535.641 Mathematical Methods Assignment 6

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1. Find all critical points of the system and for each critical point state each classification that applies: (a) stable, (b) unstable, (c) attractor, (d) repeller, (e) saddle point.

$$\frac{dx}{dt} = -(x-y)(1-x-y)$$
$$\frac{dy}{dt} = x(2+y)$$

Ans: To find the critical points of the system

$$\frac{dx}{dt} = 0 \Rightarrow -(x-y)(1-x-y) = 0 \quad \text{—①}$$

$$\frac{dy}{dt} = 0 \Rightarrow \quad x(2+y) = 0 \quad \text{—②}$$

From second equation

$x = 0$ or

$y = -2$

On substituting in ~~equation~~ the first equation

Case 1 $x = 0$

$-(0-y)(1-0-y) = 0$

$y(1-y) = 0$

This gives

$y = 0, y = 1$

This gives two critical points:

$(0,0)$ and $(0,1)$

Case 2: $y = -2$

$-(x-(-2))(1-x-(-2)) = 0$

$-(x+2)(3-x) = 0$

i.e $x = -2, x = 3$

This gives two more critical points

$(-2,-2)$ and $(3,-2)$

Linearization and classification:

To classify critical points, we first compute the Jacobinan matrix of the system.

$$f(x,y) = \frac{dx}{dt}$$

$$g(x,y) = \frac{dy}{dt}$$

$$f(x,y) = -(x-y)(1-x-y)$$
$$= -(x-y-x^2 \cancel{+xy} + xy - xy + y^2)$$
$$= -(x-y-x^2+y^2)$$
$$= x^2 - x - y^2 + y$$

$$g(x,y) = 2x + xy$$

The Jacobian matrix $J(x,y)$ is given by

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x-1 & 1-2y \\ 2+y & x \end{bmatrix}$$

Now evaluate the Jacobian at each critical point to determin its classification.

(1) Critical point $(0,0)$:

$$J(0,0) = \begin{bmatrix} 2\cdot 0-1 & 1-2\cdot 0 \\ 2+0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$$

The eigenvalue λ can be calculated as below

$$\det(J - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$(-1-\lambda)(-\lambda) - (1)(2) = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda + 2)(\lambda - 1) = 0$$

$$\Rightarrow \lambda_1 = 1, \ \lambda_2 = -2$$

Since the eigenvalues are real and have opposite signs, the point $(0,0)$ is a saddle point. i.e classified as
→ (b) unstable
→ (e) saddle point

(2) critical point $(0,1)$

$$J(0,1) = \begin{bmatrix} 2(0) - 1 & 1 - 2(1) \\ 2+1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$$

To find eigenvalue using Characteristic equation

$$\det(J - \lambda I) = 0$$

$$(-1-\lambda)(-\lambda) - (-1)(3) = 0$$

$$\lambda^2 + \lambda + 3 = 0$$

This gives

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2}$$

$$= -\frac{1}{2} \pm i\frac{\sqrt{11}}{2}$$

Since eigenvalues are complex conjugates with a negative real part, the point (0,1) is a stable spiral. We can classify it as below

→ (a) stable
→ (c) attractor

(3) Critical point (−2, −2)

$$J(-2,-2) = \begin{bmatrix} 2(-2)-1 & 1-2(-2) \\ 2+(-2) & -2 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$$

Since this is an upper triangular matrix, the eigenvalues are the diagonal entries.

$$\lambda_1 = -5$$

$$\lambda_2 = -2$$

Because both eigenvalues are real, distinct and negative, the point (−2,−2) is a stable node. It can be classified as below

→ (a) stable
→ (c) attractor

(4) Critical point (3, -2)

$$J(3,-2) = \begin{bmatrix} 2(3)-1 & 1-2(-2) \\ 2+(-2) & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$$

This is also a upper triangular matrix, so

$\lambda_1 = 5$

$\lambda_2 = 3$

Because both eigenvalues are real, distinct and positive, the point (3, -2) is a unstable node. It can be classified as below

→ (b) unstable

→ (d) repeller

2. Find the linearization of the following system at each critical point. For each linearization find the eigenvalues and eigenvectors (if the eigenvectors are complex do not compute the eigenvectors) and sketch each one of the linearized phase portraits. Then use software to draw a phase portrait of the system itself.

$$\frac{dx}{dt} = -(x-y)(1-x-y)$$
$$\frac{dy}{dt} = x(2+y)$$

Ans: This problem is using same system as previous one therefore critical points will be same as below

$(0,0)$, $(0,1)$, $(-2,-2)$ and $(3,-2)$

Similarly, Jacobian matrix for the system is

$$J(x,y) = \begin{bmatrix} 2x-1 & 1-2y \\ 2+y & x \end{bmatrix}$$

let's analyse each points

(1) Critical point $(0,0)$

⇒ Linearization:

The linearized system at $(0,0)$ is

$$u' = Au$$

when A is Jacobian evaluated at the point

$$A = J(0,0) = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$$

⇒ Eigenvalues & Eigenvectors

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -2$

Eigenvectors:

using setup equation

$$(A-\lambda I)V=0$$

$$\left(\begin{bmatrix}-1 & 1\\ 2 & 0\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)V=0$$

$$\begin{bmatrix}-1-\lambda & 1\\ 2 & -\lambda\end{bmatrix}V=0$$

putting value of $\lambda_1=1$

$$\begin{bmatrix}-2 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=0$$

$$-2v_1+v_2=0$$

$$2v_1-v_2=0$$

$$\Rightarrow v_2=2v_1$$

simplest Eigenvector is

$$v_1=\begin{bmatrix}1\\ 2\end{bmatrix}$$

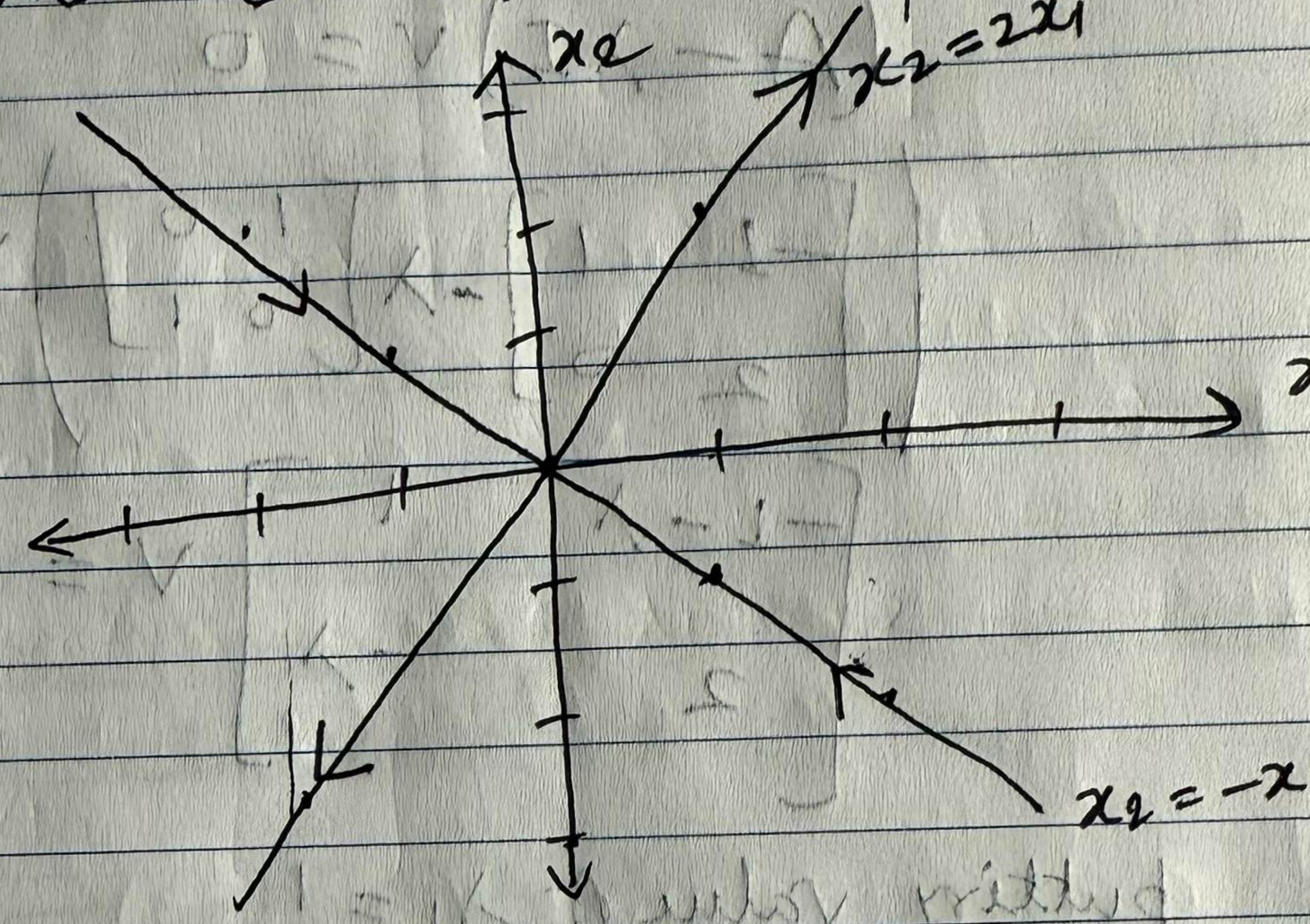
At $\lambda_2=-2$

$$\begin{bmatrix}1 & 1\\ 2 & +2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=0$$

$$\left.\begin{aligned}v_1+v_2&=0\\ 2v_1+2v_2&=0\end{aligned}\right\}$$ This gives simplified eigenvector $v_2=\begin{bmatrix}1\\ -1\end{bmatrix}$

⇒ Linearized Phase Portrait sketch

Since the eigenvalues are real with opposite signs, this is a saddle point.



This gives following solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{+t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

Trajectories move away from the origin along the eigenvector v_1 (for $\lambda_1 > 0$) and move toward the origin along the eigenvector v_2 ($\lambda_2 < 0$)

(2) Critical point (0,1)

Linearization:

$$A = J(0,1) = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

The eigenvalues are complex: $\lambda = -\frac{1}{2} \pm i \frac{\sqrt{11}}{2}$

As per instruction, we are not calculating the complex eigenvectors.

Linearized Phase Portrait sketch:

Because the eigenvalues are complex with negative real part $(-1/2)$, the point is stable spiral. The trajectories spiral inward toward the origin.

System solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t/2}\left[A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]$$