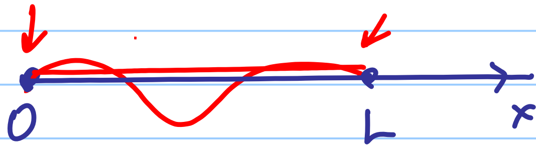
6.A MODELING THE 1-D WAVE EQUATION

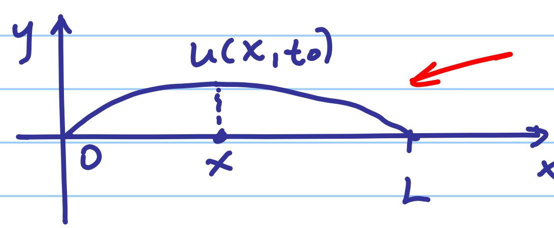
Note Title 7/23/2013

We consider a string of length L attached to fixed points with x-coordinates 0 and L.



Let $u(x,t)$ be the deflection or displacement at time t at location x. This the signed vertical distance from the x-axis at (x,t)

Snapshot at $t=t_0$



Goal: Calculate $u(x,t)$ assuming that the ends of the string are fixed and the initial displacement $u(x,0)$ and initial velocity $\frac{\partial u}{\partial t}\Big|_{t=0} = u_t(x,0)$ are given.

It turns out that $u(x,t)$ will satisfy a PDE. We need to find the "simplest", solvable meaningful PDE for $u(x,t)$. To achieve this we need some simplifying assumptions:

Simplifying Assumptions:

→ 1. The mass of the string per unit length is constant (homogeneous string)

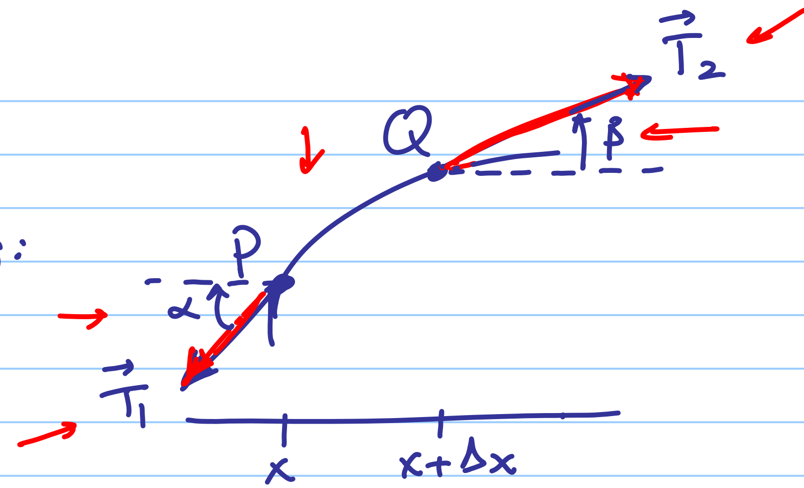
→ 2. The string is completely elastic: It does not resist bending.

→ 3. The tension caused by stretching is much greater than gravity. So the gravity force is not considered at all.

→ 4. The string performs a small transverse motion in the vertical plane, so that both the deflection $u(x,t)$ and its slope $u_x(x,t)$ are small.

Modeling the Vibrating String

Forces: Consider the forces acting on small portions of the string:



Since there is no resistance to bending, the tension is tangential to the curve of the string.

Let $\vec{T}_1$ and $\vec{T}_2$ be the tensions at P and Q

Horizontal direction: There is no horizontal motion so

$$\vec{T}_1 \cos(\alpha) + \vec{T}_2 \cos(\beta) = \vec{0}$$

Hence,

$$(1) \quad \|\vec{T}_1\| \cos(\alpha) = \|\vec{T}_2\| \cos(\beta) = T$$

Let T be the common constant

Vertical direction: In the vertical direction we have two forces:

$\longrightarrow$ $\vec{T}_1 \sin(\alpha)$, $\vec{T}_2 \sin(\beta)$. Their vector sum accounts for the motion.

In terms of components were have by Newton's 2^{nd} Law

$$\longrightarrow \quad -\|T_1\| \sin(\alpha) + \|T_2\| \sin(\beta) = \text{Mass} \times \text{Acceleration}$$

($-\sin\alpha$ because α is below the horizontal and β is above)

Let ρ be the linear density of the string. Then $\text{Mass} = \underset{\text{den}}{\rho} \cdot \underset{\text{length}}{\Delta x}$

The acceleration is $\frac{\partial^2 u}{\partial t^2}$ evaluated at some point between x and $x + \Delta x$.

$$\longrightarrow \quad \|T_2\| \sin(\beta) - \|T_1\| \sin(\alpha) = \underbrace{\rho \, \Delta x}_{\text{mass}} \, \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{accel.}} \quad \Longleftarrow$$

Using (1) we get

$$\frac{\|T_2\| \sin(\beta)}{\|T_2\| \cos(\beta)} - \frac{\|T_1\| \sin(\alpha)}{\|T_1\| \cos(\alpha)} = \rho \frac{\Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

Hence,

$$\tan(\beta) - \tan(\alpha) = \rho \frac{\Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

But $\tan(\alpha) = \left(\frac{\partial u}{\partial x}\right)_x$ and $\tan(\beta) = \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$ are the slopes at x and Δx. So,

$$\frac{1}{\Delta x}\left(\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right) = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$\lim_{\Delta x \to 0}$: $\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$

We take the limit as $\Delta x \to 0$ to get

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

$\left(c^2 = \frac{T}{\rho}\right)$

(One-dimensional Wave equation)