4.B EXPONENTIAL SHIFTING

Exponential functions of the form e^{at} play a special role in the theory of Laplace transforms. As an example, knowing $\mathcal{L}(f(t))$ yields $\mathcal{L}(e^{at}f(t))$.

Theorem: (The First Translation Theorem or Exponential Shifting)
Let $F(s) = \mathcal{L}(f(t))$. Then

$$\boxed{\mathcal{L}(e^{at}f(t)) = F(s-a) \quad \text{and} \quad \mathcal{L}^{-1}(F(s-a)) = e^{at}f(t)}$$

Example: Find $\mathcal{L}(t^3e^{5t})$

Sol^n:

$$\mathcal{L}(t^3e^{5t}) = \mathcal{L}(t^3)(s-5)$$

$$= \left.\frac{3!}{s^4}\right|_{s=s-5}$$

$$= \frac{6}{(s-5)^4}$$

Example: Find $L^{-1}\left(\frac{1}{(s+7)^3}\right)$.

Sol^n: $L^{-1}\left(\frac{1}{(s+7)^3}\right) = \frac{1}{2} L^{-1}\left(\frac{2!}{(s+7)^3}\right)$

$= \frac{1}{2} L^{-1}\left(\frac{2!}{s^3}\Big|_{s=s+7}\right)$

$= \frac{1}{2} t^2 e^{-7t}$

$L^{-1}\left(\frac{1}{s^3}\right)$

$\frac{1}{(s+7)^3} = \frac{1}{s^3}\Big|_{s=s+7}$

Example: (Completion of Square) Find $\mathcal{L}^{-1}\left(\frac{s+3}{s^2+2s+2}\right)$.

Sol^n: The quadratic in the denominator has complex roots. It cannot be factored over the reals. However, we may complete the square and use exponential shifting:

$$\mathcal{L}^{-1}\left(\frac{s+3}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left(\frac{s+3}{(s+1)^2+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1^2}\right) + \mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+1^2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s}{s^2+1^2}\Big|_{s=s+1}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s^2+1^2}\Big|_{s=s+1}\right)$$

$$= e^{-t}\cos t + 2e^{-t}\sin t$$

Existence of Laplace Transforms

Do all functions have Laplace Tranforms? NO, there are functions where the integral $\underbrace{\int_0^\infty f(t)e^{-st}dt}$ diverges.

However, if a function $f(t)$ does not grow faster than an exponential of the form Me^{kt}, it does have Laplace transform.

Theorem: Let $f(t)$ be defined for all $t \geq 0$ and be piecewise continuous on all finite subintervals of $[0,\infty)$. If for all $t \geq 0$ there are constants M and k such that

$$|f(t)| \leq Me^{kt}, \quad t \geq 0$$

Then $L(f)$ exists and it is defined for all $s > k$.

Note: $f(t)$ in the theorem is a function of exponential order.