12.C CAUCHY'S THEOREM FOR DERIVATIVES

Note Title 8/14/2013

Theorem (Cauchy's Theorem for Derivatives)

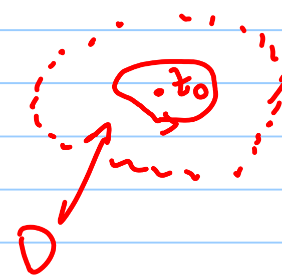
If $f(z)$ is analytic in a domain D, then $f(z)$ has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\,dz, \quad f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3}\,dz, \quad \ldots$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n = 1, 2, 3, \ldots)$$

We also have the most useful reformulation

$$\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)}$$



Example 1: For any contour enclosing πi (counterclockwise) we have

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = \underset{(n=1)}{\frac{2\pi i}{1!}} (\cos z)'\Big|_{z=\pi i}$$

$$= -2\pi i \sin \pi i = 2\pi \sinh \pi$$

Example 2: For any contour enclosing $-i$ (counterclockwise) we have

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^4}\,dz = \underset{(n=3)}{\frac{2\pi i}{3!}} (z^4-3z^2+6)'''\Big|_{z=-i} = \frac{2\pi i}{6}\left[24z\right]_{z=-i} = 8\pi$$

Example 3: For any contour for which 1 lies inside and $\pm 2i$ outside (counterclockwise).

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz = \underset{(n=1)}{\frac{2\pi i}{1!}} \left(\frac{e^z}{z^2+4}\right)'\Big|_{z=1} = 2\pi i\, \frac{e^z(z^2+4)-e^z 2z}{(z^2+4)^2}\Big|_{z=1} = \frac{6e\pi}{25} i \simeq 2.05 i$$

$= f(z)$

$+2i$

$-2i$ $\cdot 1$