Quadratic Forms

A quadratic form, $Q(\underline{x})$, is an expression,

$$Q(\underline{x}) = \underline{x}^T A \underline{x}$$

where $A \in \mathbb{R}^{n \times n}$ is referred to as the matrix of the quadratic form and is <u>symmetric</u>.

Examples

Consider the purely diagonal matrix A

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

The quadratic form is then,

$$Q(\underline{x}) = [x_1 \quad x_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [\lambda_1 x_1 \quad \lambda_2 x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \lambda_1 x_1^2 + \lambda_2 x_2^2$$

Now consider a matrix A with off-diagonal entries, but symmetric

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Then,

$$Q(\underline{x}) = [x_1 \quad x_2] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [ax_1 + bx_2 \quad bx_1 + cx_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

Change of Variable

The cross term $2bx_1x_2$ adds complication, and it may therefore be valuable to consider the diagonalization of matrix:

$$A = PDP^T$$

where D is a diagonal matrix.

We may take, $\underline{y} = P^T\underline{x}$

$$\begin{aligned}\underline{x}^T A \underline{x} &= \underline{x}^T (PDP^T)\underline{x} \\ &= \underline{y}^T D \underline{y}\end{aligned}$$

Classifying Quadratic Forms

A quadratic form $Q(\underline{x})$ is:

a. positive definite if $Q(\underline{x}) > 0 \ \forall \underline{x} \neq 0$

b. negative definite if $Q(\underline{x}) < 0 \;\forall \underline{x} \neq 0$

c. indefinite if $Q(\underline{x})$ assumes both positive & negative values.

An interesting finding then follows.

Consider the symmetric matrix $A \in \mathbb{R}^{n \times n}$ with only positive eigenvalues,

$$\lambda_1, \lambda_2, \dots, \lambda_n > 0$$

Following the diagonalization of A, the quadratic form can be written:

$$\begin{aligned} \underline{x}^T A \underline{x} &= \underline{y}^T D \underline{y} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 \\ &> 0 \end{aligned}$$

Since all eigenvalues are positive, and all $y_1^2, y_2^2, \dots, y_n^2$ are positive or zero, it follows that
$Q(\underline{x}) = \underline{x}^T A \underline{x} > 0 \quad \forall \underline{x} \neq 0$ and

is therefore positive definite.

The converse also holds true. Furthermore, without loss of generality, we may say:

For a symmetric matrix A, the quadratic form $Q(\underline{x}) = \underline{x}^T A \underline{x}$ is:

a. positive definite if and only if the eigenvalues of A are all positive.
b. negative definite if and only if the eigenvalues of A are all negative.
c. indefinite if and only if the eigenvalues of A are both positive and negative.