LINEAR INDEPENDENCE

Note Title 6/30/2013

$v_1, \dots, v_k$ n-vectors. A linear combination is a vector

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \quad , \quad \text{where } c_i \text{ are scalars.}$$

Example: (a) $v = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ a linear combination in $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$?

(b) What about $u = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$?

Sol^n (a)? Find c_1, c_2 s.t. $v = c_1 v_1 + c_2 v_2 \Rightarrow \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \underbrace{c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{matrix} c_1 = 2 \\ c_2 = -1 \end{matrix}$ YES $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ (Check!)

(b) Answer is no.

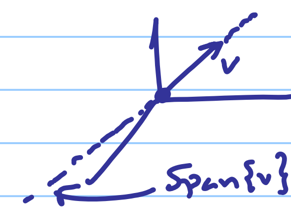
Practice Problem: Is $\begin{bmatrix} 0 \\ 10 \\ -16 \end{bmatrix}$ a lin. com. in $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix}$?

(Ans. Yes. In this case $c_1 = -1$, $c_2 = 2$, $c_3 = 3$)

The Span

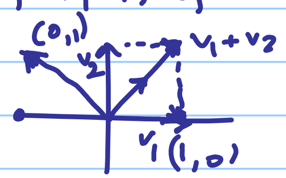
The $\text{Span}\{v_1, \dots, v_k\}$ is the set of all linear combinations in $v_1, \dots, v_k$.

- $\text{Span}\{v\} = \{cv, \ c \text{ any}\}$ Geometrically.



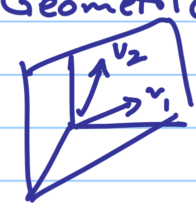
Span{v} is the line

- $\text{Span}\{v_1, v_2\}$



entire plane $\mathbb{R}^2$

plane →



Linear Dependence / Independence

$v_1, \dots, v_k$ n-vectors is <u>linearly dependent</u> (L.D.) if there are scalars $c_1, \dots, c_k$ <u>not all zero</u> s.t.

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0 \qquad (\Leftarrow \text{Linear dependence relation})$$

Special cases

1. $\{v\}$ is L.D. $\Leftrightarrow$ $v=0$ $\quad (\underset{\neq 0}{c}\, v = 0 \Rightarrow v = 0)$

2. $\{v_1, v_2\}$ is L.D. $\Leftrightarrow$ one vector is a scalar multiple of the other

   $$\Big(\overset{\text{say } \neq 0}{c_1} v_1 + c_2 v_2 = 0 \Rightarrow v_1 = \left(-\frac{c_2}{c_1}\right) v_2 \Big)$$

3. $\{v_1, \dots, v_k\}$ is L.D. $\Leftrightarrow$ at least one is a lin. comb. in the others.

__Example:__ Is $S = \left\{ \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 14 \\ 9 \end{bmatrix} \right\}$ L.D.?

Sol^n ? Find c_1, c_2, c_3 s.t. $c_1 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 14 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 3 \\ -2 & 2 & 14 \\ 3 & 7 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (hom. lin. sys.) Solve it to get $\begin{array}{l} c_1 = 4r \\ c_2 = -3r \\ c_3 = r \end{array}$ r any

If $r \neq 0$, say $r = 1$, $c_1 = 4$, $c_2 = -3$, $c_3 = 1$. __YES__ it's L.D.

$v_1, \dots, v_k$ n-vectors is linearly independent (L.I.) is it not L.D.

i.e. if $c_1 v_1 + \dots + c_k v_k = 0 \implies c_1 = c_2 = \dots = c_k = 0$

Special Cases

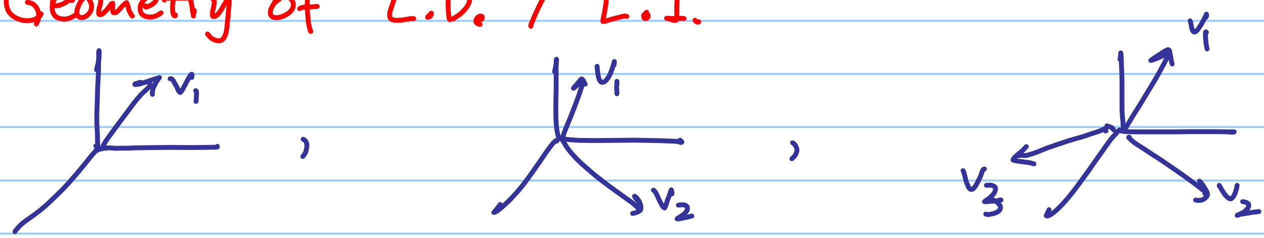
1. $\{v\}$ L.I. $\iff v \neq 0$
2. $\{v_1, v_2\}$ L.I. $\iff$ if none is a multiple of the other.

Example: Is $S=\left\{\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}5\\3\end{bmatrix}\right\}$ L.I. in $\mathbb{R}^2$?

Sol^n $c_1\begin{bmatrix}1\\-2\end{bmatrix}+c_2\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Longrightarrow$ lin. sys. $\left[\begin{array}{cc|c}1&5&0\\-2&3&0\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$ $\begin{matrix}c_1=0\\c_2=0\end{matrix}$ is L.I.

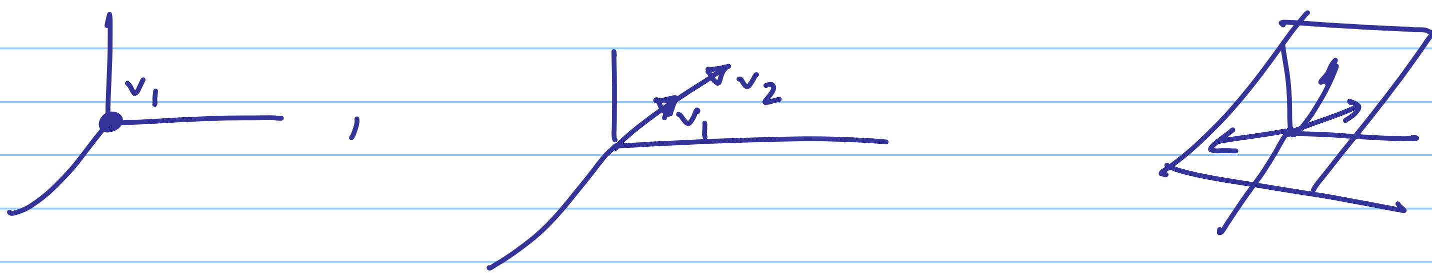
Geometry of L.D. / L.I.

L.I.



not coplanar

L.D.



coplanar