# P-IDEAL DICHOTOMY AND WEAK SQUARES

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Abstract. We answer a question of Cummings and Magidor by proving that the P-ideal dichotomy of Todorčević refutes $\square_{\kappa,\omega}$ for any uncountable $\kappa$. We also show that the P-ideal dichotomy implies the failure of $\square_{\kappa,<\mathfrak{b}}$ provided that $\mathrm{cf}(\kappa) > \omega_1$.

## 1. Introduction

The P-ideal dichotomy (PID) is a powerful combinatorial dichotomy introduced by Todorčević which has a wide variety of consequences. It is well-known that PID is a consequence of PFA, and in fact, it suffices for several applications of PFA (see [8]). An interesting feature of PID is that it is consistent with CH ([6]), and hence it provides an axiomatic route for showing that certain consequences of PFA are consistent with CH, making it possible to bypass complicated iterated forcing constructions.

A recurring theme in set theory is that forcing axioms and combinatorial principles that are their consequences tend to be incompatible with square principles. In an early application of PID, Todorčević [6] proved that it implies the failure of Jensen's square principle $\square_\kappa$, for every uncountable $\kappa$. A hierarchy of weakenings of the square principle was introduced by Schimmerling [3] and has been extensively studied since.

**Definition 1.** Let $\kappa$ and $\lambda$ be cardinals with $\kappa$ infinite. A sequence $\langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ is called a $\square_{\kappa,<\lambda}$ sequence if for each $\alpha \in \mathrm{Lim}(\kappa^+)$ the following hold:

(1) $\forall c \in \mathcal{C}_\alpha\, [c \subset \alpha \text{ is a club in } \alpha]$
(2) $0 < |\mathcal{C}_\alpha| < \lambda$
(3) $\forall c \in \mathcal{C}_\alpha \forall \beta \in \mathrm{Lim}(c) \exists c^* \in \mathcal{C}_\beta\, [c^* = c \cap \beta]$.
(4) $\forall c \in \mathcal{C}_\alpha\, [\mathrm{otp}(c) \leq \kappa]$.

A $\square_{\kappa,<\lambda^+}$ sequence is called a $\square_{\kappa,\lambda}$ sequence. $\square_{\kappa,<\lambda}$ is the statement that there exists a $\square_{\kappa,<\lambda}$ sequence. $\square_{\kappa,1}$ is equivalent to Jensen's $\square_\kappa$, and $\square_{\kappa,\kappa}$ is equivalent to the principle $\square^*_\kappa$, also introduced by Jensen. It is well-known that $\square_{\kappa,\kappa}$ is equivalent to the existence of a special Aronszajn tree on $\kappa^+$ (see for example [7]). It is easily seen that $\square_{\kappa,\kappa^+}$ is always true.

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Todorčević proved that PFA implies the failure of $\square_{\kappa,\omega_1}$ for every uncountable $\kappa$ (more precisely, this follows from the proof of Theorem 1 of [5]). Magidor proved that this result is sharp by showing that PFA is consistent with the statement $\forall \kappa \geq \omega\, [\square_{\kappa,\omega_2}$ holds]. Magidor also showed that Martin's Maximum (MM) implies the failure of $\square_{\kappa,\kappa}$ for every uncountable $\kappa$ with $\mathrm{cf}(\kappa) = \omega$. Then Cummings and Magidor [1] obtained sharp results on the greatest extent of $\square_{\kappa,\kappa}$ that is compatible with MM. Magidor also considered the influence of the axiom DPFA, the analogue of PFA for proper posets that do not add any reals. He showed that DPFA is consistent with the existence of a $\square_{\kappa,\omega_1}$ sequence at each infinite $\kappa$. On the other hand, the Mapping Reflection Principle (MRP) of Moore [2] is a consequence of DPFA, and in unpublished work, Sharon [4] showed that MRP implies the failure of $\square_{\kappa,\omega}$, for all uncountable $\kappa$. As PID is also a consequence of DPFA, this naturally led Cummings and Magidor to the following question, which we learnt from Magidor at the Conference on Mathematical Logic and Set Theory held at Chennai in August 2010.

**Question 2** (Cummings and Magidor)**.** Does PID imply the failure of $\square_{\kappa,\omega}$ for every uncountable $\kappa$?

The main result of this paper gives an affirmative answer to this question. Even though PID itself is consistent with CH, it is well-known that it becomes much more powerful when combined with an additional hypothesis like $\mathfrak{b} > \omega_1$ or $\mathfrak{p} > \omega_1$. Indeed, PID $+\, \mathfrak{p} > \omega_1$ suffices for many of the classic applications of PFA that contradict CH, such as the non-existence of $S$ spaces (see [8]). Therefore, this intuition would lead one to suspect that PID $+\, \mathfrak{p} > \omega_1$ *ought* to have the same influence on squares as PFA. In other words, the hypothesis PID $+\, \mathfrak{p} > \omega_1$ ought to imply the failure of $\square_{\kappa,\omega_1}$, for every uncountable $\kappa$, even though DPFA does not. We have not been able to fully prove this. However, in Section 4 of this paper we show that PID $+\, \mathfrak{b} > \omega_1$ implies the failure of $\square_{\kappa,\omega_1}$, *provided that* $\mathrm{cf}(\kappa) > \omega_1$.

## 2. Notation

Our notation is standard. "$a \subset b$" means $\forall x\, [x \in a \implies x \in b]$. "$\forall^\infty$" means for all but finitely many and "$\exists^\infty$" stands for there exists infinitely many. For functions $f, g \in \omega^\omega$, $f <^* g$ means $\forall^\infty n \in \omega\, [f(n) < g(n)]$. A set $F \subset \omega^\omega$ is said to be *unbounded* if there is no $g \in \omega^\omega$ such that $\forall f \in F\, [f <^* g]$. For sets $a$ and $b$, $a \subset^* b$ iff $a \setminus b$ is finite. A family $F \subset [\omega]^\omega$ is said to have the *finite intersection property (FIP)* if for any $A \in [F]^{<\omega}$, $\bigcap A$ is infinite. Recall the following cardinal invariants:

$$\mathfrak{p} = \min\{|F| : F \subset [\omega]^\omega \wedge F \text{ has the FIP } \wedge \neg \exists b \in [\omega]^\omega\, \forall a \in F\, [b \subset^* a]\}$$
$$\mathfrak{b} = \min\{|F| : F \subset \omega^\omega \wedge F \text{ is unbounded}\}$$

It is easy to see that $\mathfrak{p} \leq \mathfrak{b}$. The invariant $\mathfrak{b}$ will be used in Section 4.

We will make use of elementary submodels in Sections 3 and 4. We will simply write "$M \prec H(\theta)$" to mean "$M$ is an elementary submodel of $H(\theta)$, where $\theta$ is a regular cardinal that is large enough for the argument at hand".

## 3. Failure of $\square_{\kappa,\omega}$ under PID

**Definition 3.** Let $X$ be an uncountable set. An ideal $\mathcal{I} \subset [X]^{\leq\omega}$ is called a *P-ideal* if for every countable collection $\{x_n : n \in \omega\} \subset \mathcal{I}$, there is $x \in \mathcal{I}$ such that $\forall n \in \omega\, [x_n \subset^* x]$.

All ideals are assumed to be non-principal, meaning that $[X]^{<\omega} \subset \mathcal{I}$. Recall the P-ideal dichotomy of Todorčević [6].

**Definition 4.** *The P-ideal dichotomy* (PID) is the following statement: For any P-ideal $\mathcal{I}$ on an uncountable set $X$ *either*

(1) There is an uncountable set $Y \subset X$ such that $[Y]^{\leq\omega} \subset \mathcal{I}$

*or*

(2) There exist $\{X_n : n \in \omega\}$ such that the $X_n$ are pairwise disjoint, $X = \bigcup_{n\in\omega} X_n$, and $\forall n \in \omega\, [[X_n]^{\omega} \cap \mathcal{I} = 0]$.

In this section, we prove

**Theorem 5.** *Assume* PID. *Let $\kappa$ be an uncountable cardinal. Then $\square_{\kappa,\omega}$ fails.*

The proof is a modification of Todorčević's argument in [6] that PID implies the failure of $\square_\kappa$, for every uncountable $\kappa$. That proof used the method of minimal walks, the crucial characteristic there being the function $\rho_2$ (see [7]). Here we develop an analogue of $\rho_2$ which may be of use in other contexts where a single $c_\alpha \subset \alpha$ is replaced by several.

For the rest of this section fix $\langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$, a $\square_{\kappa,\omega}$ sequence. For each $\alpha < \kappa^+$, put $\mathcal{C}_{\alpha+1} = \{\{\alpha\}\}$. Let $\{c^i_\alpha : i < \omega\}$ be an enumeration of $\mathcal{C}_\alpha$, possibly with repetitions. Let FIN denote $[\omega]^{<\omega} \setminus \{0\}$. For a set of ordinals $X$, $\mathrm{Lim}(X) = \{\beta : X \cap \beta \text{ is unbounded in } \beta\}$.

**Definition 6.** For $F \in \mathrm{FIN}$ and $\alpha \leq \beta < \kappa^+$, define $T_F(\alpha,\beta)$ as follows. $T_F(\alpha,\alpha) = \{\langle\alpha\rangle\}$. If $\alpha < \beta$, then $T_F(\alpha,\beta) = \left\{\langle\beta\rangle^\frown\sigma : \exists i \in F\left[\sigma \in T_F(\alpha, \min(c^i_\beta \setminus \alpha))\right]\right\}$. Note that $\sigma \in T_F(\alpha,\beta)$ iff

(1) $\sigma \in (\kappa^+)^{<\omega}$ and $|\sigma| > 0$
(2) $\sigma(0) = \beta$ and $\sigma(|\sigma|-1) = \alpha$
(3) $\forall 0 < j < |\sigma|\, \exists i \in F\left[\sigma(j) = \min\left(c^i_{\sigma(j-1)} \setminus \alpha\right)\right]$.

**Definition 7.** For $F \in \mathrm{FIN}$ and $\alpha \leq \beta < \kappa^+$, define

$$S_F(\alpha,\beta) = \min\{|\sigma| : \sigma \in T_F(\alpha,\beta)\}.$$

Thus $S_F(\alpha,\beta)$ is the number of ordinals in the shortest walk from $\beta$ to $\alpha$ under the constraint that only those ladders whose index is in $F$ may be used at each step of the walk. It is not the number of steps in this walk. Therefore, $S_F(\alpha,\alpha) = 1$. This departs from the convention in [7] where $\rho_2(\alpha,\alpha) = 0$. Next, we define the P-ideal we will use.

**Definition 8.** Define $\mathcal{I}$ to be
$$\left\{X \in \left[\kappa^+\right]^{\leq\omega} : \forall F \in \mathrm{FIN}\, \forall\beta \in \mathrm{Lim}(X) \forall k \in \omega \forall^\infty \alpha \in X \cap (\beta+1)\, [S_F(\alpha,\beta) \neq k]\right\}.$$

**Lemma 9.** *Let $\alpha < \gamma$ and $\sigma \in T_F(\alpha,\gamma)$ with $|\sigma| = S_F(\alpha,\gamma)$. Then $S_F(\alpha,\sigma(1)) = S_F(\alpha,\gamma) - 1$.*

*Proof.* It is clear that $\langle\sigma(1),\dots,\sigma(|\sigma|-1)\rangle \in T_F(\alpha,\sigma(1))$. Therefore, $S_F(\alpha,\sigma(1)) \leq |\sigma|-1$. On the other hand, if there is $\tau \in T_F(\alpha,\sigma(1))$ with $|\tau| < |\sigma|-1$, then $\langle\gamma\rangle^\frown\tau \in T_F(\alpha,\gamma)$ because there is an $i \in F$ such that $\tau(0) = \sigma(1) = \min(c^i_\gamma \setminus \alpha)$. $\dashv$

Suppose $X \in \mathcal{I}$. If $Z \subset X$, then $\mathrm{Lim}(Z) \subset \mathrm{Lim}(X)$, and so for any $\beta^* \in \mathrm{Lim}(Z)$, $F^* \in \mathrm{FIN}$, and $k^* \in \omega$, $\forall^\infty\alpha \in Z \cap (\beta^*+1)\,[S_{F^*}(\alpha,\beta^*) \neq k^*]$. Therefore, $Z \in \mathcal{I}$. It will be shown below that $\mathcal{I}$ is closed under pairwise unions and that it is a P-ideal.

$\mathcal{I}$ is the correct analogue in the present context of the original ideal used by Todorčević to show that PID implies the failure of $\square_\kappa$. The ideal he uses in his proof turns out to be a P-ideal because of the following coherence property of $\rho_2$: For any $\alpha \leq \beta$, $\sup_{\xi\leq\alpha}|\rho_2(\xi,\alpha) - \rho_2(\xi,\beta)| < \infty$. While the function $S_F$ does not have such a coherence property for a fixed $F$, we do get a sort of global coherence as $F$ ranges over FIN.

**Lemma 10.** *Fix $X \in \mathcal{I}$ and $\gamma < \kappa^+$. If $\sup(X) \leq \gamma$, then for all $F \in \mathrm{FIN}$ and for all $k \in \omega$, $\forall^\infty\alpha \in X\,[S_F(\alpha,\gamma) \neq k]$.*

*Proof.* Prove by induction on $\gamma < \kappa^+$ that for each $X \in \mathcal{I}$, $F \in \mathrm{FIN}$, and $k \in \omega$, if $\sup(X) \leq \gamma$, then $\forall^\infty\alpha \in X\,[S_F(\alpha,\gamma) \neq k]$. Fix $\gamma < \kappa^+$ and assume this is true for all smaller ordinals. Let $X \in \mathcal{I}$, $F \in \mathrm{FIN}$, $k \in \omega$, and assume for a contradiction that $\sup(X) \leq \gamma$ and that there is $Y \in [X]^\omega$ such that $\forall\alpha \in Y\,[S_F(\alpha,\gamma) = k]$. For each $\alpha \in Y$ choose $\sigma_\alpha \in T_F(\alpha,\gamma)$ such that $|\sigma_\alpha| = k$. Notice that $\gamma \notin Y$, and hence that for any $\alpha \in Y$, $\alpha < \gamma$ and $\sigma_\alpha(1)$ is defined. Define a coloring $c : [Y]^2 \to 3$ as follows: for any $\xi,\alpha \in Y$ with $\xi < \alpha$, $c(\{\xi,\alpha\}) = 0$ if $\sigma_\xi(1) = \sigma_\alpha(1)$, $c(\{\xi,\alpha\}) = 1$ if $\sigma_\xi(1) < \sigma_\alpha(1)$, and $c(\{\xi,\alpha\}) = 2$ if $\sigma_\xi(1) > \sigma_\alpha(1)$. Applying Ramsey's theorem to $c$, there cannot be $Z \in [Y]^\omega$ such that $\forall\alpha_0,\alpha_1 \in Z\,[\alpha_0 < \alpha_1 \implies \sigma_{\alpha_0}(1) > \sigma_{\alpha_1}(1)]$ because an infinite strictly decreasing sequence of ordinals can be built from this. So there are two cases to consider.

Case I: There is $Z \in [Y]^\omega$ and $\beta$ such that $\forall\alpha \in Z\,[\sigma_\alpha(1) = \beta]$. In this case, notice that by Lemma 9, for any $\alpha \in Z$, $S_F(\alpha,\beta) = k-1$. However, since $Z \subset X$, $Z \in \mathcal{I}$. On the other hand, $\sup(Z) \leq \beta < \gamma$, which contradicts the induction hypothesis.

Case II: There is $Z \in [Y]^\omega$ such that $\forall\alpha_0,\alpha_1 \in Z\,[\alpha_0 < \alpha_1 \implies \sigma_{\alpha_0}(1) < \sigma_{\alpha_1}(1)]$. Now there are $Z_0 \in [Z]^\omega$ and $j \in F$ such that $\forall\alpha \in Z_0\left[\sigma_\alpha(1) = \min(c^j_\gamma \setminus \alpha)\right]$. Assume without loss of generality that $Z_0$ has no maximum and put $\beta = \sup(Z_0)$. Now, fix $\alpha_0,\alpha_1 \in Z_0$ with $\alpha_0 < \alpha_1$. If $\alpha_1 \leq \sigma_{\alpha_0}(1)$, then $\sigma_{\alpha_0}(1)$ would equal $\sigma_{\alpha_1}(1)$, contradicting choice of $Z$ and $Z_0$. So for any $\alpha_0,\alpha_1 \in Z_0$, if $\alpha_0 < \alpha_1$, then $\alpha_0 \leq \sigma_{\alpha_0}(1) < \alpha_1$. It follows from this that $\beta \in \mathrm{Lim}(c^j_\gamma)$. Therefore, there is $i \in \omega$ such that $c^i_\beta = c^j_\gamma \cap \beta$. Now, put $G = F \cup \{i\}$. It follows, again from the previous observation, that for any $\alpha \in Z_0$, $\langle\beta\rangle^\frown\langle\sigma_\alpha(1),\dots,\sigma_\alpha(k-1)\rangle \in T_G(\alpha,\beta)$, whence $S_G(\alpha,\beta) \leq k$. Therefore, for some $k_0 \leq k$, $\exists^\infty\alpha \in X \cap (\beta+1)\,[S_G(\alpha,\beta) = k_0]$. But it is clear that $\beta \in \mathrm{Lim}(Z_0) \subset \mathrm{Lim}(X)$, contradicting that hypothesis that $X \in \mathcal{I}$. $\dashv$

**Corollary 11.** *Let $X \in \mathcal{I}$. For any $\gamma < \kappa^+$, $F \in \mathrm{FIN}$, and $k \in \omega$, $\{\alpha \in X \cap (\gamma+1) : S_F(\alpha,\gamma) = k\}$ is finite.*

*Proof.* Suppose for a contradiction that $Y = \{\alpha \in X \cap (\gamma+1) : S_F(\alpha,\gamma) = k\}$ is infinite. Since $Y \subset X$, $Y \in \mathcal{I}$, and $\sup(Y) \leq \gamma$. But this contradicts Lemma 10. $\dashv$

Corollary 11 gives the necessary coherence for proving that $\mathcal{I}$ is a P-ideal.

**Lemma 12.** *$\mathcal{I}$ is a P-ideal.*

*Proof.* It is clear that $\mathcal{I}$ is closed under subsets. Let $X, Y \in \mathcal{I}$. Suppose $\beta \in \operatorname{Lim}(X \cup Y)$, $F \in \mathrm{FIN}$, and $k \in \omega$. From Corollary 11, we know that $\{\alpha \in X \cap (\beta+1) : S_F(\alpha, \beta) = k\}$ and $\{\alpha \in Y \cap (\beta+1) : S_F(\alpha, \beta) = k\}$ are finite, whence $\{\alpha \in (X \cup Y) \cap (\beta+1) : S_F(\alpha, \beta) = k\}$ is finite.

To check that $\mathcal{I}$ is a P-ideal, fix $\{X_n : n \in \omega\} \subset \mathcal{I}$. We may assume without loss of generality that the $X_n$ are pairwise disjoint and infinite. Put $Y = \bigcup_{n\in\omega} X_n$. Let $\{\beta_m : m \in \omega\}$ enumerate $\operatorname{Lim}(Y)$. By Corollary 11, for each $n \in \omega$, $m \in \omega$, $F \in \mathrm{FIN}$, and $k \in \omega$, $H(n, m, F, k) = \{\alpha \in X_n \cap (\beta_m + 1) : S_F(\alpha, \beta_m) = k\}$ is finite. Informally speaking, view the $X_n$ as pairwise disjoint copies of $\omega$ and view $\{\langle H(n, m, F, k) : n \in \omega\rangle : m \in \omega \wedge F \in \mathrm{FIN} \wedge k \in \omega\}$ as a collection of countably many functions from $\omega$ into $[\omega]^{<\omega}$. More formally, for each $n \in \omega$, let $\left\{\alpha_n^l : l \in \omega\right\}$ enumerate $X_n$. For each $m \in \omega$, $F \in \mathrm{FIN}$, and $k \in \omega$, find $f_{m,F,k} \in \omega^\omega$ such that for each $n \in \omega$, $H(n, m, F, k) \subset \left\{\alpha_n^l : l < f_{m,F,k}(n)\right\}$. As this is a countable collection of functions in $\omega^\omega$, find $f \in \omega^\omega$ so that $\forall m \in \omega \forall F \in \mathrm{FIN}\, \forall k \in \omega\, [f_{m,F,k} <^* f]$. Using this $f$, it is possible to choose for each $n \in \omega$ $H(n) \in [X_n]^{<\omega}$ such that

$$\forall m \in \omega \forall F \in \mathrm{FIN}\, \forall k \in \omega \forall^\infty n \in \omega\, [H(n, m, F, k) \subset H(n)]\,.$$

Now, put $Z = \bigcup_{n\in\omega}(X_n \setminus H(n))$. For any $F \in \mathrm{FIN}$ and $m, k \in \omega$, if $\alpha \in Z \cap (\beta_m + 1)$ and $S_F(\alpha, \beta_m) = k$, then there exists $n \in \omega$ such that $\alpha \in H(n, m, F, k)$ and $H(n, m, F, k) \not\subset H(n)$. As $\{n \in \omega : H(n, m, F, k) \not\subset H(n)\}$ is finite, it is clear that $Z \in \mathcal{I}$ and that $\forall n \in \omega\, [X_n \subset^* Z]$. $\dashv$

Now, the first alternative of PID gives an immediate contradiction.

**Lemma 13.** *There is no $X \subset \kappa^+$ with $|X| = \omega_1$ such that $[X]^{\leq\omega} \subset \mathcal{I}$*

*Proof.* Let $X \subset \kappa^+$ with $|X| = \omega_1$. Put $\gamma = \sup(X) < \kappa^+$. Choose any $F \in \mathrm{FIN}$. By the pigeon hole principle, there must be $Y \in [X]^\omega$ and $k \in \omega$ such that for all $\alpha \in Y$, $S_F(\alpha, \gamma) = k$. But by Corollary 11, this means that $Y \notin \mathcal{I}$. $\dashv$

Next, we have to work somewhat harder to get a contradiction from the second alternative. The point is that arbitrarily long walks from any subset of $\kappa^+$ of size $\kappa^+$ to any other such set can always be realized.

**Lemma 14.** *Suppose there exists $X \in [\kappa^+]^{\kappa^+}$ such that $[X]^\omega \cap \mathcal{I} = 0$. Then there exist $B, A \in [\kappa^+]^{\kappa^+}$, $F \in \mathrm{FIN}$ and $k \in \omega$ such that*

$$\forall \alpha \in A \forall \beta \in B\, [\alpha < \beta \implies S_F(\alpha, \beta) \leq k]\,.$$

*Proof.* Let $X \in [\kappa^+]^{\kappa^+}$ be such that $[X]^\omega \cap \mathcal{I} = 0$. For any $\xi < \kappa^+$, let $X(\xi)$ denote the $\xi$th element of $X$ and put $\gamma_\xi = \sup(\{X(\zeta) : \zeta < 1 + \xi\})$.

**Claim 15.** *For every $\xi < \kappa^+$ with $\mathrm{cf}(\xi) = \omega$, there is a $\xi^* < \xi$, $F \in \mathrm{FIN}$ and $k \in \omega$ such that for any $\zeta < \xi$, if $\xi^* \leq \zeta$, then $S_F(X(\zeta), \gamma_\xi) \leq k$.*

*Proof of Claim 15.* Fix $\xi < \kappa^+$ with $\mathrm{cf}(\xi) = \omega$. Fix $\{\xi_m : m \in \omega\} \subset \xi$ increasing and cofinal in $\xi$. Let $\{F_n : n \in \omega\}$ enumerate FIN. Suppose for a contradiction that the claim fails. We will produce a countably infinite subset of $X$ in $\mathcal{I}$. To this end, construct $\{\zeta_n : n \in \omega\} \subset \xi$ as follows. Given $\{\zeta_i : i < n\} \subset \xi$, find $m \geq n$ such that $\forall i < n\, [\zeta_i < \xi_m]$. Put $F = \bigcup_{i<n} F_i$. Use the supposition that the claim fails to choose $\zeta_n < \xi$ with $\xi_m \leq \zeta_n$ such that $S_F(X(\zeta_n), \gamma_\xi) > n$. Notice that for any $i < n$, if $\sigma \in T_{F_i}(X(\zeta_n), \gamma_\xi)$, then $\sigma \in T_F(X(\zeta_n), \gamma_\xi)$. So for any $i < n$, $n < S_F(X(\zeta_n), \gamma_\xi) \leq S_{F_i}(X(\zeta_n), \gamma_\xi)$. But now, it is clear that $\{X(\zeta_n) : n \in \omega\} \in [X]^\omega \cap \mathcal{I}$, contradicting our hypothesis. So the claim is proved. $\dashv$

Continuing with the proof of Lemma 14, since $\{\xi < \kappa^+ : \mathrm{cf}(\xi) = \omega\}$ is a stationary set, there are $\xi^* < \kappa^+$, $F \in \mathrm{FIN}$, $k \in \omega$, and a stationary subset of $\{\xi < \kappa^+ : \mathrm{cf}(\xi) = \omega\}$, say $B^*$, such that for each $\xi \in B^*$, $\xi^* < \xi$ and for any $\zeta < \xi$, if $\xi^* \leq \zeta$, then $S_F(X(\zeta), \gamma_\xi) \leq k$. It is clear that $B = \{\gamma_\xi : \xi \in B^*\}$, $A = \{X(\zeta) : \zeta \geq \xi^*\}$, $F$, and $k$ are as needed. $\dashv$

**Lemma 16.** *Fix $F \in \mathrm{FIN}$. For every club $C \subset \kappa^+$, there is an $\alpha \in C$ such that*

$$(\dagger) \qquad \forall \beta_0 \geq \cdots \geq \beta_l \geq \alpha \left[ C \cap \alpha \not\subset \left( \bigcup_{i \in F} c^i_{\beta_0} \right) \cup \cdots \cup \left( \bigcup_{i \in F} c^i_{\beta_l} \right) \right].$$

*Proof.* For any $\beta < \kappa^+$ and $i \in F$, $\mathrm{otp}(c^i_\beta) \leq \kappa$. So we may simply choose $\alpha \in C$ so that $\mathrm{otp}(C \cap \alpha)$ is sufficiently large. $\dashv$

**Lemma 17.** *Fix a sufficiently large regular cardinal $\theta$. Let $\langle M^l_\delta : \delta < \kappa^+ \wedge l \in \omega \rangle$ be such that*

*(1) $M^l_\delta \prec H(\theta)$, $\kappa \subset M^l_\delta$, $\left| M^l_\delta \right| = \kappa$*

*(2) $\forall l \in \omega \forall \delta < \kappa^+ \left[ \langle M^l_\xi : \xi \leq \delta \rangle \in M^l_{\delta+1} \right]$*

*(3) for each $l \in \omega$ and limit ordinal $\delta$, $M^l_\delta = \bigcup_{\xi < \delta} M^l_\xi$*

*(4) for each $l \in \omega$, $\kappa, \langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle \in M^l_0$ and $\langle M^l_\delta : \delta < \kappa^+ \rangle \in M^{l+1}_0$.*

*Fix $F \in \mathrm{FIN}$, $\kappa^+ > \beta_0 \geq \cdots \geq \beta_m$, and $l \in \omega$. Let $\xi < \kappa^+$ such that $M^l_\xi \cap \kappa^+ < \beta_m$, and $M^l_\xi \cap \kappa^+ \notin c^i_{\beta_j}$ for any $0 \leq j \leq m$ and $i \in F$. Then for every $\zeta_0 \in M^l_\xi \cap \kappa^+$, there are $\zeta \geq \zeta_0$ and $\delta < \kappa^+$ such that*

*(a) $M^0_\delta \cap \kappa^+ \leq M^l_\xi \cap \kappa^+$*

*(b) $\zeta \in M^0_\delta \cap \kappa^+$*

*(c) $\forall \alpha \in M^0_\delta \cap \kappa^+ \forall 0 \leq j \leq m \, [\zeta < \alpha \implies S_F(\alpha, \beta_j) \geq l + 3]$.*

*Proof.* We prove this by induction on $l$. First of all note that since $M^l_\xi \cap \kappa^+ \notin c^i_{\beta_j}$ for any $0 \leq j \leq m$ and $i \in F$, we can find $\zeta^* \in M^l_\xi \cap \kappa^+$ such that for any $0 \leq j \leq m$ and $i \in F$, $\sup(c^i_{\beta_j} \cap M^l_\xi \cap \kappa^+) \leq \zeta^*$. Now, if $l = 0$, then put $\zeta = \max\{\zeta^*, \zeta_0\}$ and $\delta = \xi$. Note that for any $\alpha \in M^l_\xi \cap \kappa^+$ and $0 \leq j \leq m$, if $\zeta^* \leq \zeta < \alpha$, then $\alpha < \beta_j$ and $\alpha \notin c^i_{\beta_j}$ for any $i \in F$, whence $S_F(\alpha, \beta_j) \geq 3$.

Now, suppose $l > 0$. Let $\{\beta^*_j : j \leq m^*\}$ enumerate

$$\left\{ \min\left( c^i_{\beta_j} \setminus \left( M^l_\xi \cap \kappa^+ \right) \right) : i \in F \wedge 0 \leq j \leq m \right\}.$$

Recall that $\langle M^{l-1}_\delta : \delta < \kappa^+ \rangle \in M^l_\xi$. Put

$$C = \{ M^{l-1}_\delta \cap \kappa^+ : \delta < \kappa^+ \} \setminus \max\{\zeta_0, \zeta^*\} + 1.$$

Then $C, F \in M^l_\xi$. So by Lemma 16 and the elementarity of $M^l_\xi$, there is $\alpha^* \in M^l_\xi \cap C$ such that ($\dagger$) of Lemma 16 holds. Applying ($\dagger$) to $\{\beta^*_j : j \leq m^*\}$ find $\xi^* < \kappa^+$ such that $M^{l-1}_{\xi^*} \cap \kappa^+ < M^l_\xi \cap \kappa^+$, $\max\{\zeta_0, \zeta^*\} < M^{l-1}_{\xi^*} \cap \kappa^+$, and $M^{l-1}_{\xi^*} \cap \kappa^+ \notin c^i_{\beta^*_j}$ for any $i \in F$ and $0 \leq j \leq m^*$. Applying the inductive hypothesis, we conclude that there are $\delta < \kappa^+$ and $\zeta \geq \max\{\zeta_0, \zeta^*\} \geq \zeta_0$ such that

($a'$) $M^0_\delta \cap \kappa^+ \subset M^{l-1}_{\xi^*} \cap \kappa^+ \subset M^l_\xi \cap \kappa^+$

$(b')$ $\zeta \in M^0_\delta \cap \kappa^+$
$(c')$ $\forall \alpha \in M^0_\delta \cap \kappa^+ \forall 0 \leq j \leq m^* \left[\zeta < \alpha \implies S_F(\alpha, \beta^*_j) \geq l+2\right]$.

Fix $\alpha \in M^0_\delta \cap \kappa^+$ and $0 \leq j \leq m$, and suppose that $\zeta < \alpha$. Note that for any $i \in F$, $\sup(c^i_{\beta_j} \cap M^l_\xi \cap \kappa^+) \leq \zeta^* \leq \zeta < \alpha < M^{l-1}_{\xi^*} \cap \kappa^+ < M^l_\xi \cap \kappa^+ < \beta_j$. Therefore, if $\sigma \in T_F(\alpha, \beta_j)$ such that $|\sigma| = S_F(\alpha, \beta_j)$, then $\sigma(1) = \beta^*_{j^*}$ for some $0 \leq j^* \leq m^*$. Therefore, by $(c')$ above, $S_F(\alpha, \beta^*_{j^*}) \geq l+2$. But then by Lemma 9, $S_F(\alpha, \beta_j) = S_F(\alpha, \sigma(1)) + 1 \geq l+3$. $\dashv$

**Lemma 18.** *Let* $B, A \in [\kappa^+]^{\kappa^+}$. *Fix* $F \in \mathrm{FIN}$ *and* $k \in \omega$. *Then there are* $\alpha \in A$ *and* $\beta \in B$ *with* $\alpha < \beta$ *such that* $S_F(\alpha, \beta) \geq k$.

*Proof.* This is obvious for $k = 2$. So suppose $k \geq 3$. Fix $\langle M^l_\delta : l \in \omega \wedge \delta < \kappa^+ \rangle$ satisfying $(1)-(4)$ of Lemma 17. Additionally, we also make sure that $B, A \in M^l_0$ for every $l \in \omega$. By Lemma 16, we can find $\beta \in B$ and $\xi < \kappa^+$ such that $M^{k-3}_\xi \cap \kappa^+ < \beta$ and $M^{k-3}_\xi \cap \kappa^+ \notin c^i_\beta$ for any $i \in F$. Now applying Lemma 17, find $\delta < k^+$ and $\zeta < M^0_\delta \cap \kappa^+ \leq M^{k-3}_\xi \cap \kappa^+$ such that $\forall \alpha \in M^0_\delta \cap \kappa^+ [\zeta < \alpha \implies S_F(\alpha, \beta) \geq k]$. Since $A \in M^0_\delta$, there is $\alpha \in A \cap M^0_\delta$ with $\zeta < \alpha$. $\alpha$ and $\beta$ are as needed. $\dashv$

Lemmas 14–18 together give a contradiction from the second alternative of PID. As Lemma 13 rules out the first alternative, this finishes the proof of Theorem 5.

Let $\kappa$ and $\lambda$ be cardinals with $\kappa$ infinite. Let us say that a sequence $\langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\kappa^+) \rangle$ is a *coherent* $(\kappa, < \lambda)$ *C-sequence* if it satisfies conditions (1)-(3) of Definition 1. A coherent $(\kappa, < \lambda^+)$ $C$-sequence is called a coherent $(\kappa, \lambda)$ $C$-sequence. Condition (4) of Definition 1 then imposes a non-triviality requirement on the sequence. A weaker non-triviality condition is the following. We say that a coherent $(\kappa, < \lambda)$ $C$-sequence is *threadable* if there is a club $C \subset \kappa^+$ such that for each $\alpha \in \mathrm{Lim}(C)$, $C \cap \alpha \in \mathcal{C}_\alpha$. The existence of a non-threadable coherent $(\kappa, 1)$ $C$-sequence is equivalent to the principle $\square(\kappa^+)$ studied by Todorčević. In the present context, the conclusion of Lemma 16 may be seen as giving an intermediate non-triviality requirement. We say that a coherent $(\kappa, < \lambda)$ $C$-sequence is *weakly threadable* if there exist $F_\alpha \in [\mathcal{C}_\alpha]^{<\omega} \setminus \{0\}$, for each $\alpha \in \mathrm{Lim}(\kappa^+)$, and a club $C \subset \kappa^+$ such that for every $\alpha \in C$

$$\exists \beta_0 \geq \cdots \geq \beta_l \geq \alpha \left[C \cap \alpha \subset \bigcup \{c \in F_{\beta_j} : 0 \leq j \leq l\}\right].$$

So what we have proved above is that every coherent $(\kappa, \omega)$ $C$-sequence is weakly threadable for every uncountable $\kappa$ under PID. For coherent $(\kappa, 1)$ $C$-sequences, it is easy to see that threadability is equivalent to weak threadability. If $\mathscr{C} = \langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\kappa^+) \rangle$ is a coherent $(\kappa, < \lambda)$ $C$-sequence, then

$$\mathscr{C} \text{ is a } \square_{\kappa, <\lambda} \text{ sequence} \implies \mathscr{C} \text{ is not weakly threadable} \implies \mathscr{C} \text{ is not threadable.}$$

But we do not know if weak threadability is a genuinely intermediate notion of non-triviality.

**Question 19.** Let $\kappa$ be an uncountable cardinal.

$(a)$ Suppose that every coherent $(\kappa, \omega)$ $C$-sequence is weakly threadable. Does it follow that every coherent $(\kappa, \omega)$ $C$-sequence is threadable?
$(b)$ Does PID imply that every coherent $(\kappa, \omega)$ $C$-sequence is threadable?

## 4. Failure of $\square_{\kappa,<\mathfrak{b}}$ under PID

In this section we show that the P-ideal dichotomy implies $\neg\,\square_{\kappa,<\mathfrak{b}}$ for any $\kappa$ with $\mathrm{cf}(\kappa) > \omega_1$. It is known that PID implies that $\mathfrak{b} \leq \aleph_2$ (see [8]). Therefore, it suffices to prove that PID together with the hypothesis $\mathfrak{b} > \omega_1$ implies the failure of $\square_{\kappa,\omega_1}$ whenever $\mathrm{cf}(\kappa) > \omega_1$.

It should be pointed out here that Sharon [4] has proved that Moore's Mapping Reflection Principle (MRP) together with $\mathfrak{b} > \omega_1$ implies the failure of $\square_{\kappa,\omega_1}$, for *any uncountable* $\kappa$. It would be interesting if this turns out to be a difference between PID and MRP.

**Theorem 20.** *Assume* PID *and let* $\kappa$ *be a cardinal satisfying* $\mathrm{cf}(\kappa) > \omega_1$. *Then* $\square_{\kappa,<\mathfrak{b}}$ *fails.*

*Proof.* Let $\kappa$ be a cardinal with $\mathrm{cf}(\kappa) > \omega_1$. Assume PID and $\mathfrak{b} > \omega_1$. We must show that $\square_{\kappa,\omega_1}$ fails. The proof will be very similar to the proof in Section 3. The difference will be in how the second alternative of the dichotomy is handled. In particular, we will need a stronger version of Lemma 16, and this is where that assumption that $\mathrm{cf}(\kappa) > \omega_1$ will come in. Lemmas 17 and 14 will also be appropriately modified.

Let $\langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ be a $\square_{\kappa,\omega_1}$ sequence. As before, for each $\alpha < \kappa^+$ put $\mathcal{C}_{\alpha+1} = \{\{\alpha\}\}$. For each $\alpha < \kappa^+$, let $\{c^i_\alpha : i < \omega_1\}$ enumerate $\mathcal{C}_\alpha$, possibly with repetitions. Now we use FIN to denote $[\omega_1]^{<\omega} \setminus \{0\}$. For $F \in \mathrm{FIN}$, and $\alpha \leq \beta < \kappa^+$, the definitions of $T_F(\alpha,\beta)$ and $S_F(\alpha,\beta)$ are exactly as in Definitions 6 and 7 respectively. $\mathcal{I}$ is defined exactly as in Definition 8. Lemmas 9, 10 and Corollary 11 remain valid, and $\mathcal{I}$ is still an ideal. To check that it is a P-ideal, we use the assumption that $\mathfrak{b} > \omega_1$. Let $\{X_n : n \in \omega\} \subset \mathcal{I}$ be given. Assume without loss of generality that the $X_n$ are pairwise disjoint and infinite, and put $Y = \bigcup_{n\in\omega} X_n$. Let $\{\beta_m : m \in \omega\}$ enumerate $\mathrm{Lim}(Y)$. Once again, by Corollary 11, for each $F \in \mathrm{FIN}$, $m, k, n \in \omega$, $H(F, m, k, n) = \{\alpha \in X_n \cap (\beta_m + 1) : S_F(\alpha, \beta_m) = k\}$ is a finite set. So, as in Lemma 12, we may view $\{\langle H(F, m, k, n) : n \in \omega\rangle : F \in \mathrm{FIN} \wedge m, k \in \omega\}$ as a collection of $\omega_1$ many functions in $\omega^\omega$. Since $\mathfrak{b} > \omega_1$, we may find $H(n) \in [X_n]^{<\omega}$ for each $n \in \omega$ such that

$$\forall F \in \mathrm{FIN}\, \forall m \in \omega \forall k \in \omega \forall^\infty n \in \omega\, [H(F, m, k, n) \subset H(n)]\,.$$

Now, it is clear that $Z = \bigcup_{n\in\omega}(X_n \setminus H(n))$ is as needed.

Once again, the first alternative of PID gives an immediate contradiction because we get an uncountable set and a finite to one function from that set into $\omega$. Suppose now that there is a set $X \in [\kappa^+]^{\kappa^+}$ such that $[X]^\omega \cap \mathcal{I} = 0$. Given $\alpha \leq \beta < \kappa^+$, let $T(\alpha,\beta)$ be the collection of all possible walks from $\beta$ to $\alpha$. More formally, $\sigma \in T(\alpha,\beta)$ iff

(1) $\sigma \in (\kappa^+)^{<\omega}$ and $|\sigma| > 0$
(2) $\sigma(0) = \beta$ and $\sigma(|\sigma| - 1) = \alpha$
(3) $\forall 0 < j < |\sigma|\, \exists i < \omega_1 \left[\sigma(j) = \min\left(c^i_{\sigma(j-1)} \setminus \alpha\right)\right]$.

Put $S(\alpha,\beta) = \min\{|\sigma| : \sigma \in T(\alpha,\beta)\}$. Thus $S(\alpha,\beta)$ is the shortest possible walk from $\beta$ to $\alpha$. It is clear that for any $F \in \mathrm{FIN}$, $T_F(\alpha,\beta) \subset T(\alpha,\beta)$, and therefore that $S(\alpha,\beta) \leq S_F(\alpha,\beta)$. Notice also that if $\alpha < \gamma < \kappa^+$, and $\sigma \in T(\alpha,\gamma)$ with $|\sigma| = S(\alpha,\gamma)$, then $S(\alpha,\sigma(1)) = S(\alpha,\gamma) - 1$.

Now, for each $\xi < \kappa^+$, let $X(\xi)$ denote the $\xi$th element of $X$. For each $\xi < \kappa^+$ with $\mathrm{cf}(\xi) = \omega$, put $\gamma_\xi = \sup(\{X(\zeta) : \zeta < \xi\})$. It must be the case that for each $\xi < \kappa^+$, if $\mathrm{cf}(\xi) = \omega$, then there is a $\xi^* < \xi$ and $k \in \omega$ such $\forall \zeta < \xi\, [\xi^* \leq \zeta \implies S(X(\zeta), \gamma_\xi) \leq k]$. For otherwise it is possible to produce a strictly increasing and cofinal sequence $\{\zeta_n : n \in \omega\} \subset \xi$ such that $\forall n \in \omega\, [S(X(\zeta_n), \gamma_\xi) > n]$, which would then mean that $\{X(\zeta_n) : n \in \omega\} \in [X]^\omega \cap \mathcal{I}$ contradicting the hypothesis on $X$. Again, by the pressing down lemma, there is a stationary set $B^* \subset \{\xi < \kappa^+ : \mathrm{cf}(\xi) = \omega\}$, $\xi^* < \kappa^+$, and $k \in \omega$ such that for each $\xi \in B^*$, $\xi^* < \xi$ and $\forall \zeta < \xi\, [\xi^* \leq \zeta \implies S(X(\zeta), \gamma_\xi) \leq k]$. Putting $A = \{X(\zeta) : \zeta \geq \xi^*\}$ and $B = \{\gamma_\xi : \xi \in B^*\}$, we get $A, B \in [\kappa^+]^{\kappa^+}$ and $k \in \omega$ such that $\forall \alpha \in A \forall \beta \in B\, [\alpha < \beta \implies S(\alpha, \beta) \leq k]$.

A stronger version of Lemma 16 and a modification of Lemma 17 are needed to show that it is impossible to have $A$ and $B$ in $[\kappa^+]^{\kappa^+}$ and $k \in \omega$ with these properties. First suppose that $C \subset \kappa^+$ is a club. The assumption that $\omega_1 < \mathrm{cf}(\kappa)$ implies that $S = \{\alpha \in C : \mathrm{cf}(\alpha) > \omega_1\}$ is stationary. Let $\alpha \in C$ be such that $\mathrm{otp}(S \cap \alpha) \geq \kappa \cdot \kappa$ (ordinal product), and let $X \in [\kappa^+]^{\leq \omega_1}$ be such that $\forall \beta \in X\, [\alpha \leq \beta]$. It is clear that $S \cap \alpha \not\subset \bigcup_{\langle \beta, i \rangle \in X \times \omega_1} c^i_\beta$. Therefore, for any club $C \subset \kappa^+$, there is $\alpha \in C$ such that whenever $X \in [\kappa^+]^{\leq \omega_1}$ is such that $\forall \beta \in X\, [\alpha \leq \beta]$, $\exists \xi \in C \cap \alpha \left[\omega_1 < \mathrm{cf}(\xi) \wedge \xi \notin \bigcup_{\langle \beta, i \rangle \in X \times \omega_1} c^i_\beta\right]$. Let $\langle M^l_\delta : \delta < \kappa^+ \wedge l \in \omega \rangle$ satisfy conditions (1)-(4) of Lemma 17. We prove the following claim by induction on $l$. Fix $\delta < k^+$ so that $\mathrm{cf}\left(M^l_\delta \cap \kappa^+\right) > \omega_1$. Let $X \in [\kappa^+]^{\leq \omega_1}$ be such that $\forall \beta \in X \left[M^l_\delta \cap \kappa^+ < \beta\right]$, and suppose that $M^l_\delta \cap \kappa^+ \notin \bigcup_{\langle \beta, i \rangle \in X \times \omega_1} c^i_\beta$. Fix any $\zeta < M^l_\delta \cap \kappa^+$. Then there are $\xi < \kappa^+$ and $\zeta^*$ such that

(4) $\zeta \leq \zeta^* < M^0_\xi \cap \kappa^+ \leq M^l_\delta \cap \kappa^+$

(5) $\forall \alpha < M^0_\xi \cap \kappa^+ \forall \beta \in X\, [\zeta^* \leq \alpha \implies S(\alpha, \beta) \geq l + 3]$

First note that for each $\beta \in X$ and $i < \omega_1$, $M^l_\delta \cap \kappa^+ < \beta$ and $M^l_\delta \cap \kappa^+ \notin c^i_\beta$. Since $\mathrm{cf}(M^l_\delta \cap \kappa^+) > \omega_1$, there is a $\zeta_0 < M^l_\delta \cap \kappa^+$ such that for any $\langle \beta, i \rangle \in X \times \omega_1$ and any $\alpha \in c^i_\beta \cap M^l_\delta \cap \kappa^+$, $\alpha \leq \zeta_0$. If $l = 0$, put $\xi = \delta$ and $\zeta^* = \max\{\zeta, \zeta_0 + 1\}$. Clearly, (4) is satisfied. For (5), if $\alpha < M^l_\delta \cap \kappa^+$ and $\zeta^* \leq \alpha$, then $\alpha \notin c^i_\beta$ for any $\langle \beta, i \rangle \in X \times \omega_1$, whence $S(\alpha, \beta) \geq 3$.

Now, suppose $l \geq 1$ and that the claim is true for smaller values. Let $C = \{M^{l-1}_\gamma \cap \kappa^+ : \gamma < \kappa^+\} \setminus \max\{\zeta + 1, \zeta_0 + 1\}$. $C \subset \kappa^+$ is a club and so there is $\alpha \in C$ such that whenever $Y \in [\kappa^+]^{\leq \omega_1}$ is such that $\forall \beta^* \in Y\, [\alpha \leq \beta^*]$, there exists $\xi^* \in C \cap \alpha$ such that $\omega_1 < \mathrm{cf}(\xi^*)$ and $\xi^* \notin \bigcup_{\langle \beta^*, i \rangle \in Y \times \omega_1} c^i_{\beta^*}$. Since $C \in M^l_\delta$, we can find such an $\alpha < M^l_\delta \cap \kappa^+$. Put $Y = \{\min(c^i_\beta \setminus \left(M^l_\delta \cap \kappa^+\right)) : \beta \in X \wedge i < \omega_1\}$. Note that $\alpha < \beta^*$ for every $\beta^* \in Y$. Fix $\delta^* < \kappa^+$ such that $\zeta, \zeta_0 < M^{l-1}_{\delta^*} \cap \kappa^+$, $M^{l-1}_{\delta^*} \cap \kappa^+ < \alpha$, $\omega_1 < \mathrm{cf}\left(M^{l-1}_{\delta^*} \cap \kappa^+\right)$, and $M^{l-1}_{\delta^*} \cap \kappa^+ \notin \bigcup_{\langle \beta^*, i \rangle \in Y \times \omega_1} c^i_{\beta^*}$. Applying the inductive hypothesis, there are $\xi < \kappa^+$ and $\zeta^*$ such that

(6) $\max\{\zeta, \zeta_0 + 1\} \leq \zeta^* < M^0_\xi \cap \kappa^+ \leq M^{l-1}_{\delta^*} \cap \kappa^+$

(7) $\forall \alpha < M^0_\xi \cap \kappa^+ \forall \beta^* \in Y\, [\zeta^* \leq \alpha \implies S(\alpha, \beta^*) \geq l + 2]$

It is clear that (4) is satisfied. For (5) fix $\alpha < M^0_\xi \cap \kappa^+$ with $\zeta^* \leq \alpha$ and $\beta \in X$. Fix $\sigma \in T(\alpha, \beta)$ such that $|\sigma| = S(\alpha, \beta)$. $\sigma(1) \in c^i_\beta$ for some $i < \omega_1$. Since $\zeta_0 < \sigma(1)$, $\sigma(1) \geq M^l_\delta \cap \kappa^+$. It follows that $\sigma(1) \in Y$. Therefore, $S(\alpha, \sigma(1)) \geq l + 2$, whence $S(\alpha, \beta) \geq l + 3$.

Now, arguing as in the proof of Lemma 18, find $\alpha \in A$ and $\beta \in B$ with $\alpha < \beta$ such that $S(\alpha, \beta) > k$, contradicting choice of $B, A$, and $k$. $\dashv$

If $\mathrm{cf}(\kappa) \leq \omega_1$, then the techniques developed in this paper breakdown because it is possible to have $\square_{\kappa,\omega_1}$ sequences that satisfy certain additional properties. For example, for any uncountable $\kappa$ with $\mathrm{cf}(\kappa) \leq \omega_1$, it is easy to add by forcing a $\square_{\kappa,\omega_1}$ sequence $\langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim}\,(\kappa^+) \rangle$ that has the property that for each $\alpha < \kappa^+$, if $\mathrm{cf}(\alpha) = \omega$, then

(1) $\bigcup \mathcal{C}_\alpha = \alpha$
(2) $\forall X \in [\mathcal{C}_\alpha]^{\leq \omega} \exists c \in \mathcal{C}_\alpha \forall c^* \in X\, [c^* \subset c]$.

In this case, the ideal $\mathcal{I}$ defined above is trivial, meaning that $\mathcal{I} = [\kappa^+]^{<\omega}$.

An answer to any of the following questions is of interest:

**Question 21.** (1) Does $\mathrm{PID} + \mathfrak{b} > \omega_1$ imply the failure of $\square_{\kappa,\omega_1}$ for every uncountable $\kappa$?
(2) Does $\mathrm{PID} + \mathfrak{p} > \omega_1$ imply the failure of $\square_{\kappa,\omega_1}$ for every uncountable $\kappa$?
(3) Does $\mathrm{PID} + \mathrm{MA}_{\omega_1}$ imply the failure of $\square_{\kappa,\omega_1}$ for every uncountable $\kappa$?

In the most important open case, when $\kappa = \omega_1$, this is equivalent to asking whether PID together with some fragment of $\mathrm{MA}_{\omega_1}$ kills off all special $\omega_2$ Aronszajn trees. Recall that there are no $\omega_2$ Aronszajn trees whatsoever under PFA. We do not know the answer to the following:

**Question 22.** Is "$\mathrm{PID} + \mathrm{MA}_{\omega_1} +$ there exists an $\omega_2$ Aronszajn tree" consistent?

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