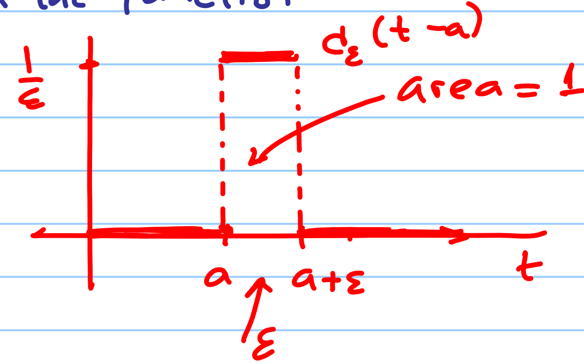
4.E DIRAC DELTA and SYSTEMS OF ODES

Note Title 7/10/2013

Dirac Delta Function

We need a function to model sudden large changes in quantities we are interested in. For example, if a weight attached to a spring is hit by a hammer, or a baseball is hit by a bat. Such changes may be modelled by using Dirac's delta function. We start with the function

$$\rightarrow \quad d_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon}, & a < t < a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$$

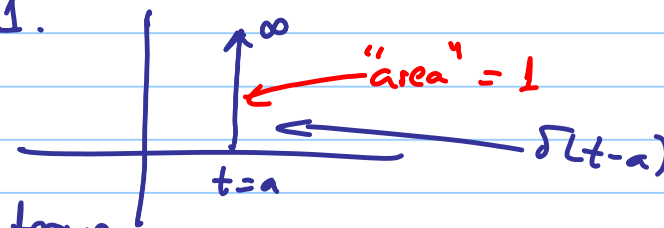


The graph of $d_\varepsilon(t-a)$ is a rectangular wave of with ε and height $1/\varepsilon$. So its area is 1.

If $\varepsilon \to 0$ the area is still one but the rectangles get slimmer and taller. The limit as $\varepsilon \to 0$ is called <u>Dirac's Delta function</u> or <u>unit impulse function</u>:

$$\rightarrow \quad \delta(t-a) = \lim_{\varepsilon \to 0} d_\varepsilon(t-a)$$

So δ has one value which is ∞ at $t=a$ and zero, if $t \neq a$ and yet the area under the graph of δ is 1.



Of course, no such function exists!
$\delta(t-a)$ is a kind of "generalized function" which can be defined in precise mathematical terms.

For our purposes we only need to know:

- $\delta(t-a) = \begin{cases} \infty, & \text{if } t=a \\ 0, & \text{if } t \neq a \end{cases}$
- $\int_0^\infty \delta(t-a)\, dt = 1$
- $\int_0^\infty f(t)\delta(t-a)\, dt = f(a)$

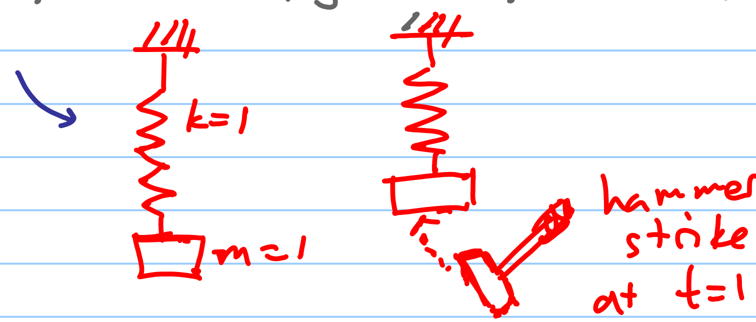
and

$$\boxed{\begin{array}{l} \bullet\ \mathcal{L}(\delta(t-a)) = e^{-as} \\ \bullet\ \mathcal{L}^{-1}(e^{-as}) = \delta(t-a) \\ \text{Special case:} \\ \bullet\ \mathcal{L}(\delta(t)) = 1 \quad (a=0) \end{array}}$$

Example: Solve the IVP.

$$y'' + y = 5\,\delta(t-1), \quad y(0) = 2, \quad y'(0) = 0$$

This models the displacement $y(t)$ of a mass of 1 mass unit attached to a spring where at time $t=1$ a hammer blow applies "five times the unit infinite force" during an infinitesimal time interval.



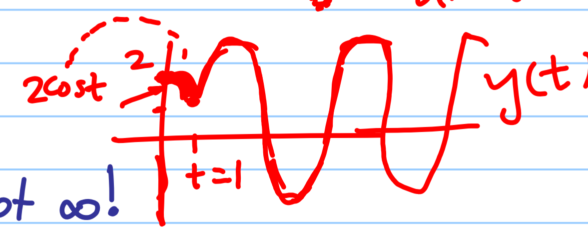
Sol^n: Apply $\mathcal{L}$:

$$s^2 Y - 2s - 0 + Y = 5e^{-s}$$

$$Y = \frac{2s}{s^2+1} + \frac{5e^{-s}}{s^2+1}$$

Thus,

$$y(t) = \underbrace{2\cos t}_{\text{no strike}} + \underbrace{5\sin(t-1)\,u(t-1)}_{\text{added hammer strike at } t=1}$$



Notice that at $t=1$ the amplitude increases but it's not ∞!

Systems of Ordinary Differential Equations by Laplace

Example: Solve the homogeneous system of differential equations for the unknown functions $y_1(t)$, $y_2(t)$, $y_3(t)$

$$\begin{cases} \dfrac{dy_1}{dt} = -y_1 + 8y_3 \\ \dfrac{dy_2}{dt} = -y_2 + y_3 \\ \dfrac{dy_3}{dt} = y_1 + y_3 \end{cases}$$

with initial conditions

$y_1(0) = -4$

$y_2(0) = 4$

$y_3(0) = -2$

Sol^n: Apply L:

$$\left.\begin{aligned} sY_1 + 4 &= -Y_1 + 8Y_3 \\ sY_2 - 4 &= -Y_2 + Y_3 \\ sY_3 + 2 &= Y_1 + Y_3 \end{aligned}\right\}$$

$Y_1 = L(y_1(t))$

Rewrite :

$$(s+1)Y_1 - 8Y_3 = -4$$
$$(s+1)Y_2 - Y_3 = 4$$
$$-Y_1 + (s-1)Y_3 = -2$$

Linear system in Y_1, Y_2, Y_3 !

Solve by Cramer's Rule to get

$$Y_1 = -\frac{4}{s-3}, \quad Y_2 = \frac{4s-14}{s^2-2s-3}, \quad Y_3 = -\frac{2}{s-3}$$

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$$\frac{9}{2(s+1)} - \frac{1}{2(s-3)}$$

Then $\mathcal{L}^{-1}$ to get :

$$y_1(t) = -4e^{3t}$$

$$y_2(t) = \frac{9}{2}e^{-t} - \frac{1}{2}e^{3t}$$

$$y_3(t) = -2e^{3t}$$