Note Title 7/3/2013

2.B BASIS AND DIMENSION

V vector space ; $v_1, \ldots, v_k \in V$. Just as in $\mathbb{R}^n$ we have:

Linear Combinations: $v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$, $c_i \in \mathbb{R}$

Span: $\text{Span}\{v_1, \ldots, v_k\}$ all linear combinations in $\{v_1, \ldots, v_k\}$

L.D. : There c_i not all zero s.t. $c_1 v_1 + \cdots + c_k v_k = 0$

L.I. : If $c_1 v_1 + \cdots + c_k v_k = 0 \implies c_1 = c_2 = \cdots = c_k = 0$

Defn If $V = \text{Span}\{v_1, \ldots, v_k\}$ we say that $\{v_1, \ldots, v_k\}$ is a spanning set of V.

This means that all vectors in V are lin. combo's in $\{v_1, \ldots, v_k\}$.

Example: Is $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$ a spanning set of $\mathbb{R}^2$? What about $K = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \end{bmatrix} \right\}$?

Sol^n: Let $v = \begin{bmatrix} a \\ b \end{bmatrix}$ be any vector in $\mathbb{R}^2$ (no restrictions on a and b)

See if there are c_1, c_2 : $\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Solve the lin. sys. $\left[\begin{array}{cc|c} 1 & 0 & a \\ 1 & -1 & b \end{array}\right] \underset{rref}{\sim} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & a-b \end{array}\right]$ Yes it's solvable for all a, b

So S is a spanning set of $\mathbb{R}^2$.

About K: The lin. sys is $\left[\begin{array}{cc|c} -1 & 4 & a \\ 2 & -8 & b \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -4 & -a \\ 0 & 0 & 2a-b \end{array}\right]$

This system has no solutions, if $2a-b \neq 0$.
For example the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not a lin. comb. in K

So K does not span $\mathbb{R}^2$.

Example: Let $p = -1 + x - 2x^2$ in P_3. Show that $p \in \text{Span}\{p_1, p_2, p_3\}$ where

$$p_1 = \underline{x - x^2 + x^3}, \quad p_2 = 1 + x + 2x^3, \quad p_3 = 1 + x$$

Sol^n Need c_1, c_2, c_3 (if they exist) s.t. $p = c_1 p_1 + c_2 p_2 + c_3 p_3$.

So,

$$\left.\begin{aligned} -1 + x - 2x^2 &= c_1 (x - x^2 + x^3) + c_2 (1 + x + 2x^3) + c_3 (1 + x) \\ \text{or} \quad -1 + x - 2x^2 &= (c_2 + c_3) + (c_1 + c_2 + c_3) x - c_1 x^2 + (c_1 + 2c_2) x^3 \end{aligned}\right\}$$

Hence: $c_2 + c_3 = -1$, $c_1 + c_2 + c_3 = 1$, $-c_1 = -2$, $c_1 + 2c_2 = 0$ LINEAR SYSTEM!

Solve to get $c_1 = 2$, $c_2 = -1$, $c_3 = 0$. YES in the span: $\underline{p = 2p_1 - p_2}$ (check)

<u>Note</u>: Aug. mat. of sys. $\left[\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -2 \\ 1 & 2 & 0 & 0 \end{array}\right]$ Anything special about the columns? $\left(\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow x - x^2 + x^3\right)$

Example: Check $\{1+x, -1+x, 4-x^2, 2+x^3\}$ for L.I. in P_3

Sol^n Let $c_1(1+x) + c_2(-1+x) + c_3(4-x^2) + c_4(2+x^3) = 0$. This yields the sys

$$\begin{array}{l} 1+x \\ \text{etc} \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \\ c_4 = 0 \end{array}$$

YES. L.I.

NOTE: Let $W \subseteq V$ (V v.s.). Another way (usually easy!) to show that W is a subspace is to prove that W is the span of vectors. This is valid because the span is always a subspace.

Example: Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 + x_2 = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

Sol^n $S = \left\{ \begin{bmatrix} x_1 \\ -x_1 \\ x_3 \end{bmatrix}, x_1, x_3 \text{ any} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_1, x_3 \text{ any} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

YES, a subspace.

BASIS

Look at the sets of vectors in $\mathbb{R}^3$.

$$\underbrace{S_1=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}}_{\text{L.I.}}\qquad \underbrace{S_2=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}}_{\text{L.I. and Spanning } \mathbb{R}^3},\qquad \underbrace{S_3=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}}_{\text{Spanning } \mathbb{R}^3}$$

(but not spanning $\mathbb{R}^3$)

$$\begin{bmatrix}a\\b\\c\end{bmatrix}=a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\end{bmatrix}$$

but not L.I.

$\begin{bmatrix}0\\1\\0\end{bmatrix}$ not a l.c. of $\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}$$

This is the best

It's a minimal spanning set

+ a $\underbrace{\text{maximal lin. ind. set}}_{\text{Basis}}$

Def A subset $\mathcal{B}$ of a v.s. space V is a basis of V, if

1. $\mathcal{B}$ is L.I.
2. $\mathcal{B}$ spans V

Fact: Every v.s. has at least one basis.

Examples:

✓ $\mathbb{R}^n$: the standard basis $\left\{ \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \overset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \dots, \overset{e_{n-1}}{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}}, \overset{e_n}{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}} \right\}$

✓ $M_{m \times n}$: the standard basis : All $E_{ij} = \begin{array}{cc} & \\ i & \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} \\ & j \end{array}$ all rest

$1 \le i \le m$
$1 \le j \le n$

P_n : the standard basis : $\{1, x, \dots, x^n\}$

Example: Show that $\mathcal{B} = \{1+x,\ -1+x,\ x^2\}$ is a basis of P_2.

Sol^n (a) Spanning: Show that every $p = a+bx+cx^2$ in P_2 is a lin. comb. in $\mathcal{B}$.

$$\Rightarrow\ a+bx+cx^2 = c_1(1+x) + c_2(-1+x) + c_3x^2$$

$$\rightarrow\ = (c_1-c_2) + (c_1+c_2)x + c_3x^2 \quad \left.\begin{matrix} c_1 - c_2 = a \\ c_1 + c_2 = b \\ c_3 = c \end{matrix}\right\}$$

Lin. Sys. $\left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 1 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(a+b) \\ 0 & 1 & 0 & \frac{1}{2}(b-a) \\ 0 & 0 & 1 & c \end{array}\right]$ Solvable for all a, b, c.

So Spanning set

(b) L.I.: $c_1(1+x) + c_2(-1+x) + c_3x^2 = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{matrix}$

So L.I.

$\mathcal{B}$ is a basis of P_2.

FACT: If a v.s V has a finite basis $\mathcal{B}$ with n elements, then all other bases of V have also n elements.

Def^n: This common number n is called the dimension of V.

$$\boxed{\dim V = n}$$

Examples:

- $\dim \mathbb{R}^n = n$
- $\dim M_{m \times n} = m \cdot n$
- $\dim P_n = n+1$

Example: Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, -5x_1 + x_2 = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

Then find a basis for S and the dimension of S.

Sol^n $S = \left\{ \begin{bmatrix} x_1 \\ 5x_1 \\ x_3 \end{bmatrix}, x_1, x_3 \text{ any} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_1, x_3 \text{ any} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

S is a subspace, since it's the span of 2 vectors

So $\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ span S. They are also L.I. so

$\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of S.

Hence,

$$\dim S = 2 \quad ✓$$