Note Title 8/14/2013

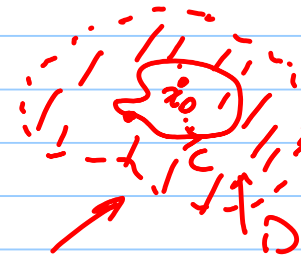
12.C CAUCHY'S INTEGRAL FORMULA

Theorem 1 (Cauchy's Integral Formula)

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D, that encloses z_0

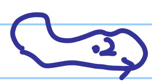
$$\boxed{\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)}$$

(counterclockwise)



Example 1:

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i\, e^z \Big|_{z=2} = 2\pi i e^2 \simeq 46.42\, i$$

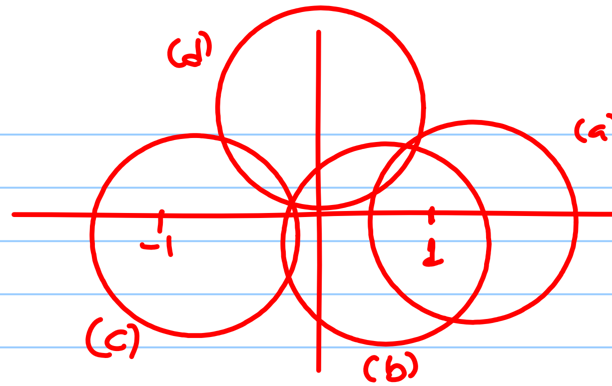


Example 2:

$$\oint_C \frac{z^3-6}{2z-i}\,dz = \oint_C \frac{\overbrace{\tfrac{1}{2}z^3-3}^{f(z)}}{z-\tfrac{1}{2}i}\,dz = 2\pi i\left[\tfrac{1}{2}z^3-3\right]_{z=i/2} = \frac{\pi}{8} - 6\pi i$$

C: $z_0 = \frac{1}{2}i$ inside C

Example 3: Integrate $g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z-1)(z+1)}$ counterclockwise over each of the regions (a), (b), (c), (d).



Sol: $g(z)$ is not analytic at -1 and 1. So, these are the points to watch for.

(a) Curve (a) contains 1, the only point enclosed by (a) where $g(z)$ is not analytic. So

$$\int_C \frac{z^2+1}{z^2-1}\,dz = \int_C \frac{\overbrace{(z^2+1)/(z+1)}^{=f(z)}}{\underbrace{z-1}_{=z_0}}\,dz = 2\pi i\, f(1) = 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i$$

(b) The answer is the same by deformation of path.

(c) $z_0 = -1$ is the only point where $g(z)$ is not analytic inside curve (c). Hence,

$$\int_C \frac{z^2+1}{z^2-1}\,dz = \int_C \frac{\overbrace{(z^2+1)/(z-1)}^{=f(z)}}{z+1}\,dz \underset{z_0=-1}{=} 2\pi i\, f(-1) = 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i$$

(d) The answer is 0, by Cauchy's Integral Theorem, because $g(z)$ is analytic on + in (d).

Example 3: $\displaystyle\int_C \frac{\tan z}{z^2-1}\,dz$

$C: |z| = 3/2$ counter clockwise

Sol^n $\tan z$ is not analytic at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$. These points are outside C.
Now $(z^2-1)^{-1} = (z-1)^{-1}(z+1)^{-1}$ is not analytic at 1 and -1. To get integrals of the form $\oint_C \frac{f(z)}{z-z_0}\,dz$ we use partial fractions to have one "bad" point at a time

$$\frac{1}{z^2-1} = \frac{1/2}{z-1} + \frac{1/2}{z+1}$$

(for z_0 is 1, the second -1)

$$\oint_C \frac{\tan z}{z^2-1}\,dz = \frac{1}{2}\left[\oint_C \frac{\tan z}{z-1}\,dz - \oint_C \frac{\tan z}{z+1}\,dz\right] = \frac{2\pi i}{2}\left[\tan 1 - \tan(-1)\right] \simeq 9.78\, i$$