N})$. In that paper, Hernández-Hernández and Hrušák asked whether $\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{d}$ (Question 3.23(a) of [7]). Their question was positively answered in [11]. Furthermore the proof in [11] also yielded the dual inequality $\mathfrak{b} \leq \mathrm{non}^*(\mathcal{Z}_0)$. We improve both of these bounds in this paper. We show that $\min\{\mathfrak{d}, \mathfrak{r}\} \leq \mathrm{non}^*(\mathcal{Z}_0)$ and that $\mathrm{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{b}, \mathfrak{s}(\mathfrak{pr})\}$, where $\mathfrak{s}(\mathfrak{pr})$ is a variant of $\mathfrak{s}$ that is not known to be distinguishable from $\mathfrak{s}$.

The second of our inequalities has implications for what types of forcings can be used to diagonalize $\mathbf{V} \cap \mathcal{Z}_0$. Recall that a forcing notion $\mathbb{P}$ in a ground model $\mathbf{V}$ is said to *diagonalize* $\mathbf{V} \cap \mathcal{Z}_0$ if there is an $\mathring{A} \in \mathbf{V}^{\mathbb{P}}$ such that $\Vdash_{\mathbb{P}} \mathring{A} \in [\omega]^{\omega}$ and for each $X \in \mathbf{V} \cap \mathcal{Z}_0$, $\Vdash_{\mathbb{P}} \left|X \cap \mathring{A}\right| < \aleph_0$. Forcings that diagonalize $\mathbf{V} \cap \mathcal{Z}_0$ tend to increase $\mathrm{cov}^*(\mathcal{Z}_0)$. A celebrated result of Laflamme from [10] is that every F_σ ideal can be diagonalized by a proper ω^ω-bounding forcing. Until the work in [11], it was unclear whether a similar result could also be proved for all $F_{\sigma\delta}$ P-ideals. The proof of the inequality $\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{d}$ from [11] shows that any proper forcing that diagonalizes $\mathbf{V} \cap \mathcal{Z}_0$ necessarily adds an unbounded real, and since $\mathcal{Z}_0$ is an $F_{\sigma\delta}$ P-ideal, it shows that Laflamme's theorem is, in a certain sense, best possible. The proof of the inequality $\mathrm{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{b}, \mathfrak{s}(\mathfrak{pr})\}$ given in Section 3 has a similar consequence. It shows that any proper forcing that diagonalizes $\mathbf{V} \cap \mathcal{Z}_0$ must either add a real that dominates $\mathbf{V} \cap \omega^\omega$ or it must add a real that is not promptly split by $\mathbf{V} \cap (\mathcal{P}(\omega))^\omega$ (this notion is introduced in Definition 2). We will also show in Section 2 that a Suslin c.c.c. forcing cannot add a real that is not promptly split by $\mathbf{V} \cap (\mathcal{P}(\omega))^\omega$, yielding the conclusion that any Suslin c.c.c. poset that diagonalizes $\mathbf{V} \cap \mathcal{Z}_0$ necessarily adds a dominating real.

The two main inequalities of this paper are obtained by analyzing certain combinatorial variants of the notion of a splitting family. The first section of this paper is devoted to introducing and studying these variants. At present, it is unclear if these variants ultimately lead to a new cardinal invariant that is distinguishable from $\mathfrak{s}$ (see Question 19).

We end this introduction by fixing some notation that will occur throughout the paper. $A \subset B$ means $\forall a\, [a \in A \implies a \in B]$. Thus the symbol "$\subset$" does not denote proper subset. The expression "$\exists^\infty x \ldots$" abbreviates the quantifier "there exist infinitely many x such that $\ldots$", and the dual expression "$\forall^\infty x \ldots$" means "for all but finitely many $x \ldots$". Given a function f and a set $X \subset \mathrm{dom}(f)$, $f''X$ denotes the image of X under f – that is, $f''X = \{f(x) : x \in X\}$. We use standard cardinal invariants such as $\mathfrak{s}$, $\mathfrak{u}$, $\mathfrak{p}$, $\mathfrak{r}$, and $\mathfrak{b}$, whose definitions may be found in [3].

2. Some variants of the splitting number

Several variations on the notion of a splitting family are studied in this section. One of these variants involves the existence of a type of strong coloring. It turns out that all of these variations ultimately lead to the same cardinal invariant, which we denote $\mathfrak{s}(\mathfrak{pr})$. It will be shown that $\mathfrak{s}(\mathfrak{pr})$ behaves very similarly to $\mathfrak{s}$. We adopt the convention that for a set $x \subset \omega$, $x^0 = x$ and $x^1 = \omega \setminus x$; this will make certain definitions easier to state.

Definition 2. Let $X = \langle x_i : i \in \omega \rangle$ be a sequence of elements of $\mathcal{P}(\omega)$. We say that X *promptly splits* a if for each $n \in \omega$ and each $\sigma \in 2^{n+1}$, $\left(\bigcap_{i<n+1} x_i^{\sigma(i)}\right) \cap a$ is infinite. A family $\mathcal{F} \subset (\mathcal{P}(\omega))^\omega$ is said to be a *promptly splitting family* if for each $a \in [\omega]^\omega$, there exists $X \in \mathcal{F}$ which promptly splits a.

Definition 3. Let $P = \langle x_i : i \in \omega \rangle$ be a partition of ω (that is, $\bigcup_{i\in\omega} x_i = \omega$ and for any $i < j < \omega$, $x_i \cap x_j = 0$). We say that P *splits* a if for each $i \in \omega$ $x_i \cap a$ is infinite. A family of partitions $\mathcal{F}$ is called a *splitting family of partitions* if for each

$a \in [\omega]^{\omega}$, there exists $P \in \mathcal{F}$ which splits a.

$$\mathfrak{s}(\mathfrak{pr}) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a splitting family of partitions}\}.$$

Lemma 4. $\mathfrak{s}(\mathfrak{pr}) = \min\{|\mathcal{F}| : \mathcal{F} \subset (\mathcal{P}(\omega))^{\omega} \wedge \mathcal{F}$ *is a promptly splitting family*$\}$.

Proof. First let $\mathcal{F} \subset (\mathcal{P}(\omega))^{\omega}$ be any promptly splitting family. Let $\{X_\alpha : \alpha < \kappa\}$ be an enumeration of $\mathcal{F}$, where $\kappa = |\mathcal{F}|$. For each $\alpha < \kappa$, write $X_\alpha = \langle x_{\alpha,i} : i < \omega\rangle$, and define $y_{\alpha,i} = x_{\alpha,i} \setminus i$. For each $n \in \omega$, define $\sigma_n \in 2^{n+1}$ as follows: for $i < n$, $\sigma_n(i) = 0$ and $\sigma_n(n) = 1$. Define $z_{\alpha,n} = \bigcap_{i<n+1} y_{\alpha,i}^{\sigma_n(i)}$. It is easy to see that if $m < n < \omega$, then $z_{\alpha,m} \cap z_{\alpha,n} = 0$. Also for any $l \in \omega$ there is a minimal $n \in \omega$ such that $l \notin y_{\alpha,n}$ because $\bigcap_{n\in\omega} y_{\alpha,n} = 0$. Then $l \in z_{\alpha,n}$, for this minimal n. Thus $P_\alpha = \langle z_{\alpha,n} : n \in \omega\rangle$ is a partition of ω. Moreover it is clear that for any $a \in [\omega]^{\omega}$, if X_α promptly splits a, then P_α splits a. Therefore $\{P_\alpha : \alpha < \kappa\}$ is a splitting family of partitions.

In the other direction, suppose that $\mathcal{F}$ is any splitting family of partitions. Let $\{P_\alpha : \alpha < \kappa\}$ enumerate $\mathcal{F}$, where $\kappa = |\mathcal{F}|$, and write $P_\alpha = \langle y_{\alpha,n} : n < \omega\rangle$, for each $\alpha < \kappa$. Fix an independent family $\langle C_i : i \in \omega\rangle$ of subsets of ω. For each $\alpha < \kappa$ and $i \in \omega$, define $x_{\alpha,i} = \bigcup_{n\in C_i} y_{\alpha,n}$. Note that $\omega \setminus x_{\alpha,i} = \bigcup_{n\in\omega\setminus C_i} y_{\alpha,n}$. Put $X_\alpha = \langle x_{\alpha,i} : i < \omega\rangle \in (\mathcal{P}(\omega))^{\omega}$. We check that for any $\alpha < \kappa$ and any $a \in [\omega]^{\omega}$, if P_α splits a, then X_α promptly splits a. This would show that $\{X_\alpha : \alpha < \kappa\}$ is a promptly splitting family and conclude the proof. Fix $\alpha < \kappa$ and $a \in [\omega]^{\omega}$. Suppose P_α splits a. Fix any $n \in \omega$ and $\sigma \in 2^{n+1}$. Since $\langle C_i : i \in \omega\rangle$ is an independent family, $\bigcap_{i<n+1} C_i^{\sigma(i)}$ is non-empty. If $m \in \bigcap_{i<n+1} C_i^{\sigma(i)}$, then $y_{\alpha,m} \subset \bigcap_{i<n+1} x_{\alpha,i}^{\sigma(i)}$. Since $y_{\alpha,m} \cap a$ is infinite, $\left(\bigcap_{i<n+1} x_{\alpha,i}^{\sigma(i)}\right) \cap a$ is also infinite, as needed. $\dashv$

Thus the "$\mathfrak{pr}$" of $\mathfrak{s}(\mathfrak{pr})$ can either stand for "partition" or for "prompt". We next show that $\mathfrak{s}(\mathfrak{pr})$ is also the least cardinal for which a certain type of strong coloring exists.

Definition 5. Let κ be any cardinal. We say that a coloring $c : \kappa \times \omega \times \omega \to 2$ is *tortuous* if for each $A \in [\omega]^{\omega}$ and each partition of κ, $\langle K_n : n \in \omega\rangle$, there exists $n \in \omega$ such that

$$\forall \sigma \in 2^{n+1} \exists \alpha \in K_n \exists k \in A \left[k > n \wedge \forall i < n+1 \left[\sigma(i) = c(\alpha, k, i)\right]\right]. \tag{1}$$

We will say that such a c is a *tortuous coloring on* κ.

It is not obvious from the definition that there are tortuous colorings. The next lemma shows that a tortuous coloring always exists on some cardinal $\leq 2^{\aleph_0}$.

Lemma 6. *Let* $\langle X_\alpha : \alpha < \kappa\rangle$ *be a promptly splitting family. There exists a tortuous coloring on* κ.

Proof. For each $\alpha < \kappa$, write $X_\alpha = \langle x_{\alpha,i} : i < \omega\rangle$. Define $c : \kappa \times \omega \times \omega \to 2$ as follows. For any $\alpha < \kappa$, $k, i \in \omega$,

$$c(\alpha, k, i) = \begin{cases} 0 & \text{if } k \in x_{\alpha,i} \\ 1 & \text{if } k \notin x_{\alpha,i}. \end{cases}$$

We check that c is a tortuous coloring. Let $A \in [\omega]^{\omega}$ and suppose $\langle K_n : n \in \omega\rangle$ is a partition of κ. Suppose $\alpha < \kappa$ is such that X_α promptly splits A. Let $n \in \omega$ be such that $\alpha \in K_n$. We check that this n has the required properties. Fix $\sigma \in 2^{n+1}$. As X_α promptly splits A, $\left(\bigcap_{i<n+1} x_{\alpha,i}^{\sigma(i)}\right) \cap A$ is infinite. Choose $k \in \left(\bigcap_{i<n+1} x_{\alpha,i}^{\sigma(i)}\right) \cap A$ with $k > n$. Now for any $i < n+1$, $c(\alpha, k, i) = 0$ iff $k \in x_{\alpha,i}$ iff $\sigma(i) = 0$. This concludes the proof. $\dashv$

By Lemma 6, there exists a κ on which a tortuous coloring exists and the least such κ is bounded above by $\mathfrak{s}(\mathfrak{pr})$. We next show that the least such κ equals $\mathfrak{s}(\mathfrak{pr})$. First we show that the definition of a tortuous coloring implies the following self strengthening. This is the strengthening we will use to prove the bound on $\mathrm{cov}^*(\mathcal{Z}_0)$.

Lemma 7. *Let κ be any cardinal and suppose that $c : \kappa \times \omega \times \omega \to 2$ is a tortuous coloring. Then for any $A \in [\omega]^\omega$ there exists $\alpha \in \kappa$ such that for each $n \in \omega$ and $\sigma \in 2^{n+1}$, $\exists^\infty k \in A \forall i < n+1\, [\sigma(i) = c(\alpha, k, i)]$.*

Proof. First fix a 1-1 and onto enumeration, $\langle \langle \sigma_k, m_k \rangle : k \in \omega \rangle$, of the set $2^{<\omega} \times \omega$ such that for each $k \in \omega$, $|\sigma_k| \leq k$ and $m_k \leq k$. Now argue by contradiction as follows. Let $A \in [\omega]^\omega$ be given and suppose that for each $\alpha \in \kappa$, there exist $n_\alpha \in \omega$ and $\sigma_\alpha \in 2^{n_\alpha + 1}$ such that

$$\exists k_\alpha \in \omega \forall k \in A\, [k \geq k_\alpha \implies \exists i < n_\alpha + 1\, [\sigma_\alpha(i) \neq c(\alpha, k, i)]]\,.$$

Let $K_n = \{\alpha \in \kappa : \sigma_\alpha = \sigma_n \wedge k_\alpha = m_n\}$, for each $n \in \omega$. Then $\langle K_n : n \in \omega \rangle$ is a partition of κ. Applying the definition of a tortuous coloring to A and $\langle K_n : n \in \omega \rangle$, find $n \in \omega$ satisfying Condition (1) of Definition 5. Note that $\sigma_n \in 2^{<\omega}$ and that $|\sigma_n| \leq n$. So there exists $\sigma \in 2^{n+1}$ such that $\sigma_n \subset \sigma$. Now we can find $\alpha \in K_n$ and $k \in A$ such that $k > n$ and $\forall i < n+1\, [\sigma(i) = c(\alpha, k, i)]$. Note that $\sigma_\alpha = \sigma_n$, $k_\alpha = m_n$, and that $|\sigma_\alpha| = n_\alpha + 1 = |\sigma_n|$. So $n_\alpha + 1 \leq n$ and for each $i < n_\alpha + 1$, $\sigma_\alpha(i) = \sigma(i)$. Moreover, $k_\alpha = m_n \leq n < k$. Thus for each $i < n_\alpha + 1$, $\sigma_\alpha(i) = \sigma(i) = c(\alpha, k, i)$. As $k \in A$ and $k > k_\alpha$, this contradicts the choice of k_α. This contradiction concludes the proof. ⊣

Lemma 8. $\mathfrak{s}(\mathfrak{pr}) = \min\{\kappa :$ *there is a tortuous coloring on* $\kappa\}$.

Proof. Let κ be the minimal cardinal on which a tortuous coloring exists. By Lemmas 4 and 6, κ exists and is $\leq \mathfrak{s}(\mathfrak{pr})$. Let $c : \kappa \times \omega \times \omega \to 2$ be a tortuous coloring. We will show that $\mathfrak{s}(\mathfrak{pr}) \leq \kappa$ by producing a promptly splitting family of size at most κ. For each $\alpha < \kappa$ and $i < \omega$, define $x_{\alpha,i} = \{k \in \omega : c(\alpha, k, i) = 0\}$, and define $X_\alpha = \langle x_{\alpha,i} : i < \omega \rangle \in (\mathcal{P}(\omega))^\omega$. We claim that $\{X_\alpha : \alpha < \kappa\}$ is promptly splitting. We will apply Lemma 7. Fix $A \in [\omega]^\omega$. Use Lemma 7 to find $\alpha \in \kappa$ such that for each $n \in \omega$ and $\sigma \in 2^{n+1}$, $\exists^\infty k \in A \forall i < n+1\, [\sigma(i) = c(\alpha, k, i)]$. We claim X_α promptly splits A. Indeed suppose $n \in \omega$ and $\sigma \in 2^{n+1}$. Then for infinitely many $k \in A$, $\forall i < n+1\, [\sigma(i) = c(\alpha, k, i)]$. It is easy to see that each of these infinitely many $k \in A$ belong to $\left(\bigcap_{i<n+1} x_{\alpha,i}^{\sigma(i)}\right) \cap A$, whence $\left(\bigcap_{i<n+1} x_{\alpha,i}^{\sigma(i)}\right) \cap A$ is infinite. ⊣

Next, we show that a very mild guessing principle implies that $\mathfrak{s} = \mathfrak{s}(\mathfrak{pr})$. The following definition introduces a parameterized version of the combinatorial principle usually denoted ⫯ (read as "*stick*"). This principle was introduced by Broverman et al. [4]. It is known to be strictly weaker than both ♣ and CH, but it is also easy to produce models where ⫯ fails, the model obtained by adding $\aleph_2$-Cohen reals being an example (see [4] for details).

Definition 9. Let κ, λ, and θ be cardinals. Then ⫯$(\kappa, \lambda, \theta)$ is the following principle: there is a family $\mathscr{C} \subset [\kappa]^{\aleph_0}$ of size λ such that for any $X \in [\kappa]^\theta$, there exists $A \in \mathscr{C}$ such that $A \subset X$.

Note that ⫯$(\aleph_1, \aleph_1, \aleph_1)$ is the same as ⫯. Several minimal instances of ⫯$(\aleph_1, \lambda, \aleph_1)$, ⫯$(\kappa, \kappa, \aleph_1)$, and ⫯$(\kappa, \kappa, \kappa)$ were studied by Fuchino et al. in [6].

Lemma 10. *If* ⫯$(\mathfrak{s}, \mathfrak{s}, \mathfrak{p})$ *holds, then* $\mathfrak{s} = \mathfrak{s}(\mathfrak{pr})$.

Proof. Fix a splitting family $\langle x_\alpha : \alpha < \mathfrak{s} \rangle$. Let $\mathscr{C} \subset [\mathfrak{s}]^{\aleph_0}$ be a family of size $\mathfrak{s}$ with the property that for any $X \in [\mathfrak{s}]^{\mathfrak{p}}$, there exists $A \in \mathscr{C}$ such that $A \subset X$. For each $A \in \mathscr{C}$ it is possible to choose $B_A \subset A$ whose order-type is ω because A is an infinite set of ordinals. Let $\langle \beta_{A,i} : i \in \omega \rangle$ be the enumeration of B_A in increasing order. Define $y_{A,0} = x_{\beta_{A,0}}$. For each $0 < i < \omega$, define $y_{A,i} = x^1_{\beta_{A,0}} \cap \cdots \cap x^1_{\beta_{A,i-1}} \cap x_{\beta_{A,i}}$. Note that for each $i \in \omega$, $y_{A,i} \subset x_{\beta_{A,i}}$ and that if $j < i$, then $y_{A,i} \cap x_{\beta_{A,j}} = 0$. So for any $j < i < \omega$, $y_{A,i} \cap y_{A,j} = 0$. Now for $i \in \omega$, if $i \in \bigcup_{j\in\omega} y_{A,j}$, then put $z_{A,i} = y_{A,i}$, else put $z_{A,i} = y_{A,i} \cup \{i\}$. Thus it is clear that $Z_A = \langle z_{A,i} : i \in \omega \rangle$ is a partition of ω. Let $\mathcal{F} = \{Z_A : A \in \mathscr{C}\}$. Then $\mathcal{F}$ is a family of partitions and $|\mathcal{F}| \leq |\mathscr{C}| = \mathfrak{s}$. We claim that it is a splitting family of partitions. To this end fix $a \in [\omega]^\omega$. We construct a set $X \in [\mathfrak{s}]^{\mathfrak{p}}$ as follows. We will build sequences $\langle \gamma_\delta : \delta < \mathfrak{p} \rangle$, $\langle c_\delta : \delta < \mathfrak{p} \rangle$, and $\langle b_\delta : \delta < \mathfrak{p} \rangle$ such that the following conditions are satisfied for each $\delta < \mathfrak{p}$:

(1) $\gamma_\delta < \mathfrak{s}$ and $\forall \xi < \delta \, [\gamma_\xi < \gamma_\delta]$;
(2) $c_\delta \in [a]^\omega$ and $\forall \xi < \delta \, [c_\delta \subset^* b_\xi]$;
(3) $b_\delta = c_\delta \cap x^1_{\gamma_\delta}$ and γ_δ is the least $\alpha < \mathfrak{s}$ such that x_α splits c_δ.

Suppose for a moment that such sequences can be constructed. By (1) each $\gamma_\delta \in \mathfrak{s}$ and $\gamma_\xi \neq \gamma_\delta$, whenever $\xi \neq \delta$. Therefore $X = \{\gamma_\delta : \delta < \mathfrak{p}\} \in [\mathfrak{s}]^{\mathfrak{p}}$. Let $A \in \mathscr{C}$ be such that $A \subset X$. We check that Z_A splits a. First $\beta_{A,0} = \gamma_\delta$, for some $\delta < \mathfrak{p}$. Since x_{γ_δ} splits c_δ by clause (3), so $x^0_{\gamma_\delta} \cap a = x^0_{\beta_{A,0}} \cap a = x_{\beta_{A,0}} \cap a = y_{A,0} \cap a$ is infinite. Next fix $0 < i < \omega$. Suppose $0 \leq j \leq i-1$. Then $\beta_{A,j} < \beta_{A,i}$, and so there are $\xi < \delta < \mathfrak{p}$ with $\beta_{A,j} = \gamma_\xi$ and $\beta_{A,i} = \gamma_\delta$. By clauses (2) and (3), $c_\delta \subset^* b_\xi \subset x^1_{\gamma_\xi} = x^1_{\beta_{A,j}}$. Therefore $c_\delta \subset^* a \cap x^1_{\beta_{A,0}} \cap \cdots \cap x^1_{\beta_{A,i-1}}$. Since x_{γ_δ} splits c_δ, we have that $a \cap x^1_{\beta_{A,0}} \cap \cdots \cap x^1_{\beta_{A,i-1}} \cap x^0_{\gamma_\delta} = a \cap x^1_{\beta_{A,0}} \cap \cdots \cap x^1_{\beta_{A,i-1}} \cap x_{\beta_{A,i}} = a \cap y_{A,i}$ is infinite. Thus we have shown that $\forall i \in \omega \, [|a \cap y_{A,i}| = \aleph_0]$, which implies that $\forall i \in \omega \, [|a \cap z_{A,i}| = \aleph_0]$ because $\forall i \in \omega \, [y_{A,i} \subset z_{A,i}]$. Hence Z_A splits a, as claimed.

To complete the proof, we show how to construct the sequences satisfying (1)–(3) by induction on $\delta < \mathfrak{p}$. Fix $\delta < \mathfrak{p}$ and assume that $\langle \gamma_\xi : \xi < \delta \rangle$, $\langle c_\xi : \xi < \delta \rangle$, and $\langle b_\xi : \xi < \delta \rangle$ satisfying (1)–(3) are given. Consider any $\zeta < \xi < \delta$. By clauses (2) and (3), $b_\xi \subset c_\xi \subset^* b_\zeta$. So the sequence $\langle b_\xi : \xi < \delta \rangle$ is $\subset^*$–descending. Also for each $\xi < \delta$, $b_\xi \in [a]^\omega$ because x_{γ_ξ} splits c_ξ and $b_\xi \subset c_\xi \subset a$. Since $\delta < \mathfrak{p}$ we can find $c_\delta \in [a]^\omega$ so that $\forall \xi < \delta \, [c_\delta \subset^* b_\xi]$. Thus clause (2) is satisfied. Let γ_δ be the least $\alpha < \mathfrak{s}$ such that x_α splits c_δ and define $b_\delta = x^1_{\gamma_\delta} \cap c_\delta$. Then $\gamma_\delta < \mathfrak{s}$ and clause (3) holds by definition. So it only remains to check that $\forall \xi < \delta \, [\gamma_\xi < \gamma_\delta]$. Fix $\xi < \delta$ and assume for a contradiction that $\gamma_\delta \leq \gamma_\xi$. Note that x_{γ_δ} splits c_ξ because $c_\delta \subset^* c_\xi$. It follows that $\gamma_\xi \leq \gamma_\delta$, whence $\gamma_\xi = \gamma_\delta$. However it now follows that x_{γ_ξ} splits $x^1_{\gamma_\xi}$ because $c_\delta \subset^* b_\xi \subset x^1_{\gamma_\xi}$, which is absurd. This contradiction completes the inductive construction.

Thus $\mathcal{F}$ is a splitting family of partitions. Since $|\mathcal{F}| \leq \mathfrak{s}$, $\mathfrak{s}(\mathfrak{pr}) \leq \mathfrak{s}$, and since $\mathfrak{s} \leq \mathfrak{s}(\mathfrak{pr})$ trivially holds, we conclude that $\mathfrak{s} = \mathfrak{s}(\mathfrak{pr})$. ⊣

We will conclude this section by establishing yet another point of similarity between $\mathfrak{s}$ and $\mathfrak{s}(\mathfrak{pr})$. We will show that a Suslin c.c.c. forcing cannot increase $\mathfrak{s}(\mathfrak{pr})$. This should be compared with the well-known result of Judah and Shelah [8] that a Suslin c.c.c. forcing cannot increase $\mathfrak{s}$ (see also [2]). Recall the following definitions.

Definition 11. A forcing notion $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}, \perp_{\mathbb{P}} \rangle$ is *Suslin c.c.c.* if it has the countable chain condition and there exist analytic sets $R_0 \subset \omega^\omega$, and $R_1, R_2 \subset \omega^\omega \times \omega^\omega$ such that

(1) $\mathbb{P} = R_0$;
(2) $\leq_{\mathbb{P}} = \{\langle q, p \rangle \in \mathbb{P} \times \mathbb{P} : q \leq_{\mathbb{P}} p\} = R_1$;
(3) $\perp_{\mathbb{P}} = \{\langle p, q \rangle \in \mathbb{P} \times \mathbb{P} : \neg \exists r \in \mathbb{P} \, [r \leq_{\mathbb{P}} p \wedge r \leq_{\mathbb{P}} q]\} = R_2$.

Analytic sets are represented as projections of trees. For any set A, if $T \subset A^{<\omega}$ is a tree, then $[T]$ denotes the set of all branches through T, that is $[T] = \{f \in A^{\omega} : \forall n \in \omega\, [f \restriction n \in T]\}$. Following standard convention, given a tree $T \subset (\omega \times \omega)^{<\omega}$ and $\sigma, \tau \in \omega^n$ for some $n \in \omega$, we will abuse notation and write $\langle \sigma, \tau \rangle \in T$ when what we mean is $\langle \langle \sigma(i), \tau(i) \rangle : i < n \rangle \in T$. In a related abuse of notation, we write $\langle f, g \rangle \in [T]$ for some $f, g \in \omega^{\omega}$ when what we mean is $\langle \langle f(i), g(i) \rangle : i \in \omega \rangle \in [T]$. Similar notational conventions apply to subtrees of $(\omega \times \omega \times \omega)^{<\omega}$. The reader may consult Kechris [9] for further details about representing analytic sets as projections of trees.

For the remainder of this section, fix a Suslin c.c.c. poset $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}, \perp_{\mathbb{P}} \rangle$. Fix also trees $T_0 \subset (\omega \times \omega)^{<\omega}$ and $T_1, T_2 \subset (\omega \times \omega \times \omega)^{<\omega}$ such that

$$\begin{aligned} &\forall p\, [p \in \mathbb{P} \iff \exists g \in \omega^{\omega}\, [\langle p, g \rangle \in [T_0]]]\,, \\ &\forall p \forall q\, [q \leq_{\mathbb{P}} p \iff \exists g \in \omega^{\omega}\, [\langle q, p, g \rangle \in [T_1]]]\,, \text{ and} \\ &\forall p \forall q\, [p \perp_{\mathbb{P}} q \iff \exists g \in \omega^{\omega}\, [\langle p, q, g \rangle \in [T_2]]]\,. \end{aligned}$$

Definition 12. Let $\mathring{A}$ be a $\mathbb{P}$-name. Suppose that $\Vdash_{\mathbb{P}} \mathring{A} \in [\omega]^{\omega}$. Choose a sequence $\mathcal{A} = \langle p_{m,n} : \langle m, n \rangle \in \omega \times \omega \rangle$ and a function $\mathcal{F} : \omega \times \omega \to 2$ such that:

(1) for each $n \in \omega$, $\{p_{m,n} : m \in \omega\} \subset \mathbb{P}$ is a maximal antichain in $\mathbb{P}$;
(2) for each $n, m \in \omega$, $p_{m,n} \Vdash_{\mathbb{P}} n \in \mathring{A}$ if and only if $\mathcal{F}(m,n) = 1$, while $p_{m,n} \Vdash_{\mathbb{P}} n \notin \mathring{A}$ if and only if $\mathcal{F}(m,n) = 0$.

Define $\mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F}) = \{\langle \check{n}, p_{m,n} \rangle : n, m \in \omega \wedge \mathcal{F}(m,n) = 1\}$. Note that $\mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F})$ is a $\mathbb{P}$-name and that $\Vdash_{\mathbb{P}} \mathring{A} = \mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F})$.

Suppose $\mathbf{W}$ is a forcing extension of the universe $\mathbf{V}$. Then $\mathbb{P}^{\mathbf{W}}$, $\leq_{\mathbb{P}}^{\mathbf{W}}$, and $\perp_{\mathbb{P}}^{\mathbf{W}}$ will denote the reinterpretations in $\mathbf{W}$ of $\mathbb{P}$, $\leq_{\mathbb{P}}$, and $\perp_{\mathbb{P}}$ respectively.

It is well-known that $\left\langle \mathbb{P}^{\mathbf{W}}, \leq_{\mathbb{P}}^{\mathbf{W}}, \mathbb{1}_{\mathbb{P}}, \perp_{\mathbb{P}}^{\mathbf{W}} \right\rangle$ is a c.c.c. forcing notion in $\mathbf{W}$ with $\mathbb{P} \subset \mathbb{P}^{\mathbf{W}}$, and also that for each $n \in \omega$, $\{p_{m,n} : m \in \omega\} \subset \mathbb{P}^{\mathbf{W}}$ is a maximal antichain in $\mathbb{P}^{\mathbf{W}}$. The reader may consult either [8] or [2] for further details. Note that $\mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F})$ is a $\mathbb{P}^{\mathbf{W}}$-name and that if H is $(\mathbf{W}, \mathbb{P}^{\mathbf{W}})$-generic, then $\mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F})[H] = \{n : n \in \omega \wedge \exists m \in \omega\, [\mathcal{F}(m,n) = 1 \wedge p_{m,n} \in H]\}$. Thus $\Vdash_{\mathbb{P}^{\mathbf{W}}} \mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F}) \subset \omega$ holds in $\mathbf{W}$.

Lemma 13. *In $\mathbf{W}$, $\Vdash_{\mathbb{P}^{\mathbf{W}}} \mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F}) \in [\omega]^{\omega}$.*

Proof. Write $\mathring{N}$ for $\mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F})$. We have remarked above that $\mathring{N}$ is a $\mathbb{P}^{\mathbf{W}}$-name and that $\Vdash_{\mathbb{P}^{\mathbf{W}}} \mathring{N} \subset \omega$ holds in $\mathbf{W}$. Now in $\mathbf{V}$, we have that for each $p \in \mathbb{P}$ and $l \in \omega$, there exist $n, m \in \omega$ so that $n > l$, $\mathcal{F}(m,n) = 1$, and $p \not\perp_{\mathbb{P}} p_{m,n}$, which can be rephrased as

$$\begin{aligned} &\forall p, g \in \omega^{\omega} \forall \langle g_{m,n} : \langle m, n \rangle \in \omega \times \omega \rangle \in (\omega^{\omega})^{\omega \times \omega} \\ &\forall l \in \omega \exists n, m \in \omega\, [\langle p, g \rangle \in [T_0] \implies (n > l \wedge \mathcal{F}(m,n) = 1 \wedge \langle p, p_{m,n}, g_{m,n} \rangle \notin [T_2])]\,. \end{aligned}$$

This statement is $\mathbf{\Pi}^1_1$, and so it holds in $\mathbf{W}$. Now in $\mathbf{W}$, suppose that $p \in \mathbb{P}^{\mathbf{W}}$ and that $l \in \omega$. Then we can find $n, m \in \omega$ and $q \in \mathbb{P}^{\mathbf{W}}$ so that $n > l$, $\mathcal{F}(m,n) = 1$, and $q \leq_{\mathbb{P}}^{\mathbf{W}} p, p_{m,n}$. Hence $\langle \check{n}, p_{m,n} \rangle \in \mathring{N}$ and so $q \Vdash_{\mathbb{P}^{\mathbf{W}}} n \in \mathring{N}$. Thus we have shown that $\forall p \in \mathbb{P}^{\mathbf{W}} \forall l \in \omega \exists n > l \exists q \leq_{\mathbb{P}}^{\mathbf{W}} p \left[q \Vdash_{\mathbb{P}^{\mathbf{W}}} n \in \mathring{N}\right]$, which implies that $\Vdash_{\mathbb{P}^{\mathbf{W}}} \mathring{N}$ is infinite. $\dashv$

Lemma 14. *Suppose $p \in \mathbb{P}$ and that $p \Vdash_{\mathbb{P}} \mathring{A}$ is not promptly split by $\mathbf{V} \cap (\mathcal{P}(\omega))^{\omega}$ holds in $\mathbf{V}$. Then in $\mathbf{W}$, $p \Vdash_{\mathbb{P}^{\mathbf{W}}} \mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F})$ is not promptly split by $\mathbf{W} \cap (\mathcal{P}(\omega))^{\omega}$.*

Proof. Write $\mathring{N}$ for $\mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F})$. In $\mathbf{V}$, we have that for each $\bar{p} \leq_{\mathbb{P}} p$ and for each $\langle x_i : i \in \omega \rangle \in (\mathcal{P}(\omega))^{\omega}$, there exist $q \leq_{\mathbb{P}} \bar{p}$, $k \in \omega$, $\sigma \in 2^{k+1}$, $l \in \omega$ such that for each

$n, m \in \omega$, if $n \geq l$, $\mathcal{F}(m,n) = 1$, and $n \in \bigcap_{i<k+1} x_i^{\sigma(i)}$, then $q \perp_{\mathbb{P}} p_{m,n}$. This can be rephrased as

$$\forall \bar{p}, \bar{g} \in \omega^\omega \forall \langle x_i : i \in \omega\rangle \in (\mathcal{P}(\omega))^\omega \exists q, g \in \omega^\omega \exists \langle g_{m,n} : \langle m,n\rangle \in \omega \times \omega\rangle \in (\omega^\omega)^{\omega\times\omega}$$
$$\exists k \in \omega \exists \sigma \in 2^{k+1} \exists l \in \omega \forall n, m \in \omega \left[\langle \bar{p}, p, \bar{g}\rangle \in [T_1] \implies \Big(\langle q, \bar{p}, g\rangle \in [T_1]\right.$$
$$\left.\wedge \left(\left(n \geq l \wedge \mathcal{F}(m,n) = 1 \wedge n \in \bigcap_{i<k+1} x_i^{\sigma(i)}\right) \implies \langle q, p_{m,n}, g_{m,n}\rangle \in [T_2]\right)\Big)\right].$$

This is $\mathbf{\Pi}^1_2$. So by Shoenfield's absoluteness, it continues to holds in $\mathbf{W}$. Now working in $\mathbf{W}$, fix any $\bar{p} \leq_{\mathbb{P}}^{\mathbf{W}} p$ and $\langle x_i : i \in \omega\rangle \in (\mathcal{P}(\omega))^\omega$. We know that there are $q \leq_{\mathbb{P}}^{\mathbf{W}} \bar{p}$, $k \in \omega$, $\sigma \in 2^{k+1}$, and $l \in \omega$ with the property that for all $n, m \in \omega$, if $n \geq l$, $\mathcal{F}(m,n) = 1$, and $n \in \bigcap_{i<k+1} x_i^{\sigma(i)}$, then $q \perp_{\mathbb{P}}^{\mathbf{W}} p_{m,n}$. We claim that $q \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k+1} x_i^{\sigma(i)}\right) \cap \mathring{N} \subset l$. Suppose not. Then let H be a $(\mathbf{W}, \mathbb{P}^{\mathbf{W}})$-generic filter such that $q \in H$ and in $\mathbf{W}[H]$, there exists an $n \in \left(\bigcap_{i<k+1} x_i^{\sigma(i)}\right) \cap \mathring{N}[H]$ with $n \notin l$. By the definition of $\mathring{N}$, $n \in \omega$ and there exists $m \in \omega$ such that $\mathcal{F}(m,n) = 1$ and $p_{m,n} \in H$. However $q \not\perp_{\mathbb{P}}^{\mathbf{W}} p_{m,n}$ because they both belong to H, which contradicts the choice of q and the fact that $n \geq l$. This contradiction proves that $q \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k+1} x_i^{\sigma(i)}\right) \cap \mathring{N} \subset l$ holds in $\mathbf{W}$, which proves that $p \Vdash_{\mathbb{P}^{\mathbf{W}}} \mathring{N}$ is not promptly split by $\mathbf{W} \cap (\mathcal{P}(\omega))^\omega$. ⊣

Recall that if $\langle \mathbb{R}, \leq_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}}\rangle$ and $\langle \mathbb{S}, \leq_{\mathbb{S}}, \mathbb{1}_{\mathbb{S}}\rangle$ are posets and if $\pi : \mathbb{R} \to \mathbb{S}$ is a complete embedding, then for any $(\mathbf{V}, \mathbb{S})$-generic filter H, $\pi^{-1}(H)$ is $(\mathbf{V}, \mathbb{R})$-generic. We can recursively define a map from the $\mathbb{R}$-names to the $\mathbb{S}$-names using π. Abusing notation, this map shall also be denoted by π. For an $\mathbb{R}$-name $\mathring{a}$, $\pi(\mathring{a}) = \{\langle \pi(\mathring{x}), \pi(p)\rangle : \langle \mathring{x}, p\rangle \in \mathring{a}\}$. If H is a $(\mathbf{V}, \mathbb{S})$-generic filter, then for any $\mathbb{R}$-name, $\mathring{a}$, $\mathring{a}\left[\pi^{-1}(H)\right] = \pi(\mathring{a})[H]$, and if $x \in \mathbf{V}$, then $\pi(\check{x}) = \check{x}$, where of course the first "$\check{x}$" is with respect to $\langle \mathbb{R}, \leq_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}}\rangle$ and the second $\check{x}$ is with respect to $\langle \mathbb{S}, \leq_{\mathbb{S}}, \mathbb{1}_{\mathbb{S}}\rangle$.

In the specific case when $\langle \mathbb{R}, \leq_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}}\rangle = \langle \mathbb{S}, \leq_{\mathbb{S}}, \mathbb{1}_{\mathbb{S}}\rangle$ and π is an automorphism, if H is any $(\mathbf{V}, \mathbb{R})$-generic filter, then $\mathbf{V}\left[\pi^{-1}(H)\right] = \mathbf{V}[H]$, and moreover for any formula $\varphi(x_1, \dots, x_n)$, any $\mathring{a}_1, \dots, \mathring{a}_n \in \mathbf{V}^{\mathbb{R}}$, and any $r \in \mathbb{R}$, $r \Vdash_{\mathbb{R}} \varphi(\mathring{a}_1, \dots, \mathring{a}_n)$ if and only if $\pi(r) \Vdash_{\mathbb{R}} \varphi(\pi(\mathring{a}_1), \dots, \pi(\mathring{a}_n))$.

Lemma 15. *Let $\langle \mathbb{R}, \leq_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}}\rangle$ be a poset that preserves ω_1. Assume that there exist sequences $\langle \mathring{x}_i : i < \omega\rangle$, $\langle \pi_{r,k} : r \in \mathbb{R} \wedge k \in \omega\rangle$, and $\langle \pi_{r,k,\alpha} : r \in \mathbb{R} \wedge k \in \omega \wedge \alpha \in \omega_1\rangle$ satisfying the following properties:*

1. *for each $i < \omega$, $\mathring{x}_i$ is an $\mathbb{R}$-name such that $\Vdash_{\mathbb{R}} \mathring{x}_i \in [\omega]^\omega$;*
2. *for each $r \in \mathbb{R}$, $k \in \omega$, and $\alpha \in \omega_1$, $\pi_{r,k,\alpha} : \mathbb{R} \to \mathbb{R}$ is an automorphism such that $\pi_{r,k,\alpha}(r) = r$ and $\forall i < k\, [\Vdash_{\mathbb{R}} \pi_{r,k,\alpha}(\mathring{x}_i) = \mathring{x}_i]$;*
3. *for each $r \in \mathbb{R}$ and $k \in \omega$, $\pi_{r,k} : \mathbb{R} \to \mathbb{R}$ is an automorphism such that $\pi_{r,k}(r) = r$, $\forall i < k\, [\Vdash_{\mathbb{R}} \pi_{r,k}(\mathring{x}_i) = \mathring{x}_i]$, and $\Vdash_{\mathbb{R}} \omega \setminus \pi_{r,k}(\mathring{x}_k) \subset^* \mathring{x}_k$;*
4. *for each $r \in \mathbb{R}$, $k \in \omega$, and $\alpha, \beta \in \omega_1$, if $\alpha \neq \beta$, then*
$$\Vdash_{\mathbb{R}} |\pi_{r,k,\alpha}(\mathring{x}_k) \cap \pi_{r,k,\beta}(\mathring{x}_k)| < \omega.$$

Then there is no $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \mathring{A}$ is not promptly split by $\mathbf{V} \cap (\mathcal{P}(\omega))^\omega$.

Proof. Assume not. Fix $p \in \mathbb{P}$ so that $p \Vdash_{\mathbb{P}} \mathring{A}$ is not promptly split by $\mathbf{V} \cap (\mathcal{P}(\omega))^\omega$. As before write $\mathring{N}$ for $\mathring{N}(\mathring{A}, \mathcal{A}, \mathcal{F})$. For the moment, fix a $(\mathbf{V}, \mathbb{R})$-generic filter G and let $\mathbf{W} = \mathbf{V}[G]$. Work inside $\mathbf{W}$. By Lemma 14 and by (1), we know that $p \Vdash_{\mathbb{P}^{\mathbf{W}}} \langle \mathring{x}_i[G] : i < \omega\rangle$ does not promptly split $\mathring{N}$. Let $k \in \omega$ be minimal with the property that there exist $\sigma \in 2^{k+1}$ and $q \leq_{\mathbb{P}}^{\mathbf{W}} p$ such that

$$q \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k+1} (\mathring{x}_i[G])^{\sigma(i)}\right) \cap \mathring{N} \text{ is finite.}$$

Choose a $\sigma \in 2^{k+1}$ witnessing this property of k. Then

$$p \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N} \text{ is infinite,}$$

where $\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}$ is taken to be ω when $k = 0$, because of the minimality of k and because $\Vdash_{\mathbb{P}^{\mathbf{W}}} \mathring{N} \in [\omega]^{\omega}$.

Claim 16. *$p \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N} \cap (\mathring{x}_k\,[G])^1$ is infinite.*

Proof. Suppose not. Then $q \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N} \cap (\mathring{x}_k\,[G])^1$ is finite, for some $q \le_{\mathbb{P}}^{\mathbf{W}} p$. In other words $q \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N} \subset^* \mathring{x}_k\,[G]$. Fix $\mathring{q} \in \mathbf{V}^{\mathbb{R}}$ with $q = \mathring{q}\,[G]$. We can find an $r \in G$ such that back in $\mathbf{V}$, $r \Vdash_{\mathbb{R}} \mathring{q} \le_{\mathbb{P}}^{\mathbf{V}[\mathring{G}]} p$ and

$$r \Vdash_{\mathbb{R}} \text{“}\mathring{q} \Vdash_{\mathbb{P}^{\mathbf{V}[\mathring{G}]}} \left(\bigcap_{i<k} \mathring{x}_i^{\sigma(i)}\right) \cap \mathring{N} \subset^* \mathring{x}_k\text{”}.$$

For each $\alpha \in \omega_1$, we have that $r \Vdash_{\mathbb{R}} \pi_{r,k,\alpha}(\mathring{q}) \le_{\mathbb{P}}^{\mathbf{V}\left[\pi_{r,k,\alpha}(\mathring{G})\right]} p$ and also that

$$r \Vdash_{\mathbb{R}} \text{“}\pi_{r,k,\alpha}(\mathring{q}) \Vdash_{\mathbb{P}^{\mathbf{V}\left[\pi_{r,k,\alpha}(\mathring{G})\right]}} \left(\bigcap_{i<k} (\pi_{r,k,\alpha}(\mathring{x}_i))^{\sigma(i)}\right) \cap \mathring{N} \subset^* \pi_{r,k,\alpha}(\mathring{x}_k)\text{”}.$$

Observe that $\pi_{r,k,\alpha}(\mathring{G})\,[G] = \mathring{G}\left[\pi_{r,k,\alpha}^{-1}(G)\right] = \pi_{r,k,\alpha}^{-1}(G)$, and so $\mathbf{V}\left[\pi_{r,k,\alpha}(\mathring{G})\,[G]\right] = \mathbf{V}\left[\pi_{r,k,\alpha}^{-1}(G)\right] = \mathbf{V}\,[G] = \mathbf{W}$. Also by Clause (2), for each $i < k$, $\pi_{r,k,\alpha}(\mathring{x}_i)\,[G] = \mathring{x}_i\,[G]$. Therefore in $\mathbf{W}$, we have that $\forall \alpha \in \omega_1 \left[\pi_{r,k,\alpha}(\mathring{q})\,[G] \le_{\mathbb{P}}^{\mathbf{W}} p\right]$ and that for each $\alpha \in \omega_1$,

$$\pi_{r,k,\alpha}(\mathring{q})\,[G] \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N} \subset^* \pi_{r,k,\alpha}(\mathring{x}_k)\,[G].$$

Furthermore by Clause (4), for each $\alpha, \beta \in \omega_1$, if $\alpha \ne \beta$, then

$$|\pi_{r,k,\alpha}(\mathring{x}_k)\,[G] \cap \pi_{r,k,\beta}(\mathring{x}_k)\,[G]| < \omega.$$

Since $p \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N}$ is infinite, it follows that $\{\pi_{r,k,\alpha}(\mathring{q})\,[G] : \alpha \in \omega_1\}$ is an antichain in $\mathbb{P}^{\mathbf{W}}$. However this means that $\mathbb{P}^{\mathbf{W}}$ is not a c.c.c. poset in $\mathbf{W}$ because $\langle \mathbb{R}, \le_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}}\rangle$ preserves ω_1 by hypothesis. This is a contradiction which proves the claim. $\dashv$

By Claim 16, we can find an $r \in G$ so that in $\mathbf{V}$,

$$r \Vdash_{\mathbb{R}} \text{“}p \Vdash_{\mathbb{P}^{\mathbf{V}[\mathring{G}]}} \left(\bigcap_{i<k} \mathring{x}_i^{\sigma(i)}\right) \cap \mathring{N} \cap \mathring{x}_k^1 \text{ is infinite”}.$$

Applying $\pi_{r,k}$, we have that in $\mathbf{V}$

$$r \Vdash_{\mathbb{R}} \text{“}p \Vdash_{\mathbb{P}^{\mathbf{V}\left[\pi_{r,k}(\mathring{G})\right]}} \left(\bigcap_{i<k} (\pi_{r,k}(\mathring{x}_i))^{\sigma(i)}\right) \cap \mathring{N} \cap (\pi_{r,k}(\mathring{x}_k))^1 \text{ is infinite”}.$$

Observe that $\mathbf{V}\left[\pi_{r,k}(\mathring{G})\,[G]\right] = \mathbf{W}$ and that for each $i < k$, $\pi_{r,k}(\mathring{x}_i)\,[G] = \mathring{x}_i\,[G]$. Therefore in $\mathbf{W}$ we have

$$p \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N} \cap (\pi_{r,k}(\mathring{x}_k)\,[G])^1 \text{ is infinite.}$$

By Clause (3), $(\pi_{r,k}(\mathring{x}_k)\,[G])^1 = \omega \setminus \pi_{r,k}(\mathring{x}_k)\,[G] \subset^* \mathring{x}_k\,[G]$. Therefore

$$p \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N} \cap \mathring{x}_k\,[G] \text{ is infinite.}$$

However this together with Claim 16 gives a contradiction because by the choice of k and σ, there exists $q \le_{\mathbb{P}}^{\mathbf{W}} p$ such that

$$q \Vdash_{\mathbb{P}^{\mathbf{W}}} \left(\bigcap_{i<k} (\mathring{x}_i\,[G])^{\sigma(i)}\right) \cap \mathring{N} \cap (\mathring{x}_k\,[G])^{\sigma(k)} \text{ is finite.}$$

This contradiction concludes the proof. $\dashv$

Theorem 17. $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}, \perp_{\mathbb{P}} \rangle$ *does not add any real that is not promptly split by* $\mathbf{V} \cap (\mathcal{P}(\omega))^{\omega}$.

Proof. If not, then there would be a $\mathbb{P}$-name $\mathring{A}$ such that $\Vdash_{\mathbb{P}} \mathring{A} \in [\omega]^{\omega}$ and a $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \mathring{A}$ is not promptly split by $\mathbf{V} \cap (\mathcal{P}(\omega))^{\omega}$. In view of Lemma 15, in order to get a contradiction, it suffices to find a c.c.c. poset $\langle \mathbb{R}, \leq_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}} \rangle$ together with sequences $\langle \mathring{x}_i : i < \omega \rangle$, $\langle \pi_{r,k} : r \in \mathbb{R} \wedge k \in \omega \rangle$, and $\langle \pi_{r,k,\alpha} : r \in \mathbb{R} \wedge k \in \omega \wedge \alpha \in \omega_1 \rangle$ satisfying Clauses (1)–(4) there. Define $\mathbb{R}$ to be the collection of all r such that r is a function, $|r| < \omega$, $\operatorname{dom}(r) \subset \omega \times \omega$, $\operatorname{ran}(r) \subset \omega$, and $\forall \langle l, i \rangle, \langle l, j \rangle \in \operatorname{dom}(r) \, [i \neq j \implies r(l,i) \neq r(l,j)]$. Define $s \leq_{\mathbb{R}} r$ if and only if $s, r \in \mathbb{R}$ and $s \supset r$, and define $\mathbb{1}_{\mathbb{R}} = \emptyset$. Obviously $\langle \mathbb{R}, \leq_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}} \rangle$ is a c.c.c. poset. Define $E = \{m \in \omega : m \text{ is even}\}$ and $O = \{m \in \omega : m \text{ is odd}\}$. Also, for each $r \in \mathbb{R}$, fix $L_r \in \omega$ with $\operatorname{ran}(r) \subset L_r$. Fix a $(\mathbf{V}, \mathbb{R})$-generic filter G for a moment. In $\mathbf{V}[G]$, $F = \bigcup G$ is a function from $\omega \times \omega$ to ω with the property that for each $\langle l, i \rangle, \langle l, j \rangle \in \omega \times \omega$, if $i \neq j$, then $F(l,i) \neq F(l,j)$. Therefore for any $l \in \omega$ and any finite $T \subset \omega$, $\{i < \omega : F(l,i) \in T\}$ is finite. For each $l \in \omega$, define $x_l = \{i \in \omega : F(l,i) \in E\}$. It is clear that $x_l \in [\omega]^{\omega}$ for every $l \in \omega$. Unfixing G, back in $\mathbf{V}$, let $\mathring{F}$ be an $\mathbb{R}$-name such that $\Vdash_{\mathbb{R}} \mathring{F} = \bigcup \mathring{G}$, and let $\langle \mathring{x}_l : l < \omega \rangle$ be a sequence of $\mathbb{R}$-names such that for each $l < \omega$, $\Vdash_{\mathbb{R}} \mathring{x}_l = \{i \in \omega : \mathring{F}(l,i) \in E\}$. Then $\Vdash_{\mathbb{R}} \mathring{x}_l \in [\omega]^{\omega}$, for all $l \in \omega$.

Now suppose that $f : \omega \to \omega$ is a permutation and that $k \in \omega$. We define a function $\pi_{f,k} : \mathbb{R} \to \mathbb{R}$ as follows. Let $r \in \mathbb{R}$ be given. Then $\pi_{f,k}(r)$ is the function such that $\operatorname{dom}(\pi_{f,k}(r)) = \operatorname{dom}(r)$ and for every $\langle l, i \rangle \in \operatorname{dom}(\pi_{f,k}(r))$, $\pi_{f,k}(r)(l,i) = f(r(l,i))$ when $k = l$, while $\pi_{f,k}(r)(l,i) = r(l,i)$ when $l \neq k$. It is easy to check that $\pi_{f,k}$ is an automorphism. Furthermore for each $r \in \mathbb{R}$, if $\forall m \in L_r \, [f(m) = m]$, then $\pi_{f,k}(r) = r$. Fix a $(\mathbf{V}, \mathbb{R})$-generic filter G. Then $\pi_{f,k}(\mathring{G})\,[G] = \mathring{G}\left[\pi_{f,k}^{-1}(G)\right] = \pi_{f,k}^{-1}(G) = \{r \in \mathbb{R} : \pi_{f,k}(r) \in G\} = \{\pi_{f^{-1},k}(s) : s \in G\}$. Therefore $\pi_{f,k}(\mathring{F})\,[G] = \bigcup \pi_{f,k}(\mathring{G})\,[G] = \bigcup \{\pi_{f^{-1},k}(s) : s \in G\}$. It follows that for any $\langle l, i \rangle \in \omega \times \omega$, $\pi_{f,k}(\mathring{F})\,[G]\,(l,i) = f^{-1}(\mathring{F}\,[G]\,(l,i))$ when $l = k$, while $\pi_{f,k}(\mathring{F})\,[G]\,(l,i) = \mathring{F}\,[G]\,(l,i)$ when $l \neq k$. So for every $l \in \omega$ with $l \neq k$, $\pi_{f,k}(\mathring{x}_l)\,[G] = \mathring{x}_l\,[G]$, and $\pi_{f,k}(\mathring{x}_k)\,[G] = \{i \in \omega : \mathring{F}\,[G]\,(k,i) \in f''E\}$. In particular, unfixing G and going back to $\mathbf{V}$, we have that for each $l \in \omega$, if $l \neq k$, then $\Vdash_{\mathbb{R}} \pi_{f,k}(\mathring{x}_l) = \mathring{x}_l$.

Now, working in $\mathbf{V}$, fix an almost disjoint family $\{A_\alpha : \alpha < \omega_1\}$ of infinite subsets of ω. Let $r \in \mathbb{R}$ and $k \in \omega$ be fixed. Let $f : \omega \to \omega$ be a permutation such that $\forall m \in L_r \, [f(m) = m]$, $f''(E \setminus L_r) = O \setminus L_r$, and $f''(O \setminus L_r) = E \setminus L_r$. Define $\pi_{r,k} = \pi_{f,k}$. Also for each $\alpha < \omega_1$, choose a permutation $f_\alpha : \omega \to \omega$ such that $\forall m \in L_r \, [f_\alpha(m) = m]$, $f_\alpha''(E \setminus L_r) = A_\alpha \setminus L_r$, and $f_\alpha''(O \setminus L_r) = \omega \setminus (A_\alpha \cup L_r)$. For each $\alpha \in \omega_1$, define $\pi_{r,k,\alpha} = \pi_{f_\alpha,k}$. In light of the observations already made, it suffices to check that $\Vdash_{\mathbb{R}} \omega \setminus \pi_{f,k}(\mathring{x}_k) \subset^* \mathring{x}_k$, and that for any $\alpha, \beta \in \omega_1$, if $\alpha \neq \beta$, then $\Vdash_{\mathbb{R}} \left|\pi_{f_\alpha,k}(\mathring{x}_k) \cap \pi_{f_\beta,k}(\mathring{x}_k)\right| < \omega$. To this end, consider an arbitrary $(\mathbf{V}, \mathbb{R})$-generic filter G. In $\mathbf{V}[G]$, it is clear that $(\omega \setminus (\pi_{f,k}(\mathring{x}_k)\,[G])) \setminus (\mathring{x}_k\,[G]) \subset \{i \in \omega : \mathring{F}\,[G]\,(k,i) \in L_r\}$, which is a finite set. Similarly, if $\alpha, \beta \in \omega_1$ and $\alpha \neq \beta$, then $\pi_{f_\alpha,k}(\mathring{x}_k)\,[G] \cap \pi_{f_\beta,k}(\mathring{x}_k)\,[G] \subset \{i \in \omega : \mathring{F}\,[G]\,(k,i) \in L_r \cup (A_\alpha \cap A_\beta)\}$. By almost disjointness, $L_r \cup (A_\alpha \cap A_\beta)$ is a finite subset of ω. Therefore $\{i \in \omega : \mathring{F}\,[G]\,(k,i) \in L_r \cup (A_\alpha \cap A_\beta)\}$ is finite as well. Hence $\pi_{f_\alpha,k}(\mathring{x}_k)\,[G] \cap \pi_{f_\beta,k}(\mathring{x}_k)\,[G]$ is a finite set. This establishes everything that is needed for the proof of the theorem. $\dashv$

It is well known that every new real that is added by a finite support iteration of Suslin c.c.c. posets is actually added by a countable fragment of the iteration, and this countable fragment itself can be coded as a Suslin c.c.c. poset (see, for example, [8] for a proof). Hence we get the following corollary to Theorem 17, which is analogous to a result of Judah and Shelah for the splitting number.

Corollary 18. *A finite support iteration of Suslin c.c.c. posets does not increase* $\mathfrak{s}(\mathfrak{pr})$.

If $\mathcal{I}$ is any analytic ideal on ω, then the Mathias and Laver forcings associated with $\mathcal{I}$ are examples of Suslin c.c.c. posets. So, in particular, finite support iterations of Mathias and Laver forcings associated with analytic ideals do not increase $\mathfrak{s}(\mathfrak{pr})$.

Question 19. *Is* $\mathfrak{s} = \mathfrak{s}(\mathfrak{pr})$*? Is* $\mathfrak{s}(\mathfrak{pr}) \leq \max\{\mathfrak{b}, \mathfrak{s}\}$*?*

3. A bound for $\operatorname{cov}^*(\mathcal{Z}_0)$

The two main inequalities of the paper saying that $\operatorname{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{b}, \mathfrak{s}(\mathfrak{pr})\}$ and $\min\{\mathfrak{d}, \mathfrak{r}\} \leq \operatorname{non}^*(\mathcal{Z}_0)$ will be proved in this section. We will need a few lemmas proved in [11] for our construction. We state these below without proof and refer the reader to [11] for details.

Lemma 20 (Lemma 12 of [11])**.** *Let I be an interval partition. Let $A \subset \omega$ be such that for each $l \geq 0$, there exists $N \in \omega$ such that for each $n \geq N$:*

(1) $\frac{|A \cap I_n|}{|I_n|} \leq 2^{-l}$;

(2) $\forall i, j \in A \cap I_n \left[i \neq j \implies |i - j| > 2^{l-1}\right]$.

Then A has density 0.

Lemma 21 (Lemma 13 of [11])**.** *Let l be a member of ω greater than 0 and let $X \subset \omega$ with $|X| = 2^l$. Then there exists a sequence $\{A_\sigma : \sigma \in 2^{\leq l}\}$ such that:*

(1) $\forall m \leq l \left[\bigcup_{\sigma \in 2^m} A_\sigma = X \wedge \forall \sigma, \tau \in 2^m \left[\sigma \neq \tau \implies A_\sigma \cap A_\tau = 0\right]\right]$;

(2) $\forall \sigma \in 2^{\leq l} \left[|A_\sigma| = 2^{l - |\sigma|}\right]$ *and* $\forall \sigma, \tau \in 2^{\leq l} \left[\sigma \subset \tau \implies A_\tau \subset A_\sigma\right]$;

(3) for each $\sigma \in 2^{\leq l}$, $\forall i, j \in A_\sigma \left[i \neq j \implies |i - j| > 2^{|\sigma| - 1}\right]$.

Definition 22 (Definition 15 of [11])**.** Let J be an interval partition such that for each $n \in \omega$ there exists $l_n \in \omega$ such that $l_n > 0$, $l_n \geq n$, and $|J_n| = 2^{l_n}$. Applying Lemma 21, fix a sequence $\bar{A} = \langle A_{n,\sigma} : n \in \omega \wedge \sigma \in 2^{\leq l_n}\rangle$ such that for each $n \in \omega$, the sequence $\{A_{n,\sigma} : \sigma \in 2^{\leq l_n}\}$ satisfies (1)–(3) of Lemma 21 with l as l_n and X as J_n. Define $\mathcal{F}_{J,\bar{A}}$ to be the collection of all functions $f \in \omega^\omega$ such that for each $n \in \omega$ and $l < l_n$, there exists $\sigma \in 2^{l+1}$ such that $f^{-1}(\{l\}) \cap J_n = A_{n,\sigma}$, and there exists $\tau \in 2^{l_n}$ such that $f^{-1}(\{l_n\}) \cap J_n = A_{n,\tau}$.

Remark 23. Observe that if $f \in \mathcal{F}_{J,\bar{A}}$, then for each $n \in \omega$ and $k \in J_n$, $f(k) \leq l_n$. Also for any $n, l \in \omega$,

$$\frac{|\{k \in J_n : f(k) \geq l\}|}{|J_n|} \leq 2^{-l},$$

and for any $i, j \in \{k \in J_n : f(k) \geq l\}$, if $i \neq j$, then $|i - j| > 2^{l-1}$. Moreover for any $f \in \mathcal{F}_{J,\bar{A}}$, $n \in \omega$, and $l \leq l_n$, there is $\sigma_{f,n,l} \in 2^l$ such that $A_{n,\sigma_{f,n,l}} = \{k \in J_n : f(k) \geq l\}$.

The next lemma is a simple variation of a standard fact. However the proof we give below is slightly more cumbersome than the standard proof because of our need to ensure Clause (2), which says that the size of each interval is equal to an exact power of 2.

Lemma 24. *There exists a family B of interval partitions such that:*

(1) $|B| \leq \mathfrak{b}$;

(2) *for each $I \in B$ and for each $n \in \omega$, there exists $l_n \in \omega$ such that $l_n > 0$, $l_n \geq n$, and $|I_n| = 2^{l_n}$;*
(3) *for any interval partition J, there exists $I \in B$ such that $\exists^\infty n \in \omega \exists k > n\,[J_k \subset I_n]$.*

Proof. For each $f \in \omega^\omega$ define an interval partition $I_f = \langle i_{f,n} : n \in \omega \rangle$ as follows. Define $i_{f,0} = 0$, and given $i_{f,n} \in \omega$, let $L = \max\{(i_{f,n}) + 1, f(n+1)\}$. Find $l_n \in \omega$ such that $l_n > 0$, $l_n \geq n$, and $2^{l_n} \geq L - i_{f,n}$. Define $i_{f,n+1} = 2^{l_n} + i_{f,n}$. Note that $i_{f,n} < (i_{f,n}) + 1 \leq L \leq i_{f,n+1}$. Note also that $f(n+1) \leq L \leq i_{f,n+1}$. This completes the definition of I_f, which is clearly an interval partition. For each $n \in \omega$, $|I_{f,n}| = i_{f,n+1} - i_{f,n} = 2^{l_n}$, for some $l_n \in \omega$ with $l_n > 0$ and $l_n \geq n$.

Now suppose $U \subset \omega^\omega$ is an unbounded family with $|U| = \mathfrak{b}$. Put $B = \{I_f : f \in U\}$. Clauses (1) and (2) hold by construction. So we verify (3). Let $J = \langle j_n : n \in \omega \rangle$ be any interval partition. For each $k \in \omega$, define $g_k \in \omega^\omega$ by $g_k(n) = j_{k+n}$, for all $n \in \omega$. Let $g \in \omega^\omega$ be such that $\forall k \in \omega\,[g_k \leq^* g]$. Since U is unbounded, find $f \in U$ such that $X = \{n \in \omega : f(n) > g(n)\}$ is infinite. We check that I_f has the required properties. Fix $N \in \omega$. Choose $m > N + 1$ such that $j_m \geq i_{f,N+1}$. Let $k = m - N - 1 \geq 1$. By choice of g, there exists $N_k \in \omega$ such that $\forall n \geq N_k\,[j_{k+n} \leq g(n)]$. Let $M = \max\{N+1, N_k\}$. Since X is infinite, there exists $n \in X$ with $n \geq M$. For any such n, $j_{k+n} \leq g(n) < f(n) \leq i_{f,n}$. So we conclude that there exists $n \geq N+1$ such that $j_{k+n} < i_{f,n}$. Let n be the minimal number with this property. Note that $N+1$ does not have this property because $j_{k+N+1} = j_m \geq i_{f,N+1}$. So $n > N+1$ and so $n - 1 \geq N + 1$. It follows by the minimality of n that $i_{f,n-1} \leq j_{k+n-1} < j_{k+n} < i_{f,n}$. Therefore, $J_{k+n-1} \subset I_{f,n-1}$. Note that $k + n - 1 > n - 1$ because $k \geq 1$ and also that $n - 1 > N$. Thus we have proved that $\forall N \in \omega \exists l > N \exists l' > l\,[J_{l'} \subset I_{f,l}]$, which establishes (3). $\dashv$

Definition 25. Let J be any interval partition such that for each $n \in \omega$, there exists $l_n \in \omega$ such that $l_n > 0$, $l_n \geq n$, and $|J_n| = 2^{l_n}$. Let $\bar{A}$ and $\mathcal{F}_{J,\bar{A}}$ be as in Definition 22. For any interval partition I, function $f \in \mathcal{F}_{J,\bar{A}}$, and $l \in \omega$, define $Z_{I,J,f,l} = \{m \in \omega : \exists k \in I_l\,[m \in J_k \wedge f(m) \geq l]\}$. Define $Z_{I,J,f} = \bigcup_{l \in \omega} Z_{I,J,f,l}$.

Lemma 26. *For any I, J, and f as in Definition 25, $Z_{I,J,f}$ has density 0.*

Proof. We apply Lemma 20 with J and $Z_{I,J,f}$ as the I and the A of Lemma 20 respectively. To check clauses (1) and (2) of Lemma 20, fix $x \geq 0$, a member of ω. Let $N = i_x \in \omega$, and suppose $n \geq N$ is given. Then by the definition of $Z_{I,J,f}$, $Z_{I,J,f} \cap J_n \subset \{m \in J_n : f(m) \geq x\}$. Hence by Remark 23, $\frac{|Z_{I,J,f} \cap J_n|}{|J_n|} \leq \frac{|\{m \in J_n : f(m) \geq x\}|}{|J_n|} \leq 2^{-x}$, as required for clause (1). Also, $\forall i, j \in Z_{I,J,f} \cap J_n\left[i \neq j \implies |i - j| > 2^{x-1}\right]$, as required for clause (2). Thus by Lemma 20, $Z_{I,J,f}$ has density 0. $\dashv$

Lemma 27. *Let J, $\bar{A}$, and $\mathcal{F}_{J,\bar{A}}$ be as in Definition 22. Let B be a family of interval partitions satisfying (1)–(3) of Lemma 24. Fix $f \in \mathcal{F}_{J,\bar{A}}$. Suppose $X \subset \omega$ is such that for each $I \in B$, $X \cap Z_{I,J,f}$ is finite. Then there exists $n \in \omega$ such that $f''X \subset n$.*

Proof. Suppose for a contradiction that for each $n \in \omega$, there exists $m \in X$ such that $f(m) \geq n$. Define an interval partition $K = \langle k_n : n \in \omega \rangle$ as follows. $k_0 = 0$ and suppose that $k_n \in \omega$ is given, for some $n \in \omega$. Define $N = \max\left(\left\{f(m) : m \in \bigcup_{k < k_n} J_k\right\} \cup \{n\}\right)$. By hypothesis, there exists $m \in X$ such that $f(m) \geq N + 1$. Choose such an $m \in X$ and let k be such that $m \in J_k$. Note that $k_n \leq k$ by the definition of N. Define $k_{n+1} = k + 1$. This completes the definition of K. Note that $\forall n \in \omega \exists k \in K_n \exists m \in X \cap J_k\,[f(m) > n]$. By clause (3) of Lemma 24, there is an interval partition $I \in B$ such that $\exists^\infty l \in \omega \exists n > l\,[K_n \subset I_l]$.

Consider any $l \in \omega$ for which there exists $n > l$ such that $K_n \subset I_l$. There exist $k \in I_l$ and $m \in X \cap J_k$ such that $f(m) > l$. It follows that $m \in X \cap Z_{I,J,f,l}$. Thus we conclude that $\exists^\infty l \in \omega\, [X \cap Z_{I,J,f,l} \neq 0]$, contradicting the hypothesis that $X \cap Z_{I,J,f}$ is finite, for all $I \in B$. $\dashv$

Definition 28. Let J and $\bar{A}$ be as in Definition 22. Suppose $C : \omega \to 2^{<\omega}$ and that for each $n \in \omega$, $\operatorname{dom}(C(n)) \geq l_n$. For each $l < l_n$, define $\sigma_{n,l} = (C(n)\restriction l)^\frown\langle 1 - C(n)(l)\rangle \in 2^{l+1}$, and define $\sigma_{n,l_n} = C(n)\restriction l_n \in 2^{l_n}$. Note that for all $l < l' \leq l_n$, $A_{n,\sigma_{n,l}} \cap A_{n,\sigma_{n,l'}} = 0$ and that $\bigcup_{l \leq l_n} A_{n,\sigma_{n,l}} = J_n$. Let $f_C : \omega \to \omega$ be defined as follows. Given $n \in \omega$ and $k \in J_n$, $f_C(k) = l$, where l is the unique number $l \leq l_n$ such that $k \in A_{n,\sigma_{n,l}}$. It is easy to check that $f_C \in \mathcal{F}_{J,\bar{A}}$

Theorem 29. *Let κ be a cardinal on which a tortuous coloring exists. Then* $\operatorname{cov}^*(\mathcal{Z}_0) \leq \max\{\kappa, \mathfrak{b}\}$.

Proof. Let $c : \kappa \times \omega \times \omega \to 2$ be a tortuous coloring. Fix any interval partition J with the property that for each $n \in \omega$, there exists $l_n \in \omega$ such that $l_n > 0$, $l_n \geq n$, and $|J_n| = 2^{l_n}$. Let $\bar{A}$ be as in Definition 22 (with respect to J). For each $\alpha \in \kappa$, define $C_\alpha : \omega \to 2^{<\omega}$ as follows. Given $n \in \omega$, $C_\alpha(n)$ is the function in 2^{l_n} such that for each $l < l_n$, $C_\alpha(n)(l) = c(\alpha, n, l)$. Define $f_\alpha = f_{C_\alpha} \in \mathcal{F}_{J,\bar{A}}$. Fix a family B of interval partitions satisfying clauses (1)–(3) of Lemma 24. For each $I \in B$ and $\alpha \in \kappa$, let $Z_{I,\alpha} = Z_{I,J,f_\alpha}$. By Lemma 26, each $Z_{I,\alpha}$ has density 0. Let $\mathcal{G} = \{Z_{I,\alpha} : I \in B \wedge \alpha \in \kappa\}$ and note that $|\mathcal{G}| \leq \max\{\kappa, \mathfrak{b}\}$. We will show that $\forall X \in [\omega]^\omega \exists I \in B \exists \alpha \in \kappa\, [|X \cap Z_{I,\alpha}| = \omega]$. Thus $\mathcal{G}$ will witness that $\operatorname{cov}^*(\mathcal{Z}_0) \leq \max\{\kappa, \mathfrak{b}\}$. Fix $X \in [\omega]^\omega$. Assume for a contradiction that $X \cap Z_{I,\alpha}$ is finite for all $I \in B$ and $\alpha \in \kappa$. $L = \{n \in \omega : J_n \cap X \neq 0\}$ is infinite because X is infinite. For each $n \in L$, there exists $\tau_n \in 2^{l_n}$ such that $X \cap A_{n,\tau_n} \neq 0$. By Lemma 27 for each $\alpha \in \kappa$ there exists $n_\alpha \in \omega$ such that $f''_\alpha X \subset n_\alpha$. Next, for each $n \in L$ let x_n be the member of 2^ω such that $x_n \restriction l_n = \tau_n$ and $\forall l \in \omega \setminus l_n\, [x_n(l) = 0]$. Find $A \in [L]^\omega$ and $x \in 2^\omega$ such that $\langle x_n : n \in A\rangle$ converges to x. Apply Lemma 7 to find $\alpha \in \kappa$ such that for each $n \in \omega$ and $\sigma \in 2^{n+1}$, $\exists^\infty k \in A \forall i < n+1\, [\sigma(i) = c(\alpha, k, i)]$. Let $n = n_\alpha$ and $\sigma = x\restriction n+1$. By convergence, there exists $k^* \in \omega$ such that $\forall k \in A\, [k \geq k^* \implies x_k\restriction n+1 = x\restriction n+1]$. Fix $k \in A$ such that $k \geq k^*$, $k > n$, and $\forall i < n+1\, [\sigma(i) = c(\alpha, k, i)]$. It is easy to see that $\tau_k\restriction n+1 = C_\alpha(k)\restriction n+1$. It follows from the definition of $f_\alpha = f_{C_\alpha}$ that for each $x \in A_{k,\tau_k\restriction n+1}$, $f_\alpha(x) > n$. However $X \cap A_{k,\tau_k\restriction n+1} \neq 0$ because $X \cap A_{k,\tau_k} \neq 0$. Therefore there exists $x \in X$ such that $f_\alpha(x) > n = n_\alpha$, contradicting the fact that $f''_\alpha X \subset n_\alpha$. This concludes the proof. $\dashv$

Corollary 30. $\operatorname{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{s}(\mathfrak{pr}), \mathfrak{b}\}$.

Suppose $\mathbf{V}$ is a ground model. Suppose that the coloring c used in the proof of Theorem 29 is defined in $\mathbf{V}$ from $\mathbf{V} \cap (\mathcal{P}(\omega))^\omega$ following the procedure of Lemma 6, and that the family of interval partitions B is defined in $\mathbf{V}$ from $\mathbf{V} \cap \omega^\omega$ via the procedure of Lemma 24. Let $\mathbf{V}[G]$ be a forcing extension of $\mathbf{V}$. If there is a set $X \in [\omega]^\omega$ in $\mathbf{V}[G]$ such that $Z \cap X$ is finite for all $Z \in \mathbf{V} \cap \mathcal{Z}_0$, then it follows from the proof of Theorem 29 that either $\mathbf{V} \cap (\mathcal{P}(\omega))^\omega$ is no longer a promptly splitting family or that $\mathbf{V} \cap \omega^\omega$ is no longer an unbounded family in $\mathbf{V}[G]$. So we get the following corollary.

Corollary 31. *Let $\mathbb{P} \in \mathbf{V}$ be a forcing notion that diagonalizes $\mathbf{V} \cap \mathcal{Z}_0$. Then either $\mathbb{P}$ adds an element of ω^ω that dominates $\mathbf{V} \cap \omega^\omega$ or it adds an element of $[\omega]^\omega$ that is not promptly split by $\mathbf{V} \cap (\mathcal{P}(\omega))^\omega$.*

If $\mathbb{P}$ is a Suslin c.c.c. poset, then the second possibility is ruled out by Theorem 17. Furthermore if $\mathbb{P} = \langle \mathbb{P}_\alpha; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \delta\rangle$ is a finite support iteration of c.c.c. posets and if each iterand preserves all unbounded families, then $\mathbb{P}$ does not increase $\mathfrak{b}$. If $\mathbb{P}$ is also not allowed to increase $\mathfrak{s}(\mathfrak{pr})$, then of course $\mathbb{P}$ cannot increase $\operatorname{cov}^*(\mathcal{Z}_0)$.

Corollary 32. *If a Suslin c.c.c. poset in* $\mathbf{V}$ *diagonalizes* $\mathbf{V} \cap \mathcal{Z}_0$*, then it necessarily adds a dominating real. If* $\mathbb{P} = \langle \mathbb{P}_\alpha ; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \delta \rangle$ *is a finite support iteration of Suslin c.c.c. posets and if each iterand preserves all unbounded families, then* $\mathbb{P}$ *does not increase* $\operatorname{cov}^*(\mathcal{Z}_0)$.

An example of a Suslin c.c.c. forcing which preserves all unbounded families is the Mathias forcing associated to an F_σ filter (see Canjar [5]). So a consequence of Corollary 32 is that finite support iterations of Mathias forcings of F_σ filters do not increase $\operatorname{cov}^*(\mathcal{Z}_0)$.

The next result dualizes Corollary 30. However we do not need any variant of $\mathfrak{r}$ because of the following fact, which says that any family of fewer than $\mathfrak{r}$ many members of $[\omega]^\omega$ can be simultaneously promptly split.

Lemma 33. *Suppose* $\mathcal{F} \subset [\omega]^\omega$ *is a family of size less than* $\mathfrak{r}$*. Then there exists a sequence* $X = \langle x_k : k < \omega \rangle \in (\mathcal{P}(\omega))^\omega$ *such that* X *promptly splits* A*, for each* $A \in \mathcal{F}$.

Proof. If $\mathcal{F}$ is empty then any $X \in (\mathcal{P}(\omega))^\omega$ vacuously satisfies the conclusion of the lemma. So we may assume that $\mathcal{F}$ is non-empty. We define a sequence $\langle y_i : i \in \omega \rangle$ as follows. Use the assumption that $\mathcal{F}$ has size less than $\mathfrak{r}$ to find $y_0 \subset \omega$ such that both $y_0 \cap A$ and $(\omega \setminus y_0) \cap A$ are infinite, for each $A \in \mathcal{F}$. Next suppose that for some $n \in \omega$, a sequence $\langle y_i : i \leq n \rangle \in \mathcal{P}(\omega)^{n+1}$ is given such that both $y_n \cap A$ and $\left(\omega \setminus \left(\bigcup_{i \leq n} y_i\right)\right) \cap A$ are infinite, for each $A \in \mathcal{F}$. As $\mathcal{F}$ is non-empty, $\left(\omega \setminus \left(\bigcup_{i \leq n} y_i\right)\right)$ is an infinite subset of ω, and $\mathcal{G} = \left\{\left(\omega \setminus \left(\bigcup_{i \leq n} y_i\right)\right) \cap A : A \in \mathcal{F}\right\}$ is a collection of infinite subsets of $\left(\omega \setminus \left(\bigcup_{i \leq n} y_i\right)\right)$ of size less than $\mathfrak{r}$. So we can find $y_{n+1} \subset \left(\omega \setminus \left(\bigcup_{i \leq n} y_i\right)\right)$ such that both $y_{n+1} \cap B$ and $\left(\left(\omega \setminus \left(\bigcup_{i \leq n} y_i\right)\right) \setminus y_{n+1}\right) \cap B$ are infinite, for each $B \in \mathcal{G}$. It is clear that $\langle y_i : i \leq n+1 \rangle$ satisfies the inductive hypothesis. This concludes the construction of $\langle y_i : i \in \omega \rangle$. Note that $\langle y_i : i \in \omega \rangle$ is a pairwise disjoint sequence. Fix a independent family $\langle C_k : k \in \omega \rangle$ of subsets of ω. For each $k \in \omega$, define $x_k = \bigcup_{i \in C_k} y_i$. This is a subset of ω, and we claim that $\langle x_k : k \in \omega \rangle$ promptly splits A, for each $A \in \mathcal{F}$. Indeed, fix $A \in \mathcal{F}$. Suppose $n \in \omega$ and $\sigma \in 2^{n+1}$. Then $\bigcap_{k < n+1} C_k^{\sigma(k)} \neq 0$. Let $i \in \bigcap_{k < n+1} C_k^{\sigma(k)}$. Since $y_i \subset \bigcap_{k < n+1} x_k^{\sigma(k)}$ and $y_i \cap A$ is infinite, $\left(\bigcap_{k < n+1} x_k^{\sigma(k)}\right) \cap A$ is infinite as well, as needed. ⊣

Lemma 34. *Let* $\kappa < \mathfrak{d}$ *be a cardinal. Suppose* $\langle I_\alpha : \alpha < \kappa \rangle$ *is a sequence of interval partitions. Then there exists an interval partition* J *such that for each* $\alpha < \kappa$, $\exists^\infty n \in \omega \exists k > n \, [I_{\alpha,k} \subset J_n]$.

Proof. This is very similar to Lemma 24. For each $\alpha < \kappa$ and $l \in \omega$, define $f_{\alpha,l}(n) = i_{\alpha,(n+l)}$, for each $n \in \omega$. Now $\{f_{\alpha,l} : \alpha < \kappa \wedge l < \omega\}$ is a family of functions of size less than $\mathfrak{d}$. So there exists $g \in \omega^\omega$ such that for each $\alpha < \kappa$ and $l < \omega$, $\exists^\infty n \in \omega \, [f_{\alpha,l}(n) < g(n)]$. Define J as follows. Put $j_0 = 0$ and suppose $j_n \in \omega$ is given for some $n \in \omega$. Define $j_{n+1} = \max\{j_n + 1, g(n+1)\}$. It is clear that J is an interval partition. We check that it is as required. So fix $\alpha < \kappa$ and $N \in \omega$. We will find $n > N$ and $k > n$ such that $I_{\alpha,k} \subset J_n$. Fix $m > N+1$ such that $i_{\alpha,m} \geq j_{N+1}$ and let $l = m - N - 1$. Note $l \geq 1$. By choice of g, there exists $M \geq N+1$ such that $g(M) > i_{\alpha,(M+l)}$. Now $j_M \geq g(M) > i_{\alpha,(M+l)}$ because $M > 0$. So we conclude that there exists M with the property that $M \geq N+1$ and $j_M > i_{\alpha,(M+l)}$. Let M be minimal with this property. Note that $N+1$ does not have this property, so $M > N+1$. Put $n = M-1$ and $k = n+l$. It follows that $n \geq N+1$ and that $j_n \leq i_{\alpha,k} < i_{\alpha,k+1} < j_{n+1}$, and so $I_{\alpha,k} \subset J_n$. Since $n > N$ and $k > n$, we are done. $\dashv$

Theorem 35. $\min\{\mathfrak{d}, \mathfrak{r}\} \leq \operatorname{non}^*(\mathcal{Z}_0)$.

Proof. Let $\mathcal{G}$ be any family of infinite subsets of ω with $|\mathcal{G}| < \min\{\mathfrak{d}, \mathfrak{r}\}$. We aim to produce a $Z \in \mathcal{Z}_0$ such that $\forall B \in \mathcal{G}\left[|B \cap Z| = \aleph_0\right]$. Fix any interval partition J such that for each $n \in \omega$, there exists $l_n \in \omega$ such that $l_n > 0$, $l_n \geq n$, and $|J_n| = 2^{l_n}$. Let $\bar{A}$ be as in Definition 22 with respect to J. Fix $B \in \mathcal{G}$. Define $L_B = \{n \in \omega : J_n \cap B \neq 0\}$. As B is infinite, L_B is infinite. For each $n \in L_B$, let $\tau_n^B \in 2^{l_n}$ be such that $A_{n,\tau_n^B} \cap B \neq 0$. For each $n \in L_B$, define x_n^B to be the element of 2^ω such that $x_n^B \restriction l_n = \tau_n^B$ and $\forall l \geq l_n\left[x_n^B(l) = 0\right]$. Now we can find $U_B \in [L_B]^\omega$ and $x^B \in 2^\omega$ such that $\langle x_n^B : n \in U_B\rangle$ converges to x^B. Unfix B and consider $\mathcal{F} = \{U_B : B \in \mathcal{G}\}$. Then $\mathcal{F} \subset [\omega]^\omega$ and $|\mathcal{F}| < \mathfrak{r}$. Therefore by Lemma 33, there exists a sequence $\langle z_k : k < \omega\rangle \in (\mathcal{P}(\omega))^\omega$ which promptly splits U_B, for each $B \in \mathcal{G}$. Now define $C : \omega \to 2^{<\omega}$ as follows. For $n \in \omega$, $C(n)$ is the function from l_n to 2 such that for each $k < l_n$, $C(n)(k) = 0$ iff $n \in z_k$. C satisfies the conditions of Definition 28. Therefore $f_C \in \mathcal{F}_{J,\bar{A}}$, where f_C is defined in Definition 28

Fix any $B \in \mathcal{G}$ and $l \in \omega$. We will produce a $y \in B$ such that $f_C(y) \geq l$. Since $\langle x_n^B : n \in U_B\rangle$ converges to x^B, there exists $N \in \omega$ such that $\forall n \in U_B\left[n \geq N \implies x_n^B \restriction (l+1) = x^B \restriction (l+1)\right]$. Also since $\langle z_k : k \in \omega\rangle$ promptly splits U_B, $\left(\bigcap_{k<l+1} z_k^{x^B(k)}\right) \cap U_B$ is infinite. Choose $n \in \left(\bigcap_{k<l+1} z_k^{x^B(k)}\right) \cap U_B$ such that $n \geq N$ and $n > l$. Note that $l_n \geq n > l$ and that for each $k < l+1$, $C(n)(k) = 0$ iff $n \in z_k$ iff $x^B(k) = 0$. Thus $C(n) \restriction (l+1) = x^B \restriction (l+1) = x_n^B \restriction (l+1) = \tau_n^B \restriction (l+1)$. For notational convenience, write $\sigma = \tau_n^B \restriction (l+1)$. Since $A_{n,\tau_n^B} \subset A_{n,\sigma}$ and since $A_{n,\tau_n^B} \cap B \neq 0$, we can choose a $y \in B \cap A_{n,\sigma}$. We claim that $f_C(y) \geq l$. By the definition of f_C, it suffices to prove that for each $l' < l$, $y \notin A_{n,\sigma_{n,l'}}$, where $\sigma_{n,l'}$ is defined in Definition 28 (with respect to C). To see this, fix any $l' < l$. Put $\eta = \sigma \restriction (l'+1)$. Then $\eta(l') = C(n)(l') \neq 1 - C(n)(l') = \sigma_{n,l'}(l')$. Thus $\sigma_{n,l'}, \eta \in 2^{l'+1}$ and $\eta \neq \sigma_{n,l'}$. Therefore $A_{n,\sigma_{n,l'}} \cap A_{n,\eta} = 0$. On the other hand $A_{n,\sigma} \subset A_{n,\eta}$ because $\eta \subset \sigma$. Hence $y \in A_{n,\eta}$, whence $y \notin A_{n,\sigma_{n,l'}}$ as claimed.

The argument of the previous paragraph shows that f_C is unbounded on every $B \in \mathcal{G}$. Now for each $B \in \mathcal{G}$ define an interval partition I_B as follows. Let $i_{B,0} = 0$ and suppose that for some $n \in \omega$, $i_{B,n} \in \omega$ is given. Define $M = \max\left(\{f_C(y) + 1 : y \in \bigcup_{m \leq i_{B,n}} J_m\} \cup \{n\}\right)$. Let $y \in B$ be such that $f_C(y) \geq M$. Let $m \in \omega$ be such that $y \in J_m$. Note that $m > i_{B,n}$. Define $i_{B,n+1} = m+1$. This concludes the definition of I_B. Note that for each $n \in \omega$, $\exists m \in I_{B,n} \exists y \in J_m \cap B\left[f_C(y) \geq n\right]$. Now $\{I_B : B \in \mathcal{G}\}$ is a family of interval partitions of size less than $\mathfrak{d}$. Therefore by Lemma 34, there is an interval partition I such that for each $B \in \mathcal{G}$, $\exists^\infty k \in \omega \exists n > k\left[I_{B,n} \subset I_k\right]$. Let $Z = Z_{I,J,f_C}$. Then Z has density 0 because $f_C \in \mathcal{F}_{J,\bar{A}}$. To complete the proof of the theorem, we show that $|Z \cap B| = \aleph_0$, for every $B \in \mathcal{G}$. To this end, fix any $B \in \mathcal{G}$. Then $Y = \{k \in \omega : \exists n > k\left[I_{B,n} \subset I_k\right]\}$ is infinite by choice of I. Consider any $k \in Y$ and let $n > k$ be such that $I_{B,n} \subset I_k$. There exist $m \in I_{B,n}$ and $y \in J_m \cap B$ with $f_C(y) \geq n$. By definition of $Z_{I,J,f_C,k}$, $y \in B \cap Z_{I,J,f_C,k}$. Thus $B \cap Z_{I,J,f_C,k} \neq 0$, for every $k \in Y$. When $k < k' < \omega$, then $Z_{I,J,f_C,k} \cap Z_{I,J,f_C,k'} = 0$. It follows that $B \cap Z$ is infinite, as claimed. $\dashv$

We point out here that it is provable in ZFC that $\min\{\mathfrak{d}, \mathfrak{r}\} = \min\{\mathfrak{d}, \mathfrak{u}\}$. This was observed by Aubrey [1]. A closely related observation was made by Mildenberger who showed that $\mathfrak{r} \geq \min\{\mathfrak{u}, \mathfrak{g}\}$. More details about Mildenberger's work may be found on Page 452 of [3].

Lemma 36 (Aubrey [1]). $\min\{\mathfrak{d}, \mathfrak{r}\} = \min\{\mathfrak{d}, \mathfrak{u}\}$.

Lemma 36 gives us the following "improvement" of Theorem 35.

Corollary 37. $\min\{\mathfrak{d}, \mathfrak{u}\} \leq \mathrm{non}^*(\mathcal{Z}_0)$.

4. Questions

An outstanding open question is about the connection between $\mathrm{cov}^*(\mathcal{Z}_0)$ and $\mathfrak{b}$, which is closely related to what forcings can diagonalize $\mathbf{V} \cap \mathcal{Z}_0$.

Question 38. *Is* $\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{b}$*? Is* $\mathfrak{d} \leq \mathrm{non}^*(\mathcal{Z}_0)$*? Is there a proper forcing which diagonalizes* $\mathbf{V} \cap \mathcal{Z}_0$ *while preserving all unbounded families?*

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