Note Title 7/6/2013

3. A DOT and INNER PRODUCT

Dot Product

The dot product $u \cdot v$ between two n-vectors $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the number

$$u \cdot v = u^T v = [u_1 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \qquad \left(\begin{matrix} > 0 \\ = 0 \\ < 0 \end{matrix} \right)$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

If $u \cdot v = 0$, then the vectors u and v are called orthogonal.

Example: Let $u = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$, $w = \begin{bmatrix} -2 \\ 1 \\ -8 \end{bmatrix}$.

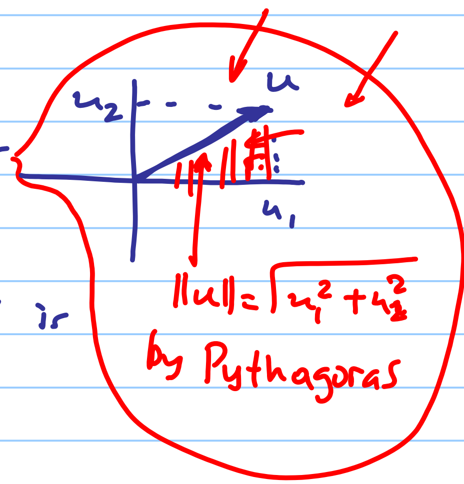
(a) Compute $u \cdot v$
(b) Are u and w orthogonal?

Sol^n: (a) $u \cdot v = (-3)(4) + (2)(-1) + (1)(5) = -9$
(b) $u \cdot w = (-3)(-2) + (2)(1) + (1)(-8) = 0$ YES

The <u>norm</u>, or <u>length</u>, or <u>magnitude</u> of $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ is the (≥ 0) number

$$\|u\| = \sqrt{u \cdot u} = \left(u_1^2 + u_2^2 + \cdots + u_n^2\right)^{\frac{1}{2}}$$

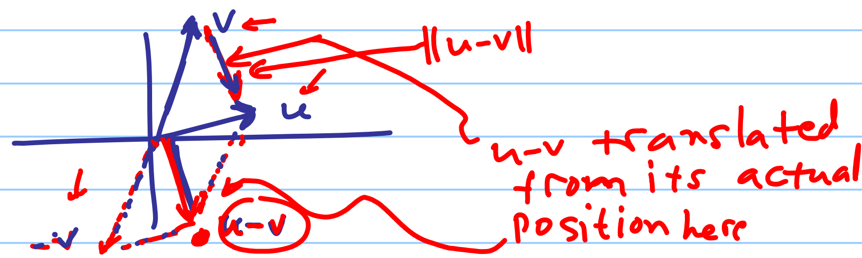
Note In 2-D (or 3-D) $\|u\|$ is the geometric length of the vector.



The <u>(Euclidean) distance</u> between two n-vectors u and v is

$$\|u - v\|$$

Note In 2-D (or 3-D) $\|u-v\|$ is the geometric distance from the tip of u to the tip of v.



A vector u with $\|u\| = 1$ is called a <u>unit</u> vector.

Example: Let $v = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$.

(a) Find the length of v.
(b) Find the distance between v and u.
(c) Is u unit?

Sol^n:

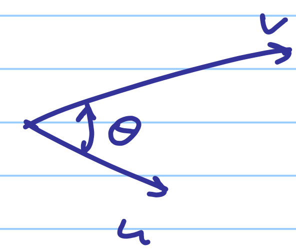
(a) $\|v\| = (1^2 + 2^2 + (-3)^2 + 1^2)^{1/2} = \sqrt{15}$ ←

(b) $\|v - u\| = \|(\frac{1}{2}, \frac{5}{2}, -\frac{7}{2}, \frac{3}{2})\| = \sqrt{21}$ ←

(c) $\|u\| = \left((\frac{1}{2})^2 + (-\frac{1}{2})^2 + (\frac{1}{2})^2 + (-\frac{1}{2})^2\right)^{1/2} = 1$ ✓. So u is a unit vector.

Note: The dot product for plane and space vectors is related to the length and angle between the vectors. Precisely, we have the important relation

$$\boxed{u \cdot v = \|u\| \, \|v\| \cos\theta}$$



Properties :

Dot Product

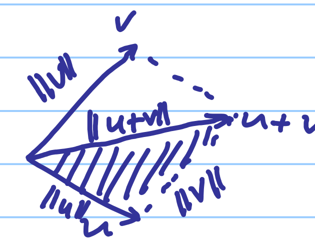
1. $u \cdot v = v \cdot u$ (Symmetry)
2. $u \cdot (v+w) = u \cdot v + u \cdot w$ (Additivity)
3. $c\,(u \cdot v) = (cu) \cdot v = u \cdot (cv)$ (Homogeneity)
4. $u \cdot u \geq 0$ and (Positive Definiteness) (Positivity

   $u \cdot u = 0 \iff u = 0$

Norm

1. $\|cu\| = |c|\,\|u\|$
2. $\|u+v\| \leq \|u\| + \|v\|$ (Triangle Inequality)
3. $\|u\| \geq 0$ and

   $\|u\| = 0 \iff u = 0$



Other Properties :

A. u and v are orthogonal $\iff \|u+v\|^2 = \|u\|^2 + \|v\|^2$ (Pythagorean Theorem)

B. $|u \cdot v| \leq \|u\|\,\|v\|$ (Cauchy-Bunyakovsky-Schwarz Inequality)

Inner Product

An inner product on a (real) vector space V is a function

$$\langle -,- \rangle : V \times V \longrightarrow \mathbb{R}, \qquad \underset{\uparrow \text{pair of vectors}}{\underline{(u,v)}} \longrightarrow \underline{\langle u,v \rangle} \leftarrow \text{number}$$

s.t.

1. $\langle u,v \rangle = \langle v,u \rangle$ (Symmetry)
2. $\langle u+w, v \rangle = \langle u,v \rangle + \langle w,v \rangle$ (Additivity)
3. $\langle cu, v \rangle = c \langle u,v \rangle$ (Homogeneity)
4. $\langle u,u \rangle \geq 0$ and $\langle u,u \rangle = 0 \Leftrightarrow u = 0$ (Positive definiteness)

Such V with an inner product is called an inner product space.

Note: Properties/Axioms 1–4 are copies of the main properties of the dot product!

Examples: 1. $V = \mathbb{R}^n$. The dot product in $\mathbb{R}^n$.

2. $V = M_{2\times 2}$: $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Define $\langle A, B \rangle = \underbrace{a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4}$

(This generalizes to $M_{m\times n}$. It is essentially the dot product)

3. $V = \mathbb{R}^n$. Let $w_1, \dots, w_n$ be positive numbers. (weights)
For the vectors $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ we define

$$\langle u, v \rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n$$ (This is the weighted dot product)

! 4. $V = C[a,b]$ the vector space of all continuous functions $f : [a,b] \to \mathbb{R}$.

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

Length and Orthogonality

Let $(V, \langle , \rangle)$ be an inner product space and let $u, v \in V$.

u, v are orthogonal if $\langle u, v \rangle = 0$

The norm (length, magnitude) of v is the ≥ 0 number

$$\|v\| = \sqrt{\langle v, v \rangle}$$

The distance $d(u,v)$ between u and v is

$$d(u,v) = \|u - v\|$$

If $\|v\| = 1$, then v is a unit vector.

Properties of Norm

1. $\|cu\| = |c|\,\|u\|$
2. $\|u+v\| \leq \|u\| + \|v\|$
3. $\|u\| \geq 0$ and $\|u\| = 0 \Leftrightarrow u = 0$

Let $(V, \langle , \rangle)$ be an inner product space. A set of vectors $\{v_1, v_2, \dots, v_k, \dots\}$ is an orthogonal set if all pairs are orthogonal. i.e., if

$$\langle v_i, v_j \rangle = 0 \qquad \text{all } i, j \; : \; i \neq j$$

Example: Is $S = \{1, \cos x, \sin x\}$ orthogonal, if (a) $V = C[-\pi, \pi]$ (b) $V = C[0, \pi]$

$\sin(k\pi) = 0$, $k = 0, \pm 1, \pm 2$

Sol^n: (a) $\langle 1, \cos x \rangle = \int_{-\pi}^{\pi} 1 \cdot \cos x \, dx = \sin x \Big|_{-\pi}^{\pi} = \sin\pi - \sin(-\pi) = 0$ ✓

$\langle 1, \sin x \rangle = \int_{-\pi}^{\pi} 1 \cdot \sin x \, dx = -\cos x \Big|_{-\pi}^{\pi} = -\cos(-\pi) - (-\cos\pi) = 1 - 1 = 0$ ✓

$\sin(2x) = 2\sin(x)\cos(x)$

$\langle \cos x, \sin x \rangle = \int_{-\pi}^{\pi} \cos x \sin x \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(2x) dx = -\frac{1}{4} \cos(2x) \Big|_{-\pi}^{\pi} = 0$ ✓

YES, S orthogonal on $[-\pi, \pi]$

(b) $\langle 1, \cos x \rangle = \int_0^{\pi} 1 \cdot \cos x \, dx = \sin x \Big|_0^{\pi} = \sin\pi - \sin 0 = 0$ ✓

$\langle 1, \sin x \rangle = \int_0^{\pi} 1 \cdot \sin x \, dx = -\cos x \Big|_0^{\pi} = -\underbrace{\cos\pi}_{=-1} - (-\underbrace{\cos 0}_{=1}) = -(-1) + 1 = \boxed{2}$ $\neq 0$

NO not on $[0, \pi]$