4.A LAPLACE and INVERSE LAPLACE TRANSFORMS

Note Title 7/7/2013

Laplace Transform

Let $f(t)$ be a function with domain all $t \geq 0$. The Laplace transform of $f(t)$ is the function $F(s)$ in s given by the integral:

$$\boxed{F(s) = L(f) = \int_0^\infty e^{-st} f(t)\, dt}$$

→ 1. Integration by Substitution

→ 2. " by Parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

→ 3. Improper integrals

$$\int_0^\infty g(t)\,dt = \lim_{k\to\infty} \int_0^k g(t)\,dt$$

Example: Show that $L(1) = \frac{1}{s}$, $s > 0$

Sol^n: $L(1) = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{k\to\infty} \frac{1}{-s} e^{-st}\Big|_{t=0}^{t=k}$

$$= \frac{1}{-s} \lim_{k\to\infty} \left(e^{-sk} - e^{0}\right)$$

$$= \frac{1}{s}, \quad s > 0$$

$1, t, \ldots, t^n$

e^{at}

$\cos(at)$

$\sin(at)$

Example: Show that $L(t) = \frac{1}{s^2}$, $s>0$

Soln: By integration by parts and the first example,

$$L(t) = \int_0^\infty e^{-st}\, t\, dt \underset{\substack{u=t\\ dv=e^{-st}dt}}{=} \frac{e^{-st}}{-s} t \Big|_0^\infty - \frac{1}{-s}\underbrace{\int_0^\infty e^{-st} dt}_{L(1)}$$

$du = dt$
$v = -\frac{1}{s}e^{-st}$

$L(1) = \frac{1}{s}$

$$\lim_{k\to\infty} \frac{e^{-st}}{-s}\cdot t \Big|_{t=0}^{t=k}$$

$$= \lim_{k\to\infty}\left(\frac{e^{-sk}}{-s}\cdot k - 0\right)$$

$$= -\frac{1}{s}\lim_{k\to\infty} \frac{k}{e^{sk}} \quad \frac{\infty}{\infty} =$$

$$= -\frac{1}{s}\lim_{k\to\infty}\frac{1}{s e^{sk}} = 0$$

$se^{sk} \to \infty$

$$= 0 + \frac{1}{s}L(1)$$

$$= \boxed{\frac{1}{s^2}} \quad (s>0)$$

Note: By iteration and induction we can get

$n! = 1\cdot 2\cdot 3 \cdots n$ $\rightarrow$

$$\boxed{L(t^n) = \frac{n!}{s^{n+1}}}, \quad s>0$$

Example: Show that $L(e^{at}) = \frac{1}{s-a}$, $s>a$, where a is constant.

Soln: $$L(e^{at}) = \int_0^\infty \underbrace{e^{-st}e^{at}}_{e^{-(s-a)t}} dt = \lim_{k\to\infty} \underbrace{-\frac{1}{s-a}}\, e^{-t(s-a)}\Big|_{t=0}^{t=k} = -\frac{1}{s-a}\lim_{k\to\infty}\left(e^{-(s-a)k} - e^{0}\right) = \frac{1}{s-a} \quad (s>a)$$

By double integration by parts (see notes!) we get

$$L(\cos(at)) = \frac{s}{s^2+a^2} \quad (s>0)$$

$$L(\sin(at)) = \frac{a}{s^2+a^2} \quad (s>0)$$

a constant

Linearity of Laplace Transform

Theorem: The Laplace transform is linear. I.e.,

$$L(c_1 f_1(t) + c_2 f_2(t)) = c_1 L(f_1(t)) + c_2 L(f_2(t)) \quad \text{for } c_1, c_2 \text{ constants.}$$

Linearity condition

Note: The linearity condition is equivalent to $\begin{cases} 1.\ L(f_1+f_2) = L(f_1)+L(f_2) \\ 2.\ L(cf) = cL(f) \end{cases}$

Linearity is very useful. It helps us compute Laplace transforms of linear combinations of functions.

Example: Find $L(7+2e^{-3t}+5t^2-\cos(6t))$

Sol^n: By linearity this equals

$$\frac{2!}{s^{2+1}}=\frac{2}{s^3}$$

$$= 7L(1)+2L(e^{-3t})+5L(t^2)-L(\cos(6t))$$

$$= \frac{7}{s}+\frac{2}{s+3}+\frac{10}{s^3}-\frac{s}{s^2+36}$$

Inverse Laplace Transform

If $F(s) = L(f)$, we say that the inverse Laplace transform of $F(s)$ is $f(t)$ and we write $L^{-1}(F(s)) = f(t)$. Hence,

$$f(t) = L^{-1}(F(s)) \iff L(f(t)) = F(s)$$

Just like Laplace tranforms, inverse Laplace tranforms are linear.

$$L^{-1}(c_1 F_1(s) + c_2 F_2(s)) = c_1 L^{-1}(F_1(s)) + c_2 L^{-1}(F_2(s))$$

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

Example: Find $L^{-1}\left(\frac{1}{s^4} + \frac{1}{s^2+4} + \frac{1}{5s-1}\right)$

$L^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{3!} L^{-1}\left(\frac{3!}{s^4}\right) = \frac{1}{3!} t^3 = \frac{t^3}{6}$

$L(t^3) = \frac{3!}{s^{3+1}}$

Sol^n: By linearity, this equals

$$\frac{1}{3!} L^{-1}\left(\frac{3!}{s^4}\right) + \frac{1}{2} L^{-1}\left(\frac{2}{s^2+4}\right) + \frac{1}{5} L^{-1}\left(\frac{1}{s-\frac{1}{5}}\right)$$

$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2} L^{-1}\left(\frac{2}{s^2+4}\right) = \sin(2t)$

$$= \frac{1}{6} t^3 + \frac{1}{2}\sin(2t) + \frac{1}{5} e^{t/5}$$

$L^{-1}\left(\frac{1}{5s-1}\right) = \frac{1}{5} L^{-1}\left(\frac{1}{s-1/5}\right)$

$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

Example: Find $L^{-1}\left(\frac{2}{(s+1)(s-1)}\right)$

Sol^n: By partial fractions $\frac{2}{(s+1)(s-1)} = \frac{1}{s-1} - \frac{1}{s+1}$. So

$$L^{-1}\left(\frac{2}{(s+1)(s-1)}\right) = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$= e^{t} - e^{-t}$$

$\frac{2}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} = \frac{A(s-1)+B(s+1)}{(s+1)(s-1)}$, with $A = -1$, $B = 1$

$A(s-1) + B(s+1) = 2$

$s=1$: $B \cdot 2 = 2 \Rightarrow B = 1$, $s=-1$: $-2A = 2 \Rightarrow A = -1$

Example: Find $L^{-1}\left(\frac{3s+1}{s^2+4}\right)$

Soln: $L^{-1}\left(\frac{3s+1}{s^2+4}\right) = 3L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{2}L^{-1}\left(\frac{2}{s^2+4}\right)$

$= 3\cos(2t) + \frac{1}{2}\sin(2t)$

Example: Find $L^{-1}\left(\frac{1}{s(s^2+1)}\right)$ $\quad \frac{1}{s} - \frac{s}{s^2+1}$

Soln: By partial fractions

$L^{-1}\left(\frac{1}{s(s^2+1)}\right) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+1}\right)$

$= 1 - \cos t$