Note Title 7/2/2013

2.A VECTOR SPACES

A vector space V is a set with two operations:

- Addition $(u,v) \longrightarrow u+v$ $(u,v \in V)$
- Scalar multiplication $(c,u) \longrightarrow cu$ $(c \in \mathbb{R},\ u \in V)$

such that for all $u,v,w \in V$ and all $c,d \in \mathbb{R}$ the following axioms hold.

(A1) $u+v \in V$ (addition is closed)

(A2) $u+v = v+u$ (commutative)

(A3) $(u+v)+w = u+(v+w)$ (Associative)

(A4) There is an element, $0 \in V$ st. $u+0=u$

(A5) For each u, there is $-u \in V$ st. $u+(-u)=0$
opposite of u

(M1) $cu \in V$ (sc. mult^n is closed)

(M2) $c(u+v) = cu+cv$ (distributive)

(M3) $(c+d)u = cu+du$ (distributive)

(M4) $c(du) = (cd)u$

(M5) $1u = u$

Motivation: Model sets after $\mathbb{R}^n$, $M_{m\times n}$.

Examples:

- $\mathbb{R}^n$ set of all n-vectors (usual addition, scalar multiplication)
- $M_{m\times n}$ set of all $m\times n$ matrices (" " , " ")
- P set of all polynomials (usual operations)

Vector spaces

- $F(R)$ set of all functions $f: \mathbb{R}\to\mathbb{R}$ is a v.s. under

  addition : $f+g : \mathbb{R}\to\mathbb{R}$, $(f+g)(x) = f(x)+g(x)$ all $x\in\mathbb{R}$

  sc. mult : $cf : \mathbb{R}\to\mathbb{R}$, $(cf)(x) = c f(x)$ all $x\in\mathbb{R}$

Example: Let $V = \mathbb{R}^2$ with the usual addition. The scalar multiplication is defined by

$$c \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \stackrel{\text{def}^n}{=} \begin{bmatrix} ca_1 \\ a_2 \end{bmatrix}.$$

Is V a vector space?

Sol^n (M3) : $(c+d)u = cu + du$

$$(c+d) \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (c+d)a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 + da_1 \\ a_2 \end{bmatrix} \neq$$

NOT a vector space

$$c \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + d \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} da_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 + da_1 \\ 2a_2 \end{bmatrix}$$

SUBSPACES

A subset W of a v.s. V is a (vector) subspace if W itself is a v.s. under the same addition and scalar multiplication as V.

Theorem (Criterion for Subspace)

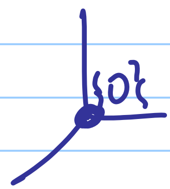
Let W be a nonempty subset of a v.s. V. Then
W is a subspace of V

$\Longleftrightarrow$ 1. If $u, v \in W \Rightarrow u+v \in W$
2. If $c \in \mathbb{R}$, $u \in W \Rightarrow cu \in W$

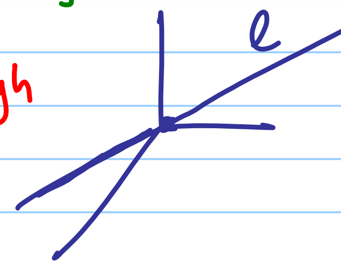
i.e. $\Longleftrightarrow$ W is closed under the two operations

1. Geometric Examples of Subspaces

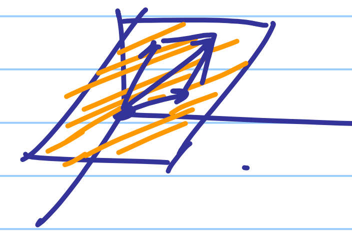
$\{0\}$



any line through the origin



any plane through the origin



2. The Span of any set of vectors

3. Other Examples

Example: Is $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1 + 3x_2 = 0 \right\}$ a subspace of $\mathbb{R}^2$? What about $K = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1 + 3x_2 = 1 \right\}$?

Sol^n S: $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $v = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in S. $x_1 + 3x_2 = 0$, $y_1 + 3y_2 = 0$

$u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$? in S $(x_1+y_1) + 3(x_2+y_2) = (x_1+3x_2) + (y_1+3y_2) = 0 + 0 = 0$. Yes it's in S

$cu = c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$? is S $(cx_1) + 3(cx_2) = c(x_1 + 3x_2) = c \cdot 0 = 0$. Yes it's in S

S is a subspace. (K : NO)

Example: Is $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 x_2 x_3 = 0 \right\}$ a subspace of $\mathbb{R}^3$?

Sol^n NO : $\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\in S} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\in S} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin S$ not closed under addition.

More Examples

- P_n all polynomials of degree $\leq n$ is a subspace of P
- $C(\mathbb{R})$ all continuous functions $f: \mathbb{R} \to \mathbb{R}$ is a subspace of $F(\mathbb{R})$