CHAINS OF P-POINTS

DILIP RAGHAVAN AND JONATHAN L. VERNER

ABSTRACT. It is proved that the Continuum Hypothesis implies that any sequence of rapid P-points of length $< \mathfrak{c}^+$ which is increasing with respect to the Rudin-Keisler ordering is bounded above by a rapid P-point. This is an improvement of a result from [**?**]. It is also proved that the notion of a δ-generic sequence is equivalent to an apparently much weaker notion. This allows the central definition used in the construction in [**?**] to be considerably simplified.

1. INTRODUCTION

The Rudin-Keisler ordering on ultrafilters, introduced in the late sixties ([**?**]; see also [**?**] and [**?**]), turned out to be a very useful tool for studying properties of ultrafilters. A variant of this ordering, the Rudin-Frolík ordering, was used by Frolík ([**?**]) to prove, in ZFC, that the space of non-principal ultrafilters on ω is non-homogeneous. Many combinatorial properties can be characterized in terms of the ordering, e.g. selective (or Ramsey) ultrafilters are precisely those which are minimal in the Rudin-Keisler ordering, Q-points are those that are minimal in the Rudin-Blass ordering, P-points are those where the Rudin-Keisler and Rudin-Blass orderings coincide.

The first comprehensive study of the Rudin-Keisler order was done by A. Blass in his thesis [**?**]. A. Blass continued his investigations by considering the lower part of the ordering viz. the ordering of P-points [**?**]. He showed that, under suitable assumptions, the ordering can be very rich. Assuming Martin's Axiom (MA), he showed that

- there are $2^{\mathfrak{c}}$ many minimal P-points
- there are no maximal P-points
- the ordering of P-points is σ-closed, both downwards and upwards
- the real line as well as ω_1 can be embedded into the P-points

These results were later extended by several authors (e.g. [**?**], [**?**], [**?**]). The results that motivated the research that went into this paper were obtained by B. Kuzeljević and D. Raghavan [**?**]. They showed

Theorem (Kuzeljević and Raghavan)**.** *Assume* MA*. The ordinal $\mathfrak{c}^+$ can be embedded into the ordering of (rapid) P-points.*

Since any ultrafilter has at most $\mathfrak{c}$-many RK-predecessors, the above is the best possible result as far as embedding of ordinals is concerned. The authors of [**?**] used the notion of a δ-generic sequence of P-points (see Definition 5.2) which allowed them to carry through an inductive construction of length $\mathfrak{c}^+$.

In this paper we improve upon their results in two ways. In the first part of the paper we show that, assuming the Continuum Hypothesis (CH), the ordering of *rapid* P-points is, in fact, $\mathfrak{c}^+$-closed:

Theorem. *Assume CH. Any increasing sequence of rapid P-points of length $< \mathfrak{c}^+$ is bounded above by a rapid P-point.*

2010 *Mathematics Subject Classification.* Primary 03E50, 03E05, 54D80.

Key words and phrases. Rudin-Keisler order, ultrafilter, P-point.

The second author was supported by the joint FWF-GAČR grant no. 17-33849L: Filters, ultrafilters and connections with forcing, by the Progres grant Q14. Krize racionality a moderní myšlení and by ...

Unlike many earlier results, this theorem is more than just an embedding result, for it provides new information about the global structure of the class of rapid P-points under the Rudin-Keisler ordering.

In the second part we show that Kuzeljević and Raghavan's notion of a generic sequence is equivalent to the notion of a weakly-generic sequence which, on its face, is a much weaker property.

We also show that the fact that we are looking at *rapid* P-points is crucial. Assuming $\diamondsuit$ (though we suspect that CH is enough), we construct an increasing sequence of P-points of length ω_1 without an upper bound.

The chains of P-points of length $\mathfrak{c}^+$ constructed in [**?**] enjoy a slightly stronger property than the long chains that can be built using the technique from Section 3 of this paper. The chains of [**?**] are all increasing in the $\leq^+_{\mathrm{RB}}$ ordering, but our technique is insufficient to ensure this property for any of the chains of length $\mathfrak{c}^+$ here. Thus the existence statement proved in [**?**] is stronger than the existence result that is implicit in the work in Section 3.

We should also comment on our assumptions. Since S. Shelah ([**?**]) showed that P-points need not exist at all, or there might be, e.g., just one (see Chapter VI of [**?**]), some assumptions which guarantee that the structure is rich is needed. For simplicity we use CH though we strongly suspect that a weaker assumption, e.g. MA (or even just $\mathfrak{b} = \mathfrak{c}$), would be sufficient.

2. Preliminaries

In this section we introduce the basic notions and state some standard facts.

Definition ([**?**]). An ultrafilter $\mathcal{U}$ on ω is a *P-point* provided that for any sequence $\langle X_n : n < \omega\rangle$ of elements of $\mathcal{U}$ there is an $X \in \mathcal{U}$ such that $|X \setminus X_n| < \omega$ for each n. The last condition will also be denoted by $X \subseteq^* X_n$.

The following is an alternate characterization which we will often use:

Fact. *An ultrafilter $\mathcal{U}$ is a P-point iff every function $f : \omega \to \omega$ is either constant or finite-to-one on some set in $\mathcal{U}$.*

Definition ([**?**]). An ultrafilter $\mathcal{U}$ on ω is *rapid* if for every $f : \omega \to \omega$ there is an $X \in \mathcal{U}$ such that its increasing enumeration e_X eventually dominates f, i.e. $(\forall^\infty n)(f(n) \leq e_X(n))$. To make notation simpler we will write $X(n)$ instead of $e_X(n)$ to denote the n-th element of X and $X[n] = X \setminus X(n)$.

Again we we will use an alternate characterization:

2.1. **Fact.** *An ultrafilter $\mathcal{U}$ is rapid iff for every partition $\{K_n : n < \omega\}$ of ω into finite sets there is $X \in \mathcal{U}$ such that $|X \cap K_n| \leq n$.*

Definition ([**?**]). The *Rudin-Keisler ordering* of ultrafilters is defined as follows. Given two ultrafilters $\mathcal{U}, \mathcal{V}$ on ω we say that $\mathcal{U}$ is *Rudin-Keisler below* (or that it is *Rudin-Keisler reducible* to) $\mathcal{V}$, denoted $\mathcal{U} \leq_{RK} \mathcal{V}$, if there is a function $f : \omega \to \omega$ such that

$$\mathcal{U} = f_*(\mathcal{V}) = \{X \subseteq \omega : f^{-1}[X] \in \mathcal{V}\}.$$

If the function is finite-to-one we say that $\mathcal{U}$ is *Rudin-Blass* below $\mathcal{V}$, $\mathcal{U} \leq_{RB} \mathcal{V}$. If the function is both finite-to-one *and nondecreasing* we write $\mathcal{U} \leq^+_{RB} \mathcal{V}$.

More information about the $\leq^+_{\mathrm{RB}}$ ordering on the ultrafilters can be found in [**?**]. A major difference between the $\leq_{\mathrm{RB}}$ and $\leq^+_{\mathrm{RB}}$ orderings, which was discovered by Laflamme and Zhu in [**?**], is that $\leq^+_{\mathrm{RB}}$ is a tree-like ordering. In other words, for any ultrafilters $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$, if $\mathcal{U} \leq^+_{\mathrm{RB}} \mathcal{W}$ and $\mathcal{V} \leq^+_{\mathrm{RB}} \mathcal{W}$, then either $\mathcal{U} \leq^+_{\mathrm{RB}} \mathcal{V}$ or $\mathcal{V} \leq^+_{\mathrm{RB}} \mathcal{U}$. This is very much false for the $\leq_{\mathrm{RB}}$ ordering even when it is restricted to the class of P-points, as was shown by Blass [**?**] who constructed a P-point with two incomparable predecessors assuming MA.

It is easy to see that being rapid and being a P-point is preserved when going down in the Rudin-Keisler ordering. Also, since Rudin-Keisler reducibility has to be witnessed by some function f :

$\omega \to \omega$ and since two RK-inequivalent ultrafilters can't be witnessed to be below a third by a single function it immediately follows that every ultrafilter has at most $\mathfrak{c}$-many RK-predecessors.

Another ordering of ultrafilters is the Tukey ordering. It was introduced by Tukey in [**?**] for comparing the cofinal type of arbitrary directed partial orders. Isbell ([**?**]) was the first who used the Tukey ordering to compare ultrafilters.

Definition ([**?**])**.** Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. We say that $\mathcal{U} \le_T \mathcal{V}$, i.e. $\mathcal{U}$ is *Tukey reducible* to $\mathcal{V}$ or $\mathcal{U}$ is *Tukey below* $\mathcal{V}$, if there is a map $\phi : \mathcal{V} \to \mathcal{U}$ such that $\forall A, B \in \mathcal{V} \left[A \subseteq B \implies \phi(A) \subseteq \phi(B)\right]$ and $\forall A \in \mathcal{U} \exists B \in \mathcal{V} \left[\phi(B) \subseteq A\right]$. We say that $\mathcal{U} \equiv_T \mathcal{V}$, i.e. $\mathcal{U}$ is *Tukey equivalent* to $\mathcal{V}$, if $\mathcal{U} \le_T \mathcal{V}$ and $\mathcal{V} \le_T \mathcal{U}$.

Recently the interest in this ordering on ultrafilters was revived by the paper [**?**]. See also [**?**] and [**?**].

Definition. Given a family $\mathscr{P}$ of functions from ω to ω we say that a function $f : \omega \to \omega$ *dominates* $\mathscr{P}$ if $g \le^* f$ for each $g \in \mathscr{P}$, where

$$g \le^* f \iff (\forall^\infty n)(g(n) \le f(n)),$$

and where $\forall^\infty n$ is a shortcut for "for all but finitely many n".

Finally, to eliminate some extraneous brackets, we will use the convenient standard shorthand $f^{-1}(n)$ to denote the preimage of $\{n\}$ instead of the formally more correct $f^{-1}[\{n\}]$. In the rest of the paper we assume CH through-out. In particular we will talk about ω_2 instead of $\mathfrak{c}^+$.

3. There is no short unbounded chain of rapid P-points

We aim to show that each RK-increasing chain of rapid P-points of length $< \omega_2$ has a rapid P-point on top. We do this by taking the chain and recursively constructing the future projections (called g in the following proposition) from the top to each ultrafilter in the sequence. If these projections commute with each of the maps witnessing the RK-relations in the chain, then the inverse images of the chain by these projections will generate a P-filter. By a relatively easy argument we can guarantee that it will be an ultrafilter (making sure that at each step we decide one set). To make it rapid we have to work more. For this purpose, we will also build a tower on the side (the Ts in the following proposition), which will generate a rapid P-point and, moreover, this P-point will be compatible with the final P-filter.

We start with a few simple observations. Below we use $Poly$ to denote the following set of polynomial functions[1]: $\{n^k : k < \omega\}$.

Observation. *If $f : \omega \to \omega$ dominates $Poly$ then so does $f'(n) = \frac{f(n)}{2n} - n^2$.*

It is easy to see that the function n in Fact 2.1 could as well have been replaced by any function tending to infinity:

3.1. **Observation.** *If $s : \omega \to \omega$ is a function tending to infinity (i.e. $\liminf s(n) = \infty$), $\pi : \omega \to \omega$ is a finite to one function, and $\mathcal{U}$ is a rapid ultrafilter then there is $X \in \mathcal{U}$ such that $(\forall n < \omega)(|\pi^{-1}(n) \cap X| \le s(n))$*

The following proposition is our main tool which we use to build the RK-maps (the g_αs) from the future upper bound of our ω_1-length sequence (the $\mathcal{U}_\alpha$s). The upper bound will be generated by a tower (the T_αs) so that it will be a P-filter. The set A in the assumption will be later used to make sure that the top filter is both an ultrafilter and rapid. The key property that will keep the induction going will be the fact that the g_αs are finite to one but *not* bounded-to-one in a very strong sense: the size of the preimages of points (we will call this somewhat imprecisely the *growth rate* of g) will dominate a function s which will in turn dominate the set $Poly$ (conditions (1&2)). Moreover the

[1] The fact that they are polynomials is not important. We could as well have chosen all functions of some countable elementary submodel of the universe; all that we need is that each function grows much faster than the previous one.

first part of condition (4) will guarantee that the maps g_α will not be bijections on some large set (otherwise our supposed upper bound would be RK-equivalent to some $\mathscr{U}_\alpha$).

Because a lot of maps between (different copies of) ω are involved we will use the following convention to hopefully make it easier for the reader to see what is going on. We imagine each $\mathscr{U}_\alpha$ lives on a separate copy of ω (a level). We will use the letter m to denote numbers on the first level (i.e. where $\mathscr{U}_0$ lives), the letter n will denote numbers living on some level $0 < \alpha < \delta$, the letter l will denote numbers living on the final level (i.e. where the top ultrafilter we will be constructing lives) and the letter k will denote numbers living on level δ. The letters i and j will be used as unrelated natural numbers.

Proposition. *Assume $\delta < \omega_1$ and $\langle U_\alpha : \alpha \le \delta\rangle$ is an RK-increasing sequence of rapid P-points as witnessed by finite-to-one maps $\Pi = \{\pi_{\alpha\beta} : \beta \le \alpha \le \delta\}$ with $\pi_{\alpha\alpha} = Id$ for each $\alpha \le \delta$. Also, let $\bar{s} = \langle s_\alpha : \alpha < \delta\rangle$ be a sequence of maps, each dominating $Poly$, $\bar{g} = \langle g_\alpha : \alpha < \delta\rangle$ a sequence of finite-to-one maps, $\bar{T} = \langle T_\alpha : \alpha < \delta\rangle$ a $\subseteq^*$-decreasing sequence of subsets of ω, $A \subseteq \omega$. Suppose, moreover, that the following conditions are satisfied:*

(1) the growth rate of g_α dominates s_α, i.e. $(\forall\alpha)(\forall^\infty n)(|g_\alpha^{-1}[\{n\}]| \ge s_\alpha(\pi_{\alpha 0}(n)))$;
(2) the sequence $\bar{s}$ is $<^*$ decreasing in the following (stronger) sense:

$$(\forall \alpha < \beta < \delta)(\forall^\infty m)\left(s_\alpha(m) \ge \frac{s_\alpha(m)}{2m} - m^2 \ge s_\beta(m)\right);$$

(3) $\Pi \cup \{g_\alpha : \alpha < \delta\}$ commute, i.e. for all $\beta \le \alpha < \delta$ there is $X \in \mathscr{U}_\alpha$ such that $g_\beta(l) = \pi_{\alpha\beta}(g_\alpha(l))$ for each $l \in g_\alpha^{-1}[X]$; and
(4) for each $\alpha < \delta$ there is $X \in \mathscr{U}_\alpha$ such that

$$\lim_{n \in X} |g_\alpha^{-1}(n) \cap T_\alpha| = \infty;$$

while also

$$|g_\alpha^{-1}(n) \cap T_\alpha| \le \sqrt{\min g_\alpha^{-1}(n) \cup \{\pi_{\alpha 0}(n)\}},$$

for each $n \in X$.

Then we can extend the sequence $\bar{g}, \bar{s}, \bar{T}$ by constructing the maps g_δ and s_δ and a set T_δ so that (corresponding modifications of) (1-4) are still satisfied and, moreover, $(\forall i)(T_\delta(i) \ge A(i))$ (recall that when X is a set, $X(i)$ denotes the i-th element of X) and T_δ decides A (i.e. $T_\delta \subseteq A$ or $T_\delta \cap A = \emptyset$).

Proof. We first introduce some notation. Fix $D \subseteq \delta$ a cofinal subset of δ of order type ω such that $0 \in D$. In our construction we will only deal with $\alpha \in D$. Given $\alpha \in D$ we write

$$\alpha^+ = \min\{\beta \in D : \alpha < \beta\}$$

for the successor of α in D. We also let

$$\#\alpha = |D \cap \alpha|,$$

i.e. α is the $\#\alpha$-th element of D. Next we use D to enumerate $Poly$ in an increasing sequence: $Poly = \{p_\alpha : \alpha \in D\}$, where $p_\alpha \le p_{\alpha^+}$.

Given $\alpha \in D$ let $c_n^\alpha = \pi_{\delta\alpha}^{-1}(n)$. Since $\mathscr{U}_\delta$ is rapid we can use Fact 2.1 to find $X_\alpha \in \mathscr{U}_\delta$ such that $|X_\alpha \cap c_n^\alpha| \le n$ for each $n < \omega$. We can also assume that[2]

$$(*) \qquad |g_\alpha^{-1}(n)| \ge s_\alpha(\pi_{\alpha 0}(n))$$

for all n such that $X_\alpha \cap c_n^\alpha \ne \emptyset$ and that if $n \in \pi_{\delta\alpha}[X_\alpha]$ then

$$(\dagger) \qquad |g_\alpha^{-1}(n) \cap T_\alpha| \le \sqrt{\min g_\alpha^{-1}(n) \cup \{\pi_{\alpha 0}(n)\}}$$

Next, choose $Y_\alpha \in \mathscr{U}_\delta$ such that

$$\{g_\beta : \beta \in D \ \&\ \beta \le \alpha^+\} \cup \{\pi_{\gamma\beta} : \gamma \le \beta \le \alpha^+, \gamma, \beta \in D\}$$

[2]otherwise throw finitely many elements of X_α away to get the first requirement and for the second intersect it with the set $\pi_{\delta\alpha}^{-1}[X]$, where X is the set guaranteed to exist by condition (4) above.

commute on Y_α; more precisely, for every $\beta \le \gamma \le \alpha^+$ all in D and every $k \in Y_\alpha$ and any $l \in g_\gamma^{-1}[\pi_{\delta\gamma}(k)]$ we have

$$\pi_{\gamma\beta}(\pi_{\delta\gamma}(k)) = \pi_{\delta\beta}(k) = g_\beta(l). \tag{‡}$$

Finally, since g_α and T_α satisfy the first part of (4), we can use Observation 3.1 to find $Z_\alpha \in \mathscr{U}_\delta$ such that

$$(\forall n)\Big(|\pi_{\delta\alpha}^{-1}(n) \cap Z_\alpha| \le \frac{|g_\alpha^{-1}(n) \cap T_\alpha|}{\#\alpha}\Big), \tag{§}$$

i.e. Z_α is $\#\alpha$-times more sparse then T_α. (Just apply Observation 3.1 to $\pi = \pi_{\delta\alpha}$, $\mathscr{U} = \mathscr{U}_\delta$, and $s(n) = |g_\alpha^{-1}(n) \cap T_\alpha|/\#\alpha$.)

Since $\mathscr{U}_\delta$ is a P-point, we can find $X \in \mathscr{U}_\delta$ which is a pseudointersection of $\{X_\alpha, Y_\alpha, Z_\alpha : \alpha \in D\}$. Recursively construct a partition $\{K_\alpha : \alpha \in D\}$ of X into finite sets and $s_\delta : \omega \to \omega$ such that

$$K_\alpha \subseteq \bigcap_{\beta \in D \cap \alpha^+} X_\beta \cap Y_\beta \cap Z_\beta,$$

and

(5) $s_\alpha(m) \ge s_\alpha(m)/m - m^2 \ge s_\delta(m) \ge p_\alpha(m)$ whenever $m = \pi_{\delta 0}(n)$ and $n \in K_\alpha$;
(6) $\pi_{\delta\alpha}^{-1}[\pi_{\delta\alpha}[K_\alpha]] \cap X \subseteq K_\alpha$; and
(7) $\#\alpha \le |A \cap \min \pi_{\delta 0}[K_\alpha]|$.

This is not hard to do: first find an increasing sequence $\{k_\alpha : \alpha \in D\}$ of natural numbers such that

$$X \setminus k_\alpha \subseteq \bigcap_{\beta \in D \cap \alpha^+} X_\beta \cap Y_\beta \cap Z_\beta;$$

and

$$s_\alpha(m) \ge \frac{s_\alpha(m)}{m} - m^2 \ge p_{\alpha^+}(m) \quad \& \quad \#\alpha \le |A \cap n|$$

for all $m = \pi_{\alpha 0}(n)$ with $n \in X \setminus k_\alpha$. Then let

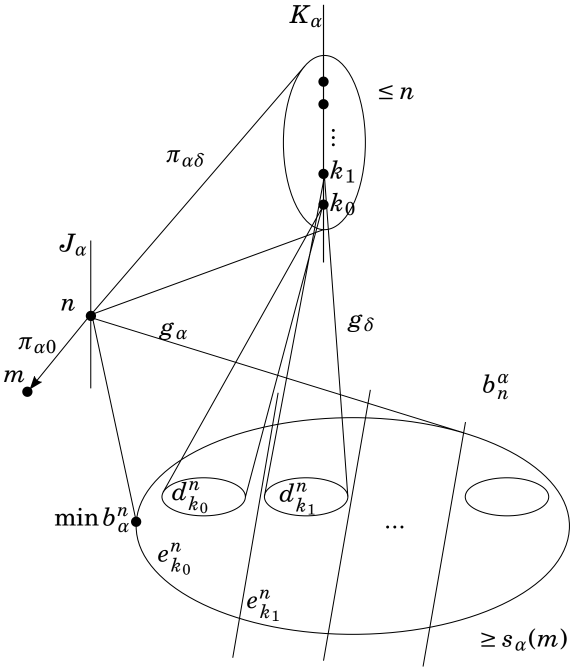
$$K_\alpha = \pi_{\delta\alpha}^{-1}[\pi_{\delta\alpha}[X \cap [k_\alpha, k_{\alpha^+})]$$

and

$$s_\delta \restriction \pi_{\delta 0}[K_\alpha] = p_\alpha.$$

(Formally, this won't be a partition, since it will not cover $X \cap [0, k_0)$; we can just throw these finitely many elements out of X).

Let $J_\alpha = \pi_{\delta\alpha}[K_\alpha]$ and $L_\alpha = g_\alpha^{-1}[J_\alpha]$. Notice that since $K_\alpha \subseteq Y_\alpha$, we can use (‡) and (6) to conclude that $L_\alpha \cap L_\beta = \emptyset$ for distinct $\alpha \ne \beta \in D$. This allows us to define g_δ separately on each L_α. For $n \in J_\alpha$ let $b_n^\alpha = g_\alpha^{-1}(n)$.



Fix n and let $m = \pi_{\alpha 0}(n)$. Then, since $K_\alpha \subseteq X_\alpha$, by (∗) and (5) we have $|b^\alpha_n| \ge s_\alpha(m) \ge s_\alpha(m)/m - m^2 \ge s_\delta(m)$. Moreover, since $K_\alpha \subseteq Z_\alpha$, by (§) and (†) we have

$$|K_\alpha \cap \pi^{-1}_{\delta\alpha}(n)| \le m,$$

and also

$$|K_\alpha \cap \pi^{-1}_{\delta\alpha}(n)| \le \frac{|b^\alpha_n \cap T_\alpha|}{\#\alpha} \le \frac{\sqrt{\min(b^\alpha_n \cup \{m\})}}{\#\alpha}$$

It follows that we can partition b^α_n into pieces $\{e^n_k : k \in K_\alpha \cap \pi^{-1}_{\delta\alpha}(n)\}$ each of size $\ge s_\delta(m) + m$ which, moreover, satisfy

$$\#\alpha \le |e^n_k \cap T_\alpha| \le \sqrt{\min b^\alpha_n \cup \{m\}}.$$

Let i be the smaller of m and $\min b^\alpha_n$. By (7) we can shrink e^n_k to a smaller set d^n_k (throwing away up to $\sqrt{m}$-many elements of $e^n_k \cap T_\alpha$) such that

$$\#\alpha \le |d^n_k \cap T_\alpha| \le \sqrt{|A \cap i|} \tag{¶}$$

Since we threw away at most $\sqrt{m}$ elements from each e^n_k, we still have $|d^n_k| \ge s_\delta(m)$. Now let $g_\delta[d^n_k] = k$ and extend g_δ to all of ω arbitrarily so that the new values are outside of X and that the requirements on g_δ are satisfied (i.e. that it is finite-to-one, its growth rate is bounded below by s_δ, etc.). This finishes the construction of g_δ. For future reference, let us note that $g_\delta \restriction T_\alpha \cap l$ is at most $\sqrt{|A \cap l|}$-to-one for any $l < \omega$.

Notice that if $\beta < \alpha \in D$, $k \in K_\alpha$ and $g_\delta(l) = k$ then $g_\beta(l) = \pi_{\alpha\beta}(g_\alpha(l))$ and so

$$\pi_{\delta\beta}(k) = \pi_{\alpha\beta}(\pi_{\delta\alpha}(k)) = g_\beta(l)$$

so (3) is satisfied for g_δ. That condition (2) for s_δ is satisfied follows from (5). That condition (1) is satisfied for s_δ and g_δ follows from the construction ($|d^n_k| \ge s_\delta(m)$).

(Formally, we have only checked the conditions for $\beta, \alpha \in D$, but this is clearly enough, since D is cofinal in δ.)

Finally we must construct T_δ. Without loss of generality we may assume that $T_\alpha \subseteq T_\beta$ for $\beta < \alpha \in D$ (otherwise we could have carried out the construction for some finite modifications of T_αs and the resulting T_δ would still work for the original T_αs). Let

$$T' = \bigcup_{\alpha \in D} g^{-1}_\delta[K_\alpha] \cap T_\alpha.$$

Then T' is a pseudointersection of $\{T_\alpha : \alpha \in D\}$. Moreover g_δ was constructed so that (see (¶))

$$\#\alpha \le |g_\delta^{-1}(k) \cap T_\alpha| \le \sqrt{\min\{m\} \cup g_\delta^{-1}(k)}$$

for each $k \in K_\alpha$ and $m = \pi_{\delta 0}(k)$. It follows that T' and g_δ satisfy (4). Consider now the map π which sends $g_\delta[l]$ to l and define $s(l) = \sqrt{|A \cap l|}$. Then by Observation 3.1 there is $X' \in \mathscr{U}_\delta$ contained in X and such that

$$|X' \cap g_\delta[l]| \le \sqrt{|A \cap l|}$$

Let $T'' = g_\delta^{-1}[X']$. Now, since $g_\delta \restriction T' \cap l$ is at most $\sqrt{|A \cap l|}$-to-1, we have

$$|T'' \cap l| \le \sqrt{|A \cap l|} \cdot |X' \cap g_\delta[l]| \le \sqrt{|A \cap l|} \cdot \sqrt{|A \cap l|} \le |A \cap l|$$

Finally notice that

$$X' = g_\delta[T''] = g_\delta[T'' \cap A] \cup g_\delta[T'' \setminus A]$$

so, since $\mathscr{U}_\delta$ is an ultrafilter we can choose $T_\delta \subseteq T''$ which decides A and still satisfies (4). This finishes the proof. □

Theorem. *Assume CH. Every RK-increasing chain of rapid P-points of length ω_1 has a top.*

Proof. Let $\langle \mathscr{U}_\alpha : \alpha < \omega_1 \rangle$ be an RK-increasing chain of rapid P-points as witnessed by finite-to-one maps $\Pi = \{\pi_{\alpha\beta} : \beta \le \alpha < \omega_1\}$. Without loss of generality we may assume that the maps commute in the sense of (3) in the previous proposition. Enumerate $[\omega]^\omega$ as $\{A_\alpha : \alpha < \omega_1\}$. Recursively build a sequence of finite-to-one maps $\langle g_\alpha : \alpha < \omega_1 \rangle$ and a decreasing tower $\langle T_\alpha : \alpha < \omega_1 \rangle$ so that T_α decides A_α and $|T_\alpha \cap n| \le |A \cap n|$. This can be done by repeatedly applying the previous proposition at each step. In the end $\langle T_\alpha : \alpha < \omega_1 \rangle$ generates a rapid P-point and the map g_α witnesses that this P-point is above $\mathscr{U}_\alpha$. □

It is likely that the above proof generalizes to the case when CH is replaced by MA. The method of proof also suggest that it might even be weakened to $\mathfrak{b} = \mathfrak{c}$, however, we have not investigated this in detail and leave it as a question for further research

Question. *Does the theorem hold if we replace CH by $\mathfrak{b} = \mathfrak{c}$? Or even $\mathfrak{d} = \mathfrak{c}$?*

4. A short unbounded chain of P-points

In this section we show, assuming ◇, that there is an RK-chain of P-points which has no P-point RK-above. We assume ◇ only for simplicity; a more involved argument using the Devlin-Shelah weak diamond can be used to construct the chain, e.g., under CH.

Definition. Let $\bar{\mathscr{U}} = \langle \mathscr{U}_\alpha : \alpha < \delta \rangle$ be a sequence of ultrafilters and $\Pi = \{\pi_{\alpha\beta} : \beta \le \alpha \le \delta\}$ an indexed family of maps from ω to ω. We say that Π *commutes with respect to* $\bar{\mathscr{U}}$, if for $\beta \le \alpha \le \gamma \le \delta$ there is $X \in \mathscr{U}_\gamma$ such that $(\forall i \in X)(\pi_{\alpha\beta}(\pi_{\gamma\alpha}(i)) = \pi_{\gamma\beta}(i))$. When the sequence $\bar{\mathscr{U}}$ is clear from the context, we just say that Π commutes. Moreover, given two indexed families Π_0, Π_1 of maps we will write $f : \Pi_0 \to \Pi_1$ to indicate that f is a map from ω to ω and for each $\alpha < \delta$ we have $\pi^0_{\delta\alpha}(n) = \pi^1_{\delta\alpha}(f(n))$ for all $n < \omega$.

Definition. Given two indexed families of maps $\Pi_i = \langle \pi^i_{\delta\alpha} : \alpha < \delta \rangle, i < 2$ we say that $\Pi_1 \prec \Pi_0$ if

$$(\forall \alpha < \delta)(\forall^\infty n < \omega)\big(|(\pi^1_{\delta\alpha})^{-1}(n)| > n \cdot |(\pi^0_{\delta\alpha})^{-1}(n)|\big)$$

Moreover, if π_0, π_1 are two maps, $\mathscr{U}$ is an ultrafilter, and $X \in [\omega]^\omega$, we write

$$\pi_1 \prec_{X,\mathscr{U}} \pi_0$$

if there is a $Y \in \mathscr{U}$ and $s : \omega \to \omega$ tending to infinity such that

$$(\forall n \in Y)\big(|\pi_1^{-1}(n) \cap X| > s(n) \cdot |\pi_0^{-1}(n) \cap X|\big).$$

4.1. **Observation.** *Assume $\bar{\mathscr{U}} = \langle U_\alpha : \alpha < \delta \rangle$ is an RK-increasing chain of P-points of length $\delta < \omega_1$ as witnessed by a family of finite-to-one maps Π. Suppose, moreover, that we are given a family of finite-to-one maps $\Pi_0 = \langle \pi^0_{\delta\alpha} : \alpha < \delta \rangle$ such that $\Pi \cup \Pi_0$ commute. Then there is an indexed family $\Pi_1 \prec \Pi_0$ such that $\Pi \cup \Pi_1$ still commutes.*

Proof. Fix an arbitrary finite-to-one π such that $|\pi^{-1}(n)]| \geq n$ and let $\pi^1_{\delta\alpha}(n) = \pi^0_{\delta\alpha}(\pi(n))$. □

Definition. Given an RK-increasing chain of P-points $\bar{\mathscr{U}}$ of length δ for some limit $\delta < \omega_1$ a witnessing indexed family of maps Π and two indexed families $\Pi_1 \prec \Pi_0$ of finite-to-one maps such that $\Pi \cup \Pi_i$ commutes for $i < 2$, we define the forcing

$$\mathbb{P}(\bar{\mathscr{U}}, \Pi_0, \Pi_1) = \left(\left\{X \in [\omega]^\omega : (\forall \alpha < \delta)(\pi^1_{\delta\alpha} \prec_{X,\mathscr{U}_\alpha} \pi^0_{\delta\alpha})\right\}, \subseteq^*\right)$$

Observation. *The forcing* $\mathbb{P}(\bar{\mathscr{U}}, \Pi_0, \Pi_1)$ *contains* ω.

4.2. **Proposition.** *The forcing* $\mathbb{P}(\bar{\mathscr{U}}, \Pi_0, \Pi_1)$ *is* σ*-closed.*

Proof. Let $\langle X_n : n < \omega\rangle$ be a descending sequence of conditions and, without loss of generality, assume $X_{n+1} \subseteq X_n$ for all $n < \omega$. Fix a cofinal subset $D \subseteq \delta$ of order type ω and, as before, write $\alpha^+ = D \setminus (\alpha + 1)$ and $\#\alpha = |D \cap \alpha|$. For each $n < \omega$ and $\alpha \in D$ fix $Y^\alpha_n \in \mathscr{U}_\alpha$ and s^α_n witnessing $\pi^1_{\delta\alpha} \prec_{X,\mathscr{U}_\alpha} \pi^0_{\delta\alpha}$. Then for each $\alpha \in D$ let $Y^\alpha \in \mathscr{U}_\alpha$ be a pseudointersection of $\{Y^\alpha_n : n < \omega\} \subseteq \mathscr{U}_\alpha$ and fix a function $s : \omega \to \omega$, tending to infinity, such that $s \leq^* s^\alpha_n$ for all $\alpha \in D, n < \omega$. For each $\alpha \in D$ fix $m_\alpha < \omega$ such that

$$\left|(\pi^1_{\delta\beta})^{-1}(m) \cap X_{\#\alpha}\right| \geq s(m) \cdot \left|(\pi^0_{\delta\beta})^{-1}(m) \cap X_{\#\alpha}\right|,$$

for each $\beta \in D \cap \alpha^+$ and $m \in Y^\alpha \setminus m_\alpha$. We also choose each m_α large enough to make sure that

$$(*) \qquad \max\left((\pi^i_{\delta\beta'})^{-1}[m']\right) < \min\left((\pi^j_{\delta\beta})^{-1}[\{m\}]\right)$$

for each $\beta, \beta' \in D \cap \alpha^+$, $i, j < 2$ and $m' < m_\alpha < m_{\alpha+1} < m$. For $\alpha \in D$ let

$$X^\alpha = \bigcup_{\beta \in D \cap \alpha^+} (\pi^1_{\delta\beta})^{-1}[m_\alpha, m_{\alpha^+}) \cap X_{\#\alpha}$$

Next, for $i < 2$, let $D_i = \{\alpha^{+i} : \alpha \in D\}$ and choose a cofinal $D' \subseteq D$ and $i < 2$ so that

$$Z^\alpha = \bigcup_{\beta \in D_i} [m_\beta, m_{\beta^+}) \cap Y^\alpha \in \mathscr{U}_\alpha.$$

Finally, define

$$X = \bigcup_{\alpha \in D_i} X^\alpha.$$

By $(*)$ it is clear that

$$(\dagger) \qquad X \cap (\pi^j_{\delta\beta})^{-1}[m_\alpha, m_{\alpha^+}) = X_{\#\alpha} \cap (\pi^j_{\delta\beta})^{-1}[m_\alpha, m_{\alpha^+})$$

for each $\alpha \in D_i$, $\beta < \alpha$, and $j < 2$. It is clear that X is a pseudointersection of the X_ns. We need to show that $X \in \mathbb{P}(\bar{\mathscr{U}}, \Pi_0, \Pi_1)$, i.e. that for each $\alpha < \delta$ we have

$$(\ddagger) \qquad \pi^1_{\delta\alpha} \prec_{X,\mathscr{U}_\alpha} \pi^0_{\delta\alpha}.$$

First assume $\alpha \in D'$. We show that $Z^\alpha \setminus m_\alpha$ witnesses $(\ddagger)$. Let $m \in Z^\alpha \setminus m_\alpha$ be arbitrary. Find $\alpha' \in D_i$ so that $m \in [m_{\alpha'}, m_{\alpha'^+})$. Then $\alpha < \alpha'$ so, in particular, we have

$$\left|(\pi^1_{\delta\alpha})^{-1}(m) \cap X_{n(\alpha')}\right| \geq s(m) \cdot \left|(\pi^0_{\delta\alpha})^{-1}(m) \cap X_{n(\alpha')}\right|$$

This, together with $(\dagger)$, shows $(\ddagger)$. Finally notice that if $\alpha < \beta < \alpha'$ and

$$\pi^1_{\delta\alpha} \prec_{X,\mathscr{U}_\alpha} \pi^0_{\delta\alpha} \;\&\; \pi^1_{\delta\alpha'} \prec_{X,\mathscr{U}_{\alpha'}} \pi^0_{\delta\alpha'}$$

then also

$$\pi^1_{\delta\beta} \prec_{X,\mathscr{U}_\beta} \pi^0_{\delta\beta}.$$

Since D' was cofinal in δ this finishes the proof of $(\ddagger)$ for all $\alpha < \delta$. □

Proposition. *If* $A \subseteq \omega$ *then the set*

$$D_A = \left\{X \in \mathbb{P}(\bar{\mathscr{U}}, \Pi_0, \Pi_1) : X \subseteq A \vee X \subseteq \omega \setminus A\right\}$$

is dense.

Proof. Notice that if

$$\pi^1_{\delta\alpha} <_{X,\mathcal{U}_\alpha} \pi^0_{\delta\alpha},$$

then either

$$\pi^1_{\delta\alpha} <_{X\cap A,\mathcal{U}_\alpha} \pi^0_{\delta\alpha},$$

or

$$\pi^1_{\delta\alpha} <_{X\setminus A,\mathcal{U}_\alpha} \pi^0_{\delta\alpha}.$$

This follows from the fact that either

$$|(\pi^1_{\delta\alpha})^{-1}(m)\cap X\cap A| \geq \frac{|(\pi^1_{\delta\alpha})^{-1}(m)\cap X|}{2}$$

or

$$|(\pi^1_{\delta\alpha})^{-1}(m)\cap X\setminus A| \geq \frac{|(\pi^1_{\delta\alpha})^{-1}(m)\cap X|}{2}$$

for $\mathcal{U}_\alpha$-many ms and that if s tends to infinity then so does $s/2$. The result then immediately follows because one of the two cases has to happen for cofinally many $\alpha<\delta$. □

4.3. **Proposition.** *If $\Pi_1 < \Pi_0$ and $f : \Pi\cup\Pi_0 \to \Pi\cup\Pi_1$ is finite-to-one, then the set*

$$D_f = \{X \in \mathbb{P}(\bar{\mathcal{U}},\Pi_0,\Pi_1) : (\exists Y\in\mathcal{U}_0)(X\cap f[(\pi^0_{\delta 0})^{-1}[Y]] = \emptyset\}$$

is dense.

Proof. Let $X\in\mathbb{P}(\bar{\mathcal{U}},\Pi_0,\Pi_1)$. For each $\alpha<\delta$ choose $Y_\alpha\in\mathcal{U}_\alpha$ and $s_\alpha:\omega\to\omega$ tending to infinity witnessing $\pi^1_{\delta\alpha} <_{X,\mathcal{U}_\alpha} \pi^0_{\delta\alpha}$. We may also assume that $\pi^1_{\delta\alpha}(f(k)) = \pi^0_{\delta\alpha}(k)$ for all $k\in(\pi^0_{\delta\alpha})^{-1}[Y_\alpha]$; $\pi^1_{\alpha 0}(\pi^1_{\delta\alpha}(k)) = \pi^1_{\delta 0}(k)$ for each $k\in(\pi^1_{\delta\alpha})^{-1}[Y_\alpha]$ and $\alpha<\delta$; and $Y_\alpha\subseteq(\pi^1_{\alpha 0})^{-1}[Y_0]$. Since $\mathcal{U}_0$ is a P-point, there is a $Y\in\mathcal{U}_0$ which is a pseudointersection of $\pi^1_{\alpha 0}[Y_\alpha]$ and let $n_\alpha<\omega$ be such that $\pi^1_{\alpha 0}[Y_\alpha\setminus n_\alpha]\subseteq Y$. Also write $Z=(\pi^0_{\delta 0})^{-1}[Y]$. Let

$$X' = \left((\pi^1_{\delta 0})^{-1}[Y]\cap X\right)\setminus f[Z\cap X]$$

Notice that for each $n\in Y_\alpha\setminus n_\alpha$ we have

$$f^{-1}\left[(\pi^1_{\delta\alpha})^{-1}(n)\right]\cap X \subseteq (\pi^0_{\delta\alpha})^{-1}(n)\cap X$$

(since $f,\pi^0_{\delta\alpha},\pi^1_{\delta\alpha}$ commute on $Y_\alpha\setminus n_\alpha$) so that

$$\left|(\pi^1_{\delta\alpha})^{-1}(n)\cap X\cap f[Z\cap X]\right| \leq \left|(\pi^0_{\delta\alpha})^{-1}(n)\cap X\right|.$$

By the choice of Y_α we also have

$$\left|(\pi^1_{\delta\alpha})^{-1}(n)\cap X\right| \geq s_\alpha(n)\cdot\left|(\pi^0_{\delta\alpha})^{-1}(n)\cap X\right|$$

Putting this together gives:

$$\left|(\pi^1_{\delta\alpha})^{-1}(n)\cap X'\right| \geq (s_\alpha(n)-1)\cdot\left|(\pi^0_{\delta\alpha})^{-1}(n)\cap X'\right|$$

Since s_α tends to infinity so does $s_\alpha-1$. This shows that $s_\alpha-1$ and $Y_\alpha\setminus n_\alpha$ witness the fact that $X'\in P(\bar{\mathcal{U}},\Pi_0,\Pi_1)$. □

We now put the previous things together and prove:

Theorem. *Assume $\diamondsuit$. There is an RK-increasing sequence of P-points with no P-point on top.*

Proof. Let $\langle\Pi^\alpha_0 : \alpha<\omega_1\rangle$ be a diamond sequence guessing sequences of functions from ω to ω, i.e. such that for every sequence of such functions $\Pi=\langle\pi_{\omega_1\alpha}:\alpha<\omega_1\rangle$ the set

$$\{\alpha\in Lim(\omega_1) : \Pi\restriction\alpha = \Pi^\alpha_0\}$$

is stationary. We recursively construct an RK-increasing sequence $\langle\mathcal{U}_\alpha:\alpha<\omega_1\rangle$ of P-points and witnessing maps Π^α_1 so that

(1) If $\delta<\omega_1$ is limit and $\Pi^\delta_1\cup\Pi^\delta_0$ commute, then $\Pi^\delta_1 < \Pi^\delta_0$;
(2) If $f:\Pi^\delta_0\to\Pi^\delta_1$ then there is $X\in\mathcal{U}_\delta$ and $Y\in\mathcal{U}_0$ such that

$$f\left[(\pi^0_{\delta 0})^{-1}[Y]\right]\cap X = \emptyset.$$

At successor steps construct an arbitrary P-point above the previous one. At limit steps first use Observation 4.1 to guarantee (1) and then recursively construct a P-filter $\mathcal{U}_\delta$ on $P(\bar{\mathcal{U}}, \Pi_0, \Pi_1)$ which hits each of ω_1-many dense sets $\{D_A, D_f : A \subseteq \omega, f : \Pi_0^\delta \to \Pi_1^\delta\}$. This can be done since the forcing is σ-closed by Proposition 4.2.

Finally notice that the chain of P-points cannot have a P-point on top. Otherwise if $\mathcal{U}$ is RK-above the chain as witnessed by finite-to-one maps Π_0, there is a limit $\delta < \omega_1$ such that $\Pi_0 \upharpoonright \delta = \Pi_0^\delta$. But then $\pi_\delta : \Pi_0^\delta \to \Pi_1^\delta$ so by (2) there is $Y \in \mathcal{U}_0$ and $X \in \mathcal{U}_\delta$ such that

$$\pi_\delta\left[(\pi^0_{\delta 0})^{-1}[Y]\right] \cap X = \emptyset.$$

contradicting the fact that π_δ witnesses that $\mathcal{U}$ is above $\mathcal{U}_\delta$. □

5. Generic is equivalent to weakly generic

It was proved in Section 3 that given a Rudin-Keisler increasing chain of rapid P-points $\langle \mathcal{U}_\alpha : \alpha < \omega_1\rangle$ together with a commuting sequence of finite-to-one witnessing maps $\langle \pi_{\beta,\alpha} : \alpha \le \beta < \omega_1\rangle$, it is possible to find a sequence of finite-to-one maps $\langle g_\alpha : \alpha < \omega_1\rangle$ together with a rapid P-point $\mathcal{V}$ such that g_α is a witness to $\mathcal{U}_\alpha \le_{\mathrm{RK}} \mathcal{V}$. However the argument in Section 3 does not guarantee that any of the g_α will be nondecreasing even when it is given that each of the maps $\pi_{\beta\alpha}$ is nondecreasing. In other words, the rapid P-point $\mathcal{V}$ may not be an $\le^+_{\mathrm{RB}}$ upper bound of the sequence $\langle \mathcal{U}_\alpha : \alpha < \omega_1\rangle$ even if that sequence is assumed to be $\le^+_{\mathrm{RB}}$-increasing. At present, it is unclear if there is an analogue of the results of Section 3 for the $\le^+_{\mathrm{RB}}$ ordering – that is, we do not know if the rapid P-points are $\mathfrak{c}^+$-closed with respect to $\le^+_{\mathrm{RB}}$.

It appears that one must fall back on the construction given in [**?**] if one wants a chain of P-points of length $\mathfrak{c}^+$ which is increasing in the $\le^+_{\mathrm{RB}}$ ordering. The ideas from Section 3 can nevertheless be used to simplify several aspects of the construction of [**?**]. In this section we show that the central definition used in [**?**] can be replaced with a significantly simpler definition. Indeed the two definitions define the same notion as we will show below that they are equivalent.

Definition. $\mathbb{P}$ denotes the collection of all functions $c : \omega \to [\omega]^{<\omega} \setminus \{0\}$ with the property that $\forall n \in \omega[|c(n)| < |c(n+1)| \wedge \max(c(n)) < \min(c(n+1))]$. We define a partial order on $\mathbb{P}$ as follows. For $c, d \in \mathbb{P}$ and $l \in \omega$, $c \le_l d$ means that $\forall m \ge l \exists n \ge m[c(m) \subseteq d(n)]$. For $c, d \in \mathbb{P}$, define $c \le d$ to mean $\exists l \in \omega[c \le_l d]$.

We borrow the following notational conventions from [**?**]. These conventions will be used through out for infinite subsets of ω and for elements of $\mathbb{P}$.

5.1. **Definition.** For any $A \in [\omega]^\omega$ and $m \in \omega$, $A(m)$ denotes the mth element of A in its increasing enumeration, and $A[m] = \{A(l) : l \ge m\}$.

For a sequence $c = \langle c(n) : n \in \omega\rangle$, let $\operatorname{set}(c) = \bigcup_{n\in\omega} c(n)$. If $m \in \omega$, then $\operatorname{set}(c)[\![m]\!] = \bigcup_{m \le n < \omega} c(n)$.

Definition. A triple $\langle \pi, \psi, c\rangle$ is called a *normal triple* if $\pi, \psi \in \omega^\omega$, $\forall l \le l' < \omega\left[\psi(l) \le \psi(l')\right]$, $\operatorname{ran}(\psi)$ is infinite, $c \in \mathbb{P}$, and $\forall l \in \omega\left[\pi'' c(l) = \{\psi(l)\}\right]$ while $\forall n \in \omega \setminus \operatorname{set}(c)[\pi(n) = 0]$.

The chains of P-points constructed in [**?**] under MA were all strictly increasing with respect to $\le^+_{\mathrm{RB}}$. The chains of P-points in [**?**] were also constructed to be strictly increasing with respect to the Tukey ordering. The notion of a δ-generic sequence was introduced in [**?**] and it played a crucial role in the construction of the chains there. We will repeat the definition here for the convenience of the reader.

5.2. **Definition.** Let $\delta \le \omega_2$ be any ordinal. We call a pair of sequences

$$\mathcal{S} = \langle\langle c_i^\alpha : i < \mathfrak{c} \wedge \alpha < \delta\rangle, \langle \pi_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$$

δ-generic if and only if:

(1) for every $\alpha < \delta$, $\langle c_i^\alpha : i < \mathfrak{c}\rangle$ is a decreasing sequence of conditions in $\mathbb{P}$; for every $\alpha \le \beta < \delta$, $\pi_{\beta,\alpha} \in \omega^\omega$;
(2) for every $\alpha < \delta$, $\mathcal{U}_\alpha = \{a \in \mathcal{P}(\omega) : \exists i < \mathfrak{c}\left[\operatorname{set}(c_i^\alpha) \subseteq^* a\right]\}$ is a rapid P-point on ω;

(3) for every $\alpha < \beta < \delta$, every normal triple $\langle \pi_1, \psi_1, b_1 \rangle$ and every $d \le b_1$, if $\pi_1'' \operatorname{set}(d) \in \mathcal{U}_\alpha$, then for every $a \in \mathcal{U}_\beta$, there are $b \in \mathcal{U}_\beta$, $\pi, \psi \in \omega^\omega$, and $d^* \le_0 d$ such that $b \subseteq^* a$ and that $\langle \pi, \psi, d^* \rangle$ is a normal triple, $\pi'' \operatorname{set}(d^*) = b$, and $\forall k \in \operatorname{set}(d^*) \left[\pi_1(k) = \pi_{\beta,\alpha}(\pi(k))\right]$;
(4) if $\alpha < \beta < \delta$, then $\mathcal{U}_\beta \not\le_T \mathcal{U}_\alpha$;
(5) for every $\alpha < \delta$, $\pi_{\alpha,\alpha} = \mathrm{id}$ and:
  (a) $\forall \alpha \le \beta < \delta \forall i < \mathfrak{c} \left[\pi''_{\beta,\alpha} \operatorname{set}(c_i^\beta) \in \mathcal{U}_\alpha\right]$;
  (b) $\forall \alpha \le \beta \le \gamma < \delta \exists i < \mathfrak{c} \forall^\infty k \in \operatorname{set}(c_i^\gamma) \left[\pi_{\gamma,\alpha}(k) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(k))\right]$;
  (c) for $\alpha < \beta < \delta$ there are $i < \mathfrak{c}$, $b_{\beta,\alpha} \in \mathbb{P}$, and $\psi_{\beta,\alpha} \in \omega^\omega$ such that $\langle \pi_{\beta,\alpha}, \psi_{\beta,\alpha}, b_{\beta,\alpha} \rangle$ is a normal triple and $c_i^\beta \le b_{\beta,\alpha}$;
(6) let $\mu < \delta$ be a limit ordinal such that $\operatorname{cf}(\mu) = \omega$; fix a countable set $X \subseteq \mu$ with $\sup(X) = \mu$, a decreasing sequence $\langle d_j : j < \omega \rangle$ of conditions in $\mathbb{P}$, and a sequence of maps $\langle \pi_\alpha : \alpha \in X \rangle \subseteq \omega^\omega$ such that:
  (a) $\forall \alpha \in X \forall j < \omega \left[\pi''_\alpha \operatorname{set}(d_j) \in \mathcal{U}_\alpha\right]$;
  (b) $\forall \alpha, \beta \in X \left[\alpha \le \beta \implies \exists j < \omega \forall^\infty k \in \operatorname{set}(d_j) \left[\pi_\alpha(k) = \pi_{\beta,\alpha}(\pi_\beta(k))\right]\right]$;
  (c) for all $\alpha \in X$, there are $j < \omega$, $b_\alpha \in \mathbb{P}$, and $\psi_\alpha \in \omega^\omega$ such that $\langle \pi_\alpha, \psi_\alpha, b_\alpha \rangle$ is a normal triple and $d_j \le b_\alpha$;

  then the collection of all $i^* < \mathfrak{c}$ such that there are $d^* \in \mathbb{P}$ and $\pi, \psi \in \omega^\omega$ satisfying:
  (d) $\forall j < \omega \left[d^* \le d_j\right]$ and $\operatorname{set}(c^\mu_{i^*}) = \pi'' \operatorname{set}(d^*)$;
  (e) $\forall \alpha \in X \forall^\infty k \in \operatorname{set}(d^*) \left[\pi_\alpha(k) = \pi_{\mu,\alpha}(\pi(k))\right]$;
  (f) $\langle \pi, \psi, d^* \rangle$ is a normal triple;

  is cofinal in $\mathfrak{c}$.

In a context where $\mathfrak{c} > \aleph_1$ and MA holds, Definition 5.2 would be naturally modified by replacing the condition $\delta \le \omega_2$ with the condition that $\delta \le \mathfrak{c}^+$, and by requiring Clause (6) to hold for any limit ordinal μ with $\operatorname{cf}(\mu) < \mathfrak{c}$, for any $X \subseteq \mu$ with $|X| < \mathfrak{c}$ and $\sup(X) = \mu$, and for any decreasing sequence of conditions in $\mathbb{P}$ of length $\operatorname{cf}(\mu)$ that satisfies conditions (6)(a)–(6)(c). The reader can find an explanation of the intuition behind Clauses (1)–(6) in [**?**]. In [**?**], it was proved that any δ-generic sequence can be extended to an ω_2-generic sequence, for any $\delta \le \omega_2$.

The main result of this section is that the notion of a δ-generic sequence is equivalent to an apparently much weaker notion to be introduced below. We begin with a lemma that is really a consequence of Fact 2.1.

5.3. **Lemma.** *Let $\mathcal{U}$ and $\mathcal{V}$ be rapid P-points on ω. Suppose that $\pi \in \omega^\omega$ and that $\forall b \in \mathcal{V} \left[\pi'' b \in \mathcal{U}\right]$. Then for every $a \in \mathcal{U}$ there exists $b \in \mathcal{V}$ such that*

(1) $\pi'' b \subseteq a$;
(2) $\forall l \in \omega [|\{x \in b : \pi(x) = a(l)\}| \le l + 2]$.

Proof. Since $\mathcal{V}$ is a P-point, there exists $c \in \mathcal{V}$ such that π is either finite-to-one or constant on c. By hypothesis, $\pi'' c \in \mathcal{U}$, and since $\mathcal{U}$ is a non-principal ultrafilter, it follows that π must be finite-to-one on c. Now define a function $g \in \omega^\omega$ by induction as follows. $g(0) = 0$ and suppose that $g(n) \in \omega$ is given for some $n \in \omega$. For each $l \in \omega$, $I_l = \{x \in c : \pi(x) = l\}$ is finite. Define $g(n+1) = \max\left(\left(\bigcup_{l \le g(n)} I_{a(l)}\right) \cup \{g(n)+1\}\right)$. Note that $g(n+1) \ge g(n) + 1 > g(n)$. So in particular, $g(n) \ge n$, for every $n \in \omega$. Now $c \cap \pi^{-1}(a) \in \mathcal{V}$ and by the rapidity of $\mathcal{V}$ there exists $b \in \mathcal{V}$ such that $b \subseteq c \cap \pi^{-1}(a)$ and $\forall n \in \omega [b(n) \ge g(n)]$. Note that (1) is satisfied by definition. For (2), fix any $l \in \omega$ and let $x \in b$ be such that $\pi(x) = a(l)$. Let $n \in \omega$ be unique so that $b(n) = x$. We have that $x \in I_{a(l)}$ and that $l \le g(l)$. So $g(n) \le b(n) = x \le g(l+1)$. Since g is strictly increasing, we conclude that $n \le l + 1$. Therefore, $\{x \in b : \pi(x) = a(l)\} \subseteq \{b(n) : n \le l + 1\}$. Since $|\{b(n) : n \le l + 1\}| = l + 2$, we are done. □

The next definition is an apparent weakening of the notion of a δ-generic sequence, where the key Clauses (3) and (6) are omitted from Definition 5.2. It will be proved however that this apparent weakening is equivalent to Definition 5.2. So the main result here is that the other clauses of Definition 5.2 automatically imply Clauses (3) and (6).

5.4. **Definition.** Let $\delta \le \omega_2$ be any ordinal. We call a pair of sequences

$$\mathcal{S} = \langle \langle c_i^\alpha : i < \mathfrak{c} \wedge \alpha < \delta \rangle, \langle \pi_{\beta,\alpha} : \alpha \le \beta < \delta \rangle \rangle$$

weakly δ-generic if and only if:

(1) for every $\alpha < \delta$, $\langle c_i^\alpha : i < \mathfrak{c}\rangle$ is a decreasing sequence of conditions in $\mathbb{P}$; for every $\alpha \le \beta < \delta$, $\pi_{\beta,\alpha} \in \omega^\omega$;
(2) for every $\alpha < \delta$, $\mathcal{U}_\alpha = \{a \in \mathcal{P}(\omega) : \exists i < \mathfrak{c}\left[\mathrm{set}(c_i^\alpha) \subseteq^* a\right]\}$ is a rapid P-point on ω;
(3) if $\alpha < \beta < \delta$, then $\mathcal{U}_\beta \not\le_T \mathcal{U}_\alpha$;
(4) for every $\alpha < \delta$, $\pi_{\alpha,\alpha} = \mathrm{id}$ and:
  (a) $\forall \alpha \le \beta < \delta \forall i < \mathfrak{c}\left[\pi''_{\beta,\alpha}\,\mathrm{set}(c_i^\beta) \in \mathcal{U}_\alpha\right]$;
  (b) $\forall \alpha \le \beta \le \gamma < \delta \exists i < \mathfrak{c} \forall^\infty k \in \mathrm{set}(c_i^\gamma)\left[\pi_{\gamma,\alpha}(k) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(k))\right]$;
  (c) for $\alpha < \beta < \delta$ there are $i < \mathfrak{c}$, $b_{\beta,\alpha} \in \mathbb{P}$, and $\psi_{\beta,\alpha} \in \omega^\omega$ such that $\langle \pi_{\beta,\alpha}, \psi_{\beta,\alpha}, b_{\beta,\alpha}\rangle$ is a normal triple and $c_i^\beta \le b_{\beta,\alpha}$.

5.5. **Lemma.** *Suppose $\pi \in \omega^\omega$. Let $a \in [\omega]^\omega$ and let $b = \pi'' a$. Assume that*

(1) $\forall n, m \in a\,[n \le m \implies \pi(n) \le \pi(m)]$;
(2) $\forall l \in \omega\,[\{x \in a : \pi(x) = l\}$ is finite$]$.

Then there exists an interval partition $I = \langle i_n : n \in \omega\rangle$ such that for each $n \in \omega$, $\pi''\{a(i) : i_n \le i < i_{n+1}\} = \{b(n)\}$.

Proof. Note that condition (2) and the fact that a is infinite imply that $b \in [\omega]^\omega$. So in particular, $b(n)$ is defined for every $n \in \omega$.

Now define an interval partition I as follows. Let $i_0 = 0$. Fix $n \in \omega$ and assume that $i_n \in \omega$ is given. Then $\{i \in \omega : \pi(a(i)) > \pi(a(i_n))\}$ is non-empty. Define $i_{n+1} = \min\{i \in \omega : \pi(a(i)) > \pi(a(i_n))\}$. So we have $\pi(a(i_{n+1})) > \pi(a(i_n))$, which by (1) implies $a(i_{n+1}) > a(i_n)$, which in turn implies $i_{n+1} > i_n$. This completes the definition of I.

We check that I is as required. Fix $n \in \omega$. If $i_n \le i < i_{n+1}$, then by (1) and by the definition of i_{n+1}, we have $\pi(a(i_n)) \le \pi(a(i)) \le \pi(a(i_n))$, whence $\pi(a(i)) = \pi(a(i_n))$. Thus

$$\pi''\{a(i) : i_n \le i < i_{n+1}\} = \{\pi(a(i)) : i_n \le i < i_{n+1}\} = \{\pi(a(i_n))\},$$

and so it suffices to show that $\pi(a(i_n)) = b(n)$, for all $n \in \omega$. To this end, define a function $e : \omega \to b$ by stipulating that $e(n) = \pi(a(i_n))$, for every $n \in \omega$. Then for every $n \in \omega$, $e(n) = \pi(a(i_n)) < \pi(a(i_{n+1})) = e(n+1)$. We claim that e is onto. To see this, consider $b(m)$, for some $m \in \omega$. There exists $i \in \omega$ such that $b(m) = \pi(a(i))$. Since I is an interval partition, there is a unique $n \in \omega$ with $i_n \le i < i_{n+1}$. By the above remarks, $e(n) = \pi(a(i_n)) = \pi(a(i)) = b(m)$. Therefore, e is onto. Since e is order-preserving and onto, it follows that for each $n \in \omega$, $b(n) = e(n) = \pi(a(i_n))$. This concludes the proof. ☐

5.6. **Lemma.** *Fix $\delta \le \omega_2$. Suppose $\mathcal{S}$ is weakly δ-generic. Then for every $\alpha < \beta < \delta$, every normal triple $\langle \pi_1, \psi_1, b_1\rangle$ and every $d \le b_1$, if $\pi_1''\,\mathrm{set}(d) \in \mathcal{U}_\alpha$, then for every $a \in \mathcal{U}_\beta$, there are $b \in \mathcal{U}_\beta$, $\pi, \psi \in \omega^\omega$, and $d^* \le_0 d$ such that $b \subseteq^* a$ and that $\langle \pi, \psi, d^*\rangle$ is a normal triple, $\pi''\,\mathrm{set}(d^*) = b$, and $\forall k \in \mathrm{set}(d^*)\left[\pi_1(k) = \pi_{\beta,\alpha}(\pi(k))\right]$.*

Proof. Fix $\alpha < \beta < \delta$. Let $\pi_1, \psi_1 \in \omega^\omega$, and $b_1, d \in \mathbb{P}$ be given. Assume that $\langle \pi_1, \psi_1, b_1\rangle$ is a normal triple, that $d \le b_1$, and that $\pi_1''\,\mathrm{set}(d) \in \mathcal{U}_\alpha$. Fix $a \in \mathcal{U}_\beta$. Let $N \in \omega$ be minimal such that $d \le_N b_1$. Note that $\pi_1''\,\mathrm{set}(d)[\![N]\!] \in \mathcal{U}_\alpha$. By Clause (4(c)) of Definition 5.4, there exists $a^* \in \mathcal{U}_\beta$ such that $\pi_{\beta,\alpha}$ is finite-to-one on a^* and $\forall x, y \in a^*\left[x \le y \implies \pi_{\beta,\alpha}(x) \le \pi_{\beta,\alpha}(y)\right]$. Define two functions from ω to ω as follows. Set $S(0) = 0$. Fix $l \in \omega$ and assume that $S(l) \in \omega$ is given. Define $S(l+1) = S(l) + l + 2$. Note that $S(l+1) > S(l)$. This completes the definition of $S : \omega \to \omega$. Next, set $\Sigma(0) = 0$. Fix $n \in \omega$ and assume that $\Sigma(n) \in \omega$ is given. Define $\Sigma(n+1) = \Sigma(n) + n + 1$. Note that $\Sigma(n+1) > \Sigma(n)$. This completes the definition of $\Sigma : \omega \to \omega$. Now define $f : \omega \to \omega$ by stipulating that for each $l \in \omega$, $f(l) = \Sigma(S(l+1))$. Since $\mathcal{U}_\alpha$ is rapid, there exists $A \in \mathcal{U}_\alpha$ such that $A \subseteq \pi_1''\,\mathrm{set}(d)[\![N]\!]$ and such that for each $i \in \omega$, $A(i) \ge \left(\pi_1''\,\mathrm{set}(d)[\![N]\!]\right)(f(i))$. It follows that for each $i \in \omega$, $\exists j \ge f(i)\left[A(i) = \left(\pi_1''\,\mathrm{set}(d)[\![N]\!]\right)(j)\right]$. Apply Lemma 5.3 to $\mathcal{U}_\alpha$, $\mathcal{U}_\beta$, $\pi_{\beta,\alpha}$, and $A \in \mathcal{U}_\alpha$ to find $a^{**} \in \mathcal{U}_\beta$ such that $\pi''_{\beta,\alpha} a^{**} \subseteq A$ and for each $l \in \omega$, $\left|\{x \in a^{**} : \pi_{\beta,\alpha}(x) = A(l)\}\right| \le l + 2$. Put $b = a \cap a^* \cap a^{**}$. Note that $b \in \mathcal{U}_\beta$ and that $b \subseteq a$. Also since $b \subseteq a^*$, it follows that $\forall x, y \in b\left[x \le y \implies \pi_{\beta,\alpha}(x) \le \pi_{\beta,\alpha}(y)\right]$ and that for all $l \in \omega$, $\{x \in b : \pi_{\beta,\alpha}(x) = l\}$ is finite. Let $B = \pi''_{\beta,\alpha} b$. Then $B \subseteq A$ because $\pi''_{\beta,\alpha} b \subseteq \pi''_{\beta,\alpha} a^{**} \subseteq A$. By Lemma 5.5, there is an interval partition $I = \langle i_n : n \in \omega\rangle$ such that for all $n \in \omega$, $\pi''_{\beta,\alpha}\{b(i) : i_n \le i < i_{n+1}\} = \{B(n)\}$. As $B \subseteq A$, for each $n \in \omega$, there exists $L_n \in \omega$ such that $A(L_n) = B(n)$. For $n < \omega$, let F_n denote

$$\left\{m < \omega : \pi''_1 d(m) = \left\{\left(\pi''_1 \operatorname{set}(d)\,[\![N]\!]\right)(n)\right\}\right\}.$$

By Lemma 2.9 of [?], $N \le \max(F_n) < \max(F_{n+1})$, for all $n < \omega$. Define $K_n = \max(F_n)$, for each $n \in \omega$. It is clear that $n \le K_n$. Now by choice of A, for each $n \in \omega$, there exists $j_n \in \omega$ such that $j_n \ge f(L_n)$ and $B(n) = A(L_n) = \left(\pi''_1 \operatorname{set}(d)\,[\![N]\!]\right)(j_n)$. Since $f(L_n) \le j_n \le K_{j_n}$, we conclude that $\left|d(K_{j_n})\right| \ge K_{j_n} + 1 \ge f(L_n) + 1$. Construct a sequence $\langle d^*(i) : i \in \omega\rangle$ such that for each $n \in \omega$, the following hold:

(1) $\forall i_n \le i < i_{n+1}\left[d^*(i) \in \left[d(K_{j_n})\right]^{i+1}\right]$;
(2) $\forall i_n \le i < i+1 < i_{n+1}[\max(d^*(i)) < \min(d^*(i+1))]$.

To show that this can be done, it suffices to show that for each $n \in \omega$,

$$\left|d(K_{j_n})\right| \ge \textstyle\sum_{i_n \le i < i_{n+1}} (i+1).$$

To this end, we first argue that for each $n \in \omega$, $i_{n+1} \le S(L_n + 1)$. Observe that for each $n \in \omega$, $|\{b(i) : i_n \le i < i_{n+1}\}| \le L_n + 2$ because $\{b(i) : i_n \le i < i_{n+1}\} \subseteq \{x \in a^{**} : \pi_{\beta,\alpha}(x) = A(L_n)\}$. It follows that $i_{n+1} \le i_n + L_n + 2$. Using this observation, we show by induction on $n \in \omega$ that $i_{n+1} \le S(L_n + 1)$. Indeed, when $n = 0$, we have that $i_{n+1} \le i_n + L_n + 2 = L_n + 2 \le S(L_n) + L_n + 2 = S(L_n + 1)$, as needed. Now fix $n \in \omega$ and assume that $i_{n+1} \le S(L_n + 1)$. Note that $L_n < L_{n+1}$ because $A(L_n) = B(n) < B(n+1) = A(L_{n+1})$. Thus $S(L_n + 1) \le S(L_{n+1})$. Putting these observation together, we have $i_{n+2} \le i_{n+1} + L_{n+1} + 2 \le S(L_n + 1) + L_{n+1} + 2 \le S(L_{n+1}) + L_{n+1} + 2 = S(L_{n+1} + 1)$. This concludes the proof that for each $n \in \omega$, $i_{n+1} \le S(L_n + 1)$. Now by the definition of Σ and of f, we have that for each $n \in \omega$,

$$\textstyle\sum_{i_n \le i < i_{n+1}} (i+1) \le \Sigma(i_{n+1}) \le \Sigma(S(L_n + 1)) = f(L_n) < f(L_n) + 1 \le \left|d(K_{j_n})\right|.$$

This shows that it is possible to construct a sequence $\langle d^*(i) : i \in \omega\rangle$ satisfying (1)-(2) above.

Since I is an interval partition, for each $i \in \omega$, there exists a unique $n \in \omega$ with $i_n \le i < i_{n+1}$. So by (1) for each $i \in \omega$, $d^*(i) \in [\omega]^{i+1}$, whence $|d^*(i)| = i + 1 < i + 2 = |d^*(i+1)|$. Next, for each $n \in \omega$ and for each $i \in \omega$, if $i_n \le i < i+1 < i_{n+1}$, then $\max(d^*(i)) < \min(d^*(i+1))$ by (2). On the other hand if $i_n \le i < i_{n+1} \le i+1 < i_{n+2}$, then $d^*(i) \subseteq d(K_{j_n})$ and $d^*(i+1) \subseteq d(K_{j_{n+1}})$. Note that $j_n < j_{n+1}$ because

$$\left(\pi''_1 \operatorname{set}(d)\,[\![N]\!]\right)(j_n) = B(n) < B(n+1) = \left(\pi''_1 \operatorname{set}(d)\,[\![N]\!]\right)(j_{n+1}).$$

Hence $K_{j_n} < K_{j_{n+1}}$, whence $\max(d^*(i)) \le \max(d(K_{j_n})) < \min(d(K_{j_{n+1}})) \le \min(d^*(i+1))$. Thus in either case for each $i \in \omega$, $\max(d^*(i)) < \min(d^*(i+1))$. Therefore $d^* \in \mathbb{P}$. Next fix $n \in \omega$ and $i_n \le i < i_{n+1}$. Then by combining some of the above established inequalities, we have that $i < i_{n+1} \le S(L_n + 1) \le \Sigma(S(L_n + 1)) = f(L_n) \le j_n \le K_{j_n}$ and also that $d^*(i) \subseteq d(K_{j_n})$. So we conclude that $\forall i \in \omega \exists k > i[d^*(i) \subseteq d(k)]$. This implies that $d^* \le_0 d$. Next, define $\pi, \psi \in \omega^\omega$ as follows. For each $i \in \omega$ define $\psi(i) = b(i)$ and for each $k \in d^*(i)$, set $\pi(k) = b(i)$. For all $k \in \omega \setminus \operatorname{set}(d^*)$, set $\pi(k) = 0$. Then for each $i \in \omega$, $\pi'' d^*(i) = \{b(i)\} = \{\psi(i)\}$, and $\psi(i) < \psi(i+1)$. Also

$$\pi'' \operatorname{set}(d^*) = \textstyle\bigcup_{i \in \omega} \pi'' d^*(i) = \bigcup_{i \in \omega} \{b(i)\} = \{b(i) : i \in \omega\} = b.$$

It is clear from the definitions that $\langle \pi, \psi, d^*\rangle$ is a normal triple. Finally fix any $k \in \operatorname{set}(d^*)$. Let $i \in \omega$ be such that $k \in d^*(i)$ and let $n \in \omega$ be such that $i_n \le i < i_{n+1}$. Then $d^*(i) \subseteq d(K_{j_n})$, $B(n) = \left(\pi''_1 \operatorname{set}(d)\,[\![N]\!]\right)(j_n)$, and $\pi''_1 d(K_{j_n}) = \left\{\left(\pi''_1 \operatorname{set}(d)\,[\![N]\!]\right)(j_n)\right\}$. Therefore

$$\pi_1(k) = \left(\pi''_1 \operatorname{set}(d)\,[\![N]\!]\right)(j_n) = B(n) = \pi_{\beta,\alpha}(b(i)) = \pi_{\beta,\alpha}(\pi(k)).$$

Therefore for each $k \in \operatorname{set}(d^*)$, $\pi_1(k) = \pi_{\beta,\alpha}(\pi(k))$. This concludes the verification that b, d^*, π, and ψ are as needed. □

Thus Lemma 5.6 shows that a weakly δ-generic sequence automatically satisfies Clause (3) of Definition 5.2. The next lemma shows the same for Clause (6) as well. The argument is quite close to the proof of Lemma 3.4 in [?].

5.7. **Lemma.** *Let $\delta \le \omega_2$ and suppose $\mathscr{S} = \langle\langle c^\alpha_i : i < \mathfrak{c} \wedge \alpha < \delta\rangle, \langle \pi_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ is weakly δ-generic. Let $\mu < \delta$ be a limit ordinal such that $\operatorname{cf}(\mu) = \omega$. Fix a countable set $X \subseteq \mu$ with $\sup(X) = \mu$, a decreasing sequence $\langle d_j : j < \omega\rangle$ of conditions in $\mathbb{P}$, and a sequence of maps $\langle \pi_\alpha : \alpha \in X\rangle \subseteq \omega^\omega$ such that:*

(1) $\forall \alpha \in X \forall j < \omega\left[\pi''_\alpha \operatorname{set}(d_j) \in \mathscr{U}_\alpha\right]$;

(2) $\forall \alpha, \beta \in X \left[\alpha \le \beta \implies \exists j < \omega \forall^\infty k \in \operatorname{set}(d_j) \left[\pi_\alpha(k) = \pi_{\beta,\alpha}(\pi_\beta(k))\right]\right]$;

(3) for all $\alpha \in X$, *there are* $j < \omega$, $b_\alpha \in \mathbb{P}$, *and* $\psi_\alpha \in \omega^\omega$ *such that* $\langle \pi_\alpha, \psi_\alpha, b_\alpha \rangle$ *is a normal triple and* $d_j \le b_\alpha$.

Then the collection of all $i^* < \mathfrak{c}$ *such that there are* $d^* \in \mathbb{P}$ *and* $\pi, \psi \in \omega^\omega$ *satisfying:*

(4) $\forall j < \omega \left[d^* \le d_j\right]$ *and* $\operatorname{set}(c^\mu_{i^*}) = \pi'' \operatorname{set}(d^*)$;

(5) $\forall \alpha \in X \forall^\infty k \in \operatorname{set}(d^*) \left[\pi_\alpha(k) = \pi_{\mu,\alpha}(\pi(k))\right]$;

(6) $\langle \pi, \psi, d^* \rangle$ *is a normal triple;*

is cofinal in $\mathfrak{c}$.

Proof. Choose a strictly increasing sequence $\langle \delta_n : n \in \omega \rangle \subseteq X$ which is cofinal in μ. For each $m \le n < \omega$ use clause (2) to choose $j_{m,n}, L_{m,n} \in \omega$ such that

$$\forall k \in \operatorname{set}(d_{j_{m,n}}) \left[\!\left[L_{m,n}\right]\!\right] \left[\pi_{\delta_m}(k) = \pi_{\delta_n,\delta_m}(\pi_{\delta_n}(k))\right].$$

For each $n < \omega$, use clause (3) to choose $j^*_n < \omega$, $b_{\delta_n} \in \mathbb{P}$, and $\psi_{\delta_n} \in \omega^\omega$ such that $\langle \pi_{\delta_n}, \psi_{\delta_n}, b_{\delta_n} \rangle$ is a normal triple and $d_{j^*_n} \le b_{\delta_n}$. Let $K_n < \omega$ be minimal such that $d_{j^*_n} \le_{K_n} b_{\delta_n}$. Define a sequence $\langle j_n : n \in \omega \rangle$ by induction on $n \in \omega$ as follows. Suppose $n \in \omega$ and suppose that $j_m \in \omega$ has been defined for all $m < n$. Then set $j_n = \max(A_n)$, where

$$A_n = \left\{j^*_n\right\} \cup \{j_m + 1 : m < n\} \cup \{j_{m,n} : m \le n\}.$$

Note that $j_{n+1} > j_n$, for all $n \in \omega$. Choose $R_n \in \omega$ so that for all $j \le j_n$, $d_{j_n} \le_{R_n} d_j$. In particular, $d_{j_n} \le_{R_n} d_j$, for all $j \in A_n$. Finally define a sequence $\langle M_n : n \in \omega \rangle$ by induction on $n \in \omega$ as follows. Suppose $n \in \omega$ and suppose that $M_l \in \omega$ has been defined for all $l < n$. Then set $M_n = \max(B_n)$, where

$$B_n = \{R_n, K_n\} \cup \{M_l + 1 : l < n\} \cup \{L_{m,n} : m \le n\}.$$

Note that $M_{n+1} > M_n$. The following claim is easy to prove and is left to the reader.

Claim 1. *For each* $n \in \omega$ *the following hold:*

(7) $\forall x, y \in \operatorname{set}(d_{j_n}) [\![M_n]\!] \left[x \le y \implies \pi_{\delta_n}(x) \le \pi_{\delta_n}(y)\right]$;

(8) $\forall m \le n \forall x \in \operatorname{set}(d_{j_n}) [\![M_n]\!] \left[\pi_{\delta_m}(x) = \pi_{\delta_n,\delta_m}(\pi_{\delta_n}(x))\right]$.

Now for each $n \in \omega$, define $E_n = \pi''_{\delta_n} \operatorname{set}(d_{j_n}) [\![M_n]\!]$ and $F_n = \pi''_{\delta_0} \operatorname{set}(d_{j_n}) [\![M_n]\!]$. Observe that $\pi''_{\delta_n,\delta_0} E_n = F_n$. Note that for any $n', n \in \omega$, if $n' \le n$, then $\operatorname{set}(d_{j_n}) [\![M_n]\!] \subseteq \operatorname{set}(d_{j_{n'}}) \left[\!\left[M_{n'}\right]\!\right]$. It is also easy to see that for each $n \in \omega$, $\forall x^*, y^* \in E_n \left[x^* \le y^* \implies \pi_{\delta_n,\delta_0}(x^*) \le \pi_{\delta_n,\delta_0}(y^*)\right]$. Let the function $S, \Sigma \in \omega^\omega$ be defined exactly as in the proof of Lemma 5.6. Define $f : \omega \to \omega$ by setting $f(l) = \Sigma(S(2l))$, for each $l \in \omega$. Using the rapidity of $\mathcal{U}_{\delta_0}$ choose for each $n \in \omega$, a set $D_n \subseteq F_n$ such that $D_n \in \mathcal{U}_{\delta_0}$ and $\forall l \in \omega [D_n(l) \ge F_n(f(l))]$. Therefore, for each $l \in \omega$, there exists $m_l \ge f(l)$ with $D_n(l) = F_n(m_l)$. Let $D \in \mathcal{U}_{\delta_0}$ be such that $\forall n \in \omega [D \subseteq^* D_n]$. Now apply Lemma 5.3 with $\mathcal{U}$ as $\mathcal{U}_{\delta_0}$, $\mathcal{V}$ as $\mathcal{U}_\mu$ and π as π_{μ,δ_0} to find $C^* \in \mathcal{U}_\mu$ such that $\pi''_{\mu,\delta_0} C^* \subseteq D$ and $\forall l \in \omega \left[\left|\{x \in C^* : \pi_{\mu,\delta_0}(x) = D(l)\}\right| \le l + 2\right]$. Let $i < \mathfrak{c}$ be given. We need to find $i \le i^* < \mathfrak{c}$ such that there exist $d^* \in \mathbb{P}$ and $\pi, \psi \in \omega^\omega$ satisfying (4)–(6). Let $i' < \mathfrak{c}$ be such that $\operatorname{set}(c^\mu_{i'}) \subseteq^* C^*$. For each $\alpha, \beta \in X$ with $\alpha \le \beta$, use clause (4)(b) of Definition 5.4 to fix $i_*(\alpha, \beta) < \mathfrak{c}$ such that $\forall^\infty x \in \operatorname{set}\left(c^\mu_{i_*(\alpha,\beta)}\right) \left[\pi_{\mu,\alpha}(x) = \pi_{\beta,\alpha}(\pi_{\mu,\beta}(x))\right]$. When $m \le n < \omega$, we will write $i(m,n)$ in place of $i_*(\delta_m, \delta_n)$. For each $n \in \omega$, use clause (4)(c) of Definition 5.4 to fix $i(n) < \mathfrak{c}$, $b_{\mu,\delta_n} \in \mathbb{P}$, and $\psi_{\mu,\delta_n} \in \omega^\omega$ such that $\langle \pi_{\mu,\delta_n}, \psi_{\mu,\delta_n}, b_{\mu,\delta_n} \rangle$ is a normal triple and $c^\mu_{i(n)} \le b_{\mu,\delta_n}$. Finally for each $n \in \omega$, find $i'(n) < \mathfrak{c}$ such that $\operatorname{set}\left(c^\mu_{i'(n)}\right) \subseteq^* \pi^{-1}_{\mu,\delta_n}(E_n)$. Now define $i^* = \sup(I)$, where

$$I = \left\{i, i'\right\} \cup \left\{i_*(\alpha, \beta) : \alpha, \beta \in X \wedge \alpha \le \beta\right\} \cup \{i(n) : n \in \omega\} \cup \left\{i'(n) : n \in \omega\right\}.$$

I is a countable subset of $\mathfrak{c}$ because X is countable. So $i \le i^* < \mathfrak{c}$. Note that $\{i(m,n) : m \le n < \omega\} \subseteq I$. Let $C = \operatorname{set}\left(c^\mu_{i^*}\right)$. Now for each $n \in \omega$, observe that the following inequalities hold: $c^\mu_{i^*} \le c^\mu_{i'}$, $\forall m \le n \left[c^\mu_{i^*} \le c^\mu_{i(m,n)}\right]$, $\forall m \le n \left[c^\mu_{i^*} \le c^\mu_{i'(m)}\right]$, and $c^\mu_{i^*} \le c^\mu_{i(n)} \le b_{\mu,\delta_n}$. Therefore we can find $T_n \in \omega$ such that the following statements hold:

- $\forall m \le n \left[C[T_n] \subseteq \pi^{-1}_{\mu,\delta_m}(E_m)\right]$;
- $\forall m \le n \forall x \in C[T_n] \left[\pi_{\mu,\delta_m}(x) = \pi_{\delta_n,\delta_m}(\pi_{\mu,\delta_n}(x))\right]$; and

- $C[T_n] \subseteq C^* \cap \mathrm{set}(b_{\mu,\delta_n})$.

Note that $\forall x, y \in C[T_n]\left[x \le y \implies \pi_{\mu,\delta_n}(x) \le \pi_{\mu,\delta_n}(y)\right]$ because $C[T_n] \subseteq \mathrm{set}(b_{\mu,\delta_n})$. Also for each $n \in \omega$, $G_n = \pi''_{\mu,\delta_n} C[T_n] \in \mathscr{U}_{\delta_n}$ because $C[T_n] \in \mathscr{U}_\mu$. It follows that π_{μ,δ_n} must be finite-to-one on $C[T_n]$ because otherwise G_n would be finite. So applying Lemma 5.5, there exists an interval partition $Z_n = \langle z_{n,k} : k \in \omega \rangle$ such that for each $k \in \omega$,

$$\pi''_{\mu,\delta_n}\{C[T_n](z) : z_{n,k} \le z < z_{n,k+1}\} = \{G_n(k)\}.$$

Note that $G_0 \subseteq D$. So for each $n \in \omega$, there exists $Q_n \in \omega$ with $G_0[Q_n] \subseteq D[Q_n] \subseteq D_n$. In particular, for any $z \ge z_{0,Q_n}$ $\left[\pi_{\mu,\delta_0}(C[T_0](z)) \in D_n\right]$. For each $n \in \omega$, define $V_n = \max\left(\left\{T_0 + z_{0,Q_n}\right\} \cup \{T_m : m \le n\}\right) \in \omega$. Then $C[V_n] \in \mathscr{U}_\mu$ and $H_n = \pi''_{\mu,\delta_n} C[V_n] \in \mathscr{U}_{\delta_n}$. In particular, we have that $H_n \subseteq E_n$ and that for each $x \in C[V_n]$, $\pi_{\delta_n,\delta_0}(\pi_{\mu,\delta_n}(x)) = \pi_{\mu,\delta_0}(x) \in D_n$. For each $n \in \omega$ and for each $l < \omega$, let $\Phi_{n,l} = \left\{m < \omega : \pi''_{\delta_0} d_{j_n}(m) = \{F_n(l)\}\right\}$. By Lemma 2.9 of [?], for each $n, l < \omega$, we have $\Phi_{n,l} \setminus M_n \ne 0$, $\left|\Phi_{n,l}\right| < \omega$, $\max(\Phi_{n,l}) < \min(\Phi_{n,l+1} \setminus M_n) \le \max(\Phi_{n,l+1})$. Similarly, for each $n \in \omega$ and $l < \omega$, let $\Lambda_{n,l} = \left\{m < \omega : \pi''_{\delta_n} d_{j_n}(m) = \{E_n(l)\}\right\}$. Then for each $n, l \in \omega$, $\Lambda_{n,l} \setminus M_n \ne 0$, $\left|\Lambda_{n,l}\right| < \omega$, and $\max(\Lambda_{n,l}) < \min(\Lambda_{n,l+1} \setminus M_n) \le \max(\Lambda_{n,l+1})$. Note that $V_0 = T_0 + z_{0,Q_0}$. For each $t \ge Q_0$, let $X_t, Y_t, Z_t \in \omega$ be such that $X_t \ge Q_0$, $Z_t \ge f(Y_t)$, and $G_0(t) = D(X_t) = D_0(Y_t) = F_0(Z_t)$. Note that for any $t \ge Q_0$,

$$\left\{C(T_0 + z) : z_{0,t} \le z < z_{0,t+1}\right\} \subseteq \left\{x \in C^* : \pi_{\mu,\delta_0}(x) = D(X_t)\right\}.$$

Therefore, $z_{0,t+1} - z_{0,t} \le X_t + 2$. We will prove the following claim.

Claim 2. *Fix $t, t' \ge Q_0$. If $X_{t'} \ge 2X_t + V_0 + 1$, then $\max(\Phi_{0,Z_{t'}}) \ge \Sigma(T_0 + z_{0,t'+1})$.*

Proof. It is easy to see by induction on t' that $T_0 + z_{0,t'+1} \le V_0 + S(X_{t'} + 1) \le S(V_0 + X_{t'} + 1)$. Moreover it is also clear that $X_{t'} - X_t \le X_{t'} - Q_0 \le Y_{t'}$. Therefore $X_{t'} + V_0 + 1 \le 2X_{t'} - 2X_t \le 2Y_{t'}$. Since for each $l < \omega$, $\max(\Phi_{0,l}) < \max(\Phi_{0,l+1})$, it follows that

$$\max(\Phi_{0,Z_{t'}}) \ge Z_{t'} \ge f(Y_{t'}) \ge \Sigma(S(2X_{t'} - 2X_t)) \ge \Sigma(S(X_{t'} + V_0 + 1)) \ge \Sigma(T_0 + z_{0,t'+1}),$$

as needed. □

Next observe that for each $n, w \in \omega$, if $w \ge T_n$, then there are $k(n,w) \in \omega$ and $l(n,w) \in \omega$ such that $T_n + z_{n,k(n,w)} \le w < T_n + z_{n,k(n,w)+1}$ and $E_n(l(n,w)) = \pi_{\mu,\delta_n}(C(w)) = G_n(k(n,w))$. This is because of the choice of the interval partition Z_n and because $C[T_n] \subseteq \pi^{-1}_{\mu,\delta_n}(E_n)$ for each $n \in \omega$. Note that for each $n \in \omega$, if $T_n \le w \le w' < \omega$, then $k(n,w) \le k(n,w')$ and $l(n,w) \le l(n,w')$.

We next make some observations that will be useful in the construction of d^*, π, and ψ. Let $t \in \omega$ be such that $t > 0$ and $t \ge Q_0$. Write $w = T_0 + z_{0,t}$. Fix $n \in \omega$ and suppose that $w \ge T_n$. Write $w' = T_n + z_{n,k(n,w)}$. Clearly, $w' \le w$. We claim that $w' = w$. Suppose for a contradiction that $w' < w$. By the definition of $k(n,w)$ and of $z_{n,k(n,w)}$, we have $\pi_{\mu,\delta_n}(C(w')) = G_n(k(n,w)) = \pi_{\mu,\delta_n}(C(w))$. Also $\pi_{\mu,\delta_0}(C(w')) = \pi_{\delta_n,\delta_0}(\pi_{\mu,\delta_n}(C(w'))) = \pi_{\delta_n,\delta_0}(\pi_{\mu,\delta_n}(C(w))) = \pi_{\mu,\delta_0}(C(w)) = G_0(t)$. Write $w'' = w - 1$. Clearly, $\pi_{\mu,\delta_n}(C(w)) = \pi_{\mu,\delta_n}(C(w''))$, and so $\pi_{\mu,\delta_0}(C(w'')) = \pi_{\delta_n,\delta_0}(\pi_{\mu,\delta_n}(C(w''))) = \pi_{\delta_n,\delta_0}(\pi_{\mu,\delta_n}(C(w))) = \pi_{\mu,\delta_0}(C(w)) = G_0(t)$. However, $z_{0,t-1} \le (z_{0,t}) - 1 < z_{0,t}$, and so $\pi_{\mu,\delta_0}(C(w'')) = \pi_{\mu,\delta_0}(C[T_0]((z_{0,t}) - 1)) = G_0(t-1)$. This is a contradiction, which shows that $w' = w$.

A simple corollary of the above observation is that if $t \in \omega$ is such that $t \ge Q_0$, and if $w \in \omega$ is such that $T_0 + z_{0,t} \le w < T_0 + z_{0,t+1}$, then for any $n \in \omega$ with $w \ge T_n$, $T_n + z_{n,k(n,w)+1} \le T_0 + z_{0,t+1}$.

Next, fix $n \in \omega$, and suppose $w \in \omega$ is such that $w \ge T_n$. Let $\zeta_{n,w}$ denote $\max\left(\Lambda_{n,l(n,w)}\right)$. Suppose $x \in d_{j_n}(\zeta_{n,w})$. Then it is not hard to see that for any $m \in \omega$ with $m \le n$, $\pi_{\delta_m}(x) = \pi_{\delta_n,\delta_m}(\pi_{\delta_n}(x)) = \pi_{\mu,\delta_m}(C(w))$.

This last observation can be used to prove the following useful fact. Fix $n, m \in \omega$ with $m < n$. Let $w, w' \in \omega$ be so that $w \ge T_m$ and $w' \ge T_n$. Let $t, t' \in \omega$ be so that $t, t' \ge Q_0$. Assume that $t < t'$ and that $T_0 + z_{0,t} \le w < T_0 + z_{0,t+1} \le T_0 + z_{0,t'} \le w' < T_0 + z_{0,t'+1}$. Then for any $x \in d_{j_m}(\zeta_{m,w})$ and any $y \in d_{j_n}(\zeta_{n,w'})$, $x < y$. For if not, then $G_0(t') = \pi_{\mu,\delta_0}(C(w')) = \pi_{\delta_0}(y) \le \pi_{\delta_0}(x) = \pi_{\mu,\delta_0}(C(w)) = G_0(t) < G_0(t')$, which is impossible.

Next suppose that $n,k,w \in \omega$ and that $T_n + z_{n,k} \le w < T_n + z_{n,k+1}$. Then $k = k(n,w)$. Moreover for any $w' \in \omega$ with $T_n + z_{n,k} \le w' < T_n + z_{n,k+1}$, $l(n,w') = l(n,w)$ and $\zeta_{n,w} = \zeta_{n,w'}$. Note also that for any $n,w,w' \in \omega$, if $w,w' \ge T_n$ and $k(n,w) < k(n,w')$, then $l(n,w) < l(n,w')$.

The following observations will help us establish an analogue of Claim 2 for all values of $n \in \omega$. It is easy to see that for any $n,l,t \in \omega$, if $\pi_{\delta_n,\delta_0}(E_n(l)) = F_n(t)$, then $l \ge t$. Now fix $n \in \omega$. Suppose $t \in \omega$ with $t \ge Q_0$ and $t \ge Q_n$. Then $X_t \ge Q_n$ and there exist $\bar{Y}_{n,t} \in \omega$ and $\bar{Z}_{n,t} \in \omega$ such that $\bar{Z}_{n,t} \ge f(\bar{Y}_{n,t})$ and $G_0(t) = D_n(\bar{Y}_{n,t}) = F_n(\bar{Z}_{n,t})$. Moreover it is not hard to see that $X_t - Q_n \le \bar{Y}_{n,t}$. Finally suppose $n,t \in \omega$, $t \ge Q_0$, $t \ge Q_n$, $w \in \omega$, $T_0 + z_{0,t} \le w < T_0 + z_{0,t+1}$, and that $w \ge T_n$. Then we have that $\pi_{\mu,\delta_0}(C(w)) = \pi_{\delta_n,\delta_0}(E_n(l(n,w))) = F_n(\bar{Z}_{n,t})$, whence $l(n,w) \ge \bar{Z}_{n,t}$. Now we establish the following version of Claim 2.

Claim 3. *Fix $n \in \omega$. Let $t,t' \in \omega$ and assume that $t,t' \ge Q_0$ and $t,t' \ge Q_n$. Assume that $X_{t'} \ge 2X_t + V_0 + 1$ and that $T_0 + z_{0,t'} \ge T_n$. Then if $T_0 + z_{0,t'} \le w < T_0 + z_{0,t'+1}$, then $\zeta_{n,w} \ge \Sigma(T_0 + z_{0,t'+1})$.*

Proof. As in Claim 2, we have that $T_0 + z_{0,t'+1} \le V_0 + S(X_{t'}+1) \le S(V_0 + X_{t'} + 1)$. The above observations imply that $V_0 + X_{t'} + 1 \le 2X_{t'} - 2X_t \le 2\bar{Y}_{n,t'}$, and hence that $\Sigma(T_0 + z_{0,t'+1}) \le \Sigma(S(V_0 + X_{t'} + 1)) \le \Sigma(S(2\bar{Y}_{n,t'})) = f(\bar{Y}_{n,t'}) \le \bar{Z}_{n,t'} \le l(n,w)$. Since for each $l \in \omega$, $\max(\Lambda_{n,l}) < \max(\Lambda_{n,l+1})$, it follows that $\zeta_{n,w} \ge l(n,w) \ge \Sigma(T_0 + z_{0,t'+1})$, as claimed. □

We are now ready to construct d^*, π, and ψ. Define an interval partition $\langle t_n : n \in \omega \rangle$ as follows. Put $t_0 = 0$. Next put $t_1 = 2X_{Q_0} + V_0 + 1$. Then $t_1 \in \omega$, $t_1 > t_0$, and $t_1 \ge Q_0$. Moreover by Claim 3, for every $t' \in \omega$ with $t' \ge t_1$, if $T_0 + z_{0,t'} \le w < T_0 + z_{0,t'+1}$, then $\zeta_{0,w} \ge \Sigma(T_0 + z_{0,t'+1})$. Now fix $n \in \omega$ with $n \ge 1$ and suppose that $t_n \in \omega$ is given. Let $t = \max\{t_n, T_n, Q_0, Q_n\}$ and define $t_{n+1} = 2X_t + V_0 + 1$. Then $t_{n+1} \in \omega$, $t_n < t_{n+1}$, $t_{n+1} \ge Q_0, Q_n$, and $T_0 + z_{0,t_{n+1}} \ge T_n$. Moreover by Claim 3, for any $t' \in \omega$ with $t' \ge t_{n+1}$, if $T_0 + z_{0,t'} \le w < T_0 + z_{0,t'+1}$, then $\zeta_{n,w} \ge \Sigma(T_0 + z_{0,t'+1})$. This concludes the definition of $\langle t_n : n \in \omega \rangle$.

Fix $n \in \omega$. For any $t \in \omega$ with $t_{n+1} \le t < t_{n+2}$, define $w_t = T_0 + z_{0,t}$ and $w'_t = T_0 + z_{0,t+1} - 1$. Then it is clear that $k(n,w_t) \le k(n,w'_t)$. Also it is not hard to see that

$$[T_0 + z_{0,t}, T_0 + z_{0,t+1}) = \bigcup_{k(n,w_t) \le k \le k(n,w'_t)} [T_n + z_{n,k}, T_n + z_{n,k+1}).$$

Now fix any $k \in \omega$ with $k(n,w_t) \le k \le k(n,w'_t)$. Define $w_{t,k} = T_n + z_{n,k}$. Then $w_{t,k} = T_n + z_{n,k} < T_n + z_{n,k+1} \le T_0 + z_{0,t+1}$ and $\eta_{t,k} = \zeta_{n,w_{t,k}} \ge \Sigma(T_0 + z_{0,t+1})$. Therefore it is possible to find a sequence $\langle d^*(w) : T_n + z_{n,k} \le w < T_n + z_{n,k+1} \rangle$ such that for each $T_n + z_{n,k} \le w < T_n + z_{n,k+1}$, $d^*(w) \in [d_{j_n}(\eta_{t,k})]^{w+1}$, and if $T_n + z_{n,k} \le w < w+1 < T_n + z_{n,k+1}$, then $\max(d^*(w)) < \min(d^*(w+1))$. Note that for each $w \in \omega$ with $T_n + z_{n,k} \le w < T_n + z_{n,k+1}$, $\zeta_{n,w_{t,k}} = \zeta_{n,w} = \eta_{t,k}$. Thus $d^*(w) \subseteq d_{j_n}(\eta_{t,k}) = d_{j_n}(\zeta_{n,w})$. Note also that for any $k' \in \omega$ with $k(n,w_t) \le k' \le k(n,w'_t)$, if $k < k'$, then $\eta_{t,k} < \eta_{t,k'}$.

Unfix n,t,k, and observe that for each $w \in \omega$, if $w \ge T_0 + z_{0,t_1}$, then there exists a unique $t \in \omega$, a unique $n \in \omega$, and a unique $k \in \omega$ such that $T_0 + z_{0,t} \le w < T_0 + z_{0,t+1}$, $t_{n+1} \le t < t_{n+2}$, $k(n,w_t) \le k \le k(n,w'_t)$, and $T_n + z_{n,k} \le w < T_n + z_{n,k+1}$. Thus the construction above defines $d^*(w) \in [\omega]^{w+1}$, for all $w \in \omega$ such that $w \ge T_0 + z_{0,t_1}$. When $w < T_0 + z_{0,t_1}$, we define $d^*(w) = [\Sigma(w), \Sigma(w+1))$. Then $d^*(w) \in [\omega]^{w+1}$, for all $w \in \omega$, and so $|d^*(w)| = w+1 < w+2 = |d^*(w+1)|$, for all $w \in \omega$. To verify that $d^* = \langle d^*(w) : w \in \omega \rangle \in \mathbb{P}$, it suffices to show that $\max(d^*(w)) < \min(d^*(w'))$ whenever $w < w' < \omega$. To this end, fix $w < w' < \omega$. Now there are several cases to consider. Suppose first that $w < w' < T_0 + z_{0,t_1}$. Then $\max(d^*(w)) < \Sigma(w+1) \le \Sigma(w') \le \min(d^*(w'))$, as needed. Next, suppose that $w < T_0 + z_{0,t_1}$ while $w' \ge T_0 + z_{0,t_1}$. Let $t', n', k' \in \omega$ be such that $T_0 + z_{0,t'} \le w' < T_0 + z_{0,t'+1}$, $t_{n'+1} \le t' < t_{n'+2}$, $k(n', w_{t'}) \le k' \le k(n', w'_{t'})$, and $T_{n'} + z_{n',k'} \le w' < T_{n'} + z_{n',k'+1}$. Since $d^*(w') \subseteq d_{j_{n'}}(\eta_{t',k'})$,

$$\min(d^*(w')) \ge \min(d_{j_{n'}}(\eta_{t',k'})) \ge \eta_{t',k'} \ge \Sigma(T_0 + z_{0,t'+1}) \ge \Sigma(T_0 + z_{0,t_1}) \ge \Sigma(w+1) > \max(d^*(w)),$$

as needed. Now suppose that we also have $w \ge T_0 + z_{0,t_1}$. Let $t', n', k' \in \omega$ be as in the previous case. Additionally, let $t,n,k \in \omega$ be such that $T_0 + z_{0,t} \le w < T_0 + z_{0,t+1}$, $t_{n+1} \le t < t_{n+2}$, $k(n,w_t) \le k \le k(n,w'_t)$, and $T_n + z_{n,k} \le w < T_n + z_{n,k+1}$. Note $t \le t'$ and $n \le n'$. Note also that $x = \max(d^*(w)) \in d_{j_n}(\zeta_{n,w})$ and that $y = \min(d^*(w')) \in d_{j_{n'}}(\zeta_{n',w'})$. Suppose first that $n < n'$. Then $t < t'$, and it immediately follows from one of our earlier observations that $x < y$, as required. So suppose next that $n = n'$. Then $k \le k'$ and if $k = k'$, then $t = t'$. Suppose first that $k < k'$. In this case, $k = k(n,w)$ and $k' = k(n,w')$. Hence $l(n,w) < l(n,w')$, whence $\zeta_{n,w} < \zeta_{n,w'}$. Therefore $x \le \max(d_{j_n}(\zeta_{n,w})) < \min(d_{j_n}(\zeta_{n,w'})) = \min(d_{j_{n'}}(\zeta_{n',w'})) \le y$, as needed. Finally assume that $k = k'$. Then we have $n = n'$,

$k = k'$, and $t = t'$. Furthermore, $T_n + z_{n,k} \le w < w' < T_n + z_{n,k+1}$, and so $\max(d^*(w)) < \min(d^*(w'))$ by the choice of $d^*(w)$ and $d^*(w')$. We have now dealt with all possible cases, and this concludes the verification that $d^* \in \mathbb{P}$.

Next we check that $\forall j \in \omega \left[d^* \le d_j\right]$. Fix $j \in \omega$. Since the sequence $\langle j_n : n \in \omega\rangle$ is strictly increasing, it is possible to find $m \in \omega$ so that $j_m \ge j$. Consider any $w \in \omega$ with $w \ge T_0 + z_{0,t_{m+1}}$. Then $w \ge T_0 + z_{0,t_1}$. Let $t, n, k \in \omega$ be such that $T_0 + z_{0,t} \le w < T_0 + z_{0,t+1}$, $t_{n+1} \le t < t_{n+2}$, $k(n, w_t) \le k \le k(n, w'_t)$, and $T_n + z_{n,k} \le w < T_n + z_{n,k+1}$. Note that $m \le n$. Also $d^*(w) \subseteq d_{j_n}(\zeta_{n,w})$, and $\zeta_{n,w} \ge M_n$ because $\Lambda_{n,l(n,w)} \setminus M_n \ne 0$. Since $d_{j_n} \le_{R_n} d_j$, there exists $\zeta \ge \zeta_{n,w}$ such that $d_{j_n}(\zeta_{n,w}) \subseteq d_j(\zeta)$. Therefore $d^*(w) \subseteq d_j(\zeta)$ and $\zeta \ge \zeta_{n,w} = \eta_{t,k} \ge \Sigma(T_0 + z_{0,t+1}) \ge \Sigma(w) \ge w$. Thus we have shown that $\forall w \ge T_0 + z_{0,t_{m+1}} \exists \zeta \ge w \left[d^*(w) \subseteq d_j(\zeta)\right]$, which proves that $d^* \le d_j$.

Next we define π and ψ in ω^ω as follows. For each $w \in \omega$, define $\psi(w) = C(w)$. Define π so that for each $w \in \omega$, $\pi'' d^*(w) = \{C(w)\} = \{\psi(w)\}$, and for all $x \in \omega \setminus \mathrm{set}(d^*)$, $\pi(x) = 0$. It is clear that $\pi''\,\mathrm{set}(d^*) = \bigcup_{w\in\omega} \pi'' d^*(w) = \bigcup_{w\in\omega} \{C(w)\} = \{C(w) : w \in \omega\} = C = \mathrm{set}\left(c^{\mu}_{i^*}\right)$, as required. It is also clear from the definitions that $\langle \pi, \psi, d^*\rangle$ is a normal triple.

It only remains to verify clause (5). To this end, fix $m \in \omega$. Consider any $w \in \omega$ with $w \ge T_0 + z_{0,t_{m+1}}$. Then $w \ge T_0 + z_{0,t_1}$. Let $t, n, k \in \omega$ be such that $T_0 + z_{0,t} \le w < T_0 + z_{0,t+1}$, $t_{n+1} \le t < t_{n+2}$, $k(n, w_t) \le k \le k(n, w'_t)$, and $T_n + z_{n,k} \le w < T_n + z_{n,k+1}$. Note that $m \le n$. Consider any $x \in d^*(w) \subseteq d_{j_n}(\zeta_{n,w})$. By one of our earlier observations it follows that $\pi_{\delta_m}(x) = \pi_{\mu,\delta_m}(C(w)) = \pi_{\mu,\delta_m}(\pi(x))$. Thus we have proved that $\forall w \ge T_0 + z_{0,t_{m+1}} \forall x \in d^*(w) \left[\pi_{\delta_m}(x) = \pi_{\mu,\delta_m}(\pi(x))\right]$. Now for the more general claim unfix m and fix any $\alpha \in X$. Let $m \in \omega$ be such that $\delta_m \ge \alpha$. Using clause (2), fix $j \in \omega$ and $W_0 \in \omega$ such that $\forall x \in \mathrm{set}(d_j) \llbracket W_0 \rrbracket \left[\pi_\alpha(x) = \pi_{\delta_m,\alpha}(\pi_{\delta_m}(x))\right]$. Next, fix $W_1 \in \omega$ with $d^* \le_{W_1} d_j$. Next let $W_2 \in \omega$ be such that $\forall x \in \mathrm{set}(d^*) \llbracket W_2 \rrbracket \left[\pi_{\delta_m}(x) = \pi_{\mu,\delta_m}(\pi(x))\right]$. Finally because of the fact that $c^{\mu}_{i^*} \le c^{\mu}_{i_*(\alpha,\delta_m)}$, there exists $W_3 \in \omega$ having the property that $\forall x \in \mathrm{set}\left(c^{\mu}_{i^*}\right)[W_3] \left[\pi_{\mu,\alpha}(x) = \pi_{\delta_m,\alpha}(\pi_{\mu,\delta_m}(x))\right]$. Now define $W = \max\{W_0, W_1, W_2, W_3\}$. Then for every $x \in \mathrm{set}(d^*) \llbracket W \rrbracket$, $\pi_\alpha(x) = \pi_{\delta_m,\alpha}(\pi_{\delta_m}(x)) = \pi_{\delta_m,\alpha}(\pi_{\mu,\delta_m}(\pi(x))) = \pi_{\mu,\alpha}(\pi(x))$, verifying (5). □

5.8. **Theorem.** *Let $\delta \le \omega_2$ be any ordinal. Then a pair of sequences*

$$\mathscr{S} = \langle \langle c^{\alpha}_i : i < \mathfrak{c} \wedge \alpha < \delta\rangle, \langle \pi_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$$

is δ-generic if and only if it is weakly δ-generic.

The proofs of Lemma 5.6 and Lemma 5.7 show that when $\mathfrak{c} > \aleph_1$ and MA holds, then for any ordinal $\delta \le \mathfrak{c}^+$, every weakly δ-generic sequence is also δ-generic in the suitably generalized sense. Combined with the results of [**?**], we get the following corollary.

5.9. **Corollary.** *If* CH *holds, then for every $\delta < \omega_2$, any weakly δ-generic sequence can be extended to a weakly ω_2-generic sequence, and if* MA *holds, then for every $\delta < \mathfrak{c}^+$, any weakly δ-generic sequence can be extended to a weakly $\mathfrak{c}^+$-generic sequence. In particular, the sequence of ultrafilters $\langle \mathscr{U}_\alpha : \alpha < \delta\rangle$ given by any weakly δ-generic sequence can be extended to a strictly $\le^+_{RB}$–increasing chain of ultrafilters $\langle \mathscr{U}_\alpha : \alpha < \mathfrak{c}^+\rangle$.*

References

E-mail address: dilip.raghavan@protonmail.com

Department of Mathematics, National University of Singapore, Singapore 119076,

URL: http://www.math.toronto.edu/raghavan

E-mail address: jonathan.verner@ff.cuni.cz

URL: http://logic.ff.cuni.cz/people/verner

Department of Logic, Faculty of Arts, Charles University, nám. Jana Palacha 2, 116 38 Praha 1,