THE ALMOST DISJOINTNESS INVARIANT FOR PRODUCTS OF IDEALS

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ABSTRACT. The almost disjointness numbers associated to the quotients determined by the transfinite products of the ideal of finite sets are investigated. A ZFC lower bound involving the minimum of the classical almost disjointness and splitting numbers is proved for these characteristics. En route, it is shown that the splitting numbers associated to these quotients are all equal to the classical splitting number. Finally, it is proved to be consistent that the almost disjointness numbers associated to these quotients are all equal to the second uncountable cardinal while the bounding number is the first uncountable cardinal. Several open problems are considered.

1. INTRODUCTION

Cardinal characteristics associated with definable ideals and their quotients have received considerable attention. Two notable examples include the papers Brendle and Shelah [5] and Hernández-Hernández and Hrušák [6]. This paper will focus on the products of the ideal of finite subsets of ω, including transfinite products, and the quotients they determine. The main result will be a ZFC lower bound on the almost disjointness numbers associated to these quotients, and as a consequence of this ZFC lower bound, it will be shown to be consistent that $\mathfrak{b} = \aleph_1$ while these almost disjointness numbers are all $\aleph_2$. The next few definitions set the basic notation.

Definition 1.1. Given an infinite set X, $\mathcal{I}$ is said to be an *ideal* on X if $\mathcal{I}$ is a subset of $\mathcal{P}(X)$ such that the following conditions hold:

(1) if $Y \subseteq X$ is finite, then $Y \in \mathcal{I}$;
(2) if $Y \in \mathcal{I}$ and $Z \subseteq Y$, then $Z \in \mathcal{I}$;
(3) if $Y \in \mathcal{I}$ and $Z \in \mathcal{I}$, then $Y \cup Z \in \mathcal{I}$;
(4) $X \notin \mathcal{I}$.

Conditions (1) and (4) are often expressed as $\mathcal{I}$ is *non-principal* and $\mathcal{I}$ is *proper* respectively. For an ideal $\mathcal{I}$ on X and A and B subsets of X define $A \equiv_{\mathcal{I}} B$ if and only if $A \triangle B \in \mathcal{I}$ and define $A \subseteq_{\mathcal{I}} B$ if $A \setminus B \in \mathcal{I}$. Then $\mathcal{P}(X)/\mathcal{I}$ is a Boolean algebra of $\equiv_{\mathcal{I}}$ equivalence classes and $[Y]_{\mathcal{I}}$ denotes the $\equiv_{\mathcal{I}}$ equivalence class of Y, for any set $Y \subseteq X$. A set $Y \in \mathcal{P}(X)$ is $\mathcal{I}$-*positive* if $Y \notin \mathcal{I}$, equivalently $[Y]_{\mathcal{I}} > 0$. This is often written as $Y \in \mathcal{I}^+$. Observe that by (1), every $\mathcal{I}$-positive set is infinite.

For $Y \in \mathcal{I}^+$, the *restriction of $\mathcal{I}$ to Y*, denoted $\mathcal{I} \restriction Y$ is $\{Z \in \mathcal{I} : Z \subseteq Y\}$. It is easy to see that $\mathcal{I} \restriction Y$ is an ideal on Y. Attention will be restricted to ideals with the property that $\mathcal{P}(X)/\mathcal{I}$ is non-atomic, which is an easy consequence (as will be seen in Proposition 1.2) of the following additional condition:

(5) for every $Y \subseteq X$, if $Y \notin \mathcal{I}$, then $\mathcal{I} \restriction Y$ is not a maximal ideal on Y.

All ideals to be considered in this paper will enjoy this property.

Definition 1.2. Let $\mathcal{B}$ be a non-atomic Boolean algebra. An *antichain in $\mathcal{B}$* is a set $A \subseteq \mathcal{B}$ such that $\forall a \in A [a > 0]$ and $\forall a, a' \in A [a \neq a' \implies a \wedge a' = 0]$. A *maximal antichain in $\mathcal{B}$* is a set $A \subseteq \mathcal{B}$ which is an antichain in $\mathcal{B}$ and which is not a proper subset of any antichain in $\mathcal{B}$. Define

$$\mathfrak{a}_{\mathcal{B}} = \min \{|A| : A \subseteq \mathcal{B} \text{ is an infinite maximal antichain in } \mathcal{B}\}.$$

Note that $\mathfrak{a}_{\mathcal{B}}$ is well-defined for any non-atomic Boolean algebra $\mathcal{B}$, for it is possible to find an infinite antichain below every $b \in \mathcal{B} \setminus \{0\}$ in view of the lack of atoms. The absence of atoms also ensures that the splitting number, define in Definition 1.3, is well-defined for $\mathcal{B}$.

Definition 1.3. $\mathcal{F} \subseteq \mathcal{B}$ is called a *splitting family in* $\mathcal{B}$ if for every $b \in \mathcal{B}$ with $b > 0$, there exists $a \in \mathcal{F}$ with $b \wedge a > 0$ and $b \wedge (1 - a) > 0$. Define

$$\mathfrak{s}_{\mathcal{B}} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{B} \text{ and } \mathcal{F} \text{ is a splitting family in } \mathcal{B}\}$$

Proposition 1.1. *If* $\mathcal{B}$ *is a non-atomic Boolean algebra, then* $\mathcal{B}$ *is a splitting family in* $\mathcal{B}$.

Conversely, if $\mathcal{B}$ has atoms, then there are no splitting families in $\mathcal{B}$ as the atoms cannot be split, and hence $\mathfrak{s}_{\mathcal{B}}$ is not well-defined. Condition (5) of Definition 1.1 ensures that $\mathcal{P}(X)/\mathcal{I}$ will be non-atomic.

Proposition 1.2. *Let* $\mathcal{I}$ *be an ideal on an infinite set* X *satisfying condition (5) of Definition 1.1. Then* $\mathcal{P}(X)/\mathcal{I}$ *is non-atomic.*

Proof. Consider $a \in \mathcal{P}(X)/\mathcal{I}$ with $a > 0$. So $a = [Y]_{\mathcal{I}}$ for some $Y \in \mathcal{P}(X)$ such that $Y \in \mathcal{I}^+$. By hypothesis $\mathcal{I}{\upharpoonright}Y$ is not a maximal ideal on Y. Let $\mathcal{J}$ be an ideal on Y so that $\mathcal{I}{\upharpoonright}Y \subsetneq \mathcal{J}$. Fix $Z \in \mathcal{J}\setminus(\mathcal{I}{\upharpoonright}Y)$. $Z \subseteq Y$ because $\mathcal{J}$ is an ideal on Y and $Z \in \mathcal{J}$. As $\mathcal{J}$ is a proper ideal, $Y \setminus Z \notin \mathcal{I}{\upharpoonright}Y$. Thus $Y \setminus Z \notin \mathcal{I}$ and $Z \notin \mathcal{I}$. Hence $0 < [Z]_{\mathcal{I}}, [Y \setminus Z]_{\mathcal{I}} \leq [Y]_{\mathcal{I}}$. Further, $[Y \setminus Z]_{\mathcal{I}} \wedge [Z]_{\mathcal{I}} = [(Y \setminus Z) \cap Z]_{\mathcal{I}} = 0$, showing that a is not an atom. $\square$

When $\mathcal{B}$ is of the form $\mathcal{P}(X)/\mathcal{I}$, $\mathfrak{a}_{\mathcal{P}(X)/\mathcal{I}}$ and $\mathfrak{s}_{\mathcal{P}(X)/\mathcal{I}}$ will sometimes be rewritten as $\mathfrak{a}_{\mathcal{I}}$ and $\mathfrak{s}_{\mathcal{I}}$. The Fubini square (see Definition 2.3) of the ideal FIN of finite subsets of ω is usually denoted as $\text{FIN} \times \text{FIN}$ and it is most naturally viewed as an ideal on $\omega \times \omega$. The quotient $\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})$ and its cardinal characteristics have been studied by several researchers. A notable difference between $\mathcal{P}(\omega)/\text{FIN}$ and $\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})$, first noticed by Szymański and Zhou [11], is that the tower number $\mathfrak{t}_{\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})}$ is provably equal to $\aleph_1$ in ZFC.

In an unpublished work [1], Brendle investigated the almost disjointness number of the quotient $\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})$. He observed that $\mathfrak{b} \leq \mathfrak{a}_{\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})} \leq \mathfrak{a}$ and by using a template style iteration along the lines of Shelah [10], he was able to show the consistency of $\mathfrak{b} = \mathfrak{d} < \mathfrak{a}_{\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})}$.

Theorem 1.1 (Brendle [1]). *It is consistent that* $\aleph_2 = \mathfrak{b} = \mathfrak{d} < \mathfrak{a}_{\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})}$.

As template iterations will not produce models with $\aleph_1 = \mathfrak{b}$, Brendle [1] asked whether $\aleph_1 = \mathfrak{b} < \mathfrak{a}_{\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})}$ is consistent. Section 2 of this paper will provide a positive answer to Brendle's question: it is consistent that $\aleph_1 = \mathfrak{b}$ and that $\mathfrak{a}_{\mathcal{P}(\omega \times \omega)/(\text{FIN} \times \text{FIN})} = \aleph_2$.

2. Products of ideals

This section begins with some preliminary results about products of ideals in general. We then focus on products that are supported on the ideal of finite subsets of the index set. The results about transfinite products of FIN are then derived as corollaries. It turns out that the splitting number of the quotients induced by such products is important to understanding the almost disjointness of these quotients.

Definition 2.1. For an indexed family $\langle A_x : x \in X \rangle$, define

$$\coprod_{x \in X} A_x = \bigcup_{x \in X} (\{x\} \times A_x).$$

Definition 2.2. For any sets A and x, define $A(x) = \{y : \langle x, y \rangle \in A\}$.

The next proposition summarizes some basic attributes of the Definitions 2.1 and 2.2. The proofs are straightforward applications of the definitions and are left to the reader.

Proposition 2.1. *The following properties hold:*

(1) $A \subseteq B \implies A(x) \subseteq B(x)$;

(2) $(A \cup B)(x) = A(x) \cup B(x)$;

(3) $(A \cap B)(x) = A(x) \cap B(x)$;

(4) $(A \setminus B)(x) = A(x) \setminus B(x)$;

(5) suppose $\langle D_x : x \in X \rangle$ is an indexed family of sets, $D = \coprod_{x \in X} D_x$, $A \subseteq X$, $\langle E_x : x \in A \rangle$ is an indexed family such that $\forall x \in A\,[E_x \subseteq D_x]$, and $E = \coprod_{x \in A} E_x$; then the following hold:

    (a) $E \subseteq D$;

    (b) $\forall x \in A\,[E_x = E(x)]$;

    (c) $\forall x \notin A\,[E(x) = \emptyset]$.

Definition 2.3. Let X be an infinite set and let $\langle D_x : x \in X \rangle$ be an indexed family of infinite sets. Suppose $\langle \mathcal{I}_x : x \in X \rangle$ is an indexed family so that $\mathcal{I}_x$ is an ideal on D_x, for all $x \in X$. Let $D = \coprod_{x \in X} D_x$.

For any, $A \subseteq D$, define $\mathrm{suppt}(A) = \{x \in X : A(x) \notin \mathcal{I}_x\}$.

Given an ideal $\mathcal{I}$ on X, define

$$\coprod_{\mathcal{I}} \mathcal{I}_x = \{A \subseteq D : \mathrm{suppt}(A) \in \mathcal{I}\}$$

Lemma 2.1. $\mathcal{J} = \coprod_{\mathcal{I}} \mathcal{I}_x$ is an ideal on the infinite set D.

Proof. As X is infinite and as each D_x is infinite, it is clear that D is infinite. By definition $\mathcal{J} \subseteq \mathcal{P}(D)$. Suppose A and B satisfy $B \subseteq A$ and $A \in \mathcal{J}$. So $A \subseteq D$ and $\mathrm{suppt}(A) \in \mathcal{I}$. As $B \subseteq A$, it is easy to see that $\mathrm{suppt}(B) \subseteq \mathrm{suppt}(A)$, whence $\mathrm{suppt}(B) \in \mathcal{I}$. Therefore, $B \in \mathcal{J}$, as required for condition (2) of Definition 1.1.

For condition (3) of Definition 1.1, fix $A, B \in \mathcal{J}$. Thus $A, B \subseteq D$ and $\mathrm{suppt}(A) \cup \mathrm{suppt}(B) \in \mathcal{I}$. Consider any $x \in \mathrm{suppt}(A \cup B)$. Then $x \in X$ and, since $\mathcal{I}_x$ is an ideal on D_x, either $A(x) \notin \mathcal{I}_x$ or $B(x) \notin \mathcal{I}_x$. It follows that $x \in \mathrm{suppt}(A) \cup \mathrm{suppt}(B)$. So $\mathrm{suppt}(A \cup B) \subseteq \mathrm{suppt}(A) \cup \mathrm{suppt}(B)$, which implies that $\mathrm{suppt}(A \cup B) \in \mathcal{I}$. Therefore, $A \cup B \in \mathcal{J}$, as wanted.

Next, note that by Proposition 2.1, $\forall x \in X\,[D_x = D(x)]$. Hence for all $x \in X$, $\mathcal{I}_x$ is a proper ideal on $D(x)$, and so $D(x) \notin \mathcal{I}_x$. Hence $\mathrm{suppt}(D) = X \notin \mathcal{I}$. Therefore, $D \notin \mathcal{J}$.

Finally suppose $E \subseteq D$ is finite. Then $\mathrm{suppt}(E) = \emptyset$. To see this suppose otherwise that $x \in \mathrm{suppt}(E)$. Then $x \in X$ and $E(x) \notin \mathcal{I}_x$. However, $E(x)$ is finite and $E(x) \subseteq D(x) = D_x$, which implies $E(x) \in \mathcal{I}_x$ because $\mathcal{I}_x$ is an ideal on D_x. This contradiction shows that $\mathrm{suppt}(E) = \emptyset \in \mathcal{I}$. Therefore, $E \in \mathcal{J}$, completing the proof. $\square$

Isomorphism between ideals is considered next. This becomes necessary in order to show that the sequence of cardinal invariants to be defined does not depend on an arbitrary choice of bijections.

Definition 2.4. Suppose X and Y are infinite sets, and $\mathcal{I}$ and $\mathcal{J}$ are ideals on X and Y respectively. We say $\mathcal{I}$ is isomorphic to Y if there is a bijection $f : X \to Y$ such that

$$\mathcal{J} = \{f''A : A \in \mathcal{I}\}.$$

f is called an isomorphism from $\mathcal{I}$ to $\mathcal{J}$.

The following proposition lists two important properties of isomorphisms which will be useful in the proof of Lemma 2.2. Their proofs are easy and left to the reader. In (1) of Proposition 2.2 and in the rest of the paper, FIN_X is the ideal of finite subsets of X.

Proposition 2.2. Let X, Y be infinite sets. Then

    (1) if $f : X \to Y$ is a bijection, then $\mathrm{FIN}_Y = \{f''A : A \in \mathrm{FIN}_X\}$;

(2) if $\mathcal{I}$ is an ideal on X, $\mathcal{J}$ is an ideal on Y, and $g : X \to Y$ is an isomorphism from $\mathcal{I}$ to $\mathcal{J}$, then for any $A \subseteq X$, if $A \notin \mathcal{I}$, then $g''A \notin \mathcal{J}$.

Lemma 2.2. *Let X, Y be infinite sets and $\langle D_x : x \in X \rangle$ and $\langle E_y : y \in Y \rangle$ be indexed families of infinite sets. Let $f : X \to Y$ be any bijection. Suppose $\langle \mathcal{I}_x : x \in X \rangle$, $\langle \mathcal{J}_y : y \in Y \rangle$, and $\langle g_x : x \in X \rangle$ are indexed families so that $\mathcal{I}_x$ is an ideal on D_x, $\mathcal{J}_y$ is an ideal on E_y, and $g_x : D_x \to E_{f(x)}$ is an isomorphism from $\mathcal{I}_x$ to $\mathcal{J}_{f(x)}$, for all $x \in X$. Then if $\mathcal{I}$ is an ideal on X, $\mathcal{J}$ is an ideal on Y, and f is an isomorphism from $\mathcal{I}$ to $\mathcal{J}$, then*

$$\coprod_{\mathcal{I}} \mathcal{I}_x \text{ is isomorphic to } \coprod_{\mathcal{J}} \mathcal{J}_y.$$

Proof. Let D denote $\coprod_{x \in X} D_x$ and let E denote $\coprod_{y \in Y} E_y$. By Lemma 2.1, D and E are infinite sets, $\coprod_{\mathcal{I}} \mathcal{I}_x$ is an ideal on D and $\coprod_{\mathcal{J}} \mathcal{J}_y$ is an ideal on E.

Define $h : D \to E$ as follows. Suppose $d \in D$ is given. Then $d = \langle x, s \rangle$, for unique $x \in X$ and $s \in D_x$. So $f(x) \in Y$ and $g_x : D_x \to E_{f(x)}$ is a bijection. Hence we may define $h(d) = \langle f(x), g_x(s) \rangle \in E$. h is 1-1 because $f : X \to Y$ is a bijection and because for any $x \in X$, $g_x : D_x \to E_{f(x)}$ is a bijection. To see that h is onto, fix $e \in E$. Then $e = \langle y, t \rangle$, for some $y \in Y$ and $t \in E_y$. As $f : X \to Y$ is a bijection, $y = f(x)$ for some $x \in X$. Thus $t \in E_{f(x)}$, and as $g_x : D_x \to E_{f(x)}$ is a bijection, $t = g_x(s)$, for some $s \in D_x$. Therefore, $d = \langle x, s \rangle \in D$ and by definition, $h(d) = \langle f(x), g_x(s) \rangle = \langle y, t \rangle = e$, showing h is onto. So h is a bijection.

Next, it is not difficult to verify that for any $A \subseteq D$ and for any $x \in X$, $g_x''A(x) = (h''A)(f(x))$. Now, consider any $A \subseteq D$. It will be verified that $\text{suppt}(h''A) = f'' \text{suppt}(A)$. In one direction, fix any $y \in \text{suppt}(h''A)$. Then $y \in Y$ and $(h''A)(y) \notin \mathcal{J}_y$. Find $x \in X$ with $f(x) = y$. By the above observation, $g_x''A(x) = (h''A)(f(x)) = (h''A)(y) \notin \mathcal{J}_y$. As $g_x : D_x \to E_{f(x)}$ is an isomorphism from $\mathcal{I}_x$ to $\mathcal{J}_{f(x)} = \mathcal{J}_y$, $A(x) \notin \mathcal{I}_x$. Therefore, $x \in \text{suppt}(A)$, and so $y \in f'' \text{suppt}(A)$. This shows $\text{suppt}(h''A) \subseteq f'' \text{suppt}(A)$. Conversely, consider any $y \in f'' \text{suppt}(A)$. Then $y = f(x)$ for some $x \in \text{suppt}(A)$. Thus $x \in X$, $A(x) \notin \mathcal{I}_x$, and $g_x : D_x \to E_{f(x)}$ is an isomorphism from $\mathcal{I}_x$ to $\mathcal{J}_{f(x)}$. By item (2) of Proposition 2.2, it follows that $g_x''A(x) \notin \mathcal{J}_{f(x)}$. By an observation made earlier, $g_x''A(x) = (h''A)(f(x))$, and so $(h''A)(f(x)) \notin \mathcal{J}_{f(x)}$. This shows that $f(x) \in \text{suppt}(h''A)$. Therefore, $y \in \text{suppt}(h''A)$, showing that $f'' \text{suppt}(A) \subseteq \text{suppt}(h''A)$ and concluding the proof that $\text{suppt}(h''A) = f'' \text{suppt}(A)$.

Now, armed with the above observation, proceed to verify that $\coprod_{\mathcal{J}} \mathcal{J}_y = \left\{ h''A : A \in \coprod_{\mathcal{I}} \mathcal{I}_x \right\}$. Suppose first that $B \in \coprod_{\mathcal{J}} \mathcal{J}_y$. Then $B \subseteq E$ and $\text{suppt}(B) \in \mathcal{J}$. As $h : D \to E$ is a bijection, there is $A \subseteq D$ with $h''A = B$. By the previous paragraph, $\text{suppt}(h''A) = f'' \text{suppt}(A)$. Thus $f'' \text{suppt}(A) = \text{suppt}(B) \in \mathcal{J}$. Applying item (2) of Proposition 2.2, it follows that $\text{suppt}(A) \in \mathcal{I}$. As $A \subseteq D$, $A \in \coprod_{\mathcal{I}} \mathcal{I}_x$, whence $B = h''A \in \left\{ h''A^* : A^* \in \coprod_{\mathcal{I}} \mathcal{I}_x \right\}$. Conversely, consider any $A \in \coprod_{\mathcal{I}} \mathcal{I}_x$. Then $A \subseteq D$, $\text{suppt}(A) \subseteq X$, and $\text{suppt}(A) \in \mathcal{I}$. Since f is an isomorphism, $f'' \text{suppt}(A) \in \mathcal{J}$. So by the previous paragraph, $\text{suppt}(h''A) = f'' \text{suppt}(A) \in \mathcal{J}$. As $h''A \subseteq E$, $h''A \in \coprod_{\mathcal{J}} \mathcal{J}_y$ by definition. This is as required to complete the proof of the lemma.   □

Definition 2.5. $\text{Lim}(\omega_1) = \{ \alpha < \omega_1 : \alpha \text{ is a limit ordinal} \}$. Let $\bar{s} = \langle s_\alpha : \alpha \in \text{Lim}(\omega_1) \rangle$ be an indexed family so that $\forall \alpha \in \text{Lim}(\omega_1) [s_\alpha : \omega \to \alpha \text{ is a bijection}]$. By induction on $\alpha < \omega_1$, define an infinite set $D(\bar{s}, \alpha)$ and an ideal $\text{FIN}(\bar{s}, \alpha)$ on $D(\bar{s}, \alpha)$ as follows:

(1) if $\alpha = 0$, then $D(\bar{s}, \alpha) = \omega$ and $\mathrm{FIN}(\bar{s}, \alpha) = \mathrm{FIN}_\omega$;

(2) when $\alpha = \xi + 1$, and an infinite set $D(\bar{s}, \xi)$ as well as an ideal $\mathrm{FIN}(\bar{s}, \xi)$ on $D(\bar{s}, \xi)$ are given, then $D(\bar{s}, \alpha) = \coprod_{n \in \omega} D(\bar{s}, \xi)$ and $\mathrm{FIN}(\bar{s}, \alpha) = \coprod_{\mathrm{FIN}_\omega} \mathrm{FIN}(\bar{s}, \xi)$; by Lemma 2.1, $D(\bar{s}, \alpha)$ is an infinite set and $\mathrm{FIN}(\bar{s}, \alpha)$ is an ideal on $D(\bar{s}, \alpha)$;

(3) when α is a limit ordinal, and for each $n \in \omega$, an infinite set $D(\bar{s}, s_\alpha(n))$ as well as an ideal $\mathrm{FIN}(\bar{s}, s_\alpha(n))$ on $D(\bar{s}, s_\alpha(n))$ are given, then $D(\bar{s}, \alpha) = \coprod_{n \in \omega} D(\bar{s}, s_\alpha(n))$ and $\mathrm{FIN}(\bar{s}, \alpha) = \coprod_{\mathrm{FIN}_\omega} \mathrm{FIN}(\bar{s}, s_\alpha(n))$; once again by Proposition 2.1, $D(\bar{s}, \alpha)$ is an infinite set and $\mathrm{FIN}(\bar{s}, \alpha)$ is an ideal on $D(\bar{s}, \alpha)$.

Lemma 2.3. *Let $\bar{s} = \langle s_\alpha : \alpha \in \mathrm{Lim}(\omega_1) \rangle$ and $\bar{t} = \langle t_\alpha : \alpha \in \mathrm{Lim}(\omega_1) \rangle$ be indexed families such that $\forall \alpha \in \mathrm{Lim}(\omega_1) \, [s_\alpha : \omega \to \alpha \text{ and } t_\alpha : \omega \to \alpha \text{ are bijections}]$. Then for each $\alpha < \omega_1$, $\mathrm{FIN}(\bar{s}, \alpha)$ is isomorphic to $\mathrm{FIN}(\bar{t}, \alpha)$.*

Proof. Proceed by induction on $\alpha < \omega_1$. If $\alpha = 0$, then by definition, $D(\bar{s}, \alpha) = \omega = D(\bar{t}, \alpha)$ and $\mathrm{FIN}(\bar{s}, \alpha) = \mathrm{FIN}_\omega = \mathrm{FIN}(\bar{t}, \alpha)$. Hence the identity map on ω is an isomorphism.

Suppose $\alpha = \xi + 1$. By the induction hypothesis, there exists $g : D(\bar{s}, \xi) \to D(\bar{t}, \xi)$ which is an isomorphism from $\mathrm{FIN}(\bar{s}, \xi)$ to $\mathrm{FIN}(\bar{t}, \xi)$. Lemma 2.2 is applicable with ω as X and Y, $\langle D(\bar{s}, \xi) : n \in \omega \rangle$ as $\langle D_x : x \in X \rangle$, $\langle D(\bar{t}, \xi) : n \in \omega \rangle$ as $\langle E_y : y \in Y \rangle$, the identity map on ω as f, $\langle \mathrm{FIN}(\bar{s}, \xi) : n \in \omega \rangle$ as $\langle \mathcal{I}_x : x \in X \rangle$, $\langle \mathrm{FIN}(\bar{t}, \xi) : n \in \omega \rangle$ as $\langle \mathcal{J}_y : y \in Y \rangle$, $\langle g : n \in \omega \rangle$ as $\langle g_x : x \in X \rangle$, and FIN_ω as $\mathcal{I}$ and $\mathcal{J}$. And it yields the conclusion that $\mathrm{FIN}(\bar{s}, \xi + 1)$ is isomorphic to $\mathrm{FIN}(\bar{t}, \xi + 1)$.

Now assume that α is a limit ordinal. $f : \omega \to \omega$ defined by $f(n) = t_\alpha^{-1}(s_\alpha(n))$ is a bijection. By item (1) of Proposition 2.2, f is an isomorphism from FIN_ω to FIN_ω. For any $n \in \omega$, by the induction hypothesis, there is $g_n : D(\bar{s}, s_\alpha(n)) \to D(\bar{t}, s_\alpha(n))$ which is an isomorphism from $\mathrm{FIN}(\bar{s}, s_\alpha(n))$ to $\mathrm{FIN}(\bar{t}, s_\alpha(n))$. Hence Lemma 2.2 is applicable with ω as X and Y, $\langle D(\bar{s}, s_\alpha(n)) : n \in \omega \rangle$ as $\langle D_x : x \in X \rangle$, $\langle D(\bar{t}, t_\alpha(n)) : n \in \omega \rangle$ as $\langle E_y : y \in Y \rangle$, f as f, $\langle \mathrm{FIN}(\bar{s}, s_\alpha(n)) : n \in \omega \rangle$ as $\langle \mathcal{I}_x : x \in X \rangle$, $\langle \mathrm{FIN}(\bar{t}, t_\alpha(n)) : n \in \omega \rangle$ as $\langle \mathcal{J}_y : y \in Y \rangle$, $\langle g_n : n \in \omega \rangle$ as $\langle g_x : x \in X \rangle$, and FIN_ω as $\mathcal{I}$ and $\mathcal{J}$. And it yields the conclusion that $\mathrm{FIN}(\bar{s}, \alpha)$ is isomorphic to $\mathrm{FIN}(\bar{t}, \alpha)$. $\square$

Therefore the choice of $\bar{s}$ is inconsequential to the properties of the quotients $\mathcal{P}(D(\bar{s}, \alpha)) / \mathrm{FIN}(\bar{s}, \alpha)$. The splitting numbers of these quotients will now be considered. The first observation is that the splitting numbers are well-defined.

Lemma 2.4. *Suppose W is a countably infinite set. Let $\langle D_w : w \in W \rangle$ be an indexed family of infinite sets and let $\langle \mathcal{I}_w : w \in W \rangle$ be an indexed family such that for each $w \in W$, $\mathcal{I}_w$ is an ideal on D_w. Suppose $E = \coprod_{w \in W} D_w$ and $\mathcal{J} = \coprod_{\mathrm{FIN}_W} \mathcal{I}_w$. Then for any $A \subseteq E$ with $A \notin \mathcal{J}$, $\mathcal{J} {\restriction} A$ is not maximal on A.*

Proof. As $A \notin \mathcal{J}$, $\mathrm{suppt}(A) \subseteq W$ is infinite. Find disjoint infinite sets X, Y with $X \cup Y = \mathrm{suppt}(A)$. Let $B = \coprod_{w \in X} A(w)$ and $C = \coprod_{w \in Y} A(w)$. Then for any $w \in X$, $B(w) = A(w) \notin \mathcal{I}_w$ and for any $w \in Y$, $C(w) = A(w) \notin \mathcal{I}_w$. It is thus clear that $X \subseteq \mathrm{suppt}(B)$ and $Y \subseteq \mathrm{suppt}(C)$. Therefore, $B, C \notin \mathcal{J}$, whence $B, C \notin \mathcal{J} {\restriction} A$. Furthermore, it is simple to check that $B \cap C = \emptyset$. Thus $C \subseteq A \setminus B$, and as $\mathcal{J} {\restriction} A$ is an ideal on A, it follows that $A \setminus B \notin \mathcal{J} {\restriction} A$. Since $B \subseteq A$, $\mathcal{J} {\restriction} A$ is an ideal on A, and $A \setminus B \notin \mathcal{J} {\restriction} A$, there is an ideal $\mathcal{K}$ on A such that $(\mathcal{J} {\restriction} A) \cup \{B\} \subseteq \mathcal{K}$. $\mathcal{J} {\restriction} A \subsetneq \mathcal{K}$ because $B \in \mathcal{K} \setminus (\mathcal{J} {\restriction} A)$, showing that $\mathcal{J} {\restriction} A$ is not a maximal ideal on A. $\square$

Corollary 2.1. *Let $\bar{s} = \langle s_\alpha : \alpha \in \mathrm{Lim}(\omega_1) \rangle$ be so that $\forall \alpha \in \mathrm{Lim}(\omega_1) \, [s_\alpha : \omega \to \alpha \text{ is a bijection}]$. The following hold:*

(1) *for each $\alpha < \omega_1$ and for each $A \subseteq D(\bar{s}, \alpha)$, if $A \notin \mathrm{FIN}(\bar{s}, \alpha)$, then $\mathrm{FIN}(\bar{s}, \alpha) {\restriction} A$ is not maximal on A;*

(2) for each $\alpha < \omega_1$ and for each $A \subseteq D(\bar{s}, \alpha)$, if $A \notin \mathrm{FIN}(\bar{s}, \alpha)$, then $\mathfrak{s}_{\mathcal{P}(A)/(\mathrm{FIN}(\bar{s},\alpha)\restriction A)}$ is well-defined.

Proof. It has already been established that each $D(\bar{s}, \alpha)$ is an infinite set and that $\mathrm{FIN}(\bar{s}, \alpha)$ is an ideal on $D(\bar{s}, \alpha)$. Item (1) is proved by induction on α. If $\alpha = 0$, then $D(\bar{s}, \alpha) = \omega$ and $\mathrm{FIN}(\bar{s}, \alpha) = \mathrm{FIN}_\omega$. So if $A \subseteq D(\bar{s}, \alpha)$ and if $A \notin \mathrm{FIN}(\bar{s}, \alpha)$, then A is an infinite subset of ω and $\mathrm{FIN}_\omega \restriction A = \mathrm{FIN}_A$, which is not maximal on A. If $\alpha > 0$, then $D(\bar{s}, \alpha)$ has the form $\coprod_{n \in \omega} D_n$ and $\mathrm{FIN}(\bar{s}, \alpha)$ has the form $\coprod_{\mathrm{FIN}_\omega} \mathcal{I}_n$, where each D_n is an infinite set and $\mathcal{I}_n$ is an ideal on D_n. Thus 2.4 yields the conclusion of item (1).

To prove item (2), fix some $\alpha < \omega_1$ and $A \subseteq D(\bar{s}, \alpha)$ with $A \notin \mathrm{FIN}(\bar{s}, \alpha)$. Then $\mathrm{FIN}(\bar{s}, \alpha) \restriction A$ is an ideal on A, and according to Propositions 1.1 and 1.2, it needs to be seen that for every $B \subseteq A$ with $B \notin \mathrm{FIN}(\bar{s}, \alpha) \restriction A$, $(\mathrm{FIN}(\bar{s}, \alpha) \restriction A) \restriction B$ is not maximal on B. This is however clear because $(\mathrm{FIN}(\bar{s}, \alpha) \restriction A) \restriction B = \mathrm{FIN}(\bar{s}, \alpha) \restriction B$, which is not maximal on B by item (1). $\square$

Lemma 2.5. *Suppose X is a countably infinite set. Suppose $\langle D_x : x \in X \rangle$ and $\langle \mathcal{I}_x : x \in X \rangle$ are indexed families so that for all $x \in X$, D_x is an infinite set and $\mathcal{I}_x$ is an ideal on D_x with the property that for each $A \subseteq D_x$, if $A \notin \mathcal{I}_x$, then $\mathcal{I}_x \restriction A$ is not maximal on A. Let $D = \coprod_{x \in X} D_x$ and $\mathcal{J} = \coprod_{\mathrm{FIN}_X} \mathcal{I}_x$. Fix $A \subseteq D$ with $A \notin \mathcal{J}$. Then the following hold:*

(1) $\mathfrak{s}_{\mathcal{P}(A)/(\mathcal{J}\restriction A)} \le \mathfrak{s}$;
(2) suppose κ is a cardinal such that $\kappa < \mathfrak{s}$ and for each $x \in \mathrm{suppt}(A)$, $\kappa < \mathfrak{s}_{\mathcal{P}(A(x))/(\mathcal{I}_x \restriction A(x))}$. Then no family $\{b_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(A)/(\mathcal{J}\restriction A)$ can be a splitting family in $\mathcal{P}(A)/(\mathcal{J}\restriction A)$.

Proof. We prove (1) first. Define $Y = \mathrm{suppt}(A)$. Y is a countably infinite set. Find a splitting family $\mathcal{G} = \left\{ [Y_\alpha]_{\mathrm{FIN}_Y} : \alpha < \mathfrak{s} \right\}$ in $\mathcal{P}(Y)/\mathrm{FIN}_Y$. For each $\alpha < \mathfrak{s}$, define $B_\alpha = \coprod_{x \in Y_\alpha} A(x)$. Then $B_\alpha \subseteq A$. Now suppose $B \subseteq A$ and $B \notin \mathcal{J}\restriction A$. Then $\mathrm{suppt}(B)$ is an infinite subset of $\mathrm{suppt}(A) = Y$. By the choice of $\mathcal{G}$, find $\alpha < \mathfrak{s}$ so that both $Y_\alpha \cap \mathrm{suppt}(B)$ and $(Y \setminus Y_\alpha) \cap \mathrm{suppt}(B)$ are infinite. It is necessary to check that $B_\alpha \cap B$ and $(A \setminus B_\alpha) \cap B$ don't belong to $\mathcal{J}\restriction A$. It suffices to see that they are not in $\mathcal{J}$. For this, it is enough to show that $\mathrm{suppt}(B_\alpha \cap B)$ and $\mathrm{suppt}((A \setminus B_\alpha) \cap B)$ are infinite. It is not difficult to verify that $Y_\alpha \cap \mathrm{suppt}(B) \subseteq \mathrm{suppt}(B_\alpha \cap B)$ and that $(Y \setminus Y_\alpha) \cap \mathrm{suppt}(B) \subseteq \mathrm{suppt}((A \setminus B_\alpha) \cap B)$, which shows that $\mathrm{suppt}(B_\alpha \cap B)$ and $\mathrm{suppt}((A \setminus B_\alpha) \cap B)$ are infinite. It now follows that $\left\{ [B_\alpha]_{\mathcal{J}\restriction A} : \alpha < \mathfrak{s} \right\}$ is a splitting family in $\mathcal{P}(A)/(\mathcal{J}\restriction A)$, which establishes (1).

To prove (2), let $\{b_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(A)/(\mathcal{J}\restriction A)$ be given. Select $B_\alpha \subseteq A$ with $[B_\alpha]_{\mathcal{P}(A)/(\mathcal{J}\restriction A)} = b_\alpha$. Define $Y = \mathrm{suppt}(A)$. As $B_\alpha(x) \subseteq A(x)$, the hypothesis on κ implies that $\left\{ [B_\alpha(x)]_{\mathcal{P}(A(x))/(\mathcal{I}_x \restriction A(x))} : \alpha < \kappa \right\}$ is not a splitting family in $\mathcal{P}(A(x))/(\mathcal{I}_x \restriction A(x))$, for each $x \in Y$. Hence for each $x \in Y$, it is possible to find $B_x \subseteq A(x)$ with $B_x \notin \mathcal{I}_x \restriction A(x)$ and with the property that for every $\alpha < \kappa$, either $B_x \subseteq_{(\mathcal{I}_x \restriction A(x))} B_\alpha(x)$ or $B_x \subseteq_{(\mathcal{I}_x \restriction A(x))} A(x) \setminus B_\alpha(x)$. Now for each $\alpha < \kappa$, let $Y_\alpha = \left\{ x \in Y : B_x \subseteq_{(\mathcal{I}_x \restriction A(x))} B_\alpha(x) \right\}$. As Y is a countably infinite set and $\kappa < \mathfrak{s}$, there is an infinite set $Z \subseteq Y$ so that for all $\alpha < \kappa$, either $Z \subseteq_{\mathrm{FIN}_Y} Y_\alpha$ or $Z \subseteq_{\mathrm{FIN}_Y} Y \setminus Y_\alpha$. Define $B = \coprod_{x \in Z} B_x$. It is clear $B \subseteq A$ and it is easy to verify that $Z \subseteq \mathrm{suppt}(B)$. Since Z is infinite, $B \notin \mathcal{J}$, whence $B \notin \mathcal{J}\restriction A$. Therefore $b = [B]_{\mathcal{P}(A)/(\mathcal{J}\restriction A)} > 0$, and it will be shown that for each $\alpha < \kappa$, either $b \le b_\alpha$ or $b \le 1 - b_\alpha$, which will show that $\{b_\alpha : \alpha < \kappa\}$ is not a splitting family in $\mathcal{P}(A)/(\mathcal{J}\restriction A)$. To see this, fix $\alpha < \kappa$. There are two cases to consider.

Case 1: $Z \subseteq_{\mathrm{FIN}_Y} Y_\alpha$. Thus $F = Z \setminus Y_\alpha \in \mathrm{FIN}_Y$. In this case, $b \le b_\alpha$ holds. In other words, $B \setminus B_\alpha \in \mathcal{J}\restriction A$. To show this, it is enough to show $B \setminus B_\alpha \in \mathcal{J}$, which is implied by showing $\mathrm{suppt}(B \setminus B_\alpha)$ is finite, which in turn is implied by showing $\mathrm{suppt}(B \setminus B_\alpha) \subseteq F$. Indeed if $x \in \mathrm{suppt}(B \setminus B_\alpha)$, then $x \in X$ and $(B \setminus B_\alpha)(x) \notin \mathcal{I}_x$. In particular, $B(x) \ne \emptyset$, which implies $x \in Z$ because if x were not in Z, then by the definition of B as $\coprod_{x \in Z} B_x$, $B(x)$ would be empty, contradicting $(B \setminus B_\alpha)(x) \notin \mathcal{I}_x$. If x were

in Y_α, then by definition of Y_α, $B_x \setminus B_\alpha(x)$ would be in $\mathcal{I}_x \restriction A(x)$. However this would be a contradiction because $(B \setminus B_\alpha)(x) \subseteq B_x \setminus B_\alpha(x)$. Therefore, $x \in Z \setminus Y_\alpha = F$. This concludes Case 1.

Case 2: $Z \subseteq_{\mathrm{FIN}_Y} (Y \setminus Y_\alpha)$. Thus $F = Z \cap Y_\alpha$ is finite. In this case $b \leq 1 - b_\alpha$ holds. In other words, $B \cap B_\alpha \in \mathcal{J} \restriction A$. For this, it suffices to show $B \cap B_\alpha \in \mathcal{J}$, which is implied by showing $\mathrm{supp} (B \cap B_\alpha)$ is finite, and this in turn is implied by showing $\mathrm{supp} (B \cap B_\alpha) \subseteq F$. Indeed if $x \in \mathrm{supp} (B \cap B_\alpha)$, then $x \in X$ and $(B \cap B_\alpha)(x) \notin \mathcal{I}_x$, whence $B(x) \cap B_\alpha(x) \notin \mathcal{I}_x$. This immediately gives $x \in Z \cap Y_\alpha = F$, concluding the proof of Case 2 and of the lemma. $\square$

Corollary 2.2. *Suppose $\bar{s} = \langle s_\alpha : \alpha \in \mathrm{Lim}(\omega_1) \rangle$ is so that $\forall \alpha \in \mathrm{Lim}(\omega_1) \, [s_\alpha : \omega \to \alpha$ is a bijection$]$. Then for any $A \subseteq D(\bar{s}, \alpha)$ with $A \notin \mathrm{FIN}(\bar{s}, \alpha)$, $\mathfrak{s}_{\mathcal{P}(A)/(\mathrm{FIN}(\bar{s},\alpha) \restriction A)} = \mathfrak{s}$.*

Proof. By the results established above, $\kappa = \mathfrak{s}_{\mathcal{P}(A)/(\mathrm{FIN}(\bar{s},\alpha) \restriction A)}$ is always well-defined for any relevant A. Now proceed by induction on α. If $\alpha = 0$, then $D(\bar{s}, \alpha) = \omega$ and $\mathrm{FIN}(\bar{s}, \alpha) = \mathrm{FIN}_\omega$. So $A \subseteq \omega$ with A infinite. Thus $\mathrm{FIN}(\bar{s}, \alpha) \restriction A = (\mathrm{FIN}_\omega) \restriction A = \mathrm{FIN}_A$. Therefore $\mathcal{P}(A)/ (\mathrm{FIN}(\bar{s}, \alpha) \restriction A) = \mathcal{P}(A)/\mathrm{FIN}_A$, which is isomorphic to $\mathcal{P}(\omega)/\mathrm{FIN}_\omega$. So $\mathfrak{s}_{\mathcal{P}(A)/(\mathrm{FIN}(\bar{s},\alpha) \restriction A)} = \mathfrak{s}$. If $\alpha > 0$, then $D(\bar{s}, \alpha) = \coprod_{n \in \omega} D_n$ and

$$\mathrm{FIN}(\bar{s}, \alpha) = \coprod_{\mathrm{FIN}_\omega} \mathcal{I}_n,$$ where each D_n has the form $D(\bar{s}, \xi)$ and $\mathcal{I}_n$ has the form $\mathrm{FIN}(\bar{s}, \xi)$ for some $\xi < \alpha$.

In particular, each D_n is an infinite set and $\mathcal{I}_n$ is an ideal on D_n with the property that for any $C \subseteq D_n$, if $C \notin \mathcal{I}_n$, then $\mathcal{I}_n \restriction C$ is not maximal on C. Hence $\kappa \leq \mathfrak{s}$ by (1) of Lemma 2.5. On the other hand, κ cannot be strictly less than $\mathfrak{s}$. For if $\kappa < \mathfrak{s}$, then by the induction hypothesis, for each $n \in \mathrm{supp}(A)$, $\kappa < \mathfrak{s} = \mathfrak{s}_{\mathcal{P}(A(n))/(\mathcal{I}_n \restriction A(n))}$. And so by (2) of Lemma 2.5, there can be no splitting family of size κ in $\mathcal{P}(A)/ (\mathrm{FIN}(\bar{s}, \alpha) \restriction A)$, contradicting the definition of κ. Therefore $\kappa = \mathfrak{s}$. $\square$

Question 2.1. What are the possible values for $\mathfrak{s}_\mathcal{I}$ relative to other cardinal invariants if $\mathcal{I}$ is an F_σ ideal? What of the specific case for the summable ideal?

Corollary 2.2 says that the splitting number is not a new cardinal invariant for the ideals from Definition 2.5 or for any of their restrictions. Their almost disjointness numbers are examined next.

Definition 2.6. For each $\alpha \in \omega_1$, define $\mathfrak{a}_\alpha = \mathfrak{a}_{\mathcal{P}(D(\bar{s},\alpha))/\mathrm{FIN}(\bar{s},\alpha)}$, where $\bar{s} = \langle s_\alpha : \alpha \in \mathrm{Lim}(\omega_1) \rangle$ is any sequence so that $\forall \alpha \in \mathrm{Lim}(\omega_1) \, [s_\alpha : \omega \to \alpha$ is a bijection$]$. By Lemma 2.3, the choice of $\bar{s}$ is immaterial.

Proposition 2.3. *If $\mathcal{J}_n$ are ideals on the infinite sets Y_n for $n \in \omega$, then $\mathfrak{a}_{\coprod_{\mathrm{FIN}_\omega} \mathcal{J}_n} \geq \mathfrak{b}$.*

Proof. Suppose that $\mathcal{A}$ is a family such that:
- $|\mathcal{A}| < \mathfrak{b}$
- $\mathcal{A} \subseteq (\coprod_{\mathrm{FIN}_\omega} \mathcal{J}_n)^+$
- if A and B are distinct elements of $\mathcal{A}$ then $A \cap B \in \coprod_{\mathrm{FIN}_\omega} \mathcal{J}_n$.

Let $\{A_n\}_{n \in \omega}$ be distinct elements of $\mathcal{A}$. It is easy to find $A_n^* \subseteq A_n$ such that $A_n^* \notin \coprod_{\mathrm{FIN}_\omega} \mathcal{J}_n$ and $A_n^* \cap A_m^* = \varnothing$ for distinct n and m. For each $A \in \mathcal{A} \setminus \{A_n\}_{n \in \omega}$ define $F_A : \omega \to \omega$ such that $A(k) \cap A_n^*(k) \in \mathcal{J}_k$ for all $k \geq F_A(n)$.

There is then $F : \omega \to \omega$ such that $F \geq^* F_A$ for all $A \in \mathcal{A} \setminus \{A_n\}_{n \in \omega}$. Define $V = \coprod_{n \in \omega} A_n^*(F(n))$. It is routine to check that $V \cap A \in \coprod_{\mathrm{FIN}_\omega} \mathcal{J}_n$ for each $A \in \mathcal{A}$ showing that $\mathcal{A}$ is not maximal. $\square$

Corollary 2.3. $\mathfrak{a}_\alpha \geq \mathfrak{b}$ *for all $\alpha \in \omega_1$.*

Proposition 2.4. *If $\mathcal{I}_n$ are ideals on the infinite sets D_n for $n \in \omega$, then $\mathfrak{a}_\mathcal{J} \leq \mathfrak{a}$, where $\mathcal{J}$ is the ideal $\coprod_{\mathrm{FIN}_\omega} \mathcal{I}_n$ on $D = \coprod_{n \in \omega} D_n$.*

Proof. Let $\{Y_\alpha : \alpha < \mathfrak{a}\}$ be a m.a.d. family on ω. Define $A_\alpha = \coprod_{n \in Y_\alpha} D_n$. Then $A_\alpha \subseteq D$ and $A_\alpha \notin \mathcal{J}$ because $Y_\alpha \subseteq \mathrm{supp}(A_\alpha)$. If $\alpha < \beta < \mathfrak{a}$ and $n \in \mathrm{supp}(A_\alpha \cap A_\beta)$, then $A_\alpha(n) \cap A_\beta(n) = (A_\alpha \cap A_\beta)(n) \notin$

$\mathcal{I}_n$, whence $n \in Y_\alpha \cap Y_\beta$. Thus $\mathrm{suppt}(A_\alpha \cap A_\beta) \subseteq Y_\alpha \cap Y_\beta$, which is finite, and so $A_\alpha \cap A_\beta \in \mathcal{J}$. Finally, suppose that $A \subseteq D$ with $A \notin \mathcal{J}$. Then $\mathrm{suppt}(A)$ is an infinite subset of ω. Find $\alpha < \mathfrak{a}$ so that $\mathrm{suppt}(A) \cap Y_\alpha$ is infinite. If $n \in \mathrm{suppt}(A) \cap Y_\alpha$, then $(A \cap A_\alpha)(n) = A(n) \cap A_\alpha(n) = A(n) \cap D_n = A(n) \notin \mathcal{I}_n$, whence $n \in \mathrm{suppt}(A \cap A_\alpha)$. Thus $\mathrm{suppt}(A) \cap Y_\alpha \subseteq \mathrm{suppt}(A \cap A_\alpha)$, and so $A \cap A_\alpha \notin \mathcal{J}$. This shows that $\left\{ [A_\alpha]_{\mathcal{P}(D)/\mathcal{J}} : \alpha < \mathfrak{a} \right\}$ is an infinite maximal almost disjoint family in $\mathcal{P}(D)/\mathcal{J}$. $\square$

Corollary 2.4. *For each $\alpha < \omega_1$, $\mathfrak{a}_\alpha \leq \mathfrak{a}$.*

Of course, $\mathfrak{a}_0$ is the classical invariant $\mathfrak{a}$. By Corollaries 2.3 and 2.4, the cardinals $\mathfrak{a}_\alpha$ stand sandwiched between $\mathfrak{b}$ and $\mathfrak{a}$. It is unknown at present whether any of the $\mathfrak{a}_\alpha$ can be distinguished from each other.

Question 2.2. Is it consistent to have $\mathfrak{a}_\alpha < \mathfrak{a}_\beta$ for some $\alpha, \beta < \omega_1$? For each $n \geq 1$, is it consistent to have $\mathfrak{b} < \mathfrak{a}_n < \cdots < \mathfrak{a}$?

The invariant $\mathfrak{a}_1$ was investigated by Brendle [1]. Brendle considered both the questions of whether or not $\mathfrak{a}_1 < \mathfrak{a}$ is consistent and weather or not $\mathfrak{b} < \mathfrak{a}_1$ is consistent. He used a template style iteration similar to the one from Shelah [10] to produce a model where $\aleph_2 = \mathfrak{b} < \mathfrak{a} = \mathfrak{a}_1 = \aleph_3$. Brendle [1] asked whether $\aleph_1 = \mathfrak{b} < \mathfrak{a}_1$ is consistent. The question of whether or not $\mathfrak{a}_1 < \mathfrak{a}$ is consistent was again implicitly raised in [8] and explicitly by Törnquist at the Fields Set Theory meeting in May 2019.

Question 2.3. What are the possible values for $\mathfrak{a}_\mathcal{I}$ relative to other cardinal invariants if $\mathcal{I}$ is an F_σ ideal? What of the specific case for the summable ideal?

The next theorem, which is the main result of this paper, provides a ZFC lower bound for $\mathfrak{a}_\alpha$ in terms of $\mathfrak{a}$ and $\mathfrak{s}$. It will shed some light on Question 2.2 by constraining possible models of $\mathfrak{a}_\alpha < \mathfrak{a}$.

Theorem 2.1. *Let X be a countably infinite set and let $\langle D_x : x \in X \rangle$ be an indexed family of infinite sets. Suppose $\langle \mathcal{I}_x : x \in X \rangle$ is an indexed family such that $\mathcal{I}_x$ is an ideal on D_x with the property that for every $A \subseteq D_x$, if $A \notin \mathcal{I}_x$, then $\mathcal{I}_x {\restriction} A$ is not maximal on A. Let κ be an infinite cardinal. Assume that for each $x \in X$ and for every $A \subseteq D_x$ with $A \notin \mathcal{I}_x$, $\kappa < \mathfrak{s}_{\mathcal{P}(A)/(\mathcal{I}_x {\restriction} A)}$ and that $\kappa < \mathfrak{a}$. Let $D = \coprod_{x \in X} D_x$ and $\mathcal{J} = \coprod_{\mathrm{FIN}_X} \mathcal{I}_x$. Then no $(\mathcal{P}(D)/\mathcal{J})$-almost disjoint sequence $\langle a_\alpha : \alpha < \kappa \rangle$ is maximal in $\mathcal{P}(D)/\mathcal{J}$.*

Proof. Choose $A_\alpha \in (\mathcal{P}(D)/\mathcal{J})^+$ so that $[A_\alpha]_{\mathcal{P}(D)/\mathcal{J}} = a_\alpha$. Let $X_\alpha = \mathrm{suppt}(A_\alpha) \subseteq X$ and note that $X_\alpha \notin \mathrm{FIN}_X$. Thus for each $\alpha < \kappa$, $|X_\alpha| = \aleph_0$. Find a family $\{Y_n : n \in \omega\}$ so that Y_n is infinite and $Y_n \subseteq X_n$, for all $n \in \omega$, and $\forall n < m < \omega [Y_n \cap Y_m = \emptyset]$. Define $Y = \bigcup_{n \in \omega} Y_n$. Fix $x \in Y$ and let n_x be the unique $n \in \omega$ so that $x \in Y_n$. Then by definition, $x \in X$, $A_{n_x}(x) \subseteq D_x$, and $A_{n_x}(x) \notin \mathcal{I}_x$. The assumption that $\kappa < \mathfrak{s}_{\mathcal{P}(A_{n_x}(x))/(\mathcal{I}_x {\restriction} A_{n_x}(x))}$ implies that $\left\{ [A_{n_x}(x) \cap A_\alpha(x)]_{\mathcal{P}(A_{n_x}(x))/(\mathcal{I}_x {\restriction} A_{n_x}(x))} : \alpha < \kappa \right\}$ is not a splitting family in $\mathcal{P}(A_{n_x}(x))/(\mathcal{I}_x {\restriction} A_{n_x}(x))$. So there is $B_x \subseteq A_{n_x}(x)$ such that $B_x \notin \mathcal{I}_x$ and for any $\alpha < \kappa$, either $B_x \subseteq_{\mathcal{I}_x} A_{n_x}(x) \cap A_\alpha(x)$ or $B_x \subseteq_{\mathcal{I}_x} A_{n_x}(x) \setminus A_\alpha(x)$. Now for each $\alpha < \kappa$, define $Z_\alpha = \{x \in Y : B_x \subseteq_{\mathcal{I}_x} A_{n_x}(x) \cap A_\alpha(x)\}$. Although the following claim is simple, it plays an important role.

Claim 1. For each $n \in \omega$, $Y_n \subseteq Z_n$.

Proof. $x \in Y_n \implies n_x = n$, and by choice of B_x, $B_x \subseteq A_n(x) = A_{n_x}(x) \cap A_n(x)$. Hence $x \in Y$ and $B_x \subseteq_{\mathcal{I}_x} A_{n_x}(x) \cap A_n(x)$, whence $x \in Z_n$ by definition. $\square$

Claim 2. For all $\alpha < \beta < \kappa$, $Z_\alpha \cap Z_\beta$ is finite.

Proof. Suppose for a contradiction that $Z_\alpha \cap Z_\beta$ is infinite. Consider $x \in Z_\alpha \cap Z_\beta$. Then $x \in X$, $B_x \subseteq_{\mathcal{I}_x} A_{n_x}(x) \cap A_\alpha(x)$ and $B_x \subseteq_{\mathcal{I}_x} A_{n_x}(x) \cap A_\beta(x)$. Hence $B_x \subseteq_{\mathcal{I}_x} A_\alpha(x) \cap A_\beta(x) \subseteq A_\alpha(x) \subseteq D_x$. As $\mathcal{I}_x$ is an ideal on D_x and $B_x \notin \mathcal{I}_x$, $A_\alpha(x) \cap A_\beta(x) \notin \mathcal{I}_x$. So $(A_\alpha \cap A_\beta)(x) \notin \mathcal{I}_x$. Thus $x \in \mathrm{suppt}(A_\alpha \cap A_\beta)$. It

has been shown that $Z_\alpha \cap Z_\beta \subseteq \mathrm{suppt}\,(A_\alpha \cap A_\beta)$, whence $\mathrm{suppt}\,(A_\alpha \cap A_\beta)$ is infinite. So $A_\alpha \cap A_\beta \notin \mathcal{J}$. However this implies $a_\alpha \wedge a_\beta = [A_\alpha]_{\mathcal{P}(D)/\mathcal{J}} \wedge [A_\beta]_{\mathcal{P}(D)/\mathcal{J}} = [A_\alpha \cap A_\beta]_{\mathcal{P}(D)/\mathcal{J}} > 0$, contradicting the almost disjointness of $\langle a_\xi : \xi < \kappa \rangle$. $\qquad \square$

Let $T = \{\alpha < \kappa : Z_\alpha \text{ is infinite}\}$. By Claim 1 and by the fact that each Y_n is infinite, $\omega \subseteq T$. Therefore by Claim 2, $\mathbb{F} = \{Z_\alpha : \alpha \in T\}$ is an infinite almost disjoint family of infinite subsets of Y with $|\mathbb{F}| \leq |T| \leq \kappa < \mathfrak{a}$. As Y is a countably infinite set, fix $Z \subseteq Y$ infinite with $|Z \cap Z_\alpha| < \aleph_0$ for all $\alpha < \kappa$. Define $B = \coprod_{x \in Z} B_x$. Now for each $x \in Z$, $B_x \notin \mathcal{I}_x$. Hence $Z \subseteq \mathrm{suppt}(B)$. As Z is infinite, $\mathrm{suppt}(B)$ is infinite as well and so $B \notin \mathcal{J}$.

Claim 3. $[B]_{\mathcal{P}(D)/\mathcal{J}}$ is $\mathcal{P}(D)/\mathcal{J}$-almost disjoint to $\{a_\alpha : \alpha < \kappa\}$.

Proof. As $B \subseteq D$ and $B \notin \mathcal{J}$, $b = [B]_{\mathcal{P}(D)/\mathcal{J}} > 0$. Fix $\alpha < \kappa$. Then $b \wedge a_\alpha = [B \cap A_\alpha]_{\mathcal{P}(D)/\mathcal{J}}$. Consider any $x \in \mathrm{suppt}\,(B \cap A_\alpha)$. Then $x \in X$ and $B(x) \cap A_\alpha(x) \notin \mathcal{I}_x$. In particular, $x \in Z$ because otherwise $B(x) = \emptyset \in \mathcal{I}_x$. Thus $x \in Y$. Now assume for a contradiction that $x \notin Z_\alpha$. Then $B_x \subseteq_{\mathcal{I}_x} A_{n_x}(x) \setminus A_\alpha(x)$. However as $B_x \subseteq A_{n_x}(x) \subseteq D_x$, it follows that $B(x) \cap A_\alpha(x) \in \mathcal{I}_x$, which is a contradiction. This contradiction shows that $x \in Z \cap Z_\alpha$. Thus it has been shown that $\mathrm{suppt}\,(B \cap A_\alpha) \subseteq Z \cap Z_\alpha$. As $\alpha < \kappa$, $Z \cap Z_\alpha$ is finite. So $\mathrm{suppt}\,(B \cap A_\alpha)$ is finite. Therefore $B \cap A_\alpha \in \mathcal{J}$, whence $b \wedge a_\alpha = 0$. $\qquad \square$

Claim 3 shows that $\langle a_\alpha : \alpha < \kappa \rangle$ is not maximal. $\qquad \square$

Corollary 2.5. *For each $\alpha < \omega_1$, $\mathfrak{a}_\alpha \geq \min\{\mathfrak{a}, \mathfrak{s}\}$.*

Proof. Let $\kappa = \mathfrak{a}_\alpha$, which is always an infinite cardinal. When $\alpha = 0$, $\kappa = \mathfrak{a} \geq \min\{\mathfrak{a}, \mathfrak{s}\}$. When $\alpha > 0$, $D(\bar{s}, \alpha)$ has the form $\coprod_{n \in \omega} D_n$ and $\mathrm{FIN}(\bar{s}, \alpha)$ has the form $\coprod_{\mathrm{FIN}_\omega} \mathcal{I}_n$, where each D_n is an infinite set and $\mathcal{I}_n$ is an ideal on D_n. Furthermore, each D_n has the form $D(\bar{s}, \xi)$ and $\mathcal{I}_n$ has the form $\mathrm{FIN}(\bar{s}, \xi)$. By Corollary 2.1, for any $A \subseteq D_n$, if $A \notin \mathcal{I}_n$, then $\mathcal{I}_n{\restriction}A$ is not maximal on A. Now assume for a contradiction that $\kappa < \min\{\mathfrak{a}, \mathfrak{s}\}$. Then by Corollary 2.2, for any $A \subseteq D_n$ with $A \notin \mathcal{I}_n$, $\kappa < \mathfrak{s} = \mathfrak{s}_{\mathcal{P}(A)/(\mathcal{I}_n{\restriction}A)}$. Therefore the hypotheses of Theorem 2.1 are satisfied and it implies that there are no maximal almost disjoint families of size κ in $\mathcal{P}(D(\bar{s}, \alpha))/\mathrm{FIN}(\bar{s}, \alpha)$, contradicting the definition of κ. $\qquad \square$

The next theorem, which easily follows from Theorem 2.1 and from the work of Shelah in [9], answers Brendle's question about the consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{a}_1 = \aleph_2$.

Theorem 2.2. *It is consistent that $\mathfrak{b} = \aleph_1$ and for each $\alpha < \omega_1$, $\mathfrak{a}_\alpha = \aleph_2$.*

Proof. In [9], Shelah produced a model in which $\mathfrak{b} = \aleph_1$ and $\mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \aleph_2$. So Theorem 2.1 says that the necessary configuration holds in Shelah's model. $\qquad \square$

In [2], Brendle found a c.c.c. forcing closely related to Shelah's creature forcing from [9] to produce models where $\mathfrak{b}$ is small while $\mathfrak{a}$ and $\mathfrak{s}$ are both larger. In fact, Brendle [2] showed that for any regular κ, there is a model with $\mathfrak{b} = \kappa < \kappa^+ = \mathfrak{a} = \mathfrak{s}$. Brendle's methods from [2] yield the following consistency result.

Corollary 2.6. *Let κ be a regular uncountable cardinal. It is consistent with ZFC that $\kappa = \mathfrak{b}$ and for each $\alpha < \omega_1$, $\mathfrak{a}_\alpha = \mathfrak{s} = \mathfrak{c} = \kappa^+$.*

Brendle and Khomskii [3] introduced the cardinal $\mathfrak{a}_{closed}$, which is the least κ such that there are κ many closed subsets of $[\omega]^\omega$ whose union is a m.a.d. family in $\mathcal{P}(\omega)/\mathrm{FIN}_\omega$. Brendle and Khomskii showed in [3] that $\mathfrak{a}_{closed} = \aleph_1 < \aleph_2 = \mathfrak{b}$ holds in the Hechler model. Since $\mathfrak{b} \leq \mathfrak{a}_\alpha$, this shows the consistency of $\mathfrak{a}_{closed} < \mathfrak{a}_\alpha$ for any $\alpha < \omega_1$. However the following seems unclear.

Question 2.4. Is there some $\alpha < \omega_1$ for which $\mathfrak{a}_\alpha < \mathfrak{a}_{closed}$ is consistent?

Brendle and Raghavan [4] adapted the arguments of Shelah [9] and Brendle [2] to produce models of $\mathfrak{b} < \mathfrak{a}_{closed}$. Their models show the following.

Corollary 2.7. *Let κ be a regular uncountable cardinal. It is consistent with* ZFC *that $\kappa = \mathfrak{b}$ and for each $\alpha < \omega_1$, $\mathfrak{a}_{closed} = \mathfrak{a}_\alpha = \mathfrak{s} = \mathfrak{c} = \kappa^+$.*

None of the presently available techniques seem to produce a model where $\mathfrak{b} = \aleph_1$ and $\mathfrak{a}$ is larger than $\aleph_2$. So the following basic question seems to be open.

Question 2.5. Is it consistent to have $\mathfrak{b} = \aleph_1$ and $\mathfrak{a} > \aleph_2$?

For $\alpha > 0$, Theorem 2.1 contains some information about the possible techniques that could be used to exhibit a model of $\mathfrak{a}_\alpha < \mathfrak{a}$. $\mathfrak{b} < \mathfrak{a}$ must hold in any such model and three essentially different techniques are presently known for getting models with $\mathfrak{b} < \mathfrak{a}$. The first class of techniques derive from Shelah's method in [9] and its c.c.c. version discovered by Brendle [1]. All of these techniques increase $\mathfrak{s}$ in addition to $\mathfrak{a}$ and they tend to produce models where $\mathfrak{a} = \mathfrak{s} = \mathfrak{c}$, and so by Theorem 2.1, they are unsuitable for $\mathfrak{a}_\alpha < \mathfrak{a}$. This class of techniques remains the only currently available one for getting $\aleph_1 = \mathfrak{b} < \mathfrak{a}$. Hence it will be necessary to solve the following open problem in order to show the consistency of $\aleph_1 = \mathfrak{a}_\alpha < \mathfrak{a}$ for some $\alpha < \omega_1$.

Question 2.6 (Brendle and Raghavan). Is it consistent to have $\aleph_1 = \mathfrak{b} = \mathfrak{s} < \mathfrak{a}$?

It is consistent to have $\aleph_2 = \mathfrak{b} = \mathfrak{s} < \mathfrak{a}$. In fact, Shelah [10] showed the consistency of $\mathfrak{d} < \mathfrak{a}$ and he invented two different techniques for this. The first method involves a measurable cardinal and the ultrapower a well chosen forcing notion. The second is the method for iteration along a template. However neither of these techniques will produce a model of $\mathfrak{a}_\alpha < \mathfrak{a}$ because the argument which shows that $\mathfrak{a}$ is increased by these forcings will also apply to $\mathfrak{a}_\alpha$. It should be noted that unlike the case for $\mathfrak{a}$, Raghavan and Shelah [7] showed that $\mathfrak{a}_{closed} = \aleph_1$ if $\mathfrak{d} = \aleph_1$.

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