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A MODEL WITH NO STRONGLY SEPARABLE ALMOST DISJOINT FAMILIES

BY

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ABSTRACT

We answer a question of Shelah and Steprāns [6] by producing a model of ZFC where there are no strongly separable almost disjoint families. The notion of a strongly separable almost disjoint family is a natural variation on the well known notion of a completely separable almost disjoint family, and is closely related to the metrization problem for countable Fréchet groups.

1. Introduction

We say that two infinite subsets a and b of ω are **almost disjoint or a.d.** if $a \cap b$ is finite. We say that a family $\mathscr{A}$ of infinite subsets of ω is **almost disjoint or a.d.** if its members are pairwise almost disjoint. A **Maximal Almost Disjoint family, or MAD family** is an infinite a.d. family that is not properly contained in a larger a.d. family.

We use $[\omega]^{\omega}$ to denote the collection of infinite subsets of ω – that is, $[\omega]^{\omega} = \{a \subset \omega : |a| = \omega\}$. Given an a.d. family $\mathscr{A} \subset [\omega]^{\omega}$, we use $\mathcal{I}(\mathscr{A})$ to denote the ideal on ω generated by $\mathscr{A}$, and $\mathcal{I}^{+}(\mathscr{A})$ denotes $\mathcal{P}(\omega) \setminus \mathcal{I}(\mathscr{A})$. An a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ is said to be **completely separable** if for any $b \in \mathcal{I}^{+}(\mathscr{A})$, there is an $a \in \mathscr{A}$ with $a \subset b$. Notice that an infinite completely separable a.d. $\mathscr{A}$

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must be MAD. The following long standing open problem is one of the most well known and well studied in set theory.

Question 1 (Erdős and Shelah [3], 1972): Does there exist a completely separable MAD family $\mathscr{A} \subset [\omega]^{\omega}$?

It is easy to see that if an a.d. $\mathscr{A} \subset [\omega]^{\omega}$ is completely separable, then for every $b \in \mathcal{I}^{+}(\mathscr{A})$, there are actually $\mathfrak{c}$ many $a \in \mathscr{A}$ with $a \subset b$. A natural strengthening of the notion of a completely separable MAD family, called a strongly separable MAD family, was introduced by Shelah and Steprāns in [6]. To motivate it, we first introduce some notation. FIN is the collection of non-empty finite subsets of ω. Given an ideal $\mathcal{I}$ on ω, we say that $P \subset \mathrm{FIN}$ is **$\mathcal{I}$-positive** if $\forall a \in \mathcal{I} \exists s \in P\,[a \cap s = 0]$. Note that for $a \subset \omega$, $a \notin \mathcal{I}$ iff $\{\{n\} : n \in a\}$ is $\mathcal{I}$-positive. Thus, we can rephrase the definition of a completely separable MAD family as follows: for every $\mathcal{I}(\mathscr{A})$-positive $P \subset \mathrm{FIN}$ which consists entirely of singletons, there are $\mathfrak{c}$ many $a \in \mathscr{A}$ such that for each of them, there is a $Q \in [P]^{\omega}$ with $a = \bigcup Q$.

Definition 2: An almost disjoint family $\mathscr{A} \subset [\omega]^{\omega}$ is called **strongly separable** if for every $\mathcal{I}(\mathscr{A})$-positive $P \subset \mathrm{FIN}$, there are $\mathfrak{c}$ many $a \in \mathscr{A}$ such that for each of them there is a $Q \in [P]^{\omega}$ consisting of pairwise disjoint sets with $\bigcup Q \subset a$. We say that an a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ is **almost strongly separable** if for each $\mathcal{I}(\mathscr{A})$-positive $P \subset \mathrm{FIN}$, there is $a \in \mathscr{A}$ and $Q \in [P]^{\omega}$ such that $\bigcup Q \subset a$.

Once again, we observe that an infinite almost strongly separable a.d. family must be MAD. Shelah and Steprāns [6] connected the notion of a strongly separable a.d. family with the Calkin algebra.

Theorem 3 (Shelah and Steprāns [6]): *If there is a strongly separable MAD family, then there is a masa in $\mathscr{C}(\mathbb{H})$ that is generated by its projections, and does not lift to a masa in $\mathscr{B}(\mathbb{H})$.*

Here, $\mathscr{B}(\mathbb{H})$ denotes the algebra of bounded operators on the Hilbert space $\mathbb{H}$, and $\mathscr{C}(\mathbb{H})$ is the **Calkin algebra**—i.e. $\mathscr{C}(\mathbb{H}) = \mathscr{B}(\mathbb{H})/\mathscr{K}(\mathbb{H})$, where $\mathscr{K}(\mathbb{H})$ is the algebra of compact operators on $\mathbb{H}$. In [6] Shelah and Steprāns asked whether a strongly separable MAD family, or at least an almost strongly separable MAD family, could be constructed in ZFC. They also asked whether the recent construction of Shelah [5] could be modified to construct an almost strongly separable MAD family from the hypothesis $\mathfrak{c} < \aleph_\omega$. The main result of this paper gives a negative answer to these questions.

Theorem 4: *There is a model of* ZFC *where there are no almost strongly separable MAD families and* $\mathfrak{c} = \aleph_2$.

The notion of a strongly separable a.d. family is also closely tied to the metrizability problem for countable Fréchet groups. Recall that a topological space X is **Fréchet** if whenever a point $p \in X$ is in the closure of a set $A \subset X$, there is a sequence of points in A converging to p. A well-known question of Malykhin asks whether every countable Fréchet topological group is metrizable. Let us say that an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is **Fréchet** if for every $\mathcal{I}$-positive $P \subset$ FIN, there is a $Q \in [P]^\omega$ consisting of pairwise disjoint sets so that $\forall a \in \mathcal{I}\,[|\bigcup Q \cap a| < \omega]$. If $\mathcal{I}$ is a Fréchet ideal which is not countably generated, then we can define a non-metrizable Fréchet topology on $\langle [\omega]^{<\omega}, \triangle \rangle$ by stipulating that

$$\{A \subset [\omega]^{<\omega} : \exists a \in \mathcal{I} \forall s \in [\omega]^{<\omega}\,[s \cap a = 0 \implies s \in A]\}$$

is a neighborhood base at 0. It is clear that if $\mathfrak{p} > \omega_1$, then *any* ideal that is $\aleph_1$ generated is Fréchet. A well-known question of Gruenhage and Szeptycki [4] asks if a Fréchet ideal that is not countably generated can be constructed in ZFC, and more specifically, whether there is an uncountable a.d. family $\mathscr{A} \subset [\omega]^\omega$ such that $\mathcal{I}(\mathscr{A})$ is Fréchet. Brendle and Hrušák [2] recently provided a partial answer to these questions by showing that it is consistent that no uncountably generated ideal on ω with fewer than $\mathfrak{c}$ generators is Fréchet. We provide one more partial result on this question in this paper. To state this, we first introduce some notation.

Definition 5: Suppose $\mathscr{A} \subset [\omega]^\omega$ is an almost disjoint family. For $b \in \mathcal{P}(\omega)$, we define $\mathscr{A}_b = \{a \in \mathscr{A} : |a \cap b| = \omega\}$. Let $\mathcal{I}_\omega(\mathscr{A}) = \{b \subset \omega : |\mathscr{A}_b| < \omega\}$.

Theorem 6: *It is consistent that for every a.d. family* $\mathscr{A} \subset [\omega]^\omega$ *with the property that for each* $b \in \mathcal{P}(\omega)$, *either* $\mathscr{A}_b$ *is finite or* $\mathscr{A}_b$ *is uncountable,* $\mathcal{I}(\mathscr{A})$ *is not Fréchet.*

As we will see in Section 3, our proof actually gives something slightly stronger than Theorem 6. The proofs of both Theorem 4 and Theorem 6 use a modification of the forcing introduced by Brendle and Hrušák in [2].

2. The Proofs

We give the proofs of Theorems 4 and 6 in this section. We first set out some notation. We will be dealing with various types of objects, and we try to use different types of letters to denote each type. Thus we will use $a, b, \dots$ to denote elements of $[\omega]^{\omega}$, $s, t, \dots$ to denote members of $[\omega]^{<\omega}$, $\sigma, \tau, \dots$ to stand for elements of $(\mathrm{FIN})^{<\omega}$, and finally $A, B, \dots$ to represent subsets of FIN. For $a \in [\omega]^{\omega}$ and $s \in [\omega]^{<\omega}$, a/s denotes $\{n \in a : \forall k \in s\,[k < n]\}$, and we write $s \sqsubset a$ to mean that s is an initial segment of a.

Given an ideal $\mathcal{I}$ on a set X, $\mathcal{I}^{+}$ denotes $\mathcal{P}(X) \setminus \mathcal{I}$, the **coideal associated with** $\mathcal{I}$. $\mathcal{I}^{*}$ will denote the **dual filter on** X **associated with** $\mathcal{I}$—i.e. $\mathcal{I}^{*} = \{X \setminus a : a \in \mathcal{I}\}$. For a filter $\mathscr{G}$ on X, $\mathscr{G}^{*}$ is the **dual ideal on** X **associated with** $\mathscr{G}$ – i.e., $\mathscr{G}^{*} = \{X \setminus a : a \in \mathscr{G}\}$, and $\mathscr{G}^{+} = \mathscr{G}^{*+}$. In this paper, we will only deal with finite support iterations of c.c.c. posets. We will use the abbreviation *FS* to stand for finite support. We also sometimes make use of elementary submodels. We will simply write "$M \prec H(\theta)$" to mean "M is an elementary submodel of $H(\theta)$, where θ is a regular cardinal that is large enough for the argument at hand".

We will be dealing with subtrees of $(\mathrm{FIN})^{<\omega}$. For a subtree $p \subset (\mathrm{FIN})^{<\omega}$ and a node $\sigma \in p$, $\mathrm{succ}_p(\sigma) = \{s \in \mathrm{FIN} : \sigma^\frown\langle s \rangle \in p\}$, the set of successors of σ in p. For a node $\sigma \in p$, $p_\sigma = \{\tau \in p : \tau \subset \sigma \vee \sigma \subset \tau\}$; it is clear that p_σ is a subtree of p. A node $\sigma \in p$ is called *the stem of* p and is denoted by $\mathrm{stem}\,(p)$ if $p_\sigma = p$ and $|\mathrm{succ}_p(\sigma)| \geq 2$; it is clear that $\mathrm{stem}\,(p)$ is unique, if it exists.

At a stage α of the iteration, we will deal with a particular a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ (chosen using a diamond sequence). We will add an $\mathcal{I}(\mathscr{A})$-positive set $P \subset \mathrm{FIN}$ which has the property that there is no $a \in \mathscr{A}$ for which there is a $Q \in [P]^{\omega}$ with $\bigcup Q \subset a$. We then ensure that at no later stage of the iteration is it possible to enlarge $\mathscr{A}$ to a bigger a.d. family $\mathscr{A}' \supset \mathscr{A}$ in such a way that P is not $\mathcal{I}(\mathscr{A}')$-positive, or in such a way that there is an $a \in \mathscr{A}'$ and a $Q \in [P]^{\omega}$ with $\bigcup Q \subset a$. For this, it is clearly enough to ensure the following:

- $(*_1)$ At every stage $\beta \geq \alpha$ it is impossible to extend $\mathscr{A}$ to an a.d. family $\mathscr{A}' \supset \mathscr{A}$ so that P is not $\mathcal{I}(\mathscr{A}')$-positive.
- $(*_2)$ At every stage $\beta \geq \alpha$ it is impossible to find a set $Q \in [P]^{\omega}$ so that $\bigcup Q$ is a.d. from $\mathscr{A}$.

The problem of how to add an $\mathcal{I}(\mathscr{A})$-positive $P \subset \mathrm{FIN}$ that satisfies $(*_2)$ was solved by Brendle and Hrušák [2]. The main contribution of this paper is to show how to add such a P that also meets the requirement given by $(*_1)$. This is done in Lemmas 9 through 17, and is done by suitably modifying the forcing of Brendle and Hrušák [2]. However, proving that the P added by our modified forcing still satisfies $(*_2)$ requires changing the argument of Brendle and Hrušák [2] a bit. This is done in Lemmas 19 through 27.

The main tool used in fulfilling requirement $(*_1)$ is Ramsey theory – more specifically, the Nash–Williams–Galvin lemma. The reader may find a proof in, among other places, Section 5 of [7].

Lemma 7 (Nash–Williams–Galvin Lemma): *For every* $F \subset [\omega]^{<\omega}$ *there is an* $a \in [\omega]^{\omega}$ *such that either* (1) *or* (2) *holds:*

(1) $[a]^{<\omega} \cap F = 0$,
(2) $\forall b \in [a]^{\omega} \exists s \sqsubset b\, [s \in F]$.

Definition 8: For $a \in [\omega]^{\omega}$, $\mathrm{FIN}\,(a) = [a]^{<\omega} \setminus \{0\}$. Thus, $\mathrm{FIN} = \mathrm{FIN}\,(\omega)$.

Lemma 9: *Let* $F : \mathrm{FIN} \to \omega$ *such that for every* $s \in \mathrm{FIN}$, $F(s) \in s$. *Then there is a set* $b \in [\omega]^{\omega}$ *such that either* (1) *or* (2) *holds:*

(1) $\forall s \in \mathrm{FIN}\,(b) \forall c \in [b/s]^{\omega} \exists t \sqsubset c\, [t \neq 0 \wedge F(s \cup t) \in t]$,
(2) $\forall c \in [b]^{\omega} \exists s \sqsubset c \left[s \neq 0 \wedge \forall t \in [b/s]^{<\omega} [F(s \cup t) \in s]\right]$.

Proof. We first build, as follows, a sequence $\omega = a_0, n_0, a_1, n_1, \ldots,$ where $a_i \in [\omega]^{\omega}$, $a_{i+1} \subset a_i/n_i$, and $n_i \in a_i$. Given $a_0, n_0, \ldots, a_k, n_k$, let $s_0, \ldots, s_l$ enumerate all non-empty $s \subset \{n_0, \ldots, n_k\}$ with $\max(s) = n_k$. Successively refine a_k/n_k to produce $a_k/n_k = a_k^{-1} \supset a_k^0 \supset \cdots \supset a_k^l = a_{k+1}$, as follows. Given a_k^i, put $\mathcal{F} = \{t \in \mathrm{FIN}\,(a_k^i) : F(s_{i+1} \cup t) \in t\}$. Applying the Nash–Williams–Galvin Lemma, we get $a_k^{i+1} \in \left[a_k^i\right]^{\omega}$ such that either $\left[a_k^{i+1}\right]^{<\omega} \cap \mathcal{F} = 0$ or for every $b \in \left[a_k^{i+1}\right]^{\omega}$, there is $t \in \mathcal{F}$ so that $t \sqsubset b$. This specifies the construction of a_{k+1}; n_{k+1} can be chosen to be any member of a_{k+1}. Now, the set $a = \{n_0 < n_1 < \cdots\}$ clearly has the property that for any $s \in \mathrm{FIN}\,(a)$, *either* $\forall b \in [a/s]^{\omega} \exists t \sqsubset b\, [t \neq 0 \wedge F(s \cup t) \in t]$ *or* $\forall t \in [a/s]^{<\omega} [F(s \cup t) \in s]$. Put $\mathcal{F} = \{s \in \mathrm{FIN}\,(a) : \forall t \in [a/s]^{<\omega} [F(s \cup t) \in s]\}$. Applying Nash–Williams–Galvin again, we get $b \in [a]^{\omega}$ which is as required. ∎

We remark here that the diagonalization procedure required to produce the set b in Lemma 9 can be carried out inside any selective ultrafilter. The sets a_k can be chosen to lie in the ultrafilter because such ultrafilters satisfy the Nash–Williams–Galvin Lemma, and the n_k can be choosen in such a way that the resulting set a also lies in the ultrafilter (see, for example, Theorem 4.5.3 of Bartoszyński and Judah [1]). We will use this remark freely in what follows.

Recall that a coideal $\mathcal{E} \subset \mathcal{P}(\omega)$ is called **selective** if for every sequence $e_0 \supset e_1 \supset \cdots$, with $e_i \in \mathcal{E}$, there is an $e = \{n_0 < n_1 < \cdots\} \in \mathcal{E}$ such that $n_0 \in e_0$ and $n_{i+1} \in e_{n_i}$ for each i. The following lemma is standard. The rest of the proof uses the selectivity of $\mathcal{P}(\omega) \setminus \mathcal{I}_\omega(\mathscr{A})$.

Lemma 10: *Suppose $\mathscr{A} \subset [\omega]^\omega$ is an almost disjoint family with the property that for every $b \in [\omega]^\omega$, $\mathscr{A}_b$ is either finite or uncountable. Then $\mathcal{P}(\omega) \setminus \mathcal{I}_\omega(\mathscr{A})$ is a selective coideal. Therefore, assuming* CH, *there is a selective ultrafilter $\mathcal{U} \subset \mathcal{P}(\omega) \setminus \mathcal{I}_\omega(\mathscr{A})$.*

Note that the hypothesis of Lemma 10 is satisfied whenever $\mathscr{A} \subset [\omega]^\omega$ is MAD. Now, assume $\mathscr{A} \subset [\omega]^\omega$ is a fixed a.d. family and that $\mathcal{U} \subset \mathcal{P}(\omega) \setminus \mathcal{I}_\omega(\mathscr{A})$ is a selective ultrafilter.

Definition 11: For $s \in \mathrm{FIN}$, $\mathrm{cone}\,(s) = \{t \in \mathrm{FIN} : s \subset t\}$.

$$\mathscr{G} = \{A \subset \mathrm{FIN} : \forall b \in \mathcal{U} \exists s \in \mathrm{FIN}\,(b)\,[\mathrm{cone}\,(s) \subset A]\}.$$

It is easily checked that $\mathscr{G}$ is a filter on FIN. The forcing notion we will use is $\mathbb{P} = \mathbb{L}(\mathscr{G})$, Laver forcing for the filter $\mathscr{G}$. That is, p is in $\mathbb{P}$ iff p is a subtree of $(\mathrm{FIN})^{<\omega}$ such that p has a stem, σ, and for every $\tau \in p$ with $\tau \supset \sigma$, $\mathrm{succ}_p(\tau) \in \mathscr{G}$. It is σ-centered, and is a special case of the forcing notion defined in Brendle and Hrušák [2]. Let $\mathring{X}$ be the $\mathbb{P}$ name for the generic function in $(\mathrm{FIN})^\omega$ added by $\mathbb{P}$. It is easily checked that for any $\bar{a} \in \mathcal{I}(\mathscr{A})$ and $p \in \mathbb{P}$, there is a $q \le p$ and $m \in \omega$ so that $q \Vdash \mathring{X}(m) \cap \bar{a} = 0$. Also, it is easy to check that for any $\bar{a} \in \mathcal{I}(\mathscr{A})$ and $p \in \mathbb{P}$, there is $q \le p$ such that $q \Vdash \forall^\infty n \in \omega[\mathring{X}(n) \not\subset \bar{a}]$.

To begin with, we need $(*_1)$ to hold in the single step extension $\mathbf{V}[G]$. If there is an a.d. family $\mathscr{A}' \supset \mathscr{A}$ in $\mathbf{V}[G]$ such that $\mathrm{ran}\,(X)$ is not $\mathcal{I}(\mathscr{A}')$-positive, then there would be an $\bar{a} \in \mathcal{I}(\mathscr{A})$ and $c \in \mathcal{I}(\mathscr{A}' \setminus \mathscr{A})$ such that for all $m \in \omega, [X(m) \cap (\bar{a} \cup c) \neq 0]$. In particular, there would be a $c \in [\omega]^\omega \cap \mathbf{V}[G]$ which is almost disjoint from $\mathscr{A}$ with the property that $X(m) \cap c \neq 0$ for every m such that $X(m) \cap \bar{a} = 0$. The next lemma shows that this cannot happen in $\mathbf{V}[G]$. Of course, we also need this to be the case not just in $\mathbf{V}[G]$, but in all further extensions of it that we make. For this it is necessary to be able to deal with countably many candidates for c at once inside $\mathbf{V}[G]$.

LEMMA 12: *Let* $\{\mathring{a}_n : n \in \omega\} \subset \mathbf{V}^{\mathbb{P}}$ *and let* $\bar{a} \in \mathcal{I}(\mathscr{A})$. *Assume that for all* $n, m \in \omega$, $\Vdash [\mathring{X}(m) \cap \bar{a} = 0 \implies \mathring{X}(m) \cap \mathring{a}_n \neq 0]$. *Then there is* $a \in \mathscr{A}$ *such that for all* $n \in \omega$, $\Vdash |a \cap \mathring{a}_n| = \omega$, *and* $|a \cap \bar{a}| < \omega$.

Proof. Put $b = \omega \setminus \bar{a} \in \mathcal{U}$. Note that $\mathcal{U} \cap [b]^\omega$ is a selective ultrafilter on b. Fix $\sigma \in (\mathrm{FIN})^{<\omega}$ and $n \in \omega$. For each $s \in \mathrm{FIN}\,(b)$, there must be $k \in s$ such that $\neg\exists q \in \mathbb{P}\,[\mathrm{stem}\,(q) = \sigma^\frown\langle s\rangle \wedge q \Vdash k \notin \mathring{a}_n]$. If not, then for each $k \in s$, there is $q_k \in \mathbb{P}$ with $\mathrm{stem}\,(q_k) = \sigma^\frown\langle s\rangle$ so that $q_k \Vdash k \notin \mathring{a}_n$. But since s is finite, there is a $q \in \mathbb{P}$ with $\mathrm{stem}\,(q) = \sigma^\frown\langle s\rangle$ so that $q \Vdash s \cap \mathring{a}_n = 0$. But since $q \Vdash \mathring{X}(|\sigma|) = s$, and since $s \cap \bar{a} = 0$, this contradicts the hypothesis of the lemma. So we may define a function $F_{\langle\sigma,n\rangle} : \mathrm{FIN}\,(b) \to b$ by $F_{\langle\sigma,n\rangle}(s)$ is the least $k \in s$ so that $\neg\exists q \in \mathbb{P}\,[\mathrm{stem}\,(q) = \sigma^\frown\langle s\rangle \wedge q \Vdash k \notin \mathring{a}_n]$. Since $\mathcal{U} \cap [b]^\omega$ is a selective ultrafilter on b, we may use Lemma 9 to find $b_{\langle\sigma,n\rangle} \in \mathcal{U} \cap [b]^\omega$ which satisfies either (1) or (2) of Lemma 9 with respect to $F_{\langle\sigma,n\rangle}$. Now, there is $a \in \mathscr{A}$ so that $\forall\langle\sigma,n\rangle\,\left[\left|a \cap b_{\langle\sigma,n\rangle}\right| = \omega\right]$, and $|a \cap \bar{a}| < \omega$. This is the a we want. Suppose, for a contradiction, that there is $n \in \omega$, and $p \in \mathbb{P}$ and $m \in \omega$ so that $p \Vdash a \cap \mathring{a}_n \subset m$. Put $\sigma = \mathrm{stem}\,(p)$.

Suppose first that $b_{\langle\sigma,n\rangle}$ satisfies (1) with respect to $F_{\langle\sigma,n\rangle}$. As $b_{\langle\sigma,n\rangle} \in \mathcal{U}$, there is $s \in \mathrm{FIN}\,(b_{\langle\sigma,n\rangle})$ so that $\mathrm{cone}\,(s) \subset \mathrm{succ}_p(\sigma)$. Put $c = (a \cap b_{\langle\sigma,n\rangle})/(s \cup m) \in \left[b_{\langle\sigma,n\rangle}/s\right]^\omega$. By (1), there is a $t \sqsubset c$, with $t \neq 0$, so that $F_{\langle\sigma,n\rangle}(s \cup t) \in t$. Put $k = F_{\langle\sigma,n\rangle}(s \cup t)$, and note that $k \in a$, and that $k \geq m$. Also, $s \cup t \in \mathrm{cone}\,(s)$, and so $\exists q \leq p$ so that $\mathrm{stem}\,(q) = \sigma^\frown\langle s \cup t\rangle$. Therefore, there is $r \leq q$ so that $r \Vdash k \in \mathring{a}_n$, which is a contradiction.

Suppose next that $b_{\langle\sigma,n\rangle}$ satisfies (2) with respect to $F_{\langle\sigma,n\rangle}$. Put $c = (a \cap b_{\langle\sigma,n\rangle}) \setminus m \in \left[b_{\langle\sigma,n\rangle}\right]^\omega$. By (2), there is $s \sqsubset c$ with $s \neq 0$ so that for all $t \in \left[b_{\langle\sigma,n\rangle}/s\right]^{<\omega}$, $F_{\langle\sigma,n\rangle}(s \cup t) \in s$. Since $b_{\langle\sigma,n\rangle} \in \mathcal{U}$, $b_{\langle\sigma,n\rangle}/s \in \mathcal{U}$ as well. So there is $t \in \mathrm{FIN}\,(b_{\langle\sigma,n\rangle}/s)$ so that $\mathrm{cone}\,(t) \subset \mathrm{succ}_p(\sigma)$. Put $k = F_{\langle\sigma,n\rangle}(s \cup t)$. Observe that $k \in a$ and $k \geq m$. Since $s \cup t \in \mathrm{cone}\,(t)$, there is $q \leq p$ with $\mathrm{stem}\,(q) = \sigma^\frown\langle s \cup t\rangle$. As above, we get $r \leq q$ with $r \Vdash k \in \mathring{a}_n$, which is a contradiction. ∎

It is clear that requirement $(*_1)$ will be met if we are able to preserve this property of X in all further extensions of $\mathbf{V}[G]$ that we make. For the sake of proving our theorems, it is enough to do this for an iteration of length ω_2, and, as we will see, this is a very easy task. However, with further applications in mind, we will take a slightly more general approach.

Definition 13: Let $\mathscr{A} \subset [\omega]^\omega$ be any uncountable a.d. family, and let $X : \omega \to \mathrm{FIN}$ be such that for any $\bar{a} \in \mathcal{I}(\mathscr{A})$, there is $m \in \omega$ such that $X(m) \cap \bar{a} = 0$. We say that X is ω **positive for** $\mathscr{A}$ if for any $\bar{a} \in \mathcal{I}(\mathscr{A})$, whenever $\{c_n : n \in \omega\} \subset [\omega]^\omega$ is a countable collection such that

$$\forall n, m \in \omega \left[X(m) \cap \bar{a} = 0 \implies X(m) \cap c_n \neq 0\right],$$

there is $a \in \mathscr{A}$ so that for all $n \in \omega$, $|c_n \cap a| = \omega$, and $|a \cap \bar{a}| < \omega$.

Thus, Lemma 12 says that our forcing $\mathbb{L}(\mathscr{G})$ adds an $X : \omega \to \mathrm{FIN}$ which is ω positive for $\mathscr{A}$ in $\mathbf{V}[G]$. We will argue that the ω positivity of X for $\mathscr{A}$ is preserved in a strong sense in future extensions.

Definition 14: Let $\mathscr{A}$ be an uncountable a.d. family, and suppose $X : \omega \to \mathrm{FIN}$. Let $\bar{a} \in \mathcal{I}(\mathscr{A})$ and let $M \prec H(\theta)$ be countable with $\mathscr{A}, X, \bar{a} \in M$. A set $a \in \mathscr{A}$ is said to **cover** M **with respect to** $(\mathscr{A}, X, \bar{a})$ if $|a \cap \bar{a}| < \omega$, and if $|a \cap c| = \omega$ for every $c \in M \cap [\omega]^\omega$ with the property that $c \cap X(m) \neq 0$ for every m such that $X(m) \cap \bar{a} = 0$.

If $X : \omega \to \mathrm{FIN}$ is ω positive for $\mathscr{A}$, then for any $\bar{a} \in \mathcal{I}(\mathscr{A})$ and countable $\mathcal{M} \prec H(\theta)$ with $\mathscr{A}, X, \bar{a} \in M$, there is $a \in \mathscr{A}$ which covers M with respect to $(\mathscr{A}, X, \bar{a})$.

Definition 15: Let $\mathscr{A}$ be an uncountable a.d. family, and suppose X is ω positive for $\mathscr{A}$. Let $\mathbb{P}$ be a c.c.c. poset. We will say that $\mathbb{P}$ **strongly preserves** X **for** $\mathscr{A}$ if for any $\bar{a} \in \mathcal{I}(\mathscr{A})$ and countable $M \prec H(\theta)$ with $\mathbb{P}, \mathscr{A}, X, \bar{a} \in M$, whenever $a \in \mathscr{A}$ covers M with respect to $(\mathscr{A}, X, \bar{a})$, $\Vdash a$ covers $M[\mathring{G}]$ with respect to $(\mathscr{A}, X, \bar{a})$.

As usual, in the above definition it suffices to consider only a club of elementary submodels $M \prec H(\theta)$. It is clear that if $\mathbb{P}$ is a c.c.c. poset that strongly preserves X for $\mathscr{A}$ and G is $(\mathbf{V}, \mathbb{P})$ generic, then in $\mathbf{V}[G]$, X remains ω positive for $\mathscr{A}$.

LEMMA 16: *Let $\mathscr{A} \subset [\omega]^\omega$ be an uncountable a.d. family, and suppose $X : \omega \to \mathrm{FIN}$ is ω positive for $\mathscr{A}$. Let $\mathbb{P}$ be any σ-centered poset. Then $\mathbb{P}$ strongly preserves X for $\mathscr{A}$.*

Proof. Fix $\bar{a} \in \mathcal{I}(\mathscr{A})$. Let $M \prec H(\theta)$ be countable with $\mathscr{A}, X, \mathbb{P}, \bar{a} \in M$. Suppose that $a \in \mathscr{A}$ covers M with respect to $(M, \mathscr{A}, \bar{a})$. Observe that this implies $|a \cap \bar{a}| < \omega$, and this cannot fail in the extension. Now, since $\mathbb{P}$ is σ-centered, there is $\langle \mathbb{P}_i : i \in \omega \rangle \in M$ such that each $\mathbb{P}_i$ is centered, and $\mathbb{P} = \bigcup \mathbb{P}_i$. Suppose for a contradiction that there are $p \in \mathbb{P}$, $\mathring{c} \in M$ and $n \in \omega$ so that $p \Vdash \forall m \in \omega\, [X(m) \cap \bar{a} = 0 \implies X(m) \cap \mathring{c} \neq 0]$, and that $p \Vdash a \cap \mathring{c} \subset n$. Suppose that $p \in \mathbb{P}_i$. Now, put $Z = \{m \in \omega : X(m) \cap \bar{a} = 0\}$, and note $Z \in M$. Define a function $K : Z \to \omega$ by stipulating that for any $m \in Z$, $K(m)$ is the least $k \in X(m)$ such that $\neg \exists q \in \mathbb{P}_i\, [q \Vdash k \notin \mathring{c}]$. Such $k \in X(m)$ must exist. For otherwise, for every $k \in X(m)$, there would be $q_k \in \mathbb{P}_i$ such that $q_k \Vdash k \notin \mathring{c}$. But then since $X(m)$ is finite and since $p \in \mathbb{P}_i$, there would be $q \leq p$ such that $q \Vdash X(m) \cap \mathring{c} = 0$, which is a contradiction. Now, since $Z, X, \mathbb{P}_i, \mathring{c} \in M$, $K \in M$, and therefore, $c = \{K(m) : m \in Z\} \in M$. But it is clear that $c \in [\omega]^{\omega}$ and that $c \cap X(m) \neq 0$ for all $m \in Z$. Therefore, $|a \cap c| = \omega$. Choose $K(m) \in a$ with $K(m) \geq n$. But since $p \in \mathbb{P}_i$, there is $q \leq p$ such that $q \Vdash K(m) \in \mathring{c}$, a contradiction. ∎

Lemma 16 is sufficient for our proof of Theorems 4 and 6 because we are doing an FS iteration of σ-centered posets of length ω_2. It is well known that the FS iteration of σ-centered posets of any length less than ω_2 is σ-centered, and hence X remains ω positive for $\mathscr{A}$ at stages of the iteration less than ω_2. And, of course, it still remains so at stage ω_2 because no new reals are added at that stage. However, with a view towards future applications, we show below that the FS iterations of arbitrary c.c.c. posets of limit length strongly preserves X for $\mathscr{A}$ provided each initial segment does.

Lemma 17: *Let $\mathscr{A} \subset [\omega]^{\omega}$ be an uncountable a.d. family and suppose $X : \omega \to \mathrm{FIN}$ is ω positive for $\mathscr{A}$. Let γ be a limit ordinal, and suppose $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a FS iteration such that $\forall \alpha < \gamma [\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha$ is c.c.c.$]$. If $\mathbb{P}_\alpha$ strongly preserves X for $\mathscr{A}$ for all $\alpha < \gamma$, then so does $\mathbb{P}_\gamma$.*

Proof. It suffices to prove the result for $\gamma = \omega$. Fix $\bar{a} \in \mathcal{I}(\mathscr{A})$, and let $M \prec H(\theta)$ be countable with $\mathscr{A}, X, \bar{a}, \langle \mathbb{P}_i, \mathring{\mathbb{Q}}_i : i \leq \omega \rangle \in M$. Suppose $a \in \mathscr{A}$ covers M with respect to $(\mathscr{A}, X, \bar{a})$. Once again, this means $|a \cap \bar{a}| < \omega$, and this remains true in the extension. Put $Z = \{m \in \omega : X(m) \cap \bar{a} = 0\}$. Suppose for a contradiction that there are $p \in \mathbb{P}_\omega$, $\mathring{c} \in M \cap \mathbf{V}^{\mathbb{P}_\omega}$ and $n \in \omega$ so that $p \Vdash_\omega \forall m \in Z\, [X(m) \cap \mathring{c} \neq 0]$, and that $p \Vdash_\omega a \cap \mathring{c} \subset n$. Suppose $\mathrm{suppt}\,(p) \subset i$. Recall that we may view $\mathring{c}$ as a $\mathbb{P}_i$ name for a $\mathbb{P}_\omega / \mathring{G}_i$ name. We use $\mathring{c}[\mathring{G}_i]$ to denote the canonical $\mathbb{P}_i$ name for a $\mathbb{P}_\omega / \mathring{G}_i$ name representing $\mathring{c}$, and we denote its valuation by a $(\mathbf{V}, \mathbb{P}_i)$ generic filter G_i by $\mathring{c}[G_i]$. Now, let G_i be a $(\mathbf{V}, \mathbb{P}_i)$ generic filter with $p \restriction i \in G_i$. Since $\operatorname{suppt}(p) \subset i$, it is clear that in $\mathbf{V}[G_i]$, both $\Vdash_{\mathbb{P}_\omega / G_i} \forall m \in Z\,[X(m) \cap \mathring{c}[G_i] \neq 0]$ and $\Vdash_{\mathbb{P}_\omega / G_i} a \cap \mathring{c}[G_i] \subset n$ hold. But since $\mathbb{P}_i, \mathbb{P}_\omega, \mathring{c} \in M$, both $\mathbb{P}_\omega / G_i$ and $\mathring{c}[G_i]$ are elements of $M[G_i]$. So there is a descending sequence of conditions in $\mathbb{P}_\omega / G_i$, $\langle p_m : m \in Z \rangle \in M[G_i]$, as well as a sequence $\langle k(m) : m \in Z \rangle \in M[G_i]$ such that for each $m \in Z$, $p_m \Vdash_{\mathbb{P}_\omega / G_i} k(m) \in X(m) \cap \mathring{c}[G_i]$. But by hypothesis a covers $M[G_i]$ with respect to $(\mathscr{A}, X, \bar{a})$. Therefore, $|a \cap \langle k(m) : m \in Z \rangle| = \omega$. Choose $k(m) \in a$ with $k(m) \geq n$. But then, $p_m \Vdash_{\mathbb{P}_\omega / G_i} k(m) \in \mathring{c}[G_i] \cap a$, contradicting $\Vdash_{\mathbb{P}_\omega / G_i} a \cap \mathring{c}[G_i] \subset n$. ∎

The above lemmas show how to ensure $(*_1)$ during the iteration. Next, we turn to ensuring $(*_2)$. We will first show that $(*_2)$ holds with $\operatorname{ran}(X)$ as P in the single step extension $\mathbf{V}[G]$, and then argue that this property of $\operatorname{ran}(X)$ is preserved in further extensions we make. This was done by Brendle and Hrušák for their forcing in [2]. Our forcing behaves differently because our filter $\mathscr{G}$ is smaller than their filter. Nevertheless, our proof of $(*_2)$ closely follows theirs, except for the very first step, where we have to argue somewhat differently.

Definition 18: Let $\mathring{A} \in \mathbf{V}^{\mathbb{P}}$ and suppose that $\Vdash \mathring{A} \in [\operatorname{ran}(\mathring{X})]^{\omega}$. We say that $\sigma \in (\mathrm{FIN})^{<\omega}$ **has rank** 0 **with respect to** $\mathring{A}$ if there is a set $X \in \mathscr{G}^{+}$ so that $\forall s \in X \neg \exists q \in \mathbb{P}[\operatorname{stem}(q) = \sigma^\frown \langle s \rangle \wedge q \Vdash s \notin \mathring{A}]$.

Lemma 19: *Let $\mathring{A} \in \mathbf{V}^{\mathbb{P}}$ and suppose that $\Vdash \mathring{A} \in [\operatorname{ran}(\mathring{X})]^{\omega}$. For every $p \in \mathbb{P}$, if $\operatorname{stem}(p) = \sigma$, then there is $\tau \in p$ with $\tau \supset \sigma$ so that τ has rank 0 with respect to $\mathring{A}$*

Proof. Suppose not. Then for each $\tau \in p$ with $\tau \supset \sigma$, there is a set $Y \subset \operatorname{succ}_p(\tau)$ with $Y \in \mathscr{G}$ such that for each $s \in Y$ there is $q \in \mathbb{P}$ with $\operatorname{stem}(q) = \tau^\frown \langle s \rangle$ satisfying $q \Vdash s \notin \mathring{A}$. Therefore, we can recursively construct a condition $q \leq p$ with $\operatorname{stem}(q) = \sigma$ satisfying the following property:

$$(*) \qquad \forall \tau \in q \left[\tau \supsetneq \sigma \implies q_\tau \Vdash \tau(|\tau| - 1) \notin \mathring{A} \right].$$

Now, since $\Vdash \mathring{A} \in [\operatorname{ran}(\mathring{X})]^{\omega}$, there are $r \leq q$, $s \in \mathrm{FIN}$, and $m \geq |\sigma|$ such that $r \Vdash s \in \mathring{A} \wedge s = \mathring{X}(m)$. If $\tau = \operatorname{stem}(r)$, then we have that $\sigma \subset \tau$, that $|\sigma| \leq m < |\tau|$, and that $\tau(m) = s$. However, it is clear that $r \leq q_\tau \leq q_{(\tau \restriction m+1)}$. But this a contradiction since $q_{(\tau \restriction m+1)} \Vdash s = \tau(m) \notin \mathring{A}$ by $(*)$. ∎

LEMMA 20: *Suppose* $\{\mathring{A}_n : n \in \omega\} \subset \mathbf{V}^{\mathbb{P}}$ *such that* $\forall n \in \omega \left[\Vdash \mathring{A}_n \in [\operatorname{ran}(\mathring{X})]^{\omega} \right]$. *Then there is* $a \in \mathscr{A}$ *such that for all* $n \in \omega$, $\Vdash \left| a \cap (\bigcup \mathring{A}_n) \right| = \omega$.

Proof. Observe that by Lemma 19, for each $p \in \mathbb{P}$ and $n \in \omega$, if $\sigma = \operatorname{stem}(p)$, then there is $\tau \in p$ with $\tau \supset \sigma$ such that τ has rank 0 with respect to $\mathring{A}_n$.

Now, notice that a set $X \subset \mathrm{FIN}$ is in $\mathscr{G}^*$ iff $\forall b \in \mathcal{U} \exists s \in \mathrm{FIN}(b)\, [\operatorname{cone}(s) \cap X = 0]$, and so a set $X \subset \mathrm{FIN}$ is in $\mathscr{G}^+$ iff $\exists b \in \mathcal{U} \forall s \in \mathrm{FIN}(b)\, [\operatorname{cone}(s) \cap X \neq 0]$. For each pair $\langle \sigma, n \rangle$ such that σ has rank 0 with respect to $\mathring{A}_n$, choose $b_{\langle \sigma, n \rangle} \in \mathcal{U}$ such that for all $s \in \mathrm{FIN}(b_{\langle \sigma, n \rangle})$, there is $t \in \operatorname{cone}(s)$ so that

$$(*) \qquad \neg \exists q \in \mathbb{P} \left[\operatorname{stem}(q) = \sigma^\frown \langle t \rangle \wedge q \Vdash t \notin \mathring{A}_n \right].$$

Choose $a \in \mathscr{A}$ such that for all pairs $\langle \sigma, n \rangle$ where σ has rank 0 with respect to $\mathring{A}_n$, $\left| a \cap b_{\langle \sigma, n \rangle} \right| = \omega$. We claim that this a does the job. Suppose for a contradiction that there is $p \in \mathbb{P}$, $n \in \omega$, and $m \in \omega$ such that $p \Vdash a \cap (\bigcup \mathring{A}_n) \subset m$. Put $\sigma = \operatorname{stem}(p)$. In view of our earlier observation, we may assume that σ has rank 0 with respect to $\mathring{A}_n$. So $b_{\langle \sigma, n \rangle}$ exists, and $\left| a \cap b_{\langle \sigma, n \rangle} \right| = \omega$. By definition of $\mathscr{G}$, there is $s \in \mathrm{FIN}(b_{\langle \sigma, n \rangle})$ such that $\operatorname{cone}(s) \subset \operatorname{succ}_p(\sigma)$. Choose $k \in a \cap b_{\langle \sigma, n \rangle}$ with $k \geq m$, and put $s' = s \cup \{k\}$. Now, $s' \in \mathrm{FIN}(b_{\langle \sigma, n \rangle})$. So by choice of $b_{\langle \sigma, n \rangle}$, there is $t \supset s' \supset s$ satisfying $(*)$ above. Now, $t \in \operatorname{succ}_p(\sigma)$. Therefore, there is $q \leq p$ with $\operatorname{stem}(q) = \sigma^\frown \langle t \rangle$. Since t satisfies $(*)$, there is $r \leq q$ such that $r \Vdash t \in \mathring{A}_n$. So $r \Vdash k \in a \cap (\bigcup \mathring{A}_n)$, a contradiction. ∎

Brendle and Hrušák [2] showed that this property of $\operatorname{ran}(X)$ is preserved in a strong sense by certain forcings of the form $\mathbb{L}(\mathscr{H})$, and also at limit stages of FS iterations of c.c.c. posets. We reproduce this here in somewhat different terminology.

Definition 21: Let $\mathscr{A} \subset [\omega]^{\omega}$ and suppose $X : \omega \to \mathrm{FIN}$ (with $|\operatorname{ran}(X)| = \omega$). We say that $\mathscr{A}$ is **ω-maximal for** X if for every $\{A_n : n \in \omega\} \subset [\operatorname{ran} X]^{\omega}$, there is $a \in \mathscr{A}$ such that for all $n \in \omega$, $|a \cap \bigcup A_n| = \omega$. Given a countable $M \prec H(\theta)$ with $\mathscr{A}, X \in M$, we say that $a \in \mathscr{A}$ **covers** M **with respect to** $(\mathscr{A}, X)$ if $|a \cap \bigcup A| = \omega$ for every $A \in M \cap [\operatorname{ran}(X)]^{\omega}$.

Definition 22: Let $\mathscr{A} \subset [\omega]^{\omega}$ be an uncountable a.d. family, and suppose $\mathscr{A}$ is ω maximal for some $X : \omega \to \mathrm{FIN}$. Let $\mathbb{P}$ be a c.c.c. poset. We will say that $\mathbb{P}$ **strongly preserves** $\mathscr{A}$ **for** X if for every countable $M \prec H(\theta)$ with $\mathscr{A}, X, \mathbb{P} \in M$, whenever $a \in \mathscr{A}$ covers M with respect to $(\mathscr{A}, X)$, $\Vdash a$ covers $M[\mathring{G}]$ with respect to $(\mathscr{A}, X)$.

As usual, it is sufficient to only consider a club of elementary submodels. Also, it is clear that if $\mathscr{A}$ is ω-maximal for X and if $\mathbb{P}$ strongly preserves $\mathscr{A}$ for X, then $\mathscr{A}$ remains ω-maximal for X in the extension by $\mathbb{P}$. It was essentially proved in Brendle and Hrušák [2] that a forcing of the form $\mathbb{L}(\mathscr{H})$ strongly preserves $\mathscr{A}$ for X as long as every set in $\mathscr{H}^+$ contains an infinite subset that is a.d. from $\mathscr{H}^*$, though they did not phrase it in terms of elementary submodels. Thus we need to check that our filter $\mathscr{G}$ has this property.

Lemma 23: *Suppose $X \in \mathscr{G}^+$; then there is $Y \in [X]^\omega$ so that for all $Z \in \mathscr{G}^*$, $|Z \cap Y| < \omega$.*

Proof. Fix $X \in \mathscr{G}^+$. Let $b \in \mathcal{U}$ be so that for all $s \in \mathrm{FIN}\,(b)$, $\mathrm{cone}\,(s) \cap X \neq 0$. Put $b = \{n_0 < n_1 < \cdots\}$. For each $k \in \omega$, find $t_k \in X$ such that $\{n_0, \ldots, n_k\} \subset t_k$. Clearly, $Y = \{t_k : k \in \omega\}$ is an infinite subset of X. Now suppose $Z \in \mathscr{G}^*$. Then there is k so that $\mathrm{cone}\,(\{n_0, \ldots, n_k\}) \cap Z = 0$, whence $t_m \notin Z$, for all $m \geq k$. ∎

Definition 24: Let $\mathscr{H}$ be a filter on a countable set C and suppose $\mathbb{P} = \mathbb{L}(\mathscr{H})$. Let $X : \omega \to \mathrm{FIN}$ and suppose $\mathring{A} \in \mathbf{V}^{\mathbb{P}}$ is such that $\Vdash \mathring{A} \in [\mathrm{ran}\,(X)]^\omega$. For each $\sigma \in C^{<\omega}$, put $Z_\sigma(\mathring{A}) = \{s \in \mathrm{FIN} : \neg\exists q \in \mathbb{P}[\mathrm{stem}\,(q) = \sigma \wedge q \Vdash s \notin \mathring{A}]\}$. We say that $\sigma \in C^{<\omega}$ has **rank 0 with respect to** $\mathring{A}$ if either $Z_\sigma(\mathring{A})$ is infinite or there is a $B \in \mathscr{H}^+$ such that for all $c \in B$, $Z_{\sigma^\frown c}(\mathring{A}) \setminus Z_\sigma(\mathring{A}) \neq 0$.

Lemma 25: *Let $\mathscr{H}$ be a filter on a countable set C and suppose $\mathbb{P} = \mathbb{L}(\mathscr{H})$. Let $X : \omega \to \mathrm{FIN}$ and suppose $\mathring{A} \in \mathbf{V}^{\mathbb{P}}$ is such that $\Vdash \mathring{A} \in [\mathrm{ran}\,(X)]^\omega$. For each $p \in \mathbb{P}$, if $\sigma = \mathrm{stem}\,(p)$, then there is a $\tau \in p$ with $\tau \supset \sigma$ which has rank 0 with respect to $\mathring{A}$.*

Proof. Suppose not. By recursion we can build a condition $q \leq p$ with $\mathrm{stem}\,(q) = \sigma$ such that $Z_\sigma(\mathring{A})$ is finite and for all $\tau \in q$, if $\tau \supset \sigma$, then $Z_\tau(\mathring{A}) \subset Z_\sigma(\mathring{A})$. But since $\Vdash \mathring{A} \in [\mathrm{ran}\,(X)]^\omega$ there are $s \in \mathrm{ran}\,(X) \setminus Z_\sigma(\mathring{A})$ and $r \leq q$ such that $r \Vdash s \in \mathring{A}$. Put $\tau = \mathrm{stem}\,(r)$. But then $\tau \in q$ and $\tau \supset \sigma$, and so $s \notin Z_\tau(\mathring{A})$, whence there is $r' \leq r$ such that $r' \Vdash s \notin \mathring{A}$, a contradiction. ∎

Lemma 26 (Brendle and Hrušák [2]): *Let $\mathscr{A} \subset [\omega]^\omega$ be an uncountable a.d. family, and suppose $\mathscr{A}$ is ω maximal for some $X : \omega \to \mathrm{FIN}$. Let $\mathscr{H}$ be a filter on a countable set C with the property that for every $A \in \mathscr{H}^+$, there is $B \in [A]^\omega$ so that for all $D \in \mathscr{H}^*$, $|B \cap D| < \omega$. Then $\mathbb{P} = \mathbb{L}(\mathscr{H})$ strongly preserves $\mathscr{A}$ for X.*

Proof. Fix $M \prec H(\theta)$ with $\mathscr{A}, X, \mathbb{P} \in M$. Suppose $a \in \mathscr{A}$ covers M with respect to $(\mathscr{A}, X)$. Suppose for a contradiction that there are $\mathring{A} \in M \cap \mathbf{V}^{\mathbb{P}}$, $p \in \mathbb{P}$, and $n \in \omega$ such that $\Vdash \mathring{A} \in [\operatorname{ran}(X)]^{\omega}$, and $p \Vdash a \cap \bigcup \mathring{A} \subset n$. By Lemma 25, we may assume that $\sigma = \operatorname{stem}(p)$ has rank 0 with respect to $\mathring{A}$. Note that since $\mathscr{H}$ and C are definable from $\mathbb{P}$, we have $\mathscr{H}, C, C^{<\omega} \in M$. Also, since $\mathring{A} \in M$, we have $\langle Z_\tau(\mathring{A}) : \tau \in C^{<\omega} \rangle \in M$.

Suppose first that $Z_\sigma(\mathring{A})$ is infinite. Note that for every $\tau \in C^{<\omega}$, $Z_\tau(\mathring{A}) \subset \operatorname{ran}(X)$. So $Z_\sigma(\mathring{A}) \in M \cap [\operatorname{ran}(X)]^{\omega}$, and since a covers M with respect to $(\mathscr{A}, X)$, $|a \cap \bigcup Z_\sigma(\mathring{A})| = \omega$. Choose $s \in Z_\sigma(\mathring{A})$ and $k \in s \cap a$ with $k \geq n$. But then there is a $q \leq p$ such that $q \Vdash s \in \mathring{A}$, whence $q \Vdash k \in a \cap \bigcup \mathring{A}$, a contradiction.

Next suppose that $\exists B \in \mathscr{H}^+ \forall c \in B [Z_{\sigma^\frown\langle c\rangle}(\mathring{A}) \setminus Z_\sigma(\mathring{A}) \neq 0]$. By elementarity, we can have $B \in M$. By the assumption on $\mathscr{H}$, there is $D \in [B]^{\omega}$ such that $\forall E \in \mathscr{H}^*, [\, |D \cap E| < \omega]$. Again, we can have $D \in M$. Therefore, there is an $A = \{s_c : c \in D\} \in M$ such that for each $c \in D$, $s_c \in Z_{\sigma^\frown\langle c\rangle}(\mathring{A}) \setminus Z_\sigma(\mathring{A})$. Observe that $A \subset \operatorname{ran}(X)$. We claim that A must be infinite. Suppose A is finite. Then there must be a s and $E \in \mathscr{H}^+ \cap [D]^{\omega}$ such that for all $c \in E$, $s_c = s$. Since $s \notin Z_\sigma(\mathring{A})$, there is $q \in \mathbb{P}$ with $\operatorname{stem}(q) = \sigma$ such that $q \Vdash s \notin \mathring{A}$. But since $\operatorname{succ}_q(\sigma) \in \mathscr{H}$ and $E \in \mathscr{H}^+$, there must be $c \in E \cap \operatorname{succ}_q(\sigma)$. But then $q_{\sigma^\frown\langle c\rangle} \leq q$ and $\operatorname{stem}\left(q_{\sigma^\frown\langle c\rangle}\right) = \sigma^\frown\langle c\rangle$. Since $s \in Z_{\sigma^\frown\langle c\rangle}(\mathring{A})$, there is $r \leq q_{\sigma^\frown\langle c\rangle} \leq q$ such that $r \Vdash s \in \mathring{A}$, a contradiction. Therefore, $A \in M \cap [\operatorname{ran}(X)]^{\omega}$, and so $|a \cap \bigcup A| = \omega$. Now, since D is almost disjoint from every member of $\mathscr{H}^*$, it must be almost contained in every member of $\mathscr{H}$. In particular $D \subset^* \operatorname{succ}_p(\sigma)$. If follows that we can find $c \in \operatorname{succ}_p(\sigma) \cap D$ and $k \in s_c \cap a$ such that $k \geq n$. But $p_{\sigma^\frown\langle c\rangle} \leq p$ and $s_c \in Z_{\sigma^\frown\langle c\rangle}(\mathring{A})$. So there is $q \leq p_{\sigma^\frown\langle c\rangle} \leq p$ such that $q \Vdash s_c \in \mathring{A}$, whence $q \Vdash k \in a \cap \bigcup \mathring{A}$, a contradiction. ∎

Lemmas 23 and 26 imply that successor steps of the iteration strongly preserve $\mathscr{A}$ for X, and the next lemma shows that an FS iteration of c.c.c. posets of limit length strongly preserves $\mathscr{A}$ for X as long as initial segments do.

Lemma 27: *Let $\mathscr{A} \subset [\omega]^{\omega}$ be an uncountable a.d. family, and suppose $\mathscr{A}$ is ω maximal for some $X : \omega \to \mathrm{FIN}$. Let γ be a limit ordinal, and suppose $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a FS iteration such that $\forall \alpha < \gamma [\Vdash_\alpha \mathring{\mathbb{Q}}_\alpha$ is c.c.c.$]$. If $\mathbb{P}_\alpha$ strongly preserves $\mathscr{A}$ for X for all $\alpha < \gamma$, then so does $\mathbb{P}_\gamma$.*

Proof. Once again, we may assume $\gamma = \omega$. Let $M \prec H(\theta)$ be countable with $\mathscr{A}, X, \langle \mathbb{P}_i, \mathring{\mathbb{Q}}_i : i \leq \omega \rangle \in M$, and suppose $a \in \mathscr{A}$ covers M with respect to $(\mathscr{A}, X)$. Suppose for a contradiction that there are $p \in \mathbb{P}_\omega$, $\mathring{A} \in M \cap \mathbf{V}^{\mathbb{P}_\omega}$, and $n \in \omega$ such that $p \Vdash_\omega \mathring{A} \in [\mathrm{ran}\,(X)]^\omega \wedge a \cap \bigcup \mathring{A} \subset n$. Suppose $\mathrm{suppt}\,(p) \subset i$. Let G_i be a $(\mathbf{V}, \mathbb{P}_i)$ generic filter with $p \restriction i \in G_i$. In $\mathbf{V}[G_i]$, we again have that $p \in \mathbb{P}_\omega / G_i$ is compatible with every element of $\mathbb{P}_\omega / G_i$. Therefore, $\Vdash_{\mathbb{P}_\omega / G_i} \mathring{A}[G_i] \in [\mathrm{ran}\,(X)]^\omega \wedge a \cap \bigcup \mathring{A}[G_i] \subset n$, where $\mathring{A}[G_i]$ is the $\mathbb{P}_\omega / G_i$ name representing $\mathring{A}$. Now, since $\mathbb{P}_\omega, \mathbb{P}_i, \mathring{A} \in M$, we have both $\mathbb{P}_\omega / G_i$ and $\mathring{A}[G_i]$ in $M[G_i]$. So there are sequences $\langle p_m : m \in \omega \rangle, \langle s_m : m \in \omega \rangle \in M[G_i]$ such that for each $m \in \omega$, $p_m \in \mathbb{P}_\omega / G_i$, $p_{m+1} \leq p_m$, $s_m \in \mathrm{ran}\,(X)$, $s_m \notin \{s_i : i < m\}$, and $p_m \Vdash_{\mathbb{P}_\omega / G_i} s_m \in \mathring{A}[G_i]$. But since, by hypothesis, a covers $M[G_i]$ with respect to $(\mathscr{A}, X)$, $a \cap \bigcup_{m \in \omega} s_m$ is infinite. So there is $k \geq n$ and $m \in \omega$ such that $k \in a \cap s_m$. But then $p_m \Vdash_{\mathbb{P}_\omega / G_i} k \in a \cap \bigcup \mathring{A}[G_i]$, contradicting $\Vdash_{\mathbb{P}_\omega / G_i} a \cap \bigcup \mathring{A}[G_i] \subset n$. ∎

Proof of Theorems 4 and 6. Put $S^2_1 = \{\alpha < \omega_2 : \mathrm{cf}\,(\alpha) = \omega_1\}$. Assume that the ground model satisfies CH, and suppose $\langle D_\alpha : \alpha \in S^2_1 \rangle$ witnesses that $\diamondsuit(S^2_1)$ holds in it. We do an FS iteration of c.c.c. posets of length $\omega_2 : \langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha : \alpha \leq \omega_2 \rangle$. At a stage $\alpha \in S^2_1$, if D_α codes a $\mathbb{P}_\alpha$ name for an a.d. family forced to have the property that for every $b \in [\omega]^\omega$, either $\mathscr{A}_b$ is finite or $\mathscr{A}_b$ is uncountable, we let $\mathring{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$ name for $\mathbb{L}(\mathscr{G})$. If α is not of this form, let $\mathring{Q}_\alpha$ be a $\mathbb{P}_\alpha$ name for the trivial poset.

For Theorem 4 note that if $\mathscr{A} \subset [\omega]^\omega$ is an almost strongly separable MAD family in $\mathbf{V}[G_{\omega_2}]$, then, by a standard argument, there is a set $C \subset S^2_1$ which is a club relative to S^2_1 such that for all $\alpha \in C$, $\mathscr{A} \cap \mathbf{V}[G_\alpha] \in \mathbf{V}[G_\alpha]$ is a MAD family in $\mathbf{V}[G_\alpha]$. Therefore, at some stage $\alpha \in C$, we would have added a $P \subset \mathrm{FIN}$ that is $\mathcal{I}(\mathscr{A} \cap \mathbf{V}[G_\alpha])$-positive, and by $(*_1)$ it would still be $\mathcal{I}(\mathscr{A})$-positive, and by $(*_2)$ there would be no $a \in \mathscr{A}$ and $Q \in [P]^\omega$ with $\bigcup Q \subset a$. This is a contradiction.

For Theorem 6, suppose $\mathscr{A} \subset [\omega]^\omega$ is an a.d. family in $\mathbf{V}[G_{\omega_2}]$ so that for each $b \in [\omega]^\omega$, either $\mathscr{A}_b$ is finite or $\mathscr{A}_b$ is uncountable. Once again, there is a $C \subset S^2_1$ which is a club relative to S^2_1 so that for each $\alpha \in C$, $\mathscr{A} \cap \mathbf{V}[G_\alpha] \in \mathbf{V}[G_\alpha]$ and has this same property in $\mathbf{V}[G_\alpha]$. But then at some $\alpha \in C$ we added a $P \subset \mathrm{FIN}$ that is $\mathcal{I}(\mathscr{A} \cap \mathbf{V}[G_\alpha])$-positive, and by $(*_1)$ it is $\mathcal{I}(\mathscr{A})$-positive, and by $(*_2)$ there is no $Q \in [P]^\omega \cap \mathbf{V}[G_{\omega_2}]$ such that $\bigcup Q$ is a.d. from $\mathscr{A} \cap \mathbf{V}[G_\alpha]$, whence $\mathcal{I}(\mathscr{A})$ is not Fréchet in $\mathbf{V}[G_{\omega_2}]$. ∎

3. Some Questions

We still have the questions of Gruenhage and Szeptycki:

Question 28: Is there an uncountable a.d. family $\mathscr{A} \subset [\omega]^\omega$ such that $\mathcal{I}(\mathscr{A})$ is Fréchet? Is there a Fréchet ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ that is not countably generated?

Regarding the first of these, it seems to us that our proof given here can be made to work as long as there is an ultrafilter $\mathcal{U} \subset \mathcal{P}(\omega) \setminus \mathcal{I}_\omega(\mathscr{A})$ with the property that for any countable collection $\{b_n : n \in \omega\} \subset \mathcal{U}$, there is an $a \in \mathscr{A}$ so that for all $n \in \omega$, $|a \cap b_n| = \omega$. Typical examples of a.d. families that fail this condition have the following form.

Definition 29: Let $\{x_n : n \in \omega\} \subset [\omega]^\omega$ be an independent family. Let $x_n^0 = \omega \setminus x_n$, and let $x_n^1 = x_n$. We say that an almost disjoint family $\mathscr{A} \subset [\omega]^\omega$ is **neat (with respect to** $\{x_n : n \in \omega\}$**)** if $\mathscr{A} = \{a_f : f \in 2^\omega\}$, where for each $f \in 2^\omega$, and $n \in \omega$, $a_f \subset^* x_n^{f(n)}$.

Treating such a.d. families seems to require a new idea as the techniques in this paper seem to breakdown. So we ask:

Question 30: Is it consistent that for every a.d. family $\mathscr{A} \subset [\omega]^\omega$ that is neat with respect to some independent family, $\mathcal{I}(\mathscr{A})$ is not Fréchet?

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