535.641 Mathematical Methods Assignment 7

Ben Minnick **Name**____________________________________

1	/25
2	/25
3	/25
4	/25
TOTAL	/100

1. Consider the damped pendulum equation

$$\frac{d^2\theta}{dt^2} + \alpha\frac{d\theta}{dt} + \beta\sin(\theta) = 0$$

for $\alpha, \beta > 0$.

(a) Find and classify the critical points

(b) Use software to plot the phase portrait of the system for $\alpha = \beta = 1$.

2. Consider the general competitive Lotka-Volterra system:

$$\begin{aligned}\frac{dx}{dt} &= x(\alpha_1 - \beta_{11}x - \beta_{12}y) \\ \frac{dy}{dt} &= y(\alpha_2 - \beta_{21}x - \beta_{22}y)\end{aligned}$$

where:

- $x(t), y(t) \geq 0$ represent the population sizes of two competing species,
- $\alpha_1, \alpha_2 > 0$ are intrinsic growth rates, and
- $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22} > 0$ represent intra- and interspecific competition coefficients

(a) Find all equilibrium points

(b) Analyze the stability of the equilibrium point when neither species exists as well as the stability of equilibrium points where only one species exists and the other goes extinct. *You do not need to analyze the stability of the equilibrium where both species coexist.*

3. The SIR model is a dynamical system that can be used to describe the spread of an infectious disease in a closed population,

$$\frac{dS}{dt} = -\beta S I$$
$$\frac{dI}{dt} = \beta S I - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

where:

- $S(t)$: number of susceptible individuals
- $I(t)$: number of infected individuals
- $R(t)$: number of recovered individuals
- $\beta > 0$: transmission rate
- $\gamma > 0$: recovery rate

(a) Show that the total population, $N = S(t) + I(t) + R(t)$ is constant

(b) What are the conditions for an outbreak (number of infected individuals increases with time)?

(c) What happens to the number of infected individuals as $t \to \infty$?

(d) Derive the expression,

$$\ln\left(\frac{S_\infty}{S_0}\right) = -\frac{\beta}{\gamma}(R_\infty - R_0)$$

where S_0, R_0 describe some initial number of susceptible and recovered individuals and S_∞, R_∞ describes the final number of susceptible and recovered individuals.

4. Use the Poincare-Bendixon Theorem as in the method of Example 3.6.3 of the instructor's notes to show that the following system has a closed trajectory. Use as a fact that $(0, 0)$ is the only equilibrium of the system.

$$\frac{dx}{dt} = x - y - x(x^2 + y^2)$$
$$\frac{dy}{dt} = x + y - y(x^2 + y^2)$$