

BOREL ORDER DIMENSION

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ABSTRACT. We introduce and study a notion of Borel order dimension for Borel quasi orders. It will be shown that this notion is closely related to the notion of Borel dichromatic number for simple directed graphs. We prove a dichotomy, which generalizes the \mathcal{G}_0 -dichotomy, for the Borel dichromatic number of Borel simple directed graphs. By applying this dichotomy to Borel quasi orders, another dichotomy that characterizes the Borel quasi orders of uncountable Borel dimension is proved. We obtain further structural information about the Borel quasi orders of countable Borel dimension by showing that they are all Borel linearizable. We then investigate the locally countable Borel quasi orders in more detail, paying special attention to the Turing degrees, and produce models of set theory where the continuum is arbitrarily large and all locally countable Borel quasi orders are of Borel dimension less than the continuum. Combining our results here with earlier work shows that the Borel order dimension of the Turing degrees is usually strictly larger than its classical order dimension.

1. INTRODUCTION

Order dimension is a measure of the complexity of a partial order. It was first considered by Dushnik and Miller in [6] and it is now well-studied in combinatorics (e.g. [21], [41], [2], [16]) and in computer science (e.g. [42], [9]). In this paper, we will consider a Borel analog of this notion for Borel quasi orders and their quotients. Our work falls into the general framework of Borel combinatorics, where one studies Borel analogs of classical combinatorial notions (see e.g. [11], [12], [19]). Like other works in this genre, one of our main aims is to prove a dichotomy theorem characterizing those Borel quasi orders whose Borel dimension is uncountable. In such dichotomy theorems, one identifies a collection of canonical basic objects together with appropriate morphisms, and shows that any object with a sufficiently complicated structure admits a morphism from one of the basic objects. A major example is the theorem of Harrington, Kechris, and Louveau [12] characterizing the non-smooth Borel equivalence relations in terms of the existence of a continuous reduction from the basic object E_0 . Another major example is the \mathcal{G}_0 -dichotomy of Kechris, Solecki, and Todorcevic [19] characterizing the analytic graphs of uncountable Borel chromatic number in terms of the existence of a continuous graph homomorphism from the basic object \mathcal{G}_0 .

Our first task here will be to provide a robust definition of the Borel order dimension of a Borel quasi order. This concept requires careful definition because, unlike the classical case, most interesting Borel quasi orders do not admit any Borel linear extensions. We will first show that an appropriate definition of this notion connects it to the Borel analog of the concept of dichromatic number of a digraph. This concept was first defined by Neumann-Lara [32] in the late 1970s

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and has since become important in finite combinatorics, where it is often seen as the appropriate generalization of the concept of chromatic number to digraphs. With the relationship between Borel order dimension and Borel dichromatic number established in Section 2, we are naturally led to seek a dichotomy theorem that characterizes Borel and analytic digraphs of uncountable Borel dichromatic number. Two such dichotomies are established in Section 3. We consider two types of morphisms: continuous digraph homomorphisms as well as those that preserve certain non-edges, namely continuous digraph homomorphisms that map every induced copy of the directed cycle of length n to another such induced copy. The latter are known as *minimal homomorphisms*. The basic objects of our dichotomies are digraphs generalizing the graph \mathcal{G}_0 of [19]. But instead of one basic object, we require a family that is uniformly parametrized by a real, so there are continuum many basic objects. Our dichotomies characterize the Borel digraphs of uncountable Borel dichromatic number in terms of the existence of a continuous minimal homomorphism from one of the basic objects. In the case of analytic digraphs of uncountable Borel dichromatic number, we can characterize them by a continuous homomorphism from one of the basic objects, though we are unable to guarantee minimality. Further, it is proved in Section 3 that the \mathcal{G}_0 -dichotomy of [19] is a special case of these dichotomies.

Having obtained a dichotomy for the Borel dichromatic number, we associate a Borel quasi order with each of its basic objects to obtain another parametrized family of continuum many basic Borel quasi orders for the third dichotomy of Section 3. Each basic object of the third dichotomy is a locally countable Borel quasi order of height 2. We are able to characterize the Borel quasi orders of uncountable Borel order dimension in terms of the existence of a continuous map from one of these basic objects that is order preserving and also preserves the incomparability of certain elements. Additionally, it is shown in Section 3 that every Borel quasi order whose Borel order dimension is countable admits a Borel linear extension.

The classical order dimension of locally countable orders, especially the Turing degrees, was investigated in [15] and [22]. Obtaining Borel analogs of some of the results from those works was one of the original motivations for this paper. Using the results in Section 3 and some ideas from the papers [15, 22, 23], we obtain several results about the Borel order dimension of locally finite and locally countable Borel quasi orders in Section 4. It was proved in [15] that the classical order dimension of the Turing degrees is usually strictly smaller than the continuum. Combining this result with the dichotomy theorems of Section 3 shows that the classical order dimension of the Turing degrees is usually strictly smaller than its Borel order dimension. In [22], the classical order dimension of the Turing degrees was characterized in terms of families of partial orders on the reals that separate countable sets of reals from points. By using a Borel analog of this notion, we obtain upper bounds on the Borel order dimension of locally finite and locally countable Borel quasi orders, in particular, it is shown that all locally finite Borel quasi orders are of countable Borel order dimension. Finally, by combining a forcing notion from [23] with Harrington's generic G_δ forcing, we show that the Borel order dimension of all locally countable Borel quasi orders can consistently be strictly smaller than the continuum. It is worth noting that non-trivial interactions between the definable and non-definable aspects of a concept are also common in the study of Tukey reducibility, which is another notion of complexity for ordered sets. Important examples where the definable aspect of the theory informs the non-definable case and vice versa can be found in [37], [34], [35], [4] among many others.

2. PRELIMINARIES

Different authors use the terms “partial order” and “quasi order” differently. We fix our terminology once and for all with the following definition.

Definition 2.1. We define the following:

- (1) \leq is a *quasi order* on P if \leq is a reflexive and transitive relation on P . $\mathcal{P} = \langle P, \leq \rangle$ is said to be a *quasi order* if \leq is a quasi order on P .
- (2) $<$ is a *partial order* on P if $<$ is an irreflexive and transitive relation on P . $\mathcal{P} = \langle P, < \rangle$ is a *partial order* if $<$ is a partial order on P .
- (3) A quasi order \leq on P is *linear* or *total* if for any $x, y \in P$, $(x \leq y \vee y \leq x)$. $\mathcal{P} = \langle P, \leq \rangle$ is a *total quasi order* or a *linear quasi order* if \leq is a total quasi order on P .
- (4) A partial order $<$ on P is *linear* or *total* if for any $x, y \in P$, $(x < y \vee y < x \vee x = y)$. $\mathcal{P} = \langle P, < \rangle$ is called a *linear order* or a *total order* if $<$ is a linear order on P .
- (5) For a quasi order \leq on P , E_{\leq} is the equivalence relation on P defined by

$$p E_{\leq} q \iff (p \leq q \wedge q \leq p).$$

- (6) For a quasi order \leq , $x < y$ means $(x \leq y \wedge y \not\leq x)$. It is easily checked that $<$ is a partial order. For a partial order $<$, $x \leq y$ means $(x < y \vee x = y)$. Then \leq is a quasi order with E_{\leq} being equality.
- (7) For a quasi order \leq on P , the relation $<$ induces a partial order on P/E_{\leq} , which will also be denoted $<$. Thus for any $x, y \in P$,

$$[x]_{E_{\leq}} < [y]_{E_{\leq}} \iff x < y.$$

When there is no risk of ambiguity, we will omit the subscript from $[x]_{E_{\leq}}$.

The main focus of this paper is on quasi orders and partial orders that are nicely definable on some Polish space. The final section of the paper will study locally finite and locally countable examples of such orders, like the Turing degrees.

Definition 2.2. A quasi order $\mathcal{P} = \langle P, \leq \rangle$ is called a *Borel (analytic) quasi order* if P is a Polish space and \leq is a Borel (analytic) subset of $P \times P$.

Definition 2.3. A quasi order $\mathcal{P} = \langle P, \leq \rangle$ is said to be *locally countable (locally finite)* if for every $x \in P$, $\{y \in P : y \leq x\}$ is countable (finite).

The following notion was first considered by Dushnik and Miller in 1941. Roughly speaking, it asks how many linear orders are necessary to resolve each incomparability in a partial order in both directions.

Definition 2.4 (Dushnik–Miller [6], 1941). Let $\mathcal{P} = \langle P, < \rangle$ be a partial order. The *order dimension* (or simply *dimension*) of \mathcal{P} , denoted $\text{odim}(\mathcal{P})$, is the smallest cardinality of a collection of linear orders on P whose intersection is $<$.

The use of the word “dimension” is justified by the following theorem proved by Dushnik and Miller in the same paper where they introduced their concept.

Theorem 2.5 (Dushnik–Miller [6], 1941). *For any partial order \mathcal{P} , $\text{odim}(\mathcal{P})$ is the minimal cardinal κ such that \mathcal{P} embeds into a product of κ many linear orders with the coordinatewise ordering on the product.*

Definition 2.6. Suppose $<_0$ and $<$ are both partial orders on P . We say $<$ *extends* $<_0$ if $<_0 \subseteq <$. In other words, $<$ extends $<_0$ if and only if for any $x, y \in P$, $x <_0 y \implies x < y$.

The following equivalent characterization of the order dimension of a partial order can be easily proved by appealing to the fact that Zorn's lemma allows every partial order to be extended to a linear order.

Lemma 2.7. *For any partial order \mathcal{P} , $\text{odim}(\mathcal{P})$ is the minimal cardinal κ such that there exists a sequence $\langle <_i : i \in \kappa \rangle$ of partial orders on P extending $<$ such that*

$$(1) \quad \forall x, y \in P \exists i \in \kappa [y <_i x \vee x < y \vee x = y].$$

As mentioned in the introduction, there has been much interest in the order dimension of finite partial orders coming from finite combinatorics and computer science. There has also been growing interest in the order dimension of infinite partial orders. The order dimension of $\langle [\kappa]^{<\aleph_0}, \subseteq \rangle$ was studied in [20]. More recently, the order dimension of locally countable orders, especially of the Turing degrees, was investigated in [15, 22]. Although Dushnik and Miller defined their notion only for partial orders, it is natural to extend their definition to quasi orders as follows.

Definition 2.8. Let $\mathcal{P} = \langle P, \leq \rangle$ be a quasi order. We define the *order dimension* (or simply *dimension*) of \mathcal{P} , denoted $\text{odim}(\mathcal{P})$, to be $\text{odim}(\langle P/E_{\leq}, < \rangle)$.

Definition 2.9. Suppose \leq_0 and \leq are both quasi orders on P . \leq is said to *extend* \leq_0 if

- (1) $x \leq_0 y \implies x \leq y$ and
- (2) $x E_{\leq_0} y \iff x E_{\leq} y$,

for all $x, y \in P$.

It is easily verified that \leq extends \leq_0 iff

- (a) $P/E_{\leq_0} = P/E_{\leq}$ and
- (b) $[x]_{\leq_0} <_0 [y]_{\leq_0} \implies [x]_{\leq} < [y]_{\leq}$, for all $x, y \in P$.

If \leq is a linear quasi order which extends \leq_0 , then we say \leq *linearizes* \leq_0 .

When $\mathcal{P} = \langle P, \leq_0 \rangle$ is a Borel quasi order, the quotient P/E_{\leq_0} is almost never a nicely definable object. Furthermore, by the work of Harrington, Marker, and Shelah [11] and of Kanovei [17], there is no Borel \leq that linearizes \leq_0 in most cases. Hence it will be useful to have a characterization of $\text{odim}(\mathcal{P})$ only in terms of quasi orders on P that extend \leq_0 in the sense of Definition 2.9. The next lemma provides such a characterization. Although its proof is simple, we include it for completeness.

Lemma 2.10. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a quasi order. Then $\text{odim}(\mathcal{P})$ is the minimal cardinal κ such that there exists a sequence $\langle \leq_i : i \in \kappa \rangle$ of quasi orders on P extending \leq such that*

$$(2) \quad \forall x, y \in P \exists i \in \kappa [y \leq_i x \vee x \leq y].$$

Proof. Let $\lambda = \text{odim}(\mathcal{P})$. By Lemma 2.7, fix a sequence $\langle <_i : i \in \lambda \rangle$ of partial orders on P/E_{\leq} extending $<$ and satisfying (1) of Lemma 2.7 for P/E_{\leq} and $<$. Define \leq_i on P by

$$x \leq_i y \iff (x \leq y \vee [x]_{E_{\leq}} <_i [y]_{E_{\leq}}),$$

for all $x, y \in P$. Then \leq_i is a quasi order on P which extends \leq and the partial order induced by \leq_i on $P/E_{\leq} = P/E_{\leq_i}$ is $<_i$. Now given any $x, y \in P$, there exists $i \in \lambda$ such that $([y]_{E_{\leq}} <_i [x]_{E_{\leq}} \vee [x]_{E_{\leq}} < [y]_{E_{\leq}} \vee [x]_{E_{\leq}} = [y]_{E_{\leq}})$. By the definitions, this is easily seen to imply $(y \leq_i x \vee x \leq y)$. Thus $\langle \leq_i : i \in \lambda \rangle$ satisfies (2) of this lemma.

On the other hand suppose that $\langle \leq_i : i \in \kappa \rangle$ is a sequence of quasi orders on P extending \leq and satisfying (2) of this lemma. Let $<_i$ be the partial order induced by \leq_i on $P/E_{\leq} = P/E_{\leq_i}$. Then $<_i$ extends $<$. Further, suppose

$$[x]_{E_{\leq}}, [y]_{E_{\leq}} \in P/E_{\leq}.$$

Let $i \in \kappa$ be so that $(y \leq_i x \vee x \leq y)$. If $(x \leq y \wedge y \leq x)$, then $[x]_{E_{\leq}} = [y]_{E_{\leq}}$. If $(x \leq y \wedge y \not\leq x)$, then $[x]_{E_{\leq}} < [y]_{E_{\leq}}$. If $(x \not\leq y \wedge y \leq_i x)$, then $[y]_{E_{\leq}} <_i [x]_{E_{\leq}}$. Thus (1) of Lemma 2.7 is satisfied by $\langle <_i : i \in \kappa \rangle$ for P/E_{\leq} and $<$. \dashv

We next provide a characterization of the order dimension of a quasi order in terms of the dichromatic number of a certain associated digraph. This apparent detour will ultimately payoff by guiding us to the correct definition of Borel order dimension and allowing us to prove a dichotomy theorem that characterizes the Borel quasi orders of uncountable Borel order dimension.

Definition 2.11. $\mathcal{G} = \langle X, R \rangle$ is called a *simple directed graph* or a *digraph* for short if $R \subseteq X \times X$ and $\langle x, x \rangle \notin R$, for every $x \in X$.

Some authors use the term digraph to allow self edges, but we do not allow this here.

Definition 2.12. A digraph $\langle X, R \rangle$ is called a *graph* if

$$\forall x, y \in X [\langle x, y \rangle \in R \leftrightarrow \langle y, x \rangle \in R].$$

Definition 2.13. A digraph $\mathcal{G} = \langle X, R \rangle$ is said to be Borel (analytic) if X is a Polish space and $R \subseteq X \times X$ is Borel (analytic).

Definition 2.14. For a digraph $\mathcal{G} = \langle X, R \rangle$ and a subset $Y \subseteq X$, an *R-path in Y* is a finite non-empty sequence $\langle y_0, \dots, y_k \rangle$ of elements of Y such that $y_i R y_{i+1}$, for all $i < k$. An *R-path* $\langle y_0, \dots, y_k \rangle$ in Y is an *R-cycle in Y* if $y_k R y_0$. The *length* of an *R-path* is one less than the length of the sequence. In other words, the length of an *R-path* $\langle y_0, \dots, y_k \rangle$ is k . Observe that for any $y \in Y$, $\langle y \rangle$ is an *R-path* of length 0 in Y , but it is not an *R-cycle* because $\langle y, y \rangle \notin R$. We will write *path* and *cycle* instead of *R-path* and *R-cycle* when R is clear from the context.

The next definition associates a digraph with every quasi order. It will be shown that in all non-trivial cases, the order dimension of a quasi order coincides with the dichromatic number of its associated digraph. This characterization will point us to the correct definition of Borel order dimension.

Definition 2.15. Let $\mathcal{P} = \langle P, \leq \rangle$ be a quasi order. Define

$$\mathcal{A}_{\mathcal{P}} = \{ \langle x, y \rangle \in P \times P : y \not\leq x \}.$$

For $\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle \in \mathcal{A}_{\mathcal{P}}$, define

$$\langle x_0, y_0 \rangle \mathcal{R}_{\mathcal{P}} \langle x_1, y_1 \rangle \iff y_0 \leq x_1.$$

Note that $\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle$ is a digraph. Further, if \mathcal{P} is a Borel quasi order, then $\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle$ is a Borel digraph.

Lemma 2.16. Suppose that $\mathcal{P} = \langle P, \leq \rangle$ is a quasi order and $X \subseteq \mathcal{A}_{\mathcal{P}}$. Let Y be the transitive closure of X in $P \times P$. For any $\langle p, q \rangle \in Y$, there is a path $\langle \langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle \rangle$ in X with $p = x_0$ and $q = y_k$.

Proof. Let $\langle p, q \rangle \in Y$. Then there are $l \geq 1$ and a sequence $\langle r_0, \dots, r_l \rangle$ of elements of P such that $p = r_0$, $q = r_l$, and $\langle r_i, r_{i+1} \rangle \in X$, for all $i < l$. The proof is by induction on this l . If $l = 1$, then $\langle \langle p, q \rangle \rangle$ is the required path in X . Assume the statement for some $l \geq 1$ and let $\langle r_0, \dots, r_l, r_{l+1} \rangle$ be a sequence of elements of P such that $p = r_0$, $q = r_{l+1}$, and $\langle r_i, r_{i+1} \rangle \in X$, for all $i < l+1$. By the induction hypothesis, there is a

path $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle$ in X with $x_0 = r_0 = p$ and $y_k = r_l$. Since $\langle r_l, r_{l+1} \rangle \in X$ and $y_k \leq r_l$, $\langle x_k, y_k \rangle \mathcal{R}_{\mathcal{P}} \langle r_l, r_{l+1} \rangle$ holds. Thus $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle, \langle r_l, r_{l+1} \rangle\rangle$ is the required path in X , completing the induction. \dashv

Lemma 2.17. *Suppose that $\mathcal{P} = \langle P, \leq \rangle$ is a quasi order and $X \subseteq \mathcal{A}_{\mathcal{P}}$. Let Z be the transitive closure of $\leq \cup X$ in $P \times P$. For any $\langle p, q \rangle \in Z \setminus \leq$, there is a path $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle$ in X with $p \leq x_0$ and $y_k \leq q$.*

Proof. Let Y be the transitive closure of X in $P \times P$. Then Z is the transitive closure of $\leq \cup Y$ in $P \times P$. Let $\langle p, q \rangle \in Z \setminus \leq$. Then there are $l \geq 1$ and a sequence $\langle r_0, \dots, r_l \rangle$ of elements of P such that $p = r_0$, $q = r_l$, and either $\langle r_i, r_{i+1} \rangle \in \leq$ or $\langle r_i, r_{i+1} \rangle \in Y$, for all $i < l$. The proof is by induction on this l . First suppose $l = 1$. Then since $\langle p, q \rangle \notin \leq$, $\langle p, q \rangle \in Y$. So by Lemma 2.16, there is a path $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle$ in X with $p \leq p = x_0$ and $y_k = q \leq q$. This is as required. Now assume the statement for some $l \geq 1$ and let $\langle r_0, \dots, r_l, r_{l+1} \rangle$ be a sequence of elements of P such that $p = r_0$, $q = r_{l+1}$, and either $\langle r_i, r_{i+1} \rangle \in \leq$ or $\langle r_i, r_{i+1} \rangle \in Y$, for all $i < l + 1$. Then $\langle r_0, r_l \rangle \in Z$. First suppose that $\langle r_0, r_l \rangle \in \leq$. If $\langle r_l, r_{l+1} \rangle \in \leq$, then $p = r_0 \leq r_l \leq r_{l+1} = q$, contradicting $\langle p, q \rangle \notin \leq$. So $\langle r_l, r_{l+1} \rangle \in Y$. By Lemma 2.16, there is a path $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle$ in X with $p = r_0 \leq r_l = x_0$ and $y_k = r_{l+1} = q \leq q$. This is as needed. Now suppose that $\langle r_0, r_l \rangle \notin \leq$. Then the induction hypothesis applies and implies there is a path $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle$ in X with $p = r_0 \leq x_0$ and $y_k \leq r_l$. If $\langle r_l, r_{l+1} \rangle \in \leq$, then $y_k \leq r_l \leq r_{l+1} = q$, and this is as required. So suppose that $\langle r_l, r_{l+1} \rangle \in Y$. Then by Lemma 2.16, there is a path $\langle\langle w_0, z_0 \rangle, \dots, \langle w_m, z_m \rangle\rangle$ in X with $r_l = w_0$ and $r_{l+1} = z_m$. Since $y_k \leq r_l = w_0$, $\langle x_k, y_k \rangle \mathcal{R}_{\mathcal{P}} \langle w_0, z_0 \rangle$ holds. Therefore, $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle \frown \langle\langle w_0, z_0 \rangle, \dots, \langle w_m, z_m \rangle\rangle$ is a path in X . Since $p \leq x_0$ and $z_m = r_{l+1} = q \leq q$, this is as required, completing the induction. \dashv

Lemma 2.18. *Suppose that $\mathcal{P} = \langle P, \leq \rangle$ is a quasi order and $X \subseteq \mathcal{A}_{\mathcal{P}}$. Let \preceq be the transitive closure of $\leq \cup X$ in $P \times P$. If $E_{\preceq} \neq E_{\leq}$, then X contains a cycle.*

Proof. Note that \preceq is a quasi order on P with $\leq \subseteq \preceq$. Hence E_{\preceq} is an equivalence relation with $E_{\leq} \subseteq E_{\preceq}$. Suppose $p, q \in P$ and that $p \preceq q \not\leq p$. Assume first that $p \not\preceq q$ and $q \not\preceq p$. Then by Lemma 2.17 there exist paths $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle$ and $\langle\langle w_0, z_0 \rangle, \dots, \langle w_l, z_l \rangle\rangle$ in X with $p \leq x_0$, $y_k \leq q$, $q \leq w_0$ and $z_l \leq p$. Thus $y_k \leq w_0$ and $z_l \leq x_0$, and so, $\langle x_k, y_k \rangle \mathcal{R}_{\mathcal{P}} \langle w_0, z_0 \rangle$ and $\langle w_l, z_l \rangle \mathcal{R}_{\mathcal{P}} \langle x_0, y_0 \rangle$ hold. It follows that $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle \frown \langle\langle w_0, z_0 \rangle, \dots, \langle w_l, z_l \rangle\rangle$ is a cycle in X . Next, suppose that $p \not\preceq q$, but $q \leq p$. Then there exists a path $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle$ in X with $p \leq x_0$ and $y_k \leq q$. Thus $y_k \leq q \leq p \leq x_0$, and so $\langle x_k, y_k \rangle \mathcal{R}_{\mathcal{P}} \langle x_0, y_0 \rangle$ holds. It follows that $\langle\langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle\rangle$ is a cycle in X . Finally, suppose that $p \leq q$, but $q \not\preceq p$. Then there exists a path $\langle\langle w_0, z_0 \rangle, \dots, \langle w_l, z_l \rangle\rangle$ in X with $q \leq w_0$ and $z_l \leq p$. Thus $z_l \leq p \leq q \leq w_0$, and so $\langle w_l, z_l \rangle \mathcal{R}_{\mathcal{P}} \langle w_0, z_0 \rangle$ holds. It follows that $\langle\langle w_0, z_0 \rangle, \dots, \langle w_l, z_l \rangle\rangle$ is a cycle in X . Therefore if X contains no cycles, then necessarily, $p \leq q \leq p$. This shows $E_{\preceq} \subseteq E_{\leq}$. \dashv

The dichromatic number of a digraph was defined by Neumann-Lara [32] (see also [7, pp. 17–20]). It has become a well-studied notion in finite combinatorics (e.g. [33], [13], [31]) with important connections to various other notions in graph theory. For instance, the well-known conjecture of Erdős and Hajnal about cliques or anti-cliques in graphs with forbidden subgraphs is known, by [1], to be equivalent to a statement about the dichromatic number of certain tournaments. Soukup [39] has studied the dichromatic number of uncountable digraphs. In a recent work, Higgins [14] has proved a Brooks type theorem for the Borel dichromatic number of Borel digraphs.

The next definition collects the various notions of chromatic number and dichromatic number that will be considered in this paper together. We define both a weak notion and a strong notion of Borel dichromatic number. The weak notion only requires a cover of the digraph by Borel cycle-free subsets, while the strong notion asks for a Borel mapping into a Polish space such that the preimage of each point is cycle-free. The difference between the two notions boils down to this: the weak notion sees cardinals between \aleph_0 and 2^{\aleph_0} whereas the strong notion does not. More precisely, the weak notion can consistently take values between \aleph_0 and 2^{\aleph_0} (see Section 4), while the strong notion is equal to 2^{\aleph_0} as soon as it is uncountable. Nevertheless, there is a very straightforward relationship between these two notions. The weak notion is countable if and only if the strong one is, and in this case the two notions coincide. The weak notion is uncountable if and only if the strong one is equal to 2^{\aleph_0} . We will formulate most of our results, including our dichotomy theorems, in terms of the weaker notion, but given the simple relationship between the two, the statements of our results would not change if the stronger notion were substituted everywhere. In the classical case, there is no difference between the strong notion and the weak one.

Definition 2.19. Let $\mathcal{G} = \langle X, R \rangle$ be a digraph. The *dichromatic number* of \mathcal{G} , denoted $\vec{\chi}(\mathcal{G})$, is the minimal cardinal κ such that there exists a collection $\{X_\alpha : \alpha < \kappa\}$ such that $X = \bigcup_{\alpha < \kappa} X_\alpha$ and for each $\alpha < \kappa$, there are no R -cycles in X_α .

For a graph $\mathcal{G} = \langle X, R \rangle$, the *chromatic number* of \mathcal{G} , denoted $\chi(\mathcal{G})$, is the minimal cardinal κ such that there exists a collection $\{X_\alpha : \alpha < \kappa\}$ such that $X = \bigcup_{\alpha < \kappa} X_\alpha$ and $\forall \alpha < \kappa \forall x, y \in X_\alpha [\langle x, y \rangle \notin R]$.

If $\mathcal{G} = \langle X, R \rangle$ is an analytic digraph, then the *weak Borel dichromatic number* of \mathcal{G} , denoted $\vec{\chi}_B^w(\mathcal{G})$, is the minimal cardinal κ such that there exists a collection $\{X_\alpha : \alpha < \kappa\}$ of Borel subsets of X such that $X = \bigcup_{\alpha < \kappa} X_\alpha$ and for each $\alpha < \kappa$, there are no R -cycles in X_α .

If $\mathcal{G} = \langle X, R \rangle$ is an analytic graph, then the *weak Borel chromatic number* of \mathcal{G} , denoted $\chi_B^w(\mathcal{G})$, is the minimal cardinal κ such that there exists a collection $\{X_\alpha : \alpha < \kappa\}$ of Borel subsets of X such that $X = \bigcup_{\alpha < \kappa} X_\alpha$, $\forall \alpha < \beta < \kappa [X_\alpha \cap X_\beta = \emptyset]$, and $\forall \alpha < \kappa \forall x, y \in X_\alpha [\langle x, y \rangle \notin R]$.

For an analytic digraph $\mathcal{G} = \langle X, R \rangle$, the *Borel dichromatic number* of \mathcal{G} , denoted $\vec{\chi}_B(\mathcal{G})$, is the minimal κ such that there exist a Polish space M and a Borel function $\varphi : X \rightarrow M$ such that $|M| = \kappa$ and for every $m \in M$, $\varphi^{-1}(\{m\})$ does not contain any R -cycles.

For an analytic graph $\mathcal{G} = \langle X, R \rangle$, the *Borel chromatic number* of \mathcal{G} , denoted $\chi_B(\mathcal{G})$, is the minimal κ such that there exist a Polish space M and a Borel function $\varphi : X \rightarrow M$ such that $|M| = \kappa$ and $\forall x, y \in X [\varphi(x) = \varphi(y) \implies \langle x, y \rangle \notin R]$.

Several remarks are in order. The definitions of $\vec{\chi}(\mathcal{G})$ and $\chi(\mathcal{G})$ would not change if we added the requirement that $\{X_\alpha : \alpha < \kappa\}$ be a pairwise disjoint family. Clearly, $\vec{\chi}(\mathcal{G}) \leq \vec{\chi}_B^w(\mathcal{G}) \leq \vec{\chi}_B(\mathcal{G})$, for any analytic digraph \mathcal{G} and $\chi(\mathcal{G}) \leq \chi_B^w(\mathcal{G}) \leq \chi_B(\mathcal{G})$, for any analytic graph \mathcal{G} . For any graph \mathcal{G} , $\chi(\mathcal{G}) = \vec{\chi}(\mathcal{G})$ because not containing cycles is the same as not containing edges. Clearly, any family of at most \aleph_1 many Borel sets can be refined to a pairwise disjoint family of Borel sets with the same union. Therefore, for any analytic graph \mathcal{G} , if $\vec{\chi}_B^w(\mathcal{G}) \leq \aleph_1$, then $\vec{\chi}_B(\mathcal{G}) = \chi_B^w(\mathcal{G})$. In particular, for any analytic graph \mathcal{G} , $\vec{\chi}_B^w(\mathcal{G}) > \aleph_0$ if and only if $\chi_B^w(\mathcal{G}) > \aleph_0$. Lastly, $\vec{\chi}_B(\mathcal{G})$ and $\chi_B(\mathcal{G})$ can only take on these values: finite, \aleph_0 , or 2^{\aleph_0} . We will see in Section 4 that $\vec{\chi}_B^w(\mathcal{G})$ and $\chi_B^w(\mathcal{G})$ have a much larger spectrum of possible values. Observe, however, that $\vec{\chi}_B^w(\mathcal{G}) > \aleph_0$ if and only if $\vec{\chi}_B(\mathcal{G}) = 2^{\aleph_0}$, for an analytic digraph \mathcal{G} , and $\chi_B^w(\mathcal{G}) > \aleph_0$ if and only if $\chi_B(\mathcal{G}) = 2^{\aleph_0}$, for an analytic graph \mathcal{G} . Furthermore, if $\vec{\chi}_B^w(\mathcal{G}) \leq \aleph_0$, then $\vec{\chi}_B^w(\mathcal{G}) = \vec{\chi}_B(\mathcal{G})$, and if $\chi_B^w(\mathcal{G}) \leq \aleph_0$, then $\chi_B^w(\mathcal{G}) = \chi_B(\mathcal{G})$.

Lemma 2.20. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a quasi order. Let \preceq be a quasi order on P that extends \leq . Define*

$$X(\preceq, \leq) = \{\langle x, y \rangle \in P \times P : y \not\preceq x \wedge x \preceq y\}.$$

Then $X(\preceq, \leq)$ is a subset of $\mathcal{A}_{\mathcal{P}}$ that does not contain any cycles.

Proof. It is clear from the definitions that $X(\preceq, \leq) \subseteq \mathcal{A}_{\mathcal{P}}$. Suppose for a contradiction that $\langle \langle x_0, y_0 \rangle, \dots, \langle x_k, y_k \rangle \rangle$ is a cycle in $X(\preceq, \leq)$. Then $x_0 \preceq y_0 \leq \dots \leq x_k \preceq y_k \leq x_0$. It follows that $x_0 E_{\preceq} y_0$. Since $E_{\preceq} = E_{\leq}$, $x_0 E_{\leq} y_0$. However, this is a contradiction because $y_0 \not\preceq x_0$. \dashv

Lemma 2.21. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a quasi order such that $|P/E_{\leq}| > 1$. Then $\text{odim}(\mathcal{P}) = \bar{\chi}(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle)$.*

Proof. Suppose $\langle \leq_i : i \in \kappa \rangle$ is a sequence of quasi orders on P extending \leq such that (2) of Lemma 2.10 is satisfied. For each $i \in \kappa$, define $X_i = X(\leq_i, \leq)$. By Lemma 2.20, $X_i \subseteq \mathcal{A}_{\mathcal{P}}$ and it does not contain any cycles. Suppose $\langle x, y \rangle \in \mathcal{A}_{\mathcal{P}}$. Then there is $i \in \kappa$ so that $x \leq_i y$, whence $\langle x, y \rangle \in X_i$. Therefore $\mathcal{A}_{\mathcal{P}} = \bigcup_{i \in \kappa} X_i$. It follows that $\bar{\chi}(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle) \leq \text{odim}(\mathcal{P})$.

For the other direction, suppose $\mathcal{A}_{\mathcal{P}} = \bigcup_{i \in \kappa} X_i$, where each X_i contains no cycles. The hypothesis that $|P/E_{\leq}| > 1$ implies that $\mathcal{A}_{\mathcal{P}} \neq \emptyset$. Therefore $0 < \kappa$. Define \leq_i to be the transitive closure of $\leq \cup X_i$ in $P \times P$. By Lemma 2.18, each \leq_i is a quasi order on P that extends \leq . Let $x, y \in P$. If $x \leq y$, then $i = 0$ fulfills (2) of Lemma 2.10. If $x \not\leq y$, then $\langle y, x \rangle \in \mathcal{A}_{\mathcal{P}}$, whence $\langle y, x \rangle \in X_i$, for some $i \in \kappa$. Then $y \leq_i x$ and (2) of Lemma 2.10 is fulfilled by i . It follows that $\text{odim}(\mathcal{P}) \leq \bar{\chi}(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle)$. \dashv

Lemma 2.21 suggests that the Borel order dimension of a Borel quasi order \mathcal{P} ought to be $\bar{\chi}_{\mathbb{B}}^w(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle)$, or $\bar{\chi}_{\mathbb{B}}(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle)$ if a strong version is needed. In fact, we will now prove that this is exactly what we get if we take the characterization of $\text{odim}(\mathcal{P})$ given by Lemma 2.10 and add the requirement that the witnessing sequence consist entirely of Borel quasi orders. The proof of this will use Corollary 2.25, which will also play a key role in the proofs of the dichotomies in Section 3. The proof of Corollary 2.25, in turn, will use Theorem 2.23, which can be found in Kechris [18].

Definition 2.22. Let X be a Polish space and $\Phi \subseteq \mathcal{P}(X)$. We say that Φ is Π_1^1 on Σ_1^1 if for any Polish space Y and any $A \subseteq Y \times X$ which is Σ_1^1 , the set $A_{\Phi} = \{y \in Y : A_y \in \Phi\}$ is Π_1^1 . Here $A_y = \{x \in X : \langle y, x \rangle \in A\}$, for every $y \in Y$.

Theorem 2.23 (First Reflection Theorem). *Let X be a Polish space and let $\Phi \subseteq \mathcal{P}(X)$ be Π_1^1 on Σ_1^1 . Then for any Σ_1^1 set $A \in \Phi$, there is a Borel set $B \subseteq X$ such that $A \subseteq B$ and $B \in \Phi$.*

Lemma 2.24. *Let $\mathcal{G} = \langle X, R \rangle$ be an analytic digraph. Let*

$$\Phi = \{A \subseteq X : A \text{ does not contain a cycle}\}.$$

Then Φ is Π_1^1 on Σ_1^1 .

Proof. Let Y be Polish and let $A \subseteq Y \times X$ be Σ_1^1 . Fix Polish spaces M and N and continuous functions $f : M \rightarrow X \times X$ and $g : N \rightarrow Y \times X$ with $f''M = R$ and $g''N = A$. For $k \in \omega$ define

$$C_k = \left\{ \langle y, x_0, \dots, x_k, n_0, \dots, n_k, m_0, \dots, m_k \rangle \in Y \times X^{k+1} \times N^{k+1} \times M^{k+1} : \right. \\ \left. \left(\bigwedge_{i \leq k} g(n_i) = \langle y, x_i \rangle \right) \wedge \left(\bigwedge_{i < k} f(m_i) = \langle x_i, x_{i+1} \rangle \right) \wedge f(m_k) = \langle x_k, x_0 \rangle \right\}.$$

Clearly C_k is a closed subset of $Y \times X^{k+1} \times N^{k+1} \times M^{k+1}$, and so $D_k = \text{Proj}_Y(C_k)$ is a Σ_1^1 subset of Y . Therefore $D = \bigcup_{k \in \omega} D_k$ is a Σ_1^1 subset of Y and $E = Y \setminus D$ is a Π_1^1 subset of Y . It is easily verified that $E = \{y \in Y : A_y \in \Phi\} = A_{\Phi}$. \dashv

Corollary 2.25. *Let $\mathcal{G} = \langle X, R \rangle$ be an analytic digraph. If $A \subseteq X$ is Σ_1^1 and does not contain cycles, then there exists a Borel set $B \subseteq X$ such that $A \subseteq B$ and B does not contain cycles.*

Lemma 2.26. *Suppose $\mathcal{P} = \langle P, \leq \rangle$ is a Borel quasi order. If \leq_0 is a Σ_1^1 quasi order on P that extends \leq , then there exists a Borel quasi order \leq_1 on P that extends \leq_0 .*

Proof. As noted in Definition 2.15, $\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle$ is a Borel digraph. Build sequences $\langle \preceq_n : n \in \omega \rangle$ and $\langle X_n : n \in \omega \rangle$ satisfying the following:

- (1) \preceq_0 is \leq_0 and \preceq_n is a Σ_1^1 quasi order on P extending \leq ;
- (2) \preceq_{n+1} extends \preceq_n ;
- (3) $X_n \subseteq \mathcal{A}_{\mathcal{P}}$ is Borel and does not contain cycles;
- (4) $X(\preceq_n, \leq) \subseteq X_n$ and \preceq_{n+1} is the transitive closure of $\leq \cup X_n$ in $P \times P$.

To construct the sequences, let \preceq_0 be \leq_0 . Given \preceq_n , by Lemma 2.20, $X(\preceq_n, \leq) \subseteq \mathcal{A}_{\mathcal{P}}$ and it does not contain cycles. Since \preceq_n is Σ_1^1 and \leq is Borel, $X(\preceq_n, \leq)$ is Σ_1^1 . By Corollary 2.25, there exists a Borel set $X_n \subseteq \mathcal{A}_{\mathcal{P}}$ such that $X(\preceq_n, \leq) \subseteq X_n$ and X_n does not contain cycles. Let \preceq_{n+1} be the transitive closure of $\leq \cup X_n$ in $P \times P$. By Lemma 2.18, \preceq_{n+1} is a quasi order on P extending \leq . Since \leq and X_n are Borel sets, \preceq_{n+1} is Σ_1^1 . Note that $E_{\preceq_{n+1}} = E_{\leq} = E_{\preceq_n}$. Let $x, y \in P$ and suppose $x \preceq_n y$. If $y \leq x$, then $x E_{\preceq_n} y$, whence $x E_{\preceq_{n+1}} y$ and $x \preceq_{n+1} y$. If $y \not\leq x$, then $\langle x, y \rangle \in X(\preceq_n, \leq) \subseteq X_n$, whence $x \preceq_{n+1} y$. Therefore \preceq_{n+1} extends \preceq_n and (1)–(4) are satisfied, finishing the construction.

Define $\leq_1 = \bigcup_{n \in \omega} \preceq_n$. By (1) and (2), \leq_1 is a quasi order on P that extends \leq_0 . Define $X = \bigcup_{n \in \omega} X_n$ and $R = \leq \cup X$. Then R is Borel. It will be verified that $R = \leq_1$. If $\langle x, y \rangle \in \leq$, then $\langle x, y \rangle \in \leq_0 = \preceq_0 \subseteq \leq_1$. If $\langle x, y \rangle \in X_n$, for some $n \in \omega$, then $\langle x, y \rangle \in \preceq_{n+1} \subseteq \leq_1$. This shows $R \subseteq \leq_1$. For the other direction, consider $\langle x, y \rangle \in \preceq_n$, for some $n \in \omega$. If $y \leq x$, then $y \preceq_n x$, whence $x E_{\preceq_n} y$, and so $x E_{\leq} y$ and $\langle x, y \rangle \in \leq \subseteq R$. If $y \not\leq x$, then $\langle x, y \rangle \in X(\preceq_n, \leq) \subseteq X_n \subseteq X \subseteq R$. Therefore, $R = \leq_1$ and \leq_1 is Borel. \dashv

As our official definition of Borel order dimension, we will take the one derived from Definition 2.8 and Lemma 2.10, and we will show that it agrees with the one we would expect from Lemma 2.21.

Definition 2.27. Let $\mathcal{P} = \langle P, \leq \rangle$ be a Borel quasi order. A partial order \prec on P/E_{\leq} is said to be *Borel* if there is a Borel quasi order \preceq on P such that $E_{\preceq} = E_{\leq}$ and \prec is the partial order on $P/E_{\preceq} = P/E_{\leq}$ induced by \preceq .

Definition 2.28. Let $\mathcal{P} = \langle P, \leq \rangle$ be a Borel quasi order. The *Borel order dimension* (or simply *Borel dimension*) of \mathcal{P} , denoted $\text{odim}_B(\mathcal{P})$, is the minimal cardinal κ such that there exists a sequence $\langle <_i : i \in \kappa \rangle$ of Borel partial orders on P/E_{\leq} extending $<$ such that

$$(3) \quad \forall x, y \in P \exists i \in \kappa [[y] <_i [x] \vee [x] < [y] \vee [x] = [y]].$$

Lemma 2.29. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a Borel quasi order. Then $\text{odim}_B(\mathcal{P})$ is the minimal cardinal κ such that there exists a sequence $\langle \leq_i : i \in \kappa \rangle$ of Borel quasi orders on P extending \leq such that*

$$(4) \quad \forall x, y \in P \exists i \in \kappa [y \leq_i x \vee x \leq y].$$

Proof. Suppose $\langle <_i : i \in \kappa \rangle$ is a sequence of Borel partial orders on P/E_{\leq} extending $<$ and satisfying (3) of Definition 2.28. Then for every $i \in \kappa$, there is a Borel quasi order \leq_i on P so that $E_{\leq_i} = E_{\leq}$ and $<_i$ is the partial order on $P/E_{\leq_i} = P/E_{\leq}$ induced by \leq_i . Each \leq_i extends \leq . Given $x, y \in P$, let $i \in \kappa$

be such that $([y] <_i [x] \vee [x] < [y] \vee [x] = [y])$. If $[y] <_i [x]$, then $y \leq_i x$, while if $([x] < [y] \vee [x] = [y])$, then $x \leq y$. Thus (4) holds.

Conversely, suppose $\langle \leq_i : i \in \kappa \rangle$ is a sequence of Borel quasi orders on P extending \leq and satisfying (4). For each $i \in \kappa$, define $<_i$ to be the partial order on $P/E_\leq = P/E_{\leq_i}$ induced by \leq_i . Each $<_i$ is a Borel partial order on P/E_\leq . Each $<_i$ extends $<$. Given $x, y \in P$, there exists $i \in \kappa$ with $(y \leq_i x \vee x \leq y)$. If $[x] \neq [y]$, then $([y] <_i [x] \vee [x] < [y])$. Therefore, (3) of Definition 2.28 is satisfied. \dashv

Theorem 2.30. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a Borel quasi order such that $|P/E_\leq| > 1$. Then $\text{odim}_B(\mathcal{P}) = \vec{\chi}_B^w(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle)$.*

Proof. Suppose $\langle \leq_i : i \in \kappa \rangle$ is a sequence of Borel quasi orders on P extending \leq such that (4) of Lemma 2.29 is satisfied. For each $i \in \kappa$, define $X_i = X(\leq_i, \leq)$. By the proof of Lemma 2.21, $X_i \subseteq \mathcal{A}_\mathcal{P}$, X_i does not contain any cycles, and $\mathcal{A}_\mathcal{P} = \bigcup_{i \in \kappa} X_i$. Further, since both \leq and \leq_i are Borel, X_i is a Borel set. It follows that $\vec{\chi}_B^w(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle) \leq \text{odim}_B(\mathcal{P})$.

For the other direction, suppose $\mathcal{A}_\mathcal{P} = \bigcup_{i \in \kappa} X_i$, where each X_i is Borel and contains no cycles. The hypothesis that $|P/E_\leq| > 1$ implies that $\mathcal{A}_\mathcal{P} \neq \emptyset$. Therefore $0 < \kappa$. Define \preceq_i to be the transitive closure of $\leq \cup X_i$ in $P \times P$. By Lemma 2.18, each \preceq_i is a quasi order on P that extends \leq . Since \leq and X_i are both Borel, \preceq_i is Σ_1^1 . By Lemma 2.26, there is a Borel quasi order \leq_i on P that extends \preceq_i . Let $x, y \in P$. If $x \leq y$, then $i = 0$ fulfills (4) of Lemma 2.29. If $x \not\leq y$, then $\langle y, x \rangle \in \mathcal{A}_\mathcal{P}$, whence $\langle y, x \rangle \in X_i$, for some $i \in \kappa$. Then $y \preceq_i x$ and $y \leq_i x$, and (4) of Lemma 2.29 is fulfilled by i . It follows that $\text{odim}_B(\mathcal{P}) \leq \vec{\chi}_B^w(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle)$. \dashv

In view of Theorem 2.30, we could think of $\text{odim}_B(\mathcal{P})$ as the weak Borel order dimension of \mathcal{P} . The strong notion would then be defined as $\vec{\chi}_B(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle)$ in the case when $|P/E_\leq| > 1$, and as 0 or 1 in the cases when P/E_\leq is empty or a singleton respectively. Replacing our weaker definition with the stronger notion will not make any difference to the dichotomy theorems in Section 3.

Definition 2.31. A Borel quasi order $\mathcal{P} = \langle P, \leq \rangle$ is said to be *Borel linearizable* if there exists a Borel linear quasi order \preceq on P extending \leq .

Our definition of $\mathcal{A}_\mathcal{P}$ considers all pairs $\langle x, y \rangle$ where $y \not\leq x$. If we considered only the induced sub-digraph of $\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle$ on the collection of pairs $\langle x, y \rangle$ where x and y are incomparable, then we would be off from the Borel dichromatic number of \mathcal{P} by at most 1. We will conclude this section by proving this fact, which will be useful in the proof of one of the dichotomies to be presented in the next section.

Definition 2.32. Let $\mathcal{P} = \langle P, \leq \rangle$ be a Borel quasi order. Define

$$\begin{aligned} \mathcal{B}_\mathcal{P} &= \{ \langle x, y \rangle \in \mathcal{A}_\mathcal{P} : x \not\leq y \}; \\ \mathcal{S}_\mathcal{P} &= \mathcal{R}_\mathcal{P} \cap (\mathcal{B}_\mathcal{P} \times \mathcal{B}_\mathcal{P}). \end{aligned}$$

It is clear that $\langle \mathcal{B}_\mathcal{P}, \mathcal{S}_\mathcal{P} \rangle$ is a Borel digraph.

Lemma 2.33. *For any Borel quasi order $\mathcal{P} = \langle P, \leq \rangle$, $\vec{\chi}_B^w(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle) \leq 1 + \vec{\chi}_B^w(\langle \mathcal{B}_\mathcal{P}, \mathcal{S}_\mathcal{P} \rangle)$.*

Proof. Let $\kappa = \vec{\chi}_B^w(\langle \mathcal{B}_\mathcal{P}, \mathcal{S}_\mathcal{P} \rangle)$ and let $\langle Y_\alpha : \alpha < \kappa \rangle$ witness this. Define $X_0 = \mathcal{A}_\mathcal{P} \setminus \mathcal{B}_\mathcal{P}$. Suppose $\langle \langle p_0, q_0 \rangle, \dots, \langle p_k, q_k \rangle \rangle$ is an $\mathcal{R}_\mathcal{P}$ -cycle in X_0 . Then we have $p_0 \leq q_0 \leq \dots \leq p_k \leq q_k \leq p_0$, whence $q_0 \leq p_0$, contradicting $\langle p_0, q_0 \rangle \in \mathcal{A}_\mathcal{P}$. Thus X_0 is a Borel subset of $\mathcal{A}_\mathcal{P}$ without any $\mathcal{R}_\mathcal{P}$ -cycles. For each $\alpha < \kappa$, define $X_{1+\alpha} = Y_\alpha$. It is easy to check that each $X_{1+\alpha}$ is a Borel subset of $\mathcal{A}_\mathcal{P}$ without any $\mathcal{R}_\mathcal{P}$ -cycles. Since $\mathcal{A}_\mathcal{P} = X_0 \cup \mathcal{B}_\mathcal{P} = \bigcup_{\beta < 1+\kappa} X_\beta$, $\langle X_\beta : \beta < 1+\kappa \rangle$ witnesses that $\vec{\chi}_B^w(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle) \leq 1 + \kappa = 1 + \vec{\chi}_B^w(\langle \mathcal{B}_\mathcal{P}, \mathcal{S}_\mathcal{P} \rangle)$. \dashv

Corollary 2.34. *For a Borel quasi order $\mathcal{P} = \langle P, \leq \rangle$, if $\text{odim}_B(\mathcal{P}) > \aleph_0$, then $\bar{\chi}_B^w(\langle \mathcal{B}_\mathcal{P}, \mathcal{S}_\mathcal{P} \rangle) > \aleph_0$.*

Proof. Since $\text{odim}_B(\mathcal{P}) > \aleph_0$, $|P/E_\leq| > 1$, and $\bar{\chi}_B^w(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle) > \aleph_0$. Since by Lemma 2.33, $\bar{\chi}_B^w(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle) \leq 1 + \bar{\chi}_B^w(\langle \mathcal{B}_\mathcal{P}, \mathcal{S}_\mathcal{P} \rangle)$, $\bar{\chi}_B^w(\langle \mathcal{B}_\mathcal{P}, \mathcal{S}_\mathcal{P} \rangle) > \aleph_0$. \dashv

3. THREE DICHOTOMIES

This section contains the main results of this paper. These results take the form of three dichotomy theorems that characterize the Borel and analytic digraphs of uncountable weak Borel dichromatic number as well as the Borel quasi orders of uncountable Borel order dimension in terms of a basis. The first two dichotomies generalize the celebrated \mathcal{G}_0 -dichotomy of Kechris, Solecki, and Todorcevic [19]. Kechris, Solecki, and Todorcevic identified a particular Borel graph, called \mathcal{G}_0 , and showed that an arbitrary analytic graph \mathcal{G} has uncountable weak Borel chromatic number if and only if there is a continuous graph homomorphism from \mathcal{G}_0 to \mathcal{G} . The \mathcal{G}_0 -dichotomy has found numerous applications in descriptive set theory (see Miller [30]) and has inspired an impressive body of work in Borel combinatorics (e.g. [29], [26], [5], [40]).

The basic object \mathcal{G}_0 of [19] is an acyclic graph defined on 2^ω . The basic objects in the dichotomies about to be presented will be digraphs defined on finitely branching trees, where the nodes of height n branch into $f(n)$ successors, for some $f \in \omega^\omega$ that satisfies $f(n) \geq 2$. Thus instead of one basic object, we get a family of continuum many basic objects parametrized by a real in a uniform way. In the \mathcal{G}_0 dichotomy, the natural morphisms of interest are continuous graph homomorphisms, and it is known that, in general, one cannot ask for an embedding. In the category of digraphs, it is natural to consider homomorphisms that map every induced copy of the directed cycle C_n to an induced copy of C_n (see, for example, [38] and [8]). In particular, such digraph homomorphisms preserve specific non-edges as well. Our first dichotomy will show that if \mathcal{G} is a Borel digraph with $\bar{\chi}_B^w(\mathcal{G}) > \aleph_0$, then one can always ask for a continuous digraph homomorphism from one of our basic objects to \mathcal{G} that preserves all induced copies of C_n .

Definition 3.1. Define $\mathcal{F} = \{f \in \omega^\omega : \forall k \in \omega [f(k) \geq 2]\}$. For $l \in \omega$, define $\mathcal{F}_l = \{s \in \omega^l : \forall k \in l [s(k) \geq 2]\}$. Define $\mathcal{F}_{<\omega} = \bigcup_{l < \omega} \mathcal{F}_l$.

Definition 3.2. Let X be any set. $T \subseteq X^{<\omega}$ is called a *subtree* if T is downwards closed. In other words, $T \subseteq X^{<\omega}$ is a subtree if and only if $\forall t \in T \forall l' \leq \text{dom}(t) [t \upharpoonright l' \in T]$. For a subtree $T \subseteq X^{<\omega}$, the n th level of T , denoted $\text{Lev}_n(T)$, is $\{s \in T : \text{dom}(s) = n\}$, for all $n \in \omega$. For a subtree $T \subseteq X^{<\omega}$ and $s \in T$, define

$$\begin{aligned} T\langle s \rangle &= \{t \in T : s \subseteq t \vee t \subseteq s\}; \\ \text{succ}_T(s) &= \{x : s \frown \langle x \rangle \in T\}. \end{aligned}$$

For a subtree $T \subseteq X^{<\omega}$, $[T]$ denotes the collection of all infinite branches through T . In other words,

$$[T] = \{f \in X^\omega : \forall l \in \omega [f \upharpoonright l \in T]\}.$$

Definition 3.3. For $\sigma \in \mathcal{F}_{<\omega}$, define

$$\begin{aligned} T_{[\sigma]} &= \prod_{k \in \text{dom}(\sigma)} \sigma(k) \\ T_\sigma &= \bigcup_{l \leq \text{dom}(\sigma)} \prod_{k \in l} \sigma(k) \end{aligned}$$

Note $T_\sigma \subseteq \omega^{\leq \text{dom}(\sigma)} \subseteq \omega^{<\omega}$ is a subtree and $T_{[\sigma]} \subseteq T_\sigma$.

Definition 3.4. For $f \in \mathcal{F}$, define

$$T_f = \bigcup_{l \in \omega} \prod_{k \in l} f(k)$$

Note $T_f \subseteq \omega^{<\omega}$ is a subtree and that $[T_f] = \prod_{k \in \omega} f(k)$.

The definition of the graph \mathcal{G}_0 of [19] depends on a parameter D , which is a dense subset of $2^{<\omega}$ that intersects every level exactly once. The choice of D does not affect the properties of \mathcal{G}_0 , but in our case, the choice of a dense set will determine the function f , which specifies the extent of the branching at each level. Hence, we take a general approach and define the following.

Definition 3.5. A function $E : \mathcal{F}_{<\omega} \rightarrow \omega^{<\omega}$ is called a *selector* if for every $\sigma \in \mathcal{F}_{<\omega}$, $E(\sigma) \in T_{[\sigma]}$. A selector E is *dense* if for every $f \in \mathcal{F}$, the set $\{E(f \upharpoonright l) : l \in \omega\}$ is dense in T_f . In other words, a selector E is dense if and only if

$$\forall f \in \mathcal{F} \forall s \in T_f \exists l \in \omega [s \subseteq E(f \upharpoonright l)].$$

It is an easy exercise to see that dense selectors exist. We will outline a construction for completeness. Let $\{s_l : l \in \omega\}$ be an enumeration of $\omega^{<\omega}$ such that $|s_l| \leq l$, for every $l \in \omega$. Define $E : \mathcal{F}_{<\omega} \rightarrow \omega^{<\omega}$ as follows. Given $\sigma \in \mathcal{F}_{<\omega}$, let $l = |\sigma|$. If $s_l \in T_\sigma$, then $E(\sigma) = \tau$, where $\tau \in T_{[\sigma]}$ is such that $\tau \upharpoonright |s_l| = s_l$ and $\tau(i) = 0$, for all $|s_l| \leq i < |\sigma|$. If $s_l \notin T_\sigma$, then $E(\sigma) = \tau$, where $\tau \in T_{[\sigma]}$ is such that $\tau(i) = 0$, for all $i < |\sigma|$. To see that this works, suppose $f \in \mathcal{F}$ and $s \in T_f$. Then $s = s_l$, for some $l \in \omega$. Let $\sigma = f \upharpoonright l$. Since $|s| \leq l$ and $s \in T_f$, $s \in T_\sigma$. So, by definition, $s \subseteq E(\sigma) = E(f \upharpoonright l)$.

We now define a Borel digraph $\mathcal{G}_0(E, f)$ for every selector E and every $f \in \mathcal{F}_{<\omega}$. It will be shown that the family $\{\mathcal{G}_0(E, f) : f \in \mathcal{F} \text{ and } E \text{ is a dense selector}\}$ characterizes the Borel and analytic digraphs of uncountable weak Borel dichromatic number. In fact, we will show more. We will show that if \mathcal{G} is any analytic digraph whose weak Borel dichromatic number is uncountable, then for every E there exist f and a continuous homomorphism Π from $\mathcal{G}_0(E, f)$ to \mathcal{G} , with the additional feature that Π can be made to preserve copies of C_n when \mathcal{G} is Borel. Moreover, if there exists k so that \mathcal{G} does not contain copies of C_n for any $n \geq k$, then f is bounded by the constant function $k - 1$. Thus, our dichotomy directly generalizes the \mathcal{G}_0 dichotomy (see the proof of Corollary 3.28).

Definition 3.6. Let E be a selector and let $\sigma \in \mathcal{F}_{<\omega}$. Define $R_{E, \sigma} \subseteq T_{[\sigma]} \times T_{[\sigma]}$ as follows. Given $\langle s, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, $\langle s, t \rangle \in R_{E, \sigma}$ if and only if there exist $k < \text{dom}(\sigma)$ and $x \in \omega^{<\omega}$ such that $t(k) = s(k) + 1 \pmod{\sigma(k)}$, $s = E(\sigma \upharpoonright k) \hat{\ } \langle s(k) \rangle \hat{\ } x$, and $t = E(\sigma \upharpoonright k) \hat{\ } \langle t(k) \rangle \hat{\ } x$. Define $\mathcal{K}(E, \sigma) = \langle T_{[\sigma]}, R_{E, \sigma} \rangle$. Note that $\mathcal{K}(E, \sigma)$ is a digraph.

Let E be a selector let $f \in \mathcal{F}$. Define $R_{E, f} \subseteq [T_f] \times [T_f]$ as follows. Given $\langle y, z \rangle \in [T_f] \times [T_f]$, $\langle y, z \rangle \in R_{E, f}$ if and only if there exist $l \in \omega$ and $x \in \omega^\omega$ such that $z(l) = y(l) + 1 \pmod{f(l)}$, $y = E(f \upharpoonright l) \hat{\ } \langle y(l) \rangle \hat{\ } x$, and $z = E(f \upharpoonright l) \hat{\ } \langle z(l) \rangle \hat{\ } x$. Define $\mathcal{G}_0(E, f) = \langle [T_f], R_{E, f} \rangle$. Note that $\mathcal{G}_0(E, f)$ is a Borel digraph.

Note that $\mathcal{G}_0(E, f)$ is a tournament when $f(n) > 2$, for every n . Also, if f is constantly 2, then $\mathcal{G}_0(E, f)$ is a copy of the graph \mathcal{G}_0 .

Definition 3.7. Recall the following cardinal characteristic known as the *covering number of the meager ideal*, or *covering of meager* for short.

$$\begin{aligned} \mathcal{M} &= \{M \subseteq 2^\omega : M \text{ is Borel and meager}\}, \\ \text{cov}(\mathcal{M}) &= \min \left\{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{M} \wedge 2^\omega = \bigcup \mathcal{F} \right\}. \end{aligned}$$

Lemma 3.8. *For every dense selector E and every $f \in \mathcal{F}$, $\bar{\chi}_B^w(\mathcal{G}_0(E, f)) \geq \text{cov}(\mathcal{M})$.*

Proof. For any $f \in \mathcal{F}$, $[T_f]$ is a perfect Polish space and hence it cannot be covered by fewer than $\text{cov}(\mathcal{M})$ many Borel meager sets. Therefore it suffices to show that if $X \subseteq [T_f]$ is Borel and non-meager, then X contains a cycle. Indeed, we will argue that if $X \subseteq [T_f]$ is Baire measurable and non-meager, then X contains a cycle. By Baire measurability, there is a non-empty open subset $U \subseteq [T_f]$ such that X is comeager in U . By the hypothesis that E is a dense selector, we may assume that there exists $l \in \omega$ so that, letting $s = E(f|l)$, $U = \{x \in [T_f] : s \subseteq x\}$. Define $\psi : U \rightarrow U$ by

$$\psi(s \smallfrown \langle i \rangle \smallfrown y) = s \smallfrown \langle i+1 \bmod f(l) \rangle \smallfrown y,$$

for each i and y satisfying $i \in f(l)$, $y \in \omega^\omega$, and $s \smallfrown \langle i \rangle \smallfrown y \in [T_f]$. Then ψ and ψ^{-1} are auto-homeomorphisms of U . For $k \in \omega$, define the iterates $\psi^k : U \rightarrow U$ and $\psi^{-k} : U \rightarrow U$ by induction as usual. In other words, define ψ^0 as the identity function on U , and given ψ^k and ψ^{-k} , define ψ^{k+1} as $\psi \circ \psi^k$ and ψ^{-k-1} as $\psi^{-1} \circ \psi^{-k}$. Then for each k , $\psi^{-k}(U \cap X)$ is comeager in U . So $Y = \bigcap_{k < f(l)} \psi^{-k}(U \cap X)$ is comeager, in particular it is non-empty. Choose $x \in Y$. Then $\psi^k(x) \in U \cap X$, for every $k < f(l)$, and by the definitions of ψ and of $R_{E,f}$, $\langle x, \psi(x), \dots, \psi^{f(l)-1}(x) \rangle$ is a cycle in X . \dashv

As remarked after Definition 2.19, this implies that $\bar{\chi}_B(\mathcal{G}_0(E, f)) = 2^{\aleph_0}$, for every dense selector E and $f \in \mathcal{F}$.

Definition 3.9. Let $\langle X, R \rangle$ be a digraph and $Y \subseteq X$. A cycle $\langle x_0, \dots, x_n \rangle$ in Y is said to be *minimal* if

$$\langle x_i, x_j \rangle \in R \iff j = i+1 \bmod n+1,$$

for all $i, j < n+1$. Observe that if $\langle x_0, \dots, x_n \rangle$ is a minimal cycle, then $x_i \neq x_j$, for all $i \neq j$.

Observe that a minimal cycle of length n is just an isomorphic copy of the directed cycle C_n of length n – in other words, the induced digraph on $\{x_0, \dots, x_n\}$ is isomorphic to C_n . We will be interested in homomorphisms of digraphs that preserve copies of C_n .

Definition 3.10. Let $\mathcal{G} = \langle X, R \rangle$ and $\mathcal{H} = \langle Y, E \rangle$ be digraphs. We will say that Π is a *homomorphism from \mathcal{G} to \mathcal{H}* if $\Pi : X \rightarrow Y$ is a function such that $\forall \langle x, y \rangle \in R [\langle \Pi(x), \Pi(y) \rangle \in E]$.

We will say that Π is a *minimal homomorphism from \mathcal{G} to \mathcal{H}* if $\Pi : X \rightarrow Y$ is a function such that the following hold:

- (1) $\forall \langle x, y \rangle \in R [\langle \Pi(x), \Pi(y) \rangle \in E]$;
- (2) for every minimal R -cycle $\langle z_0, \dots, z_m \rangle$ in X and every $i, j < m+1$ such that $\langle z_i, z_j \rangle \notin R$, $\langle \Pi(z_i), \Pi(z_j) \rangle \notin E$.

A minimal homomorphism is merely a homomorphism which maps every copy of C_n to a copy of C_n . Such homomorphisms are of interest in the theory of digraphs, for example, see [38, 8].

The proofs of our dichotomies are based on the same ideas as the proof of the \mathcal{G}_0 -dichotomy in Miller [30] and in Kechris, Solecki, and Todorcevic [19], but our notation and presentation will follow Bernshteyn [3].

Notation 3.11. Fix the following notation from now until the end of the proof of Lemma 3.22. Let $\langle X, R \rangle$ be a fixed Borel digraph. Then both R and $(X \times X) \setminus R$ are Σ_1^1 . Fix Polish spaces M and N as well as continuous functions $p : M \rightarrow X \times X$

and $q : N \rightarrow X \times X$ with $p''M = R$ and $q''N = (X \times X) \setminus R$. Fix also a selector E . These parameters will be suppressed for as long as they remain fixed. In particular, we will write R_σ instead of $R_{E,\sigma}$, for $\sigma \in \mathcal{F}_{<\omega}$.

Definition 3.12. For any $\sigma \in \mathcal{F}_{<\omega}$, S_σ will denote the Polish space

$$X^{T_{[\sigma]}} \times M^{(T_{[\sigma]} \times T_{[\sigma]})} \times N^{(T_{[\sigma]} \times T_{[\sigma]})}.$$

We will say that $\langle \pi, \varphi, \psi \rangle \in S_\sigma$ is a *minimal S -homomorphism* if the following hold:

- (1) $\forall \langle s, t \rangle \in R_\sigma [p(\varphi(\langle s, t \rangle)) = \langle \pi(s), \pi(t) \rangle]$;
- (2) for every minimal cycle $\langle s_0, \dots, s_m \rangle$ in $T_{[\sigma]}$ and for every $i, j < m + 1$ such that $i \neq j$ and $\langle s_i, s_j \rangle \notin R_\sigma$, $q(\psi(\langle s_i, s_j \rangle)) = \langle \pi(s_i), \pi(s_j) \rangle$.

Define $\text{Hom}(\sigma) = \{ \langle \pi, \varphi, \psi \rangle \in S_\sigma : \langle \pi, \varphi, \psi \rangle \text{ is a minimal } S\text{-homomorphism} \}$.

Note that $\text{Hom}(\sigma)$ is a closed subset of the Polish space S_σ .

Definition 3.13. For $\sigma \in \mathcal{F}_{<\omega}$, $H \subseteq \text{Hom}(\sigma)$, and $s \in T_{[\sigma]}$, $H(s)$ denotes the set $\{ \pi(s) : \exists \varphi \exists \psi [\langle \pi, \varphi, \psi \rangle \in H] \} \subseteq X$. For $\langle s, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, $H_M(s, t)$ denotes the set $\{ \varphi(\langle s, t \rangle) : \exists \pi \exists \psi [\langle \pi, \varphi, \psi \rangle \in H] \} \subseteq M$, and $H_N(s, t)$ denotes the set $\{ \psi(\langle s, t \rangle) : \exists \pi \exists \varphi [\langle \pi, \varphi, \psi \rangle \in H] \} \subseteq N$.

Observe that if H is a Borel subset of $\text{Hom}(\sigma)$, then $H(s)$, $H_M(s, t)$, and $H_N(s, t)$ are all Σ_1^1 .

Definition 3.14. For $\sigma \in \mathcal{F}_{<\omega}$, $2 \leq n < \omega$, and $i < n$, define $r_{n,i,\sigma} : S_{\sigma \frown \langle n \rangle} \rightarrow S_\sigma$ as follows. Given $\langle \pi, \varphi, \psi \rangle \in S_{\sigma \frown \langle n \rangle}$, $r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) = \langle \pi', \varphi', \psi' \rangle$, where for each $s \in T_{[\sigma]}$, $\pi'(s) = \pi(s \frown \langle i \rangle)$, and for each $\langle s, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, $\varphi'(\langle s, t \rangle) = \varphi(\langle s \frown \langle i \rangle, t \frown \langle i \rangle \rangle)$, and $\psi'(\langle s, t \rangle) = \psi(\langle s \frown \langle i \rangle, t \frown \langle i \rangle \rangle)$.

It is clear that each $r_{n,i,\sigma}$ is continuous.

Definition 3.15. Let $\sigma \in \mathcal{F}_{<\omega}$, $2 \leq n < \omega$, and $H \subseteq \text{Hom}(\sigma)$. Define

$$H^{\otimes n} = \{ \langle \pi, \varphi, \psi \rangle \in \text{Hom}(\sigma \frown \langle n \rangle) : \forall i < n [r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) \in H] \}.$$

If H is Borel, then so is $H^{\otimes n}$ because $r_{n,i,\sigma}$ is continuous, for each $i < n$.

Definition 3.16. Let $\sigma \in \mathcal{F}_{<\omega}$. $H \subseteq \text{Hom}(\sigma)$ is called *tiny* if there exists $s \in T_{[\sigma]}$ such that $H(s)$ does not contain any cycles. $H \subseteq \text{Hom}(\sigma)$ is called *small* if H can be covered by countably many tiny Borel subsets of $\text{Hom}(\sigma)$. $H \subseteq \text{Hom}(\sigma)$ is called *large* if H is not small.

Note that a union of countably many small subsets of $\text{Hom}(\sigma)$ is small. Note also that if $H = \emptyset$, then H is tiny.

Lemma 3.17. If $\vec{\chi}_B^w(\langle X, R \rangle) > \aleph_0$, then $\text{Hom}(\emptyset)$ is large.

Proof. The hypothesis implies that $R \neq \emptyset$. Fix some $\langle x^*, y^* \rangle \in R$ as well as $\mathfrak{m} \in M$ and $\mathfrak{n} \in N$ with $p(\mathfrak{m}) = \langle x^*, y^* \rangle$ and $q(\mathfrak{n}) = \langle x^*, x^* \rangle$. Note that $T_{[\emptyset]} = \{\emptyset\}$ and $T_{[\emptyset]} \times T_{[\emptyset]} = \{\langle \emptyset, \emptyset \rangle\}$. Thus, for any $x \in X$, if we define $\langle \pi, \varphi, \psi \rangle \in S_\emptyset$ by $\pi(\emptyset) = x$, $\varphi(\langle \emptyset, \emptyset \rangle) = \mathfrak{m}$, and $\psi(\langle \emptyset, \emptyset \rangle) = \mathfrak{n}$, then $\langle \pi, \varphi, \psi \rangle \in \text{Hom}(\emptyset)$. Now assume for a contradiction that there are Borel tiny subsets $\{H_n : n \in \omega\}$ of $\text{Hom}(\emptyset)$ with $\text{Hom}(\emptyset) = \bigcup_{n \in \omega} H_n$. As H_n is tiny, $H_n(\emptyset)$ is a Σ_1^1 subset of X that does not contain any cycles. By Corollary 2.25, find a Borel set $H_n(\emptyset) \subseteq B_n \subseteq X$ such that B_n does not contain any cycles. For any $x \in X$, there are $n \in \omega$ and $\langle \pi, \varphi, \psi \rangle \in H_n$ with $\pi(\emptyset) = x$, whence $x \in H_n(\emptyset) \subseteq B_n$. Therefore $X = \bigcup_{n \in \omega} B_n$, contradicting $\vec{\chi}_B^w(\langle X, R \rangle) > \aleph_0$. \dashv

Lemma 3.18. Let $\sigma \in \mathcal{F}_{<\omega}$ and write $s = E(\sigma) \in T_{[\sigma]}$. Suppose $H \subseteq \text{Hom}(\sigma)$ and $\langle x_0, \dots, x_m \rangle$ is a minimal cycle in $H(s)$. Then there exists $\langle \pi, \varphi, \psi \rangle \in H^{\otimes(m+1)}$ such that $\pi(s \frown \langle i \rangle) = x_i$, for each $i < m + 1$.

Proof. Write $n = m + 1$ and note that since there are no cycles of length 0, $m \geq 1$, and so $n \geq 2$. For each $i < n$, choose $\langle \pi_i, \varphi_i, \psi_i \rangle \in H$ with $\pi_i(s) = x_i$. For each $i, j < n$ such that $i \neq j$, if $j = i + 1 \pmod n$, then choose $m_i \in M$ with $p(m_i) = \langle x_i, x_j \rangle$, while if $j \neq i + 1 \pmod n$, then choose $n_{i,j} \in N$ with $q(n_{i,j}) = \langle x_i, x_j \rangle$. Define $\pi : T_{[\sigma \frown \langle n \rangle]} \rightarrow X$ by setting $\pi(u \frown \langle i \rangle) = \pi_i(u)$, for each $u \in T_{[\sigma]}$ and $i < n$. Define $\varphi : T_{[\sigma \frown \langle n \rangle]} \times T_{[\sigma \frown \langle n \rangle]} \rightarrow M$ and $\psi : T_{[\sigma \frown \langle n \rangle]} \times T_{[\sigma \frown \langle n \rangle]} \rightarrow N$ as follows. Suppose $u, t \in T_{[\sigma]}$ and $i, j < n$ are given. Unless $u = t = s$ and $i \neq j$, set $\varphi(\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle) = \varphi_i(\langle u, t \rangle)$ and $\psi(\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle) = \psi_i(\langle u, t \rangle)$. If $u = t = s$ and $j = i + 1 \pmod n$, then set $\varphi(\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle) = m_i$ and $\psi(\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle) = \psi_i(\langle u, t \rangle)$. If $u = t = s$, $i \neq j$, and $j \neq i + 1 \pmod n$, then set $\psi(\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle) = n_{i,j}$ and $\varphi(\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle) = \varphi_i(\langle u, t \rangle)$. This concludes the definition of $\langle \pi, \varphi, \psi \rangle$. Note that by definition, $r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) = \langle \pi_i, \varphi_i, \psi_i \rangle \in H$, for each $i < n$. Also, for each $i < n$, $\pi(s \frown \langle i \rangle) = \pi_i(s) = x_i$. Hence we only need to verify that $\langle \pi, \varphi, \psi \rangle \in \text{Hom}(\sigma \frown \langle n \rangle)$. Suppose $u, t \in T_{[\sigma]}$, $i, j < n$, and $\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle \in R_{\sigma \frown \langle n \rangle}$. There are 2 cases to consider. The first possibility is that $u = t = s$ and $j = i + 1 \pmod n$. Then $p(\varphi(\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle)) = p(m_i) = \langle x_i, x_j \rangle = \langle \pi(u \frown \langle i \rangle), \pi(t \frown \langle j \rangle) \rangle$, as required. The other possibility is that $\langle u, t \rangle \in R_\sigma$ and $i = j$. Then $p(\varphi(\langle u \frown \langle i \rangle, t \frown \langle j \rangle \rangle)) = p(\varphi_i(\langle u, t \rangle)) = \langle \pi_i(u), \pi_i(t) \rangle = \langle \pi_i(u), \pi_j(t) \rangle = \langle \pi(u \frown \langle i \rangle), \pi(t \frown \langle j \rangle) \rangle$, as needed. Next consider some minimal cycle $\langle u_0, \dots, u_k \rangle$ in $T_{[\sigma \frown \langle n \rangle]}$, and fix $i, j < k + 1$ such that $i \neq j$ and $\langle u_i, u_j \rangle \notin R_{\sigma \frown \langle n \rangle}$. Again there are 2 possibilities to consider. The first possibility is that there exist $i', j' < n$ such that $i' \neq j'$, $j' \neq i' + 1 \pmod n$, and $u_i = s \frown \langle i' \rangle$ and $u_j = s \frown \langle j' \rangle$. Then $q(\psi(\langle u_i, u_j \rangle)) = q(\psi(\langle s \frown \langle i' \rangle, s \frown \langle j' \rangle \rangle)) = q(n_{i',j'}) = \langle x_{i'}, x_{j'} \rangle = \langle \pi(s \frown \langle i' \rangle), \pi(s \frown \langle j' \rangle) \rangle = \langle \pi(u_i), \pi(u_j) \rangle$, as required. The other possibility is that for some minimal cycle $\langle t_0, \dots, t_k \rangle$ in $T_{[\sigma]}$ and for some $i' < n$, $u_l = t_l \frown \langle i' \rangle$, for every $l < k + 1$. In this case, $\langle t_i, t_j \rangle \notin R_\sigma$ and since $\langle \pi_{i'}, \varphi_{i'}, \psi_{i'} \rangle \in \text{Hom}(\sigma)$, $q(\psi(\langle u_i, u_j \rangle)) = q(\psi(\langle t_i \frown \langle i' \rangle, t_j \frown \langle i' \rangle \rangle)) = q(\psi_{i'}(\langle t_i, t_j \rangle)) = \langle \pi_{i'}(t_i), \pi_{i'}(t_j) \rangle = \langle \pi(t_i \frown \langle i' \rangle), \pi(t_j \frown \langle i' \rangle) \rangle = \langle \pi(u_i), \pi(u_j) \rangle$, as required. This verifies that $\langle \pi, \varphi, \psi \rangle \in \text{Hom}(\sigma \frown \langle n \rangle)$ and concludes the proof. \dashv

Lemma 3.19. *Let $\sigma \in \mathcal{F}_{<\omega}$, $2 \leq n < \omega$, and $H \subseteq \text{Hom}(\sigma)$. The following hold:*

- (1) *for each $i < n$, for each $s \in T_{[\sigma]}$, $H^{\otimes n}(s \frown \langle i \rangle) \subseteq H(s)$, and for each $\langle s, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, $H_M^{\otimes n}(s \frown \langle i \rangle, t \frown \langle i \rangle) \subseteq H_M(s, t)$ and $H_N^{\otimes n}(s \frown \langle i \rangle, t \frown \langle i \rangle) \subseteq H_N(s, t)$;*
- (2) *if H is tiny, then for every $i < n$,*

$$H^{\otimes(i,n)} = \{ \langle \pi, \varphi, \psi \rangle \in \text{Hom}(\sigma \frown \langle n \rangle) : r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) \in H \}$$

is tiny;

- (3) *if H is small, then so is $H^{\otimes(i,n)}$, for every $i < n$;*
- (4) *if $G \subseteq H$, $H \setminus G$ is small, and $G^{\otimes n}$ is small, then so is $H^{\otimes n}$.*

Proof. For (1): for $\langle \pi, \varphi, \psi \rangle \in H^{\otimes n}$ and $i < n$, let $\langle \pi_i, \varphi_i, \psi_i \rangle = r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) \in H$. Then for all $s \in T_{[\sigma]}$, $\pi(s \frown \langle i \rangle) = \pi_i(s) \in H(s)$. Similarly, for all $\langle s, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, $\varphi(\langle s \frown \langle i \rangle, t \frown \langle i \rangle \rangle) = \varphi_i(\langle s, t \rangle) \in H_M(s, t)$ and $\psi(\langle s \frown \langle i \rangle, t \frown \langle i \rangle \rangle) = \psi_i(\langle s, t \rangle) \in H_N(s, t)$. (1) follows from this.

For (2): fix $s \in T_{[\sigma]}$ so that $H(s)$ does not contain any cycles. Then $s \frown \langle i \rangle \in T_{[\sigma \frown \langle n \rangle]}$, and if $x \in H^{\otimes(i,n)}(s \frown \langle i \rangle)$, then for some $\langle \pi, \varphi, \psi \rangle \in H^{\otimes(i,n)}$, $x = \pi(s \frown \langle i \rangle)$. Since $\langle \pi', \varphi', \psi' \rangle = r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) \in H$, then $x = \pi(s \frown \langle i \rangle) = \pi'(s) \in H(s)$. Therefore, $H^{\otimes(i,n)}(s \frown \langle i \rangle) \subseteq H(s)$, and so $H^{\otimes(i,n)}(s \frown \langle i \rangle)$ does not contain any cycles. This shows $H^{\otimes(i,n)}$ is tiny.

For (3): let $\{G_l : l \in \omega\}$ be Borel tiny subsets of $\text{Hom}(\sigma)$ so that $H \subseteq \bigcup_{l \in \omega} G_l$. Fix $i < n$. By (2), $G_l^{\otimes(i,n)} \subseteq \text{Hom}(\sigma \frown \langle n \rangle)$ is tiny. Since $r_{n,i,\sigma}$ is continuous and G_l is Borel, $G_l^{\otimes(i,n)}$ is also Borel. If $\langle \pi, \varphi, \psi \rangle \in H^{\otimes(i,n)}$, then $\langle \pi, \varphi, \psi \rangle \in \text{Hom}(\sigma \frown \langle n \rangle)$ and

$r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) \in H$, whence for some $l \in \omega$, $r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) \in G_l$ and $\langle \pi, \varphi, \psi \rangle \in G_l^{\otimes \langle i, n \rangle}$. Therefore, $H^{\otimes \langle i, n \rangle} \subseteq \bigcup_{l \in \omega} G_l^{\otimes \langle i, n \rangle}$, and so $H^{\otimes \langle i, n \rangle}$ is small.

For (4): write $S = H \setminus G$. By (3), $S^{\otimes \langle i, n \rangle}$ is small for every $i < n$. Therefore $G^{\otimes n} \cup (\bigcup_{i < n} S^{\otimes \langle i, n \rangle})$ is a small subset of $\text{Hom}(\sigma \frown \langle n \rangle)$. Suppose $\langle \pi, \varphi, \psi \rangle \in H^{\otimes n}$. Then $\langle \pi, \varphi, \psi \rangle \in \text{Hom}(\sigma \frown \langle n \rangle)$ and for each $i < n$, $r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) \in H$. If for some $i < n$, $r_{n,i,\sigma}(\langle \pi, \varphi, \psi \rangle) \in S$, then $\langle \pi, \varphi, \psi \rangle \in S^{\otimes \langle i, n \rangle}$. Otherwise $\langle \pi, \varphi, \psi \rangle \in G^{\otimes n}$. Therefore, $H^{\otimes n} \subseteq G^{\otimes n} \cup (\bigcup_{i < n} S^{\otimes \langle i, n \rangle})$. So $H^{\otimes n}$, being a subset of a small set, is small. \dashv

Lemma 3.20. *Let $\sigma \in \mathcal{F}_{<\omega}$ and write $s = E(\sigma) \in T_{[\sigma]}$. Suppose $H \subseteq \text{Hom}(\sigma)$ is Borel. Let $X \subseteq \omega \setminus 2$ be so that for every $n \in X$, $H^{\otimes n}$ is small. Then there exists $G \subseteq H$ such that G is Borel, $H \setminus G$ is small, and for every $n \in X$, $G(s)$ does not contain a minimal cycle of length $n - 1$.*

Proof. For each $n \in X$, we have that $H^{\otimes n} \subseteq \bigcup_{l < \omega} K_{n,l}$, where each $K_{n,l}$ is a Borel tiny subset of $\text{Hom}(\sigma \frown \langle n \rangle)$. For each $n \in X$ and $l < \omega$, choose $s_{n,l} \in T_{[\sigma]}$ and $i_{n,l} < n$ so that $K_{n,l}(s_{n,l} \frown \langle i_{n,l} \rangle)$ does not contain any cycles. Recall that since $K_{n,l}$ is Borel, $K_{n,l}(s_{n,l} \frown \langle i_{n,l} \rangle)$ is a Σ_1^1 subset of X . So by Corollary 2.25, there exists a Borel set $K_{n,l}(s_{n,l} \frown \langle i_{n,l} \rangle) \subseteq B_{n,l} \subseteq X$ such that $B_{n,l}$ does not contain any cycles. Define $T_{n,l} = \{ \langle \pi', \varphi', \psi' \rangle \in \text{Hom}(\sigma) : \pi'(s_{n,l}) \in B_{n,l} \}$. Since $B_{n,l}$ is Borel, $T_{n,l}$ is Borel, and since $T_{n,l}(s_{n,l}) \subseteq B_{n,l}$, $T_{n,l}$ is tiny. Therefore, $S = \bigcup \{ T_{n,l} : n \in X \wedge l < \omega \}$ is Borel and small. Define $G = H \setminus S$. Then $G \subseteq H$, G is Borel, and $H \setminus G \subseteq S$, so $H \setminus G$ is small. Suppose for a contradiction that for some $n \in X$, $G(s)$ contains a minimal cycle of length $n - 1$. Using Lemma 3.18, find $\langle \pi, \varphi, \psi \rangle \in G^{\otimes n} \subseteq H^{\otimes n}$. Fix $l < \omega$ with $\langle \pi, \varphi, \psi \rangle \in K_{n,l}$, and note that $\langle \pi', \varphi', \psi' \rangle = r_{n,i_{n,l},\sigma}(\langle \pi, \varphi, \psi \rangle) \in G$. However, $\pi'(s_{n,l}) = \pi(s_{n,l} \frown \langle i_{n,l} \rangle) \in K_{n,l}(s_{n,l} \frown \langle i_{n,l} \rangle) \subseteq B_{n,l}$. This implies $\langle \pi', \varphi', \psi' \rangle \in T_{n,l} \subseteq S$, contradicting $\langle \pi', \varphi', \psi' \rangle \in G$. \dashv

Corollary 3.21. *Let $\sigma \in \mathcal{F}_{<\omega}$ and let $H \subseteq \text{Hom}(\sigma)$ be Borel and large. Then there exists $2 \leq n < \omega$ such that $H^{\otimes n}$ is large. Furthermore, if $s = E(\sigma)$ and $2 \leq n < \omega$ is minimal such that $H^{\otimes n}$ is large, there exists $G \subseteq H$ such that G is Borel, $H \setminus G$ is small, $G^{\otimes n}$ is large, and $G(s)$ does not contain any cycles of length less than $n - 1$.*

Proof. Let $X = \{ 2 \leq n < \omega : H^{\otimes n} \text{ is small} \}$. Apply Lemma 3.20 to find $G \subseteq H$ such that G is Borel, $H \setminus G$ is small, and $G(s)$ does not contain a minimal cycle of length $n - 1$, for any $n \in X$. G cannot be tiny because H is large and $H \setminus G$ is small. So $G(s)$ contains a cycle, and hence, a minimal cycle of some length $1 \leq m < \omega$. Hence $2 \leq n = m + 1 < \omega$ and $n \notin X$, whence $H^{\otimes n}$ is large.

Suppose $2 \leq n < \omega$ is minimal so that $H^{\otimes n}$ is large. If $G^{\otimes n}$ were small, then by (4) of Lemma 3.19, $H^{\otimes n}$ would also be small, contradicting the choice of n . So $G^{\otimes n}$ is large. Finally, suppose $G(s)$ contains a cycle of some length less than $n - 1$. Then it contains a minimal cycle of some length $1 \leq m < n - 1$. Hence $2 \leq l = m + 1 < n$ and $l \notin X$, whence $H^{\otimes l}$ is large. But this is a contradiction as $l < n$. \dashv

Lemma 3.22. *Suppose d , d_1 , and d_2 are compatible complete metrics on X , M , and N respectively. Suppose $\sigma \in \mathcal{F}_{<\omega}$ and $H \subseteq \text{Hom}(\sigma)$ is Borel and large. Then for every $\varepsilon > 0$, there exists $G \subseteq H$ so that G is Borel and large, for every $t \in T_{[\sigma]}$, $\text{diam}_d(\overline{G(t)}) < \varepsilon$, and for every $\langle u, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, $\text{diam}_{d_1}(\overline{G_M(u, t)}) < \varepsilon$ and $\text{diam}_{d_2}(\overline{G_N(u, t)}) < \varepsilon$.*

Proof. This easily follows from the fact the Polish spaces X , M , and N can each be covered by closed sets of diameter $< \varepsilon$, the fact that the union of countably many small sets is small, and the fact that the functions $\langle \pi, \varphi, \psi \rangle \mapsto \pi(t)$, $\langle \pi, \varphi, \psi \rangle \mapsto$

$\varphi(\langle u, t \rangle)$, and $\langle \pi, \varphi, \psi \rangle \mapsto \psi(\langle u, t \rangle)$ are all continuous, for every $t \in T_{[\sigma]}$ and every $\langle u, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$. \dashv

Theorem 3.23. *Let $\langle X, R \rangle$ be any Borel digraph. Exactly one of the following alternatives holds:*

- (1) $\bar{\chi}_B^w(\langle X, R \rangle) \leq \aleph_0$;
- (2) *for every selector E , there exists $f \in \mathcal{F}$ such that both of the following hold:*
 - (a) $\forall n, m \in \omega [E(f \upharpoonright m) \subseteq E(f \upharpoonright n) \implies f(m) \leq f(n)]$;
 - (b) *there is a continuous minimal homomorphism Π from $\mathcal{G}_0(E, f)$ to $\langle X, R \rangle$.*

Proof. Suppose E is some dense selector and $\Pi : [T_f] \rightarrow X$ is a continuous homomorphism. If $X = \bigcup_{l \in \omega} X_l$, where each X_l is Borel and contains no cycles, then $[T_f] = \bigcup_{l \in \omega} \Pi^{-1}(X_l)$, where each $\Pi^{-1}(X_l)$ is Borel and contains no cycles because Π is a homomorphism. Since this would contradict Lemma 3.8 and since dense selectors are easily seen to exist, it follows that (1) and (2) are mutually exclusive.

Now assume that $\bar{\chi}_B^w(\langle X, R \rangle) > \aleph_0$. Fix M, N, p, q as in Notation 3.11. Fix also a selector E . As before, these parameters will be suppressed for the remainder of this proof. Also, fix compatible complete metrics d, d_1 , and d_2 on X, M , and N respectively. By Lemma 3.17, $\text{Hom}(\emptyset)$ is large. Construct by induction $f \in \mathcal{F}$ and $\langle H_n : n \in \omega \rangle$ such that:

- (3) $H_n \subseteq \text{Hom}(\sigma_n)$ is Borel and large, where $\sigma_n = f \upharpoonright n$, and $H_n(s_n)$ does not contain any cycles of length less than $f(n) - 1$, where $s_n = E(\sigma_n)$;
- (4) $H_n^{\otimes f(n)}$ is large and $H_{n+1} \subseteq H_n^{\otimes f(n)}$;
- (5) for each $t \in T_{[\sigma_n]}$, $\text{diam}_d(\overline{H_n(t)}) < 2^{-n}$ and for each $\langle u, t \rangle \in T_{[\sigma_n]} \times T_{[\sigma_n]}$, $\text{diam}_{d_1}(\overline{(H_n)_M(u, t)}) < 2^{-n}$ and $\text{diam}_{d_2}(\overline{(H_n)_N(u, t)}) < 2^{-n}$;
- (6) for any $m < n$, $s \in T_{[\sigma_m]}$ and $x \in \omega^{<\omega}$ such that $s \frown x \in T_{[\sigma_n]}$,

$$H_n(s \frown x) \subseteq H_m(s),$$

and for any $\langle s, t \rangle \in T_{[\sigma_m]} \times T_{[\sigma_m]}$ and $x \in \omega^{<\omega}$ such that $\langle s \frown x, t \frown x \rangle \in T_{[\sigma_n]} \times T_{[\sigma_n]}$,

$$\begin{aligned} (H_n)_M(\langle s \frown x, t \frown x \rangle) &\subseteq (H_m)_M(\langle s, t \rangle) \text{ and} \\ (H_n)_N(\langle s \frown x, t \frown x \rangle) &\subseteq (H_m)_N(\langle s, t \rangle). \end{aligned}$$

It is easily seen that such f and $\langle H_n : n \in \omega \rangle$ can be inductively constructed starting from $\text{Hom}(\emptyset)$ and applying Lemma 3.22 and Corollary 3.21 at every stage. Item (6) is satisfied because of Item (1) of Lemma 3.19.

To see that (2)(a) holds, suppose $s_m \subseteq s_n$. If $m = n$, then there is nothing to show, so assume $m < n$ and let $x \in \omega^{<\omega}$ be such that $s_n = s_m \frown x$. As $H_{n+1} \neq \emptyset$, choose $\langle \pi, \varphi, \psi \rangle \in H_{n+1}$. Then $\langle \pi(s_n \frown \langle 0 \rangle), \dots, \pi(s_n \frown \langle f(n) - 1 \rangle) \rangle$ is a cycle and each $\pi(s_n \frown \langle i \rangle) \in H_{n+1}(s_n \frown \langle i \rangle) \subseteq H_n(s_n) \subseteq H_m(s_m)$. So there is a cycle of length $f(n) - 1$ in $H_m(s_m)$, whence $f(m) - 1 \leq f(n) - 1$ and $f(m) \leq f(n)$, as claimed.

For (2)(b), define $\Pi : [T_f] \rightarrow X$ as follows. For $z \in [T_f]$, by (5), (6), and the completeness of the metric d , there is a unique element in $\bigcap_{n \in \omega} \overline{H_n(z \upharpoonright n)}$. Define $\Pi(z)$ to be this unique element of $\bigcap_{n \in \omega} \overline{H_n(z \upharpoonright n)}$. It is clear that Π is continuous. To see that Π is a homomorphism, suppose $\langle y, z \rangle \in R_f$. Then for some $m \in \omega$, $\langle u, t \rangle \in T_{[\sigma_m]} \times T_{[\sigma_m]}$, and $x \in \omega^\omega$, we have that $y = u \frown x$, $z = t \frown x$, and $\forall n \in \omega [\langle u \frown x \upharpoonright n, t \frown x \upharpoonright n \rangle \in R_{\sigma_{m+n}}]$. By (5), (6), and the completeness of the metric d_1 , there exists $m \in M$ with

$$\{\mathfrak{m}\} = \bigcap_{n \in \omega} \overline{(H_{m+n})_M(u \frown x \upharpoonright n, t \frown x \upharpoonright n)}.$$

For each $n \in \omega$, choose $\mathfrak{m}_n \in (H_{m+n})_M(u \frown x \upharpoonright n, t \frown x \upharpoonright n)$ with $d_1(\mathfrak{m}, \mathfrak{m}_n) < 2^{-m-n}$ and then choose $\langle \pi, \varphi, \psi \rangle \in H_{m+n}$ such that $\varphi(\langle u \frown x \upharpoonright n, t \frown x \upharpoonright n \rangle) = \mathfrak{m}_n$. Observe that $p(\mathfrak{m}_n) = \langle \pi(u \frown x \upharpoonright n), \pi(t \frown x \upharpoonright n) \rangle$. Observe also that $d(\pi(u \frown x \upharpoonright n), \Pi(y)) < 2^{-m-n}$ and $d(\pi(t \frown x \upharpoonright n), \Pi(z)) < 2^{-m-n}$ because $\pi(u \frown x \upharpoonright n) \in H_{m+n}(u \frown x \upharpoonright n)$ and $\pi(t \frown x \upharpoonright n) \in H_{m+n}(t \frown x \upharpoonright n)$. By the continuity of p , $p(\mathfrak{m}) = \langle \Pi(y), \Pi(z) \rangle$. And since $p''M = R$, $\langle \Pi(y), \Pi(z) \rangle \in R$ as needed. The verification that Π is minimal is similar. Let $\langle z_0, \dots, z_m \rangle$ be a minimal cycle in $[T_f]$ and suppose $i, j < m+1$ such that $\langle z_i, z_j \rangle \notin R_f$. We may assume $i \neq j$, for otherwise $\langle \Pi(z_i), \Pi(z_i) \rangle \notin R$ by our definition of a digraph. Then for some $l \in \omega$, $\langle u_0, \dots, u_m \rangle$ in $T_{[\sigma_l]}$, and $x \in \omega^\omega$, we have that $\forall 0 \leq k \leq m$ $[z_k = u_k \frown x]$, and for each $n \in \omega$, $\langle u_0 \frown x \upharpoonright n, \dots, u_m \frown x \upharpoonright n \rangle$ is a minimal cycle in $T_{[\sigma_{l+n}]}$ and $\langle u_i \frown x \upharpoonright n, u_j \frown x \upharpoonright n \rangle \notin R_{\sigma_{l+n}}$. By (5), (6), and the completeness of the metric d_2 , there exists $\mathfrak{n} \in N$ with

$$\{\mathfrak{n}\} = \bigcap_{n \in \omega} \overline{(H_{l+n})_N(u_i \frown x \upharpoonright n, u_j \frown x \upharpoonright n)}.$$

For each $n \in \omega$, choose $\mathfrak{n}_n \in (H_{l+n})_N(u_i \frown x \upharpoonright n, u_j \frown x \upharpoonright n)$ with $d_2(\mathfrak{n}, \mathfrak{n}_n) < 2^{-l-n}$ and choose $\langle \pi, \varphi, \psi \rangle \in H_{l+n}$ such that $\psi(\langle u_i \frown x \upharpoonright n, u_j \frown x \upharpoonright n \rangle) = \mathfrak{n}_n$. By (2) of Definition 3.12, $q(\mathfrak{n}_n) = \langle \pi(u_i \frown x \upharpoonright n), \pi(u_j \frown x \upharpoonright n) \rangle$. Observe that $d(\pi(u_i \frown x \upharpoonright n), \Pi(z_i)) < 2^{-l-n}$ and $d(\pi(u_j \frown x \upharpoonright n), \Pi(z_j)) < 2^{-l-n}$ because $\pi(u_i \frown x \upharpoonright n) \in H_{l+n}(u_i \frown x \upharpoonright n)$ and $\pi(u_j \frown x \upharpoonright n) \in H_{l+n}(u_j \frown x \upharpoonright n)$. By the continuity of q , $q(\mathfrak{n}) = \langle \Pi(z_i), \Pi(z_j) \rangle$. And since $q''N = (X \times X) \setminus R$, $\langle \Pi(z_i), \Pi(z_j) \rangle \notin R$ as needed. This verifies all of (2) and concludes the proof. \dashv

Definition 3.24. A digraph $\langle X, R \rangle$ is said to be k -uniform if every minimal cycle in X has length at most k .

Theorem 3.25. Let $\langle X, R \rangle$ be any analytic digraph. Exactly one of the following alternatives holds:

- (1) $\bar{\chi}_B^w(\langle X, R \rangle) \leq \aleph_0$;
- (2) for every selector E , there exists $f \in \mathcal{F}$ such that both of the following hold:
 - (a) $\forall n, m \in \omega$ $[E(f \upharpoonright m) \subseteq E(f \upharpoonright n) \implies f(m) \leq f(n)]$;
 - (b) there is a continuous homomorphism Π from $\mathcal{G}_0(E, f)$ to $\langle X, R \rangle$.
Furthermore, if $\langle X, R \rangle$ is k -uniform, then $f(n) \leq k+1$, for every $n \in \omega$.

Proof. Just as in the proof of Theorem 3.23, alternatives (1) and (2) are mutually exclusive by an application of Lemma 3.8 and by the existence of dense selectors.

Now assume that $\bar{\chi}_B^w(\langle X, R \rangle) > \aleph_0$. Fix a selector E . Repeat the proof of Theorem 3.23 without any reference to N , q , d_2 , or the minimality of S -homomorphisms. More explicitly, let M be a Polish space and let $p : M \rightarrow X \times X$ be a continuous function with $p''M = R$. For any $\sigma \in \mathcal{F}_{<\omega}$, S_σ denotes the Polish space $X^{T_{[\sigma]}} \times M^{(T_{[\sigma]} \times T_{[\sigma]})}$. A pair $\langle \pi, \varphi \rangle \in S_\sigma$ is an S -homomorphism if

$$\forall \langle s, t \rangle \in R_\sigma \ [p(\varphi(\langle s, t \rangle)) = \langle \pi(s), \pi(t) \rangle].$$

Define $\text{Hom}(\sigma) = \{\langle \pi, \varphi \rangle \in S_\sigma : \langle \pi, \varphi \rangle \text{ is an } S\text{-homomorphism}\}$, and as before, $\text{Hom}(\sigma)$ is a closed subset of S_σ . For $\sigma \in \mathcal{F}_{<\omega}$, $H \subseteq \text{Hom}(\sigma)$, and $s \in T_{[\sigma]}$, define $H(s) = \{\pi(s) : \exists \varphi [\langle \pi, \varphi \rangle \in H]\} \subseteq X$. And for $\langle s, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, define $H_M(s, t) = \{\varphi(\langle s, t \rangle) : \exists \pi [\langle \pi, \varphi \rangle \in H]\} \subseteq M$. If H is a Borel subset of $\text{Hom}(\sigma)$, then $H(s)$ and $H_M(s, t)$ are both Σ_1^1 . For $\sigma \in \mathcal{F}_{<\omega}$, $2 \leq n < \omega$, and $i < n$, define $r_{n,i,\sigma} : S_{\sigma \frown \langle n \rangle} \rightarrow S_\sigma$ so that for every $\langle \pi, \varphi \rangle \in S_{\sigma \frown \langle n \rangle}$, $r_{n,i,\sigma}(\langle \pi, \varphi \rangle) = \langle \pi', \varphi' \rangle$, where for each $s \in T_{[\sigma]}$, $\pi'(s) = \pi(s \frown \langle i \rangle)$, and for each $\langle s, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, $\varphi'(\langle s, t \rangle) = \varphi(\langle s \frown \langle i \rangle, t \frown \langle i \rangle \rangle)$. Each $r_{n,i,\sigma}$ is continuous. For $\sigma \in \mathcal{F}_{<\omega}$, $2 \leq n < \omega$, and $H \subseteq \text{Hom}(\sigma)$, define

$$H^{\otimes n} = \{\langle \pi, \varphi \rangle \in \text{Hom}(\sigma \frown \langle n \rangle) : \forall i < n \ [r_{n,i,\sigma}(\langle \pi, \varphi \rangle) \in H]\}.$$

If H is Borel, then so is $H^{\otimes n}$. Now, the notions of *tiny*, *small*, and *large* can be defined verbatim as in Definition 3.16. Using the same arguments as in the proofs of Lemma 3.17 to Lemma 3.22, the following can be proved:

- (3) if $\bar{\chi}_B^w(\langle X, R \rangle) > \aleph_0$, then $\text{Hom}(\emptyset)$ is large;
- (4) if $\sigma \in \mathcal{F}_{<\omega}$, $s = E(\sigma)$, $H \subseteq \text{Hom}(\sigma)$, and $\langle x_0, \dots, x_m \rangle$ is any cycle in $H(s)$, then there exists $\langle \pi, \varphi \rangle \in H^{\otimes(m+1)}$ such that $\pi(s \smallfrown \langle i \rangle) = x_i$, for each $i < m + 1$;
- (5) if $\sigma \in \mathcal{F}_{<\omega}$, $s = E(\sigma)$, and $H \subseteq \text{Hom}(\sigma)$ is Borel and large, then there exist $2 \leq n < \omega$ and $G \subseteq H$ such that G is Borel, $H \setminus G$ is small, $G^{\otimes n}$ is large, and $G(s)$ does not contain any cycles of length less than $n - 1$;
- (6) if d and d_1 are compatible complete metrics on X and M respectively, $\sigma \in \mathcal{F}_{<\omega}$, and $H \subseteq \text{Hom}(\sigma)$ is Borel and large, then for every $\varepsilon > 0$, there exists $G \subseteq H$ so that G is Borel and large, for every $t \in T_{[\sigma]}$, $\text{diam}_d(\overline{G(t)}) < \varepsilon$, and for every $\langle u, t \rangle \in T_{[\sigma]} \times T_{[\sigma]}$, $\text{diam}_{d_1}(\overline{G_M(u, t)}) < \varepsilon$.

Fixing compatible complete metrics d and d_1 on X and M respectively, it is possible to use (3)–(6) to construct $f \in \mathcal{F}$ and $\langle H_n : n \in \omega \rangle$ satisfying:

- (7) $H_n \subseteq \text{Hom}(\sigma_n)$ is Borel and large, where $\sigma_n = f \restriction n$, and $H_n(s_n)$ does not contain any cycles of length less than $f(n) - 1$, where $s_n = E(\sigma_n)$;
- (8) $H_n^{\otimes f(n)}$ is large and $H_{n+1} \subseteq H_n^{\otimes f(n)}$;
- (9) for each $t \in T_{[\sigma_n]}$, $\text{diam}_d(\overline{H_n(t)}) < 2^{-n}$ and for each $\langle u, t \rangle \in T_{[\sigma_n]} \times T_{[\sigma_n]}$, $\text{diam}_{d_1}(\overline{(H_n)_M(u, t)}) < 2^{-n}$;
- (10) for any $m < n$, $s \in T_{[\sigma_m]}$ and $x \in \omega^{<\omega}$ such that $s \smallfrown x \in T_{[\sigma_n]}$,

$$H_n(s \smallfrown x) \subseteq H_m(s),$$

and for any $\langle s, t \rangle \in T_{[\sigma_m]} \times T_{[\sigma_m]}$ and $x \in \omega^{<\omega}$ such that $\langle s \smallfrown x, t \smallfrown x \rangle \in T_{[\sigma_n]} \times T_{[\sigma_n]}$,

$$(H_n)_M(\langle s \smallfrown x, t \smallfrown x \rangle) \subseteq (H_m)_M(\langle s, t \rangle).$$

Now, the verification of (2)(a) is identical to the corresponding argument in the proof of Theorem 3.23. Also, the map $\Pi : [T_f] \rightarrow X$ can be defined exactly as in the proof of Theorem 3.23 and the verification of (2)(b) is similar to the verification of the first half of (2)(b) of Theorem 3.23. To see the final sentence of (2), assume that $\langle X, R \rangle$ is k -uniform. For each $n \in \omega$, H_{n+1} is a non-empty subset of $H_n^{\otimes f(n)}$, so consider $\langle \pi, \varphi \rangle \in H_{n+1}$. Then $\langle \pi(s_n \smallfrown \langle 0 \rangle), \dots, \pi(s_n \smallfrown \langle f(n) - 1 \rangle) \rangle$ is a cycle of length $f(n) - 1$ in $H_n(s_n)$. Since $H_n(s_n)$ does not contain any cycle of length less than $f(n) - 1$, $\langle \pi(s_n \smallfrown \langle 0 \rangle), \dots, \pi(s_n \smallfrown \langle f(n) - 1 \rangle) \rangle$ must be a minimal cycle, and so by k -uniformity, $f(n) - 1 \leq k$, as claimed. \dashv

As noted after Definition 2.19, Item (1) of Theorems 3.23 and 3.25 is equivalent to the condition that $\bar{\chi}_B(\langle X, R \rangle) \leq \aleph_0$. We do not know whether the homomorphism Π can be chosen to be minimal in the case when $\langle X, R \rangle$ is analytic, but not Borel.

Question 3.26. Suppose $\langle X, R \rangle$ is an analytic digraph with $\bar{\chi}_B^w(\langle X, R \rangle) > \aleph_0$. Is it true that for every selector E , there exist $f \in \mathcal{F}$ and a continuous minimal homomorphism Π from $\mathcal{G}_0(E, f)$ to $\langle X, R \rangle$?

Next, we will show that the \mathcal{G}_0 dichotomy of Kechris, Solecki, and Todorcevic is a consequence of Theorem 3.25.

Definition 3.27. $D \subseteq 2^{<\omega}$ is said to be *dense* if $\forall s \in 2^{<\omega} \exists t \in D [s \subseteq t]$.

For $D \subseteq 2^{<\omega}$, $\mathcal{G}_0(D)$ is the graph $\langle X, R \rangle$, where $X = 2^\omega$ and

$$R = \{ \langle s \smallfrown \langle i \rangle \smallfrown x, s \smallfrown \langle i + 1 \mod 2 \rangle \smallfrown x \rangle : s \in D \wedge i \in 2 \wedge x \in 2^\omega \}.$$

Corollary 3.28 (G_0 -Dichotomy [19]). *Let $\langle X, R \rangle$ be any analytic graph. Exactly one of the following alternatives holds:*

- (1) $\chi_B^w(\langle X, R \rangle) \leq \aleph_0$;
- (2) for every $D \subseteq 2^{<\omega}$ with the property that $\forall n \in \omega [|D \cap 2^n| = 1]$, there is a continuous homomorphism from $\mathcal{G}_0(D)$ to $\langle X, R \rangle$.

Proof. The argument of Lemma 3.8 shows that if $D \subseteq 2^{<\omega}$ is dense and has the property that $\forall n \in \omega [|D \cap 2^n| = 1]$, then $\chi_B^w(\mathcal{G}_0(D)) \geq \text{cov}(\mathcal{M})$. It follows from this and from the existence of a dense $D \subseteq 2^{<\omega}$ such that $\forall n \in \omega [|D \cap 2^n| = 1]$ that the alternatives (1) and (2) are mutually exclusive.

Assume that $\chi_B^w(\langle X, R \rangle) > \aleph_0$. This implies $\bar{\chi}_B^w(\langle X, R \rangle) > \aleph_0$. Fix any $D \subseteq 2^{<\omega}$ so that $\forall n \in \omega [|D \cap 2^n| = 1]$. Define a selector $E : \mathcal{F}_{<\omega} \rightarrow \omega^{<\omega}$ as follows. If $\sigma \in \mathcal{F}_{<\omega}$ is such that $\forall k \in \text{dom}(\sigma) [\sigma(k) = 2]$, then define $E(\sigma)$ to be the unique element of $D \cap 2^{\text{dom}(\sigma)}$. Otherwise, define $E(\sigma)$ to be an arbitrary element of $T_{[\sigma]}$. By Theorem 3.25, there exists $f \in \mathcal{F}$ and a continuous homomorphism from $\mathcal{G}_0(E, f)$ to $\langle X, R \rangle$. Furthermore, since $\langle X, R \rangle$ is 1-uniform, $f(k) \leq 2$ and since $f \in \mathcal{F}$, $f(k) = 2$, for all $k \in \omega$. Therefore $\mathcal{G}_0(E, f) = \mathcal{G}_0(D)$ and the conclusion follows. \dashv

Once again, as mentioned after Definition 2.19, Item (1) of Corollary 3.28 is equivalent to the assertion that $\chi_B(\langle X, R \rangle) \leq \aleph_0$. Next, we associate a Borel quasi order with every Borel digraph as follows.

Definition 3.29. Let $\mathcal{G} = \langle X, R \rangle$ be a digraph. Define $P_{\mathcal{G}} = 2 \times X$ and $\leq_{\mathcal{G}} = \{ \langle \langle i, x \rangle, \langle j, y \rangle \rangle \in P_{\mathcal{G}} \times P_{\mathcal{G}} : (\langle i, x \rangle = \langle j, y \rangle) \vee (i = 0 \wedge j = 1 \wedge \langle x, y \rangle \in R) \}$. Define $\mathcal{P}(\mathcal{G}) = \langle P_{\mathcal{G}}, \leq_{\mathcal{G}} \rangle$. It is clear that $\mathcal{P}(\mathcal{G})$ is a quasi-order and that $E_{\leq_{\mathcal{G}}}$ is equality. Furthermore, if \mathcal{G} is a Borel digraph, then $\mathcal{P}(\mathcal{G})$ is a Borel quasi order.

Lemma 3.30. *Suppose $\mathcal{G} = \langle X, R \rangle$ is a Borel digraph. Then there is a Borel homomorphism from \mathcal{G} to $\langle \mathcal{A}_{\mathcal{P}(\mathcal{G})}, \mathcal{R}_{\mathcal{P}(\mathcal{G})} \rangle$. In particular, if $X \neq \emptyset$, then $\text{odim}_B(\mathcal{P}(\mathcal{G})) \geq \bar{\chi}_B^w(\mathcal{G})$.*

Proof. Define $f : X \rightarrow \mathcal{A}_{\mathcal{P}(\mathcal{G})}$ by $f(x) = \langle \langle 1, x \rangle, \langle 0, x \rangle \rangle$, for every $x \in X$. Note that $f(x) \in \mathcal{A}_{\mathcal{P}(\mathcal{G})}$ because $\langle x, x \rangle \notin R$ and that f is a Borel map with respect to any Polish topology on $\mathcal{A}_{\mathcal{P}(\mathcal{G})}$ which extends the topology on $\mathcal{A}_{\mathcal{P}(\mathcal{G})}$ inherited from $P_{\mathcal{G}} \times P_{\mathcal{G}}$. Further, for any $x, y \in X$, $\langle x, y \rangle \in R$ if and only if $\langle 0, x \rangle \leq_{\mathcal{G}} \langle 1, y \rangle$ if and only if $f(x) \mathcal{R}_{\mathcal{P}(\mathcal{G})} f(y)$. It follows that $\bar{\chi}_B^w(\langle \mathcal{A}_{\mathcal{P}(\mathcal{G})}, \mathcal{R}_{\mathcal{P}(\mathcal{G})} \rangle) \geq \bar{\chi}_B^w(\mathcal{G})$. If $x \in X$, then $\langle 0, x \rangle, \langle 1, x \rangle \in P_{\mathcal{G}}$, and $[\langle 0, x \rangle]_{E_{\leq_{\mathcal{G}}}} \neq [\langle 1, x \rangle]_{E_{\leq_{\mathcal{G}}}}$. Therefore, if $X \neq \emptyset$, then $|P_{\mathcal{G}}/E_{\leq_{\mathcal{G}}}| > 1$, and so by Theorem 2.30, $\text{odim}_B(\mathcal{P}(\mathcal{G})) = \bar{\chi}_B^w(\langle \mathcal{A}_{\mathcal{P}(\mathcal{G})}, \mathcal{R}_{\mathcal{P}(\mathcal{G})} \rangle) \geq \bar{\chi}_B^w(\mathcal{G})$. \dashv

Corollary 3.31. *For every dense selector E and for every $f \in \mathcal{F}$, we have $\text{odim}_B(\mathcal{P}(\mathcal{G}_0(E, f))) \geq \text{cov}(\mathcal{M})$.*

We are now ready to state and prove our third and final dichotomy. The basic objects of this dichotomy are the Borel quasi orders of the form $\mathcal{P}(\mathcal{G}_0(E, f))$. The morphisms will be continuous order preserving maps that also preserve the order incomparability of certain elements. The dichotomy characterizes the Borel quasi orders of uncountable Borel order dimension in terms of the existence of such a morphism from one of the basic objects.

Theorem 3.32. *For any Borel quasi order $\mathcal{P} = \langle P, \leq \rangle$, exactly one of the following holds:*

- (1) $\text{odim}_B(\mathcal{P}) \leq \aleph_0$;
- (2) for every selector E , there exist $f \in \mathcal{F}$ and a continuous $\varphi : 2 \times [T_f] \rightarrow P$ such that both of the following hold:

- (a) $\forall x, y \in [T_f] [\langle 0, x \rangle \leq_{\mathcal{G}_0(E, f)} \langle 1, y \rangle \implies \varphi(\langle 0, x \rangle) \leq \varphi(\langle 1, y \rangle)];$
 (b) for every $x \in [T_f]$, $\varphi(\langle 0, x \rangle)$ and $\varphi(\langle 1, x \rangle)$ are \leq -incomparable.

Proof. Let us first see that (1) and (2) are mutually exclusive. Assume (2). Fix a dense selector E . Let $f \in \mathcal{F}$ and $\varphi : 2 \times [T_f] \rightarrow P$ be a continuous map satisfying (2)(a) and (2)(b). Define $\psi : [T_f] \rightarrow \mathcal{A}_{\mathcal{P}}$ by $\psi(x) = \langle \varphi(\langle 1, x \rangle), \varphi(\langle 0, x \rangle) \rangle$, for all $x \in [T_f]$. Note that $\psi(x) \in \mathcal{A}_{\mathcal{P}}$ because of (2)(b) and that ψ is a Borel map with respect to any Polish topology on $\mathcal{A}_{\mathcal{P}}$ which extends the topology on $\mathcal{A}_{\mathcal{P}}$ inherited from $P \times P$. Further, if $\langle x, y \rangle \in R_{E, f}$, then $\langle 0, x \rangle \leq_{\mathcal{G}_0(E, f)} \langle 1, y \rangle$, which by (2)(a) implies $\varphi(\langle 0, x \rangle) \leq \varphi(\langle 1, y \rangle)$, which in turn implies $\langle \psi(x), \psi(y) \rangle \in \mathcal{R}_{\mathcal{P}}$. By (2)(b), $|P/E_{\leq}| > 1$. It follows that $\text{odim}_B(\mathcal{P}) = \bar{\chi}_B^w(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle) \geq \bar{\chi}_B^w(\mathcal{G}_0(E, f)) > \aleph_0$.

Now assume that $\text{odim}_B(\mathcal{P}) > \aleph_0$. By Corollary 2.34, $\bar{\chi}_B^w(\langle \mathcal{B}_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}} \rangle) > \aleph_0$. Let E be any selector. Using Theorem 3.23, fix $f \in \mathcal{F}$ and a continuous minimal homomorphism Π from $\mathcal{G}_0(E, f)$ to $\langle \mathcal{B}_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}} \rangle$. Define $\varphi : 2 \times [T_f] \rightarrow P$ as follows. Given $x \in [T_f]$, $\Pi(x) = \langle p_x, q_x \rangle \in \mathcal{B}_{\mathcal{P}}$. By the definition of $\mathcal{B}_{\mathcal{P}}$, $p_x \not\leq q_x$ and $q_x \not\leq p_x$. Define $\varphi(\langle 0, x \rangle) = q_x$ and $\varphi(\langle 1, x \rangle) = p_x$. It is easily verified that φ is continuous and (2)(b) holds by definition. For (2)(a), assume $\langle 0, x \rangle \leq_{\mathcal{G}_0(E, f)} \langle 1, y \rangle$. Then $\langle x, y \rangle \in R_{E, f}$, whence $\langle \Pi(x), \Pi(y) \rangle \in \mathcal{S}_{\mathcal{P}}$, whence $\langle \Pi(x), \Pi(y) \rangle \in \mathcal{R}_{\mathcal{P}}$, whence $q_x \leq p_y$, whence $\varphi(\langle 0, x \rangle) \leq \varphi(\langle 1, y \rangle)$, as needed. \dashv

Corollary 3.33. *For any Borel quasi order \mathcal{P} , $\text{odim}_B(\mathcal{P}) \leq \aleph_0$ or $\text{odim}_B(\mathcal{P}) \geq \text{cov}(\mathcal{M})$.*

Proof. If $\text{odim}_B(\mathcal{P}) > \aleph_0$, then (2) of Theorem 3.32 holds. If (2) of Theorem 3.32 holds, then the argument in the first paragraph of the proof of Theorem 3.32 shows that $\text{odim}_B(\mathcal{P}) \geq \bar{\chi}_B^w(\mathcal{G}_0(E, f))$, for some dense selector E and $f \in \mathcal{F}$. By Lemma 3.8, $\bar{\chi}_B^w(\mathcal{G}_0(E, f)) \geq \text{cov}(\mathcal{M})$, whence $\text{odim}_B(\mathcal{P}) \geq \text{cov}(\mathcal{M})$. \dashv

Our next task is to obtain further structural information about Borel quasi orders of countable Borel order dimension. We will show that every such order has a Borel linearization. Kanovei [17] proved a dichotomy that characterized the Borel quasi orders that are Borel linearizable. The basic object of his dichotomy is the following quasi order.

Definition 3.34. Recall that for $x, y \in 2^\omega$, $x E_0 y$ if and only if $\{n \in \omega : x(n) \neq y(n)\}$ is finite. On 2^ω , define the quasi ordering \leq_0 by $x \leq_0 y$ if and only if $x = y$, or $x \neq y$ and $x E_0 y$ and for the maximal $n \in \omega$ such that $x(n) \neq y(n)$, we have $x(n) < y(n)$. Clearly, $\langle 2^\omega, \leq_0 \rangle$ is a Borel quasi order. It is easily verified that E_{\leq_0} is equality and elements in distinct E_0 classes are incomparable.

Kanovei proved that a Borel quasi order has a Borel linearization if and only if there is no continuous injection from $\langle 2^\omega, \leq_0 \rangle$ that is order preserving and maps E_0 -inequivalent elements to incomparable ones (see Theorem 3.37 below).

Lemma 3.35. *Suppose \preceq is a Borel quasi order on 2^ω extending \leq_0 . Then \preceq is a meager subset of $2^\omega \times 2^\omega$.*

Proof. Suppose not. There exist $s, t \in 2^{<\omega}$ such that $|s| = |t|$ and $\preceq \cap (U \times V)$ is comeager in $U \times V$, where $U = \{z \in 2^\omega : s \frown \langle 1 \rangle \subseteq z\}$ and $V = \{z \in 2^\omega : t \frown \langle 0 \rangle \subseteq z\}$. Define $\psi : U \times V \rightarrow U \times V$ by $\psi(\langle s \frown \langle 1 \rangle \frown x, t \frown \langle 0 \rangle \frown y \rangle) = \langle s \frown \langle 1 \rangle \frown y, t \frown \langle 0 \rangle \frown x \rangle$, for every $x, y \in 2^\omega$. Then ψ is an auto-homeomorphism of $U \times V$, and hence $\preceq \cap (U \times V) \cap \psi^{-1}(\preceq \cap (U \times V))$ is non-empty. Find $x, y \in 2^\omega$ so that $s \frown \langle 1 \rangle \frown x \preceq t \frown \langle 0 \rangle \frown y$ and $s \frown \langle 1 \rangle \frown y \preceq t \frown \langle 0 \rangle \frown x$. Since $|s| = |t|$, we have $s \frown \langle 1 \rangle \frown y \preceq t \frown \langle 0 \rangle \frown x \leq_0 s \frown \langle 1 \rangle \frown x \preceq t \frown \langle 0 \rangle \frown y \leq_0 s \frown \langle 1 \rangle \frown y$. This means $\langle s \frown \langle 1 \rangle \frown y, t \frown \langle 0 \rangle \frown x \rangle \in E_{\preceq} = E_{\leq_0}$, which is impossible as E_{\leq_0} is equality. \dashv

Lemma 3.36. $\text{odim}_B(\langle 2^\omega, \leq_0 \rangle) > \aleph_0$.

Proof. Suppose $\langle \preceq_i : i < \omega \rangle$ is a sequence of Borel quasi orders on 2^ω extending \leq_0 . Write $M = \bigcup_{i < \omega} \preceq_i$ and $N = \{\langle y, x \rangle : \langle x, y \rangle \in M\}$. By Lemma 3.35, $M \cup N$ is a meager subset of $2^\omega \times 2^\omega$. Choose any $\langle y, x \rangle \in (2^\omega \times 2^\omega) \setminus (M \cup N)$. Then $x \not\preceq_0 y$, and yet $y \not\preceq_i x$, for any $i < \omega$. Therefore $\langle \preceq_i : i < \omega \rangle$ cannot witness $\text{odim}_B(\langle 2^\omega, \leq_0 \rangle) \leq \aleph_0$. \dashv

Theorem 3.37 (Kanovei [17]). *Suppose $\langle P, \leq \rangle$ is a Borel quasi order. Then exactly one of the following two conditions is satisfied:*

- (1) $\langle P, \leq \rangle$ is Borel linearizable;
- (2) there is a continuous 1-1 map $F : 2^\omega \rightarrow P$ satisfying both of the following:
 - (a) $a \leq_0 b \implies F(a) \leq F(b)$;
 - (b) $a \not\leq_0 b \implies F(a)$ and $F(b)$ are \leq -incomparable.

Theorem 3.38. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a Borel quasi order. If $\text{odim}_B(\mathcal{P}) \leq \aleph_0$, then \mathcal{P} is Borel linearizable.*

Proof. Assume for a contradiction that \mathcal{P} is not Borel linearizable. By Theorem 3.37 fix a continuous 1-1 map $F : 2^\omega \rightarrow P$ satisfying (2)(a) and (2)(b) of Theorem 3.37. Fix any selector E . By Lemma 3.36 and Theorem 3.32, find $f \in \mathcal{F}$ and a continuous $\varphi : 2 \times [T_f] \rightarrow 2^\omega$ such that (2)(a) and (2)(b) of Theorem 3.32 are satisfied with respect to \leq_0 . Write $\psi = F \circ \varphi$. Then $\psi : 2 \times [T_f] \rightarrow P$ is continuous. For $x, y \in [T_f]$, if $\langle 0, x \rangle \leq_{\mathcal{G}_0(E, f)} \langle 1, y \rangle$, then $\varphi(\langle 0, x \rangle) \leq_0 \varphi(\langle 1, y \rangle)$, whence $\psi(\langle 0, x \rangle) \leq \psi(\langle 1, y \rangle)$. For any $x \in [T_f]$, $\varphi(\langle 0, x \rangle)$ and $\varphi(\langle 1, x \rangle)$ are \leq_0 -incomparable. Note that if $\varphi(\langle 0, x \rangle) E_0 \varphi(\langle 1, x \rangle)$, then, by the definition of \leq_0 , $\varphi(\langle 0, x \rangle)$ and $\varphi(\langle 1, x \rangle)$ would be \leq_0 -comparable. Hence $\varphi(\langle 0, x \rangle) \not E_0 \varphi(\langle 1, x \rangle)$, whence $\psi(\langle 0, x \rangle)$ and $\psi(\langle 1, x \rangle)$ are \leq -incomparable. Thus (2) of Theorem 3.32 holds for \mathcal{P} , which means that $\text{odim}_B(\mathcal{P}) > \aleph_0$, a contradiction. \dashv

By a celebrated theorem of Harrington, Marker, and Shelah [11], Borel linearizability provides a good deal of structural information about a Borel quasi order. In fact, if \mathcal{P} is Borel linearizable, then \mathcal{P} can be embedded into 2^α equipped with the lexicographical ordering, where α is some countable ordinal. In Theorem 3.40 and Corollary 3.41 below \leq_{lex} denotes the lexicographical ordering on 2^α . The result of Harrington, Marker, and Shelah applies to all the so called thin quasi orders.

Definition 3.39 (Harrington, Marker, and Shelah [11]). \mathcal{P} is *thin* if there is no perfect set of pairwise incomparable elements.

Theorem 3.40 (Harrington, Marker, and Shelah [11]). *If $\mathcal{P} = \langle P, \leq \rangle$ is a thin Borel quasi order, then for some $\alpha < \omega_1$, there is a Borel $f : P \rightarrow 2^\alpha$ such that*

- (1) $x \leq y \implies f(x) \leq_{\text{lex}} f(y)$ and
- (2) $x E_\leq y \iff f(x) = f(y)$, for all $x, y \in P$.

Theorem 3.40 has a number of important consequences for Borel quasi orders and linear orders. Among other things, it implies that there are no Borel realizations of certain combinatorial counterexamples such as Suslin trees or Suslin lattices. See [36] for further details. By combining Theorem 3.38 with Theorem 3.40 we get the following corollary which essentially says that every Borel quasi order of countable Borel order dimension is Borel embeddable into $\langle 2^\alpha, \leq_{\text{lex}} \rangle$, for some countable α .

Corollary 3.41. *Suppose $\mathcal{P} = \langle P, \leq \rangle$ is a Borel quasi order with $\text{odim}_B(\mathcal{P}) \leq \aleph_0$. Then for some $\alpha < \omega_1$, there is a Borel $f : P \rightarrow 2^\alpha$ such that*

- (1) $x \leq y \implies f(x) \leq_{\text{lex}} f(y)$;
- (2) $x E_\leq y \iff f(x) = f(y)$,

for all $x, y \in P$.

We will end this section with the following, somewhat ambitious, problem.

Question 3.42. *Is there a classification of all the Borel quasi orders of countable Borel order dimension? How about those of finite or bounded Borel order dimension?*

4. BOREL ORDER DIMENSION OF LOCALLY COUNTABLE BOREL QUASI ORDERS

In this section, we will use the dichotomies proved in Section 3 to obtain more information about locally countable Borel quasi orders. Recall this definition.

Definition 4.1. A quasi order $\mathcal{P} = \langle P, \leq \rangle$ is said to be *locally countable* (*locally finite*) if for every $x \in P$, $\{y \in P : y \leq x\}$ is countable (finite).

Many important naturally occurring examples of Borel quasi orders are locally countable, for instance, the Turing degrees.

Definition 4.2. For $x, y \in 2^\omega$, $x \leq_T y$ will mean that x is Turing reducible to y . Define $\mathcal{D} = \langle 2^\omega, \leq_T \rangle$.

Observe that if E is a dense selector and $f \in \mathcal{F}$, then $\mathcal{P}(\mathcal{G}_0(E, f))$ is locally countable. This is because for any $z \in [T_f]$, for each $l \in \omega$, there are at most finitely many y such that $z \upharpoonright l = E(f \upharpoonright l) = y \upharpoonright l$ and $y R_{E, f} z$. Therefore, $\{y \in [T_f] : y R_{E, f} z\}$ is countable, hence $\{\langle 0, y \rangle \in P_{\mathcal{G}_0(E, f)} : \langle 0, y \rangle \leq_{\mathcal{G}_0(E, f)} \langle 1, z \rangle\}$ is countable. So Theorem 3.32 says that for any Borel quasi order \mathcal{P} , if $\text{odim}_B(\mathcal{P}) > \aleph_0$, then this fact is witnessed by an order-preserving continuous map from some locally countable Borel quasi order.

The classical order dimension of \mathcal{D} was investigated in detail in [15] and in [22]. Kumar and Raghavan [22] showed that \mathcal{D} has the largest order dimension among all locally countable orders of size at most continuum. Higuchi, Lempp, Raghavan, and Stephan [15] proved that if $2^{\aleph_0} = \kappa^+$, where $\text{cf}(\kappa) > \omega$, then $\text{odim}(\mathcal{D}) \leq \kappa$. Hence we obtain the following corollary to the results of Section 3 that is worth explicitly stating.

Corollary 4.3. *If PFA holds, then $\text{odim}(\mathcal{D}) = \aleph_1 < \aleph_2 = \text{odim}_B(\mathcal{D}) = 2^{\aleph_0}$.*

Proof. Since PFA implies $2^{\aleph_0} = \aleph_2$, Proposition 4.3 and Theorem 3.11 of [15] imply that $\text{odim}(\mathcal{D}) = \aleph_1$. Since $\text{odim}_B(\mathcal{D}) \geq \text{odim}(\mathcal{D})$, Corollary 3.32 implies that $\text{odim}_B(\mathcal{D}) \geq \text{cov}(\mathcal{M})$. Under PFA, $\text{cov}(\mathcal{M}) = \aleph_2 = 2^{\aleph_0}$, so the result follows. \dashv

The following notions were crucial for the results in Kumar and Raghavan [22]. They were able to use the notion of separation defined below to provide a purely combinatorial characterization of $\text{odim}(\mathcal{D})$.

Definition 4.4. Let P be a set and λ a cardinal. A family $\mathcal{F} \subseteq \mathcal{P}(P)$ *separates points from elements of $[P]^{<\lambda}$* if for every $A \in [P]^{<\lambda}$ and $p \in P \setminus A$, there exists $X \in \mathcal{F}$ such that $p \in X$ and $A \cap X = \emptyset$.

A sequence $\langle <_i : i \in I \rangle$ of partial orders on P *separates points from elements of $[P]^{<\lambda}$* if for every $A \in [P]^{<\lambda}$ and $p \in P \setminus A$, there exists $i \in I$ so that $\forall q \in A [q <_i p]$.

We will now show that the notion of separation by Borel sets or by Borel partial orders can be used to provide upper bounds on $\text{odim}_B(\mathcal{P})$, where \mathcal{P} is any locally countable Borel quasi order.

Definition 4.5. Let X be a Polish space and λ, κ cardinals. We say that $\dagger(X, \lambda, \kappa)$ *holds* if there is a sequence $\langle <_i : i < \kappa \rangle$ of Borel partial orders on X which separates points from elements of $[X]^{<\lambda}$.

Lemma 4.6. *Let X be a Polish space and \preceq a Borel quasi order on X with $\langle X, \preceq \rangle$ being locally countable. Let $<$ be a Borel partial order on X . Then there exists a Borel quasi order \trianglelefteq on X that extends \preceq and has the property that for any $x, y \in X$, if $\forall u \preceq x [u < y]$, then $x \trianglelefteq y$.*

Proof. Define $R = \{\langle y, x \rangle \in X \times X : \exists u \preceq x \forall v \preceq y [v < u]\}$. Define

$$\trianglelefteq = \{\langle x, y \rangle \in X \times X : \langle x, y \rangle \in \preceq \vee \langle x, y \rangle \in R\}.$$

It is an easy exercise to verify that \trianglelefteq is a quasi order on X which is an extension of \preceq in the sense of Definition 2.9. If $x, y \in X$ are such that $\forall u \preceq x [u < y]$, then $\langle x, y \rangle \in R$ by definition, whence $x \trianglelefteq y$. Therefore to complete the proof, it suffices to verify that \trianglelefteq is a Borel subset of $X \times X$, and since it is known that \preceq is a Borel subset of $X \times X$, it suffices for this to check that R is Borel.

To this end, write $Q = \{\langle y, x \rangle \in X \times X : x \preceq y\}$. Q is Borel and by the hypothesis that $\langle X, \preceq \rangle$ is locally countable, $Q_y = \{x \in X : \langle y, x \rangle \in Q\}$ is countable for every $y \in X$. By the Luzin–Novikov theorem write $Q = \bigcup_{n \in \omega} Q_n$, where each Q_n is Borel and is the graph of a function. Thus for each $n \in \omega$, $\text{dom}(Q_n)$ is a Borel subset of X . We write $Q_n(x)$ to denote the unique $y \in X$ with $\langle x, y \rangle \in Q_n$, for every $n \in \omega$ and $x \in \text{dom}(Q_n)$. For $n, m \in \omega$, define

$$B_{n,m} = \{\langle x, y \rangle \in \text{dom}(Q_n) \times \text{dom}(Q_m) : \langle Q_n(x), Q_m(y) \rangle \in <\}.$$

$B_{n,m}$ is Borel. For each $n, m \in \omega$, define

$$C_{n,m} = B_{n,m} \cup ((X \setminus \text{dom}(Q_n)) \times \text{dom}(Q_m)),$$

and note that $C_{n,m}$ is also Borel. Therefore, $\bigcup_{m \in \omega} \bigcap_{n \in \omega} C_{n,m}$ is Borel, and it is easily checked that $R = \bigcup_{m \in \omega} \bigcap_{n \in \omega} C_{n,m}$. \dashv

Lemma 4.7. *Suppose X is a Polish space, κ is a cardinal, $\lambda \in \{\aleph_0, \aleph_1\}$, and that $\dagger(X, \lambda, \kappa)$ holds. If $\langle X, \preceq \rangle$ is a Borel quasi order with the property that for every $x \in X$, $|\{y \in X : y \preceq x\}| < \lambda$, then $\text{odim}_B(\langle X, \preceq \rangle) \leq \kappa$.*

Proof. Note that the hypothesis on $\langle X, \preceq \rangle$ implies that it is locally countable, hence Lemma 4.6 is applicable. By $\dagger(X, \lambda, \kappa)$, fix a sequence $\langle <_i : i < \kappa \rangle$ of Borel partial orders on X which separates points from elements of $[X]^{<\lambda}$. By Lemma 4.6, for every $i < \kappa$, let \trianglelefteq_i be a Borel quasi order on X that extends \preceq and has the property that for any $x, y \in X$, if $\forall u \preceq x [u <_i y]$, then $x \trianglelefteq_i y$. Consider $x, y \in X$ with $y \not\trianglelefteq x$. Then $A = \{u \in X : u \preceq x\} \in [X]^{<\lambda}$ and $y \in X \setminus A$. Hence there exists $i < \kappa$ such that $\forall u \in A [u <_i y]$, whence $x \trianglelefteq_i y$. Therefore, $\langle \trianglelefteq_i : i < \kappa \rangle$ is a witness that $\text{odim}_B(\langle X, \preceq \rangle) \leq \kappa$. \dashv

Lemma 4.8. *For any Polish space X , $\dagger(X, \aleph_0, \aleph_0)$ holds.*

Proof. Fix a closed $F \subseteq \omega^\omega$ and a continuous bijection $f : F \rightarrow X$. For $s \in \omega^{<\omega}$, let $F_s = F \cap \{r \in \omega^\omega : s \subseteq r\}$. Since f is 1-1 and continuous, $B_s = f''F_s$ is a Borel subset of X . Define $<_s = \{\langle x, y \rangle \in X \times X : x \notin B_s \wedge y \in B_s\}$. Then $<_s$ is Borel and it is easily seen to be a partial order on X . Now suppose $A \subseteq X$ is finite and $y \in X \setminus A$. Then $f^{-1}(A) \subseteq F$ is finite and if $u \in F$ is unique so that $f(u) = y$, then $u \notin f^{-1}(A)$. Hence there exists $s \in \omega^{<\omega}$ such that $u \in F_s$, but $f^{-1}(A) \cap F_s = \emptyset$. It follows that $y \in B_s$, but $A \cap B_s = \emptyset$, whence $\forall x \in A [x <_s y]$. Therefore, $\langle <_s : s \in \omega^{<\omega} \rangle$ witnesses $\dagger(X, \aleph_0, \aleph_0)$. \dashv

It was pointed out in [15] that a theorem of Kierstead and Milner [20] implies that for every locally finite partial order $\langle P, < \rangle$ of size at most $2^{2^{\aleph_0}}$, $\text{odim}(\langle P, < \rangle) \leq \aleph_0$. It turns out that the Borel analog of this true as well.

Theorem 4.9. *If $\langle X, \preceq \rangle$ is a locally finite Borel quasi order, then $\text{odim}_B(\langle X, \preceq \rangle) \leq \aleph_0$.*

Proof. By combining Lemmas 4.7 and 4.8. \dashv

Corollary 4.10. *Every locally finite Borel quasi order has a Borel linearization.*

Next we use ideas from [23] to show that the Borel order dimension of every locally countable Borel quasi order may consistently be any regular cardinal below the continuum, with the continuum being arbitrarily large. Thus ZFC does not prove that there is some locally countable Borel quasi order whose Borel order dimension is equal to 2^{\aleph_0} . In this model, we also get that for some fixed $\lambda \leq 2^{\aleph_0}$, $\chi_{\mathbb{B}}^w(\mathcal{G}_0(E, f)) = \lambda$, for every $f \in \mathcal{F}$ and dense selector E . Geschke [10] has obtained consistency results on the weak Borel chromatic number of Borel graphs.

Harrington introduced a c.c.c. forcing for adding a generic G_δ set. Miller [27, 28] introduced a modification of Harrington's forcing and used it to obtain consistency results about Borel hierarchies. A new c.c.c. forcing notion, which is an amalgam of a forcing from Kumar and Raghavan [23] and the one from Miller [28], is introduced below. Our forcing generically adds a family of \aleph_1 many G_δ sets that separates countable sets in the ground model from points.

Definition 4.11. For ordinals α, δ , define $\alpha^{<\delta} = \{X \subseteq \alpha : \text{otp}(X) < \delta\}$.

Definition 4.12. Let κ be a cardinal and $\delta < \omega_1$ be an indecomposable ordinal. Define

$$\begin{aligned} \mathcal{G}_{\kappa, \delta}^* &= \left\{ g : g \text{ is a function } \wedge \text{dom}(g) \in \kappa^{<\delta} \wedge \text{ran}(g) \subseteq [\omega]^{<\aleph_0} \right\}, \\ \mathcal{G}_{\kappa, \delta} &= \left\{ g \in \mathcal{G}_{\kappa, \delta}^* : \forall n \in \omega [|\{\alpha \in \text{dom}(g) : n \in g(\alpha)\}| < \aleph_0] \right\}. \end{aligned}$$

For $g_1, g_2 \in \mathcal{G}_{\kappa, \delta}$, define $g_2 \leq_{\mathcal{G}} g_1$ if and only if $\text{dom}(g_2) \supseteq \text{dom}(g_1)$ and $\forall \alpha \in \text{dom}(g_1) [g_2(\alpha) \supseteq g_1(\alpha)]$. It is clear that $\leq_{\mathcal{G}}$ is a quasi order.

Definition 4.13. Define

$$\mathcal{F} = \left\{ f : f \text{ is a function } \wedge \text{dom}(f) \in [\omega^{<\omega}]^{<\aleph_0} \wedge \text{ran}(f) \subseteq [\omega]^{<\aleph_0} \right\}.$$

For $f_1, f_2 \in \mathcal{F}$ define $f_2 \leq_{\mathcal{F}} f_1$ if and only if $\text{dom}(f_2) \supseteq \text{dom}(f_1)$ and $\forall s \in \text{dom}(f_1) [f_2(s) \supseteq f_1(s)]$. Clearly, $\leq_{\mathcal{F}}$ is a quasi order.

Definition 4.14. Let $E : 2^{\aleph_0} \rightarrow \omega^\omega$ be a bijection. Let $\delta < \omega_1$ be indecomposable. Define $\mathbb{R}_{E, \delta}$ to be the collection of all $\langle f, g, h \rangle \in \mathcal{F} \times \mathcal{G}_{2^{\aleph_0}, \delta} \times [2^{\aleph_0}]^{<\aleph_0}$ satisfying the following conditions:

- (1) $h \cap \text{dom}(g) = \emptyset$;
- (2) $\forall \alpha \in \text{dom}(g) \forall l \in \omega [E(\alpha) \upharpoonright l \in \text{dom}(f) \implies g(\alpha) \cap f(E(\alpha) \upharpoonright l) = \emptyset]$.

Given $p = \langle f_p, g_p, h_p \rangle \in \mathbb{R}_{E, \delta}$ and $q = \langle f_q, g_q, h_q \rangle \in \mathbb{R}_{E, \delta}$, define $q \leq_{\mathbb{R}} p$ if and only if $f_q \leq_{\mathcal{F}} f_p$, $g_q \leq_{\mathcal{G}} g_p$, and $h_q \supseteq h_p$. Then it is clear that $\langle \mathbb{R}_{E, \delta}, \leq_{\mathbb{R}} \rangle$ is a forcing notion, and when $p = \langle f, g, h \rangle \in \mathbb{R}_{E, \delta}$, we will write f_p, g_p , and h_p to denote f, g , and h respectively.

Define \mathbb{S}_E to be the finite support product of the $\mathbb{R}_{E, \delta}$ as δ ranges over the countable indecomposable ordinals. In other words, \mathbb{S}_E is the collection of all p such that p is a function, $\text{dom}(p)$ is a finite subset of $\{\delta < \omega_1 : \delta \text{ is indecomposable}\}$, and $\forall \delta \in \text{dom}(p) [p(\delta) \in \mathbb{R}_{E, \delta}]$. And for $p, q \in \mathbb{S}_E$, $q \leq_{\mathbb{S}} p$ if and only if $\text{dom}(q) \supseteq \text{dom}(p)$ and $\forall \delta \in \text{dom}(p) [q(\delta) \leq_{\mathbb{R}} p(\delta)]$. Then $\langle \mathbb{S}_E, \leq_{\mathbb{S}} \rangle$ is a forcing notion.

Lemma 4.15. *Let $E : 2^{\aleph_0} \rightarrow \omega^\omega$ be a bijection. Let $\delta < \omega_1$ be indecomposable. Then the following hold:*

- (1) *for each $n \in \omega$, $D_n = \{q \in \mathbb{R}_{E, \delta} : \exists s \in \text{dom}(f_q) [n \in f(s)]\}$ is dense;*
- (2) *for each $p \in \mathbb{R}_{E, \delta}$, $\beta \in h_p$, and $n \in \omega$,*

$$D_{p, \beta, n} = \{r \leq_{\mathbb{R}} p : \exists l \in \omega [E(\beta) \upharpoonright l \in \text{dom}(f_r) \wedge n \in f_r(E(\beta) \upharpoonright l)]\}$$

is dense below p ;

(3) for each $p \in \mathbb{R}_{E,\delta}$ and $\alpha \in \text{dom}(g_p)$, $D_{p,\alpha} = \{r \leq_{\mathbb{R}} p : g_r(\alpha) \neq \emptyset\}$ is dense below p .

Proof. For (1): fix $p \in \mathbb{R}_{E,\delta}$. Then $X = \{E(\alpha) : \alpha \in \text{dom}(g_p) \wedge n \in g_p(\alpha)\} \subseteq \omega^\omega$ is finite. Hence it is possible to find $s \in \omega^{<\omega} \setminus \text{dom}(f_p)$ such that $s \not\subseteq x$, for all $x \in X$. Define $f_q = f_p \cup \{\langle s, \{n\} \rangle\}$, $g_q = g_p$, and $h_q = h_p$. To see that (2) of Definition 4.14 holds, suppose $\alpha \in \text{dom}(g_q)$, $l \in \omega$, and that $E(\alpha)|l \in \text{dom}(f_q)$. If $E(\alpha)|l \in \text{dom}(f_p)$, then $g_q(\alpha) \cap f_q(E(\alpha)|l) = g_p(\alpha) \cap f_p(E(\alpha)|l) = \emptyset$. Otherwise, $E(\alpha)|l = s$, which means $E(\alpha) \notin X$, and so $n \notin g_p(\alpha)$. Hence $g_q(\alpha) \cap f_q(E(\alpha)|l) = g_p(\alpha) \cap \{n\} = \emptyset$. Therefore, $q \leq_{\mathbb{R}} p$ and $q \in D_n$, as required.

For (2): fix any $q \leq_{\mathbb{R}} p$. Once again, $X = \{E(\alpha) : \alpha \in \text{dom}(g_q) \wedge n \in g_q(\alpha)\}$ is a finite set. Further, since $\beta \in h_p \subseteq h_q$ and $h_q \cap \text{dom}(g_q) = \emptyset$, $E(\beta) \notin X$. Therefore, there exists $l \in \omega$ such that $E(\beta)|l \in \omega^{<\omega} \setminus \text{dom}(f_q)$ and $E(\beta)|l \not\subseteq x$, for all $x \in X$. Define $f_r = f_q \cup \{\langle E(\beta)|l, \{n\} \rangle\}$, $g_r = g_q$, and $h_r = h_q$. As before, suppose $\alpha \in \text{dom}(g_r)$, $m \in \omega$, and that $E(\alpha)|m \in \text{dom}(f_r)$. If $E(\alpha)|m \in \text{dom}(f_q)$, then $g_r(\alpha) \cap f_r(E(\alpha)|m) = g_q(\alpha) \cap f_q(E(\alpha)|m) = \emptyset$, while if not, then $E(\beta)|l \subseteq E(\alpha)$, whence $n \notin g_q(\alpha)$ and $g_r(\alpha) \cap f_r(E(\alpha)|m) = g_q(\alpha) \cap \{n\} = \emptyset$. Thus $r \leq_{\mathbb{R}} q$ and $r \in D_{p,\beta,n}$, as needed.

For (3): fix any $q \leq_{\mathbb{R}} p$. Then $\bigcup \text{ran}(f_q)$ is a finite subset of ω . Choose $n \in \omega \setminus (\bigcup \text{ran}(f_q))$. Define g_r to be the function such that $\text{dom}(g_r) = \text{dom}(g_q)$, $g_r(\alpha) = g_q(\alpha) \cup \{n\}$, and $g_r(\beta) = g_q(\beta)$, for all $\beta \in \text{dom}(g_q) \setminus \{\alpha\}$. Note $g_r \in \mathcal{G}_{2^{\aleph_0},\delta}$ and $g_r \leq_{\mathcal{G}} g_q$. Define $f_r = f_q$ and $h_r = h_q$. Note $h_r \cap \text{dom}(g_r) = h_q \cap \text{dom}(g_q) = \emptyset$. Suppose $\beta \in \text{dom}(g_r)$, $l \in \omega$, and that $E(\beta)|l \in \text{dom}(f_r)$. Then $g_r(\beta) \cap f_r(E(\beta)|l) \subseteq (g_q(\beta) \cup \{n\}) \cap f_q(E(\beta)|l) = g_q(\beta) \cap f_q(E(\beta)|l) = \emptyset$ because $n \notin f_q(E(\beta)|l)$. Thus $r \leq_{\mathbb{R}} q$ and $r \in D_{p,\alpha}$, as needed. \dashv

Definition 4.16. Suppose \mathbf{V} is a transitive model of a sufficiently large fragment of ZFC. In \mathbf{V} , suppose that $E : 2^{\aleph_0} \rightarrow \omega^\omega$ is a bijection and that $\delta < \omega_1$ is indecomposable. Suppose G is a $(\mathbf{V}, \mathbb{R}_{E,\delta}^{\mathbf{V}})$ -generic filter. In $\mathbf{V}[G]$, define the following sets. For each $n \in \omega$, $U_{G,n} = \bigcup \{[s] : \exists p \in G [s \in \text{dom}(f_p) \wedge n \in f_p(s)]\}$, where $[s] = \{x \in \omega^\omega : s \subseteq x\}$, for every $s \in \omega^{<\omega}$. Define $\mathcal{F}_G = \bigcap_{n \in \omega} U_{G,n}$. Define $\mathcal{G}_G = \{E(\alpha) : \exists p \in G [\alpha \in \text{dom}(g_p)]\}$ and $\mathcal{H}_G = \{E(\alpha) : \exists p \in G [\alpha \in h_p]\}$.

Lemma 4.17. \mathcal{F}_G is a G_δ set such that $\mathcal{H}_G \subseteq \mathcal{F}_G$ and $\mathcal{G}_G \cap \mathcal{F}_G = \emptyset$.

Proof. It is clear from the definition that \mathcal{F}_G is a G_δ set. Suppose $p \in G$, $\alpha \in h_p$ and $n \in \omega$. By (2) of Lemma 4.15, $D_{p,\alpha,n}$ is dense below p , and so, there exist $q \in G$ and $l \in \omega$ with $E(\alpha)|l \in \text{dom}(f_q)$ and $n \in f_q(E(\alpha)|l)$, whence $E(\alpha) \in U_{G,n}$. As this is for every $n \in \omega$, $E(\alpha) \in \mathcal{F}_G$. This shows $\mathcal{H}_G \subseteq \mathcal{F}_G$. Next, suppose for a contradiction that $p \in G$, $\alpha \in \text{dom}(g_p)$, and that $E(\alpha) \in \mathcal{F}_G$. By (3) of Lemma 4.15, there exist $q \in G$ and $n \in \omega$ with $q \leq_{\mathbb{R}} p$ and $n \in g_q(\alpha)$. Since we have assumed $E(\alpha) \in \mathcal{F}_G$, there exist $r \in G$ and $l \in \omega$ so that $E(\alpha)|l \in \text{dom}(f_r)$ and $n \in f_r(E(\alpha)|l)$. Find $r^* \in G$ with $r^* \leq_{\mathbb{R}} q, r$. Then we have $\alpha \in \text{dom}(g_{r^*})$, $l \in \omega$, $E(\alpha)|l \in \text{dom}(f_{r^*})$, and $n \in g_{r^*}(\alpha) \cap f_{r^*}(E(\alpha)|l) = \emptyset$ because r^* satisfies (2) of Definition 4.14. This contradiction shows $\mathcal{G}_G \cap \mathcal{F}_G = \emptyset$. \dashv

Definition 4.18. Suppose $\delta < \omega_1$ is indecomposable and that $\aleph_0 \leq \kappa$ is a cardinal. Define $\mathbb{Q}_{\kappa,\delta}$ to be the collection of all p such that p is a function, $\text{dom}(p) \in \kappa^{<[\delta]}$, $\text{ran}(p) \subseteq 2$, and $\{\xi \in \text{dom}(p) : p(\xi) = 1\}$ is finite. For $p, q \in \mathbb{Q}_{\kappa,\delta}$, define $q \leq_{\mathbb{Q}} p$ if and only if $q \supseteq p$.

Define \mathbb{P}_κ to be the finite support product of the $\mathbb{Q}_{\kappa,\delta}$ as δ ranges over the indecomposable countable ordinals. In other words, \mathbb{P}_κ is the collection of all p such that p is a function, $\text{dom}(p)$ is a finite subset of $\{\delta < \omega_1 : \delta \text{ is indecomposable}\}$, and $\forall \delta \in \text{dom}(p) [p(\delta) \in \mathbb{Q}_{\kappa,\delta}]$. For $p, q \in \mathbb{P}_\kappa$, $q \leq_{\mathbb{P}} p$ if and only if $\text{dom}(q) \supseteq \text{dom}(p)$ and $\forall \delta \in \text{dom}(p) [q(\delta) \leq_{\mathbb{Q}} p(\delta)]$.

For an indecomposable $\delta < \omega_1$ and a bijection $E : 2^{\aleph_0} \rightarrow \omega^\omega$ define $j_\delta : \mathbb{R}_{E,\delta} \rightarrow \mathbb{Q}_{2^{\aleph_0},\delta}$ as follows. Given $p \in \mathbb{R}_{E,\delta}$, $j_\delta(p) = q$, where q is the function such that $\text{dom}(q) = \text{dom}(g_p) \cup h_p$, $q^{-1}(\{1\}) = h_p$, and $q^{-1}(\{0\}) = \text{dom}(g_p)$. Define $j : \mathbb{S}_E \rightarrow \mathbb{P}_{2^{\aleph_0}}$ as follows. Given $p \in \mathbb{S}_E$, $j(p) = q$, where q is the function such that $\text{dom}(q) = \text{dom}(p)$ and $\forall \delta \in \text{dom}(q) [q(\delta) = j_\delta(p(\delta))]$.

The forcings $\mathbb{Q}_{\kappa,\delta}$ and \mathbb{P}_κ were first defined by Kumar and Raghavan [23]. They proved that \mathbb{P}_κ is c.c.c. for all infinite κ . We will now use this fact to prove that \mathbb{S}_E is c.c.c.

Lemma 4.19. *Suppose $E : 2^{\aleph_0} \rightarrow \omega^\omega$ is a bijection and $\delta < \omega_1$ is indecomposable. For any $p, q \in \mathbb{R}_{E,\delta}$, if $f_p = f_q$ and $j_\delta(p) \not\leq_{\mathbb{Q}} j_\delta(q)$, then $p \not\leq_{\mathbb{R}} q$.*

Proof. Define g_r to be the function such that $\text{dom}(g_r) = \text{dom}(g_p) \cup \text{dom}(g_q)$, $\forall \alpha \in \text{dom}(g_p) \setminus \text{dom}(g_q) [g_r(\alpha) = g_p(\alpha)]$, $\forall \alpha \in \text{dom}(g_q) \setminus \text{dom}(g_p) [g_r(\alpha) = g_q(\alpha)]$, and $\forall \alpha \in \text{dom}(g_p) \cap \text{dom}(g_q) [g_r(\alpha) = g_p(\alpha) \cup g_q(\alpha)]$. Note that $g_r \in \mathcal{G}_{2^{\aleph_0},\delta}$ and that $g_r \leq_{\mathcal{G}} g_p, g_q$. Define $f_r = f_p = f_q$ and $h_r = h_p \cup h_q$. Since $j_\delta(p) \not\leq_{\mathbb{Q}} j_\delta(q)$, it follows that $\text{dom}(g_p) \cap h_q = \emptyset = \text{dom}(g_q) \cap h_p$. Therefore, $\text{dom}(g_r) \cap h_r = \emptyset$. Next, fix $\alpha \in \text{dom}(g_r)$, $l \in \omega$, and assume that $E(\alpha) \upharpoonright l \in \text{dom}(f_r) = \text{dom}(f_p) = \text{dom}(f_q)$. Then if $\alpha \in \text{dom}(g_p)$, then $g_p(\alpha) \cap f_p(E(\alpha) \upharpoonright l) = \emptyset$ and if $\alpha \in \text{dom}(g_q)$, then $g_q(\alpha) \cap f_q(E(\alpha) \upharpoonright l) = \emptyset$. It follows that $g_r(\alpha) \cap f_r(E(\alpha) \upharpoonright l) = \emptyset$. Thus we have verified that $r \in \mathbb{R}_{E,\delta}$. Since $r \leq_{\mathbb{R}} p, q$, $p \not\leq_{\mathbb{R}} q$, as claimed. \dashv

Lemma 4.20. *\mathbb{S}_E is c.c.c.*

Proof. For a contradiction, assume that $\langle p_\gamma : \gamma < \omega_1 \rangle$ is an antichain in \mathbb{S}_E . Write $I = \{\delta < \omega_1 : \delta \text{ is indecomposable}\}$ and $D_\gamma = \text{dom}(p_\gamma) \in [I]^{<\aleph_0}$. Also, for each $\gamma < \omega_1$ and each $\delta \in D_\gamma$, write $p_\gamma(\delta) = \langle f_{\gamma,\delta}, g_{\gamma,\delta}, h_{\gamma,\delta} \rangle$ instead of $\langle f_{p_\gamma(\delta)}, g_{p_\gamma(\delta)}, h_{p_\gamma(\delta)} \rangle$ for ease of notation. Find $A \in [\omega_1]^{\aleph_1}$ and $D \in [I]^{<\aleph_0}$ such that $\langle D_\gamma : \gamma \in A \rangle$ is a Δ -system with root D . Since \mathcal{F}^D is a countable set, find $\langle f_\delta : \delta \in D \rangle \in \mathcal{F}^D$ and $B \in [A]^{\aleph_1}$ such that $\forall \gamma \in B \forall \delta \in D [f_{\gamma,\delta} = f_\delta]$. For each $\gamma \in B$, define $q_\gamma = j(p_\gamma) \in \mathbb{P}_{2^{\aleph_0}}$ and note that $\text{dom}(q_\gamma) = \text{dom}(p_\gamma) = D_\gamma$ and that $\forall \delta \in D [q_\gamma(\delta) = j_\delta(p_\gamma(\delta))]$. Now consider any $\gamma, \gamma^* \in B$ with $\gamma \neq \gamma^*$. By hypothesis, $p_\gamma \perp_{\mathbb{S}} p_{\gamma^*}$. So there exists $\delta \in D$ where $p_\gamma(\delta) \perp_{\mathbb{R}} p_{\gamma^*}(\delta)$. Since $f_{\gamma,\delta} = f_\delta = f_{\gamma^*,\delta}$, Lemma 4.19 implies that $j_\delta(p_\gamma(\delta)) \perp_{\mathbb{Q}} j_\delta(p_{\gamma^*}(\delta))$, in other words that $q_\gamma(\delta) \perp_{\mathbb{Q}} q_{\gamma^*}(\delta)$. Therefore, $q_\gamma \perp_{\mathbb{P}} q_{\gamma^*}$. Thus we have shown that $\langle q_\gamma : \gamma \in B \rangle$ is an uncountable antichain in $\mathbb{P}_{2^{\aleph_0}}$. However, it is proved in [23][Lemma 4.2. (1), pp. 8-9] that \mathbb{P}_κ is c.c.c. for all $\kappa \geq \omega$. This contradiction concludes the proof. \dashv

Theorem 4.21. *Let \mathbf{V} be a transitive model of a sufficiently large fragment of ZFC. In \mathbf{V} , suppose that $E : 2^{\aleph_0} \rightarrow \omega^\omega$ is a bijection. Let G be a $(\mathbf{V}, \mathbb{S}_E^{\mathbf{V}})$ -generic filter. Then in $\mathbf{V}[G]$ there exists a family \mathcal{D} of at most \aleph_1 many G_δ subsets of ω^ω such that for any $A \in \mathbf{V} \cap [\omega^\omega]^{<\aleph_1}$ and any $x \in (\mathbf{V} \cap \omega^\omega) \setminus A$, there exists $D \in \mathcal{D}$ with $x \in D$ and $A \cap D = \emptyset$.*

Proof. Note that since \mathbb{S}_E is c.c.c. in \mathbf{V} , all cardinals are preserved. For $p \in \mathbb{S}_E$ and $\delta \in \text{dom}(p)$, write $p(\delta) = \langle f_{p,\delta}, g_{p,\delta}, h_{p,\delta} \rangle$ instead of $\langle f_{p(\delta)}, g_{p(\delta)}, h_{p(\delta)} \rangle$. For each indecomposable $\delta < \omega_1$, define $G(\delta) = \{p(\delta) : p \in G\}$. Then $G(\delta)$ is a $(\mathbf{V}, \mathbb{R}_{E,\delta}^{\mathbf{V}})$ -generic filter, and let $\mathcal{F}_{G(\delta)}^{\mathbf{V}[G(\delta)]}$, $\mathcal{G}_{G(\delta)}$, and $\mathcal{H}_{G(\delta)}$ be as in definition 4.16. By Lemma 4.17, $\mathcal{F}_{G(\delta)}^{\mathbf{V}[G(\delta)]}$ is a G_δ subset of $(\omega^\omega)^{\mathbf{V}[G(\delta)]}$ such that $\mathcal{H}_{G(\delta)} \subseteq \mathcal{F}_{G(\delta)}^{\mathbf{V}[G(\delta)]}$ and $\mathcal{G}_{G(\delta)} \cap \mathcal{F}_{G(\delta)}^{\mathbf{V}[G(\delta)]} = \emptyset$. Define $D_\delta = \mathcal{F}_{G(\delta)}^{\mathbf{V}[G]}$. Then D_δ is a G_δ subset of $(\omega^\omega)^{\mathbf{V}[G]}$ and we claim that the family $\mathcal{D} = \{D_\delta : \delta < \omega_1 \wedge \delta \text{ is indecomposable}\}$ has the required properties. Working in \mathbf{V} , fix $A \in [\omega^\omega]^{<\aleph_1}$ and $x \in \omega^\omega \setminus A$. Let

$B = E^{-1}(A) \in [2^{\aleph_0}]^{<\aleph_1}$ and let $\alpha \in 2^{\aleph_0}$ be unique such that $E(\alpha) = x$. Note that $\alpha \notin B$. Define

$$D_{A,x} = \{p \in \mathbb{S}_E : \exists \delta \in \text{dom}(p) [\alpha \in h_{p,\delta} \wedge B \subseteq \text{dom}(g_{p,\delta})]\}.$$

We will verify that $D_{A,x}$ is dense in \mathbb{S}_E . To see this consider $p^* \in \mathbb{S}_E$ and find $\delta < \omega_1$ such that δ is indecomposable, $\delta \notin \text{dom}(p^*)$, and $\text{otp}(B) < \delta$. Define $f_{p,\delta} = \emptyset$. Define $g_{p,\delta}$ to be the function such that $\text{dom}(g_{p,\delta}) = B$, and $\forall \beta \in B [g_{p,\delta}(\beta) = \emptyset]$. Define $h_{p,\delta} = \{\alpha\}$. Then $\langle f_{p,\delta}, g_{p,\delta}, h_{p,\delta} \rangle \in \mathbb{R}_{E,\delta}$. Define $p = p^* \cup \{\langle \delta, \langle f_{p,\delta}, g_{p,\delta}, h_{p,\delta} \rangle \rangle\}$. Then $p \in D_{A,x}$ and $p \leq_{\mathbb{S}} p^*$, showing that $D_{A,x}$ is dense. Since G is $(\mathbf{V}, \mathbb{S}_E)$ -generic, there exists $p \in G \cap D_{A,x}$. Let $\delta \in \text{dom}(p)$ be so that $\alpha \in h_{p,\delta}$ and $B \subseteq \text{dom}(g_{p,\delta})$. Then by definition $x \in \mathcal{H}_{G(\delta)}$ and $A \subseteq \mathcal{G}_{G(\delta)}$, whence $x \in \mathcal{F}_{G(\delta)}^{\mathbf{V}[G(\delta)]}$ and $A \cap \mathcal{F}_{G(\delta)}^{\mathbf{V}[G(\delta)]} = \emptyset$. Since this is absolute, $x \in D_\delta$ and $A \cap D_\delta = \emptyset$, as required. \dashv

Theorem 4.22. *Suppose \mathbf{V}_0 is any transitive model of a sufficiently large fragment of ZFC. In \mathbf{V}_0 , suppose that $\aleph_1 \leq \lambda \leq \kappa$ are cardinals so that λ is regular and $\kappa^{\aleph_0} = \kappa$. Then there is a c.c.c. forcing extension in which all of the following hold:*

- (1) $2^{\aleph_0} = \kappa$;
- (2) for every locally countable Borel quasi order \mathcal{P} , either $\text{odim}_B(\mathcal{P}) \leq \aleph_0$ or $\text{odim}_B(\mathcal{P}) = \lambda$;
- (3) for every dense selector E and every $f \in \mathcal{F}$, $\bar{\chi}_B^w(\mathcal{G}_0(E, f)) = \lambda$.

Proof. First, produce a c.c.c. forcing extension $\mathbf{V} \supseteq \mathbf{V}_0$ so that $2^{\aleph_0} = \kappa$ holds in \mathbf{V} (e.g. force with $\text{Fn}(\kappa \times \omega, 2)$ over \mathbf{V}_0 , see [24][Lemma 5.14]). Now work in \mathbf{V} and define a finite support iteration $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha : \alpha \leq \lambda \rangle$ of c.c.c. forcings such that for each $\alpha < \lambda$, $\dot{\mathbb{Q}}_\alpha$ is a full \mathbb{P}_α -name such that

$$\Vdash_{\mathbb{P}_\alpha} \exists E [E : 2^{\aleph_0} \rightarrow \omega^\omega \text{ is a bijection and } \dot{\mathbb{Q}}_\alpha = \mathbb{S}_E].$$

By Lemma 4.20, $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is c.c.c., for every $\alpha < \lambda$, so \mathbb{P}_λ is c.c.c. and all cofinalities and cardinals are preserved. Further, for each $\alpha < \lambda$, $\Vdash_{\mathbb{P}_\alpha} |\dot{\mathbb{Q}}_\alpha| = 2^{\aleph_0}$, so by standard arguments, $\Vdash_{\mathbb{P}_\lambda} 2^{\aleph_0} = \kappa$. Therefore, (1) holds.

Suppose G is a $(\mathbf{V}, \mathbb{P}_\lambda)$ -generic filter. For $\alpha < \lambda$, define $G(\alpha) = \{p \restriction \alpha : p \in G\}$. By Theorem 4.21, there is a family \mathcal{D}_α of at most \aleph_1 many G_δ subsets of $(\omega^\omega)^{\mathbf{V}[G(\alpha+1)]}$ with the property that for every $A \in \mathbf{V}[G(\alpha)] \cap [\omega^\omega]^{<\aleph_1}$ and $x \in (\mathbf{V}[G(\alpha)] \cap \omega^\omega) \setminus A$, there exists $D \in \mathcal{D}_\alpha$ such that $x \in D^{\mathbf{V}[G(\alpha+1)]}$ and $A \cap D^{\mathbf{V}[G(\alpha+1)]} = \emptyset$. Define $\mathcal{D} = \{D^{\mathbf{V}[G]} : \exists \alpha < \lambda [D \in \mathcal{D}_\alpha]\}$. Then \mathcal{D} is a family of at most λ many G_δ subsets of $(\omega^\omega)^{\mathbf{V}[G]}$. If $A \in \mathbf{V}[G] \cap [\omega^\omega]^{<\aleph_1}$ and $x \in (\mathbf{V}[G] \cap \omega^\omega) \setminus A$, then since \mathbb{P}_λ is c.c.c. and $\text{cf}(\lambda) > \omega$, there exists $\alpha < \lambda$ such that $A, x \in \mathbf{V}[G(\alpha)]$, whence for some $D \in \mathcal{D}_\alpha$, $x \in D^{\mathbf{V}[G(\alpha+1)]}$ and $A \cap D^{\mathbf{V}[G(\alpha+1)]} = \emptyset$. Since this is absolute, $x \in D^{\mathbf{V}[G]}$ and $A \cap D^{\mathbf{V}[G]} = \emptyset$.

Now, working in $\mathbf{V}[G]$, let X be an arbitrary Polish space. Fix a closed $F \subseteq \omega^\omega$ and a continuous bijection $\varphi : F \rightarrow X$. For each $L \in \mathcal{D}$, $B_L = \varphi''(F \cap L)$ is a Borel subset of X because φ is 1-1 and continuous. Define

$$<_L = \{\langle x, y \rangle \in X \times X : x \notin B_L \wedge y \in B_L\}.$$

Then $<_L$ is a Borel partial order on X . If $A \in [X]^{<\aleph_1}$ and $u \in X \setminus A$, then $\varphi^{-1}(A) \subseteq F$ and $|\varphi^{-1}(A)| < \aleph_1$. If $x \in F$ is unique with $\varphi(x) = u$, then $x \in \omega^\omega \setminus \varphi^{-1}(A)$, and so, for some $L \in \mathcal{D}$, $x \in L$, but $L \cap \varphi^{-1}(A) = \emptyset$. It follows that $u \in B_L$ and that $A \cap B_L = \emptyset$, whence $\forall v \in A [v <_L u]$. Thus $\{<_L : L \in \mathcal{D}\}$ witnesses $\dagger(X, \aleph_1, \lambda)$ in $\mathbf{V}[G]$.

Now suppose in $\mathbf{V}[G]$ that \mathcal{P} is a locally countable Borel quasi order with $\text{odim}_B(\mathcal{P}) > \aleph_0$. By Lemma 4.7, $\text{odim}_B(\mathcal{P}) \leq \lambda$. On the other hand, by Corollary 3.33, $\text{odim}_B(\mathcal{P}) \geq \text{cov}(\mathcal{M})$. Since \mathbb{P}_λ is a finite support iteration of non-trivial forcing notions, a well-known argument (see [24][Chapter VIII, Exercise J3, pp. 299]) shows that Cohen reals are added at each stage $\alpha < \lambda$ with $\text{cf}(\alpha) = \omega$. Since λ is regular and \mathbb{P}_λ is c.c.c., it follows that $\text{cov}(\mathcal{M}) \geq \lambda$ holds in $\mathbf{V}[G]$. Therefore, $\text{odim}_B(\mathcal{P}) = \lambda$, as required for (2).

Finally, suppose in $\mathbf{V}[G]$ that E is a dense selector and that $f \in \mathcal{F}$. It is clear that $\mathcal{P}(\mathcal{G}_0(E, f))$ is a locally countable Borel quasi order. By (2) and by Lemma 3.30, $\vec{\chi}_B^w(\mathcal{G}_0(E, f)) \leq \text{odim}_B(\mathcal{P}(\mathcal{G}_0(E, f))) \leq \lambda$. On the other hand, by Lemma 3.8, $\lambda \leq \text{cov}(\mathcal{M}) \leq \vec{\chi}_B^w(\mathcal{G}_0(E, f))$. Therefore, $\vec{\chi}_B^w(\mathcal{G}_0(E, f)) = \lambda$. This concludes the proof of the theorem. \dashv

It was mentioned earlier that Kumar and Raghavan [22] showed that \mathcal{D} has the largest order dimension among the locally countable quasi orders of size continuum. We do not know if the Borel analog of this result is provable in ZFC.

Question 4.23. *Is it provable in ZFC that for every locally countable Borel quasi order \mathcal{P} , $\text{odim}_B(\mathcal{P}) \leq \text{odim}_B(\mathcal{D})$?*

It is worth pointing out that Lutz [25] has proved that not all Borel quasi orders are Borel embeddable in \mathcal{D} .

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