535.641 Mathematical Methods Assignment 4

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1. Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order on $[0, \infty)$. Define their convolution as:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$$

Using the definition of the Laplace Transform:

$$\mathcal{L}\{h(t)\}(s) = \int_0^\infty e^{-st} h(t)\, dt,$$

prove the convolution property of the Laplace Transform:

$$\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{g(t)\}(s)$$

Ans: Let

$$F(s) = \mathcal{L}\{f(t)\}(s)$$

$$G(s) = \mathcal{L}\{g(t)\}(s)$$

Apply the definition of the Laplace Transform:

According to the definition

$$h(t) = (f * g)(t)$$

Therefore,

$$\mathcal{L}\{h(t)\}(s) = \mathcal{L}\{(f*g)(t)\}$$

$$= \int_{t=0}^{\infty} e^{-st} (f*g)(t)\, dt$$

$$= \int_{t=0}^{\infty} e^{-st} \left[\int_{\tau=0}^{t} f(\tau) g(t-\tau)\, d\tau \right] dt$$

This gives double integration in $t-\tau$ plane.

Change the order of Integration:

Integration region is defined by

$$0 \le \tau \le t$$

$$0 \le t < \infty$$

If we change order of integration from $d\tau\, dt$ to $dt\, d\tau$

This gives new limits:

$\rightarrow \; 0 \le \tau < \infty$

$\rightarrow \; \tau \le t < \infty$

Let's rewrite integral

$$\mathcal{L}\{(f*g)(t)\} = \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} e^{-st} f(\tau) g(t-\tau)\, dt\, d\tau$$

Since $f(\tau)$ does not depend on t

$$\mathcal{L}\{(f*g)(t)\} = \int_{\tau=0}^{\infty} f(\tau) \left[\int_{t=\tau}^{\infty} e^{-st} g(t-\tau)\, dt \right] d\tau$$

Solve the inner integral:

$$\int_{t=\tau}^{\infty} e^{-st} g(t-\tau)\, dt$$

Using substitution: $u = t - \tau$

$\rightarrow t = \tau, \; u = t - \tau = 0$

$\rightarrow t = \infty, \; u = \infty$

$\rightarrow du = dt$

$\rightarrow t = u + \tau$

This gives

$$\int_{u=0}^{\infty} e^{-s(u+\tau)} g(u)\,du$$

$$= \int_{u=0}^{\infty} e^{-su} \cdot e^{-s\tau} g(u)\,du$$

$$= e^{-s\tau} \int_{u=0}^{\infty} e^{-su} g(u)\,du$$

$$= e^{-s\tau} G(s)$$

substitute this result back to main equation:

$$\mathcal{L}\{(f * g)(t)\} = \int_{\tau=0}^{\infty} f(\tau)\left[e^{-s\tau} G(s)\right] d\tau$$

$$= G(s) \int_{\tau=0}^{\infty} e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s) \cdot F(s)$$

This completes the proof

$$\mathcal{L}\{f*g\}(s) = \mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{g(t)\}(s)$$

2. Consider the linear non-homogeneous initial value problem,

$$y'' + y = \sum_{k=1}^{\infty} (-1)^k \delta(t - ak), \quad y(t=0) = 1, \quad y'(t=0) = 0$$

where $a \in \mathbb{R}$ with $a > 0$. Solve for $y(t)$ and sketch the solution for $a = \pi$ and $a = 2\pi$.

Ans: This is application of Ordinary Differential Equation. Let

$$Y(s) = \mathcal{L}\{y(t)\}$$

Let's take Laplace Transform of given equation

~~LHS~~ As per first derivative

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$
$$= sY(s) - 1$$

As per second derivative

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$
$$= s^2Y(s) - s \times 1 - 0$$
$$= s^2Y(s) - s$$

LHS : $\mathcal{L}\{y'' + y\} = \mathcal{L}\{y''\} + \mathcal{L}\{y\}$

$$= s^2Y(s) - s + Y(s)$$
$$= Y(s)(s^2 + 1) - s$$

RHS: To calculate, we will apply Direc Delta

$$\mathcal{L}\{\delta(t-c)\} = \sum_{t=0}^{\infty} e^{-st} \cdot \delta(t-c)$$

Let's rewrite

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs} \cdot \mathcal{L}\{\sum_{k=1}^{\infty} (-1)^k \delta(t-ak)\}$$

$$= \sum_{k=1}^{\infty} (-1)^k \mathcal{L}\{\delta(t-ak)\}$$

$$= \sum_{k=1}^{\infty} (-1)^k e^{-aks}$$

Now solve $Y(s)$

$$Y(s)(s^2+1) - s = \sum_{k=1}^{\infty} (-1)^k e^{-aks}$$

$$Y(s)(s^2+1) = s + \sum_{k=1}^{\infty} (-1)^k e^{-aks}$$

$$Y(s) = \frac{s}{s^2+1} + \sum_{k=1}^{\infty} (-1)^k \frac{e^{-aks}}{s^2+1}$$

As per inverse Laplace

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = f(t-c)\,u(t-c)$$

Here, $F(s) = \frac{1}{s^2+1}$, $c = ak$

The inverse of $F(s)$ is $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$

This gives

$$\mathcal{L}^{-1}\left\{\frac{e^{-aks}}{s^2+1}\right\} = \sin(t-ak)\, u(t-ak)$$

This gives full solution $y(t)$

$$y(t) = \cos(t) + \sum_{k=1}^{\infty} (-1)^k \sin(t-ak)\, u(t-ak)$$

⇒ Sketch for $a=\pi$

$$y(t) = \cos(t) + \sum_{k=1}^{\infty} (-1)^k \sin(t-k\pi)\, u(t-k\pi)$$

As per

$$\sin(A-B) = \sin(A)\cos B - \cos A \cdot \sin B$$

$$\sin(t-k\pi) = \sin t \cdot \underbrace{\cos k\pi}_{(-1)^k} - \cos t \cdot \underbrace{\sin k\pi}_{0}$$

$$= \sin t\, (-1)^k$$

Back to equation

$$y(t) = \cos t + \sum_{k=1}^{\infty} (-1)^k \left[(-1)^k \sin t\right] u(t-k\pi)$$

$$= \cos t + \sin t \sum_{k=1}^{\infty} (-1)^{2k} u(t-k\pi)$$

$$= \cos t + \sin t \sum_{k=1}^{\infty} u(t-k\pi)$$

Let's analyze the solution in intervals

→ $0 \le t \le \pi$

$$y(t) = \cos t$$

→ $\pi \le t < 2\pi$

$$y(t) = \cos t + \sin(t)$$

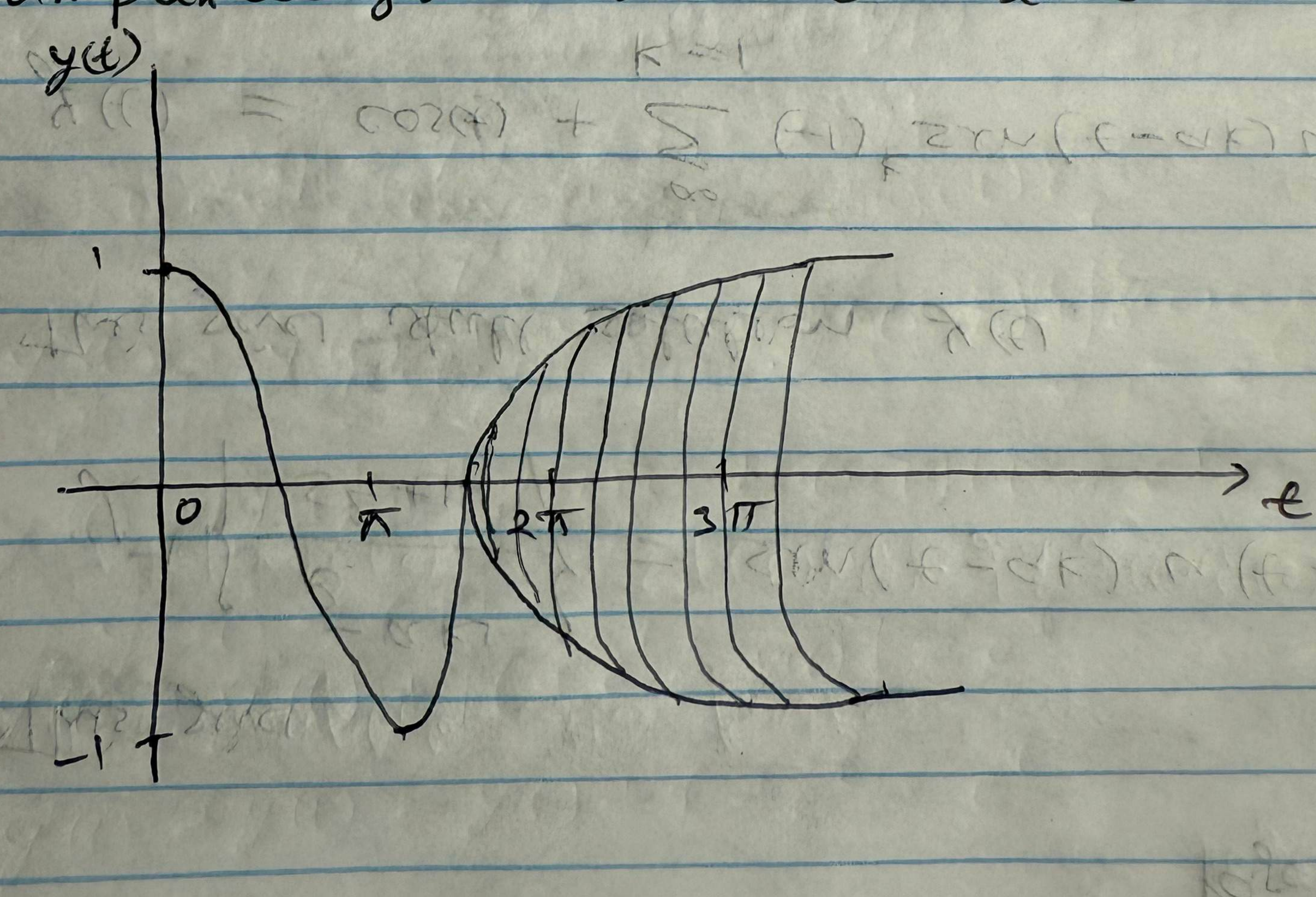
→ $2\pi \le t < 3\pi$

$$y(t) = \cos t + 2\sin t$$

→ $n\pi \le t < (n+1)\pi$

$$y(t) = \cos t + n\sin t$$

The graph is a continuous wave whose amplitude grows with each intervals.



$\Rightarrow$ Sketch at $a = 2\pi$

$$y(t) = \cos(t) + \sum_{k=1}^{\infty} (-1)^k \sin(t-2k\pi)\, u(t-2k\pi)$$

Let's simplify it:

$\rightarrow$ since $\sin(t)$ is periodic with 2π

$$\sin(t-2k\pi) = \sin(t)$$

Therefore

$$y(t) = \cos t + \sum_{k=1}^{\infty} (-1)^k \sin(t)\, u(t-2k\pi)$$

$$= \cos t + \sin t \sum_{k=1}^{\infty} (-1)^k u(t-2k\pi)$$

Let's solve in intervals:

$\rightarrow$ $0 \le t < 2\pi$

$$y(t) = \cos t$$

$\rightarrow$ $2\pi \le t < 4\pi$

$$y(t) = \cos t - \sin t$$

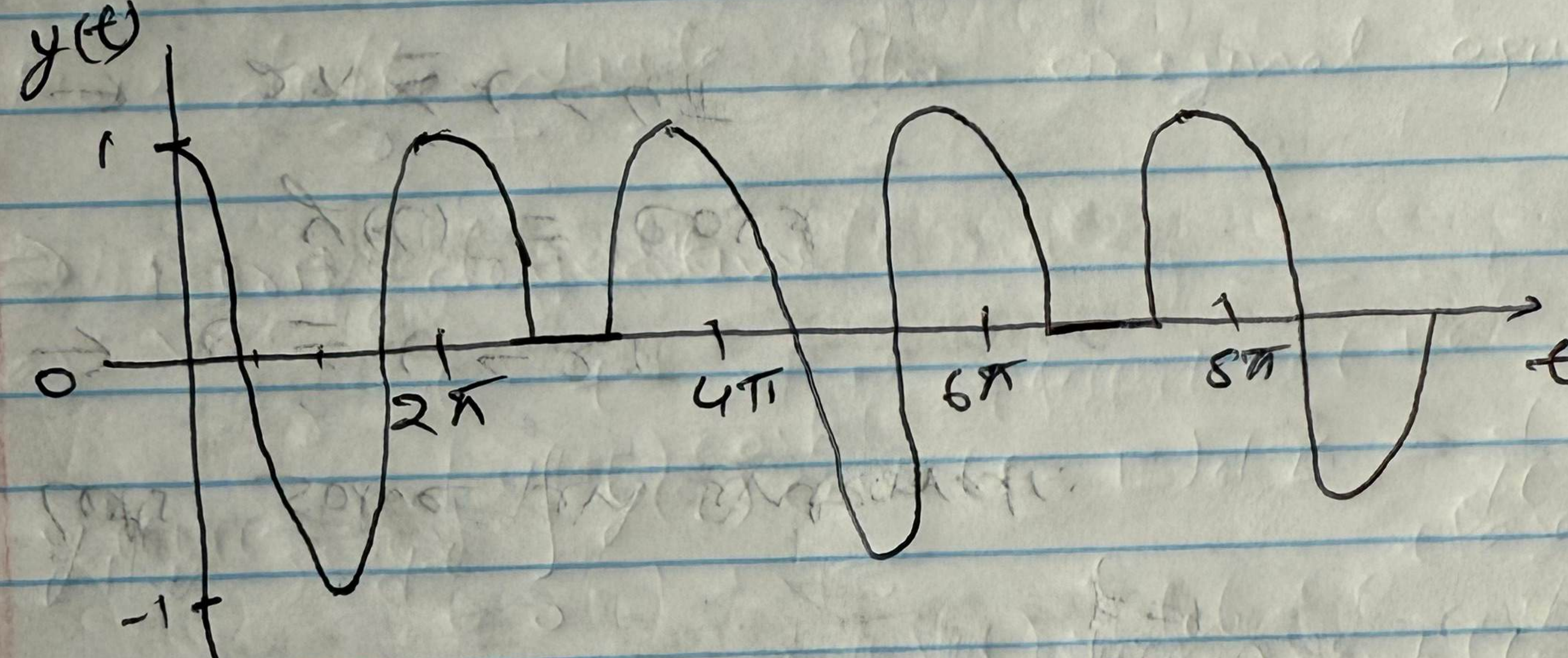
$\rightarrow$ $4\pi \le t < 6\pi$

$$y(t) = \cos t$$

$\rightarrow$ $6\pi \le t < 8\pi$

$$y(t) = \cos t - \sin t$$

The graph is continuous, periodic wave that looks like cos t for one full period then change to cos t - sin t for the next period and repeats.



3. Consider the integral equation for $y(t)$,

$$y(t) + \int_0^t y(\tau)\cosh(t-\tau)\, d\tau = t + e^t$$

(a) Solve the integral equation for $y(t)$ using Laplace transforms.

(b) Convert the integral equation into an initial value problem by taking two derivatives with respect to t, then solve this ODE and verify your solution in part (a).

Ans:

(a): We are given

$$y(t) + \int_0^t y(\tau)\cosh(t-\tau)\, d\tau = t+e^t$$

As per convolution of $y(t)$ and $\cosh(t)$

$$(y * \cosh)(t) = \int_0^t y(\tau)\cosh(t-\tau)\, d\tau$$

This gives

$$y(t) + (y * \cosh)(t) = t + e^t$$

Apply Laplace on both sides

Let $Y(s) = \mathcal{L}\{y(t)\}$

As per convolution theorem

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$$

$$LHS = \mathcal{L}\{y(t) + (y*\cosh)(t)\}$$

$$= \mathcal{L}\{y(t)\} + \mathcal{L}\{(y*\cosh)(t)\}$$

$$= Y(s) + \mathcal{L}\{y(t)\}\cdot\mathcal{L}\{\cosh(t)\}$$

$$= Y(s) + Y(s)\cdot\frac{s}{s^2-1}$$

RHS:

$$\mathcal{L}\{t+e^t\} = \mathcal{L}\{t\} + \mathcal{L}\{e^t\}$$

$$= \frac{1}{s^2} + \frac{1}{s-1}$$

This gives final equation as

$$Y(s) + Y(s)\left(\frac{s}{s^2-1}\right) = \frac{1}{s^2} + \frac{1}{s-1}$$

$$Y(s)\left(\frac{\cancel{s^2+s-1}}{\cancel{(s-1)}(s+1)}\right) = \frac{\cancel{s^2+s-1}}{s^2\cancel{(s-1)}}$$

$$Y(s) = \frac{s+1}{s^2}$$

Apply the inverse Laplace Transform

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 1 + t$$

(b)

Rule to differentiate the integral

$$\frac{d}{dt}\int_0^t f(t,\tau)\,d\tau = f(t,t) + \int_0^t \frac{\partial}{\partial t} f(t,\tau)\,d\tau$$

Now, differentiate the original equation

$\Rightarrow$ First derivative

$$y'(t) + \frac{d}{dt}\int_0^t y(\tau)\cosh(t-\tau)\,d\tau = \frac{d}{dt}(t+e^t)$$

$$y'(t) + y(t)\cosh(t-t) + \int_0^t y(\tau)\sinh(t-\tau)\,d\tau = 1+e^t$$

$$y'(t) + y(t) + \int_0^t y(\tau)\sinh(t-\tau)\,d\tau = 1+e^t$$

$\Rightarrow$ second derivative

$$y''(t) + y'(t) + y(t)\sinh(t-\tau) + \int_0^t y(\tau)\cosh(t-\tau)\,d\tau = e^t$$

$$y''(t) + y'(t) + \int_0^t y(\tau)\cosh(t-\tau)\,d\tau = e^t$$

$$y''(t) + y'(t) + [t+e^t - y(t)] = e^t$$

$$y''(t) + y'(t) - y(t) + t + e^t = e^t$$

$$y''(t) + y'(t) - y(t) = -t$$

As per initial condition:

$\Rightarrow y(0)$: set $t = 0$ in original equation

$$y(0) + \int_0^0 y(\tau)\cosh(t-\tau)\,d\tau = 0 + e^0$$

$$y(0) + 0 = 1$$

$$y(0) = 1$$

$\rightarrow y'(0)$: set $t = 0$ in first derivative

$$y'(t) + y(t) + \int_0^t y(\tau)\sinh(t-\tau)\,d\tau = 1 + e^t$$

$$y'(0) + y(0) + \int_0^0 \cdots\, d\tau = 1 + e^0$$

$$y'(0) + 1 + 0 = 1 + 1$$

$$y'(0) = 1$$

Solve the initial Value Problem:

At $t=0$, second derivative

$$y'' + y' - y = -t$$

$$y'' + y' - 1 = 0$$

Characteristic equation:

$$r^2 + r - 1 = 0$$

Using quadratic formula

$$r = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$y_h(t) = c_1 e^{\frac{-1+\sqrt{5}}{2}t} + c_2 e^{\frac{-1-\sqrt{5}}{2}t}$$

→ Particular solution (y_p)

$$y_p = At + B$$

$$y'_p = A \quad y''_p$$

As per ODE

$$A = 1, \; B = 1$$

$$y_p(t) = t + 1$$

General solⁿ

$$y(t) = y_h(t) + y_p(t)$$

$$= c_1 e^{\frac{-1+\sqrt{5}}{2}t} + c_2 e^{\frac{-1-\sqrt{5}}{2}t} + t + 1$$

After applying condition

$$c_1 = 0, \quad c_2 = 0$$

This gives

$$y(t) = t + 1$$

This match the both a & b solution.

4. Use the Laplace transform to solve for $y_1(t)$ and $y_2(t)$ that satisfies the coupled differential equation,

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with initial condition, $y_1(0) = 1$ and $y_2(0) = 1$.

Ans: Write matrix equation as a system of two linea ODE

$$y_1' = -2y_1 - y_2$$
$$y_2' = y_1 - 2y_2$$

⇒ Take the laplace transformation

$$\mathcal{L}\{y_1(t)\} = Y_1(s)$$
$$\mathcal{L}\{y_2(t)\} = Y_2(s)$$

with derivative rule

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$

Equation 1

$$\mathcal{L}\{y_1'\} = \mathcal{L}\{-2y_1 - y_2\}$$

On plugin $y_1(0) = 1$

$$sY_1(s) - y_1(0) = -2Y_1(s) - Y_2(s)$$

$$(s+2)Y_1(s) + Y_2(s) = 1 \quad \text{— A}$$

Equation 2

$$\mathcal{L}\{y_2'\} = \mathcal{L}\{y_1 - 2y_2\}$$
$$sY_2(s) - y_2(0) = Y_1(s) - 2Y_2(s)$$
$$sY_2(s) - 1 = Y_1(s) - 2Y_2(s)$$
$$-Y_1(s) + (s+2)Y_2(s) = 1 \quad \text{— B}$$

Solve the Algebraic system:

→ From equation A

$$Y_2 = 1 - (s+2)Y_1$$

→ From equation B

$$-Y_1 + (s+2)\left[1-(s+2)Y_1\right] = 1$$

$$-Y_1 + (s+2) - (s+2)^2 Y_1 = 1$$

$$Y_1(-1 - s^2 - 4s - 4) = -s - 1$$

$$Y_1(s) = \frac{s+1}{s^2+4s+5}$$

This gives

$$Y_2(s) = 1 - (s+2)Y_1$$

$$= 1 - (s+2)\frac{s+1}{s^2+4s+5}$$

$$= \frac{s+3}{s^2+4s+5}$$

⇒ Find the inverse Laplace Transform

$$s^2+4s+5 = (s+2)^2+1$$

Let's use exponential shifting with a=-2 and b=

Find $y_1(t)$

$$Y_1(s) = \frac{s+1}{(s+2)^2+1}$$

Rewrite numerator s+1 in terms of the shift (s+2)

$$s+1 = (s+2)-1$$

Therefore

$$Y_1(s) = \frac{(s+2)-1}{(s+2)^2+1}$$

$$= \frac{s+2}{(s+2)^2+1} - \frac{1}{(s+2)^2+1}$$

Now we take the inverse transform of each part

→ $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}$ is the shifted form of $\frac{s}{s^2+1}$

which is $e^{-2t}\cos t$

→ $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$ is the shifted form of

$\frac{1}{s^2+1}$, which is $e^{-2t}\sin(t)$

$$y_1(t) = e^{-2t}\cos(t) - e^{-2t}\sin(t)$$

⇒ Find $y_2(t)$

$$Y_2(s) = \frac{s+3}{(s+2)^2+1}$$

$$= \frac{(s+2)+1}{(s+2)^2+1}$$

$$= \frac{s+2}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}$$

Take the inverse transform

$$y_2(t) = e^{-2t}\cos t + e^{-2t}\sin t$$