$8x + 4y = 0 \Rightarrow 2x + y = 0$ — ①

$4x + 2y = 0 \Rightarrow 2x + y = 0$ — ②

This gives $y = -2x$

The eigenvector

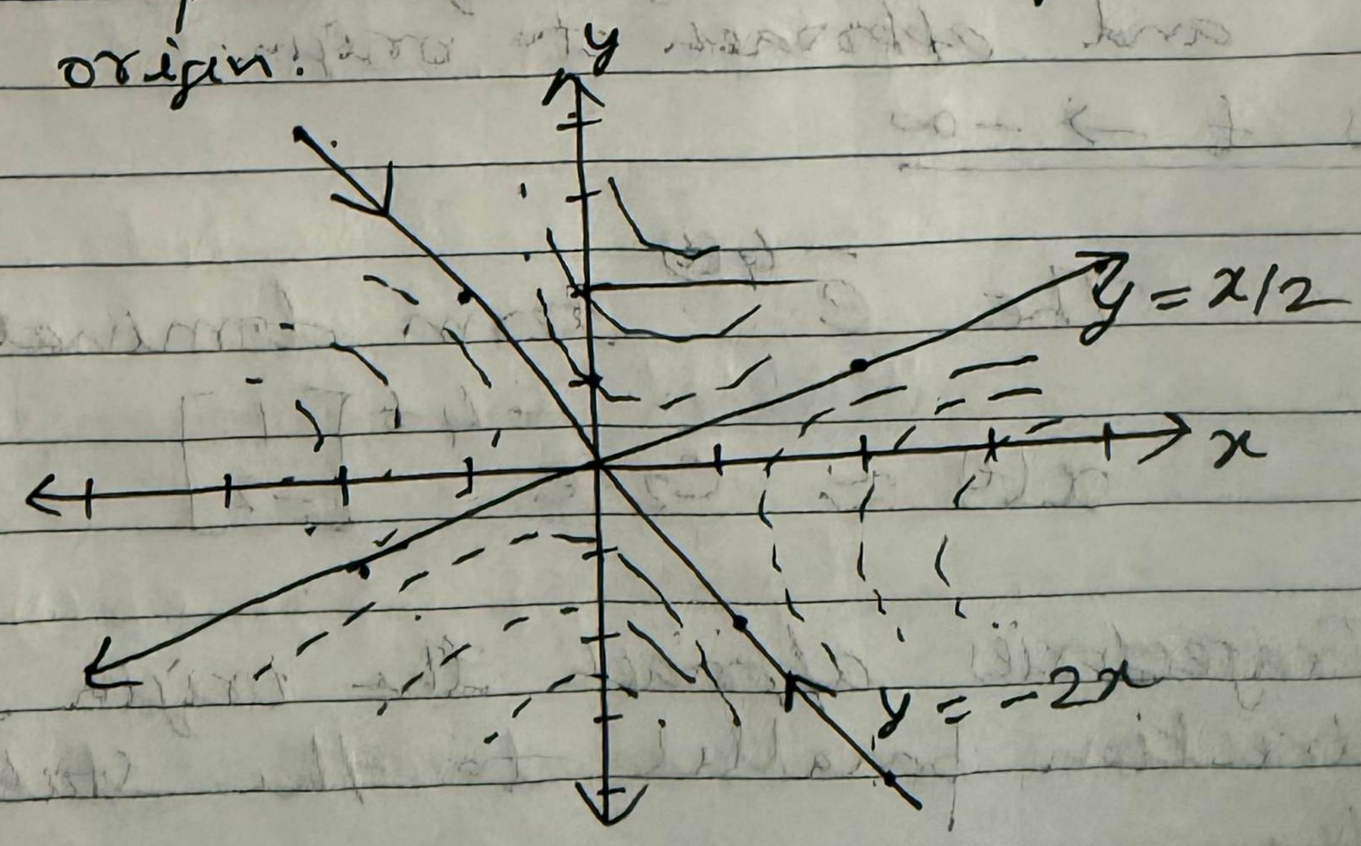
$$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The general solution is a linear combination of the straight-line solution

$$x(t) = c_1 e^{6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b) Phase Portrait

The phase portrait is a saddle point centered at the origin.



At $\lambda_1 = 6$ (>0) therefore trajectories moves away from origin

At $\lambda_2 = -4$ (<0) therefore trajectories move towards the origin.

(c) Trajectories

<u>Shape:</u>

The trajectories are shape like hyperbolas. The straight lines solution along the eigenvectors act as asymptotes for all other trajectories.

<u>As $t \to +\infty$</u>

$$x(t) = c_1 e^{6t}\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{-4t}\begin{bmatrix}1\\-2\end{bmatrix}$$

e^{6t} term dominates, therefore

$$x(t) \approx c_1 e^{6t}\begin{bmatrix}2\\1\end{bmatrix}$$

Unless $c_1 = 0$, all trajectories move away from the origin and become parallel to v_1. At $c_1 = 0$, trajectory lies on stable manifold and approach to origin.

<u>As $t \to -\infty$</u>

The $e^{-4(t)}$ term dominates

$$x(t) \approx c_2 e^{-4t}\begin{bmatrix}1\\-2\end{bmatrix}$$

Trajectories approach the origin from a direction parallel to the stable eigenvector v_2.

Second system:

$$A = \begin{bmatrix} 4 & 4 \\ -4 & 4 \end{bmatrix}$$

(a) General solution:

Find eigenvalues using characteristic equation

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 4 & 4 \\ -4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 4-\lambda & 4 \\ -4 & 4-\lambda \end{bmatrix}\right) = 0$$

$$(4-\lambda)(4-\lambda) + 16 = 0$$

$$(4-\lambda)^2 = -16$$

$$4 - \lambda = \pm 4i$$

$$\lambda = 4 \mp 4i$$

The eigenvalues are complex conjugates. The positive real part ($\alpha = 4$) indicates the origin is an unstable spiral.

Find eigenvector for on $\lambda = 4 + 4i$

$$(A - \lambda I)v = 0$$

$$(A - (4+4i)I)v = 0$$

$$\left(\begin{bmatrix} 4 & 4 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} 4+4i & 0 \\ 0 & 4+4i \end{bmatrix}\right) \cdot v = 0$$

$$\begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4ix + 4y = 0 \quad \text{— (1)}$$

$$-4x - 4iy = 0 \Rightarrow -4ix + 4y = 0 \quad \text{— (2)}$$

Both gives

$$-4ix + 4y = 0$$

$$\Rightarrow y = ix$$

The complex eigenvector is

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This gives general real solution as

$$x_c(t) = e^{(4+4i)t} v$$

$$= e^{4t}(\cos 4t + i \sin(4t)) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

The real and imaginary part of this function form two independent real solution

$$x(t) = c_1 e^{4t} \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix}$$

(b) Phase portrait:

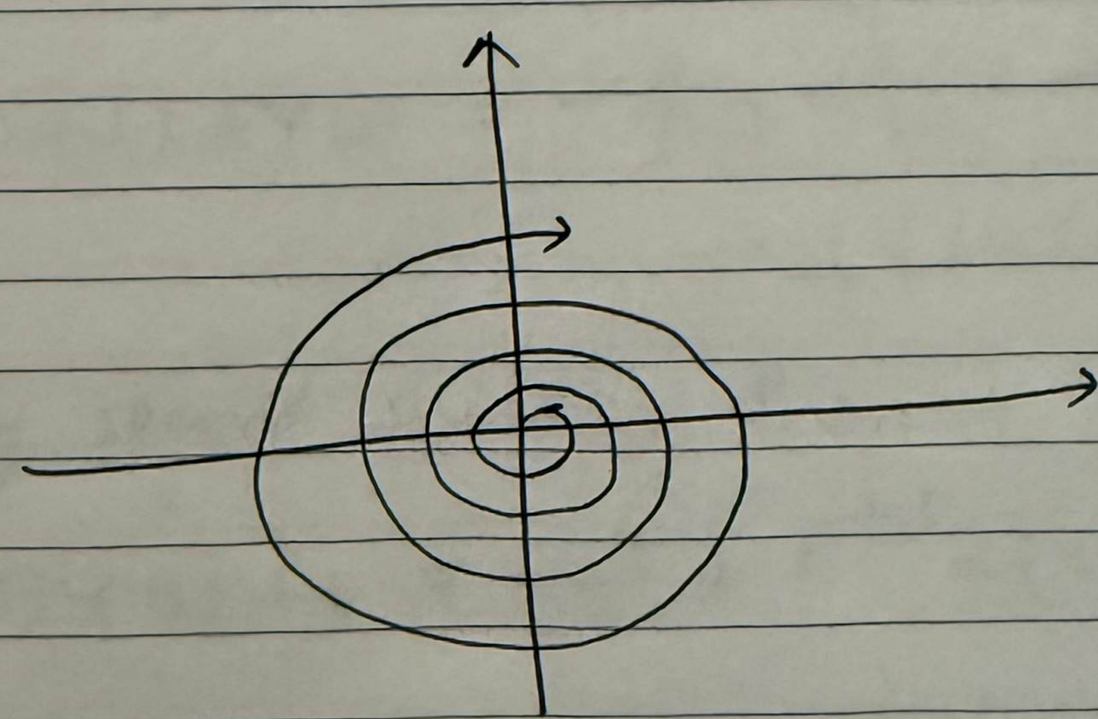
The phase portrait is a spiral source centered at the origin.

To determin the direction, test at (1,0)

$$\dot{x} = Ax$$
$$= \begin{bmatrix} 4 & 4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

At the point (1,0) on the positive x-axis, the velocity vector points down and to the right.

This mean spiral is rotating clockwise



(c) Trajectories

<u>shape</u>: As discussed above, it is spiral and moves away from origin ~~that~~ due to positive real part.

<u>As $t \to +\infty$</u>

The term e^{4t} grows exponentially, causing amplitude of spiral to increase without bound.

As $t \to -\infty$

The e^{4t} term approaches to zero.
The trajectories spiral inward towards origin.

5. Please answer each question.

(a) Use the Laplace transform to solve the initial value problem. **Here you must use unit step functions. Any other method will not earn credit.** Please show all steps.

$$\frac{dy}{dt} + y = f(t), \quad y(0) = -1, \quad \text{where } f(t) = \begin{cases} -2 & \text{if } 0 \le t < 4 \\ t & \text{if } t \ge 4 \end{cases}$$

Ans: Express $f(t)$ using unit step function

The forcing function is given as a piecewise function

$$f(t) = \begin{cases} -2 & \text{if } 0 \le t < 4 \\ t & \text{if } t \geq 4 \end{cases}$$

We can express this using Heaviside unit ~~sel~~ step function $u(t-a)$

$$f(t) = -2 + (t-(-2))\, u(t-4)$$
$$= -2 + (t+2)\, u(t-4)$$

To use second shifting theorem of Laplace transform

$$\mathcal{L}\{g(t-a)\, u(t-a)\} = e^{-as} G(s)$$

we must express the term $u(t-4)$ as a function $(t-4)$

$$t+2 = (t-4)+6$$

(b) Compute each of the following. Please show all work.

(i) $L(e^{-2t} t u(t-4))$

(ii) $L^{-1}\left(\dfrac{3s-8}{s^2+2s+5}\right)$

Function becomes

$$f(t) = -2 + [(t-4)+6]\,u(t-4)$$

$$= -2 + (t-4)\,u(t-4) + 6u(t-4)$$

Now take the Laplace transform of Differential Equation:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

Using Laplace transform properties

$$(sY(s) - y(0)) + Y(s) = \mathcal{L}\{-2 + (t-4)u(t-4) + 6u(t-4)\}$$

At $y(0) = -1$

$$(sY(s)+1) + Y(s) = -\frac{2}{s} + \mathcal{L}\{(t-4)u(t-4)\} + \mathcal{L}\{6u(t-4)\}$$

Applying second shifting theorem

$$(s+1)Y(s) + 1 = -\frac{2}{s} + e^{-4s}\mathcal{L}\{t\} + e^{-4s}\mathcal{L}\{6\}$$

$$(s+1)Y(s) + 1 = -\frac{2}{s} + e^{-4s}\left(\frac{1}{s^2}\right) + e^{-4s}\left(\frac{6}{s}\right)$$

Now solve for $Y(s)$

$$(s+1)Y(s) = -1 - \frac{2}{s} + e^{-4s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$$

$$(s+1)Y(s) = \frac{s+2}{s} + e^{-4s}\left(\frac{1+6s}{s^2}\right)$$

$$Y(s) = -\frac{s+2}{s(s+1)} + e^{-4s}\left(\frac{6s+1}{s^2(s+1)}\right)$$

Now find the inverse Laplace transform:

For each term seperately:

$$G(s) = -\frac{s+2}{s(s+1)}$$

$$\frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow \quad s+2 = A(s+1) + Bs$$

if $s=0$ then

$$2 = A \Rightarrow A = 2$$

if $s=-1$,

$$1 = B(-1) \Rightarrow B = -1$$

This gives

$$G(s) = -\left(\frac{2}{s} - \frac{1}{s+1}\right) = \frac{1}{s+1} - \frac{2}{s}$$

The inverse transform

$$\mathcal{L}^{-1}\{G(s)\} = e^{-t} - 2$$

second term:

$$H(s) = \frac{6s+1}{s^2(s+1)}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$\Rightarrow 6s+1 = As(s+1) + B(s+1) + Cs^2$$

At $s=0$, $B=1$

At $s=-1$, $C=-5$

s^2 coefficient: $A+C=0 \Rightarrow A=-C=5$

This gives

$$H(s) = \frac{5}{s} + \frac{1}{s^2} - \frac{5}{s+1}$$

The inverse transform of $H(s)$

$$\mathcal{L}^{-1}\{H(s)\} = 5 + t - 5e^{-t}$$

Now apply second shifting theorem to find

$$\mathcal{L}^{-1}\{e^{-4s}H(s)\} = h(t-4)u(t-4)$$

$$h(t-4) = 5 + (t-4) - 5e^{-(t-4)}$$

$$= t+1-5e^{-(t-4)}$$

$$\mathcal{L}^{-1}\{e^{-4s}H(s)\} = \left(t+1-5e^{-(t-4)}\right)u(t-4)$$

This gives final solution

$$y(t) = \left(e^{-t}-2\right) + \left(t+1-5e^{-(t-4)}\right)u(t-4)$$

(b) (i) $\mathcal{L}\left(e^{-2t}\,t\,u(t-4)\right)$

Rewrite function in terms of $(t-4)$

As per second shifting theorem

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,U(s)$$

Apply following ~~sot~~ substitution

(1) $t = (t-4)+4$

(2) $e^{-2t} = e^{-2((t-4)+4)} = e^{-2(t-4)}\cdot e^{-8}$

This gives

$$e^{-2t}\,t\cdot u(t-4) = \left(e^{-8}e^{-2(t-4)}\right)\left((t-4)+4\right)u(t-4)$$

$$= e^{-8}\left[(t-4)e^{-2(t-4)} + 4e^{-2(t-4)}\right]u(t-4)$$

$$= e^{-8}\, g(t-4)\, u(t-4)$$

where

$$g(t-4) = (t-4)e^{-2(t-4)} + 4e^{-2(t-4)}$$

As per unshifted function

$$g(t) = te^{-2t} + 4e^{-2t}$$

Laplace transform

$$G(s) = \mathcal{L}\{g(t)\}$$

Using shifting theorem

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Gives:

$$G(s) = \mathcal{L}\{te^{-2t}\} + 4\mathcal{L}\{e^{-2t}\}$$

→ For the first term

$$f(t) = t, \quad F(s) = \frac{1}{s^2}$$

shifting gives,

$$\mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2}$$

→ For the second term

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$$

This gives

$$U(s) = \frac{1}{(s+2)^2} + \frac{4}{s+2}$$

Applying the second shifting theorem

t shift with $a=4$

$$\mathcal{L}\{e^{-8} g(t-4) u(t-4)\} = e^{-8} \cdot e^{-4s} U(s)$$

$$= e^{-8} e^{-4s} \left[\frac{1}{(s+2)^2} + \frac{4}{s+2}\right]$$

~~Combining it gives~~

$$= e^{-8(4s+8)} \left(\frac{1+4(s+2)}{(s+2)^2}\right)$$

$$\mathcal{L}\{e^{-2t} t u(t-4)\} = e^{-(4s+8)} \frac{4s+9}{(s+2)^2}$$

Inverse transform of $\dfrac{3s-8}{s^2+2s+5}$

Denominator:

$$s^2+2s+5 = (s^2+2s+1) - 1 + 5$$
$$= (s+1)^2 + 4$$
$$= (s+1)^2 + 2^2$$

This indicates shifting $a = -1$, frequency $b = 2$

Rewrite numerator:

$$3s - 8 = 3(s+1) - 3 - 8$$
$$= 3(s+1) - 11$$

This gives

$$\frac{3s-8}{s^2+2s+5} = \frac{3(s+1) - 11}{(s^2+1)^2 + 2^2}$$

$$= \frac{3(s+1)}{(s+1)^2+2^2} - \frac{11}{(s+1)^2+2^2}$$

$$= \frac{3(s+1)}{(s+1)^2+2^2} - \frac{11}{2} \cdot \frac{2}{(s+1)^2+2^2}$$

Inverse transform:

For each term:

$$\mathcal{L}^{-1}\left\{3\,\frac{s+1}{(s+1)^2+2^2}\right\} = 3e^{-t}\cos 2t$$

$$\mathcal{L}^{-1}\left\{-\frac{11}{2}\,\frac{2}{(s+1)^2+2^2}\right\} = -\frac{11}{2}\,e^{-t}\sin 2t$$

This gives final answer

$$\mathcal{L}^{-1}\left(\frac{3s-8}{s^2+2s+5}\right) = 3e^{-t}\cos(2t) - \frac{11}{2}\,e^{-t}\sin(2t)$$