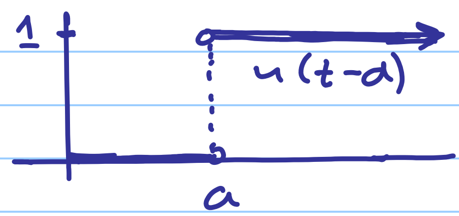
4.D UNIT STEP FUNCTION

Note Title 7/10/2013

The unit step function (or Heaviside function), $u(t-a)$, about $t=a$ $(a \geq 0)$ is the function in t that has a unit jump at $t=a$. I.e.,

$$u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$$

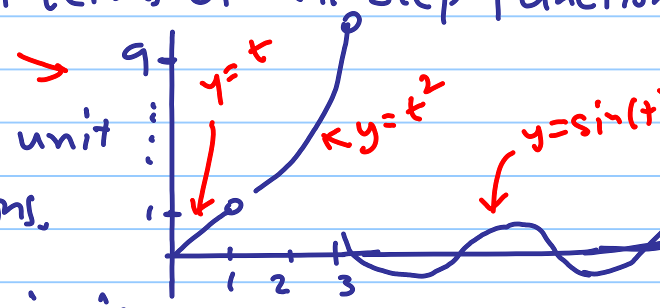


For $a=0$ we just write $u(t)$.

Theorem: (Laplace of Unit Step Function)

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}, \quad \mathcal{L}^{-1}\left(\frac{e^{-as}}{s}\right) = u(t-a)$$

Piecewise continuous functions can be written in terms of unit step functions. This can be very useful.



Example: Write $f(t) = \begin{cases} t, & 0<t<1 \\ t^2, & 1<t<3 \\ \sin t, & t>3 \end{cases}$ in terms of unit step functions.

Solⁿ: We see the points 1, 3 as "switches". At the beginning we "turn on" t by starting with t. Then at "time" $t=1$ we "turn off" t by subtracting $t\,u(t-1)$ and "turn on" t^2 by adding $t^2u(t-1)$. We continue like this to get

$$f(t) = t - t\,u(t-1) + t^2u(t-1) - t^2u(t-3) + \sin(t)\,u(t-3)$$

Theorem: (Second Translation Theorem)

Let $L(f) = F(s)$. Then

$$\boxed{\begin{aligned} L\big(f(t-a)\,u(t-a)\big) &= e^{-as}F(s) \\ L^{-1}\big(e^{-as}F(s)\big) &= f(t-a)\,u(t-a) \end{aligned}}$$

Example 1: Find $L\big((2t-1)\,u(t-1)\big)$.

$\big(L(f(t-a)\,u(t-a)) = e^{-as}F(s)\big)$

Sol^n: We have $a=1$ and $f(t-1) = 2t-1$. Hence, $f(t) = 2(t+1)-1 = 2t+1$.

$$L\big((2t-1)\cdot u(t-1)\big) = e^{-s}L(2t+1) = e^{-s}\left(\frac{2}{s^2}+\frac{1}{s}\right)$$

Example 2: Find $\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2+9}\right)$ $\left(\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)u(t-a)\right)$

Sol^n: $a=5$. Need a function $f(t)$ with Laplace $\frac{1}{s^2+9}$.

Ans. $f(t) = \frac{1}{3}\sin(3t)$.

So $\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2+9}\right) = \frac{1}{3}\sin(3(t-5))\,u(t-5)$

$$= \frac{1}{3}\sin(3t-15)\,u(t-5)$$

$$= \begin{cases} \frac{1}{3}\sin(3t-15), & t>5 \\ 0, & t<5 \end{cases}$$

(optional step)

Example 3: Find $g(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

Sol^n:

$g(t) = e^t - e^t u(t-1) + 0 \cdot u(t-1)$. Hence,

$$L(g(t)) = L(e^t) - L(e^t u(t-1))$$

$$= \frac{1}{s-1} - e^{-s} L(e \cdot e^t)$$

$$= \frac{1}{s-1} - e^{-s} \cdot e \cdot L(e^t)$$

$$= \frac{1}{s-1} - \frac{e^{1-s}}{s-1}$$

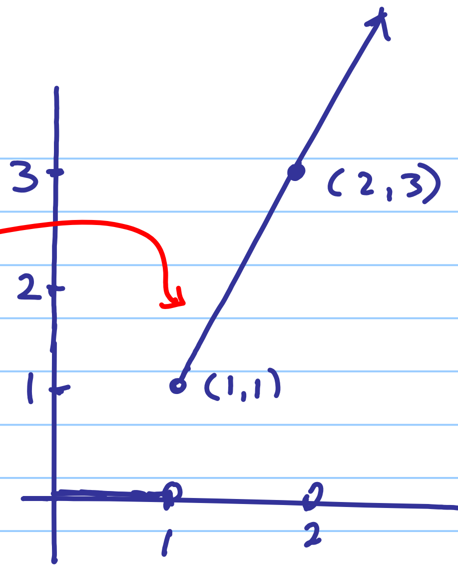
$$= \frac{1 - e^{1-s}}{s-1}$$

$\Big(f(t-1) = e^t \Rightarrow f(t) = e^{t+1} = e \cdot e^t \Big)$

constant

Example 4: Solve the initial value problem.

$y' + y = g(t)$, $y(0) = 0$ where $g(t)$



Sol^n: $g(t) = \begin{cases} 0 , & 0 < t < 1 \\ 2t-1 , & t > 1 \end{cases}$

$g(t) = (2t-1)\,u(t-1)$. By Example 1,

$L(g) = e^{-s}\left(\frac{2}{s^2} + \frac{1}{s}\right)$

Apply L to diff. eq. $(Y = L(y))$:

$sY - 0 + Y = e^{-s}\left(\frac{2}{s^2} + \frac{1}{s}\right)$

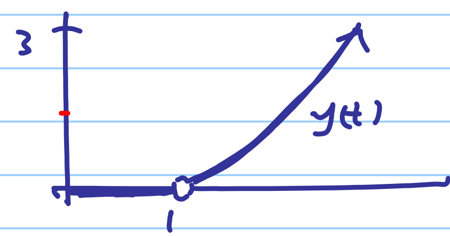
$Y = e^{-s}\frac{s+2}{s^2(s+1)} = e^{-s}\left(\frac{1}{s+1} - \frac{1}{s} + \frac{2}{s^2}\right)$

$$Y = \frac{e^{-s}}{s+1} - \frac{e^{-s}}{s} + \frac{2e^{-s}}{s^2}$$

Take L^{-1}:

$$y(t) = e^{-(t-1)}u(t-1) - u(t-1) + 2(t-1)u(t-1)$$

$$y(t) = \begin{cases} 0, & t < 1 \\ e^{1-t} + 2t - 3, & t > 1 \end{cases}$$



$$L^{-1}\left(\frac{e^{-s}}{s+1}\right) = e^{-(t-1)}u(t-1)$$

$a = 1$

$$\frac{1}{s+1} \xrightarrow{?} f(t) = e^{-t}$$

$$L^{-1}(\quad) = f(t-a)u(t-a)$$