6.C 1-D WAVE EQUATION: EXAMPLE

Note Title 7/24/2013

Recall the 1-D IBVP

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{(PDE)} \\ u(0,t) = 0 \quad , t \geq 0 & (BC_1) \\ u(L,t) = 0 & (BC_2) \\ u(x,0) = f(x) \quad , 0 \leq x \leq L & (IC_1) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = g(x) & (IC_2) \end{cases}$$

with solution

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)}$$

where,

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots$$

$$b_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

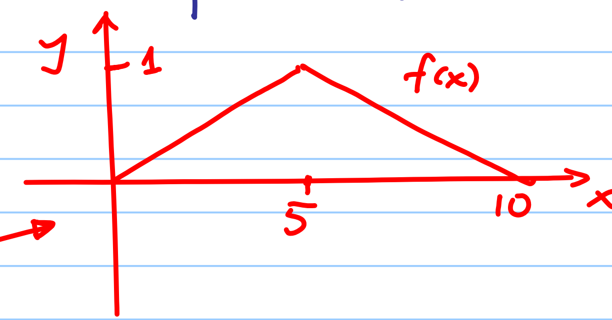
Example: (a) Solve the fixed ends vibrating string problem:

$$u_{tt} = \overset{=c^2}{4}\, u_{xx} \qquad \text{PDE} \qquad c=2$$

$$u(0,t) = u(10,t) = 0 \qquad \text{BCs} \qquad L=10$$

with zero initial velocity and with initial displacement the "tent" function

$$u(x,0) = f(x) = \begin{cases} \dfrac{x}{5}, & 0<x<5 \\ \dfrac{10-x}{5}, & 5\le x<10 \end{cases}$$



(b) Use the first 3 nonzero terms of the answer to approximate the displacement u at location 2 units and time 1 unit.

Sol^n: We have $L=10$, $c=2$. Also, $b_n = 0$, since the initial velocity is zero. We are left with

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{10}\right) \sin\left(\frac{n\pi x}{10}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{5}\right) \sin\left(\frac{n\pi x}{10}\right)$$

$$\int_0^{10} = \int_0^5 + \int_5^{10}$$

$$a_n = \frac{2}{10}\int_0^{10} f(x) \sin\left(\frac{n\pi x}{10}\right) dx, \qquad n = 1, 2, \ldots$$

We have

$$a_n = \frac{1}{5}\int_0^5 \frac{x}{5} \sin\left(\frac{n\pi x}{10}\right) dx + \frac{1}{5}\int_5^{10} \frac{10-x}{5} \sin\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{25}\int_0^5 x \sin\left(\frac{n\pi x}{10}\right) dx + \frac{1}{25}\int_5^{10} (10-x) \sin\left(\frac{n\pi x}{10}\right) dx$$

We integrate by parts by using $u = x$ for the first integral and $u = 10 - x$ for the second to get

$\int u\,dv = uv - \int v\,du$

$$a_n = \frac{1}{25}\left(-\frac{10}{n\pi}\, x \cos\left(\frac{n\pi x}{10}\right)\Big|_0^5 - \left(\frac{-10}{n\pi}\right)\int_0^5 \cos\left(\frac{n\pi x}{10}\right) dx \right) +$$

$$+ \frac{1}{25}\left(-\frac{10}{n\pi}(10-x)\cos\left(\frac{n\pi x}{10}\right)\Big|_5^{10} - \left(\frac{-10}{n\pi}\right)\int_5^{10} (-1)\cos\left(\frac{n\pi x}{10}\right) dx \right)$$

$$= \frac{1}{25}\left(\frac{-50}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right)\Big|_0^5 \right) +$$

$$+ \frac{1}{25}\left(\frac{50}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right)\Big|_5^{10} \right)$$

$$= \underbrace{\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)}_{a_n}$$

Keep in mind:

$\sin(n\pi) = 0$, n integer

$\cos(n\pi) = (-1)^n$, n integer

Hence,

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi t}{5}\right)\sin\left(\frac{n\pi x}{10}\right)$$

(b) Expanding the above sum yields

$$u(x,t) = \frac{8}{\pi^2}\left(\frac{1}{1^2}\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi t}{5}\right)\sin\left(\frac{\pi x}{10}\right) + \frac{1}{2^2}\underbrace{\sin(\pi)}_{=0}\cos\left(\frac{2\pi t}{5}\right)\sin\left(\frac{\pi x}{5}\right)\right.$$

$$+ \frac{1}{3^2}\sin\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi t}{5}\right)\sin\left(\frac{3\pi x}{10}\right) + \frac{1}{4^2}\underbrace{\sin(2\pi)}_{=0}\cos\left(\frac{4\pi t}{5}\right)\sin\left(\frac{2\pi x}{5}\right)$$

$$\left.+ \frac{1}{5^2}\sin\left(\frac{5\pi}{2}\right)\cos(\pi t)\sin\left(\frac{\pi x}{2}\right) + \cdots\right)$$

So

$$u(x,t) \simeq \underbrace{\frac{8}{\pi^2}\left(\cos\left(\frac{\pi t}{5}\right)\sin\left(\frac{\pi x}{10}\right) - \frac{1}{9}\cos\left(\frac{3\pi t}{5}\right)\sin\left(\frac{3\pi x}{10}\right)\right.} \left. + \frac{1}{25}\cos(\pi t)\sin\left(\frac{\pi x}{2}\right)\right)$$

Hence,

$$u(\underset{x}{2},\underset{t}{1}) \simeq \frac{8}{\pi^2}\left(\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{10}\right) - \frac{1}{9}\cos\left(\frac{3\pi}{5}\right)\sin\left(\frac{6\pi}{10}\right) + \frac{1}{25}\cos(\pi)\sin(\pi)\right)$$

$$\simeq 0.41192 \text{ height units.}$$