UNIT - III

RANDOM PROCESS

2M **Definition:**

Let 's' be a sample space of a random experiment. A random process is a mapping x which assigns to every outcome $s \in S$ a real valued function of time, $x(t,s)$ i.e.,

$$x(s) = x(t,s)$$

The family of all such time functions is denoted by $x(t,s)$ and is called a random process.

Example:

If $S = \{H, T\}$ then x assigning H into $x(t,H) = \sin t$ and T into $x(t,T) = \cos t$

4M **Classifications of Random process:**

→ Random processes can be classified into four types.

1. **Discrete Random process:**

If S is discrete and T is continuous then the random process is called discrete random process.

Example:

Let $x(t)$ denotes the no. of telephone calls received in the interval $(0,t)$.

Here $S = \{1, 2, 3 ---\}$

$T = \{t ; t \geq 0\}$

Here $x(t)$ is a discrete random process.

2. **Continuous Random process:**

If both S and T are continuous then the random process is called continuous random process.

Example:

Let $x(t)$ = maximum temperature at a place in the interval $(0,t)$. Here S is a continuous set (representing temperature values) and T is a continuous set (representing time values).

$x(t)$ is a continuous random process.

3. Discrete Random Sequence:

If both S and T are discrete, the random process is called a discrete random sequence.

Example:

Let X_n denote the outcome of the n^{th} toss of a fair die.

Here $S = \{1, 2, 3, 4, 5, 6\}$

$T = \{1, 2, 3, ------\}$

So $\{X_n, n = 1, 2, 3, ---\}$ is a discrete random sequence.

4. Continuous Random Sequence:

If S is a continuous and T is discrete then the random process is called continuous random sequence.

Example:

Let X_n = Temperature at the end of the n^{th} hour of a day.

Then S is a continuous set (all possible values of temperature).

$T = \{1, 2, ---- 24\}$

$\therefore \{X_n, n = 1, 2, 3, --- 24\}$ is a continuous random sequence.

Mean:

The mean of the random process, $x(t)$ is the expected value of the random variable $x(t)$ at any time 't' i.e.,

$$\mu_x(t) = E[x(t)]$$

2M Auto Correlation:

The auto correlation of the process $\{x(t) : t \in T\}$ defined by $R_x(t_1, t_2)$ or $R(t_1, t_2)$ is the expected value of the product of $x(t_1)$ & $x(t_2)$ of the process.

$$R(t_1, t_2) = E[x(t_1) \cdot x(t_2)]$$

2M Auto covariance:

The auto covariance of the process $\{x(t) ; t \in T\}$ denoted by $C_x(t_1, t_2)$ or $C(t_1, t_2)$ is defined as,

$$C(t_1, t_2) = R(t_1, t_2) - \mu(t_1) \cdot \mu(t_2)$$

2M **Auto Correlation Coefficient:**

The auto correlation coefficient of the process $\{x(t); t \in T\}$ denoted by,

$P_x(t_1, t_2)$ or $\rho(t_1, t_2)$ is defined as;

$$\rho(t_1, t_2) = \frac{c(t_1, t_2)}{\sqrt{c(t_1, t_1)(t_2, t_2)}}.$$

Stationary process:

A random process is said to be stationary if its mean, variance are constants. Other process is called non-stationary.

Problems:

4M

1. A random process $x(t)$ is given by $x(t) = y\cos 2\pi t, t>0$ & where 'y' is a random variable with $E(y) = 1$ is the process is stationary.

Soln: Given,

$$x(t) = y\cos 2\pi t; \ t>0 \ ; \ E(y)=1$$

$$E(x(t)) = E[y\cos 2\pi t]$$
$$= E[y]\cos 2\pi t$$
$$= 1.\cos 2\pi t$$
$$E[x(t)] = \cos 2\pi t \neq \text{constant}$$

$\therefore$ It is non-stationary process.

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2. If the random process $x(t)$ takes the value $[-1]$ with probability - $1/3$ & takes the value $[1]$ with probability - $2/3$. Find $x(t)$ is a stationary or not.

Soln: The random process $x(t)$ with probability,

$x(t) = n$	-1	1
$P_n(t)$	$1/3$	$2/3$

$$E[x(t)] = \Sigma n.P_n(t)$$
$$= (-1)\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right)$$
$$= -\frac{1}{3} + \frac{2}{3}$$
$$= 1/3$$

$E[x(t)] = 1/3 = \text{constant}$

$E[x^2(t)] = \sum n^2 P_n(t)$

$= (-1)^2\left(\frac{1}{3}\right) + (1)^2\left(\frac{2}{3}\right)$

$= \frac{1}{3} + \frac{2}{3}$

$= \frac{3}{3} = 1$

$E[x^2(t)] = 1 = \text{constant}$

$\therefore$ Variance $= E(x^2(t)) - [E[x(t)]]^2$

$= 1 - \left[\frac{1}{3}\right]^2$

$= 1 - 1/9$

Variance $= 8/9 = \text{constant}$

$\therefore$ It is a stationary process.

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3. 12M The process $\{x(t)\}$ whose probability distribution is given by, $p[x(t) = n] = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1,2,3. \\ \frac{at}{1+at}, & n = 0 \end{cases}$

Soln: The probability distribution of $x(t)$ is,

$x(t) = n$	0	1	2	3
$P_n(t)$	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$

Mean $E[x(t)] = \sum_{n=0}^{\infty} n P_n(t)$

$= 0 + \frac{1}{(1+at)^2} + 2\left(\frac{at}{(1+at)^3}\right) + 3\left(\frac{(at)^2}{(1+at)^4}\right)$

$= \frac{1}{(1+at)^2}\left[1 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2\right]$

$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{(1+at)}\right]^{-2}$ $\quad [\because [1-x]^{-2} = 1 + 2x + 3x^2 \ldots]$

$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$

$= \frac{1}{\cancel{(1+at)^2}}\left[\cancel{(1+at)^2}\right]$

$= 1$

$\therefore E[x(t)] = 1$

$$E[x^2(t)] = \sum_{n=0}^{\infty} n^2 \cdot P_n(t)$$

$$= \sum_{n=0}^{\infty} [n(n+1)-n] P_n(t)$$

$$= \sum_{n=0}^{\infty} n(n+1) P_n(t) - \sum_{n=0}^{\infty} n \cdot P_n(t)$$

$$= \sum_{n=0}^{\infty} n(n+1) \cdot P_n(t) - 1$$

$$= \left[0 + 2\left(\frac{1}{(1+at)^2}\right) + 6\left(\frac{at}{(1+at)^3}\right) + 12\left(\frac{(at)^2}{(1+at)^4}\right)\right] - 1$$

$$= \left[\frac{2}{(1+at)^2}\left(1 + 3\left(\frac{at}{1+at}\right) + 6\left(\frac{(at)}{(1+at)}\right)^2\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-3} - 1 \qquad \left[\because [1-x]^{-3} = 1+3x+6x^2+\ldots\right]$$

$$= \frac{2}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{\cancel{(1+at)^2}}[1+at]^{\cancel{3}} - 1$$

$$= 2[1+at] - 1$$

$$= 2 + 2at - 1$$

$$E[x^2(t)] = 1 + 2at$$

$$\text{Variance}[x(t)] = E[x^2(t)] - [E(x(t))]^2$$

$$= 1 + 2at - (1)^2$$

$$= \cancel{1} + 2at - \cancel{1}$$

$$\text{var}[x(t)] = 2at$$

$\therefore$ It isn't constant.

$\therefore$ It is a non-stationary process.

** 4. 4M If the random process $x(t) = \cos(t+\phi)$ where 'ϕ' is a random variable with density function $f(\phi) = 1/\pi$, where $(-\frac{\pi}{2} < \phi < \pi/2)$. Check whether the process is stationary or not.

Soln: Given, $x(t) = \cos(t+\phi)$

$$f(\phi) = \frac{1}{\pi}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}.$$

$$E[x(t)] = E[\cos(t+\phi)]$$

$$= \int_{-\pi/2}^{\pi/2} \cos(t+\phi) \cdot f(\phi)\, d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \cos(t+\phi) \cdot \frac{1}{\pi}\, d\phi$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\, d\phi$$

$$= \frac{1}{\pi} \left[\sin(t+\phi)\right]_{-\pi/2}^{\pi/2} \qquad [\because \sin(\pi/2+\theta) = \cos\theta]$$

$$= \frac{1}{\pi}\left[\sin(t+\pi/2) - \sin(t-\pi/2)\right]$$

$$= \frac{1}{\pi}\left(\sin(\pi/2+t) - \sin-(\pi/2-t)\right)$$

$$= \frac{1}{\pi}\left[\sin(\pi/2+t) + \sin(\pi/2-t)\right]$$

$$= \frac{1}{\pi}\left[\cos t + \cos t\right]$$

$$= \frac{1}{\pi}\left[2\cos t\right]$$

$$E[x(t)] = \frac{2}{\pi}\cos t$$

$\therefore$ It isn't a constant value.

$\therefore$ It is a non-stationary process.

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5. HM If the random process $x(t) = \sin(t+\phi)$ where 'ϕ' is a random variable with density function $f(\phi) = \frac{1}{\pi}$, where $\left(-\frac{\pi}{2} < \phi < \pi/2\right)$. Check whether the process is stationary or not.

Soln: Given, $x(t) = \sin(t+\phi)$

$$f(\phi) = \frac{1}{\pi}, \left(\frac{-\pi}{2} < \phi < \frac{\pi}{2}\right)$$

$$E[x(t)] = E[\sin(t+\phi)]$$

$$= \int_{-\pi/2}^{\pi/2} \sin(t+\phi) \cdot f(\phi)\, d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \sin(t+\phi) \cdot \frac{1}{\pi}\, d\phi$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin(t+\phi)\, d\phi$$

$$= \frac{1}{\pi}\left[-\cos(t+\phi)\right]_{-\pi/2}^{\pi/2} \qquad [\because \cos(\pi/2+\theta) = \sin\theta]$$

$$= \frac{1}{\pi}\left[-\cos(t+\pi/2) + \cos(t-\pi/2)\right]$$

$= \frac{1}{\pi}\left[-\cos(\pi/2+t) + \cos-(\pi/2-t)\right]$

$= \frac{1}{\pi}\left[-\cos(\pi/2+t) + (-\cos(\pi/2-t))\right]$

$= \frac{-1}{\pi}\left[\sin t + \sin t\right]$

$= \frac{-1}{\pi}\left[2\sin t\right]$

$E[x(t)] = \frac{-2}{\pi}\sin t$

∴ It isn't a constant value.

∴ It is a non-stationary process.

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* Wide Sense Stationary Process :

A random process $x(t)$ is said to be wide sense stationary (WSS), if the following conditions are satisfied.

* $\mu_x = E[x(t)] \rightarrow$ Mean is constant

* $E[x(t)+(t+\tau)] = R_{xx}(\tau)$

Then the auto correlation process function only depends on the time difference.

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6. 4M If the process $x(t) = V\cos t + (V+1)\sin t$ then $E(u) = E(v) = 0$ & $E(u^2) = E(v^2) = 1$. Find the process is wide sense stationary process or not.

Soln: Given,

$x(t) = V\cos t + (V+1)\sin t$

$E(u) = E(v) = 0$

$E(u^2) = E(v^2) = 1$

$E[x(t)] = E[V\cos t + (V+1)\sin t]$

$= E[u\cos t + V\sin t + \sin t]$

$= E(u)\cos t + E(v)\sin t + \sin t$

$= 0 + 0 + \sin t$

$E[x(t)] = \sin t$

∴ It isn't constant value.

∴ It is not a wide sense stationary process.

*** 7. 12M

Show that the random process, $x(t) = A\cos(\omega t+\theta)$ is a wide sense stationary, where A & ω are constants & 'θ' is uniformly distributed on the interval $[0, 2\pi]$

Soln: Given, $x(t) = A\cos(\omega t+\theta)$

- The probability density function of $f(\theta) = \frac{1}{b-a}$

$$f(\theta) = \frac{1}{2\pi - 0}$$

$$\boxed{f(\theta) = \frac{1}{2\pi}}$$

$$E[x(t)] = E[A\cos(\omega t+\theta)]$$

$$= \int_0^{2\pi} A\cos(\omega t+\theta) \cdot \tfrac{1}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi}\int_0^{2\pi} \cos(\omega t+\theta)\, d\theta$$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\cos(A-B) = \cos A \cos B + \sin A \sin B$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$
$\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$

$$= \frac{A}{2\pi}\left[\sin(\omega t+\theta)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi}\left[\sin(\omega t+2\pi) - \sin(\omega t)\right]$$

$\sin(2\pi+\theta) = \sin\theta$
$\sin(4\pi+\theta) = \sin\theta$

$$= \frac{A}{2\pi}\left[\sin\omega t - \sin\omega t\right]$$

$$E[x(t)] = 0$$

$$E[x(t)+(t+\tau)] = E[A\cos(\omega t+\theta) \cdot A\cos(\omega(t+\tau)+\theta)]$$

$$= E[A^2\cos(\omega t+\theta) \cdot \cos(\omega t+\theta+\omega\tau)]$$

$$= \int_0^{2\pi} A^2\cos(\omega t+\theta) \cdot \cos(\omega t+\theta+\omega\tau)] \cdot \frac{1}{2\pi} \cdot d\theta$$

$$= \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}\left[\cos(\omega t+\theta+\omega t+\theta+\omega\tau) + \cos(\cancel{\omega t}+\cancel{\theta}-\cancel{\omega t}-\cancel{\theta}-\omega\tau)\right] d\theta$$

$$= \frac{A^2}{4\pi}\int_0^{2\pi}[\cos(2\omega t+2\theta+\omega\tau) + \cos\omega\tau)] \cdot d\theta$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin(2\omega t+2\theta+\omega\tau)}{2} + \cos\omega\tau\theta\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[\left(\frac{\sin 2\omega t+4\pi+\omega\tau)}{2} + 2\pi\cos\omega\tau\right) - \frac{\sin 2\omega t+\omega\tau}{2}\right]$$

$$= \frac{A^2}{4\pi}\left[\cancel{\frac{\sin(2\omega t+\omega\tau)}{2}} + 2\pi\cos\omega\tau - \cancel{\frac{\sin(2\omega t+\omega\tau)}{2}}\right]$$

$$= \frac{A^2}{\cancel{4}_2\cancel{\pi}}\left[\cancel{2}\cancel{\pi}\cos\omega\tau\right]$$

$$= \frac{A^2}{2}(\cos\omega\tau)$$

8. Show that the random process $X(t) = A\sin(\omega t+\theta)$ is a wide sense stationary, where A & ω are constants & 'θ' is uniformly distributed on the interval $[0, 2\pi]$

Soln: Given, $X(t) = A\sin(\omega t+\theta)$

The probability density function of $f(\theta) = \frac{1}{b-a}$

$$f(\theta) = \frac{1}{2\pi - 0}$$

$$\boxed{f(\theta) = 1/2\pi}$$

$$E[X(t)] = E[A\sin(\omega t+\theta)]$$

$$= \int_0^{2\pi} A\sin(\omega t+\theta)\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{A}{2\pi}\int_0^{2\pi}\sin(\omega t+\theta)\,d\theta$$

$$= \frac{A}{2\pi}\left[-\cos(\omega t+\theta)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi}\left[-\cos(\omega t+2\pi)+\cos(\omega t)\right]$$

$$= \frac{A}{2\pi}\left[-\cos\omega t+\cos\omega t\right]$$

$$E[X(t)] = 0$$

$$E[X(t)+(t+\tau)] = E[A\sin(\omega t+\theta)\cdot A\sin(\omega(t+\tau)+\theta)]$$

$$= E[A^2\sin(\omega t+\theta)\cdot\sin(\omega t+\theta+\omega\tau)]$$

$$= \int_0^{2\pi} A^2\sin(\omega t+\theta)\cdot\sin(\omega t+\theta+\omega\tau)]\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{A^2}{2\pi}\int_0^{2\pi}\frac{1}{2}\left[\cos(\cancel{\omega t}+\cancel{\theta}-\cancel{\omega t}-\cancel{\theta}-\omega\tau)-\cos(\omega t+\theta+\omega t+\theta+\omega\tau)\right]d\theta$$

$$\left[\sin A\,\sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]\right]$$

$$= \frac{A^2}{2\pi}\int_0^{2\pi}\frac{1}{2}\left[\cos\omega\tau-\cos(2\omega t+2\theta+\omega\tau)\right]d\theta$$

$$= \frac{A^2}{4\pi}\left[-\frac{\sin(2\omega t+2\theta+\omega\tau)}{2}+\cos\omega\tau\,\theta\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[\left(\cos\omega\tau(2\pi)-\frac{\sin(2\omega t+4\pi+\omega\tau)}{2}\right)-\left(-\frac{\sin(2\omega t+\omega\tau)}{2}\right)\right]$$

$$= \frac{A^2}{4\pi}\left[2\pi\cos\omega\tau-\frac{\sin(2\omega t+\omega\tau)}{\cancel{2}}+\frac{\sin(2\omega t+\omega\tau)}{\cancel{2}}\right]$$

$$= \frac{A^2}{\cancel{4\pi}_2}\left[\cancel{2\pi}\cos\omega\tau\right]$$

$$= \frac{A^2}{2}(\cos\omega\tau)$$

* **Poisson's Distribution Process:**

If $x(t)$ represents the no. of occurance of a certain even in $(0,t)$ then the discrete random process $\{x(t)\}$ is called poisson process.

$$p[x(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

* **Properties of poisson process:**

→ The poisson process is a markovprocess.

→ The sum of two independent poisson processes is again a poisson process.

→ The difference of two independent poisson processes is not a poisson process.

* **Theorem 1:**

q. Proove that the difference of two independent poisson processes isn't a poisson process.

Proof:

Let $x_1(t)$ & $x_2(t)$ is two poisson process with mean λ_1 & λ_2 respectively.

$$x(t) = x_1(t) - x_2(t)$$

$$E[x(t)] = E[x_1(t) - x_2(t)]$$
$$= E[x_1(t)] - E[x_2(t)]$$
$$= \lambda_1 t - \lambda_2 t$$
$$E[x(t)] = (\lambda_1 - \lambda_2)t$$

$var[x(t)] = E[x^2(t)] - E[x(t)]^2$

$E[x^2(t)] = var[x(t)] + E[x(t)]^2$

$E[x_1^2(t)] = \lambda_1 t + \lambda_1^2 t^2$

$E[x_2^2(t)] = \lambda_2 t + \lambda_2^2 t^2$

$$E[x^2(t)] = E[(x_1(t) - x_2(t))]^2$$
$$= E[(x_1(t))^2 + (x_2(t))^2 - 2x_1(t).x_2(t)]$$
$$= E[x_1(t)]^2 + E[x_2(t)]^2 - 2x_1(t).x_2(t)$$
$$= \lambda_1 t + \lambda_1^2 t^2 + \lambda_2 t + \lambda_2^2 t^2 - 2\lambda_1 t . \lambda_2 t$$
$$= (\lambda_1 + \lambda_2)t + (\lambda_1^2 + \lambda_2^2)t^2 - 2\lambda_1 t \lambda_2 t$$
$$E[x^2(t)] = (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2 \neq (\lambda_1 - \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

∴ Hence, proved.

∴ The difference of two independent poisson processes isn't a poisson process.

2M*
10. Is the poisson process is stationary?

Soln: Let x(t) be a poisson process,

$$E[x(t)] = \lambda t \neq \text{constant}$$

$\therefore$ It is a not a stationary process.

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11. Suppose the customer arraive at a bank, according to a poisson process with mean rate of 3 per min. Find the probability that during a time interval of 2 min.

(i) Exactly 4 customers arraived.

(ii) Greater than 4 customers arraived.

(iii) Fewer than 4 customers arraived.

Soln: Let x(t) be the customer arraived at a bank, in the time interval 't'. x(t) follows the poisson's process.

(i) $P[x(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ $[\lambda=3, t=2]$

$n=4;$ $= \frac{e^{-3(2)}[3(2)]^4}{4!}$

$= \frac{e^{-6}(6)^4}{24}$

$= \frac{e^{-6}\cdot 6\times \cancel{6}^{3}\times \cancel{6}^{3}\times \cancel{6}}{\cancel{24}\,\cancel{4}\cancel{2}}$

$= (0.0024)(54)$

$= 0.13$ //

(ii) $P[x(t)>4] \Rightarrow [1-P(x(t)\leq 4]$

$$= 1-\left[\frac{e^{-3(2)}(6)^0}{0!}+\frac{e^{-6}(6)^1}{1!}+\frac{e^{-6}(6)^2}{2!}+\frac{e^{-6}(6)^3}{3!}+\right]\frac{e^{-6}(6)^4}{4!}\Bigg]$$

$$= 1-\left(\frac{e^{-6}(1)}{1}+\frac{e^{-6}(6)}{1}+\frac{e^{-6}(\cancel{36}^{18})}{\cancel{2}}+\frac{e^{-6}(6\times 6\times \cancel{6}^{2})}{\cancel{3}\times\cancel{2}\times 1}+\frac{e^{-6}(6\times\cancel{6}^{3}\times\cancel{6}^{3}\times\cancel{6})}{{}_{2}\cancel{4}\times\cancel{3}\times\cancel{2}\times 1}\right)$$

$$= 1-\left[\frac{e^{-6}}{1}+\frac{6e^{-6}}{1}+18e^{-6}+36e^{-6}+54e^{-6}\right]$$

$$= 1-[e^{-6}+6e^{-6}+18e^{-6}+36e^{-6}+54e^{-6}]$$

$= 1-115e^{-6}$

$= 1-115(0.002)$

$= 1-0.23$

$= 0.77$ //

(iii) $P[x(t)<4] = \left[\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!}\right]$

$= \left(\frac{e^{-6}(1)}{1} + \frac{e^{-6}(6)}{1} + \frac{e^{-6}(\overset{18}{\cancel{36}})}{\cancel{2}} + \frac{e^{-6}(6\times6\times\cancel{6})}{\cancel{3}\times\cancel{2}\times1}\right)$

$= \left[e^{-6} + 6e^{-6} + 18e^{-6} + 36e^{-6}\right]$

$= 61e^{-6}$

$= 61(0.0024)$

$= 0.1464$ //

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12. Queries presented in a computer database are following a poisson's process a mean rate, $\lambda=6$, queries per min. An experiment consist of monitoring a database for 'm' minutes & recording 'n' the no. of queries presented.

(i) What is the probability that no queries arraive in 1 min arrival.

(ii) What is the probability that exactly 6 queries arraive in 1 min interval

(iii) What is the probability that less than 3 queries arraive in 1/2 min interval.

Soln: Let $x(t)$ be the queries presented in the time arrival 't'.

$x(t)$ follows the poisson's process.

(i) $P[x(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ $[\lambda=6, t=1]$

$n=0$; $= \frac{e^{-6(1)}(6(1))^0}{0!}$

$= \frac{e^{-6}(6)^0}{0!}$

$= \frac{e^{-6}(1)}{1} = e^{-6}$

$= 0.0024$ //

(ii) $[P(x(t))=6] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ $[\lambda=6, t=1][n=6]$

$= \frac{e^{-6(1)}(6(1))^6}{6!}$

$= \frac{e^{-6}(6)^6}{6!} = \frac{e^{-6}[6\times6\times\overset{3}{\cancel{6}}\times\overset{3}{\cancel{6}}\times\cancel{6}\times\cancel{6}]}{\cancel{6}\times5\times\underset{2}{\cancel{4}}\times\cancel{3}\times\cancel{2}\times1}$

$= \frac{e^{-6}(324)}{5} = 64.8e^{-6} = 64.8(0.0024) = 0.155$ //

(iii) $P[x(t)<3] = \left[\frac{e^{-6(\frac{1}{2})}\,(6(\frac{1}{2}))^n}{n!}\right]$

$[\lambda=6;\ t=\frac{1}{2}]$

$$= \left[\frac{e^{-3}(3)^0}{0!} + \frac{e^{-3}(3)^1}{1!} + \frac{e^{-3}(3)^2}{2!}\right] + \frac{e^{-3}(3)^3}{3!}$$

$$= \left[\frac{e^{-3}(1)}{1} + \frac{e^{-3}(3)}{1} + \frac{e^{-3}(9)}{2}\right]$$

$$= [e^{-3} + 3e^{-3} + 4.5e^{-3}]$$

$$= 8.5e^{-3}$$

$$= 8.5(0.05)$$

$$= 0.425$$

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* **Markov Process:**

→ It is a random process in which future behaviour of the process depends only on the present state not on the states in the past.

Ex: (i) The probability of raining today depends only on the previous conditions existed for the last 2 days & not on the past weather conditions.

(ii) Snake & ladder game.

* **Markov Chain:**

→ We define the markov chain as follows, if

$P.\{x_n = a_n / x_{n-1} = a_{n-1},\ x_{n-2} = a_{n-2},\ x_n = a_0\}$

$P\{x_n = a_n / x_{n-1} = a_{n-1}\}$ for all n, then the process $\{x_n\}$, $n=0,1,2\ldots$ is called as markov chain, where $a_1, a_2, \ldots a_n$ are called state of the markov chain.

* **Transition Probability Matrix [TPM]:**

→ If $P=[P_{ij}]$ is a square matrix & is called the TPM of the chain. If the no. of states is finite say 'm' then the matrix 'p' will be a $n \times n$ square matrix. Otherwise the matrix will be infinite.

* **State Chapman - Kolmogorov Theorem:**

→ If 'p' is the TPM of a markov chain, then the 'n' step TPM;

$p^{(n)} = p^n$ i.e., $[P_{ij}^{(n)}] = [P_{ij}]^n$

Imp

Steady State Distribution:

(i) $\pi p = \pi$

(ii) $\pi_1+\pi_2+\pi_3 = 1$; where $\pi = [\pi_1\ \pi_2]$ or

(or) $\pi = [\pi_1\ \pi_2\ \pi_3]$

$\pi_1+\pi_2 = 1$

12M vvImp

13. The 3 state markov chain is given by transition probability matrix [TPM] where, $p = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$. Find the steady state distribution.

Soln: Given, $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

$\pi = [\pi_1\ \pi_2\ \pi_3]$

(i) $\pi p = \pi$

$$[\pi_1\ \pi_2\ \pi_3]\begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} = [\pi_1\ \pi_2\ \pi_3]$$

$$\left[0+\frac{\pi_2}{2}+0 \quad \frac{2\pi_1}{3}+\frac{\pi_3}{2} \quad \frac{\pi_1}{3}+\frac{\pi_2}{2}+\frac{\pi_3}{2}\right] = [\pi_1\ \pi_2\ \pi_3]$$

$$\left[\frac{\pi_2}{2} \quad \frac{2\pi_1}{3}+\frac{\pi_3}{2} \quad \frac{\pi_1}{3}+\frac{\pi_2+\pi_3}{2}\right] = [\pi_1\ \pi_2\ \pi_3]$$

$\frac{\pi_2}{2} = \pi_1$; $\frac{2\pi_1}{3}+\frac{\pi_3}{2} = \pi_2$; $\frac{\pi_1}{3}+\frac{\pi_2+\pi_3}{2} = \pi_3$

$\pi_2 = 2\pi_1 \rightarrow (1)$

$4\pi_1+3\pi_3 = 6\pi_2 \rightarrow (2)$

$2\pi_1+3\pi_2+3\pi_3 = 6\pi_3$

$2\pi_1+3\pi_2+3\pi_3-6\pi_3 = 0$

$2\pi_1+3\pi_2-3\pi_3 = 0 \rightarrow (3)$

(ii) $\pi_1+\pi_2+\pi_3 = 1$

Sub (1) in above eqn

$\pi_1+2\pi_1+\pi_3 = 1$

$3\pi_1+\pi_3 = 1$

$\boxed{\pi_3 = 1-3\pi_1} \rightarrow (4)$

Sub (4) in (2)

$4\pi_1+3(1-3\pi_1) = 6\pi_2$

$4\pi_1+3-9\pi_1 = 6\pi_2$

$4\pi_1-9\pi_1-6\pi_2 = -3$

$-5\pi_1-6\pi_2 = -3$

$-5\pi_1-6(2\pi_1) = -3$

$-5\pi_1-12\pi_1 = -3$

$\cancel{+}17\pi_1 = \cancel{+}3$

$\boxed{\pi_1 = 3/17} \rightarrow (5)$

Substitute (5) in (1)

$\pi_2 = 2(3/17)$

$\boxed{\pi_2 = 6/17}$

Substitute (5) in (4)

$\pi_3 = 1-3(3/17)$

$\pi_3 = 1-9/17$

$\boxed{\pi_3 = 8/17}$

$\therefore [\pi_1\ \pi_2\ \pi_3] = [3/17\ \ 6/17\ \ 8/17]$

$\therefore \pi_1 + \pi_2 + \pi_3 = 1$

$\frac{3}{17} + \frac{6}{17} + \frac{8}{17} = 1$

$\frac{3+6+8}{17} = 1$

$17/17 = 1$

Hence, verified//

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14. The 2 state markov chain is given by TPM, where $P = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$ Find steady state distribution.

Soln:

$\pi = [\pi_1 \ \pi_2]$

(i) $\pi P = \pi$

$[\pi_1 \ \pi_2]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [\pi_1 \ \pi_2]$

$[0.3\pi_1 + 0.4\pi_2 \quad 0.7\pi_1 + 0.6\pi_2] = [\pi_1 \ \pi_2]$

$0.3\pi_1 + 0.4\pi_2 = \pi_1$; $0.7\pi_1 + 0.6\pi_2 = \pi_2$

$0.4\pi_2 = \pi_1 - 0.3\pi_1$

$0.7\pi_1 + 0.6\pi_2 - \pi_2 = 0$

$0.4\pi_2 = 0.7\pi_1$

$0.7\pi_1 - 0.4\pi_2 = 0$

$\boxed{\pi_2 = \frac{0.7}{0.4}\pi_1} \rightarrow (1)$

Sub $\pi_1 = \frac{4}{11}$ in above eqn

$0.7\left(\frac{4}{11}\right) - 0.4\pi_2 = 0$

(ii) $\pi_1 + \pi_2 = 1$

$\frac{2.8}{11} - 0.4\pi_2 = 0$

$\pi_1 + \frac{0.7}{0.4}\pi_1 = 1$

$\frac{2.8}{11} = 0.4\pi_2$

$\pi_1 + \frac{7}{4}\pi_1 = 1$

$\boxed{\pi_2 = \frac{7}{11}}$

$4\pi_1 + 7\pi_1 = 4$

$11\pi_1 = 4$

$\boxed{\pi_1 = \frac{4}{11}}$

$\therefore [\pi_1 \ \pi_2] = \left[\frac{4}{11}, \frac{7}{11}\right]$

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15. Probability distribution based on the initial distribution; if the initial state probability distribution of a markov chain is $p^{(0)} = \left[\frac{5}{6}, \frac{1}{6}\right]$ & the TPM of the chain is $P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$. Find the probability distribution of the chain after 2 steps.

Soln: Given, $p^{(0)} = \left[\frac{5}{6}, \frac{1}{6}\right]$; $P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$p^{(1)} = p^{(0)} \cdot p$

$p^{(1)} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$p^{(1)} = \begin{bmatrix} 0 + \frac{1}{12} & \frac{5}{6} + \frac{1}{12} \end{bmatrix}$

$p^{(1)} = \begin{bmatrix} \frac{1}{12} & \frac{11}{12} \end{bmatrix}$

$p^{(2)} = p^{(1)} \cdot p$

$p^{(2)} = \begin{bmatrix} \frac{1}{12} & \frac{11}{12} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$p^{(2)} = \begin{bmatrix} 0 + \frac{11}{24} & \frac{1}{12} + \frac{11}{24} \end{bmatrix}$

$p^{(2)} = \begin{bmatrix} \frac{11}{24} & \frac{13}{24} \end{bmatrix}$

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16. If the initial state probability distribution of a markov chain is $p^{(0)} = [1 \; 0]$ & the TPM of $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$. Find $p^{(2)}$.

Soln: $p^{(0)} = [1 \; 0]$; $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$

$p^{(1)} = p^{(0)} \cdot p$

$p^{(1)} = (1 \; 0) \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$

$p^{(1)} = (0.6+0 \quad 0.4+0)$

$p^{(1)} = [0.6 \; 0.4]$

; $p^{(2)} = p^{(1)} \cdot p$

$p^{(2)} = [0.6 \; 0.4] \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$

$p^{(2)} = [0.36+0.08 \quad 0.24+0.32]$

$p^{(2)} = [0.44 \quad 0.56]$

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17. The initial process of the markov TPM is given by,

$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$ with initial probabilities $p_1^{(0)} = 0.4$, $p_2^{(0)} = 0.3$, $p_3^{(0)} = 0.8$. Find $p_1^{(1)}$, $p_2^{(1)}$ & $p_3^{(1)}$.

Soln: Given,

$p^{(0)} = [0.4 \; 0.3 \; 0.8]$; $p = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$

$p^{(1)} = p^{(0)} \cdot p$

$p^{(1)} = [0.4 \; 0.3 \; 0.8] \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$

$p^{(1)} = [0.08+0.03+0.48 \quad 0.12+0.06+0.24 \quad 0.2+0.21+0.08]$

$p^{(1)} = [0.59 \quad 0.42 \quad 0.49]$

$\therefore p_1^{(1)} = 0.59$

$p_2^{(1)} = 0.42$

$p_3^{(1)} = 0.49$ //