3.C EIGENVALUES

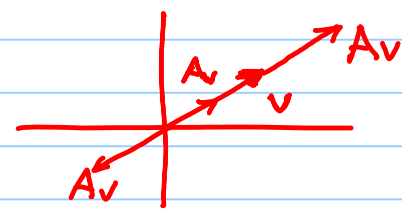
Let A be a square matrix, say $n \times n$. A nonzero vector v is an eigenvector of A, if for some scalar λ

(EVE)

$$\boxed{Av = \lambda v}$$

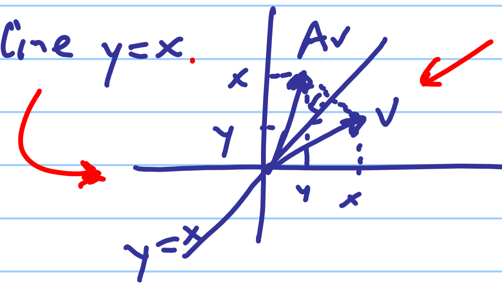
The scalar λ (which may be zero) is called an eigenvalue (EVA) of A corresponding to (or associated with) the eigenvector v.

Note: If v is an EVE of A, then v and Av are on the same line through the origin.



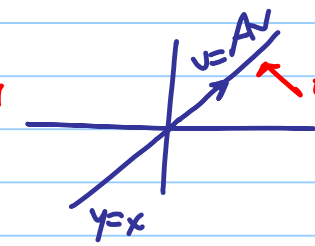
Geometric Example: Find the EVAs and EVEs of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ geometrically.

Sol^n $Av = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. This is reflection about the line $y=x$.



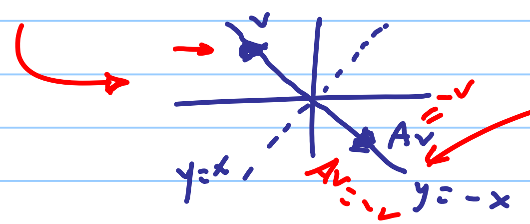
Observe: The only vectors that remain on the same line after reflection are the vectors along the lines $y=x$ and $y=-x$

- Along $y=x$ we have $Av = 1v$, so v is an EVE with EVA $\lambda = 1$.



EVES with $\lambda = 1$ (except the origin)

- Along $y=-x$ we have $Av = (-1)v$, so v is an EVE with EVA $\lambda = -1$.



EVEs with $\lambda = -1$ (except the origin)

Computation of EVAs + EVEs

Theorem: Let A be a square matrix.

1. v is an EVE of A with EVA λ $\iff$ v is a nontrivial solution of the homogeneous linear system

$$(A-\lambda I)v = 0 \qquad (A-\lambda I : \text{Characteristic Matrix})$$

2. A scalar λ is an EVA of A $\iff$

$$\underbrace{\det(A-\lambda I)}_{\text{(characteristic polynomial)}} = 0 \qquad \text{(Characteristic Equation)}$$

Note: If A is size $n \times n$, then $\det(A-\lambda I)$ is a polynomial in λ of degree n. So the EVA's are the roots of the char. poly. If λ has multiplicity k as a root we say that the EVA λ has algebraic multiplicity k.

Remark The solutions x (in $\mathbb{R}^n$) of $Ax=0$ is a (vector) subspace of $\mathbb{R}^n$.
This is not true for $Ax=b\,(\neq 0)$!

Pf x_1, x_2 solutions
$\Rightarrow Ax_1+Ax_2=0$
$A(x_1+x_2)=0$
so x_1+x_2 also a solution
Likewise, with cx

Justification of the Theorem

1. v ($\neq 0$) EVE of A. So, $Av=\lambda v$ for some scalar λ

$$\Leftrightarrow Av=\lambda I v$$
$$\Leftrightarrow Av-\lambda I v=0$$
$$\Leftrightarrow (A-\lambda I)v=0$$
$\Leftrightarrow$ v is nontrivial solution of $(A-\lambda I)x=0$

2. $(A-\lambda I)v=0$ has nontrivial solutions $\Leftrightarrow \det(A-\lambda I)=0$

Note: Since the EVE's are $(\neq 0)$ solutions of the hom. sys. $(A-\lambda I)x=0$ if we add 0 by the remark above we get a subspace of $\mathbb{R}^n$ called the eigenspace $\boxed{E_\lambda}$ of λ. Its dimension is the geometric multiplicity of λ.

Examples

In all examples below compute the EVAs, EVEs, the alg./geom. multiplities and find bases for the eigenspaces of the given matrix A.

Example 1: $A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 0 & 4 \\ -2 & 6 & -2 \end{bmatrix}$

Sol^{n}: Char. Eq^{n}: $\underbrace{\begin{vmatrix} 1-\lambda & -1 & -1 \\ -2 & 0-\lambda & 4 \\ -2 & 6 & -2-\lambda \end{vmatrix}}_{\det(A-\lambda I)} = -\lambda^3-\lambda^2+30\lambda = \underbrace{-\lambda(\lambda-5)(\lambda+6)}_{\text{Ch. poly.}} \overset{\text{set}}{=} 0.$

EVAs: $\lambda_1=0$, $\lambda_2=5$, $\lambda_3=-6$

Computing all EVAs

$(A-\lambda I)\vec{x}=0$

$\lambda_1=0$:
- Solving $[A-0I\,|\,0] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & 0 & 4 & 0 \\ -2 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Solution}} \begin{bmatrix} 2r \\ r \\ r \end{bmatrix}, r\in\mathbb{R}$

EVEs

- $E_0 = \left\{\begin{bmatrix} 2r \\ r \\ r \end{bmatrix}, r\in\mathbb{R}\right\} = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}$
- So $\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for E_0.
- $\lambda_1=0$ has alg. mult. 1 and geom. mult 1.

$\underline{\lambda_2 = 5}$

- $[A-5I \mid 0] = \left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ -2 & -5 & 4 & 0 \\ -2 & 6 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{solution}} \begin{bmatrix} -r/2 \\ r \\ r \end{bmatrix}, \ r \text{ any}$

- $E_5 = \left\{ \begin{bmatrix} -r/2 \\ r \\ r \end{bmatrix}, \ r \text{ any} \right\} = \text{Span} \left\{ \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

  if we don't like fractions!

- Basis $\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

- $\lambda_2 = 5$ has alg. mult. 1 and geom mult. 1.

$\underline{\lambda_3 = -6}$

- Likewise we get $E_{-6} = \left\{ \begin{bmatrix} r/20 \\ -13r/20 \\ r \end{bmatrix}, \ r \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1/20 \\ -13/20 \\ 1 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -13 \\ 20 \end{bmatrix} \right\}$

- Basis $\left\{ \begin{bmatrix} 1 \\ -13 \\ 20 \end{bmatrix} \right\}$

- $\lambda_3 = -6$ has alg mult 1 and geom. mult 1.

Example 2: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol^n Char. eq. $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda)^2 = 0 \Rightarrow \left.\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = \lambda_3 = 1 \end{array}\right\}$ EVA's

$\lambda_1 = 0$ $[A - 0I \mid 0] = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{solution}} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$,

- $E_0 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ basis
- $\lambda_1 = 0$ has alg. mult 1, geom. mult 1.

$\lambda_2 = \lambda_3 = 1$ • $[A - 1I \mid 0] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Sol}^n} \begin{bmatrix} r \\ s \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $r, s \in \mathbb{R}$

- $E_1 = \left\{ \begin{bmatrix} r \\ s \\ r \end{bmatrix}, r, s \text{ any} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$
- Basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$
- $\lambda_2 = \lambda_3 = 1$ has alg. mult 2 and geom mult 2.