5.B ORTHOGONAL SETS OF FUNCTIONS 1

Note Title 7/21/2013

We consider continuous real-valued functions on an interval $[a,b]$. They are in $C[a,b]$ which is an inner product vector space with inner product

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

and norms:

$$\| f \| = \left(\int_a^b (f(x))^2\, dx \right)^{\frac{1}{2}}$$

Let $S = \{ g_1(x), g_2(x), \ldots, g_n(x), \ldots \}$ be a subset of $C[a,b]$

We say that S is orthogonal on $[a,b]$ if

$\rightarrow$ $\langle g_n, g_m \rangle = 0$ for all m, n s.t. $m \neq n$

We say that S is orthogonormal on $[a,b]$, if

1. S is orthogonal on $[a,b]$

and

2. All $g_n(x)$ in S are unit (have norm 1) : $\|g_n\| = 1$

So, equivalently,

1. $\langle g_m, g_n \rangle = 0$ for all $m \neq n$

and 2. $\|g_n\| = 1$ for all n

Note: Since $\|g_n\| = 1 \iff \|g_n\|^2 = \langle g_n, g_n \rangle = 1^2 = 1$

we may combine 1 & 2 into the condenced

$$S \text{ is orthonormal} \iff \langle g_m, g_n \rangle = \begin{cases} 0 \,, & m \neq n \\ 1 \,, & m = n \end{cases}$$

Note: If we have an orthogonal set $S = \{ g_1(x), \ldots, g_n(x), \ldots \}$ of nonzero functions, then we can turn it to orthonormal by diving by the norms

$$S_1 = \left\{ \frac{g_1}{\|g_1\|}, \frac{g_2}{\|g_2\|}, \ldots, \frac{g_n}{\|g_n\|}, \ldots \right\}$$

(Just as we make a vector $v_{\neq 0}$ a unit vector $u = \dfrac{v}{\|v\|}$)

So the more important notion here is orthogonality.

ASSUMPTIONS:
1. all functions we discuss are bounded on $[a,b]$.
2. their integrals over $[a,b]$ are finite
3. their norms are nonzero.

Examples

Example 1: Let $S = \{\sin(x), \sin(2x), \ldots, \underbrace{\sin(nx)}_{g_n}, \ldots\}$

Check for orthogonality and if orthogonal compute the norms on

(a) $[0, \pi]$ (b) $[-\pi, \pi]$ (c) $[0, 1]$

Sol^n: (a) $\langle g_m, g_n \rangle_{m \neq n} = \int_0^{\pi} \sin(mx)\sin(nx)\,dx \underset{\text{Trig. Id.}}{=}$ $\left(\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b)) \right)$

$$= \frac{1}{2}\int_0^{\pi} \left(\cos((m-n)x) - \cos((m+n)x)\right)dx$$

$$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n \neq 0}\Big|_0^{\pi} - \frac{\sin((m+n)x)}{m+n}\Big|_0^{\pi} \right]$$

$(\sin(k\pi) = 0 \quad k \text{ integer})$

$$= 0 + 0 = 0 \qquad \text{ORTHOGONAL}$$

Norm $\quad \|g_n\|^2 = \int_0^{\pi} \sin^2(nx)\,dx =$

(Trig. id. Half-angle formulas

$\sin^2(a) = \frac{1}{2}(1-\cos(2a))$

$\cos^2(a) = \frac{1}{2}(1+\cos(2a))$)

$$= \frac{1}{2}\int_0^{\pi} (1-\cos(2nx))\,dx$$

$$= \frac{1}{2}\left(x - \frac{\sin(2nx)}{2n}\right)\Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

So $\|g_n\| = \sqrt{\frac{\pi}{2}}$

(6) Orthogonality is valid: Just as before but the limits: $\int_{-\pi}^{\pi}$

Norm: $\|g_n\|^2 = \int_{-\pi}^{\pi} \cdots = \frac{1}{2}\left(x - \frac{\sin(2nx)}{2n}\right)\Big|_{-\pi}^{\pi} = \pi$

So $\|g_n\| = \sqrt{\pi}$

(c) $\langle g_m, g_n \rangle = \int_0^1 \cdots =$

$$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{\underbrace{m-n}_{\neq 0}} \Bigg|_0^1 - \frac{\sin((m+n)x)}{m+n} \Bigg|_0^1 \right]$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)}{m-n} - \frac{\sin(m+n)}{m+n} \right] \neq 0$$

$\left.\begin{matrix} m-n \\ m+n \end{matrix}\right\}$ different integers

Not orthogonal

E.g. Take $m=2$, $n=1$

$$\frac{1}{2}\left(\frac{\sin(1)}{1} - \frac{\sin(3)}{3}\right) \neq 0$$

Example 2: An identical computation as in Example 1 shows that

$$S = \left\{ \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \ldots, \sin\left(\frac{n\pi x}{L}\right), \ldots \right\} \qquad L > 0 \text{ constant}$$

is

(a) Orthogonal on $[0, L]$

norms : $\left\| \sin\left(\frac{n\pi x}{L}\right) \right\| = \sqrt{\frac{L}{2}}$

(b) Orthogonal on $[-L, L]$

norms : $\left\| \sin\left(\frac{n\pi x}{L}\right) \right\| = \sqrt{L}$

Special case :
$L = \pi$
get Example 1