535.641 Mathematical Methods Assignment 5

Ben Minnick **Name**__________________________________

1	/25
2	/25
3	/25
4	/25
TOTAL	/100

1. Rewrite the n^{th} order differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + ... + a_0 y(t) = b_0 u(t) + b_1 \dot{u}(t)$$

as a dimension-n linear state equation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$$

Hint: Let $x_n(t) = y^{(n-1)}(t) - b_1 u(t)$

2. Consider the system $\dot{\mathbf{x}} = A\mathbf{x}$. Sketch a phase portrait for each A matrix:

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

3. Consider the linear system:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + Bu, \quad \mathbf{x} \in \mathbb{R}^2, \quad u \in \mathbb{R},$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ be a state feedback gain such that a forcing, $u(t)$, is prescribed according to,

$$u(t) = -K\mathbf{x}(t).$$

So that the closed-loop system is:

$$\frac{d\mathbf{x}}{dt} = (A - BK)\mathbf{x}.$$

(a) Show that the open-loop system (with $u = 0$) is unstable.

(b) Determine conditions on the feedback gain $K = [k_1 \quad k_2]$ such that the closed-loop system is stable. *Hint: For a 2 × 2 matrix, all eigenvalues have negative real parts if and only if the trace is negative and the determinant is positive.*

(c) Among all such stabilizing gains K, find the one that minimizes the Euclidean norm

$$\min_K \|K\|_2 = \sqrt{k_1^2 + k_2^2}$$

4. Alfvén waves describe the evolution of small amplitude perturbations in an electrically conducting fluid in the presence of a strong background magnetic field and can be modeled using the following equations

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \mathrm{i}k\ \mathrm{j},$$
$$\frac{\mathrm{d}\mathrm{j}}{\mathrm{d}t} = \mathrm{i}k\ \omega,$$

where ω is the vorticity, j is the induced current density, k is the wavenumber of the fluid disturbance, and $\mathrm{i} = \sqrt{-1}$.

(a) Cast the set of equations in the form, $\dot{\mathbf{x}} = A\mathbf{x}$, and clearly identify $\mathbf{A}$ and $\mathbf{x}$.

(b) Classify the A matrix, (Hermitian/Skew-Hermitian/Unitary)?

(c) Diagonalize the matrix if possible.

(d) Obtain the second order ODEs for $\omega(t), \mathrm{j}(t)$ from the given first order ODEs. Solve them as an initial value problem to find their solution. The initial conditions are $\omega(t = 0) = \omega_0$, $\mathrm{j}(t = 0) = 0$.

(e) Confirm your solution in part (d) by expanding the solution vector $\mathbf{x}$ using the eigen solutions of matrix A and initial conditions given in (d).