Note Title 7/5/2013

2.C DETERMINANTS AND CRAMER'S RULE

Determinants

A 2x2 determinant of a matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - cb \leftarrow \left(\begin{matrix} >0 \\ <0 \\ =0 \end{matrix} \right)$$

cross multiply ; take the difference

A 3x3 determinant of $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is the number written as 2x2 det's

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \quad \left(\begin{matrix} >0 \\ <0 \\ =0 \end{matrix} \right)$$

signs alternate
(in addition to the sign of a, b, c, ...)

Similarly, we define the det of any size square matrix.

Example : Find $\begin{vmatrix} -1 & 1 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$

Sol^n The det equals $+(-1)\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} - (1)\begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = -2-(-4)+3(-1)$

$= \boxed{-1}$

Note The way we computed the determinant is called Cofactor Expansion about the first row.

$\boxed{\text{COFACTOR EXPANSION}}$ We may compute any determinant about any row and column by cofactor expansion and get the same answer.

Here is how it works :

- pick a row or column : $\begin{vmatrix} \cdots & a & \cdots \\ \cdots & b & \cdots \\ \cdots & c & \\ & \vdots & \end{vmatrix}$
- Use the entries $a, b, c, \ldots$
- Introduce extra signs in front of $a,b,c,\ldots$ according to the pattern $\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & & \ddots \end{vmatrix}$ checker board pattern
- Each signed new coefficient is multiplied the lower order det obtained by deleting the current row and column.

Example: Find $\begin{vmatrix} -1 & 1 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$ by cofactor expansion about the 2nd column

Sol'n
$$\begin{array}{ccc} + & - & \\ \end{array}$$
$$\begin{vmatrix} -1 & 1 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = +4 - 5 + 0 = \boxed{-1}$$

Note We can do this for any size and any row or column but it is NOT EFFICIENT.

Triangular Det: $\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = a.d.f$ $\left(= a\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} = a.d.f \right)$

|| product of diagonal entries

In practice we compute determinants by correct Gauss elimination that brings them to diagonal form.

For this we need 3 basic properties

1. $A \xrightarrow{\boxed{R_i + cR_j \to R_i}} B \implies \det B = \det A$ — not multiplied!
2. $A \xrightarrow{\boxed{cR_i \to R_i}} B \implies \det B = c \cdot \det A$
3. $A \xrightarrow{\boxed{R_i \leftrightarrow R_j}} B \implies \det B = -\det A$

Example:

$$\begin{vmatrix} 1 & 2 & 3 & -1 & 8 \\ 0 & 0 & 4 & 2 & -1 \\ 0 & -5 & 5 & 3 & 7 \\ 0 & 0 & 0 & 1 & 6 \\ 1 & 2 & 3 & -2 & -9 \end{vmatrix} \underset{\boxed{-R_1 + R_5 \to R_5}}{=} \begin{vmatrix} 1 & 2 & 3 & -1 & 8 \\ 0 & 0 & 4 & 2 & -1 \\ 0 & -5 & 5 & 3 & 7 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 & -17 \end{vmatrix} \underset{\boxed{R_2 \leftrightarrow R_3}}{=} - \begin{vmatrix} 1 & 2 & 3 & -1 & 8 \\ 0 & -5 & 5 & 3 & 7 \\ 0 & 0 & 4 & 2 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 & -17 \end{vmatrix}$$

$$\underset{\boxed{R_4 + R_5 \to R_5}}{=} - \begin{vmatrix} 1 & 2 & 3 & -1 & 8 \\ 0 & -5 & 5 & 3 & 7 \\ 0 & 0 & 4 & 2 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & -11 \end{vmatrix} = -(1)(-5)(4)(1)(-11) = \boxed{-220}$$

Triangular

Cramer's Rule

It applies to square systems $Ax=b$ with $\det A \neq 0$.

If $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then

$$\boxed{x_i = \frac{\det A_i}{\det A}}$$

Cramer's Rule

where A_i is obtained from A by replacing its ith column with b.

Example: Use Cramer's Rule to solve

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$$

Sol^n: $\det A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4$ $\quad \det A_1 = \begin{vmatrix} 2 & 1 & -1 \\ 3 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} = -10$

$\det A_2 = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ -1 & 4 & 1 \end{vmatrix} = -12$ $\quad \det A_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & 4 \end{vmatrix} = -14$

$$\left.\begin{aligned} x_1 &= \frac{\det A_1}{\det A} = \frac{5}{2} \\ x_2 &= \frac{\det A_2}{\det A} = 3 \\ x_3 &= \frac{\det A_3}{\det A} = \frac{7}{2} \end{aligned}\right.$$