COFINAL TYPES OF ULTRAFILTERS

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Abstract. We study Tukey types of ultrafilters on ω, focusing on the question of when Tukey reducibility is equivalent to Rudin-Keisler reducibility. We give several conditions under which this equivalence holds. We show that there are only $\mathfrak{c}$ many ultrafilters that are Tukey below any basically generated ultrafilter. The class of basically generated ultrafilters includes all known ultrafilters that are not Tukey above $[\omega_1]^{<\omega}$. We give a complete characterization of all ultrafilters that are Tukey below a selective. A counterexample showing that Tukey reducibility and RK reducibility can diverge within the class of P-points is also given.

1. Introduction

We say that a poset $\langle D, \leq \rangle$ is *directed* if any two members of D have an upper bound in D. A set $X \subset D$ is *unbounded in* D if it doesn't have an upper bound in D. A set $X \subset D$ is said to be *cofinal in* D if $\forall y \in D \exists x \in X\, [y \leq x]$. Given directed sets D and E, a map $f : D \to E$ is called a *Tukey map* if the image (under f) of every unbounded subset of D is unbounded in E. A map $g : E \to D$ is called a *convergent map* if the image (under g) of every cofinal subset of E is cofinal in D. It is easy to see that there is a Tukey map $f : D \to E$ iff there exists a convergent $g : E \to D$. When this situation obtains, we say that D is *Tukey reducible* to E, and we write $D \leq_T E$. The relation $\leq_T$ is a quasi order, and induces an equivalence relation in the usual way: $D \equiv_T E$ iff both $D \leq_T E$ and $E \leq_T D$ hold. If $D \equiv_T E$, we say that D and E are *Tukey equivalent* or have the same *cofinal type*, and this is intended to capture the idea that D and E have "the same cofinal structure". As support for this, it can be shown that $D \equiv_T E$ iff there is a directed set R into which both D and E embed as cofinal subsets, so that D and E contain the "same information" about the cofinal type of R.

These notions first arose in the Moore–Smith theory of convergence studied by general topologists. They were introduced by Tukey [17], and further studied by Ginsburg and Isbell [7] and Isbell [8]. The topological significance is that if $D \leq_T E$, then any D-net on a topological space contains an E-subnet.

The notion of Tukey reducibility has proved to be useful in many contexts (see [5], [6]). For example, some of the inequalities in the Cichoń diagram are best understood in terms of this notion (see [2] and [3]). Moreover the notion of Tukey reducibility provides a reasonable context for a rough classification of partially ordered sets where the isomorphism relation is too fine for giving us any general result

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(see, for example, [15] and [16]). By "rough classification" we mean any classification that is done modulo a similarity type which is coarser than isomorphism type. The most informative classification theorems provide a small and complete list of all the isomorphism types in some class of mathematical structures. However, rough classification theorems become useful when one is dealing with a class containing "too many" isomorphism types, so that there can be no meaningful classification results modulo isomorphism type for that class. A frequent starting point of rough classification is the consideration of a quasiorder $\prec$ on some class $\mathcal{K}$ of structures, with the idea being that if $A \prec B$, where $A, B \in \mathcal{K}$, then the structure of A is "simpler than" or "reducible to" that of B. This quasi order then gives rise to an equivalence relation $\equiv$ which is expected to capture some essential similarity between the structures in $\mathcal{K}$. A rough classification theorem then classifies these structures modulo $\equiv$ by assigning to each structure in $\mathcal{K}$ some "simple" complete invariant up to $\equiv$. One prominent way to do this is to show that $\langle \mathcal{K}, \prec \rangle$ is well quasiordered. Such a result shows that each structure in $\mathcal{K}$ may be assigned a complete invariant up to $\equiv$ that is only slightly more complicated than an ordinal. A recent illustration of this is the theorem that the Proper Forcing Axiom implies that the class of Aronszajn lines is well quasiordered under embeddability [10].

There are in fact older structure theorems as well as non-structure theorems due to Todorcevic concerning the possible cofinal types of uncountable directed sets and posets in general ([15], [16]). In the non-structure direction, Todorcevic showed that there are $2^{\mathfrak{c}}$ pairwise Tukey inequivalent directed sets of size $\mathfrak{c}$. On the other hand, his structure theorem states that the Proper Forcing Axiom (PFA) implies that there are only five cofinal types of size at most $\aleph_1$: $1, \omega, \omega_1, \omega \times \omega_1$, and $[\omega_1]^{<\omega}$. Here, the ordering on $\omega \times \omega_1$ is the product ordering, and $[\omega_1]^{<\omega}$ is ordered by inclusion.

An ultrafilter $\mathcal{U}$ on ω may be naturally viewed as the directed poset $\langle \mathcal{U}, \supset \rangle$. When this is done, Tukey reducibility turns out to be a coarser quasiorder on ultrafilters than the well studied Rudin-Keisler (RK) reducibility. Recall the following.

Definition 1. Let $\mathcal{F}$ and $\mathcal{G}$ be filters on ω. We say that $\mathcal{F}$ is *Rudin-Keisler(RK) reducible to* $\mathcal{G}$ or *Rudin-Keisler(RK) below* $\mathcal{G}$, and we write $\mathcal{F} \leq_{RK} \mathcal{G}$, if there is a map $f : \omega \to \omega$ such that for each $a \subset \omega$, $a \in \mathcal{F}$ iff $f^{-1}(a) \in \mathcal{G}$. We say that $\mathcal{F}$ is *Rudin-Blass(RB) reducible to* $\mathcal{G}$ or *Rudin-Blass(RB) below* $\mathcal{G}$, and we write $\mathcal{F} \leq_{RB} \mathcal{G}$, if there is a finite-to-one map $f : \omega \to \omega$ such that for each $a \subset \omega$, $a \in \mathcal{F}$ iff $f^{-1}(a) \in \mathcal{G}$.

There is another motivation for considering the cofinal types in this class of structures. By Todorcevic's non-structure result mentioned above, there is no hope of classifying *all* cofinal types of size $\mathfrak{c}$. There are two natural ways to restrict this class. One approach is to demand that the posets be "nicely definable" directed sets, and this line, suggested in [16], was already pursued in a series of recent papers (see, for example, [9] and [14]; in fact, we shall import here something from the definable setting, the notion of a *basically generated ultrafilter* on ω that comes from the key notion of a *basic poset* from [14]). Another, orthogonal, approach is to impose additional structure on the directed sets, like maximality. Indeed, while it is easy to construct an ultrafilter that is Tukey equivalent to $[\mathfrak{c}]^{<\omega}$, it is not known how to build an ultrafilter realizing any other cofinal type in ZFC. Note that $[\mathfrak{c}]^{<\omega}$ is the maximal cofinal type for directed sets of size at most $\mathfrak{c}$. $[\kappa]^{<\omega} \not\leq_T \mathcal{U}$ means that $\forall X \in [\mathcal{U}]^{\kappa} \exists A \in [X]^{\omega} \left[\bigcap A \in \mathcal{U}\right]$, which in turn means that if $\mathcal{U}$ realizes a cofinal type different from $[\mathfrak{c}]^{<\omega}$, then $\mathcal{U}$ is "sometimes a P-point". And clearly, if $\mathcal{U}$ is a P-point, then $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$. Thus the following long standing question of Isbell [8] is an indication that it may be consistent to have only a few cofinal types of ultrafilters.

Question 2 (Isbell [8], 1965)**.** *Is it consistent that for every ultrafilter $\mathcal{U}$ on ω, $\langle \mathcal{U}, \supset \rangle \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subset \rangle$?*

A positive result would be striking because it would say that consistently, all ultrafilters on ω are the same in the sense of cofinal type, and it would strengthen the celebrated result of Shelah [13] on the consistency of no P-points. Whereas a negative solution would show how to build an ultrafilter with a certain degree of "P-point-ness" in ZFC. Tukey types of ultrafilters on ω have recently been studied by Milovich [11] and Dobrinen and Todorcevic [4]. More precisely, the paper [11] mostly looks at ultrafilters on ω as directed sets under the ordering $\supseteq^*$ while the paper [4] considers only the Tukey theory of ultrafilters on ω as directed sets with the ordering $\supseteq$ and it is this approach that we will follow below.

Some simplifications to the basic definitions of the theory occur when we restrict ourselves to ultrafilters. Let D and E be directed sets. A map $f : E \to D$ is called *monotone* if $\forall e_0, e_1 \in E\, [e_0 \leq e_1 \implies f(e_0) \leq f(e_1)]$. f is said to be *cofinal in* D if $\forall d \in D \exists e \in E\, [d \leq f(e)]$. It is clear that if f is monotone and cofinal in D, then f is convergent. It can be checked that if $\mathcal{U}$ is an ultrafilter and D is any directed set such that $\mathcal{U} \leq_T D$, then there is a map from D to $\mathcal{U}$ which is monotone and cofinal in $\mathcal{U}$.

In this paper, we focus on the question of when Tukey reducibility is actually equivalent to RK reducibility. This is similar in spirit to the study of "liftings" in set theory and measure theory. For instance, the question of when an automorphism of $\mathcal{P}(\omega)/[\omega]^{<\omega}$ is induced by a permutation of ω was a famous problem in the history of set theory, and our question has much the same flavor. Notice that if $\mathcal{U} \equiv_T [\mathfrak{c}]^{<\omega}$, then every ultrafilter is Tukey below $\mathcal{U}$, but most ultrafilters are not RK below $\mathcal{U}$, since only $\mathfrak{c}$ of them can be. It is natural to suspect that putting Ramsey-like restrictions on $\mathcal{U}$, like requiring it to be selective or a P-point, may force $\mathcal{V} \leq_T \mathcal{U}$ to imply $\mathcal{V} \leq_{RK} \mathcal{U}$ because Ramsey theory helps with canonizing monotone maps defined on $\mathcal{U}$. However, there is an obstruction to this even when $\mathcal{U}$ is selective because of product ultrafilters. In Section 5, we show that this is the only obstruction: if $\mathcal{U}$ is selective and $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V}$ must be RK equivalent to some power of $\mathcal{U}$.

It turns out that it is more informative to consider what happens when only mild restrictions are placed on $\mathcal{U}$, such as $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$, or even just $[\mathfrak{c}]^{<\omega} \not\leq_T \mathcal{U}$, but $\mathcal{V}$ is required to be Ramsey-like. This approach helps us to analyze the Tukey orbits of ultrafilters that are not Tukey above $[\omega_1]^{<\omega}$ (or even those not above $[\mathfrak{c}]^{<\omega}$). There are three interrelated questions here, and each of them will be a theme of our investigations. If $\mathcal{U} \equiv_T [\mathfrak{c}]^{<\omega}$, then the Tukey orbit of $\mathcal{U}$ has size $2^{\mathfrak{c}}$. So the first question is whether it is possible that all other Tukey orbits are "more tractable". More precisely, does it follow from forcing axioms that if $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$ (or $[\mathfrak{c}]^{<\omega} \not\leq_T \mathcal{U}$), then the Tukey orbit of $\mathcal{U}$ has size $\mathfrak{c}$? Second, does it follow from forcing axioms that if $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$ (or $[\mathfrak{c}]^{<\omega} \not\leq_T \mathcal{U}$) and $\mathcal{V} \leq_T \mathcal{U}$, then there is a "nicely definable" $\phi : \mathcal{U} \to \mathcal{V}$ that is monotone and cofinal in $\mathcal{V}$? "Nicely definable" may mean Borel or something more general. Obviously, since there are only $\mathfrak{c}$

"nicely definable" maps ϕ on $\mathcal{U}$, and since ϕ and $\mathcal{U}$ determine $\mathcal{V}$, a positive answer to the second question gives a positive answer to the first. The third question asks whether forcing axioms imply that if $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$ (or $[\mathfrak{c}]^{<\omega} \not\leq_T \mathcal{U}$), and if $\mathcal{V}$ is a selective ultrafilter such that $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V} \leq_{RK} \mathcal{U}$. The idea here is that given a "nice enough" $\phi : \mathcal{U} \to \mathcal{V}$, the Ramsey property of $\mathcal{V}$ should allow us to "lift" it up to an RK reduction. Recall that in the case of the automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\omega}$, forcing axioms say that any automorphism has a nice lifting to $\mathcal{P}(\omega)$, and it is a theorem of ZFC that any such nice lifting must be induced by a permutation of ω.

We deal with the first and second questions in Section 4, where positive answers are obtained for a large class of relevant $\mathcal{U}$. The third question is dealt with in Sections 3 and 7. In Section 6, we give a counterexample related to the third question showing that it is not enough for $\mathcal{V}$ to be a P-point.

Since we are dealing here with the directed set $\langle \mathcal{U}, \supset \rangle$, we adopt the following standing convention throughout the paper. *If $\mathcal{X} \subset \mathcal{P}(\omega)$ is treated as a directed set, then it will be understood that the ordering is "$\supset$".*

This paper is composed by joining the works of two authors as follows. The results of sections 3 and 4 are due to the first author. The results of Section 5 are due to the second author. The example appearing in Section 6 is due to both authors. The results of Section 7 are due to the first author. The first author acknowledges his debt to the second author for many enlightening conversations on these topics.

2. Notation

We establish here some notation that will be used throughout the paper. We will deal a lot with the *Fubini product* of ultrafilters. Intuitively, this is the product measure on $\omega \times \omega$ obtained by integrating an ω–sequence of ultrafilters on ω with respect to another ultrafilter on ω. We will be iterating the process of taking Fubini products. Therefore, even though we usually think of Fubini products as ultrafilters on $\omega \times \omega$, our formal definition will make them ultrafilters on ω. To facilitate this translation between ω and $\omega \times \omega$, *fix once and for all a bijection $\pi : \omega \times \omega \to \omega$.* This π will be used throughout the paper. So when we use an expression like "view $\mathcal{U}$ as an ultrafilter on $\omega \times \omega$", where $\mathcal{U}$ is some ultrafilter on ω, we mean that we are considering the ultrafilter $\{\pi^{-1}(a) : a \in \mathcal{U}\}$.

Definition 3. Let $\pi : \omega \times \omega \to \omega$ be our fixed bijection. Given an ultrafilter $\mathcal{V}$ and a sequence of ultrafilters $\langle \mathcal{U}_n : n \in \omega \rangle$ on ω define

$$\bigotimes_{\mathcal{V}} \mathcal{U}_n = \{\pi'' a : a \subset \omega \times \omega \wedge \{n \in \omega : \{m \in \omega : \langle n, m \rangle \in a\} \in \mathcal{U}_n\} \in \mathcal{V}\}.$$

If $\forall n \in \omega\, [\mathcal{U}_n = \mathcal{U}]$, then we will write $\mathcal{V} \otimes \mathcal{U}$ for $\bigotimes_{\mathcal{V}} \mathcal{U}_n$.

The symbol "$\bigotimes$" will be used for Fubini products, while "$\times$" is reserved for the Cartesian product of arbitrary directed posets with the coordinatewise ordering. That is, if $\langle D_0, \leq_0 \rangle$ and $\langle D_1, \leq_1 \rangle$ are directed posets, $D_0 \times D_1$ denotes $\langle D_0 \times D_1, \leq \rangle$, where $\langle d_0, d_1 \rangle \leq \langle e_0, e_1 \rangle$ iff $[d_0 \leq_0 e_0 \wedge d_1 \leq_1 e_1]$. The distinction becomes important in Section 7 where we will consider both types of products.

Next, let $a \subset \omega \times \omega$. For $n \in \omega$, define $a(n) = \{m \in \omega : \langle n, m \rangle \in a\}$, and put $\mathrm{dom}\,(a) = \{n \in \omega : a(n) \neq 0\}$.

In Section 7 we will deal with elements of $\mathcal{P}(\omega)^{n+1}$. We will use symbols like $\bar{a}, \bar{b}, \bar{c}, \ldots$ for members of $\mathcal{P}(\omega)^{n+1}$. We treat such an $\bar{a}$ both as a function with domain $n+1$ and also as an ordered $n+1$ tuple. Thus, for any $i < n+1$, $\bar{a}(i)$ denotes the ith coordinate of $\bar{a}$, and $\bar{a} = \langle \bar{a}(0), \ldots, \bar{a}(n) \rangle$. Given $\bar{a} \in \mathcal{P}(\omega)^{n+1}$ and $b \in \mathcal{P}(\omega)$, $\bar{a}^\frown \langle b \rangle = \langle \bar{a}(0), \ldots, \bar{a}(n), b \rangle$.

Definition 4. Let $\bar{a}, \bar{b} \in \mathcal{P}(\omega)^{n+1}$, where $n \in \omega$. We write $\bar{a} \subset \bar{b}$ to mean $\forall i \leq n \left[\bar{a}(i) \subset \bar{b}(i) \right]$, and $\bar{a} \cap \bar{b}$ denotes $\langle \bar{a}(0) \cap \bar{b}(0), \ldots, \bar{a}(n) \cap \bar{b}(n) \rangle$. For $m \in \omega$, we use $\bar{a} \cap m$ to denote $\langle \bar{a}(0) \cap m, \ldots, \bar{a}(n) \cap m \rangle$.

For $a \subset \omega$ and $n \in \omega$, $a/n = \{m \in a : m > n\}$. Given, $n, m \in \omega$, $[n, m) = \{i \in \omega : n \leq i < m\}$. Lastly, we also define

Definition 5. Let $\mathcal{U}$ be an ultrafilter on ω. A set $\mathcal{B} \subset \mathcal{U}$ is said to be a *filter base* for $\mathcal{U}$ if $\mathcal{B}$ is cofinal in $\mathcal{U}$ and if $\forall b_0, b_1 \in \mathcal{B} \left[\, b_0 \cap b_1 \in \mathcal{B} \right]$.

3. Continuity and Rudin–Blass Reducibility

In this section we show that the existence of a continuous, monotone, and cofinal map from an *arbitrary* ultrafilter $\mathcal{U}$ into a Q-point $\mathcal{V}$ implies that $\mathcal{V}$ is Rudin-Blass reducible to $\mathcal{U}$. This is of some interest for several reasons. Firstly, the result works even when $\mathcal{U} \equiv_T [\mathfrak{c}]^{<\omega}$. Secondly, Dobrinen and Todorcevic [4] have shown that a witnessing continuous, monotone, and cofinal map can always be found when $\mathcal{U}$ is a P-point and $\mathcal{V}$ is any ultrafilter Tukey below it.

Theorem 6 (Dobrinen and Todorcevic)**.** *Suppose $\mathcal{U}$ is a P-point and $\mathcal{V}$ is an arbitrary ultrafilter such that $\mathcal{V} \leq_T \mathcal{U}$. Then there is a continuous $\phi : \mathcal{U} \to \mathcal{V}$ that is monotone and cofinal in $\mathcal{V}$.*

Thirdly, our result is similar in spirit to well known results asserting that every nicely definable (Baire measurable) automorphism of $\mathcal{P}(\omega)/[\omega]^{<\omega}$ is induced by a permutation of ω. We do not know whether the result in this section can be extended to all Borel, monotone, and cofinal maps from an arbitrary ultrafilter into a selective ultrafilter (or Q-point).

The key to the proof is the analysis of the following monotone map derived from an arbitrary map from a subset of $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)$. Variations on this idea will play an important role throughout this paper.

Definition 7. Let $\mathcal{X} \subset \mathcal{P}(\omega)$ and let $\phi : \mathcal{X} \to \mathcal{P}(\omega)$. Define $\psi_\phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ by $\psi_\phi(a) = \{k \in \omega : \forall b \in \mathcal{X} \, [a \subset b \implies k \in \phi(b)]\} = \bigcap \{\phi(b) : b \in \mathcal{X} \wedge a \subset b\}$, for each $a \in \mathcal{P}(\omega)$.

It is easy to see that ψ_ϕ is always monotone, and that if ϕ is monotone, then $\psi_\phi \restriction \mathcal{X} = \phi$. Moreover, if $b \in \mathcal{X}$ and $a \subset b$, then $\psi_\phi(a) \subset \phi(b)$.

Lemma 8. *Suppose $\mathcal{X} \subset \mathcal{P}(\omega)$. Let $\mathcal{V}$ be an ultrafilter and assume that $\phi : \mathcal{X} \to \mathcal{V}$ is monotone and cofinal in $\mathcal{V}$. Then there is a set $a \in \mathcal{X}$ such that $\forall s \in [a]^{<\omega}$ [$\psi_\phi(s)$ is finite].*

Proof. Consider $\mathcal{F} = \{\psi_\phi(s) : s \in [\omega]^{<\omega} \wedge |\psi_\phi(s)| = \omega\}$. This is a countable sub collection of $[\omega]^\omega$. So there is an $e \in [\omega]^\omega$ such that $|\psi_\phi(s) \cap e| = |\psi_\phi(s) \cap (\omega \setminus e)| = \omega$ for every $\psi_\phi(s) \in \mathcal{F}$. Since $\mathcal{V}$ is an ultrafilter either $e \in \mathcal{V}$ or $\omega \setminus e \in \mathcal{V}$. Suppose without loss that $\omega \setminus e \in \mathcal{V}$. Choose $a \in \mathcal{X}$ such that $\phi(a) \subset (\omega \setminus e)$. Now suppose, for a contradiction, that there is $s \in [a]^{<\omega}$ such that $|\psi_\phi(s)| = \omega$. But then $\psi_\phi(s) \in \mathcal{F}$, and so $|\psi_\phi(s) \cap e| = \omega$, which is a contradiction because $\psi_\phi(s) \subset \phi(a) \subset (\omega \setminus e)$. $\dashv$

We also point out that if $\mathcal{X} \subset \mathcal{P}(\omega)$, $\mathcal{V}$ is an ultrafilter, and $\phi : \mathcal{X} \to \mathcal{V}$ is cofinal in $\mathcal{V}$, then $\psi_\phi(0) = 0$. For otherwise, if $k \in \psi_\phi(0)$, then there is no $b \in \mathcal{X}$ with $\phi(b) \subset \omega \setminus \{k\}$, contradicting the cofinality of ϕ in $\mathcal{V}$.

The next lemma will be useful not only in proving the main theorem of this section but also in proving the main theorem of Section 7.

Lemma 9. *Let D be a directed set and let $\mathcal{V}$ be a Q-point. Suppose $\phi : D \to \mathcal{V}$ is monotone and cofinal in $\mathcal{V}$. Let $\mathcal{U}$ be an arbitrary ultrafilter and suppose $\pi^* : D \to \mathcal{U}$ is monotone and cofinal in $\mathcal{U}$. Assume there is a map $\psi^* : [\omega]^{<\omega} \to [\omega]^{<\omega}$ such that*

(1) $\forall s, t \in [\omega]^{<\omega} [s \subset t \implies \psi^*(s) \subset \psi^*(t)]$

(2) For each $m \in \omega$ and $j \in \omega$, if $j \notin \psi^*(m)$, then there exists $n(j, m) > m$ such that $\forall t \in [\omega]^{<\omega} [t \cap [m, n(j, m)) = 0 \implies j \notin \psi^*(m \cup t)]$

(3) $\forall d \in D \exists s \in [\pi^*(d)]^{<\omega} [\psi^*(s) \cap \phi(d) \neq 0]$.

Then $\mathcal{V} \leq_{RB} \mathcal{U}$.

Proof. Define $g \in \omega^\omega$ as follows. $g(0) = 0$. Given $g(n)$, choose $g(n + 1) > g(n)$ such that for each $m \leq g(n)$ and $j \leq g(n)$, if $j \notin \psi^*(m)$, then $n(j, m) < g(n + 1)$ and $\psi^*(n(j, m)) \subset g(n + 1)$. Since $\mathcal{V}$ is a Q-point, there is $e_0 \in \mathcal{V}$ such that $|e_0 \cap [g(n), g(n + 1))| = 1$ for each $n \in \omega$. Also, since $\mathcal{V}$ is a ultrafilter either $\bigcup_{n\in\omega}[g(2n), g(2n + 1)) \in \mathcal{V}$ or $\bigcup_{n\in\omega}[g(2n + 1), g(2n + 2)) \in \mathcal{V}$. Assume without loss of generality that $\bigcup_{n\in\omega}[g(2n), g(2n + 1)) \in \mathcal{V}$. Let $\{k_0 < k_1 < \cdots\} \in \mathcal{V}$ enumerate $e_0 \cap \bigcup_{n\in\omega}[g(2n), g(2n + 1))$. Notice that $\{k_i\} = e_0 \cap [g(2i), g(2i + 1))$. Since $\psi^*(0) \in [\omega]^{<\omega}$, find $i_0 \in \omega$ such that $\forall i \geq i_0 [k_i \notin \psi^*(0)]$. Now, define $h \in \omega^\omega$ satisfying the following properties:

(a) $k_{i_0+i} \notin \psi^*(h(i))$

(b) $h(i) \leq g(2i_0 + 2i + 1)$

(c) $\forall s \in [\omega]^{<\omega} [k_{i_0+i} \in \psi^*(s) \implies s \cap [h(i), h(i + 1)) \neq 0]$.

Let $h(0) = 0$. Notice that since $k_{i_0} \notin \psi^*(0)$ by hypothesis, (a) is satisfied. Now, given $h(i)$, observe that $k_{i_0+i} < g(2i_0 + 2i + 1)$, that $h(i) \leq g(2i_0 + 2i + 1)$, and that $k_{i_0+i} \notin \psi^*(h(i))$. Therefore, $n(k_{i_0+i}, h(i)) > h(i)$ exists. Moreover, $n(k_{i_0+i}, h(i)) < g(2(i_0 + i + 1))$ and $\psi^*(n(k_{i_0+i}, h(i))) \subset g(2(i_0 + i + 1))$. Set $h(i + 1) = n(k_{i_0+i}, h(i))$. Note $h(i + 1) > h(i)$. Also, $\psi^*(h(i + 1)) \subset g(2(i_0 + i + 1)) \leq k_{i_0+i+1}$, and so $k_{i_0+i+1} \notin \psi^*(h(i + 1))$. Additionally, $h(i + 1) < g(2i_0 + 2i + 2) < g(2i_0 + 2i + 3)$. So (a) and (b) hold. To check (c), fix $s \in [\omega]^{<\omega}$ and assume that $s \cap [h(i), h(i + 1)) = 0$. Put $t = s \cap [h(i + 1), \omega)$. Since $t \cap [h(i), n(k_{i_0+i}, h(i)) = 0$, $k_{i_0+i} \notin \psi^*(h(i) \cup t)$. But since $s \subset t \cup h(i)$, it follows that $k_{i_0+i} \notin \psi^*(s)$.

Now, define $f \in \omega^\omega$ so that for each $i \in \omega$, $f'' [h(i), h(i + 1)) = \{k_{i_0+i}\}$. f is clearly finite to one. We claim f witnesses $\mathcal{V} \leq_{RB} \mathcal{U}$. If not, then there is $a \in \mathcal{U}$ such that $f''a \notin \mathcal{V}$. Choose $d_0, d_1 \in D$ such that $\pi^*(d_0) \subset a$ and $\phi(d_1) \subset (\omega \setminus f''a) \cap \{k_{i_0} < k_{i_0+1} < \cdots\}$. Since D is directed, let $d \in D$ with $d \geq d_0$ and $d \geq d_1$. Then $\pi^*(d) \subset a$ and $\phi(d) \subset (\omega \setminus f''a) \cap \{k_{i_0} < k_{i_0+1} < \cdots\}$. Now, choose $s \in [\pi^*(d)]^{<\omega}$ such that $\psi^*(s) \cap \phi(d) \neq 0$. Choose $k_{i_0+i} \in \phi(d) \cap \psi^*(s)$. Note that $k_{i_0+i} \notin f''a$. However, since $k_{i_0+i} \in \psi^*(s)$, $s \cap [h(i), h(i + 1)) \neq 0$. But if $l \in s \cap [h(i), h(i + 1))$, then $f(l) = k_{i_0+i}$, and since $s \subset a$, $k_{i_0+i} \in f''a$, a contradiction. $\dashv$

Theorem 10. *Let $\mathcal{U}$ be an arbitrary ultrafilter and let $\mathcal{V}$ be a Q-point. Suppose that $\phi^* : \mathcal{U} \to \mathcal{V}$ is continuous, monotone, and cofinal in $\mathcal{V}$. Then $\mathcal{V} \leq_{RB} \mathcal{U}$.*

Proof. We will apply Lemma 9. Put $\psi = \psi_{\phi^*}$. Use Lemma 8 to fix $a \in \mathcal{U}$ such that $\forall s \in [a]^{<\omega}\,[\psi(s)$ is finite$]$. Let $D = \mathcal{U} \cap [a]^{\omega}$. Let $\phi = \phi^* \restriction D$. Let $\pi^* : D \to \mathcal{U}$ simply be the identity and define $\psi^* : [\omega]^{<\omega} \to [\omega]^{<\omega}$ by $\psi^*(s) = \psi(a \cap s)$. It is clear that (1) of Lemma 9 is satisfied. For (3), fix $d \in \mathcal{U} \cap [a]^{\omega}$ and $k \in \phi(d)$. By continuity and monotonicity of ϕ^*, there exists $n \in \omega$ such that $\forall b \in \mathcal{U}\,[d \cap n \subset b \implies k \in \phi^*(b)]$, whence $k \in \psi(d \cap n)$. So $k \in \psi^*(d \cap n) \cap \phi(d)$. So letting $s = d \cap n \in [\pi^*(d)]^{<\omega}$ makes (3) true. For (2) of Lemma 9, fix $m, j \in \omega$, and assume that $j \notin \psi^*(m)$. By definition of ψ^*, choose $b \in \mathcal{U}$ such that $m \subset b$ and $j \notin \phi^*(b)$. Put $d = b \cap a$. Note $j \notin \phi^*(d)$. Again by continuity and monotonicity of ϕ^*, find $n(j, m) > m$ with the property that $\forall c \in \mathcal{U}\,[c \cap n(j, m) \subset d \cap n(j, m) \implies j \notin \phi^*(c)]$. Now, suppose that $t \in [\omega]^{<\omega}$ and that $t \cap [m, n(j, m)) = 0$. Assume for a contradiction that $j \in \psi^*(m \cup t) = \psi((a \cap m) \cup (a \cap t))$. Now, put $c = d \cup (a \cap t) \in \mathcal{U}$. Since $m \subset b$, $m \cap a \subset b \cap a = d$. Therefore, $(a \cap m) \cup (a \cap t) \subset c$, whence $j \in \phi^*(c)$. On the other hand, $a \cap t \cap n(j, m) \subset m \cap a \cap n(j, m) \subset d \cap n(j, m)$. Therefore, $c \cap n(j, m) = (d \cap n(j, m)) \cup (a \cap t \cap n(j, m)) = d \cap n(j, m)$, whence $j \notin \phi^*(c)$, a contradiction. ⊣

As an immediate corollary to Theorems 10 and 6 is that if $\mathcal{U}$ is a P-point and $\mathcal{V}$ is a Q-point such that $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V} \leq_{RB} \mathcal{U}$. Another corollary is a negative result. Dobrinen and Todorcevic asked whether their result quoted above could be extended to cover the case when $\mathcal{U}$ is a Fubini product of P-points, or more generally when $\mathcal{U}$ is strictly Tukey below $[\mathfrak{c}]^{<\omega}$. The next corollary shows that their result fails even for the Fubini product of two P-points, and even when the ultrafilter Tukey below that product is selective.

Corollary 11. *Let $\mathcal{U}$ and $\mathcal{V}$ be non-isomorphic selective ultrafilters. There is no continuous $\phi : \mathcal{U} \otimes \mathcal{V} \to \mathcal{U}$ which is monotone and cofinal in $\mathcal{U}$.*

Proof. View $\mathcal{U} \otimes \mathcal{V}$ as an ultrafilter on $\omega \times \omega$. Since $\mathcal{U}$ and $\mathcal{V}$ are non-isomorphic selective ultrafilters, it is easy to see that for any pair of functions $f : \omega \times \omega \to \omega$ and $g : \omega \times \omega \to \omega$, *either* there is a set $a \in \mathcal{U} \otimes \mathcal{V}$ such that $\forall \langle n, m \rangle \in a\,[f(\langle n, m \rangle) = g(\langle n, m \rangle)]$, *or* there is a set $a \in \mathcal{U} \otimes \mathcal{V}$ such that $f''a \cap g''a = 0$.

Now, let $f : \omega \times \omega \to \omega$ be the projection map – that is, $f(\langle n, m \rangle) = n$, for every $\langle n, m \rangle \in \omega \times \omega$. f witnesses that $\mathcal{U} \leq_{RK} \mathcal{U} \otimes \mathcal{V}$. In particular, $f''a \in \mathcal{U}$, for any $a \in \mathcal{U} \otimes \mathcal{V}$. Now, suppose for a contradiction that there is a continuous $\phi : \mathcal{U} \otimes \mathcal{V} \to \mathcal{U}$ which is monotone and cofinal in $\mathcal{U}$. Then by Theorem 10, $\mathcal{U} \leq_{RB} \mathcal{U} \otimes \mathcal{V}$. Let $g : \omega \times \omega \to \omega$ be a finite-to-one map witnessing this. In particular, $g''a \in \mathcal{U}$, for any $a \in \mathcal{U} \otimes \mathcal{V}$, and so, $f''a \cap g''a \neq 0$, for any $a \in \mathcal{U} \otimes \mathcal{V}$. It follows that there is $a \in \mathcal{U} \otimes \mathcal{V}$ such that $\forall \langle n, m \rangle \in a\,[f(\langle n, m \rangle) = g(\langle n, m \rangle)]$. However, there is no set in $\mathcal{U} \otimes \mathcal{V}$ on which f is finite-to-one. ⊣

Corollary 12. *Suppose $\mathcal{U}$ is a P-point and $\mathcal{V}$ is a Q-point. If $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V}$ is automatically selective.*

4. A canonical form for monotone cofinal maps

In this section, we show for a wide range of ultrafilters $\mathcal{U}$, which are strictly Tukey below $[\mathfrak{c}]^{<\omega}$, that any Tukey reduction from $\mathcal{U}$ to any other ultrafilter $\mathcal{V}$ may be viewed as an RK reduction from a suitably chosen *filter* into $\mathcal{V}$.

Definition 13. Let $\mathcal{U}$ be an ultrafilter on ω. We say that $\mathcal{U}$ is *basically generated* if there is a filter base $\mathcal{B} \subset \mathcal{U}$ with the property that for every $\langle b_n : n \in \omega \rangle \subset \mathcal{B}$ and

$b \in \mathcal{B}$, if $\langle b_n : n \in \omega \rangle$ converges to b (with respect to the usual topology on $\mathcal{P}(\omega)$), then there exists $X \in [\omega]^{\omega}$ such that $\bigcap_{n \in X} b_n \in \mathcal{U}$.

It is easy to see that if $\mathcal{U}$ is basically generated, then $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$. It is also easy to show (see Section 7) that every member of the class of ultrafilters gotten by closing off the P-points under arbitrary countable Fubini products is basically generated. In fact, the class of basically generated ultrafilters includes all examples of ultrafilters that are strictly Tukey below $[\mathfrak{c}]^{<\omega}$ which are currently known.

In this section we show that there are only $\mathfrak{c}$ ultrafilters that are Tukey below any given basically generated ultrafilter. As mentioned in Section 1, one motif of our investigations is the analysis of Tukey orbits of ultrafilters that are not Tukey equivalent to $[\mathfrak{c}]^{<\omega}$. So our results here show that many such ultrafilters have "small" Tukey orbits. We do not know whether this result can be extended (consistently) to all ultrafilters that are not Tukey above $[\omega_1]^{<\omega}$ or even to all that are not Tukey equivalent to $[\mathfrak{c}]^{<\omega}$.

Question 14. *Assume* PFA. *Let $\mathcal{U}$ be an ultrafilter such that $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$. Is it true that $|\{\mathcal{V} : \mathcal{V} \leq_T \mathcal{U}\}| \leq \mathfrak{c}$? Is it consistent with* ZFC + CH *that for every $\mathcal{U}$ such that $\mathcal{U} <_T [\omega_1]^{<\omega}$, $|\{\mathcal{V} : \mathcal{V} \leq_T \mathcal{U}\}| \leq \mathfrak{c}$?*

Definition 15. Let $\mathcal{U}$ be an ultrafilter on ω, and let $P \subset \mathrm{FIN}$. We define $\mathcal{U}(P) = \{A \subset P : \exists a \in \mathcal{U} \left[P \cap [a]^{<\omega} \subset A\right]\}$.

If $\forall a \in \mathcal{U}\left[\left|P \cap [a]^{<\omega}\right| = \omega\right]$, then $\mathcal{U}(P)$ is a proper, non-principal filter on P. It is usually not an ultrafilter on P. The next theorem says that for any basically generated $\mathcal{U}$, any Tukey reduction from $\mathcal{U}$ can be replaced with an RK reduction from some $\mathcal{U}(P)$. The heart of the matter is in the following lemma, which will be of use later also.

Lemma 16. *Let $\mathcal{U}$ be basically generated by $\mathcal{B} \subset \mathcal{U}$. Let $\phi : \mathcal{B} \to \mathcal{P}(\omega)$ be a monotone map such that $\phi(b) \neq 0$ for every $b \in \mathcal{B}$. Let $\psi = \psi_{\phi}$. Then for every $b \in \mathcal{B}$, $\bigcup_{s \in [b]^{<\omega}} \psi(s) \neq 0$.*

Proof. Towards a contradiction, assume that there is $b \in \mathcal{B}$ such that

$$\bigcup_{s \in [b]^{<\omega}} \psi(s) = 0.$$

Thus, for each $n \in \omega$, $\psi(b \cap n) = 0$. Therefore, by the definition of $\psi = \psi_{\phi}$, for each $n \in \omega$ and each $i \leq n$, there is $b_n^i \in \mathcal{B}$ such that $b \cap n \subset b_n^i$, but $i \notin \phi(b_n^i)$. Put $b_n = b \cap b_n^0 \cap \cdots \cap b_n^n$. As $\mathcal{B}$ is closed under finite intersections, $b_n \in \mathcal{B}$. Moreover, $b_n \cap n = b \cap n$. Notice also that by monotonicity of ϕ, $i \notin \phi(b_n)$ for any $n \in \omega$ and $i \leq n$. Thus $\langle b_n : n \in \omega \rangle$ is a sequence of elements of $\mathcal{B}$ converging to $b \in \mathcal{B}$. So there is $X \in [\omega]^{\omega}$ such that $\bigcap_{n \in X} b_n \in \mathcal{U}$. Since $\mathcal{B}$ is cofinal in $\mathcal{U}$, there is $c \in \mathcal{B}$ with $c \subset \bigcap_{n \in X} b_n$. But then by monotonicity of ϕ, $\phi(c) = 0$, a contradiction. $\dashv$

Theorem 17. *Let $\mathcal{U}$ be basically generated by $\mathcal{B} \subset \mathcal{U}$. Let $\mathcal{V}$ be an arbitrary ultrafilter so that $\mathcal{V} \leq_T \mathcal{U}$. Then there is $P \subset \mathrm{FIN}$ such that*

(1) $\forall t, s \in P\, [t \subset s \implies t = s]$
(2) $\mathcal{U}(P) \equiv_T \mathcal{U}$
(3) $\mathcal{V} \leq_{RK} \mathcal{U}(P)$

Proof. Let $\phi : \mathcal{U} \to \mathcal{V}$ be a map which is monotone and cofinal in $\mathcal{V}$. Note that $\phi \restriction \mathcal{B} : \mathcal{B} \to \mathcal{V}$ satisfies the hypotheses of Lemma 16. Set $\psi = \psi_{(\phi \restriction \mathcal{B})}$. Define $N = \{s \in [\omega]^{<\omega} : \psi(s) \neq 0\}$, and put

$$P = \{s \in N : s \text{ is minimal in } N \text{ with respect to } \subset\}.$$

Note that since $\psi(0) = 0$, $0 \notin N$, and so $P \subset \mathrm{FIN}$. It is also clear that P satisfies (1) by definition. Next, for any $a \in \mathcal{U}$, $\bigcup(P \cap [a]^{<\omega}) \in \mathcal{U}$. To see this, fix $a \in \mathcal{U}$, and suppose that $a \setminus (\bigcup(P \cap [a]^{<\omega})) \in \mathcal{U}$. Choose $b \in \mathcal{B}$ with $b \subset a \setminus (\bigcup(P \cap [a]^{<\omega}))$. By Lemma 16, there is $s \in [b]^{<\omega}$ with $\psi(s) \neq 0$. So $s \in N$. But clearly, there is $t \in P$ with $t \subset s$, whence $t = 0$, an impossibility. It follows that for each $a \in \mathcal{U}$, $P \cap [a]^{<\omega}$ is infinite.

Next, verify that $\mathcal{U}(P) \equiv_T \mathcal{U}$. Define $\chi : \mathcal{U} \to \mathcal{U}(P)$ by $\chi(a) = P \cap [a]^{<\omega}$, for each $a \in \mathcal{U}$. This map is clearly monotone and cofinal in $\mathcal{U}(P)$. So χ is a convergent map. On the other hand, χ is also Tukey. To see this, fix $\mathcal{X} \subset \mathcal{U}$, unbounded in $\mathcal{U}$. Assume that $\{\chi(a) : a \in \mathcal{X}\}$ is bounded in $\mathcal{U}(P)$. So there is $b \in \mathcal{U}$ such that $P \cap [b]^{<\omega} \subset P \cap [a]^{<\omega}$ for each $a \in \mathcal{X}$. But $c = \bigcup(P \cap [b]^{<\omega}) \in \mathcal{U}$. Now, it is clear that $c \subset a$, for each $a \in \mathcal{X}$, a contradiction.

Next, check that $\mathcal{V} \leq_{RK} \mathcal{U}(P)$. Define $f : P \to \omega$ by $f(s) = \min(\psi(s))$ for each $s \in P$. This makes sense because $P \subset N$, and so $\psi(s) \neq 0$. Fix $e \subset \omega$, and suppose first that $f^{-1}(e) \in \mathcal{U}(P)$. Fix $a \in \mathcal{U}$ with $P \cap [a]^{<\omega} \subset f^{-1}(e)$. If $e \notin \mathcal{V}$, then $\omega \setminus e \in \mathcal{V}$, and there exists $c \in \mathcal{U}$ with $\phi(c) \subset \omega \setminus e$. Fix $b \in \mathcal{B}$ with $b \subset a \cap c$. By Lemma 16 there exists $s \in [b]^{<\omega}$ such that $\psi(s) \neq 0$. Fix $t \subset s$ with $t \in P$. Let $k = \min(\psi(t)) = f(t)$. As $t \subset s \subset b \subset a$, $t \in P \cap [a]^{<\omega} \subset f^{-1}(e)$. Thus $k \in e$. On the other hand, since $b \in \mathcal{B}$, and $t \subset b$, $\psi(t) \subset \phi(b)$. So $k \in \phi(b) \subset \phi(c) \subset \omega \setminus e$, a contradiction.

Next, suppose that $e \in \mathcal{V}$. By cofinality of ϕ, there is $a \in \mathcal{U}$ such that $\phi(a) \subset e$. Fix $b \in \mathcal{B}$ with $b \subset a$. Now, if $s \in P \cap [b]^{<\omega}$, then $\psi(s) \subset \phi(b) \subset \phi(a) \subset e$. Therefore, $f(s) = \min(\psi(s)) \in e$. Therefore, $P \cap [b]^{<\omega} \subset f^{-1}(e)$, whence $f^{-1}(e) \in \mathcal{U}(P)$. ⊣

Note that given $P \subset \mathrm{FIN}$ and $f : P \to \omega$ witnessing $\mathcal{V} \leq_{RK} \mathcal{U}(P)$ as in Theorem 17, the map $\chi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ defined by $\chi(a) = \{f(s) : s \in P \cap [a]^{<\omega}\}$ is a monotone Baire class one map, and its restriction to $\mathcal{U}$ goes into $\mathcal{V}$ and is cofinal in $\mathcal{V}$. Therefore, the Tukey reducibility of some ultrafilter to a basically generated ultrafilter is always witnessed by a Baire class one, monotone, and cofinal map. In Section 3 we showed that Theorem 6 proved by Dobrinen and Todorcevic fails even for the Fubini product of two P-points. But Theorem 17 says that the next best thing holds for a wide class of ultrafilters that are not Tukey above $[\omega_1]^{<\omega}$.

Corollary 18. *(1) Every $\leq_T$ chain of ultrafilters that are basically generated by a base closed under finite intersections has cardinality $\leq \mathfrak{c}^+$.*

(2) Every family $\mathcal{F}$ of such ultrafilters of cardinality $> \mathfrak{c}$ contains a subfamily $\mathcal{F}_0 \subset \mathcal{F}$ of equal size such that $\mathcal{U} \not\leq_T \mathcal{V}$ whenever $\mathcal{U} \neq \mathcal{V}$ are in $\mathcal{F}_0$.

5. A characterization of ultrafilters Tukey below a selective

Let $\mathcal{U}$ be a selective ultrafilter. Consider the class $\mathcal{C}(\mathcal{U})$ of ultrafilters obtained from $\mathcal{U}$ as follows. Put $\mathcal{C}_0(\mathcal{U}) = \{\mathcal{U}\}$. Given $\mathcal{C}_\alpha(\mathcal{U})$, let $\mathcal{C}_{\alpha+1}(\mathcal{U}) = \{\bigotimes_{\mathcal{U}} \mathcal{U}_n : \langle \mathcal{U}_n : n \in \omega \rangle \subset \mathcal{C}_\alpha(\mathcal{U})\}$. For a limit ordinal α, $\mathcal{C}_\alpha(\mathcal{U}) = \bigcup_{\beta<\alpha} \mathcal{C}_\beta(\mathcal{U})$. Finally, put $\mathcal{C}(\mathcal{U}) = \mathcal{C}_{\omega_1}(\mathcal{U})$. It is not difficult to prove that every ultrafilter in $\mathcal{C}(\mathcal{U})$ is Tukey equivalent to $\mathcal{U}$. The main result of this section is that if $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V}$ is RK equivalent to some member of $\mathcal{C}(\mathcal{U})$. This provides a complete characterization of all ultrafilters that are Tukey below a given selective ultrafilter.

It it well known that the only ultrafilters that are RK below a selective $\mathcal{U}$ are the ones that are RK equivalent to $\mathcal{U}$. Our results show that there is a similarly simple description of the ultrafilters that are Tukey below a selective $\mathcal{U}$: they are the ones that are RK equivalent to some countable Fubini power of $\mathcal{U}$. Our main tool here is canonical Ramsey theory.

Definition 19. Let $a \in [\omega]^{\omega}$ and let $B \subset [a]^{<\omega}$. We say that B is a barrier on a if:

(1) $\forall s, t \in B\,[s \neq t \implies (s \not\subset t \wedge t \not\subset s)]$
(2) $\forall b \in [a]^{\omega} \exists s \in B\,[s \sqsubset b]$.

Definition 20. Fix $a \in [\omega]^{\omega}$. For $B \subset [a]^{<\omega} \setminus \{0\}$ and $n \in a$, we define $B_{\{n\}} = \{s \setminus \{n\} : s \in B \wedge n = \min(s)\}$. By induction on $\alpha < \omega_1$ we define the notion of an α-uniform barrier on an element of $[\omega]^{\omega}$:

(1) $B \subset [a]^{<\omega}$ is a 0-uniform barrier on a iff $B = \{0\}$
(2) If $\alpha = \beta + 1$, then $B \subset [a]^{<\omega}$ is an α-uniform barrier on a iff $B \subset [a]^{<\omega} \setminus \{0\}$ and for every $n \in a$, $B_{\{n\}}$ is a β-uniform barrier on a/n.
(3) If α is a limit, then $B \subset [a]^{<\omega}$ is an α-uniform barrier on a iff $B \subset [a]^{<\omega} \setminus \{0\}$ and there is an increasing sequence $\langle \alpha_n : n \in a \rangle$ converging to α such that for every $n \in a$, $B_{\{n\}}$ is an α_n-uniform barrier on a/n.

Definition 21. By induction on $1 < \alpha < \omega_1$ we define $\mathcal{U}_B$ for any ultrafilter $\mathcal{U}$ on a set $a \in [\omega]^{\omega}$ and an α-uniform barrier B on a. If $\alpha = 1$, then $\mathcal{U}_B = \{\{\{n\} : n \in b\} : b \in \mathcal{U}\}$. If $\alpha > 1$, then a set $A \subset B$ is a member of $\mathcal{U}_B$ iff $\left\{n \in a : A_{\{n\}} \in (\mathcal{U} \cap [a/n]^{\omega})_{B_{\{n\}}}\right\} \in \mathcal{U}$.

It is easy to check that $\mathcal{U}_B$ is always an ultrafilter on the countable set B. We won't check this here because in the case that interests us – when $\mathcal{U}$ is selective – there is a simple description of a generating set for $\mathcal{U}_B$.

Lemma 22. *Let $\mathcal{U}$ be a selective ultrafilter on ω and suppose $B \subset [\omega]^{<\omega} \setminus \{0\}$ is an α-uniform barrier on ω for some $\alpha \geq 1$. Then $\mathscr{F} = \{B \restriction a : a \in \mathcal{U}\}$ generates an ultrafilter on B, where $B \restriction a$ denotes $\{s \in B : s \subset a\}$. In fact, $\mathcal{U}_B$ is the ultrafilter $\mathscr{F}$ generates.*

Proof. First, given $a, b \in \mathcal{U}$, $(B \restriction a) \cap (B \restriction b) = B \restriction (a \cap b) \in \mathscr{F}$. Next, let $A \subset B$ be arbitrary. Since $\mathcal{U}$ is selective, there is an $a \in \mathcal{U}$ such that either $B \restriction a \subset A$ or $B \restriction a \subset B \setminus A$ (apply the Nash-Williams–Galvin Lemma twice, first to A, then to $B \setminus A$). Finally, if $\{s_0, \ldots, s_k\} \subset B$, then $a = \omega \setminus (s_0 \cup \cdots \cup s_k) \in \mathcal{U}$, and $B \restriction a \subset B \setminus \{s_0, \ldots, s_k\}$ because $0 \notin B$. Therefore, we have checked that $\mathscr{F}$ generates a non principal ultrafilter on B.

We next check that $\mathcal{U}_B$ is the ultrafilter that is generated by $\mathscr{F}$. We must show for each $A \subset B$ that $A \in \mathcal{U}_B$ iff there is an $a \in \mathcal{U}$ such that $B \restriction a \subset A$. If $\alpha = 1$, this is clear. So assume $\alpha > 1$ and that this is true for all $1 \leq \beta < \alpha$. Fix $A \subset B$. Suppose first that $A \in \mathcal{U}_B$. There is an $a \in \mathcal{U}$ such that either $B \restriction a \subset A$ or $B \restriction a \subset B \setminus A$, and we argue $B \restriction a \not\subset B \setminus A$. Choose $n \in a$ such that $A_{\{n\}} \in (\mathcal{U} \cap [\omega/n]^{\omega})_{B_{\{n\}}}$. By the inductive hypothesis there is a $b \in \mathcal{U} \cap [\omega/n]^{\omega}$ such that $B_{\{n\}} \restriction b \subset A_{\{n\}}$. As $B_{\{n\}}$ is a barrier on ω/n, there is an $s \sqsubset a \cap b$ with $s \in B_{\{n\}}$. Therefore, $s \in A_{\{n\}}$, and so $s \cup \{n\} \in A$. We conclude that $s \cup \{n\} \in A$ and $s \cup \{n\} \in B \restriction a$ because $n \in a$, and so $B \restriction a \not\subset B \setminus A$. Conversely, if there is an $a \in \mathcal{U}$ such that $B \restriction a \subset A$ and if $A \notin \mathcal{U}_B$, then choose $n \in a$ such that $B_{\{n\}} \setminus A_{\{n\}} \in (\mathcal{U} \cap [\omega/n]^\omega)_{B_{\{n\}}}$. By the inductive hypothesis, choose $b \in \mathcal{U} \cap [\omega/n]^\omega$ with $B_{\{n\}} \restriction b \subset B_{\{n\}} \setminus A_{\{n\}}$. Now, choose $s \in B$ with $s \sqsubset \{n\} \cup (a \cap b)$. Note that $s \in B \restriction a$ because $n \in a$, and so $s \in A$. On the other hand, $n = \min(s)$ and so $s \setminus \{n\} \in A_{\{n\}}$, and $s \setminus \{n\} \in B_{\{n\}} \restriction b$. This is a contradiction which shows that $A \in \mathcal{U}_B$. ⊣

It is easy to see, by induction on α, that if B is α-uniform, then $\mathcal{U}_B$ is actually RK equivalent to a member of $\mathcal{C}(\mathcal{U})$. We will show that if $\mathcal{U}$ is selective and if $\mathcal{V} \leq_T \mathcal{U}$, then there is a set $a \in \mathcal{U}$ and an α-uniform barrier B on a such that $\mathcal{V}$ is Rudin-Keisler equivalent to $(\mathcal{U} \cap [a]^\omega)_B$. By Theorem 6, when $\mathcal{U}$ is selective, the relation $\mathcal{V} \leq_T \mathcal{U}$ is witnessed by a continuous, monotone map ϕ. The Pudlák–Rödl Theorem of [12] gives us a canonical form for the map $a \mapsto \min(\phi(a))$. We state this theorem below in the form in which we use it.

Theorem 23 (Pudlák and Rödl [12]. See also [1])**.** *Let $\phi_0 : [\omega]^\omega \to \omega$ be continuous. Then there is a $b \in [\omega]^\omega$, a β-uniform barrier C on b, a one to one map $\psi : C \to \omega$, and a map $f : [b]^\omega \to C$ such that:*

(1) $\forall c \in [b]^\omega\, [f(c) \subset c]$
(2) $\forall c \in [b]^\omega\, [\phi_0(c) = \psi(f(c))]$.

Theorem 24. *Let $\mathcal{U}$ be a selective ultrafilter. Suppose $\mathcal{V}$ is an ultrafilter such that $\mathcal{V} \leq_T \mathcal{U}$. There is a $b \in \mathcal{U}$ and a β-uniform barrier C on b such that $\mathcal{V} \equiv_{RK} (\mathcal{U} \cap [b]^\omega)_C$.*

Proof. As $\mathcal{U}$ is selective, there is a monotone, continuous map $\phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that

(a) $\forall a \in \mathcal{U}\, [\phi(a) \in \mathcal{V}]$
(b) $\forall e \in \mathcal{V} \exists a \in \mathcal{U}\, [\phi(a) \subset e]$.

Using selectivity again, there is an $a \in \mathcal{U}$ such that *either* $\forall b \in [a]^\omega\, [\phi(b) \neq 0]$ *or* $\forall b \in [a]^\omega\, [\phi(b) = 0]$, and clearly the latter does not happen. So we may define $\phi_0 : [a]^\omega \to \omega$ by $\phi_0(b) = \min(\phi(b))$ for each $b \in [a]^\omega$. This is continuous because ϕ is continuous. Applying Theorem 23 and the selectivity of $\mathcal{U}$, we may find a $b \in \mathcal{U} \cap [a]^\omega$, a β-uniform barrier C on b, a one to one map $\psi : C \to \omega$, and a map $f : [b]^\omega \to C$ satisfying (1) and (2) of Theorem 23. We claim that ψ witnesses that $\mathcal{V} \equiv_{RK} (\mathcal{U} \cap [b]^\omega)_C$. As ψ is one to one, it is enough to show that for each $A \subset C$ with $A \in (\mathcal{U} \cap [b]^\omega)_C$, $\psi'' A \in \mathcal{V}$. Suppose not. Fix $c \in \mathcal{U} \cap [b]^\omega$ with $\omega \setminus \psi''(C \restriction c) \in \mathcal{V}$. By (b) above choose $d \in \mathcal{U} \cap [c]^\omega$ such that $\phi(d) \subset \omega \setminus \psi''(C \restriction c)$. But then $\phi_0(d) = \psi(f(d)) \notin \psi''(C \restriction c)$. But since $f(d) \subset d \subset c$, $f(d) \in C \restriction c$, and so $\psi(f(d)) \in \psi''(C \restriction c)$, a contradiction. ⊣

Note that by Lemma 22, when $\mathcal{U}$ is selective, $\mathcal{U}_B$ is of the form $\mathcal{U}(P)$ of Definition 15, and in fact, it is also possible to derive Theorem 24 from Theorem 17.

6. A Counterexample for P-Points

A negative result is presented in this section. If $\mathcal{V}$ is a selective ultrafilter, then $\mathcal{V} \otimes \mathcal{V} \equiv_T \mathcal{V}$, and since $\mathcal{V} \otimes \mathcal{V}$ is not a P-point, it follows that $\mathcal{V} \otimes \mathcal{V} \not\leq_{RK} \mathcal{V}$. So there exist ultrafilters $\mathcal{U}$ and $\mathcal{V}$ so that neither $\mathcal{U}$ nor $\mathcal{V}$ is Tukey above $[\omega_1]^{<\omega}$ and such that $\mathcal{U} \leq_T \mathcal{V}$, but $\mathcal{U} \not\leq_{RK} \mathcal{V}$. But this example nevertheless leaves open the possibility that Rudin-Keisler theory and Tukey theory may still coincide within the class of P-points. The next theorem shows that this is consistently not the case. But we actually get a strong counterexample because we have $\mathcal{V} <_{RK} \mathcal{U}$. So RK reducibility and Tukey reducibility can diverge in a strong sense even within the class of P-points. Thus in Theorem 10, the hypothesis that $\mathcal{V}$ is a Q-point cannot be dropped or replaced with the hypothesis that it is a P-point. By this same theorem, the $\mathcal{U}$ constructed in this section must be far from a Q-point.

Theorem 25. *Assume* CH. *There exist P-points $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{V} <_{RK} \mathcal{U}$, but $\mathcal{V} \equiv_T \mathcal{U}$.*

CH can be replaced here with $\mathfrak{p} = \mathfrak{c}$. We will build $\mathcal{U}$ on $\omega \times \omega$. $\mathcal{V}$ will be the projection of $\mathcal{U}$ onto ω. If we make sure that $\mathcal{U}$ is a P-point, then $\mathcal{V}$ will also be a P-point and $\mathcal{V} \leq_{RK} \mathcal{U}$. To get $\mathcal{V} <_{RK} \mathcal{U}$, we will ensure that every member of $\mathcal{U}$ has unbounded intersection with the vertical columns of ω.

Definition 26. Define

$$\mathcal{E}_0 = \{E \subset \omega \times \omega : \forall k \in \omega \exists^\infty n \in \omega\, [\, |E(n)| > k]\}.$$

Thus if $\mathcal{U}$ is an ultrafilter on $\omega \times \omega$, then the projection of $\mathcal{U}$ onto ω is simply $\{\mathrm{dom}\,(E) : E \in \mathcal{U}\}$. The following easy fact will help ensure that $\mathcal{V} <_{RK} \mathcal{U}$.

Lemma 27. *If $\mathcal{U}$ is a P-point with $\mathcal{U} \subset \mathcal{E}_0$, and if $\mathcal{V} = \{\mathrm{dom}\,(E) : E \in \mathcal{U}\}$, then $\mathcal{U} \not\leq_{RK} \mathcal{V}$.*

To get $\mathcal{V} \equiv_T \mathcal{U}$, we fix a monotone continuous map $\phi : [\omega]^\omega \to [\omega \times \omega]^\omega$ ahead of time and ensure during the construction that $\{\phi(\mathrm{dom}(E)) : E \in \mathcal{U}\}$ is cofinal in $\mathcal{U}$.

Definition 28. Let $f \in \omega^\omega$ such that $\forall i \in \omega \left[\left|f^{-1}(\{i\})\right| = \omega\right]$. Define $\phi_f : [\omega]^\omega \to [\omega \times \omega]^\omega$ as follows: for any $a \in [\omega]^\omega$,

$$\forall \langle n, m\rangle \in \omega \times \omega\, [\langle n, m\rangle \in \phi_f(a) \iff (n, m \in a) \wedge (m < n) \wedge (f(m) = f(n))]\,.$$

Lemma 29. *Let f and ϕ_f be as in Definition 28. Let $a, b \in [\omega]^\omega$. If $a \subset b$, then $\phi_f(a) \subset \phi_f(b)$.*

Proof. If $\langle n, m\rangle \in \phi_f(a)$, then $n, m \in a \subset b$, $m < n$ and $f(n) = f(m)$, whence $\langle n, m\rangle \in \phi_f(b)$ ⊣

It makes no difference to us which f we use. So let us fix once and for all a $f \in \omega^\omega$ as in Definition 28 and put $\phi = \phi_f$.

Definition 30. We define $\mathcal{E}_1 = \{a \subset \omega : \forall k \in \omega \exists^\infty i \in \omega \left[\left|f^{-1}(\{i\}) \cap a\right| > k\right]\}$. And we put $\mathcal{E} = \{E \subset \omega \times \omega : \exists a \in \mathcal{E}_1\, [\, a \subset \mathrm{dom}\,(E) \wedge \phi(a) \subset E]\}$.

Lemma 31. $\mathcal{E} \subset \mathcal{E}_0$.

Proof. Suppose $E \in \mathcal{E}$. Fix $a \in \mathcal{E}_1$ such that $a \subset \mathrm{dom}\,(E)$ and $\phi(a) \subset E$. Fix $k \in \omega$ and $n_0 \in \omega$. We much find $n \geq n_0$ such that $|E(n)| > k$. By the definition of $\mathcal{E}_1$, $\exists^\infty i \left[\left|a \cap f^{-1}(\{i\})\right| > k + 1\right]$. Since the sets $f^{-1}(\{i\})$ are pairwise disjoint, we can choose $i \in \omega$ such that $f^{-1}(\{i\}) \cap n_0 = 0$ and $\left|a \cap f^{-1}(\{i\})\right| > k + 1$. We may choose $n \in a \cap f^{-1}(\{i\})$ so that $\left|n \cap a \cap f^{-1}(\{i\})\right| > k$. Now, for any $m \in n \cap a \cap f^{-1}(\{i\})$, we have $n, m \in a$, $m < n$, and $f(m) = i = f(n)$. Therefore, $\langle n, m\rangle \in \phi(a) \subset E$. We conclude that $n \cap a \cap f^{-1}(\{i\}) \subset E(n)$, whence $|E(n)| > k$. Notice that since $n \in f^{-1}(\{i\})$ and $f^{-1}(\{i\}) \cap n_0 = 0$, $n \geq n_0$. ⊣

We will construct $\mathcal{U}$ so that it is contained in $\mathcal{E}$. Lemmas 27 and 31 will ensure that $\mathcal{V} <_{RK} \mathcal{U}$. To be able to construct a P-point inside $\mathcal{E}$ (assuming CH) we need to know that $\mathcal{E}$ is a P-coideal. We verify this next. The next lemma is useful in this context because it tells us that subsets of domains of elements of $\mathcal{E}$ constructed in a particular manner are elements of $\mathcal{E}_1$.

Lemma 32. *Let $a \in \mathcal{E}_1$. Let $\langle i_k : k \in \omega \rangle$ and $\langle b_{i_k} : k \in \omega \rangle$ be two sequences such that*

(1) $i_{k+1} > i_k$

(2) b_{i_k} is a finite subset of $a \cap f^{-1}(\{i_k\})$ with $|b_{i_k}| > k + 1$.

Then $b = \bigcup (b_{i_k} \setminus \{min(b_{i_k})\}) \in \mathcal{E}_1$ and is a subset of a.

Proof. It is clear that b is a subset of a. Fix $k \in \omega$. We need to check that $\exists^\infty i \in \omega \left[\left|f^{-1}(\{i\}) \cap b\right| > k\right]$. It suffices to check that $\forall j \geq k \left[\left|f^{-1}(\{i_j\}) \cap b\right| > k\right]$. Since $b_{i_j} \setminus \{min(b_{i_j})\} \subset b \cap f^{-1}(\{i_j\})$, it is enough to show that $\left|b_{i_j} \setminus \{min(b_{i_j})\}\right| > k$. But $\left|b_{i_j}\right| > j + 1 \geq k + 1$, and so $\left|b_{i_j} \setminus \{min(b_{i_j})\}\right| > k$. ⊣

Lemma 33. *$\mathcal{E}$ is a P-coideal.*

Proof. We first check that $\mathcal{E}$ is upwards closed. Let $E_0 \in \mathcal{E}$ and suppose $E_0 \subset E_1$. There is an $a \in \mathcal{E}_1$ with $a \subset \mathrm{dom}\,(E_0)$ such that $\phi(a) \subset E_0$. But notice that $\mathrm{dom}\,(E_0) \subset \mathrm{dom}\,(E_1)$. So the same a witnesses that $E_1 \in \mathcal{E}$.

Next, suppose $E_0 \cup E_1 \in \mathcal{E}$. We need to check that either E_0 or E_1 is in $\mathcal{E}$. Choose $a \subset \mathrm{dom}\,(E_0 \cup E_1)$ such that $a \in \mathcal{E}_1$ and $\phi(a) \subset E_0 \cup E_1$. Fix $i \in \omega$. We know that for any $m, n \in a \cap f^{-1}(\{i\})$ satisfying $m < n$, $\langle n, m \rangle \in \phi(a) \subset E_0 \cup E_1$. So we can define a partition $p_i : \left[a \cap f^{-1}(\{i\})\right]^2 \to 2$ by

$$p_i(\{m < n\}) = 0 \iff \langle n, m \rangle \in E_0.$$

Clearly, we also have $p_i(\{m < n\}) = 1 \iff \langle n, m \rangle \in E_1$. It follows from Ramsey's theorem that there is a $j \in 2$ such that

$$\forall k \in \omega \exists^\infty i \in \omega \exists b_i \subset f^{-1}(\{i\}) \cap a \left[|b_i| > k \wedge p_i''[b_i]^2 = \{j\}\right]. \tag{1}$$

Now, suppose without loss of generality that (1) above holds with $j = 0$. We argue that $E_0 \in \mathcal{E}$. Build sequences $\langle i_k : k \in \omega \rangle$ and $\langle b_{i_k} : k \in \omega \rangle$ as follows. Given i_{k-1} and $b_{i_{k-1}}$, use (1) above (with $j = 0$) to obtain $i_k > i_{k-1}$ and a finite $b_{i_k} \subset f^{-1}(\{i_k\}) \cap a$ such that $|b_{i_k}| > k + 1$ and $p_{i_k}''[b_{i_k}]^2 = \{0\}$. Put $b = \bigcup (b_{i_k} \setminus \{min(b_{i_k})\})$. By Lemma 32, $b \in \mathcal{E}_1$. We show that $b \subset \mathrm{dom}\,(E_0)$ and that $\phi(b) \subset E_0$, thereby proving that $E_0 \in \mathcal{E}$. Suppose $n \in b$ and fix k such that $n \in b_{i_k} \setminus \{min(b_{i_k})\}$. Put $m = min(b_{i_k})$. Then we have $m < n$ and $m, n \in b_{i_k}$. Therefore, $p_{i_k}(\{m, n\}) = 0$, whence $\langle n, m \rangle \in E_0$. So $n \in \mathrm{dom}\,(E_0)$. Next, suppose that $\langle n, m \rangle \in \phi(b)$. Then $n, m \in b$, $m < n$ and $f(n) = f(m)$. Since $b_{i_k} \subset f^{-1}(\{i_k\})$, there must be a single k such that both n and m are members of $b_{i_k} \setminus \{min(b_{i_k})\}$. But then, again, $p_{i_k}(\{m, n\}) = 0$, whence $\langle n, m \rangle \in E_0$.

Finally, we check that the P-property of $\mathcal{E}$. Fix $E_0 \supset E_1 \supset \cdots$ with $E_n \in \mathcal{E}$. We must find $E \in \mathcal{E}$ such that $\forall n \in \omega\, [E \subset^* E_n]$. For each $n \in \omega$ choose $a_n \in \mathcal{E}_1$ such that $a_n \subset \mathrm{dom}\,(E_n)$ and $\phi(a_n) \subset E_n$. Define sequences $\langle i_k : k \in \omega \rangle$, $\langle b_{i_k} : k \in \omega \rangle$, and $\langle e_{i_k} : k \in \omega \rangle$ as follows. Given i_{k-1}, $b_{i_{k-1}}$, $e_{i_{k-1}}$, choose $i_k > i_{k-1}$ such that $\left|f^{-1}(\{i_k\}) \cap a_k\right| > k+1$. Choose a finite set $b_{i_k} \subset f^{-1}(\{i_k\}) \cap a_k$ with $|b_{i_k}| > k+1$. Put $e_{i_k} = \{\langle n, m \rangle : m < n \wedge m, n \in b_{i_k}\}$. Note that $e_{i_k} \subset b_{i_k}^2$, and since b_{i_k} is finite, e_{i_k} is finite too. Now, put $E = \bigcup e_{i_k}$ and $b = \bigcup (b_{i_k} \setminus \{min(b_{i_k})\})$. We first show that $\forall k \in \omega\, [E \subset^* E_k]$. Indeed, $E \setminus E_k \subset \bigcup_{j<k} e_{i_j}$, which is a finite set. To see this, suppose $\langle n, m \rangle \in e_{i_j}$ for some $j \geq k$. By definition of e_{i_j}, we have $m < n$ and $m, n \in b_{i_j}$. Since $b_{i_j} \subset f^{-1}(\{i_j\}) \cap a_j$, we get that $m, n \in a_j$ and that $f(m) = f(n) = i_j$. Hence $\langle n, m \rangle \in \phi(a_j) \subset E_j \subset E_k$.

Next, by Lemma 32 (apply it with $a = \bigcup a_k$), $b \in \mathcal{E}_1$. We argue that $b \subset \operatorname{dom}(E)$ and that $\phi(b) \subset E$, proving that $E \in \mathcal{E}$. Suppose $n \in b$ and fix k so that $n \in b_{i_k} \setminus \{min(b_{i_k})\}$. Put $m = min(b_{i_k})$. Then $m < n$ and $m, n \in b_{i_k}$. So by definition of e_{i_k}, $\langle n, m \rangle \in e_{i_k} \subset E$, whence $n \in \operatorname{dom}(E)$. To see that $\phi(b) \subset E$, fix some $\langle n, m \rangle \in \phi(b)$. This means that $n, m \in b$, that $m < n$, and that $f(m) = f(n)$. Since $b_{i_k} \subset f^{-1}(\{i_k\})$, there must be a single k such that $n, m \in b_{i_k} \setminus \{min(b_{i_k})\}$. But then, by the definition of e_{i_k}, $\langle n, m \rangle \in e_{i_k} \subset E$. $\dashv$

Lemma 34. *Suppose $E \in \mathcal{E}$ and that $a \subset \operatorname{dom}(E)$ such that $a \in \mathcal{E}_1$ and $\phi(a) \subset E$. Put $E \restriction a = \{\langle n, m \rangle \in E : n \in a\}$. Then $E \restriction a \in \mathcal{E}$ and $\operatorname{dom}(E \restriction a) = a$.*

Proof. It is clear that $\operatorname{dom}(E \restriction a) = a$. We again define sequences $\langle i_k : k \in \omega \rangle$ and $\langle b_{i_k} : k \in \omega \rangle$ as follows. Given i_{k-1} and $b_{i_{k-1}}$, choose $i_k > i_{k-1}$ such that $\left| f^{-1}(\{i_k\}) \cap a \right| > k + 1$. Choose a finite $b_{i_k} \subset f^{-1}(\{i_k\}) \cap a$ with $|b_{i_k}| > k + 1$. By Lemma 32, $b \subset a = \operatorname{dom}(E \restriction a)$ and $b \in \mathcal{E}_1$. We will check that $\phi(b) \subset E \restriction a$, proving that $E \restriction a \in \mathcal{E}$. By the monotonicity of ϕ (Lemma 29), $\phi(b) \subset \phi(a) \subset E$. Thus if $\langle n, m \rangle \in \phi(b)$, then $\langle n, m \rangle \in E$, and since by definition of $\phi(b)$, $n, m \in b \subset a$, $\langle n, m \rangle \in E \restriction a$. $\dashv$

Proof of Theorem 25. We will build a tower in $[\omega \times \omega]^\omega$ that generates $\mathcal{U}$. Let $\langle F_\alpha : \alpha < \omega_1 \rangle$ enumerate $\mathcal{P}(\omega \times \omega)$. We will build a sequence $\langle E_\alpha : \alpha < \omega_1 \rangle$ such that

(1) $E_\alpha \in \mathcal{E}$
(2) $\forall \beta \leq \alpha < \omega_1\, [E_\alpha \subset^* E_\beta]$
(3) either $E_\alpha \subset F_\alpha$ or $E_\alpha \subset (\omega \times \omega) \setminus F_\alpha$
(4) $\forall \beta < \omega_1 \exists \beta \leq \alpha < \omega_1\, [\phi(\operatorname{dom}(E_\alpha)) \subset E_\beta]$.

$\mathcal{U}$ is the filter generated by $\langle E_\alpha : \alpha < \omega_1 \rangle$, and $\mathcal{V}$ is the projection of $\mathcal{U}$ – i.e. $\mathcal{V} = \{\operatorname{dom}(F) : F \in \mathcal{U}\}$. Items (ii) and (iii) ensure that $\mathcal{U}$ is a P-point. Hence $\mathcal{V}$ is also a P-point and item (i) ensures that $\mathcal{V} <_{RK} \mathcal{U}$. Item (iv) gives us $\mathcal{V} \equiv_T \mathcal{U}$. To see this, suppose $F_\beta \in \mathcal{U}$. Then $E_\beta \subset F_\beta$, and by condition (iv), there is $\alpha \geq \beta$ such that $\phi(\operatorname{dom}(E_\alpha)) \subset E_\beta \subset F_\beta$. Since $\operatorname{dom}(E_\alpha) \in \mathcal{V}$, the image of $\mathcal{V}$ under the monotone map ϕ is cofinal in $\mathcal{U}$, whence $\mathcal{V} \equiv_T \mathcal{U}$.

To construct the sequence $\langle E_\alpha : \alpha < \omega_1 \rangle$, suppose that $\langle E_\beta : \beta < \alpha \rangle$ is given to us. By (i), (ii) and the fact that $\mathcal{E}$ is a P-coideal (Lemma 33), we can find $E \in \mathcal{E}$ such that $\forall \beta < \alpha\, [E \subset^* E_\beta]$. Next, to deal with (iii), note that either $E \cap F_\alpha \in \mathcal{E}$ or $E \cap (\omega \times \omega) \setminus F_\alpha \in \mathcal{E}$. Assume without loss that $E^* = E \cap F_\alpha \in \mathcal{E}$. To take care of (iv), put $S_\alpha = \{\beta < \alpha : \neg\, \exists \beta \leq \gamma < \alpha\, [\phi(\operatorname{dom}(E_\gamma)) \subset E_\beta]\}$. If S_α is empty, then simply set $E_\alpha = E^*$. Else let β be the least element of S_α. Let $E^{**} = E^* \cap E_\beta$. As $E^* \subset^* E_\beta$, $E^{**} \in \mathcal{E}$. Choose $a \subset \operatorname{dom}(E^{**})$ with $a \in \mathcal{E}_1$ such that $\phi(a) \subset E^{**}$. By Lemma 34, $E^{**} \restriction a \in \mathcal{E}$ and $\operatorname{dom}(E^{**} \restriction a) = a$. Put $E_\alpha = E^{**} \restriction a$. Then $\phi(\operatorname{dom}(E_\alpha)) = \phi(a) \subset E^{**} \subset E_\beta$. Notice that once E_β is taken care of, $\beta \notin S_\xi$ for any $\xi > \alpha$. $\dashv$

7. What is Tukey above a selective ultrafilter?

Let $\mathcal{K}$ be the class of ultrafilters obtained by closing the P-points under arbitrary countable Fubini products. The main result of this section (Corollary 56) is that a selective ultrafilter is Tukey below an ultrafilter in $\mathcal{K}$ iff it is RK below that ultrafilter. We saw in Section 3 that there may be no continuous, monotone, and cofinal maps witnessing a Tukey reduction from an ultrafilter in $\mathcal{K}$ to a selective ultrafilter. So Theorem 10 does not apply here.

In Section 5 we gave a complete characterization of all ultrafilters that are Tukey below a selective ultrafilter. So the results here may be seen as an attempt to say which ultrafilters are Tukey *above* a selective. We know that if $\mathcal{U} \equiv_T [\mathfrak{c}]^{<\omega}$, then every selective ultrafilter is Tukey below $\mathcal{U}$. Since $2^{\mathfrak{c}}$ selective ultrafilters can be constructed under mild hypotheses, we cannot hope to prove that the only ultrafilters that are Tukey above a selective ultrafilter are the ones that are RK above it. We do not know if the next best thing is consistent.

Question 35. *Assume* PFA. *Let $\mathcal{U}$ be an ultrafilter such that $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$ and let $\mathcal{V}$ be a selective ultrafilter such that $\mathcal{V} \leq_T \mathcal{U}$. Is it true that $\mathcal{V} \leq_{RK} \mathcal{U}$? Is it consistent with* $\mathrm{ZFC} + \mathrm{CH}$ *that for every $\mathcal{V} \leq_T \mathcal{U} <_T [\omega_1]^{<\omega}$, if $\mathcal{V}$ is selective, then $\mathcal{V} \leq_{RK} \mathcal{U}$?*

The result of this section shows (in ZFC) that such RK maps can be obtained for a wide range of ultrafilters $\mathcal{U}$ that satisfy $\mathcal{U} <_T [\omega_1]^{<\omega}$. Every ultrafilter in the class $\mathcal{K}$ is basically generated, and we do not know whether the result in this section can be extended to this wider class. In Section 4 a canonical form for monotone, cofinal maps from a basically generated ultrafilter into an arbitrary ultrafilter was obtained. We do not know whether the existence of such a canonical map already implies the desired conclusion.

Question 36. *Suppose $\mathcal{U}$ is an arbitrary ultrafilter and $\mathcal{V}$ is selective. Let $P \subset \mathrm{FIN}$ be such that $\forall a \in \mathcal{U}\left[\left|P \cap [a]^{<\omega}\right| = \omega\right]$. If $\mathcal{V} \leq_{RK} \mathcal{U}(P)$, then is $\mathcal{V} \leq_{RK} \mathcal{U}$? If $\mathcal{U}$ is basically generated and $\mathcal{V}$ is selective, then does $\mathcal{V} \leq_T \mathcal{U}$ imply $\mathcal{V} \leq_{RK} \mathcal{U}$?*

En route to proving Corollary 56 we will develop some machinery necessary for analyzing monotone maps from finite products of ultrafilters. We hope that some of these lemmas will have further applications.

Definition 37. By induction on $\alpha < \omega_1$, we define two classes of ultrafilters, $\mathcal{K}_\alpha$ and $\mathcal{C}_\alpha$, as follows.

1. $\mathcal{C}_0 = \mathcal{K}_0 = \{\mathcal{U} : \mathcal{U} \text{ is a P-Point}\}$.
2. Given $\mathcal{K}_\alpha$, $\mathcal{K}_{\alpha+1} = \{\bigotimes_{\mathcal{V}} \mathcal{U}_n : \mathcal{V} \in \mathcal{K}_\alpha \wedge \{\mathcal{U}_n : n \in \omega\} \subset \mathcal{K}_\alpha\}$. Given $\mathcal{C}_\alpha$, $\mathcal{C}_{\alpha+1} = \{\bigotimes_{\mathcal{V}} \mathcal{U}_n : \mathcal{V} \in \mathcal{C}_0 \wedge \{\mathcal{U}_n : n \in \omega\} \subset \mathcal{C}_\alpha\}$.
3. If α is a limit ordinal, then given $\mathcal{K}_\beta$ for every $\beta < \alpha$, $\mathcal{K}_\alpha = \bigcup_{\beta<\alpha} \mathcal{K}_\beta$. Given $\mathcal{C}_\beta$ for every $\beta < \alpha$, $\mathcal{C}_\alpha = \bigcup_{\beta<\alpha} \mathcal{C}_\beta$.

$\mathcal{K} = \bigcup_{\alpha<\omega_1} \mathcal{K}_\alpha$. $\mathcal{C} = \bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha$. For $\mathcal{W} \in \mathcal{K}$, define $\mathrm{rank}_{\mathcal{K}}(\mathcal{W})$ to be the least $\alpha < \omega_1$ such that $\mathcal{W} \in \mathcal{K}_\alpha$. For $\mathcal{W} \in \mathcal{C}$, define $\mathrm{rank}_{\mathcal{C}}(\mathcal{W})$ to be the least $\alpha < \omega_1$ such that $\mathcal{W} \in \mathcal{C}_\alpha$.

Note that the rank is always either 0 or a successor ordinal. It appears as if $\mathcal{K}$ is a bigger class than $\mathcal{C}$. But the next lemma shows that the two classes are the same.

Lemma 38. *For each $\mathcal{W} \in \mathcal{K}$, there exists $\mathcal{W}^* \in \mathcal{C}$ such that $\mathcal{W} \equiv_{RK} \mathcal{W}^*$.*

Proof. The proof is by double induction. First induct on $\operatorname{rank}_{\mathcal{K}}(\mathcal{W})$. If $\operatorname{rank}_{\mathcal{K}}(\mathcal{W}) = 0$, then there is nothing to prove. So suppose that $\operatorname{rank}_{\mathcal{K}}(\mathcal{W}) = \alpha + 1$ for some $\alpha < \omega_1$, and that the claim holds for all $\beta \leq \alpha$. By definition, $\mathcal{W} = \bigotimes_{\mathcal{V}} \mathcal{U}_n$, where the $\mathcal{U}_n$ and $\mathcal{V}$ are members of $\mathcal{K}_\alpha$. So by the inductive hypothesis, there exists $\mathcal{V}^* \in \mathcal{C}$ such that $\mathcal{V} \equiv_{RK} \mathcal{V}^*$. Therefore, there is a permutation $f : \omega \to \omega$ such that $\mathcal{W} = \bigotimes_{\mathcal{V}} \mathcal{U}_n \equiv_{RK} \bigotimes_{\mathcal{V}^*} \mathcal{U}_{f(n)}$.

Now, by induction on $\gamma < \omega_1$, we show that for any $\mathcal{V}^* \in \mathcal{C}$ and $\langle \mathcal{U}_n : n \in \omega \rangle \subset \mathcal{K}_\alpha$, if $\operatorname{rank}_{\mathcal{C}}(\mathcal{V}^*) = \gamma$, then there exists $\mathcal{W}^* \in \mathcal{C}$ such that $\bigotimes_{\mathcal{V}^*} \mathcal{U}_n \equiv_{RK} \mathcal{W}^*$, which will complete the proof. Fix $\mathcal{V}^* \in \mathcal{C}$ and $\langle \mathcal{U}_n : n \in \omega \rangle \subset \mathcal{K}_\alpha$. Suppose that $\operatorname{rank}_{\mathcal{C}}(\mathcal{V}^*) = 0$. By the main inductive hypothesis, there are $\mathcal{U}_n^* \in \mathcal{C}$ such that $\mathcal{U}_n \equiv_{RK} \mathcal{U}_n^*$ for each $n \in \omega$. So $\bigotimes_{\mathcal{V}^*} \mathcal{U}_n \equiv_{RK} \bigotimes_{\mathcal{V}^*} \mathcal{U}_n^* \in \mathcal{C}$. Now, suppose that $\operatorname{rank}_{\mathcal{C}}(\mathcal{V}^*) = \gamma + 1$. By definition, $\mathcal{V}^* = \bigotimes_{\mathcal{V}^{**}} \mathcal{U}_i^{**}$, where $\mathcal{V}^{**} \in \mathcal{C}_0$ and the $\mathcal{U}_i^{**}$ are in $\mathcal{C}_\gamma$. Now, it is clear that $\bigotimes_{\mathcal{V}^*} \mathcal{U}_n \equiv_{RK} \bigotimes_{\mathcal{V}^{**}} \mathcal{W}_i$, where $\mathcal{W}_i = \bigotimes_{\mathcal{U}_i^{**}} \mathcal{U}_{\pi(\langle i,j \rangle)}$, and where π is our fixed bijection from $\omega \times \omega$ to ω (see Definition 3). Since $\mathcal{U}_i^{**} \in \mathcal{C}_\gamma$ and $\left\{ \mathcal{U}_{\pi(\langle i,j \rangle)} : j \in \omega \right\} \subset \mathcal{K}_\alpha$, we know by the secondary inductive hypothesis that there is $\mathcal{W}_i^* \in \mathcal{C}$ such that $\mathcal{W}_i \equiv_{RK} \mathcal{W}_i^*$. Therefore, $\bigotimes_{\mathcal{V}^*} \mathcal{U}_n \equiv_{RK} \bigotimes_{\mathcal{V}^{**}} \mathcal{W}_i \equiv_{RK} \bigotimes_{\mathcal{V}^{**}} \mathcal{W}_i^* \in \mathcal{C}$. $\dashv$

The following is a standard fact giving a sufficient condition for verifying that an ultrafilter is RK reducible to a Fubini product. The proof is straightforward. We will use it in the proof of the Theorem 55

Lemma 39. *Let $\mathcal{V}$, $\langle \mathcal{U}_m^* : m \in \omega \rangle$, and $\mathcal{V}^*$ be ultrafilters. Suppose that $\{m \in \omega : \mathcal{V} \leq_{RK} \mathcal{U}_m^*\} \in \mathcal{V}^*$. Then $\mathcal{V} \leq_{RK} \bigotimes_{\mathcal{V}^*} \mathcal{U}_m^*$.*

We now consider monotone maps from subsets of $\mathcal{P}(\omega)^{n+1}$ into $\mathcal{P}(\omega)$. Definition 40 is the analogue of Definition 7.

Definition 40. Let $\mathcal{X}_0, \dots, \mathcal{X}_n \subset \mathcal{P}(\omega)$. Let $\phi : \mathcal{X}_0 \times \cdots \times \mathcal{X}_n \to \mathcal{P}(\omega)$. Define $\psi_\phi : \mathcal{P}(\omega)^{n+1} \to \mathcal{P}(\omega)$ by $\psi_\phi(\bar{a}) = \left\{ k \in \omega : \forall \bar{b} \in \mathcal{X}_0 \times \cdots \times \mathcal{X}_n \left[\bar{a} \subset \bar{b} \implies k \in \psi(\bar{b}) \right] \right\}$, for each $\bar{a} \in \mathcal{P}(\omega)^{n+1}$.

Once again, it is clear that ψ_ϕ is monotone, and that if ϕ is monotone, then $\psi_\phi \restriction (\mathcal{X}_0 \times \cdots \times \mathcal{X}_n) = \phi$. Also, for any $\bar{a} \in \mathcal{P}(\omega)^{n+1}$ if $\bar{a} \subset \bar{b} \in \mathcal{X}_0 \times \cdots \times \mathcal{X}_n$, then $\psi_\phi(\bar{a}) \subset \phi(\bar{b})$. The next lemma is the analogue of Lemma 16, and we leave its proof, which is nearly identical, to the reader. In what follows, we will apply this lemma not only to monotone maps into $\mathcal{P}(\omega)$ but also to monotone maps into $\mathcal{P}\left([\omega]^{<\omega}\right)$.

Lemma 41. *Let $\mathcal{U}_0, \dots, \mathcal{U}_n$ be basically generated by $\mathcal{B}_0 \subset \mathcal{U}_0, \dots, \mathcal{B}_n \subset \mathcal{U}_n$ respectively. Let $\phi : \mathcal{B}_0 \times \cdots \times \mathcal{B}_n \to \mathcal{P}(\omega)$ be a monotone map such that $\forall \bar{b} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n \left[\phi(\bar{b}) \neq 0 \right]$. Then for every $\bar{b} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n$, there exists $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{b}$ such that $\psi_\phi(\bar{s}) \neq 0$.*

The next lemma is the analogue of Lemma 8. Its proof is essentially the same and is left to the reader.

Lemma 42. *Let $\mathcal{U}_0, \dots, \mathcal{U}_n$ be basically generated by $\mathcal{B}_0 \subset \mathcal{U}_0, \dots, \mathcal{B}_n \subset \mathcal{U}_n$ respectively. Suppose $\mathcal{V}$ is an ultrafilter and that $\phi : \mathcal{B}_0 \times \cdots \times \mathcal{B}_n \to \mathcal{V}$ is monotone and cofinal in $\mathcal{V}$. Let $\psi = \psi_\phi$. Then for each $\bar{a} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$, there exists $\bar{a}^* \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$ with $\bar{a}^* \subset \bar{a}$ such that $\forall \bar{s} \in \left([\omega]^{<\omega}\right)^{n+1} [\bar{s} \subset \bar{a}^* \implies \psi(\bar{s})$ is finite$]$.*

Definition 43. For $\mathcal{U} \in \mathcal{C}$, define $\mathcal{B}_{\mathcal{U}} \subset \mathcal{U}$ by induction on $\mathrm{rank}_{\mathcal{C}}(\mathcal{U})$ as follows. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}) = 0$, then $\mathcal{B}_{\mathcal{U}} = \mathcal{U}$. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}) = \alpha + 1$, then first fix $\mathcal{V} \in \mathcal{C}_0$ and $\langle \mathcal{U}_n : n \in \omega \rangle \subset \mathcal{C}_\alpha$ such that $\mathcal{U} = \bigotimes_{\mathcal{V}} \mathcal{U}_n$. Now, define

$$\mathcal{B}_{\mathcal{U}} = \{\pi'' a : a \subset \omega \times \omega \wedge \exists e \in \mathcal{V} \, [\forall n \in e \, [a(n) \in \mathcal{B}_{\mathcal{U}_n}] \wedge \forall n \notin e \, [a(n) = 0]]\} .$$

Let $\langle \mathcal{U}_n : n \in \omega \rangle$, $\langle \mathcal{U}_n^* : n \in \omega \rangle$, and $\mathcal{V}$ be ultrafilters. If $\{n \in \omega : \mathcal{U}_n = \mathcal{U}_n^*\} \in \mathcal{V}$, then $\bigotimes_{\mathcal{V}} \mathcal{U}_n = \bigotimes_{\mathcal{V}} \mathcal{U}_n^*$. Therefore, for $\mathcal{U} \in \mathcal{C}$ with $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}) = \alpha + 1$, there is no unique choice of $\mathcal{V} \in \mathcal{C}_0$ and $\langle \mathcal{U}_n : n \in \omega \rangle \subset \mathcal{C}_\alpha$ witnessing $\mathcal{U} = \bigotimes_{\mathcal{V}} \mathcal{U}_n$. Thus in the definition of $\mathcal{B}_{\mathcal{U}}$, a specific choice must be made. We take Definition 43 as fixing this choice once and for all.

The next lemma is easy to prove. It was first proved by Dobrinen and Todorcevic in [4]. We refer the reader to their paper for more details.

Lemma 44 (Dobrinen and Todorcevic[4])**.** *For each $\mathcal{U} \in \mathcal{C}$, $\mathcal{U}$ is basically generated by $\mathcal{B}_{\mathcal{U}} \subset \mathcal{U}$.*

Definition 45. For $\mathcal{U} \in \mathcal{C}$ and $s \in [\omega]^{<\omega}$, define by induction on $\mathrm{rank}_{\mathcal{C}}(\mathcal{U})$ a collection $\mathcal{F}(\mathcal{U}, s)$ of ultrafilters in $\mathcal{C}$ as follows. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}) = 0$, then for any $s \in [\omega]^{<\omega}$, $\mathcal{F}(\mathcal{U}, s) = \{\mathcal{U}\}$. Suppose $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}) = \alpha + 1$. Let $\mathcal{V} \in \mathcal{C}_0$ and $\langle \mathcal{U}_n : n \in \omega \rangle \subset \mathcal{C}_\alpha$ be the ultrafilters chosen in Definition 43 such that $\mathcal{U} = \bigotimes_{\mathcal{V}} \mathcal{U}_n$. Given $s \in [\omega]^{<\omega}$, let $t = \{n \in \omega : \pi^{-1}(s)(n) \neq 0\}$. Define $\mathcal{F}(\mathcal{U}, s) = \{\mathcal{U}\} \cup \{\mathcal{V}\} \cup \{\mathcal{U}_n : n \in t\} \cup \bigcup_{n \in t} \mathcal{F}(\mathcal{U}_n, \pi^{-1}(s)(n))$.

Note that any $\mathcal{V} \in \mathcal{F}(\mathcal{U}, s)$ is in $\mathcal{C}$ and $\mathrm{rank}_{\mathcal{C}}(\mathcal{V}) \leq \mathrm{rank}_{\mathcal{C}}(\mathcal{U})$. Note also that $\mathcal{U} \in \mathcal{F}(\mathcal{U}, s)$, and that for any $\mathcal{V} \in \mathcal{F}(\mathcal{U}, s)$, $\mathcal{V} \neq \mathcal{U}$ implies that $\mathrm{rank}_{\mathcal{C}}(\mathcal{V}) < \mathrm{rank}_{\mathcal{C}}(\mathcal{U})$.

Definition 46. For $\mathcal{X} \subset \mathcal{P}(\omega)$ and $s \in [\omega]^{<\omega}$, define $\mathcal{X}(s) = \{a \in \mathcal{X} : s \subset a\}$.

The proof of Corollary 56 involves analyzing directed sets of the form $\langle \mathcal{B}_{\mathcal{U}}(s), \supset \rangle$ where $\mathcal{U} \in \mathcal{C}$ and $s \in [\omega]^{<\omega}$. By the next lemma these are Tukey equivalent to finite products of ultrafilters in $\mathcal{F}(\mathcal{U}, s)$. This is the reason we are forced to consider such finite products in this section.

Lemma 47. *Let $\mathcal{U} \in \mathcal{C}$ and $s \in [\omega]^{<\omega}$. Then there exist $\mathcal{U}_0, \dots, \mathcal{U}_l \in \mathcal{F}(\mathcal{U}, s)$ such that*

(1) $\langle \mathcal{B}_{\mathcal{U}}(s), \supset \rangle \equiv_T \langle \mathcal{U}_0 \times \cdots \times \mathcal{U}_l, \supset \rangle$.
(2) $\exists! i \, [0 \leq i \leq l \wedge \mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = \mathrm{rank}_{\mathcal{C}}(\mathcal{U})]$.

Proof. The proof is by induction on $\mathrm{rank}_{\mathcal{C}}(\mathcal{U})$. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}) = 0$, then $\mathcal{B}_{\mathcal{U}} = \mathcal{U}$. So if $\phi : \mathcal{B}_{\mathcal{U}}(s) \to \mathcal{U}$ is defined by $\phi(a) = a \setminus s$, for each $a \in \mathcal{B}_{\mathcal{U}}(s)$, then ϕ is Tukey, monotone and cofinal in $\mathcal{U}$, showing that $\mathcal{B}_{\mathcal{U}}(s) \equiv_T \mathcal{U}$.

Suppose that $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}) = \alpha + 1$ and let $\mathcal{V}_0 \in \mathcal{C}_0$ and $\langle \mathcal{U}_n : n \in \omega \rangle \subset \mathcal{C}_\alpha$ be the ultrafilters fixed in Definition 43 such that $\mathcal{U} = \bigotimes_{\mathcal{V}} \mathcal{U}_n$. If $s = 0$, then $\mathcal{B}_{\mathcal{U}}(s) = \mathcal{B}_{\mathcal{U}}$, and so $\mathcal{B}_{\mathcal{U}}(s) \equiv_T \mathcal{U}$. Now, assume that $s \neq 0$. Viewing $\mathcal{U}$ as an ultrafilter on $\omega \times \omega$ and s as a subset of $\omega \times \omega$, put $t = \{n \in \omega : s(n) \neq 0\}$. Since $s \neq 0$, $t \neq 0$. Let $\{n_0 < \cdots < n_k\}$ enumerate t. Now, it is clear that

$$\langle \mathcal{B}_{\mathcal{U}}(s), \supset \rangle \equiv_T \langle \mathcal{B}_{\mathcal{U}_{n_0}}(s(n_0)) \times \cdots \times \mathcal{B}_{\mathcal{U}_{n_k}}(s(n_k)) \times \mathcal{U}, \supset \rangle$$

Now, applying the inductive hypothesis to $\mathcal{B}_{\mathcal{U}_{n_i}}(s(n_i))$, we may find ultrafilters $\mathcal{U}_0^i, \dots \mathcal{U}_{l_i}^i \in \mathcal{F}(\mathcal{U}_{n_i}, s(n_i))$, for each $0 \leq i \leq k$ such that

$$\langle \mathcal{B}_{\mathcal{U}}(s), \supset \rangle \equiv_T \langle \mathcal{U}_0^0 \times \cdots \times \mathcal{U}_{l_0}^0 \times \cdots \times \mathcal{U}_0^k \times \cdots \times \mathcal{U}_{l_k}^k \times \mathcal{U}, \supset \rangle.$$

Since $\mathcal{U} \in \mathcal{F}(\mathcal{U}, s)$ and $\mathcal{F}(\mathcal{U}_{n_i}, s(n_i)) \subset \mathcal{F}(\mathcal{U}, s)$, (1) is satisfied. Since for each $0 \leq i \leq k$ and $0 \leq j \leq l_i$, $\mathcal{U}^i_j \in \mathcal{F}(\mathcal{U}_{n_i}, s(n_i))$, $\operatorname{rank}_{\mathcal{C}}(\mathcal{U}^i_j) \leq \operatorname{rank}_{\mathcal{C}}(\mathcal{U}_{n_i}) \leq \alpha < \alpha + 1 = \operatorname{rank}_{\mathcal{C}}(\mathcal{U})$, and so (2) is satisfied. $\dashv$

Lemma 48. *Let $\mathcal{U}_0, \dots, \mathcal{U}_n$ be basically generated by $\mathcal{B}_0 \subset \mathcal{U}_0, \dots, \mathcal{B}_n \subset \mathcal{U}_n$ respectively. Let $\mathcal{V}$ be a P-point and suppose that $\phi : \mathcal{B}_0 \times \cdots \times \mathcal{B}_n \to \mathcal{V}$ is monotone and cofinal in $\mathcal{V}$. Let $\bar{a} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$. Then there exists $\bar{b} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n$ such that $\bar{b} \subset \bar{a}$ and for each $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$, if $\bar{s} \subset \bar{b}$, then $\forall e \in \mathcal{V} \exists \bar{c} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n \left[\bar{s} \subset \bar{c} \subset \bar{b} \wedge \phi(\bar{c}) \setminus \psi_\phi(\bar{s}) \subset e\right]$.*

Proof. For ease of reading, put $\psi = \psi_\phi$. Say that $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ is *bad* if $\bar{s} \subset \bar{a}$ and there exists $e \in \mathcal{V}$ such that $\forall \bar{c} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n \left[\bar{s} \subset \bar{c} \implies \phi(\bar{c}) \setminus \psi(\bar{s}) \not\subset e\right]$. For each $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ that is bad, choose $e_{\bar{s}} \in \mathcal{V}$ such that $\forall \bar{c} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n \left[\bar{s} \subset \bar{c} \implies \phi(\bar{c}) \setminus \psi(\bar{s}) \not\subset e_{\bar{s}}\right]$. Choose $e \in \mathcal{V}$ such that $e \subset^* e_{\bar{s}}$ for every bad $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$. Choose $\bar{b} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n$ such that $\bar{b} \subset \bar{a}$ and $\phi(\bar{b}) \subset e$. To prove the lemma, it suffices to show that no $\bar{s} \subset \bar{b}$ is bad. Fix $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{b}$, and suppose that $\bar{s}$ is bad. Then there is $m \in \omega$ such that $\phi(\bar{b}) \setminus m \subset e_{\bar{s}}$. Observe that if $m \setminus \psi(\bar{s}) = 0$, then $\phi(\bar{b}) \setminus \psi(\bar{s}) \subset e_{\bar{s}}$, and since $\bar{b} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n$ and since $\bar{s} \subset \bar{b}$, this contradicts the choice of $e_{\bar{s}}$. So $m \setminus \psi(\bar{s}) \neq 0$. For each $k \in m$, if $k \notin \psi(\bar{s})$, then by definition of ψ_ϕ, there is $\bar{d}_k \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n$ with $\bar{s} \subset \bar{d}_k$ such that $k \notin \phi(\bar{d}_k)$. Now, put $\bar{c} = \bar{b} \cap \bigcap\{\bar{d}_k : k \in m \setminus \psi(\bar{s})\}$. Note, $\bar{c} \in \mathcal{B}_0 \times \cdots \times \mathcal{B}_n$ and that $\bar{s} \subset \bar{c}$. Since $\phi(\bar{c}) \subset \phi(\bar{b})$, if $k \in \phi(\bar{c}) \setminus \psi(\bar{s})$, then $k \notin m$, and so $k \in e_{\bar{s}}$, again contradicting the choice of $e_{\bar{s}}$. $\dashv$

Strictly speaking, the next lemma is not needed for the proof of the main result of this section. All we need is the statement gotten by replacing "$\leq_T$" everywhere in the lemma with "$\leq_{RK}$", which is easily seen to hold.

Lemma 49. *Let $\mathcal{U} \in \mathcal{C}$. Let $\mathcal{V}$ be an arbitrary ultrafilter such that $\mathcal{V} \not\leq_T \mathcal{U}$. Then $\forall a \in \mathcal{U} \exists b \in \mathcal{B}_{\mathcal{U}} \cap [a]^\omega \forall s \in [b]^{<\omega} \forall \mathcal{W} \in \mathcal{F}(\mathcal{U}, s) \left[\mathcal{V} \not\leq_T \mathcal{W}\right]$.*

Proof. The proof is by induction on $\operatorname{rank}_{\mathcal{C}}(\mathcal{U})$. If $\operatorname{rank}_{\mathcal{C}}(\mathcal{U}) = 0$, then there is nothing to prove. So suppose $\operatorname{rank}_{\mathcal{C}}(\mathcal{U}) = \alpha + 1$. Let $\mathcal{V}^* \in \mathcal{C}_0$ and $\langle \mathcal{U}_n : n \in \omega \rangle \subset \mathcal{C}_\alpha$ be the ultrafilters fixed in Definition 43 such that $\mathcal{U} = \bigotimes_{\mathcal{V}^*} \mathcal{U}_n$. Fix $a \in \mathcal{U}$ and $c \in \mathcal{B}_{\mathcal{U}} \cap [a]^\omega$. Viewing $\mathcal{U}$ as an ultrafilter on $\omega \times \omega$ and c as a subset of $\omega \times \omega$, fix $e \in \mathcal{V}^*$ such that $\forall n \in e \left[c(n) \in \mathcal{B}_{\mathcal{U}_n}\right] \wedge \forall n \notin e \left[c(n) = 0\right]$. Since $\mathcal{V} \not\leq_T \mathcal{U}$, $\mathcal{V} \not\leq_T \mathcal{V}^*$. Now, consider $e^* = \{n \in e : \mathcal{V} \leq_T \mathcal{U}_n\}$. Suppose for a moment that $e^* \in \mathcal{V}^*$. For $n \in e^*$ fix a Tukey map $\psi_n : \mathcal{V} \to \mathcal{U}_n$. Now, define $\psi : \mathcal{V} \to \mathcal{U}$ by $\psi(v) = \bigcup_{n \in e^*} \{n\} \times \psi_n(v)$. for each $v \in \mathcal{V}$. Suppose that $\mathcal{X} \subset \mathcal{V}$ is unbounded in $\mathcal{V}$. Assume that $\{\psi(v) : v \in \mathcal{X}\}$ is bounded by $d \in \mathcal{U}$. By definition of $\mathcal{U}$, there exists $n \in e^*$ such that $d(n) \in \mathcal{U}_n$. But since $d(n) \subset \psi_n(v)$ for each $v \in \mathcal{X}$, it follows that $\{\psi_n(v) : v \in \mathcal{X}\}$ is bounded by $d(n) \in \mathcal{U}_n$, contradicting that ψ_n is Tukey. Hence we conclude that $\{n \in e : \mathcal{V} \not\leq_T \mathcal{U}_n\}$ is in $\mathcal{V}^*$. Now *relabel* $e^* = \{n \in e : \mathcal{V} \not\leq_T \mathcal{U}_n\}$. Using the inductive hypothesis, find $b_n \in \mathcal{B}_{\mathcal{U}_n} \cap [c(n)]^\omega$ for each $n \in e^*$ such that for each $s \in [b_n]^{<\omega}$ and each $\mathcal{W} \in \mathcal{F}(\mathcal{U}_n, s)$, $\mathcal{V} \not\leq_T \mathcal{W}$. Let $b = \bigcup_{n \in e^*} \{n\} \times b_n$, and note that $b \in \mathcal{B}_{\mathcal{U}} \cap [a]^\omega$. Now, fix $s \in [b]^{<\omega}$. Put $t = \{n \in \omega : s(n) \neq 0\}$. Notice that $t \subset e^*$, and recall that $\mathcal{F}(\mathcal{U}, s) = \{\mathcal{U}\} \cup \{\mathcal{V}^*\} \cup \{\mathcal{U}_n : n \in t\} \cup \bigcup_{n \in t} \mathcal{F}(\mathcal{U}_n, s(n))$. Now, $\mathcal{V} \not\leq_T \mathcal{U}$ and $\mathcal{V} \not\leq_T \mathcal{V}^*$. Next, since $t \subset e^*$, $\mathcal{V} \not\leq_T \mathcal{U}_n$ for any $n \in t$. Finally, $s(n) \in [b(n)]^{<\omega}$. But for $n \in t$, $b(n) = b_n$, and b_n was chosen in such a way that for each $\mathcal{W} \in \mathcal{F}(\mathcal{U}_n, s(n))$, $\mathcal{V} \not\leq_T \mathcal{W}$. $\dashv$

Definition 50. Let $\mathcal{U}_0, \dots, \mathcal{U}_n \in \mathcal{C}$. For $\bar{a} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$, $0 \le i \le n$, and $t \in [\omega]^{<\omega}$, define $\bar{a}\restriction(i,t) \in \mathcal{P}(\omega)^{n+1}$ as follows. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = 0$, then $\bar{a}\restriction(i,t)(i) = \bar{a}(i) \cap t$, and $\bar{a}\restriction(i,t)(j) = \bar{a}(j)$ for any $0 \le j \le n$ with $i \ne j$. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = \alpha + 1$, then $\bar{a}\restriction(i,t)(i) = \pi''d$, where $d = \bigcup_{n \in t} \{n\} \times \pi^{-1}(\bar{a}(i))(n)$, and $\bar{a}\restriction(i,t)(j) = \bar{a}(j)$ for any $0 \le j \le n$ with $i \ne j$.

Given $\bar{a} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$ and $0 \le i \le n$ define $\pi^*(\bar{a}(i))$ as follows. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = 0$, then $\pi^*(\bar{a}(i)) = \bar{a}(i)$. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = \alpha + 1$, then $\pi^*(\bar{a}(i)) = \{m \in \omega : \pi^{-1}(\bar{a}(i))(m) \ne 0\}$.

Observe that $\bar{a}\restriction(i,t) \subset \bar{a}$ and that if $t \subset s$, then $\bar{a}\restriction(i,t) \subset \bar{a}\restriction(i,s)$. Also, if $\bar{a} \subset \bar{b}$, then $\bar{a}\restriction(i,t) \subset \bar{b}\restriction(i,t)$. It is also clear that $\bar{a}\restriction(i,0)(i) = 0$.

It is easy to check that for $\bar{a}, \bar{b} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$ and $0 \le i \le n$, if $\bar{a} \subset \bar{b}$, then $\pi^*(\bar{a}(i)) \subset \pi^*(\bar{b}(i))$. Moreover, if $\bar{a} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ and $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{a}$, then there exists $t \in [\pi^*(\bar{a}(i))]^{<\omega}$ such that $\bar{s} \subset \bar{a}\restriction(i,t)$.

Lemma 51. *Let $\mathcal{U}_0, \dots, \mathcal{U}_n \in \mathcal{C}$, and let $\mathcal{V}$ be a P-point. Assume that $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_0) \le \cdots \le \mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n)$. Suppose $\phi : \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n} \to \mathcal{V}$ is monotone and cofinal in $\mathcal{V}$. Put $\psi = \psi_\phi$. Then one of (1)–(3) holds:*

(1) $\exists \bar{b} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n} \forall t \in [\omega]^{<\omega} \left[\psi(\bar{b}\restriction(n,t)) \notin \mathcal{V}\right]$.

(2) There exists $\alpha < \omega_1$ such that $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n) = \alpha + 1$, and for each $\bar{a} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$, there are $k \in \omega$, $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{a}$, and ultrafilters $\left\{\mathcal{U}_0^i, \dots, \mathcal{U}_{l_i}^i\right\}$ for $i < n + k + 1$ such that

(a) $\mathcal{V} \le_T \prod_{i < n+k+1, l^* \le l_i} \mathcal{U}_{l^*}^i$

(b) $\forall i < n \forall l^* \le l_i \left[\mathcal{U}_{l^*}^i \in \mathcal{F}(\mathcal{U}_i, \bar{s}(i))\right]$

(c) $\forall j < k + 1 \forall l^* \le l_{n+j} \left[\mathcal{U}_{l^*}^{n+j} \in \mathcal{F}(\mathcal{U}_n, \bar{s}(n))\right]$.

(d) $\forall i < n \exists! 0 \le l^* \le l_i \left[\mathrm{rank}_{\mathcal{C}}\left(\mathcal{U}_{l^*}^i\right) = \mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i)\right]$

(e) $\forall j < k + 1 \forall l^* \le l_{n+j} \left[\mathrm{rank}_{\mathcal{C}}\left(\mathcal{U}_{l^*}^{n+j}\right) < \mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n)\right]$.

(3) $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n) = 0$, *and* $n > 0$, *and* $\mathcal{V} \le_T \mathcal{U}_0 \times \cdots \times \mathcal{U}_{n-1}$.

Proof. Assume that for every $\bar{b} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$, there exists $t \in [\omega]^{<\omega}$ such that $\psi(\bar{b}\restriction(n,t)) \in \mathcal{V}$. Consider the map $\chi : \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n} \to \mathcal{P}\left([\omega]^{<\omega}\right)$ defined by $\chi(\bar{b}) = \{t \in [\omega]^{<\omega} : \psi(\bar{b}\restriction(n,t)) \in \mathcal{V}\}$, for each $\bar{b} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$. By hypothesis, $\chi(\bar{b}) \ne 0$ for every $\bar{b} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$. To see that χ is monotone, fix $\bar{b}, \bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ with $\bar{c} \subset \bar{b}$. Take any $t \in \chi(\bar{c})$, and observe that since $\bar{c}\restriction(n,t) \subset \bar{b}\restriction(n,t)$ and since $\psi(\bar{c}\restriction(n,t)) \in \mathcal{V}$, $\psi(\bar{b}\restriction(n,t)) \in \mathcal{V}$ and $t \in \chi(\bar{b})$. Therefore, Lemma 41 applied to the map χ tells us that for any $\bar{b} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$, there is $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{b}$ and $t \in [\omega]^{<\omega}$ such that

$$(*) \qquad \forall \bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n} \left[\bar{s} \subset \bar{c} \implies \psi(\bar{c}\restriction(n,t)) \in \mathcal{V}\right].$$

Note that for $t, t^* \in [\omega]^{<\omega}$ and $\bar{c} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$, $t \subset t^*$ implies that $\bar{c}\restriction(n,t) \subset \bar{c}\restriction(n,t^*)$. Therefore, we may always assume that $\forall i < n+1 \left[\bar{s}(i) \ne 0\right]$ and that $\{m \in \omega : \pi^{-1}(\bar{s}(n))(m) \ne 0\} \subset t$.

Assume first that $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n) = \alpha + 1$, and fix $\bar{a} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$. First use Lemma 42 to fix $\bar{a}^* \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$ with $\bar{a}^* \subset \bar{a}$ such that for all $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ if $\bar{s} \subset \bar{a}^*$, then $\psi(\bar{s})$ is finite. Use Lemma 48 to find $\bar{b} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ with $\bar{b} \subset \bar{a}^*$ such that for every $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$, if $\bar{s} \subset \bar{b}$, then $\forall e \in \mathcal{V} \exists \bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times$

$\cdots \times \mathcal{B}_{\mathcal{U}_n}\left[\bar{s} \subset \bar{c} \subset \bar{b} \wedge \phi(\bar{c}) \setminus \psi(\bar{s}) \subset e\right]$. Now, find $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{b}$ and $t \in [\omega]^{<\omega}$ such that $(*)$ holds. Let $\mathcal{V}^* \in \mathcal{C}_0$ and $\langle \mathcal{U}^*_m : m \in \omega\rangle \subset \mathcal{C}_\alpha$ be the ultrafilters fixed in Definition 43 such that $\mathcal{U}_n = \bigotimes_{\mathcal{V}^*}\mathcal{U}^*_m$. Viewing $\bar{b}(n)$ and $\bar{s}(n)$ as subsets of $\omega \times \omega$, put $t^* = \{m \in \omega : \bar{s}(n)(m) \neq 0\}$. As observed earlier, we may assume that $\bar{s}(n) \neq 0$ and that $t^* \subset t$. Since $\bar{s}(n) \neq 0$, $t^* \neq 0$. Now, suppose that $\bar{c}$ is any member of $\mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ with $\bar{s} \subset \bar{c}$. Define $\bar{d}$ as follows. For $i < n$, $\bar{d}(i) = \bar{c}(i)$. For each $m \in t \setminus t^*$, $\bar{d}(n)(m) = 0$, and for all $m \notin t \setminus t^*$, $\bar{d}(n)(m) = \bar{c}(n)(m)$. It is clear that $\bar{d} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ and that $\bar{s} \subset \bar{d}$. So by $(*)$, $\psi(\bar{d}\restriction(n,t)) \in \mathcal{V}$. But since $t^* \subset t$ and since $\bar{d}(n)(m)$ is equal to $\bar{c}(n)(m)$ for $m \in t^*$ and equal to 0 for $m \in t \setminus t^*$, it follows that $\bar{d}\restriction(n,t) = \bar{c}\restriction(n,t^*)$. Therefore, $\psi(\bar{c}\restriction(n,t^*)) \in \mathcal{V}$. So the pair $\bar{s}, t^*$ satisfy $(*)$. Notice that $\bar{s} \subset \bar{a}$.

Let $\{m_0 < \ldots < m_k\}$ enumerate t^*. Put $l = n + k + 1$. Now let $\mathcal{D}$ denote:

$$\left\{\bar{d} \in \mathcal{P}(\omega)^l : \forall i < n\left[\bar{d}(i) \in \mathcal{B}_{\mathcal{U}_i}(\bar{s}(i))\right] \wedge \forall j < k+1\left[\bar{d}(n+j) \in \mathcal{B}_{\mathcal{U}^*_{m_j}}(\bar{s}(n)(m_j))\right]\right\}.$$

Given $\bar{d} \in \mathcal{D}$, define $\bar{c}_{\bar{d}}$ as follows. For $i < n$, $\bar{c}_{\bar{d}}(i) = \bar{d}(i)$. For n, define $\bar{c}_{\bar{d}}(n)(m_j) = \bar{d}(n+j)$ for $0 \leq j \leq k$, and $\bar{c}_{\bar{d}}(n)(m) = \bar{b}(n)(m)$ for $m \notin t^*$. Since $\bar{d}(i) \in \mathcal{B}_{\mathcal{U}_i}(\bar{s}(i))$ for $i < n$ and $\bar{d}(n+j) \in \mathcal{B}_{\mathcal{U}^*_{m_j}}(\bar{s}(n)(m_j))$ for $j < k+1$, and $\bar{b}(n) \in \mathcal{B}_{\mathcal{U}_n}$ it follows that $\bar{c}_{\bar{d}} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ and that $\bar{s} \subset \bar{c}_{\bar{d}}$. Therefore, $\psi(\bar{c}_{\bar{d}}\restriction(n,t^*)) \in \mathcal{V}$. Note also that $\bar{s} \subset \bar{b} \subset \bar{a}^*$, and so $\psi(\bar{s})$ is finite. Therefore, it is possible to define a map $\theta : \langle \mathcal{D} \supset\rangle \to \mathcal{V}$ by $\theta(\bar{d}) = \psi(\bar{c}_{\bar{d}}\restriction(n,t^*)) \setminus \psi(\bar{s})$, for each $\bar{d} \in \mathcal{D}$. Now, it is easy to check that for $\bar{d}, \bar{d}^* \in \mathcal{D}$, if $\bar{d} \subset \bar{d}^*$, then $\bar{c}_{\bar{d}} \subset \bar{c}_{\bar{d}^*}$, and hence $\bar{c}_{\bar{d}}\restriction(n,t^*) \subset \bar{c}_{\bar{d}^*}\restriction(n,t^*)$. Therefore, θ is monotone. To see that θ is cofinal in $\mathcal{V}$, fix $e \in \mathcal{V}$. We know that there exists $\bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ with $\bar{s} \subset \bar{c} \subset \bar{b}$ such that $\phi(\bar{c}) \setminus \psi(\bar{s}) \subset e$. Now define $\bar{d}$ as follows. For $i < n$, $\bar{d}(i) = \bar{c}(i)$. For $j < k+1$, $\bar{d}(n+j) = \bar{c}(n)(m_j)$. It is easy to check that $\bar{d} \in \mathcal{D}$. Now, it is clear that for $i < n$, $\bar{c}_{\bar{d}}\restriction(n,t^*)(i) = \bar{c}\restriction(n,t^*)(i)$. For n, $\bar{c}_{\bar{d}}\restriction(n,t^*)(n) = \bigcup_{0\leq j\leq k}\{m_j\} \times \bar{c}_{\bar{d}}(n)(m_j) = \bigcup_{0\leq j\leq k}\{m_j\} \times \bar{d}(n+j) = \bigcup_{0\leq j\leq k}\{m_j\} \times \bar{c}(n)(m_j) = \bar{c}\restriction(n,t^*)(n)$. Therefore, we have shown $\theta(\bar{d}) = \psi(\bar{c}_{\bar{d}}\restriction(n,t^*)) \setminus \psi(\bar{s}) = \psi(\bar{c}\restriction(n,t^*)) \setminus \psi(\bar{s}) \subset \psi(\bar{c}) \setminus \psi(\bar{s}) \subset \phi(\bar{c}) \setminus \psi(\bar{s}) \subset e$. So we conclude that $\mathcal{V} \leq_T \langle \mathcal{D}, \supset\rangle$.

Applying Lemma 47 to $\mathcal{B}_{\mathcal{U}_i}(\bar{s}(i))$ for $i < n$ and to $\mathcal{B}_{\mathcal{U}^*_{m_j}}(\bar{s}(n)(m_j))$ for $j < k+1$, find $\left\{\mathcal{U}^i_0, \ldots, \mathcal{U}^i_{l_i}\right\} \subset \mathcal{F}(\mathcal{U}_i, \bar{s}(i))$, for each $i < n$, and $\left\{\mathcal{U}^{n+j}_0, \ldots, \mathcal{U}^{n+j}_{l_{n+j}}\right\} \subset \mathcal{F}\left(\mathcal{U}^*_{m_j}, \bar{s}(n)(m_j)\right)$, for $j < k+1$ such that

(a') $\mathcal{V} \leq_T \prod_{i<l, l^* \leq l_i} \mathcal{U}^i_{l^*}$

(b') $\forall i < n \exists! 0 \leq l^* \leq l_i \left[\operatorname{rank}_{\mathcal{C}}\left(\mathcal{U}^i_{l^*}\right) = \operatorname{rank}_{\mathcal{C}}(\mathcal{U}_i)\right]$.

(c') $\forall j < k+1 \forall 0 \leq l^* \leq l_{n+j}\left[\operatorname{rank}_{\mathcal{C}}\left(\mathcal{U}^{n+j}_{l^*}\right) < \operatorname{rank}_{\mathcal{C}}(\mathcal{U}_n)\right]$.

(c') holds because for each $j < k+1$ and $0 \leq l^* \leq l_{n+j}$, $\mathcal{U}^{n+j}_{l^*} \in \mathcal{F}\left(\mathcal{U}^*_{m_j}, \bar{s}(n)(m_j)\right)$, and so $\operatorname{rank}_{\mathcal{C}}\left(\mathcal{U}^{n+j}_{l^*}\right) \leq \operatorname{rank}_{\mathcal{C}}\left(\mathcal{U}^*_{m_j}\right) \leq \alpha < \operatorname{rank}_{\mathcal{C}}(\mathcal{U}_n)$. Notice also that for each $j < k+1$, $\mathcal{F}\left(\mathcal{U}^*_{m_j}, \bar{s}(n)(m_j)\right) \subset \mathcal{F}(\mathcal{U}_n, \bar{s}(n))$, and so we have proved (a)–(e) of (2).

Assume next that $\operatorname{rank}_{\mathcal{C}}(\mathcal{U}_n) = 0$. This means that $\operatorname{rank}_{\mathcal{C}}(\mathcal{U}_i) = 0$ for all $0 \leq i \leq n$. Once again, given any $\bar{a} \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$ use Lemmas 42 and 48 to fix $\bar{a}^* \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$ with $\bar{a}^* \subset \bar{a}$ such that for each $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$, if $\bar{s} \subset \bar{a}^*$, then $\psi(\bar{s})$ is finite, and $\bar{b} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ with the property that $\bar{b} \subset \bar{a}^*$ and for each $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$, if $\bar{s} \subset \bar{b}$, then $\forall e \in \mathcal{V} \exists \bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n} \left[\bar{s} \subset \bar{c} \subset \bar{b} \wedge \phi(\bar{c}) \setminus \psi(\bar{s}) \subset e\right]$. Now, fix $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{b}$ and $t \in [\omega]^{<\omega}$ such that $(*)$ holds. If $n = 0$, then $(*)$ tells us that $\psi(\bar{b}\restriction(0,t)) = \psi(\langle \bar{b}(0) \cap t\rangle) \in \mathcal{V}$. However, since t is finite, $\psi(\langle \bar{b}(0) \cap t\rangle)$ is finite. Therefore, $n > 0$. Define $\theta : \mathcal{B}_{\mathcal{U}_0}(\bar{s}(0)) \times \cdots \times \mathcal{B}_{\mathcal{U}_{n-1}}(\bar{s}(n-1)) \to \mathcal{V}$ as follows. First define $\bar{s}^* \in \left([\omega]^{<\omega}\right)^{n+1}$ as follows. For $0 \le i \le n-1$, $\bar{s}^*(i) = \bar{s}(i)$, and $\bar{s}^*(n) = \bar{b}(n) \cap t$. Now, given $\bar{d} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_{n-1}}$ with $\bar{s}(i) \subset \bar{d}(i)$ for every $0 \le i \le n-1$, let $\bar{c} = \bar{d}^\frown \langle \bar{b}(n)\rangle$. Clearly, $\bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$, and $\bar{s} \subset \bar{c}$. Therefore, $\psi(\bar{c}\restriction(n,t)) \in \mathcal{V}$. Moreover, since $\bar{s}^* \subset \bar{b} \subset \bar{a}^*$, $\psi(\bar{s}^*)$ is finite. So $\psi(\bar{c}\restriction(n,t)) \setminus \psi(\bar{s}^*) \in \mathcal{V}$. So we set $\theta(\bar{d}) = \psi(\bar{c}\restriction(n,t)) \setminus \psi(\bar{s}^*)$. It is clear that θ is monotone. Since $\bar{s}^* \subset \bar{b}$, for any $e \in \mathcal{V}$, there exists $\bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ with $\bar{s}^* \subset \bar{c} \subset \bar{b}$ such that $\phi(\bar{c}) \setminus \psi(\bar{s}^*) \subset e$. Put $\bar{d} = \langle \bar{c}(0), \ldots, \bar{c}(n-1)\rangle$. It is clear that $\bar{d} \in \mathcal{B}_{\mathcal{U}_0}(\bar{s}(0)) \times \cdots \times \mathcal{B}_{\mathcal{U}_{n-1}}(\bar{s}(n-1))$. And $\theta(\bar{d}) = \psi(\langle \bar{c}(0), \ldots, \bar{c}(n-1), \bar{b}(n) \cap t\rangle) \setminus \psi(\bar{s}^*) \subset \psi(\bar{c}) \setminus \psi(\bar{s}^*) \subset \phi(\bar{c}) \setminus \psi(\bar{s}^*) \subset e$, because $\bar{b}(n) \cap t = \bar{s}^*(n) \subset \bar{c}(n)$. Since $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = 0$ for each $0 \le i \le n-1$, $\mathcal{B}_{\mathcal{U}_i}(\bar{s}(i)) \equiv_T \mathcal{U}_i$, and so we get $\mathcal{V} \le_T \mathcal{U}_0 \times \cdots \times \mathcal{U}_{n-1}$. $\dashv$

Definition 52. Let $\mathcal{U}$ be an ultrafilter. The *P-point game on $\mathcal{U}$* is a two player game in which Players I and II alternatively choose sets a_n and s_n respectively, where $a_n \in \mathcal{U}$ and $s_n \in [a_n]^{<\omega}$. Together they construct the sequence

$$a_0, s_0, a_1, s_1, \ldots$$

Player I wins iff $\bigcup_{n \in \omega} s_n \notin \mathcal{U}$.

A proof of the following useful characterization of P-points in terms of the P-point game can be found in Bartoszyński and Judah [2].

Theorem 53. *An ultrafilter $\mathcal{U}$ is a P-point iff Player I does not have a winning strategy in the P-point game on $\mathcal{U}$.*

Lemma 54. *Let $\mathcal{U}_0, \ldots, \mathcal{U}_n \in \mathcal{C}$ and let $\mathcal{V}$ be a P-point. Suppose $\phi : \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n} \to \mathcal{V}$ is monotone and cofinal in $\mathcal{V}$. Let $\psi = \psi_\phi$. Suppose there exists $\bar{b} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ with the property that $\forall t \in [\omega]^{<\omega} \left[\psi(\bar{b}\restriction(n,t)) \notin \mathcal{V}\right]$. Then there is a map $\psi^* : [\omega]^{<\omega} \to [\omega]^{<\omega}$ such that*

(1) $\forall s, t \in [\omega]^{<\omega} [s \subset t \implies \psi^*(s) \subset \psi^*(t)]$

(2) For each $m \in \omega$ and $j \in \omega$, if $j \notin \psi^*(m)$, then there exists $n(j,m) > m$ such that $\forall t \in [\omega]^{<\omega} [t \cap [m, n(j,m)) = 0 \implies j \notin \psi^*(m \cup t)]$

(3) $\forall \bar{a} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n} \exists t \in [\pi^*(\bar{a}(n))]^{<\omega} [\psi^*(t) \cap \phi(\bar{a}) \neq 0]$.

Proof. First choose $e \in \mathcal{V}$ such that $\forall t \in [\omega]^{<\omega} \left[e \subset^* \omega \setminus \psi(\bar{b}\restriction(n,t))\right]$, and choose $\bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ such that $\bar{c} \subset \bar{b}$ and $\phi(\bar{c}) \subset e$. Now, for a fixed $t \in [\omega]^{<\omega}$, since $\bar{c}\restriction(n,t) \subset \bar{b}\restriction(n,t)$ and $\bar{c}\restriction(n,t) \subset \bar{c}$, $\psi(\bar{c}\restriction(n,t)) \subset \psi(\bar{b}\restriction(n,t)) \cap \psi(\bar{c}) \subset \psi(\bar{b}\restriction(n,t)) \cap \phi(\bar{c}) \subset \psi(\bar{b}\restriction(n,t)) \cap e$, which is finite. Thus we have produced $\bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ with the property that $\psi(\bar{c}\restriction(n,t))$ is finite for each $t \in [\omega]^{<\omega}$.

Now, if $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n) = 0$, then let $\mathcal{V}^* = \mathcal{U}_n$. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n) = \alpha + 1$, then let $\mathcal{V}^* \in \mathcal{C}_0$ and $\langle \mathcal{U}^*_m : m \in \omega\rangle \subset \mathcal{C}_\alpha$ be the ultrafilters fixed in Definition 43 such that $\mathcal{U}_n = \bigotimes_{\mathcal{V}^*} \mathcal{U}^*_m$. In either case $\mathcal{V}^*$ is a P-point. Notice also that for any $\bar{a} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$, $\pi^*(\bar{a}(n)) \in \mathcal{V}^*$. Define a strategy for Player I in the P-point game on $\mathcal{V}^*$ as follows. For $k \in \omega$, suppose that $d_l \in \mathcal{V}^*$ and $t_l \in [d_l]^{<\omega}$ have been given for $l < k$. Suppose also that $\bar{c}_l \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ is given for $l < k$ with the property that for each $i < n$, $\bar{c}_l(i) = \bar{c}(i)$. For each $m \le \max\left(\bigcup_{l<k} t_l\right)$ define $\bar{d}_m \in \mathcal{P}(\omega)^{n+1}$ by stipulating that $\bar{d}_m(i) = \bigcup_{l<k} \bar{c}_l\restriction(n, t_l \cap m)(i)$ for each $i < n+1$. For each $j < k$

and $m \leq \max\left(\bigcup_{l<k} t_l\right)$, if $j \notin \psi(\bar{d}_m)$, then choose $\bar{c}_{\langle m,j,k\rangle} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ such that $\bar{d}_m \subset \bar{c}_{\langle m,j,k\rangle}$ and $j \notin \phi(\bar{c}_{\langle m,j,k\rangle})$. Put $\bar{c}_k = \left(\bigcap_{l<k} \bar{c}_l\right) \cap \left(\bigcap_{\langle m,j\rangle \in F} \bar{c}_{\langle m,j,k\rangle}\right) \cap \bar{c}$, where $F = \left\{\langle m,j\rangle : m \leq \max\left(\bigcup_{l<k} t_l\right) \wedge j < k \wedge j \notin \psi(\bar{d}_m)\right\}$. Note that since $\bar{c}_l, \bar{c}_{\langle m,j,k\rangle}, \bar{c} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$, $\bar{c}_k \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ as well, and that $\pi^*(\bar{c}_k(n)) \in \mathcal{V}^*$. Notice also that for each $i < n$, $\bar{c}_k(i) = \bar{c}(i)$. Now, Player I plays $d_k = \pi^*(\bar{c}_k(n)) / \max\left(\bigcup_{l<k} t_l\right)$. Since this is not a winning strategy for Player I, there is a play

$$d_0, t_0, d_1, t_1, \ldots$$

in which Player I plays according to this strategy and looses. So $e = \bigcup_{l\in\omega} t_l \in \mathcal{V}^*$. Define $\bar{d} \in \mathcal{P}(\omega)^{n+1}$ by $\bar{d}(i) = \bigcup_{l\in\omega} \bar{c}_l \restriction (n, t_l)(i)$ for each $i < n+1$.

We first check that $\bar{d} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$. For $i < n$, $\bar{d}(i) = \bigcup_{l\in\omega} \bar{c}_l(i) = \bigcup_{l\in\omega} \bar{c}(i) = \bar{c}(i) \in \mathcal{B}_{\mathcal{U}_i}$. For $i = n$, there are two cases to consider. Suppose first that $\operatorname{rank}_{\mathcal{C}}(\mathcal{U}_n) = 0$. Then for any $l \in \omega$, $\bar{c}_l \restriction (n, t_l)(n) = \bar{c}_l(n) \cap t_l$. Since $t_l \subset d_l \subset \pi^*(\bar{c}_l(n)) = \bar{c}_l(n)$, $\bar{c}_l \restriction (n, t_l)(n) = t_l$. Therefore, $\bar{d}(n) = \bigcup_{l\in\omega} t_l = e \in \mathcal{U}_n = \mathcal{B}_{\mathcal{U}_n}$. Suppose next that $\operatorname{rank}_{\mathcal{C}}(\mathcal{U}_n) = \alpha + 1$. Recall that $\mathcal{U}_n = \bigotimes_{\mathcal{V}^*} \mathcal{U}^*_m$. Let us view $\bar{d}(n)$ and $\bar{c}_l(n)$ as subsets of $\omega \times \omega$. Since $e \in \mathcal{V}^*$, it is enough to show that for each $m \in e$, $\bar{d}(n)(m) \in \mathcal{B}_{\mathcal{U}^*_m}$ and that for each $m \notin e$, $\bar{d}(n)(m) = 0$. Observe that for any $m \in \omega$, if $i \in \bar{d}(n)(m)$, then there exists unique $l \in \omega$ such that $m \in t_l$ and $i \in \bar{c}_l(n)(m)$. Therefore, if $i \in \bar{d}(n)(m)$, then $m \in e$, whence $m \notin e \implies \bar{d}(n)(m) = 0$. Next suppose that $m \in e$. There is a unique $l \in \omega$ such that $m \in t_l$. Since $t_l \subset d_l \subset \pi^*(\bar{c}_l(n)) = \{m \in \omega : \bar{c}_l(n)(m) \neq 0\}$ and since $\bar{c}_l(n) \in \mathcal{B}_{\mathcal{U}_n}$, it follows that $\bar{c}_l(n)(m) \in \mathcal{B}_{\mathcal{U}^*_m}$. We will show that $\bar{d}(n)(m) = \bar{c}_l(n)(m)$. The uniqueness of l ensures that $\bar{d}(n)(m) \subset \bar{c}_l(n)(m)$. On the other hand, if $i \in \bar{c}_l(n)(m)$, then $\langle m, i\rangle \in \{m\} \times \bar{c}_l(n)(m)$, and since $m \in t_l$, $\{m\} \times \bar{c}_l(n)(m) \subset \bar{c}_l \restriction (n, t_l)(n) \subset \bar{d}(n)$, whence $i \in \bar{d}(n)(m)$.

Next, we show that $\bar{d} \subset \bar{c}$. Indeed, if $i < n$, then $\bar{d}(i) = \bar{c}(i)$. For n, notice that $\bar{c}_l \restriction (n, t_l) \subset \bar{c}_l \subset \bar{c}$. Therefore, $\bar{d}(n) = \bigcup_{l\in\omega} \bar{c}_l \restriction (n, t_l)(n) \subset \bar{c}(n)$. Thus, for any $s \in [\omega]^{<\omega}$, $\bar{d} \restriction (n, s) \subset \bar{c} \restriction (n, s)$, and thus $\psi(\bar{d} \restriction (n, s))$ is finite. Therefore, we may define $\psi^* : [\omega]^{<\omega} \to [\omega]^{<\omega}$ by $\psi^*(s) = \psi(\bar{d} \restriction (n, s))$. It is easy to check that for any $s \in [\omega]^{<\omega}$ and any $0 \leq i \leq n$, $\bar{d} \restriction (n, s)(i) = \bigcup_{l\in\omega} \bar{c}_l \restriction (n, t_l \cap s)(i)$. Moreover, if $s \subset t$, then $\bar{d} \restriction (n, s) \subset \bar{d} \restriction (n, t)$, and so $\psi^*(s) \subset \psi^*(t)$, whence (1) holds.

To check that (2) holds, fix $m, j \in \omega$ with $j \notin \psi^*(m)$. Fix $k \in \omega$ such that $m \leq \max(\bigcup_{l<k} t_l)$ and $j < k$. Note that $0 < k$. Put $n(j, m) = \max\left(\bigcup_{l<k} t_l\right) + 1$. Note that $m < n(j, m)$. Fix $t \in [\omega]^{<\omega}$ such that $t \cap [m, n(j, m)) = 0$. We must show that $j \notin \psi(\bar{d} \restriction (n, m \cup t))$. For any $i < n$, $\bar{d} \restriction (n, m)(i) = \bar{d}(i) = \bar{c}(i) = \bigcup_{l<k} \bar{c}_l \restriction (n, t_l \cap m)(i)$ because $0 < k$. And $\bar{d} \restriction (n, m)(n) = \bigcup_{l\in\omega} \bar{c}_l \restriction (n, t_l \cap m)(n) = \bigcup_{l<k} \bar{c}_l \restriction (n, t_l \cap m)(n)$ because for $l \geq k$, $t_l \cap m = 0$ and so $\bar{c}_l \restriction (n, t_l \cap m)(n) = \bar{c}_l \restriction (n, 0)(n) = 0$. So we conclude that for any $i < n+1$, $\bar{d} \restriction (n, m)(i) = \bigcup_{l<k} \bar{c}_l \restriction (n, t_l \cap m)(i)$. Since $j \notin \psi(\bar{d} \restriction (n, m))$ and since $\bar{d}_m(i)$ as computed at stage k is equal to $\bigcup_{l<k} \bar{c}_l \restriction (n, t_l \cap m)(i)$, $\bar{c}_{\langle m,j,k\rangle}$ exists. Moreover, $\bar{c}_k \subset \bar{c}_{\langle m,j,k\rangle}$ and $j \notin \psi\left(\bar{c}_{\langle m,j,k\rangle}\right)$. It suffices to show that $\bar{d} \restriction (n, m \cup t) \subset \bar{c}_{\langle m,j,k\rangle}$. Fix $i < n+1$ and note that $\bar{d} \restriction (n, m \cup t)(i) = \bigcup_{l\in\omega} \bar{c}_l \restriction (n, (t_l \cap m) \cup (t_l \cap t))(i)$. When $l < k$, $(t_l \cap m) \cup (t_l \cap t) = t_l \cap m$. For otherwise, there would be $i^* \notin t_l \cap m$ such that $i^* \in t_l \cap t$. Therefore, $i^* \geq m$, and $i^* \leq \max\left(\bigcup_{l<k} t_l\right) < n(j, m)$, contradicting the assumption that $t \cap [m, n(j, m)) = 0$. Therefore, $\bar{d} \restriction (n, m \cup t)(i) = \bigcup_{l<k} \bar{c}_l \restriction (n, t_l \cap m)(i) \cup \bigcup_{l\geq k} \bar{c}_l \restriction (n, (t_l \cap m) \cup (t_l \cap t))(i)$. Note that $\bar{d}_m \subset \bar{c}_{\langle m,j,k\rangle}$ and that $\bar{d}_m(i) = \bigcup_{l<k} \bar{c}_l \restriction (n, t_l \cap m)(i)$. Next, for each $l \geq$

k, $(t_l \cap m) \cup (t_l \cap t) \subset t_l$, and so $\bar{c}_l \restriction (n, (t_l \cap m) \cup (t_l \cap t)) \subset \bar{c}_l \restriction (n, t_l) \subset \bar{c}_l \subset \bar{c}_k \subset \bar{c}_{\langle m,j,k \rangle}$. So $\bar{d} \restriction (n, m \cup t)(i) \subset \bar{c}_{\langle m,j,k\rangle}(i)$, as needed.

Next, to verify (3), fix $\bar{a} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$. Since $\bar{d} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$, $\bar{a} \cap \bar{d} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$. Applying Lemma 41, choose $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{a} \cap \bar{d}$ such that $\psi(\bar{s}) \neq 0$. Choose $t \in \left[\pi^*\left(\left(\bar{a} \cap \bar{d}\right)(n)\right)\right]^{<\omega}$ such that $\bar{s} \subset \left(\bar{a} \cap \bar{d}\right) \restriction (n, t)$. Since $\pi^*\left(\left(\bar{a} \cap \bar{d}\right)(n)\right) \subset \pi^*(\bar{a}(n))$, $t \in [\pi^*(\bar{a}(n))]^{<\omega}$. Since $\left(\bar{a} \cap \bar{d}\right) \restriction (n,t) \subset \bar{d} \restriction (n,t)$, $\bar{s} \subset \bar{d} \restriction (n,t)$, and so, $\psi(\bar{s}) \subset \psi(\bar{d} \restriction (n,t)) = \psi^*(t)$. Also, since $\bar{s} \subset \bar{a} \cap \bar{d} \subset \bar{a}$, $\psi(\bar{s}) \subset \psi(\bar{a}) \subset \phi(\bar{a})$. Hence $\psi^*(t) \cap \phi(\bar{a}) \neq 0$, as needed. $\dashv$

Theorem 55. *Let $\mathcal{U}_0, \dots, \mathcal{U}_n \in \mathcal{C}$ and let $\mathcal{V}$ be a selective ultrafilter. Suppose that $\mathcal{V} \leq_T \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$. Then $\mathcal{V} \leq_{RK} \mathcal{U}_i$ for some $0 \leq i \leq n$.*

Proof. We may assume that $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_0) \leq \cdots \leq \mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n)$. The proof is by double induction, first on $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n)$, and then on

$$m = \left|\{i < n+1 : \mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = \mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n)\}\right|.$$

Fix $\phi : \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n} \to \mathcal{V}$ that is monotone and cofinal in $\mathcal{V}$. Put $\psi = \psi_\phi$ and note that the hypotheses of Lemma 51 are satisfied. Therefore, one of (1)–(3) of Lemma 51 holds. Assume first that (3) holds. Then $\forall i < n+1\, [\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = 0]$, $n > 0$, and $\mathcal{V} \leq_T \mathcal{U}_0 \times \cdots \times \mathcal{U}_{n-1}$. Now, $|\{i < n : \mathrm{rank}_{\mathcal{C}}(\mathcal{U}_i) = 0\}| = n < n+1 = m$. So by the inductive hypothesis, there is $i < n$ such that $\mathcal{V} \leq_{RK} \mathcal{U}_i$.

Next, suppose that (1) of Lemma 51 holds. Note that this means that the hypotheses of Lemma 54 are satisfied. Fix a map $\psi^* : [\omega]^{<\omega} \to [\omega]^{<\omega}$ that satisfies (1)–(3) of Lemma 54. Now, put $D = \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n) = 0$, then let $\mathcal{V}^* = \mathcal{U}_n$. If $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n) = \alpha + 1$, then let $\mathcal{V}^* \in \mathcal{C}_0$ and $\langle \mathcal{U}^*_m : m \in \omega \rangle \subset \mathcal{C}_\alpha$ be the ultrafilters fixed in Definition 43 such that $\mathcal{U}_n = \bigotimes_{\mathcal{V}^*} \mathcal{U}^*_m$. Note that in either case $\mathcal{V}^* \leq_{RK} \mathcal{U}_n$. Define $\chi^* : D \to \mathcal{V}^*$ by $\chi^*(\bar{a}) = \pi^*(\bar{a}(n))$ for each $\bar{a} \in D$. χ^* is monotone and cofinal in $\mathcal{V}^*$. So $\phi : D \to \mathcal{V}$, $\chi^* : D \to \mathcal{V}^*$, and $\psi^* : [\omega]^{<\omega} \to [\omega]^{<\omega}$ satisfy the hypotheses of Lemma 9, whence $\mathcal{V} \leq_{RB} \mathcal{V}^* \leq_{RK} \mathcal{U}_n$.

Finally, suppose that (2) of Lemma 51 holds. Then $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}_n) = \alpha + 1$. So let $\mathcal{V}^* \in \mathcal{C}_0$ and $\langle \mathcal{U}^*_m : m \in \omega \rangle \subset \mathcal{C}_\alpha$ be the ultrafilters fixed in definition 43 such that $\mathcal{U}_n = \bigotimes_{\mathcal{V}^*} \mathcal{U}^*_m$. Assume for a contradiction that for each $i < n+1$, $\mathcal{V} \not\leq_{RK} \mathcal{U}_i$. By the inductive hypothesis and Lemma 39, this means that $\forall i < n\, [\mathcal{V} \not\leq_T \mathcal{U}_i]$, that $e = \{m \in \omega : \mathcal{V} \not\leq_T \mathcal{U}^*_m\} \in \mathcal{V}^*$, and that $\mathcal{V} \not\leq_T \mathcal{V}^*$. Using Lemma 49 find $\bar{a} \in \mathcal{B}_{\mathcal{U}_0} \times \cdots \times \mathcal{B}_{\mathcal{U}_n}$ such that

(1) $\forall i < n \forall s \in [\bar{a}(i)]^{<\omega} \forall \mathcal{W} \in \mathcal{F}(\mathcal{U}_i, s)\, [\mathcal{V} \not\leq_T \mathcal{W}]$.
(2) $\forall m \in e \left[\pi^{-1}(\bar{a}(n))(m) \in \mathcal{B}_{\mathcal{U}^*_m}\right]$.
(3) $\forall m \in e \forall s \in \left[\pi^{-1}(\bar{a}(n))(m)\right]^{<\omega} \forall \mathcal{W} \in \mathcal{F}(\mathcal{U}^*_m, s)\, [\mathcal{V} \not\leq_T \mathcal{W}]$.
(4) $\forall m \notin e \left[\pi^{-1}(\bar{a}(n))(m) = 0\right]$.

Now apply (2) of Lemma 51 to find $k \in \omega$, $\bar{s} \in \left([\omega]^{<\omega}\right)^{n+1}$ with $\bar{s} \subset \bar{a}$, and $\{\mathcal{U}^i_{l^*} : i < n+k+1 \wedge l^* \leq l_i\} \subset \mathcal{C}$, where $l_i \in \omega$, satisfying (2)(a)–(2)(e). Observe that for any $i < n$ and $l^* \leq l_i$, $\mathcal{U}^i_{l^*} \in \mathcal{F}(\mathcal{U}_i, \bar{s}(i))$. Since $\bar{s}(i) \in [\bar{a}(i)]^{<\omega}$, $\mathcal{V} \not\leq_T \mathcal{U}^i_{l^*}$. Next, recall that $\mathcal{F}(\mathcal{U}_n, \bar{s}(n)) = \{\mathcal{U}_n\} \cup \{\mathcal{V}^*\} \cup \{\mathcal{U}^*_m : m \in t\} \cup \bigcup_{m \in t} \mathcal{F}(\mathcal{U}^*_m, \pi^{-1}(\bar{s}(n))(m))$, where $t = \{m \in \omega : \pi^{-1}(\bar{s}(n))(m) \neq 0\}$. Note that if $m \in t$, then since $\pi^{-1}(\bar{s}(n))(m) \subset \pi^{-1}(\bar{a}(n))(m)$, $m \in e$. Therefore, for each $m \in t$, $\mathcal{V} \not\leq_T \mathcal{U}^*_m$, and for each $\mathcal{W} \in \mathcal{F}(\mathcal{U}^*_m, \pi^{-1}(\bar{s}(n))(m))$, $\mathcal{V} \not\leq_T \mathcal{W}$ because $\pi^{-1}(\bar{s}(n))(m) \in \left[\pi^{-1}(\bar{a}(n))(m)\right]^{<\omega}$. Note that for each $j < k+1$ and $l^* \leq l_{n+j}$, $\mathcal{U}^{n+j}_{l^*} \in \mathcal{F}(\mathcal{U}_n, \bar{s}(n))$, and $\mathrm{rank}_{\mathcal{C}}(\mathcal{U}^{n+j}_{l^*}) <$

$\operatorname{rank}_{\mathcal{C}}(\mathcal{U}_n)$. So since $\mathcal{V} \not\leq_T \mathcal{V}^*$, it follows that $\mathcal{V} \not\leq_T \mathcal{U}^i_{l^*}$ for any $i < n+k+1$ and $l^* \leq l_i$. However, put $\beta = \max\{\operatorname{rank}_{\mathcal{C}}(\mathcal{U}^i_{l^*}) : i < n+k+1 \wedge l^* \leq l_i\}$. Clearly, $\beta \leq \alpha+1$. If $\beta = \alpha + 1$, then $\left|\{\langle i, l^*\rangle : i < n+k+1 \wedge l^* \leq l_i \wedge \operatorname{rank}_{\mathcal{C}}(\mathcal{U}^i_{l^*}) = \alpha+1\}\right| < m$. Therefore, the induction hypothesis applies, and implies that $\mathcal{V} \leq_{RK} \mathcal{U}^i_{l^*}$ for some $i < n+k+1$ and $l^* \leq l_i$. This a contradiction, which completes the proof. $\dashv$

Corollary 56. *Let $\mathcal{U} \in \mathcal{K}$ and let $\mathcal{V}$ be a selective ultrafilter. If $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V} \leq_{RK} \mathcal{U}$.*

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