535.641 Mathematical Methods Assignment 2

Ben Minnick **Name** DILIP KUMAR

1	/20
2	/20
3	/20
4	/20
TOTAL	/80

1. In the following problems check to see if the set S is a subspace of the corresponding vector space. If it is not, explain why not. If it is, then find a basis and the dimension.

(a)

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ 2x_1 + x_2 = 1 \right\} \subset \mathbb{R}^3$$

(b)

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ x_1 \leq x_2 \right\} \subset \mathbb{R}^2$$

(c)

$$S = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \ \forall a \in \mathbb{R} \right\} \subset \mathbb{R}^{2\times 2}$$

(d)

$$S = \left\{ f(x), \ \frac{df}{dx} = A\sin(2x), \forall A \in \mathbb{R} \right\} \subset \mathbb{R}$$

Ans: For a set to be subspace, it must satisfy three conditions

(1) The zero vector test
(2) closure under addition
(3) closure under scalar multiplication

Let's apply these rules on the given problems.

(a) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ 2x_1 + x_2 = 1 \right\} \subset R^3$

Let's apply the subspace rules

(1) The zero vector Test

zero vector in R^3 is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Let's verify if this satisfy the condition

$$2x_1 + x_2 = 1$$
$$2\times 0 + 0 = 1$$
$$0 = 1$$

since $0=1$ is a false statement, the zero vector is not in the set S.

Therefore the set S is not a subspace of R^3

(b) ~~Let's apply~~

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1 \leq x_2 \right\} \subset R^2$$

Let's apply the subspace rules

(1) The zero vector test:

The zero vector $= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 \leq x_2 \Rightarrow 0 \leq 0$ which is true

(2) Closure under Addition:

Let's have two vector

$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ with $a_1 \leq a_2$, $b_1 \leq b_2$

$$a+b = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix}$$

since $a_1 \leq a_2$
$b_1 \leq b_2$

Therefor $a_1+b_1 \leq a_2+b_2$

we can say that $a+b$ vector also satisfy the given condition.

(3) Closure Under scalar Multiplication:

Let's take following vector

$v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, sin $2 < 5$ therefore a valid one

Now let's multiply by scalar $c = -1$

$cv = -1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

To see if cv is in S, let's verify the condition

$x_1 \leq x_2$

$-2 \leq -5$

which is not true.

Therefore, S is not closed under scalar multiplication and therefore is not a subspace.

(c) $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \forall a \in R \right\} \subset R^{2\times 2}$

Let's apply the subspace rules.

(1) The zero vector test:

The zero matrix $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

At $a = 0$, we have

$\begin{bmatrix} 0 & 0 \\ 0 & -0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, The zero matrix is in S.

(2) Closed Under Addition:

Let's take two matrix

$$V_1 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad V_2 = \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}$$

$$V_1 + V_2 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}$$

$$= \begin{bmatrix} a+b & 0 \\ 0 & -a-b \end{bmatrix}$$

$$= \begin{bmatrix} a+b & 0 \\ 0 & -(a+b) \end{bmatrix}$$

The resulting matrix fits the required condition. i.e set is closed under addition.

(3) Closed Under scalar Multiplication:

Let's take following matrix

$$M = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$$

Now multiply by a scalar c

$$cM = c\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} = \begin{bmatrix} ca & 0 \\ 0 & -ca \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}$$

when $k = ca$, it means set is closed under scalar multiplication.

We can conclude that this set S is a subspace.

Find Basis:

A basis is the smallest possible set of "building block" vectors for a subspace. It must satisfy following

(1) It must span the subspace

(2) It must be linearly independent.

let's check both.

(1) Find the spanning set:

Any matrix in the given set S can be written as

$$\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This shows that every matrix in S is simply a scaler multiple of the single matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Therefore, this single matrix span the entire set S.

(2) check for Linear Independence:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

since this is not a zero matrix therefore it is linearly independent.

we can say that $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}$ is a basis for S.

Find Dimension:

Dimension = dimension of subspace

= 1

(d) $S = \left\{ f(x),\ \frac{df}{dx} = A\sin(2x),\ \forall A \in R \right\} \subset R$

Let's apply subspace three conditions

(1) Contains the zero function

zero function is $f(x) = 0$

Does it satisfy the condition?

$\frac{df}{dx} = \frac{d\,0}{dx} = 0 = A\sin(2x)$ if we choose $A = 0$

We can say that, the zero function is in S.

(2) Closed Under Addition:

Let's take two function that are in S

$f_1(x)$ with $\frac{df_1}{dx} = A_1\sin(2x)$

$f_2(x)$ with $\frac{df_2}{dx} = A_2\sin(2x)$

Now, $f_3(x) = f_1(x) + f_2(x)$

Lets validate the condition

$$\frac{df_3}{dx} = \frac{d}{dx}(f_1 + f_2) = \frac{df_1}{dx} + \frac{df_2}{dx}$$

$$= A_1\sin(2x) + A_2\sin(2x)$$

$$= (A_1 + A_2)\sin(2x)$$

$$= A\sin(2x), \text{ when } A = A_1 + A_2$$

We can say that derivative of the sum is also of the form $A\sin(2x)$. Therefore, the set is closed under addition.

(3) closed under scalar Multiplication

Let's take a function from S

$f_1(x)$ with $\frac{df_1}{dx} = A_1 \sin(2x)$

Applying scalar multiplication with c,

$f_2(x) = c \cdot f_1(x)$

Let's verify the condition

$$\frac{df_2}{dx} = \frac{d}{dx} c \cdot f_1(x)$$

$$= c \cdot \frac{d}{dx} f_1(x)$$

$$= c \cdot A_1 \sin(2x)$$

$$= (cA_1) \sin(2x)$$

$$= A \sin(2x), \text{ when } A = c \cdot A_1$$

Therefore, the set S is closed under scalar multiplication.

Since the set S satisfies all three conditions, it is a subspace.

<u>Find Basis:</u>

<u>Step 1:</u> Find the general form of the function in S

Condition for $f(x)$ to be in set S is

$$\frac{df}{dx} = A \sin(2x)$$

To find the function $f(x)$, we need to integrate

$$f(x) = \int A \sin(2x)\,dx$$
$$= A \int \sin(2x)\,dx$$
$$= A\left(-\frac{1}{2}\cos(2x)\right) + C$$
$$= -\frac{A}{2}\cos(2x) + C$$
$$= B\cos(2x) + C, \text{ where } B = -\frac{A}{2}$$

So any function in the subspace S can be written in the form:

$$f(x) = B \cdot \cos(2x) + C \cdot 1$$

<u>Step 2:</u> Identify the Basis

The general form $f(x) = B\cos(2x) + C \cdot 1$ is a linear combination of two functions: $\cos 2(x)$ and the constant function 1.

(1) Spanning set:
Since every function in S can be created by combining these two functions, the set

$\{\cos(2x), 1\}$ spans the subspace.

(2) Linear Independence:
The $\cos 2x$ and 1 is linear independent as one is not scalar multiplication of other.

Therefore, basis for S is

$$\{\cos(2x), 1\}$$

<u>Dimension:</u>

Since there are two functions in the basis, therefore dimension of the subspace is 2.

2. Use Cramer's rule to compute the solutions of the system

$$2x_1 + x_2 = 7$$
$$-3x_1 + x_3 = -8$$
$$x_2 + 2x_3 = -3$$

Ans: Let's write the system in matrix form

$$Ax = b$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ -8 \\ -3 \end{bmatrix}$$

As per Cramer's rule

$$x_i = \frac{\det(A_i)}{\det(A)},$$ where A_i is the matrix A with ith column replaced by b.

Determinant calculation:

(1) $$\det(A) = 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -3 & 1 \\ 0 & 2 \end{vmatrix} + 0$$

$$= 2(0-1) - 1(-6-0) = -2+6 = 4$$

(2) $$\det(A_1) = \det\left(\begin{bmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix}\right)$$

$$= 7\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 1\cdot\begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix} + 0$$

$$= 7(0-1) - (-16+3) = -7+13 = 6$$

(3) $\det(A_2) = \det\left(\begin{bmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{bmatrix}\right)$

$$= 2\begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix} - 7\begin{vmatrix} -3 & 1 \\ 0 & 2 \end{vmatrix} + 0$$

$$= 2(-16+3) - 7(-6-0)$$

$$= -26 + 42 = 16$$

(4) $\det(A_3) = \det\left(\begin{bmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{bmatrix}\right)$

$$= 2\begin{vmatrix} 0 & -8 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} -3 & -8 \\ 0 & -3 \end{vmatrix} + 7\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 2(0+8) - (9-0) + 7(-3-0)$$

$$= 16 - 9 - 21 = 16 - 30 = -14$$

Final solution

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{6}{4} = \frac{3}{2}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{16}{4} = 4$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-14}{4} = -\frac{7}{2}$$

Page - 4

3. *Read section 1.15 of the instructor's notes.* You are tasked with: 1. developing a model for an elastic beam supported on the edges, and 2. applying this model for a specific application, i.e. find the loading given a displacement. You decide to discretize the beam so that there are three nodes (P_1, P_2, P_3). Assume Hooke's law applies.

(a) You have tested the beam under unit forces applied to each node and respectfully found the following displacements:

$$\mathbf{y}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where $\mathbf{y}_1$ is the displacement that results from loading $\mathbf{F}_1 = [1,\ 0,\ 0]^{\mathrm{T}}$, $\mathbf{y}_2$ is the displacement from loading $\mathbf{F}_2 = [0,\ 1,\ 0]^{\mathrm{T}}$, and $\mathbf{y}_3$ is the displacement from loading $\mathbf{F}_3 = [0,\ 0,\ 1]^{\mathrm{T}}$.

Given this information, build the model and construct the flexibility matrix.

(b) Using the model you have just derived, determine the loading that produces the displacement

$$\mathbf{y} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

Ans: We have following equation

$$AF = y$$

where, A = Flexibility matrix

F = vector of forces

y = vector of displacement

(a) Following are given

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) $AF = y$

$A = F^{-1} \cdot y$

since given F is an identity matrix of 3x3 therefore F^{-1} will also by identity matrix of 3x3. That mean A equal y.

$$A = y = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Determining the loading for given displacements

Following is give

$$y = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

$AF = y$

$F = A^{-1} \cdot y$

<u>Calculate A^{-1}:</u>

(i) Calculate $|A|$

$$|A| = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} + 0 = -2$$

(2) Find the Adjoint of A.

sign for cofactors

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

given A

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

calculate matrix of cofactors as below

$$C = \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0-1 & 0 & 2-0 \\ 0 & 0 & -(2) \\ 2-0 & -(2) & 2-4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & -2 \\ +2 & -2 & -2 \end{bmatrix}$$

We transpose it to get the adjoint

$$\text{adj}(A) = C^T = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & -2 \\ 2 & -2 & -2 \end{bmatrix}$$

(3) Find the inverse of A (Stiffness Matrix)

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{-2}\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & -2 \\ 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

(4) Calculate the force vector F

$$F = A^{-1} \cdot y = \begin{bmatrix} 1/2 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\times 4 + 0\times 5 - 1\times 3 \\ 0 + 0 + 1\times 3 \\ -1\times 4 + 1\times 5 + 1\times 3 \end{bmatrix} = \begin{bmatrix} 2-3 \\ 3 \\ -4+5+3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

The loading that produces the given displacement is

$$F = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

4. Consider the (x, y) vertices of an equilateral triangle in $\mathbb{R}^2$ given by,

$$(0,0)\ ,\ (1,0)\ ,\ (.5, \sqrt{3}/2)$$

Perform the following actions in order using matrix transformations:

(a) Reflect the triangle over the x-axis,

(b) Compress the triangle along the y-axis by a factor of 2,

(c) Rotate the triangle counterclockwise by $\pi/6$ radians about the origin, and

(d) Shear the triangle along the x-axis by a factor of 0.5.

Clearly show the matrix transformations used and compute the final set of vertices of the triangle.

Ans: Let's find the transformation matrix for each:

The standard basis vectors are

$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ unit vector in x-axis

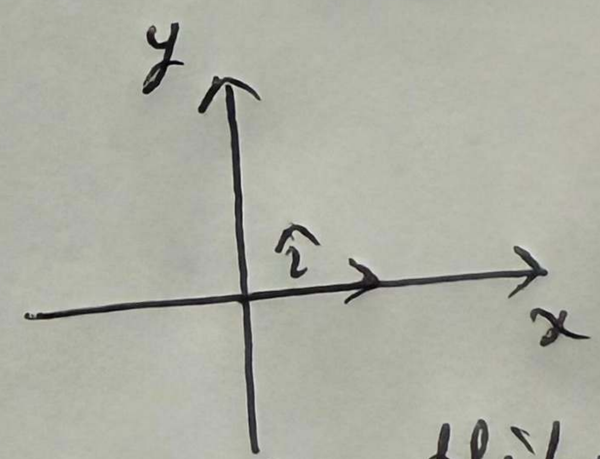
$\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ unit vector in y-axis

(a) Reflection over the x-axis

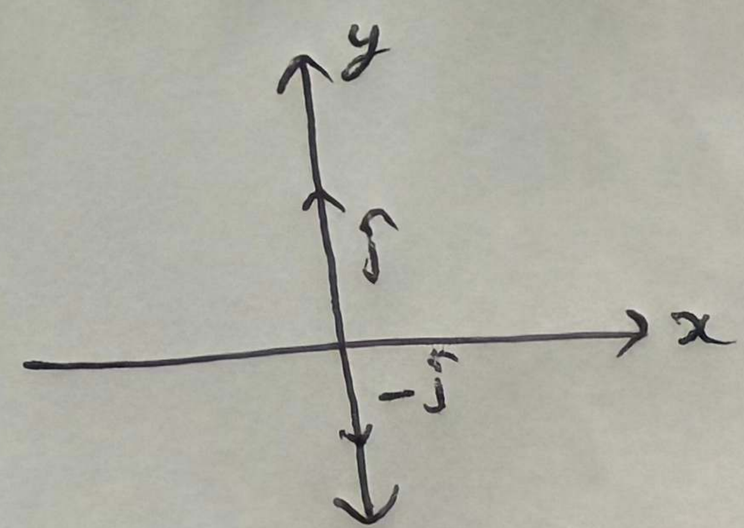
Action on $\hat{i}$: Reflection over x-axis

therefore $\hat{i}$ does not move

$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



Action on $\hat{j}$: Reflection over x-axis flips to negative y-axis.



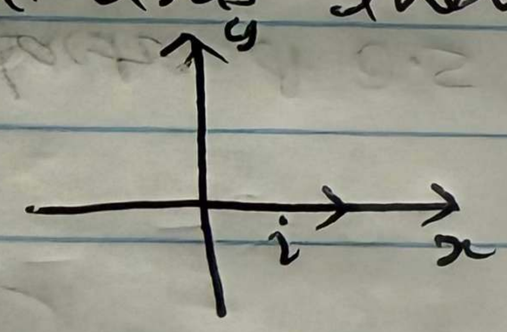
$\hat{j} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Resulting transform matrix:

$$T_a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

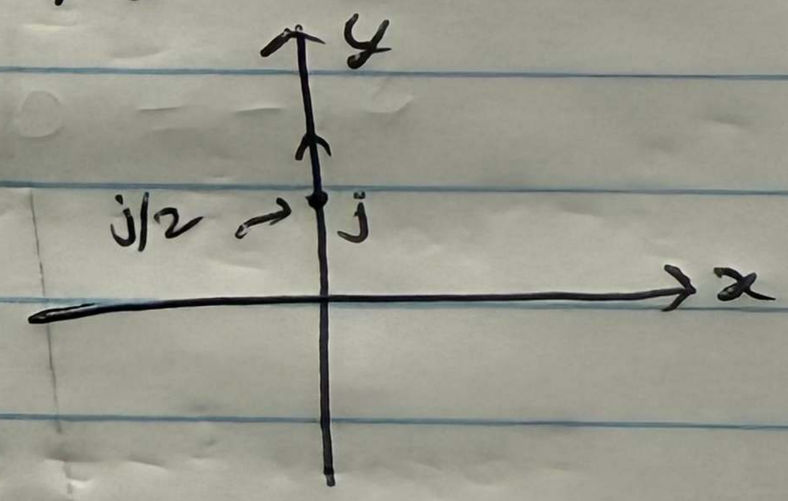
(b) Compress along y-axis by a factor of 2

<u>Action on $\hat{i}$</u>: compression ~~along~~ the space vertically does not affect x-axis therefore no change in $\hat{i}$



$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

<u>Action on $\hat{j}$</u>: y component is compressed by a factor of 2 (i.e. multiply by 0.5)



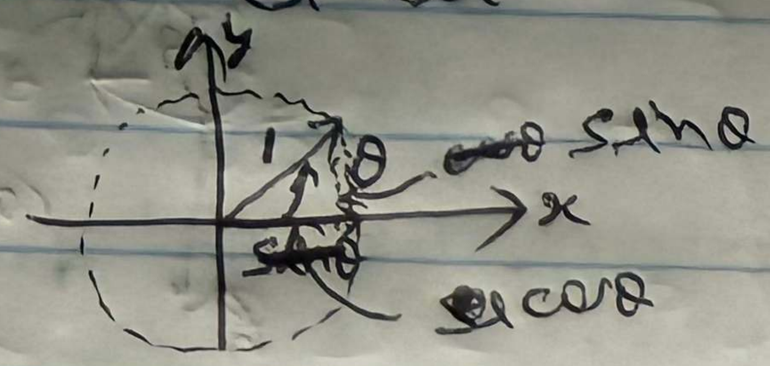
$$\hat{j} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

Resulting matrix

$$T_b = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

(c) Counter clockwise rotation by $\pi/6$

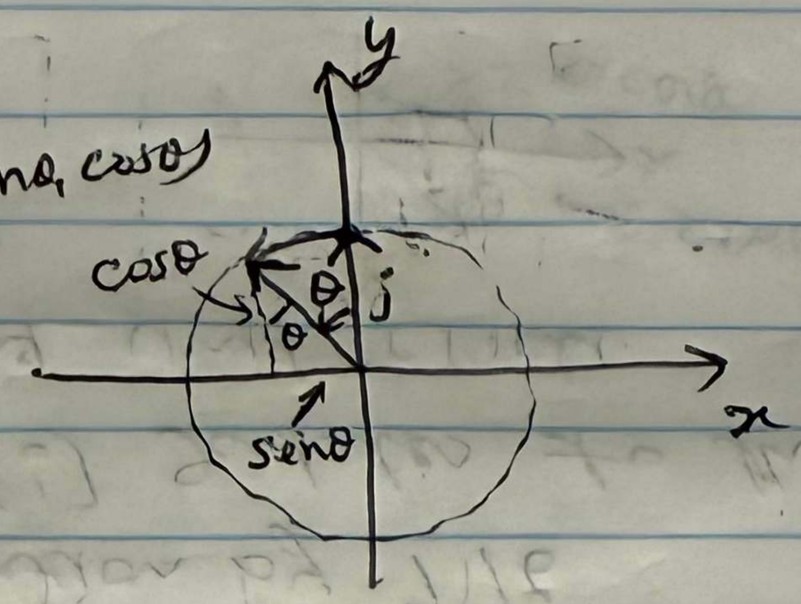
<u>Action on $\hat{i}$</u>: The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates to the point $(\cos\theta, \sin\theta)$ on that unit circle.



$$\hat{i} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Action on $\hat{j}$:

The vector $\begin{bmatrix}0\\1\end{bmatrix}$ rotates to $(-\sin\theta, \cos\theta)$



$$\hat{j} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

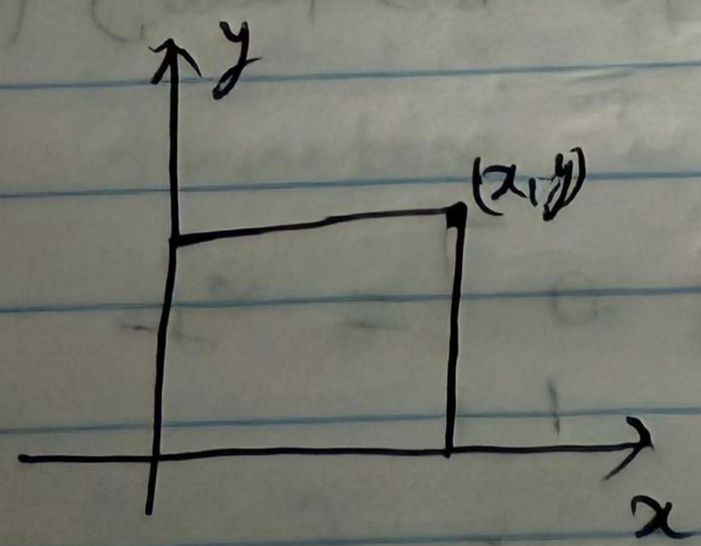
Resulting matrix at $\theta = \pi/6$

$$T_c = \begin{bmatrix}\cos\pi/6 & -\sin\pi/6\\ \sin\pi/6 & \cos\pi/6\end{bmatrix}$$

$$= \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

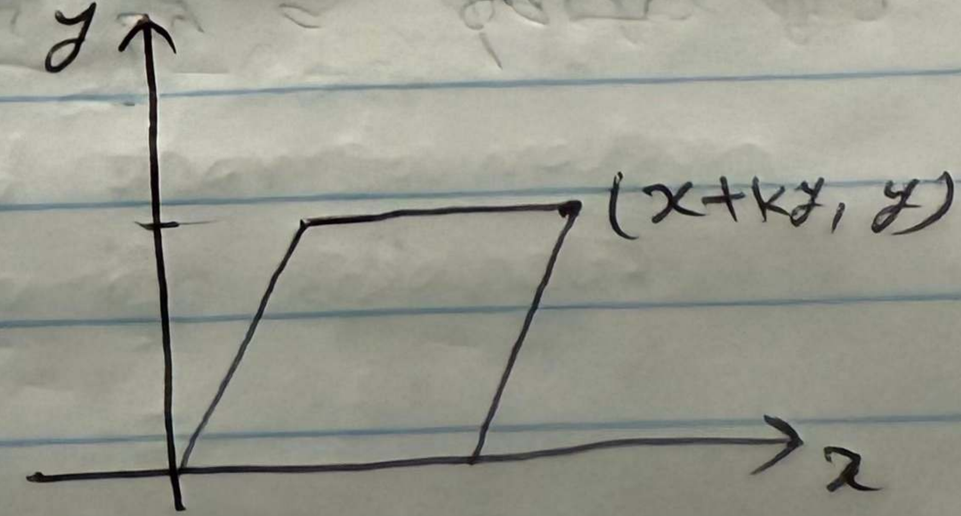
(d) sheer along the x-axis by a factor of 0.5

An x-shear with factor k transform a point (x, y) to $(x+ky, y)$



Original object

x-shear $\Rightarrow$



Object after x-shear

Action on $\hat{i}$:

The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has y-component of 0, so the shear term ky is zero. It means vector does not move.

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Action on $\hat{j}$:

The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has a y-component of 1. It is shifted horizontally by $k \cdot 1 = k$ with $k = 0.5$, we have

$$\hat{j} = \begin{bmatrix} 0 + 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

The resulting matrix

$$T_d = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Combined Transformation:

To perform actions in the given order

$T = T_d \cdot T_c \cdot T_b \cdot T_a$ (we multiply matrix from right to left)

$$T = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1+0=1 & 0+0=0 \\ 0+0=0 & 0-0.5 \\ & =-0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 + 0 & 0 - \frac{1}{2} \times -0.5 \approx \frac{1}{4} \\ \frac{1}{2} + 0 & 0 + \frac{\sqrt{3}}{2} \times -0.5 \approx -\frac{\sqrt{3}}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} + 0.5 \times \frac{1}{2} & \frac{1}{4} + 0.5 \times -\frac{\sqrt{3}}{4} \\ 0 + \frac{1}{2} & 0 + 1 \times -\frac{\sqrt{3}}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2\sqrt{3}+1}{4} & \frac{2-\sqrt{3}}{8} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} \end{bmatrix}$$

Apply the transformation:

We apply the combined transformation matrix T to each vertex of triangle.

For $(0,0)$:

$$\begin{bmatrix} \frac{2\sqrt{3}+1}{4} & \frac{2-\sqrt{3}}{8} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $(1,0)$:

$$\begin{bmatrix} \frac{2\sqrt{3}+1}{4} & \frac{2-\sqrt{3}}{8} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt{3}+1}{4} \\ \frac{1}{2} \end{bmatrix}$$

For $(0.5, \sqrt{3}/2)$:

$$\begin{bmatrix} \frac{2\sqrt{3}+1}{4} & \frac{2-\sqrt{3}}{8} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \cancel{\frac{2\sqrt{3}+1}{4} \times \frac{1}{2} + \frac{2-\sqrt{3}}{8} \times \frac{\sqrt{3}}{2}}$$

$$= \begin{bmatrix} \frac{2\sqrt{3}+1}{4} \times \frac{1}{2} + \frac{2-\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} \\ \frac{1}{2} \times \frac{1}{2} - \frac{\sqrt{3}}{4} \times \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{4\sqrt{3}+2+2\sqrt{3}-3}{16} \\ \frac{1}{4} - \frac{3}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6\sqrt{3}-1}{16} \\ -\frac{1}{8} \end{bmatrix} \approx \begin{bmatrix} 0.587 \\ -0.125 \end{bmatrix}$$