6.B 1-DIMENSIONAL WAVE EQUATION

Note Title 7/24/2013

Solving the 1-D wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(PDE)}$$ ← The PDE

subject to the fixed end boundary conditions:

fixed ends

$$u(0,t) = 0 \qquad \text{(BC}_1\text{)}$$
$$u(L,t) = 0 \quad , t \geq 0 \qquad \text{(BC}_2\text{)}$$

← Boundary conditions

and initial conditions satisfying an initial deflection $f(x)$ and initial velocity $g(x)$, for x s.t. $0 \leq x \leq L$.

Initial displacement
$$u(x,0) = f(x) \qquad 0 \leq x \leq L \qquad \text{(IC}_1\text{)}$$

initial velocity
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \qquad \text{(IC}_2\text{)}$$

← Initial conditions

Note: PDE + BCs = Boundary value Problem (BVP)
PDE + ICs = Initial Value Problem (IVP)
PDE + BCs + ICs = IBVP

$$\boxed{\begin{array}{ll} u_{tt} = c^2 u_{xx} & \text{(PDE)} \\ u(0,t) = 0 \quad (t \geq 0) & \text{(BCs)} \\ u(L,t) = 0 & \\ u(x,0) = f(x) \quad (0 \leq x \leq L) & \\ u_t(x,0) = g(x) & \text{(ICs)} \end{array}}$$

Method of Solution:

STAGE 1: SEPARATION OF VARIABLES

At this stage we only use the PDE + the BCs.
We seek solutions of the form

$$\longrightarrow \quad u(x,t) = X(x)\,T(t)$$

$x^2 \cdot t$, t^3 , x^4 $\quad \cancel{x+t} \quad \cancel{x^2+t^2}$

where $X(x)$ is a function in x only and $T(t)$ is a function in t only

$u = XT$ $\qquad \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

Substitution into the PDE:

$$\underline{XT'' = c^2 X'' T}$$

(Here $X' = \dfrac{dX}{dx}$, $T' = \dfrac{dT}{dt}$, etc.)

we separate the variables by dividing by c^2XT to get

$$\frac{T''}{c^2T} = \frac{X''}{X}$$

Now x and t are completely independent variables, one being location and the other time. So the only way the functions $\frac{T''}{c^2T}$ of t and $\frac{X''}{X}$ of x can be equal is if they are both the same constant, say, $-\lambda$. So,

$$\frac{T''}{c^2T} = \frac{X''}{X} = -\lambda$$

Therefore, $\qquad T'' + c^2\lambda T = 0$ and $X'' + \lambda X = 0$

$u_{tt} = c^2 u_{xx}$
$u(0,t) = 0 = u(L,t)$

These are simple 2nd order linear homogeneous ODEs with constant coefficients. They can be solved easily, provided that we know λ. The possible values from λ will emerge from the boundary conditions.

We have $X'' + \lambda X = 0$. This has obviously the trivial solution as one of its solutions. We look for other solutions. Start with the BCs:

BC_1 : $u(0,t) = X(0)\,T(t) \overset{\text{set}}{=} 0$ for all $t \geq 0$

But $T(t)$ cannot be 0 for all t or else u would the trivial solution.

So

$X(0) = 0$

BC_2: $u(L,t) = X(L)\,T(t) \overset{\text{set}}{=} 0 \Rightarrow$ (Likewise) $X(L) = 0$

So we have the following simple S-L problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases}$$

That we have essentially solved before

Let us take another look.

(Examples done before $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$ and $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$)

Case 1: $\lambda < 0$, $\lambda = -v^2$ $(v > 0)$

$$X'' - v^2 X = 0 \Rightarrow r^2 - v^2 = 0 \Rightarrow r = \pm v$$

$\rightarrow$ $X(x) = c_1 e^{vx} + c_2 e^{-vx}$

$\rightarrow$ $X(0) = c_1 + c_2 \overset{\text{set}}{=} 0 \Rightarrow c_2 = -c_1$

$\rightarrow$ $X(L) = c_1 e^{vL} + c_2 e^{-vL} = 0 \Rightarrow c_1 \underbrace{(e^{vL} - e^{-vL})}_{\neq 0} = 0 \Rightarrow \left.\begin{matrix} c_1 = 0 \\ c_2 = 0 \end{matrix}\right\}$ Trivial solution

Case 2: $\lambda = 0$, $\quad X'' = 0 \Rightarrow X(x) = c_1 x + c_2$

$$\left.\begin{array}{l} X(0) = c_1 \cdot 0 + c_2 \overset{\text{set}}{=} 0 \Rightarrow c_2 = 0 \\ X(L) = \underset{\not 0}{c_1 L} \underset{\text{set}}{=} 0 \Rightarrow c_1 = 0 \end{array}\right\} \text{Trivial solution}$$

Case 3: $\lambda > 0$, say $\lambda = v^2$ $\quad (v > 0)$

$X'' + v^2 X = 0 \Rightarrow r^2 + v^2 = 0 \Rightarrow r = \pm iv$

$X(x) = \cancel{c_1 \cos(vx)} + c_2 \sin(vx)$

$X(0) = c_1 \overset{\text{set}}{=} 0 \Rightarrow X(x) = c_2 \sin(vx)$

$X(L) \underset{\text{if } \not 0}{=} c_2 \sin(vL) = 0 \Rightarrow \underline{\sin(vL) = 0} \Rightarrow vL = n\pi$ $\quad$ (n integer)

$v = \frac{n\pi}{L}$, $\quad \lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$ $\quad (n = 1, 2, \dots)$

So $\quad \nu = \dfrac{n\pi}{L}$, $\quad \lambda_n = \left(\dfrac{n\pi}{L}\right)^2$

$X(x) = X_n(x) = \sin\left(\dfrac{n\pi}{L}x\right)$

up to constant coefficients

So we have figured out λ and X. Next, we turn to T:

$T'' + c^2\lambda T = 0 \Rightarrow T'' + \left(\dfrac{cn\pi}{L}\right)^2 T = 0$ $\qquad r^2 + \left(\dfrac{cn\pi}{L}\right)^2 = 0$

Now we can solve for in one step since $\left(\dfrac{cn\pi}{L}\right)^2$ is known (and >0)

$T_n(t) = a_n \cos\left(\dfrac{cn\pi}{L}t\right) + b_n \sin\left(\dfrac{cn\pi}{L}t\right)$

Together:

$u_n(x,t) = X_n(x)T_n(t) = \left(a_n \cos\left(\dfrac{cn\pi}{L}t\right) + b_n \sin\left(\dfrac{cn\pi}{L}t\right)\right)\sin\left(\dfrac{n\pi}{L}x\right)$

$(n = 1, 2, 3, \dots)$

Note: Any sum of u_n is also a solution to BVP.

So $\sum_{m=1}^{n} u_m$ is again a solution.

Since we have infinitely many n, in order to not loose any solution we use the infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} u_n$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

The question that remains is how to compute the a_n, b_n's.

Stage 2 Fourier Analysis

Now is time to factor in the initial conditions.

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{\overbrace{\left(a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)\right)}} \sin\left(\frac{n\pi}{L} x\right)$$

(IC$_1$) $u(x,0) = \underbrace{\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)}_{} \overset{\text{set}}{=} \underline{f(x)}$ This is FSS for $f(x)$ on $[0,L]$

we know the coefficients of FSS : $a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

(IC$_2$) $\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \underbrace{\left(b_n \frac{cn\pi}{L}\right)}_{} \sin\left(\frac{n\pi x}{L}\right) \underset{\text{set}}{=} g(x)$ FSS for $g(x)$ on $[0,L]$

The Fourier coefficients are $b_n \frac{cn\pi}{L}$ so

$$b_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Therefore,

$$b_n = \frac{2}{cn\pi}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Let us collect the formulas to get the following unique solution as a series.

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right) \\ \text{where, } \quad a_n &= \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots \\ b_n &= \frac{2}{cn\pi}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}}$$