Weakly Represented Families in the Context of Reverse Mathematics

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Abstract. It is common pratice in computability theory to represent a family of objects by a single object; this is typically done in such a way that the individual members of the family can be derived in a uniformly effective way from the single object. For example, a single set A could represent the family of sets consisting of the *rows* of A. This approach is called *uniform* and is relatively restricted in what families of objects it allows to represent.

In this article we study the reverse mathematics proof strength of second order logic principles that make statements about families of sets and functions. To allow us to make these statements more expressive, we would like to formulate them in such a way that they talk about a larger class of families of sets and functions than can be represented in the uniform way. To enable this, we define a more general way of representing families, the so-called weak representation of families of functions and sets. Using this tool, we can then state and investigate the second order logic principles we want to study.

Specifically, we investigate the Domination Principle DOM, the Avoidance Principle AVOID, the Meeting Principle MEET, and the Hyperimmunity Principle HI. Furthermore, we define the Cohesion Principle for weakly represented families COHW and separate it from the previously known Cohesion Principle COH.

The results obtained witness that the notion of weakly represented families is a useful and robust tool in reverse mathematics.

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1 Introduction

The study of cardinal invariants of the continuum is an important and well-studied branch of set-theory. A cardinal invariant is a cardinal that lies between ω_1 and the continuum $2^{\aleph_0}$. Their study has been important both for forcing theory and for the development of techniques for constructing certain special sets of real numbers in ZFC.

In this work we try to formulate analogues of some of these cardinal invariants in the context of models of second order arithmetic and reverse mathematics. Consider a model of second order arithmetic $(M, S, +, \cdot, 0, 1)$. The basic idea of the present study is that if a suitably "nice" coding of a set of subsets of M satisfying certain combinatorial properties is present in the second order part S of this model, then this corresponds to the set-theoretic statement that a certain cardinal invariant of the continuum is small. The notion of "nice coding" that we will use is that of weakly represented families, the definition of which will be made precise in Definition 4.

In the next subsection we give a short introduction to reverse mathematics, which will then allow us to formulate the second order arithmetical principles that we wish to study in Subsection 1.2. In Subsection 1.3 we can then discuss the connections with cardinal invariants.

1.1 Second order arithmetic and its base system

Second order arithmetic is the two-sorted strengthening of first order logic, that is, it is obtained as follows: We introduce set variables in addition to the number variables existing in first order logic. The function and relation symbols "$\cdot$", "$+$", "$=$" and "$<$" of the language of first order logic remain unchanged, and are supplemented by a new relation symbol "$\in$".

Adopting the convention of Simpson [20], we let $\mathcal{L}_2$ denote the language of second order arithmetic. In the following, without explicit mention, we will let capital letters denote set variables while lower-case letters will denote number variables.

Definition 1 (Second order arithmetic). The axioms of second order arithmetic consist of the universal closure of the following $\mathcal{L}_2$-formulas.

1. Basic Axioms:
   - $n + 1 \neq 0$
   - $m + 1 = n + 1 \rightarrow m = n$
   - $m + 0 = m$
   - $m + (n + 1) = (m + n) + 1$
   - $m \cdot 0 = 0$
   - $m \cdot (n + 1) = (m \cdot n) + m$
   - $\neg(m < 0)$
   - $m < n + 1 \rightarrow (m < n \vee m = n)$
   - $\neg(n \in m)$

- $\neg(X \in n)$
- $\neg(X \in Y)$

2. Induction Axiom: $(0 \in X \wedge \forall n\,(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n\,(n \in X)$
3. Comprehension Axioms:

$$\exists X\ \forall n\ (n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is any $\mathcal{L}_2$ formula in which X does not occur freely.

In the context of reverse mathematics, in order to investigate the strength of different axiom systems, we need to first agree on a base system, that is, on the basic logical facts that we take for granted.

Definition 2 (Induction schemes). Given a set of formulae $\mathcal{B}$, the $\mathcal{B}$-induction scheme consists of all axioms of the form

$$(\varphi(0) \wedge \forall n\ (\varphi(n) \rightarrow \varphi(n+1)))\ \rightarrow\ \forall n\ (\varphi(n))$$

for any formula $\varphi(n) \in \mathcal{B}$ in which X does not occur freely.

Definition 3 (Base system RCA_0). RCA_0 is the subsystem of second order arithmetic consisting of the Basic Axioms as in Definition 1 (1), the Σ^0_1-induction scheme as in Definition 2, and the Comprehension Axioms as in Definition 1 (3) restricted to the class of Δ^0_1 formulas.

It is reasonable to use RCA_0 as base system for the investigation of stronger axiom systems in the context of reverse mathematics as it captures the effective aspects of mathematics. Additionally, it was shown that a fair number of mathematical theories can be developed relying solely on RCA_0; for details, see Simpson [20]. In this article we will also follow this established pratice unless otherwise stated.

It is common to informally refer to different base systems as different logical *principles*, and we will employ this expression frequently in the following.

1.2 Some second order combinatorial principles

A model of a set of second order arithmetical principles in general takes the form $\mathcal{M} = (M, S, +, \cdot, 0, 1)$ where M is the first order part of the structure and S is the second order part. If we decide not to require that all of the axioms of Definiton 1 hold, but only a subset of them, such as RCA_0, then it is not guaranteed anymore that a model of such an axiom set has $S = \mathcal{P}(M)$; typically, S will be much smaller.

The major textbook of reverse mathematics, Simpson [20], describes the five major axiom systems of reverse mathematics that cover many branches of mathematics, such as algebra, analysis, etc. Hirschfeldt and Shore [7] studied some further combinatorial principles which are now at the center of attention of many logicians. In particular, Ramsey's Theorem for Pairs has been an interesting subject of study for a large number of researchers for decades.

Before we can define the principles that we will study in this article, we need the following definitions.

Definition 4 (Weakly represented partial functions). A partial function f is said to be weakly represented by a set A if, for every x and y, there exists a z with $\langle x, y, z\rangle \in A$ iff

1. $x \in \mathrm{dom}(f) \wedge f(x) = y$, and (representation)
2. $\forall x, y, y', z, z' \, [(\langle x, y, z\rangle \in A \wedge \langle x, y', z'\rangle \in A) \rightarrow y = y']$, and (consistency)
3. $\forall x, y, y', z, z' \, [(\langle x, y, z\rangle \in A \wedge z < z') \rightarrow \langle x, y, z'\rangle \in A]$, and (monotonicity)
4. $\exists z \, \langle x, y, z\rangle \in A \rightarrow \forall t < x \, \exists y' \, \exists z' \, \langle t, y', z'\rangle \in A$. (downward closure)

Definition 5 (Weakly represented families of functions). Let $A \in S$ be given and write $A_e = \{n \colon \langle e, n\rangle \in A\}$, $e \in M$, for its rows. For each e, write f_e for the (possibly partial) function weakly represented by A_e.

Then a set of total functions $\mathcal{F}$ is said to be a weakly represented family of functions represented by A if we have that $\mathcal{F}$ contains exactly those f_e, $e \in M$, that are total.

Note that all functions in a weakly represented family are by definition total. Rows A_e of A that do not represent such a function are ignored.

Definition 6 (Weakly represented families of sets). A set of sets $\mathcal{S}$ is said to be a weakly represented family of sets if their corresponding characteristic functions form a weakly represented family of functions.

Definition 7. $\mathcal{F}$ is said to be a uniform family of sets represented by A if

$$\mathcal{F} = \{A_e \colon e \in M\}$$

where $A_e = \{n \colon \langle e, n\rangle \in A\}$, $e \in M$.

Remark 8. It is easy to see that every uniform family of sets represented by some A is also a weakly represented family of sets represented by some A' where $A =_{\mathrm{T}} A'$.

One motivation for introducing weakly represented families is that the set of all partial recursive functions is a weakly represented family of functions. Similarly, it can easily be seen that in the classical setting the collection of all recursive sets is a weakly represented family of sets. This is because the class of characteristic functions of recursive sets

$$\mathcal{F} = \{\varphi_e \colon \varphi_e \text{ is total} \wedge \mathrm{range}(\varphi_e) \subseteq \{0, 1\}\}$$

can be weakly represented by a recursive set in any model of RCA_0.

These are examples of how the notion of weakly represented families enables us to talk about more and larger sets of functions; and this new ability then allows us to define new reverse mathematics principles, as we will now see.

For any set $A \subseteq M$, write $\bar{A}$ for $M \setminus A$.

Definition 9 (Cohesive set). Given a set of sets $\mathcal{F} \subseteq M^M$, a set G is said to be $\mathcal{F}$-cohesive if for any $A \in \mathcal{F}$, either $G \subseteq^* A$ or $G \subseteq^* \bar{A}$.

If $\mathcal{F}$ is the collection of all recursive sets, then G is called r-cohesive.

Statement 10 (Cohesion Principle COH). *For every uniform family $\mathcal{F}$ of sets, there exists an $\mathcal{F}$-cohesive set.*

The principle COH has already been studied by Jockusch and Stephan [12] as well as by Cholak, Jockusch, and Slaman [2]. In this article we will also study COHW, a variant of COH that takes advantage of the new possibilities introduced with the notion of weakly represented families of sets.

Statement 11 (Cohesion for weakly represented families COHW). *For every weakly represented family $\mathcal{F}$ of sets, there exists an $\mathcal{F}$-cohesive set.*

By Remark 8, COHW trivially implies COH. But we will show that the other implication does not hold, not even over ω-models.

Statement 12 (Domination Principle DOM). *Given any weakly represented family of functions $\mathcal{F}$, there exists a function g such that for every $f \in \mathcal{F}$ there is some $b \in M$ such that $g(x) > f(x)$ for all $x > b$.*

We will show that over RCA_0 and $\mathsf{I}\Sigma_2$, DOM implies COH and even COHW. It is unknown whether the implication still holds over weaker base systems.

Statement 13 (Hyperimmunity Principle HI). *Given any weakly represented family of functions $\mathcal{F}$, there exits a function g such that for each $f \in \mathcal{F}$ and each $b \in M$ we have $g(x) > f(x)$ for some $x > b$.*

HI is weaker than DOM.

For $f, g \in M^M$ we write $f <^* g$ to express that $\{n \in M : g(n) \leq f(n)\}$ is finite. The symbol "$\leq^*$" is defined accordingly. A subset $\mathcal{F} \subseteq M^M$ is called *bounded* if there exists $g \in M^M$ such that for all $f \in \mathcal{F}$ we have $f <^* g$. Otherwise $\mathcal{F}$ is said to be *unbounded*.

Statement 14 (Meeting Principle MEET). *Given any weakly represented family of functions $\mathcal{F}$, there exits a function g such that for each $f \in \mathcal{F}$ the set $\{n \in M : f(n) = g(n)\}$ is infinite.*

We will show that HI and MEET are equivalent.

Definition 15. We say that a function g avoids a function f if

$$\{n \in M : f(n) = g(n)\}$$

is finite.

Statement 16 (Avoidance Principle AVOID). *Given any weakly represented family of functions $\mathcal{F}$, there exits a function g avoiding all $f \in \mathcal{F}$.*

Two subsets of M, A and B, are said to be *almost disjoint* if $A \cap B$ is finite. A set $\mathcal{F} \subseteq M^M$ is called *almost disjoint* if any two distinct elements of $\mathcal{F}$ are almost disjoint. A set $\mathcal{F} \subseteq M^M$ is called *maximal almost disjoint* if it is infinite and almost disjoint and is not properly contained in any larger almost disjoint set. Any infinite almost disjoint set can be extended to a maximal almost disjoint set by Zorn's Lemma.

Statement 17 (Maximal Almost Disjoint Family Principle MAD). *There exists a weakly represented family $\mathcal{F}$ of infinite sets such that*

- *if $A, B \in \mathcal{F}$ are pairwise different, then $A \cap B$ is infinite, and*
- *for every infinite set C there is a $D \in \mathcal{F}$ such that $C \cap D$ is infinite.*

For a set $A \subseteq M$, let us temporarily write A^0 for A and A^1 for $\bar{A}$. A family $\mathcal{F} \subseteq \mathcal{P}(M)$ is said to be *independent* if for any $n \geq 1$, any collection $\{A_0, \dots, A_{n-1}\} \subseteq \mathcal{F}$, and any string $\sigma \in 2^n$, the set $\bigcap_{i<n} A_i^{\sigma(i)}$ is infinite. A *maximal independent* family is an independent family that can not be extended to a strictly larger independent family. Zorn's Lemma also guarantees the existence of maximal independent families.

Statement 18 (Maximal Independent Family Principle MIND). *There exists a weakly represented family of infinite sets that is maximal independent.*

Statement 19 (Biimmunity Principle BI). *For every weakly represented family $\mathcal{F}$ of infinite sets there is a set $B \in S$ such that there is no set $A \in \mathcal{F}$ with $A \subseteq B$ or $A \subseteq \overline{B}$.*

1.3 Cardinal invariants

DILIP: We need to have a better explanation of what the cardinal invariants have to do with the reverse mathematics principles that we study. In the following text, repeatedly a correspondance is claimed, but this is never explained. Maybe it is enough to explain once in the beginning what the relationship, or maybe every correspondence needs to be explained individually.

We now discuss the nine cardinal invariants of the continuum that are considered in this paper, the most basic being the cardinality of the continuum.

Definition 20. $\mathfrak{c} = 2^{\aleph_0} = |\mathbb{R}|$.

Recall that the Continuum Hypothesis CH is the statement that $\mathfrak{c} = \aleph_1$. The analogue of CH in a model $(M, S, \cdot, +, 0, 1)$ is the statement that there is a weakly represented family of sets $A \in S$ such that the characteristic function of every member of S appears in A. The simplest example of this is the case where S consists exactly of the recursive sets.

Recall the partial order $\langle M^M, <^* \rangle$ from the previous subsection. $\mathcal{F}$ is *dominating* if for all $g \in M^M$ there exists an $f \in \mathcal{F}$ with $g <^* f$. It is clear that every dominating set is unbounded. Based on these definitions, we define the following two cardinal invariants.

Definition 21.
$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \mathcal{F} \text{ is unbounded}\}$$
$$\mathfrak{d} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \mathcal{F} \text{ is dominating}\}$$

DILIP: Is it correct to use ω^ω here, or does it have to be M^M? Similar for all the other definitions of cardinal invariants.

It is easy to prove that $\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$ Dilip: "cf" is undefined.. It is also a classical theorem of Hechler that these are the only restrictions that are provable in ZFC. In a second order model $(M, S, \cdot, +, 0, 1)$, the statement $\mathfrak{b} = \aleph_1$ corresponds to the statement that there is a weakly represented family $\mathcal{F} \in S$ of functions such that no function in S dominates all the members of $\mathcal{F}$. This is the negation of the principle DOM. So DOM is the analogue of $\mathfrak{b} > \aleph_1$. Similarly HI corresponds to $\mathfrak{d} > \omega_1$.

Another important pair of cardinals come from the notion of splitting. Let $a, b \subseteq M$. We say that A *splits* B is both $B \cap A$ and $B \cap \bar{A}$ are infinite. A set $\mathcal{F} \subseteq \mathcal{P}(M)$ is called a *splitting family* if $\forall B \in M^M\ \exists A \in \mathcal{F}\ [A$ splits $B]$. A set $A \subseteq M$ is said to *reap* a family $\mathcal{F} \subseteq \mathcal{P}(M)$ if for all $B \in \mathcal{F}$ we have that A splits B. A family $\mathcal{F} \subseteq M^M$ is *unreaped* if there is no $A \in \mathcal{P}(M)$ which reaps $\mathcal{F}$. The following cardinals correspond to the notions of splitting and reaping.

Definition 22. $\mathfrak{s} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{P}(\omega) \wedge \mathcal{F} \text{ is a splitting family}\}$

$$\mathfrak{r} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \wedge \mathcal{F} \text{ is an unreaped family}\}$$

Dilip: What is $[\omega]^\omega$? Appears in other places in article.

It is not difficult to prove that $\mathfrak{s} \leq \mathfrak{d}$ and this proof dualizes to show that $\mathfrak{b} \leq \mathfrak{r}$. Blass and Shelah constructed a model with $\aleph_1 = \mathfrak{r} < \mathfrak{s} = \aleph_2$ and $\aleph_1 = \mathfrak{s} < \mathfrak{b} = \aleph_2$ holds in the Laver model. The notion of a cohesive set in recursion theory is related to the notion of splitting. To say that G is $\mathcal{F}$-cohesive is the same as saying that G is not split by any member of $\mathcal{F}$. So the principle COHW corresponds to the statement $\mathfrak{s} > \aleph_1$. COH is a bit stronger. The principle BI corresponds to the statement $\mathfrak{r} > \aleph_1$. The reverse mathematical analogue of the ZFC theorem $\mathfrak{s} \leq \mathfrak{d}$ would be the statement that COHW implies HI. However in contrast with Blass and Shelah's result that $\aleph_1 = \mathfrak{b} = \mathfrak{r} < \mathfrak{s} = \aleph_2$ is consistent with ZFC, we can show that COHW implies DOM. Given that DOM is stronger than HI this last statement is more than one might expect by analogy with ZFC.

The next group of cardinals that we will define stem from the context of categoricity. Recall that a set $X \subseteq \mathbb{R}$ is called *nowhere dense* if the interior of its closure is empty. A subset of $\mathbb{R}$ is *meager* if it is the union of countably many nowhere dense sets. We define the following cardinals.

Definition 23. $\mathrm{cov}(\mathcal{M}) = \min\left\{ |\mathcal{F}| : \begin{array}{c} \mathcal{F} \text{ consists of meager subsets of } \mathbb{R} \\ \text{and } \bigcup \mathcal{F} = \mathbb{R} \end{array} \right\}$

$$\mathrm{non}(\mathcal{M}) = \min\{|M| : M \text{ is a non-meager subset of } \mathbb{R}\}$$

These topologically defined cardinals have purely combinatorial characterizations, as the following theorem shows.

Theorem 24 (Miller Dilip: Insert citation.).

1. *$\mathrm{cov}(\mathcal{M})$ is the minimal cardinal κ such that there exists an $\mathcal{F} \subseteq \omega^\omega$ with $|\mathcal{F}| = \kappa$ and such that for all $g \in \omega^\omega$ there is an $f \in \mathcal{F}$ such that $\{n \in \omega\colon f(n) = g(n)\}$ is finite.*

2. non$(\mathcal{M})$ *is the minimal cardinal* κ *such that there exists an* $\mathcal{F} \subseteq \omega^\omega$ *with* $|\mathcal{F}| = \kappa$ *and such that for all* $g \in \omega^\omega$ *there is an* $f \in \mathcal{F}$ *such that* $\{n \in \omega \colon g(n) = f(n)\}$ *is infinite.*

This theorem allows us to formulate analogues of these topological invariants in any model of second order arithmetic. The principle MEET corresponds to the statement that $\mathrm{cov}(\mathcal{M}) > \aleph_1$ and AVOID is the analogue of $\mathrm{non}(\mathcal{M}) > \aleph_1$. As it is easy to prove in ZFC that $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}$, one would expect MEET to imply HI, and indeed this is easy to check. But somewhat unexpectedly we will prove that MEET and HI are equivalent — at least for ω-models. This contrasts with the fact that $\mathrm{cov}(\mathcal{M}) = \aleph_1 < \aleph_2 = \mathfrak{b} = \mathfrak{d}$ holds in the Laver model DILIP: Insert citation.. As a result, in the classical ZFC context, we do not even have that DOM implies MEET. Dualizing the equivalence of MEET and HI one would expect AVOID to be equivalent to DOM DILIP: Is this true?. Also it is consistent that $\mathfrak{b} = \aleph_1 < \aleph_2 = \mathrm{cov}(\mathcal{M})$; in fact this holds in the Cohen model DILIP: Insert citation.. This is reflected by the fact that MEET does not imply DOM DILIP: Is this correct?. Next, regarding $\mathrm{non}(\mathcal{M})$, it can be proved in ZFC that $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$, and, accordingly, DOM implies AVOID. It is also easy to prove in ZFC that $\mathfrak{s} \leq \mathrm{non}(\mathcal{M})$. This is only partially true in the reverse mathematical context. We will prove that COHW implies AVOID in ω-models. However this is not true in all non-ω-models, as we will show. Finally, in the classical ZFC context, $\mathfrak{d}$ and $\mathrm{non}(\mathcal{M})$ are independent, meaning that while it is consistent to have $\aleph_1 = \mathfrak{d} < \mathrm{non}(\mathcal{M}) = \aleph_2$ it is also consistent to have $\aleph_1 = \mathrm{non}(\mathcal{M}) < \mathfrak{d} = \aleph_2$. This is reflected by the independence of AVOID and MEET, even in ω-models.

We also considered cardinal invariants associated with almost disjointness and independence.

Definition 25.
$$\mathfrak{a} = \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^\omega \text{ is a maximal almost disjoint family}\}$$
$$\mathfrak{i} = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathcal{P}(\omega) \text{ is a maximal independent family}\}$$

DILIP: Should the first thing not also be defined with the power set?
The principle MAD corresponds to $\mathfrak{a} = \aleph_1$ and MIND corresponds to $\mathfrak{i} = \aleph_1$.

We prove that, at least for ω-models, MAD holds iff DOM fails. Since in ZFC the inequality $\mathfrak{b} \leq \mathfrak{a}$ holds, it does not come as a surprise that DOM implies $\neg$MAD. However, $\aleph_1 = \mathfrak{b} < \mathfrak{a} = \aleph_2$ is consistent by a theorem of Shelah DILIP: Insert citation., so the fact that $\neg$DOM implies MAD is indeed unexpected.

Similarly, we prove that, at least for ω-models, MIND holds iff BI fails. One can prove in ZFC that $\mathfrak{r} \leq \mathfrak{i}$ and so the direction from BI to $\neg$MIND is unsurprising. But once again the consistency of $\mathfrak{r} = \aleph_1 < \aleph_2 = \mathfrak{i}$ was proved by Shelah DILIP: Insert citation., making the implication from $\neg$BI to MIND unexpected.

2 Cohesion Principles

We need the following definitions.

Definition 26. Let A and B be sets. A is PA-complete with respect to B (written as $A \gg B$) if for every partial B-recursive $\{0,1\}$-valued function f, there exists an A-recursive total extension g of f. In this definition we can replace sets by degrees in the canonical way.

Definition 27. Let A and B be sets. A is hyperimmune-free with respect to B if every function recursive in $B \oplus A$ is dominated by some B-recursive function.

Theorem 28. *Over* RCA_0, COH *does not imply* COHW. *This even holds for* ω*-models.*

To proof the non-implication for ω-models, we need the following lemmata and theorem. The first lemma establishes a relationship between two 1-generic sets and their join. It is the genericity analogue of van Lambalgen's Theorem 41.

Lemma 29 (Yu [22]). *The following are equivalent for* $n \geq 1$.

1. $A \oplus B$ *is* n*-generic.*
2. A *is* n*-generic and* B *is* n*-generic relative to* A.
3. B *is* n*-generic and* A *is* n*-generic relative to* B.

The following theorem is a reformulation and slight variation of a result of Jockusch and Stephan [12, Theorem 2.1]; the proof is largely identical and omitted here.

Theorem 30. *Let* $\mathcal{F}$ *be a uniform family represented by* A. *If* $B' \gg A'$ *then there is a* B*-recursive* $\mathcal{F}$*-cohesive set.*

Lemma 31. *There exists a sequence of sets* $(A_i : i \in \omega)$ *such that* A_0 *is* 1*-generic and such that, for every* $i \in \omega$, A_{i+1} *is* 1*-generic,* $A_{i+1} \geq_{\mathrm{T}} A_i$, *and* $A'_{i+1} \gg A'_i$ *and* A'_{i+1} *is hyperimmune-free relative to* A'_i.

Proof. Jockusch and Soare [11] showed that every recursive tree has a path which is hyperimmune-free. Computably in $\emptyset'$ we can build a tree each path of which is a complete extension of PA relative to $\emptyset'$ (see Odifreddi [18]). By the hyperimmune-free basis theorem (for example, see Downey and Hirschfeldt [4]) there exists a set A that is PA-complete relative to $\emptyset'$ and hyperimmune-free relative to $\emptyset'$; in particular $A \geq_{\mathrm{T}} \emptyset'$. By the Jump Inversion Theorem of Soare [21] we can find a 1-generic set A_0 such that $A'_0 =_{\mathrm{T}} A$. This finishes the base case construction.

Suppose inductively that for $i \geq 0$ we have found an A_i as required. To obtain A_{i+1}, take a set $B \gg A'_i$ that is hyperimmune-free relative to A'_i, then apply the Jump Inversion Theorem relative to A_i and obtain C_{i+1} such that $C'_{i+1} = B$ and such that C_{i+1} is 1-generic relative to A_i. Let $A_{i+1} = C_{i+1} \oplus A_i$ and by Lemma 29 and using that A_i is 1-generic by induction hypothesis we have that A_{i+1} is 1-generic.

By repeating this process infinitely often, we obtain $(A_i : i \in \omega)$. ☐

Now let $S = \{A \subseteq \omega : A \leq_{\mathrm{T}} A_0 \oplus \cdots \oplus A_n \text{ for some } n \in \omega\}$, where the sets A_i, $i \in \omega$, are as in Lemma 31. The following lemmata show that the ω-model $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$ satisfies COH and RCA_0, but not COHW.

Lemma 32. $\mathcal{M} \models \mathsf{COH} + \mathsf{RCA}_0$

Proof. First note that S is closed under Turing reducibility and join, thus $\mathcal{M}$ is a model of RCA_0.

To see that $\mathcal{M}$ is also a model of COH, let a uniform familiy $\mathcal{F}$ represented by $A \leq_{\mathrm{T}} A_0 \oplus \cdots \oplus A_n =_{\mathrm{T}} A_n$ for large enough n be given. Then, by construction, $A'_{n+1} \gg A'_n$, and we can apply Theorem 30 with A_{n+1} substituted for B and A_n substituted for A to see that there is an A_{n+1}-recursive $\mathcal{F}$-cohesive set. $\square$

Lemma 33. *For any set $X \in S$, X is not high, that is, $X' \not\geq_{\mathrm{T}} \emptyset''$.*

Proof. We show this by induction. A'_0 is hyperimmune-free with respect to $\emptyset'$, therefore, for each function g recursive in A'_0, there exists a function f recursive in $\emptyset'$ such that g is dominated by f. By relativizing Martin's characterization of high degrees in terms of domination [16] to $\emptyset'$, we have that $A''_0 \not\geq_{\mathrm{T}} \emptyset'''$, therefore $A'_0 \not\geq_{\mathrm{T}} \emptyset''$, and so A_0 is not high.

Now suppose inductively that A_n is not high. As A'_{n+1} is hyperimmune-free relative to A'_n, we have that for each g recursive in A'_{n+1} there exists an f_0 recursive in A'_n such that f_0 dominates g. Similarly, there exists an f_1 recursive in A'_{n-1} such that f_1 dominates f_0 thus g. Continuing in this fashion, we finally obtain a function f_{n+1} recursive in $\emptyset''$ such that f_{n+1} majorizes g. Thus as before, $A''_{n+1} \not\geq_{\mathrm{T}} \emptyset'''$ so $A'_{n+1} \not\geq_{\mathrm{T}} \emptyset''$, so A_{n+1} it is not high. $\square$

Lemma 34. *No 1-generic set computes a diagonally non-recursive function.*

Proof. Suppose otherwise and let $f = \varphi_e^G$ where G is 1-generic and f is diagonally non-recursive. Let $W_f = \{\sigma\colon \exists n\ \exists t\ \varphi_{n,t}(n)\downarrow\ \wedge\ \varphi^{\sigma}_{e,t}(n) = \varphi_n(n)\}$. If there is a $\sigma \prec G$ with $\sigma \in W_f$, then f is not diagonally non-recursive. If there exists $\sigma \prec G$ such that $\forall \tau \succeq \sigma\ [\tau \notin W_f]$, then we can let $h(n) = \varphi_e^{\tau}(n)$ where τ is the minimal extension of σ such that $\varphi_e^{\tau}(n)\downarrow$. Such a τ exists for every n as by assumption φ_e^G is total. However, the function h defined this way is diagonally non-recursive and it is also recursive by definition, contradiction. Therefore no 1-generic set can compute a diagonally non-recursive function. $\square$

Corollary 35. $\mathcal{M} \not\models \mathsf{COHW}$

Proof. By Lemma 33, no set in S is high. Suppose for the sake of contradiction that there exists an r-cohesive set S. By Jockusch and Stephan [12], each non-high r-cohesive set is cohesive. However, again by Jockusch and Stephan [12], each cohesive set computes a diagonally non-recursive function f. By definition of S there exists an $i \in \omega$ such that f is computed by the 1-generic set $A_0 \oplus \cdots \oplus A_i$. This contradicts Lemma 34. $\square$

This concludes the proof of Theorem 28, separating COH and COHW over RCA_0.

Theorem 36. COH *does not imply* AVOID. *This even holds for ω-models.*

Proof. We again use the above ω-model $\mathcal{M}$ with second order part S. Observe that S is a downward closure of non-high 1-generic sets. In particular, by Lemma 34, all sets in S are neither diagonally non-recursive nor high. By a result of Kjos-Hanssen, Merkle, and Stephan [13, Theorem 5.1 ($\neg(3) \Rightarrow \neg(1)$)], this implies that no $A \in S$ computes a function avoiding all total recursive function. As the set of all total recursive functions is a weakly represented family, this contradicts AVOID. $\square$

The next theorem illustrates once more the difference in reverse mathematics strength between COH and COHW.

Theorem 37. $\mathsf{COHW} \vdash \mathsf{AVOID}$ *for ω-models.*

More precisely, given any r-cohesive set G, one can recursively produce a total function g such that $\{n \in \omega \colon g(n) = \varphi_e(n)\}$ is finite for every total recursive function φ_e.

Proof. Let $\mathcal{F}$ be the collection of all total recursive functions. We will show that there exists a function $g \in S$ such that for every $f \in \mathcal{F}$ we have that $\{n \in \omega \colon f(n) = g(n)\}$ is finite. Then the general case follows by relativization.

Let $\mathcal{F}'$ be the collection of all recursive sets. COHW ensures the existence of an $\mathcal{F}'$-cohesive set, say G. If G is high, then by Martin [16], there exists a function g recursive in G that dominates every total recursive function, and we are done.

If G is not high, then by Jockusch and Stephan [12] there exists an effectively immune set A recursive in G. Here we call A effectively immune if there is a recursive function p such that for any r.e. set W_e we have $W_e \subseteq A \to |W_e| < p(e)$. Fix this p and assume without loss of generality that it is increasing.

Let f be the total recursive function such that

$$W_{f(e,i)} = \begin{cases} W_{\varphi_i(e)} & \text{if } \varphi_i(e)\downarrow, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let g be the total recursive function such that $W_{g(e)}$ consists of the first

$$p(\max\{f(i,e) \colon i \leq e\}) + 1$$

elements of A.

Claim. For all $i \leq e$ we have $g(e) \neq \varphi_i(e)$.

Proof. Suppose otherwise, then $g(e) = \varphi_i(e)$ for some $i \leq e$. Then $\varphi_i(e)\downarrow$ and $W_{f(e,i)} = W_{\varphi_i(e)} = W_{g(e)} \subseteq A$, so

$$p(f(e,i)) < p(\max\{f(e,j) \colon j \leq e\}) + 1 = |W_{g(e)}| = |W_{f(e,i)}| < p(f(e,i)),$$

which is a contradiction. $\diamond$

Then g is the required function. $\square$

Given that the previous proof was carried out in the standard model, it is natural to ask how COHW interacts with AVOID in non-standard models.

Question 38. *In models of fragments of second order arithmetic with restricted induction, does COHW imply AVOID? If so, what level of induction is needed?*

3 The Meeting and Hyperimmunity Principles

Theorem 39. *Over* RCA_0, MEET *and* HI *are equivalent.*

Proof. $\mathsf{MEET} \vdash \mathsf{HI}$: If g is as in the statement of MEET, then HI holds with $g+1$ substituted for g.

$\mathsf{HI} \vdash \mathsf{MEET}$: Let an arbitrary model $\mathcal{M} = (M, S, +, \cdot, 0, 1)$ be given. Let $\mathcal{F}$ be a weakly represented family represented by A and let A_e and f_e, for $e \in M$, be as in Definition 5. Define a function $\widetilde{f}_e$ via $x \mapsto n_x$ for $x \in M$, where n_x is the first number of the form $\langle\langle e, x\rangle, y, z\rangle$ inside A_e, if it exists; note that $y = f_e(\langle e, x\rangle)$ whenever n_x exists. Then $\widetilde{f}_e$ is total iff f_e is total.

Note that the set $\mathcal{F}' = \{\widetilde{f}_e : \widetilde{f}_e \text{ is total}\}$ is again a weakly represented family. By applying HI to $\mathcal{F}'$ we obtain a function $\widetilde{g}$ such that for each total $\widetilde{f}_e$ there are infinitely many x with $\widetilde{g}(x) > \widetilde{f}_e(x)$.

Then define $g(\langle e, x\rangle)$ as follows: If there is a number m of the form $\langle\langle e, x\rangle, y, z\rangle$ in A_e such that $m < \widetilde{g}(x)$ then $g(\langle e, x\rangle) = y$; else $g(\langle e, x\rangle) = 0$. The function g is in S and is total; furthermore, whenever $\widetilde{g}(x) > \widetilde{f}_e(x)$ then we have $g(\langle e, x\rangle) = f_e(\langle e, x\rangle)$ and thus for all total f_e there are infinitely many n with $g(n) = f(n)$. This implies MEET. □

Our next result shows that AVOID is incomparable with HI. As essential tools we employ the following two well-known results.

Lemma 40. *There exists a hyperimmune-free Martin-Löf random set.*

Proof. Apply the hyperimmune-free basis theorem (for example, see Downey and Hirschfeldt [4]) to the complement of the first component of the universal Martin-Löf test. □

Theorem 41 (van Lambalgen [15]). *The following are equivalent.*

1. *$A \oplus B$ is n-random.*
2. *A is n-random and B is n-random relative to A.*
3. *B is n-random and A is n-random relative to B.*

Theorem 42. AVOID *does not imply* HI. *This even holds for ω-models.*

Proof. Let A be a hyperimmune-free Martin-Löf random, as in Lemma 40. For $i \in \omega$, let $A_i = \{x : \langle i, x\rangle \in A\}$. Then by Theorem 41, for every $i \in \omega$, A_{i+1} is Martin-Löf random relative to $A_0 \oplus \cdots \oplus A_i$. Fix the model $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$ with second order part $S = \{B \subseteq \omega : B \leq_{\mathrm{T}} A_0 \oplus \cdots \oplus A_i \text{ for some } i \in \omega\}$.

Let a weakly represented family $\mathcal{F}$ represented by $B \leq_{\mathrm{T}} A_0 \oplus \cdots \oplus A_i$ with $i \in \omega$ large enough be given. Fix a computably bijective map $\nu : \{0,1\}^* \to \omega$, and let g be the function $n \mapsto \nu(A_{i+1}(0) \ldots A_{i+1}(n))$. Fix any $f \in \mathcal{F}$; trivially, $f \leq_{\mathrm{T}} B$. Assume that g does not avoid f. Then there are infinitely many n such that the Kolmogorov complexity relative to B of $A_{i+1}(0) \ldots A_{i+1}(n)$ is less than $2\log(n)$, which contradicts that A_{n+1} is Martin-Löf random relative to $A_0 \oplus \cdots \oplus A_i$. Therefore $g \leq_{\mathrm{T}} A_{i+1} \in S$ is a function as required by AVOID.

On the other hand, for every $C \in S$ we have $C \leq_{\mathrm{T}} A$, and since A is hyperimmune-free, C is hyperimmune-free as well. As C was arbitrary, this implies that $\mathcal{M}$ does not satisfy HI. □

We now turn to the other direction.

Theorem 43. HI *does not imply* AVOID.

Proof. We again use the model $\mathcal{M}$ defined on page 9. As shown in the proof of Theorem 36, $\mathcal{M}$ does not satisfy AVOID.

To see that $\mathcal{M}$ satisfies HI, let a weakly represented family $\mathcal{F}$ represented by $A \leq_{\mathrm{T}} A_n$ be given. As A_{n+1} is by construction 1-generic relative to A_n, it is in particular hyperimmune relative to A. Then it computes a function g that is infinitely often larger than any function $f \leq_{\mathrm{T}} A$, and in particular g is for $\mathcal{F}$ as required by HI. □

4 The Domination Principle

Theorem 44. *Over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma_2$, DOM *implies* COH.

Proof. Let $\mathcal{M} = (M, S, +, \cdot, 0, 1)$ be a model of DOM and let $\mathcal{F}$ be a uniform family of sets represented by $A \in S$. For $e \in M$ let A_e be as in Definition 7 and let $\widetilde{f}_{e,x}(y)$ be the first $z > y$ with $\forall d \leq e\ [A_d(z) = A_d(x)]$ and let $\widetilde{\mathcal{F}}$ be the weakly represented family of those $\widetilde{f}_{e,x}$ which are total. By DOM there is a function $g \in S$ which dominates all members of $\widetilde{\mathcal{F}}$. Define an infinite set $G = \{x_0, x_1, \ldots\} \in S$ as follows:

- $x_0 = 0$, and
- Let $X_n = \{x_n + 1, x_n + 2, \ldots, x_n + g(x_n)\}$ and define x_{n+1} as the minimal $y \in X_n$ such that

  $$A_0(y)A_1(y)\ldots A_{x_n}(y) = \max\{A_0(z)A_1(z)\ldots A_{x_n}(z) \colon z \in X_n\},$$

  where the maximum is with respect to $\leq_{\mathrm{lex}}$, the lexigraphic ordering on strings.

Let $\Psi(e, x)$ be the statement

$$x \in G \ \wedge\ \forall y \geq x\ [y \in G \to A_0(y)A_1(y)\ldots A_{e-1}(y) = A_0(x)A_1(x)\ldots A_{e-1}(x)].$$

Claim. For all e, $\exists x\ (\Psi(e, x))$ holds.

Proof. As $\exists x\ \Psi(e, x)$ is a Σ^0_2 statement, using $\mathsf{I}\Sigma_2$, we can prove it by induction over $e \in M$.

The statement $\Psi(0, x)$ holds vacuously for all $x \in G$. So assume by induction that for a given $e \in M$, $\Psi(e, x')$ is true for some $x' \in G$. We distinguish two cases:

Case 1. $G \cap A_e$ is finite. Then there exists an $x'' \geq x'$ with $x'' \in G$ and $x'' > \max(A_e \cap G)$. Then for all $y \in G$ with $y \geq x''$, we have $A_e(y) = A_e(x'') = 0$ on the one hand; and by the induction hypothesis

$$A_0(y)A_1(y)\ldots A_{e-1}(y) = A_0(x'')A_1(x'')\ldots A_{e-1}(x'')$$

on the other hand. Thus $\Psi(e+1, x'')$ holds and $\exists x\ \Psi(e+1, x)$ is satisfied.

Case 2. $G \cap A_e$ is infinite. Then let x'' be any element of $G \cap A_e$ with $x'' \geq x'$. For all such x'' the function $\widetilde{f}_{e,x''}$ is the same and thus one can, without loss of generality, assume that x'' is large enough that $g(y) > \widetilde{f}_{e,x''}(y)$ for all $y \geq x''$ and $x'' > e+1$. Now let $n \in M$ be arbitrary such that $x_n \geq x''$. Then

$$A_0(x_{n+1})A_1(x_{n+1})\ldots A_{x_n}(x_{n+1}) \geq_{\mathrm{lex}} A_0(x'')A_1(x'')\ldots A_e(x'')0^{x_n-e-1};$$

and thus,

$$A_0(x_{n+1})A_1(x_{n+1})\ldots A_e(x_{n+1}) = A_0(x'')A_1(x'')\ldots A_e(x'').$$

As $n \in M$ was arbitrary with $x_n \geq x''$ it follows that $\Psi(e+1, x'')$ holds and that $\exists x\ \Psi(e+1, x)$ is satisfied. $\diamond$

Thus $\exists x\ \Psi(e, x)$ holds for all e, and in particular for each e there is an $x \in G$ with $A_e(y) = A_e(x)$ for all $y \geq x$ with $y \in G$. Thus COH is satisfied. $\square$

Question 45. *Is it true that* $\mathsf{DOM} + \mathsf{B}\Sigma_2 \vdash \mathsf{COH}$*? What about even weaker assumptions?*

4.1 DOM does not imply SRT^2_2

Lemma 46. *Let A be Martin-Löf random relative to Ω. Then A does not compute any infinite subset of Ω or $\overline{\Omega}$.*

Proof. Without loss of generality, assume that A computes an infinite subset G of Ω; the case $\overline{\Omega}$ is symmetric. By Theorem 41 we have that Ω is Martin-Löf random relative to A, and since $G \leq_{\mathrm{T}} A$, Ω is Martin-Löf random relative to G. Let $(b_i : i \in \omega)$ be a strictly monotone listing of the elements of G. Then it is easy to see that the sequence $(U_n : n \in \omega)$ defined via

$$U_n = [\{\sigma \in 2^{b_n+1} : \sigma(b_i) = 1 \text{ for all } 0 \leq i \leq n\}]$$

is a G-Martin-Löf test covering Ω, contradiction. $\square$

Theorem 47. DOM *does not imply* SRT^2_2.

Proof. We construct an ω-model of $\mathsf{DOM} + \neg\mathsf{SRT}^2_2$. To achieve this, we will use a result of Chong, Lempp, and Yang [3] and Cholak, Jockusch, and Slaman [2] who proved that SRT^2_2 is equivalent to the following principle D^2_2:

For every Δ^0_2 $G \subseteq \omega$ exists an infinite $A \subseteq \omega$ such that $A \subseteq G$ or $A \subseteq \overline{G}$.

To ensure $\neg\mathsf{SRT}^2_2$ it is therefore enough to ensure $\neg\mathsf{D}^2_2$. To this end, for all $n \in \omega$, let $A_n = \Omega^{\emptyset'} \oplus \Omega^{\emptyset''} \oplus \cdots \oplus \Omega^{\emptyset^{(n+1)}}$, where $\emptyset^{(i)}$ is the i-th Turing jump for $i \in \omega$. Now let

$$S = \{A \subseteq \omega \colon A \leq_{\mathrm{T}} A_n \text{ for some } n \in \omega\}$$

and let $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$. As $\emptyset' =_{\mathrm{T}} \Omega$ we have that $\Omega^{\emptyset'}$ is Martin-Löf random relative to Ω, and by repeated application of Theorem 41 it follows that, for any $n \in \omega$, A_n is Martin-Löf random relative to Ω. By Lemma 46 we obtain that no set in S computes an infinite subset of Ω or $\overline{\Omega}$. But since Ω is Δ^0_2 this implies $\neg\mathsf{D}^2_2$.

To see that DOM is satisfied by $\mathcal{M}$ let a weakly represented family $\mathcal{F}$ represented by $A \leq_{\mathrm{T}} A_i$ for $i \in \omega$ large enough be given. Note that A_{i+1} is high relative to A_i, and that it therefore computes a function g dominating all functions computable from A, in particular g dominates all $f \in \mathcal{F}$. As $\mathcal{F}$ was arbitrary, this establishes DOM. □

4.2 Π^1_1-conservation of DOM over RCA_0

In this section we will prove that given any Π^1_1 sentence φ, if $\mathsf{DOM} + \mathsf{RCA}_0 \vdash \varphi$, then $\mathsf{RCA}_0 \vdash \varphi$. Note that Theorem 47 is a corollary of the result of this subsection. We begin by introducing the following concept.

Definition 48. The model $\mathcal{M} = (M, S, +, \times, 0, 1, <)$ is an $\omega-$submodel of $\mathcal{M}' = (M', S', +', \times', 0', 1', <')$ if $M = M'$, $+ = +'$, $\times = \times'$, $0 = 0'$, $1 = 1'$, $< = <'$ and $S \subseteq S'$.

By a result of Cholak, Jockusch, and Slaman [2, Lemma 6.6] it suffices to show that every countable model of RCA_0 is an $\omega-$submodel of some countable model of $\mathsf{RCA}_0 + \mathsf{DOM}$ to establish that $\mathsf{RCA}_0 + \mathsf{DOM}$ is Π^1_1-conservative over RCA_0. Accordingly, the goal of this subsection is to prove the following theorem and corollary.

Definition 49. Given a structure $\mathcal{M} = (M, S, +, \times, 0, 1, <)$ of second order arithmetic and $g \subseteq M$, let $\mathcal{M}_g$ be the $\mathcal{L}_2$-structure $(M, S \cup \{g\}, +, \times, 0, 1, <)$ and $\mathcal{M}[g]$ be the $\mathcal{L}_2$-structure $(M, \Delta^0_1(\mathcal{M}_g), +, \times, 0, 1, <)$ where

$$\Delta^0_1(\mathcal{M}_g) = \{X \subseteq M : X \text{ is } \Delta^0_1 \text{ definable over } \mathcal{M}_g\}.$$

Remark 50. For any $\mathcal{L}_2$-structure $\mathcal{M}$ and $g \subseteq M$, if $\mathcal{M}_g \models \mathsf{I}\Sigma_1 +$ basic axioms, then $\mathcal{M}[g]$ is a model of RCA_0.

Theorem 51. *Given a countable model $\mathcal{M}$ of RCA_0 with second order part S and given $X \in S$, there exists a function $g \colon M \to M$ such that g dominates all $\Delta^{0,X}_1$ functions in S and $\mathcal{M}[g] \models \mathsf{RCA}_0$.*

Corollary 52. *Given a countable model $\mathcal{M}$ of RCA_0, there exists a model $\mathcal{M}'$ of DOM and RCA_0 such that $\mathcal{M}$ is an ω-submodel of $\mathcal{M}'$.*

Proof (Corollary 52). Appealing to Theorem 52 we will recursively define a sequence of models of RCA_0 $\langle \mathcal{M}_i : i \in \omega \rangle$ such that for every i we have that $\mathcal{M}_i$ is an ω-submodel of $\mathcal{M}_{i+1}$ and such that for every $\Delta_1^{0,X}$ function h coded in $\mathcal{M}_i$ for some $X \in \mathcal{M}_i$ there exists $j > i$ and a function $g \in \mathcal{M}_j$ such that g dominates h, namely $\mathcal{M}_j \models \exists b\, \forall a > b\, [g(a) > h(a)]$. Note that this will continue to hold in $\mathcal{M}_k$ for any $k > j$. This is possible since each model being constructed is countable.

Let $\mathcal{M}' = \bigcup_{i \in \omega} \mathcal{M}_i$. Then we show that $\mathcal{M}' \models \mathsf{RCA}_0 + \mathsf{DOM}$. To see that, note $\mathsf{I}\Sigma_1$ and Δ_1^0-comprehension only involve finitely many parameters so are already determined in $\mathcal{M}_j$ for large enough j (note that the first order parts of the models are the same). For each recursive $h \in \mathcal{M}'$, $\exists p \in \omega\ h \in \mathcal{M}_p$ so by construction it is dominated by a function in $\mathcal{M}_{p'}$ for some $p' > p$. □

To prove Theorem 51 we will use Hechler Forcing, a forcing notion that is similar to the Mathias forcing used in Cholak, Jockusch, and Slaman [2]; see Jech [8] for other set theoretic applications of Hechler Forcing. The results obtained in this section owe much to a discussion of Dorais [5].

The forcing conditions we will use are of the form $(s, f) \in M^{<M} \times M^M$ where $M^{<M}$ denotes the collection of $\mathcal{M}$-finite sets, and M^M denotes functions in the second order part of $\mathcal{M}$, while the pre-conditions are $(s, f) \in M^{<M} \times M^{<M}$. The extensional relationship between conditions is defined by $(t, f) \leq (s, g)$ iff

- $t \supseteq s$, and
- $\forall x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)\ [f(x) \geq g(x)]$, and
- $t(y) \geq g(y)$ for $|s| \leq y < |t|$.

We will define the standard forcing interpretation now. The formulae in the forcing language will have a new predicate $\dot{g}$ (roughly speaking it is the name of the generic object we are adding) in addition to the usual first and second order parameters from $\mathcal{M}$. For convenience, we might suppress the mention of parameters from $\mathcal{M}$ in formulae in the forcing language.

Definition 53. (Forcing for Δ_0^0 formulae)

The following definition is done in $\mathcal{M}$. Given (s, f) a condition or pre-condition, the following defines the forcing relation $\Vdash$ in $\mathcal{M}$.

- $(s, f) \Vdash \dot{g}(x) = z$ iff $x < |s|$ and $s(x) = z$; $(s, f) \Vdash \dot{g}(x) \neq z$ iff $x < |s|$ and $s(x) \neq z$.
- For other atomic formula θ with no occurrence of $\dot{g}$, $(s, f) \Vdash \theta$ iff $\mathcal{M} \models \theta$.
- For other cases of $\theta(g)$ where θ is Δ_0^0 the forcing relation follows from induction. More precisely, if $\theta(\dot{g})$ is
  - $\psi(\dot{g}) \wedge \phi(\dot{g})$, then $(s, f) \Vdash \theta$ iff $(s, f) \Vdash \psi(\dot{g})$ and $(s, f) \Vdash \phi(\dot{g})$.
  - $\neg\phi(\dot{g})$, then $(s, f) \Vdash \theta$ iff $(s, f) \nVdash \phi(\dot{g})$.
  - $\forall x < a\, \phi(x, \dot{g})$, then $(s, f) \Vdash \theta$ iff $\forall x < a\ (s, f) \Vdash \phi(x, \dot{g})$.

Definition 54. (Forcing for Π_1^0 and Σ_1^0 formulae)
Let $\varphi(g)$ be $\forall x\, \theta(x, g)$ with $\theta \in \Delta_0^0$.

– $(s,h) \Vdash \varphi(g)$ iff for any pre-condition $(t,f) \leq (s,h)$ and any $w \in M$ we have that $(t,f) \Vdash \theta(w,g)$ holds.
– $(s,h) \Vdash \neg\varphi(g)$ iff there exist $w \in M$ and pre-condition (t,f) such that $(s,h) \leq (t,f)$ and $(t,f) \Vdash \neg\theta(w,g)$.

Note that the predicates above are $\Pi^{0,h}_1$ and $\Sigma^{0,h}_1$ respectively.

Lemma 55. *Given a condition (s,h) and a Π^0_1 formula $\forall x\, \varphi(x,\dot{g})$, if*

$$(s,h) \nVdash \forall x\, \varphi(x,\dot{g}),$$

then there exists a condition $(s',h') \leq (s,h)$ satisfying that

$$(s',h') \Vdash \neg(\forall x\, \varphi(x,\dot{g})).$$

Furthermore, if h is recursive, we can choose h' recursive as well.

Proof. Since $(s,h) \nVdash \forall x\, \varphi(x,\dot{g})$, there exists a precondition $(s',h'') \leq (s,h)$ and $w \in M$ such that $(s',h'') \Vdash \neg\varphi(w,\dot{g})$. Hence, by definition,

$$(s',h'') \Vdash \neg(\forall x\, \varphi(x,\dot{g})).$$

We let $h'(x) = h''(x)$ for $x \in \mathrm{dom}(h'')$ and $h'(x) = h(x)+1$ otherwise. It is easy to check that if h is recursive, then h' is recursive; this follows from the fact that h'' is a finite function. □

Lemma 56. *Given a Π^0_1 formula $\forall x\, \varphi(x,\dot{g})$ and a condition (or a pre-condition) (s,f) such that $(s,f) \Vdash \forall x\, \varphi(x,\dot{g})$, then for any $s \sqsubseteq g \subseteq M$* JING: *See below. Also $\sqsubseteq$ is undefined.*, $\mathcal{M}[g] \models \forall x\, \varphi(x,g)$.

Proof. First we verify the theorem for Δ^0_0 formulae by induction. For atomic formulae with no mention of g, it follows trivially by the way the forcing relation is defined and by the absoluteness of atomic formulae between second-order structures with the same first-order universe.

If $(s,f) \Vdash \dot{g}(x) = z$, then $x \in \mathrm{dom}(s), s(x) = z$. Hence the theorem follows. The same idea applies for $\dot{g}(x) \neq z$. The inductive steps for Δ^0_0 are exactly by the definition of forcing relation, hence we omit the details. Now suppose we have the forcing and truth theorem for Δ^0_0 formulae, let us check it for Π^0_1 formulae.

By definition for any pre-condition $(t,g) \leq (s,f)$ JING: Clarify, whether g is the same as in the statement of the lemma; and if so, then g in the statement needs to be in M^M. and $w \in M$ we have $(t,g) \Vdash \varphi(w,\dot{g})$, so we know $(s,f) \Vdash \varphi(w,\dot{g})$ for all $w \in M$. But then by the theorem for Δ^0_0 formulae we have $\mathcal{M}[g] \models \forall w\, \varphi(w,g)$. □

Proof (Theorem 51). We prove the unrelativised version, that is, $X = \emptyset$. It is routine to relativise everything to arbitrary X.

Fix a list $\{\psi_i : i \in \omega\}$ of all total recursive functions coded in $\mathcal{M}$ and a list $\{\varphi_j(x,g) : j \in \omega\}$ of all Π^0_1 formulae with x, g being the only free variables with potential first-order or second-order parameters from $\mathcal{M}$ (other cases can easily be reduced to this case). For notational simplicity, we will write (s,i) for (s,ψ_i) for a condition in the forcing poset.

Let $h:\omega\to M$ be a bijection. We wish to fulfill the following requirements, for $e\in\omega$:

R_{3e}: g is defined at $h(e)$.
R_{3e+1}: g lies in the neighborhood of (s,e) for some finite function s, that is, $s\subseteq g$ and g dominates ψ_e.
R_{3e+2}: Either $\mathcal{M}[g]\models\forall x\,\varphi_e(x,g)$ or there exists $b\in M$ such that

$$\mathcal{M}[g]\models\neg\varphi_e(b,g)\ \wedge\ \forall a<b\,[\varphi_e(a,g)].$$

Next we need to argue that at any stage of the construction, and given a condition (g_s,i_s), we can extend it to satisfy all of the above requirements, that is, the respective sets are dense in the forcing poset ordering.

- Suppose we want to meet R_{3e}. If $h(e)\in\mathrm{dom}(g_s)$ then we are done; otherwise, let g_{s+1} be $g_s\cup\{\langle h(e),\psi_{i_s}(e)+1\rangle\}$. Let $i_{s+1}=i_s$. Then $(g_{s+1},i_{s+1})\leq(g_s,i_s)$ is the desired condition.
- Suppose we want to meet R_{3e+1}. Since ψ_e and ψ_{i_s} are both total recursive, there exists $i_{s+1}\in\omega$ such that $\psi_{i_{s+1}}=\psi_{i_s}+\psi_e$. Let $g_{s+1}=g_s$. Then (g_{s+1},i_{s+1}) is the desired condition.
- Suppose we want to meet R_{3e+2}. Write $\varphi_e(x,g)=\forall m\ \psi(e,x,g)$ where $\psi(e,x,g)$ is Δ^0_0. Consider the set

$$X:=\{x\in M\colon (g_s,i_s)\not\Vdash\varphi_e(x,\dot g)\}.$$

If X is empty then $(g_{s+1},i_{s+1})=(g_s,i_s)$ and we are done by Lemma 56. Otherwise note that X is $\Sigma^{0,\psi_{i_s}}_1$ and since ψ_{i_s} is total recursive, this set is thus Σ^0_1. Thus by $\mathsf{I}\Sigma_1$ in $\mathcal{M}$, there exists a least element $b\in M$ such that for all $a<b$ we have $(g_s,i_s)\Vdash\varphi_e(a,\dot g)$. Since $(g_s,i_s)\not\Vdash\varphi_e(b,\dot g)$, by Lemma 55, we can find a condition $(g_{s+1},i_{s+1})\leq(g_s,i_s)$ such that $(g_{s+1},i_{s+1})\Vdash\neg\varphi_e(b,\dot g)$. Therefore,

$$(g_{s+1},i_{s+1})\Vdash\neg\varphi_e(b,\dot g)\wedge\forall a<b\ \varphi_e(a,\dot g).$$

Hence, for any $g\sqsupseteq g_{s+1}$ Jing: $\sqsubseteq$ undefined., $\mathcal{M}[g]$ satisfies the requirement.

Thus $g=\bigcup_{s\in\omega}g_s$ is the desired dominating function. By construction we have $\mathcal{M}_g\models\mathsf{I}\Sigma^0_1$, so $\mathcal{M}[g]\models\mathsf{RCA}_0$. ☐

Corollary 57. $\mathsf{DOM}+\mathsf{RCA}_0$ *is* Π^1_1*-conservative over* RCA_0. ☐

5 Almost Disjointness and Independence

Theorem 58. *An* ω*-model satisfies* MAD *iff it does not satisfy* DOM.

Proof. ($\Rightarrow$): Let $\mathcal{F}$ be a weakly represented family of sets represented by $A \in S$ that is almost disjoint. Suppose that DOM holds; we will show that this implies $\neg\mathsf{MAD}$.

Assume without loss of generality that for the characteristic function f of every set in $\mathcal{F}$ there is a unique $e \in \omega$ such that f is weakly represented by A_e (where A_e is as in Definition 5). Indeed, this can be achieved by replacing A with a set A' derived from it, where A' and $A'_e = \{n\colon \langle e, n\rangle \in A'\}$ are such that whenever f'_e (the function weakly represented by A'_e) looks identical to f'_d for some $d < e$, the enumeration of elements into A'_e is suspended; this way, should there indeed be a $d < e$ with $f_e = f_d$ in the limit, then f'_e will become non-total, and A'_e will not weakly represent any function in $\mathcal{F}$.

As a consequence of the previous assumption, if, for some $d \neq e$, A_d and A_e weakly represent the characteristic functions of sets $F \in \mathcal{F}$ and $G \in \mathcal{F}$ respectively, then $F \cap G$ is finite.

Let $(\varphi_e^A\colon e \in \omega)$ be an enumeration of all A-recursive functions.

Claim. There is a function $g \in S$ that dominates every A-recursive function in the following sense: For every total φ_e^A and almost all n it holds that

$$g(n+1) > \varphi_e^A(g(n)).$$

Proof. Consider the weakly represented family of all A-recursive functions, and apply DOM to obtain a function $\widehat{g}$ dominating it. Without loss of generality, assume that $\widehat{g}$ is strictly increasing, let $g(0) = 0$ and define for all n that $g(n+1) = \widehat{g}(g(n))$. $\diamond$

Let $h(x, n)$ be the least number d such that either $\varphi_d^A(x)[g(n+2)]{\downarrow} = 1$ or $d = n$. Let B be the set consisting of numbers $b_n \in \omega$, $n \in \omega$, with $g(n) < b_n < g(n+1)$ and $h(b_n, n) \geq h(x, n)$ for all x with $g(n) < x < g(n+1)$.

Informally, for an element x, the value $h(x)$ tells us that x does not seem to show up in those sets that have characteristic functions who have an A-recursive index up to $h(x)$; of course this can only be determined given an enumeration timebound, which is provided by the dominating function g here. Then B picks elements where this number is as large as possible.

More formally, note that by the choice of g, if φ_d^A coincides with the characteristic function f_e of a set in $\mathcal{F}$, then for almost all n there is an x with $g(n) < x < g(n+1)$ such that $f_e(x)[g(n+2)]{\downarrow} = 1$ and, due to the almost disjointness, for all $d < e$ either $f_d(x){\uparrow}$ or $f_d(x){\downarrow} = 0$. As by construction B consists only of numbers of this type, for almost all n it holds that $b_n \notin A_d$ for $d < e$ and therefore the set B has finite intersection with every $C \in \mathcal{F}$. Thus MAD is not satisfied.

($\Leftarrow$): Let $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$ be an ω-model of $\neg\mathsf{DOM}$. We assume without loss of generality that S contains no high set; otherwise carry out the construction below relative to an oracle relative to which no high set in S exists.

The fact that we don't know which indeces e describe total recursive functions φ_e is a complication in the construction that follows. To circumvent this issue, we take advantage of the possibilities that the concept of weakly represented families offer, namely that partial information about functions is ignored when defining such a family; only total functions are considered a member of the family. Using this, we build a recursive numbering of partial-recursive functions such that the total functions appearing in it are all $\{0,1\}$-valued and when interpreted as characteristic functions of sets, the collection of these sets is a maximal almost disjoint family.

Let $(\varphi_e\colon e \in \omega)$ be an enumeration of all recursive functions. First we build the uniformly recursive helper procedures $\psi_{c_0,c_1,\ldots,c_e}$ for all $e \in \omega$ with $c_d \in \{0,1,\ldots,\infty\}$ for $d \leq e$. We call $(c_0,c_1,\ldots,c_e)$ *true parameters* if, for all $d \leq e$, c_d is the minimal i such that $\varphi_d(i)\uparrow$ if such an i exists, and $c_d = \infty$ if φ_d is total.

The procedure $\psi_{c_0,c_1,\ldots,c_e}$ has three states: *wait*, *success*, and *aborted*. When we define the enumeration of the characteristic function of A_e below, we will only enumerate a new function value whenever $\psi_{c_0,c_1,\ldots,c_e}$ is in state *success*. The idea is that this will only happen infinitely often, when $(c_0,c_1,\ldots,c_e)$ are the true parameters. If $(c_0,c_1,\ldots,c_e)$ are not true parameters, then $\psi_{c_0,c_1,\ldots,c_e}$ will either be stuck in state *wait* forever, or it will enter state *aborted* and stay in it forever. Then the true parameters will be the only parameters used to define A_e.

To achieve what we just described, we proceed as follows: $\psi_{c_0,\ldots,c_e}$ starts in state *wait* and runs the following $e+1$ parallel procedures:

- For all $d \leq e$, the computations $\varphi_d(c_d)$, $d \leq e$, are run in parallel. If one of them ever terminates, then by definition $(c_0,c_1,\ldots,c_e)$ are not true parameters. Then $\psi_{c_0,\ldots,c_e}$ stops all computations, enters state *aborted*, and remains in this state permanently.
- In a single procedure, all computations $\varphi_d(c)$ with $d \leq e$ and $c < c_d$ are run *sequentially* and in order ascending with $\langle d,c\rangle$. While one of the computations runs, $\psi_{c_0,\ldots,c_e}$ is in state *wait*. Every time one of the computations $\varphi_d(c)$ terminates, $\psi_{c_0,\ldots,c_e}$ enters state *success*. If (d,c) was the last pair of parameters as above (which can only happen if all c_d, $d \leq e$, are finite) then remain in state *success* permanently. Otherwise enter state *wait* again, and continue with the next pair (d',c'), that is, with the smallest pair as above such that $\langle d',c'\rangle > \langle d,c\rangle$.

Note that this arrangement ensures that $\psi_{c_0,\ldots,c_e}$ is in state *success* at infinitely many stages if and only if $(c_0,c_1,\ldots,c_e)$ are the true parameters.

We can now describe how to produce a maximal almost disjoint family. In parallel, for all $e \in \omega$ and all possible sets of parameters $(c_0,c_1,\ldots,c_e)$ we run the following procedure.[5]

[5] Note that to simplify notation, we do not explicitly define total characteristic functions of the sets A_e, $e \in \omega$, or the enumeration of a set that represents these functions as a weakly represented family. But since the elements of every A_e are enumerated in increasing order by the given procedure, it is easy to convert it into one defining the enumeration of such a set.

Run $\psi_{c_0,\dots,c_e}$ step by step.
At every stage, check if $\psi_{c_0,\dots,c_e}$ is currently in state *success*.
If so, let m be the smallest number not in $A_0 \cup A_1 \cup \ldots \cup A_{e-1}$, and check whether

$$m = n + \varphi_e(0) + \varphi_e(1) + \ldots + \varphi_e(n) \text{ for some } n. \qquad (*)$$

If not, enumerate m into A_e.

Note that if $(c_0, c_1, \ldots, c_e)$ are the true parameters, then checking $(*)$ is recursive, and the procedure never gets stuck. This finishes the construction.

We need to prove that the weakly represented family $\{A_n : n \in \omega\}$ constructed by this procedure is maximal almost disjoint. First note that for each $e \in \omega$ the complement of $A_0 \cup \ldots \cup A_e$ is infinite and contains at most n elements below $\varphi_e(n)$. Furthermore, A_e is disjoint with all A_d for $d < e$. As a consequence, $\{A_n : n \in \omega\}$ is almost disjoint.

It remains to show that $\{A_n : n \in \omega\}$ is also maximal almost disjoint. To see this let B be an infinite non-high set. Then there is a recursive function φ_e such that, for infinitely many n, there are more than $2n$ elements of B below $\varphi_e(n)$. It follows that the intersection of B with $A_0 \cup \ldots \cup A_e$ is infinite and therefore $B \cap A_d$ is infinite for some $d \leq e$. This completes the proof. □

Theorem 59. *An ω-model satisfies* MIND *iff it does not satisfy* BI.

Proof. For the purpose of the simplicity of notation, we show the following unrelativized result:

> An ω-model has a weakly represented family of sets that is maximal independent *iff* it does not contain a biimmune set.

As in the second half of the previous proof, the general implications follow by relativization.

$(\Rightarrow)$: As before, for a set $C \subseteq \omega$, let us write C^0 for C and C^1 for $\omega \setminus C$.

Let a weakly represented family $\mathcal{F}$ of sets represented by $A \leq_{\mathrm{T}} \emptyset$ be given. Also fix any collection $\{A_0, \ldots, A_{n-1}\} \subseteq \mathcal{F}$, and any string $\sigma \in 2^n$, as well as a biimmune set B. Observe that the set $\widehat{A} := \bigcap_{i<n} A_i^{\sigma(i)}$ is recursive. Then B's biimmunity implies that both $\widehat{A} \cap B$ and $\widehat{A} \cap \overline{B}$ are infinite.

As $\{A_0, \ldots, A_{n-1}\}$ was arbitrary, it follows that $\mathcal{F} \cup \{B\}$ is still an independent family, which contradicts the assumption that $\mathcal{F}$ is maximal independent.

$(\Leftarrow)$: Similarly to the proof of the previous theorem, we work with a list of parameters where true parameters define sets in the maximal independent family that we are building.

FRANK: The proof is much too vague, many notions are undefined, etc. I cannot follow. It needs to be made much more formal, and needs to use variable names instead of formulations like "the family we built above" to improve readability etc. The idea of the proof is to begin with a stream FRANK: What's that? $x_n^\varepsilon = n$ Frank: for all $n \in \omega$? What is ε? and to split each stream x^σ into two disjoint infinite strings Frank: What's the difference between a stream and a string? $x^{\sigma 0}$ and $x^{\sigma 1}$ which are both infinite.

Then the set A_e will be the union of the ranges Frank: Undefined. of all streams $x^{\sigma 1}$ with $|\sigma| = e$. It is clear from this set-up that the sets A_e formed from the true parameters will form an independent family; furthermore, all false parameters will result in finite functions. The parameter for A_e consists of a vector of 2^e values from $\{0, 1, \ldots, \infty\}$, the indices of these parameters are σ (from the stream to be split) where $|\sigma| = e$; furthermore, one also needs the parameters for the sets A_d with $d < e$ which are all the c_σ with $|\sigma| = d$.

Now for σ of length e and stream $x_0^\sigma, x_1^\sigma, \ldots$, $c_\sigma = \infty$ in the case that φ_e is on the stream $\{0, 1\}$-valued and total and takes both values 0 and 1 infinitely often and $c_\sigma = n$ in the case that n is the least number such that there is no $m \geq n$ for which $\varphi_e(x_k)\downarrow\{0, 1\}$ for all $k \leq m + 1$ and $\varphi_e(x_m) \neq \varphi_e(x_{m+1})$. The streams $x^{\sigma a}$ of σ of length e will be either terminated Frank: What does "stream gets terminated" mean? (if a mistake in the parameters is found witnessing that they are not true Frank: Vague.) or get stuck (if some parameter is claimed to be ∞ although the true value is some finite number) or gets stuck because some wrong lower level parameter causes the corresponding streams to be finite. If all parameters c_τ with $|\tau| \leq |\sigma|$ are the true parameters, then each stream $x^{\sigma a}$ is an infinite substream of x^σ selected by $x_n^{\sigma a} = x_{2n+a}^\sigma$ in the case that c_σ is a finite number and the stream $x_n^{\sigma a}$ is the substream of all x_m^σ which satisfy that $\varphi_e(x_m) = a$ in the case that $c_\sigma = \infty$.

The partial-recursive function $\psi_{c_\varepsilon, c_0, c_1, \ldots, c_{11\ldots 1}}$ with the true parameters being all the c_σ with $|\sigma| \leq e$ will be the union of the ranges of the streams $x_n^{\sigma 1}$ for all $\sigma \in \{0, 1\}^e$. Note that the $\psi_{\ldots}$ are total recursive functions iff all supplied parameters are the true parameters and the corresponding Frank: Vague. set is called A_e.

Let $B \in S$ be any set. Without loss of generality, assume that B is not biimmune. Then there is an infinite recursive set with characteristic function φ_e which is either disjoint or a subset of B, say the first. So there is one stream x_n^σ which has an infinite intersection with the set of characteristic function φ_e and therefore almost all elements of the substream $x_n^{\sigma 1}$ are contained in the recursive set and therefore disjoint from B. This substream is a finite Boolean combination of $A_0, \ldots, A_e$ and therefore B cannot be used to augment the independent family. Thus the above constructed weakly represented family witnesses the axiom MIND for the given second order model. □

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