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535.641 Mathematical Methods Assignment 1

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1. Find the solutions (x_1, x_2, x_3) for the linear system with various parameters (a, b).

$$2x_1 + 3x_2 + ax_3 = 1$$
$$2x_1 + 3x_2 + 2x_3 = b$$
$$4x_1 + 3x_2 + 2x_3 = 4$$

(a) $a = 4, \forall b \in \mathbb{R}$
(b) $a = 2, b = 1$
(c) $a = 2, \forall b \in \mathbb{R} \ni b \neq 1$

Perform row reduction starting with the augmented matrix and ending with reduced row echelon form. Show each row reduction step.

If the system has no solutions, explain why. If the system is consistent, find all solutions. In the case of infinitely many solutions, use parameters.

Ans:

⇒ General Row Reduction (Row Echelon Form)

Initial augmented matrix for the given linear system

$$\begin{bmatrix} 2 & 3 & a & 1 \\ 2 & 3 & 2 & b \\ 4 & 3 & 2 & 4 \end{bmatrix}$$

Step 1: $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 2 & 3 & a & 1 \\ 0 & 0 & 2-a & b-1 \\ 4 & 3 & 2 & 4 \end{bmatrix}$$

Step 2: $R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 2 & 3 & a & 1 \\ 0 & 0 & 2-a & b-1 \\ 0 & -3 & 2-2a & 2 \end{bmatrix}$$

Step 3: $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & 3 & a & 1 \\ 0 & -3 & 2-2a & 2 \\ 0 & 0 & 2-a & b-1 \end{bmatrix}$$

Above is the row echelon form for the general system.

(a) Solution for $a=4, \forall b \in R$

Echelon form at $a=4$

$$\begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & 2 \\ 0 & 0 & -2 & b-1 \end{bmatrix}$$

Rank of coefficient matrix$(A) = 3$

Rank of augmented matrix$([A|b]) = 3$

~~Rank is equal to~~ number of variable $= 3$

Since all three are equal therefore system is consistent with a unique solution.

Let's perform Reduced Row Echelon Form (RREF):

Step 4: $R_3 \leftarrow -\frac{1}{2}R_3$

$$\begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & 2 \\ 0 & 0 & 1 & \frac{1-b}{2} \end{bmatrix}$$

Step 5: $R_2 \leftarrow R_2 + 6R_3$

$R_1 \leftarrow R_1 - 4R_3$

$$\begin{bmatrix} 2 & 3 & 0 & 2b-1 \\ 0 & -3 & 0 & 5-3b \\ 0 & 0 & 1 & \frac{1-b}{2} \end{bmatrix}$$

Step 6: $R_2 \leftarrow -\frac{1}{3}R_2$

$$\begin{bmatrix} 2 & 3 & 0 & 2b-1 \\ 0 & 1 & 0 & \frac{3b-5}{3} \\ 0 & 0 & 1 & \frac{1-b}{2} \end{bmatrix}$$

Step 7: $R_1 \leftarrow R_1 - 3R_2$

$$\begin{bmatrix} 2 & 0 & 0 & 4-b \\ 0 & 1 & 0 & \frac{3b-5}{3} \\ 0 & 0 & 1 & \frac{1-b}{2} \end{bmatrix}$$

Step 3: $R_1 \leftarrow \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4-b}{2} \\ 0 & 1 & 0 & \frac{3b-5}{3} \\ 0 & 0 & 1 & \frac{1-b}{2} \end{bmatrix}$$

The unique solution is

$$x_1 = \frac{4-b}{2}$$

$$x_2 = \frac{3b-5}{3}$$

$$x_3 = \frac{1-b}{2}$$

(b) Solution for $a=2, b=1$

The echelon form:

$$\begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -3 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = \text{Rank}([A|b]) = 2 < 3$

It means system is consistent with infinitely many solutions.

Let's now find RREF.

Step 4: $R_2 \leftarrow -\frac{1}{3} R_2$

$$\begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>step 5:</u> $R_1 \leftarrow R_1 - 3R_2$

$$\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>step 6:</u> $R_1 \leftarrow \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let the free variable $x_3 = t$. Then the solution set will be

$$x_1 = 3/2$$

$$x_2 = -\frac{2}{3} - \frac{2}{3}t$$

$$x_3 = t$$

(c) Solution for $a = 2$ $\forall b \in R \ni b \neq 1$

At $a=2$, the echelon form will be

$$\begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -3 & -2 & 2 \\ 0 & 0 & 0 & b-1 \end{bmatrix}$$

For the last row

$$0 = b - 1$$

since $b \neq 1$, therefore this is a contradiction.

Therefore system has no solutions.

Also,

$$\text{rank}(A) = 2 < 3$$

$$\text{rank}(A|b) = 3$$

Therefore inconsistent system i.e. no solution.

2. Let $A \in \mathbb{R}^{3\times 3}$ be an invertible matrix, and let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be column vectors. Define:

$$B = A + \mathbf{u}\mathbf{v}^T$$

Suppose A^{-1} is known and $1 + \mathbf{v}^T A^{-1}\mathbf{u} \neq 0$.

The **Sherman–Morrison identity** states:

$$\left(A + \mathbf{u}\mathbf{v}^T\right)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}}$$

This identity is useful to compute the inverse of a rank-1 update to a matrix whose inverse has previously been computed which for large matrices can result in considerable computational savings.

(a) Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(i) Compute the rank of A

(ii) Compute the rank of $\mathbf{u}\mathbf{v}^T$

(b)

(i) Compute A^{-1}

(ii) Use the Sherman–Morrison identity to compute B^{-1}

(iii) Compute B^{-1} directly by first computing B, then inverting it. Verify that the result matches part (ii).

Ans:

(a) Rank computations

(i) Compute the rank of A

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Let's reduce it to row echelon form:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - \frac{2}{3}R_2} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

The echelon form has three non-zero row.

Therefore, the rank of A is 3.

(ii) Compute the rank of uv^T

$$u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$uv^T = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}^T \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [2 \ 1 \ 0] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

→ The second row is all zeros

→ Third row is multiple of the first row.

→ This means there is only one linearly independent row.

Therefore, the rank of uv^T is 1

(b) Inverse computations

(i) Compute A^{-1}

<u>Step 1</u>: setup the augmented matrix

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Step 2: Perform row operations

→ $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

→ $R_2 \leftarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

→ $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & -2 & 1 & -2 & 0 \end{array}\right]$$

→ $R_1 \leftarrow R_1 - 2R_2$

$R_3 \leftarrow R_3 + 3R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 1 & -2 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & -2 & 3 \end{array}\right]$$

$\rightarrow R_3 \leftarrow (1/4) R_3$

$$\left[\begin{array}{cccccc} 1 & 0 & -3 & 0 & 1 & -2 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right]$$

$\rightarrow R_1 \leftarrow R_1 + 3R_3$
$R_2 \leftarrow R_2 - 2R_3$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right]$$

This gives right side as inverse of A

$$A^{-1} = \begin{bmatrix} 3/4 & -1/2 & 1/4 \\ -1/2 & 1 & -1/2 \\ 1/4 & -1/2 & 3/4 \end{bmatrix} \approx \frac{1}{4}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

(ii) Use the Sherman-Morrison identity to compute B^{-1}

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}$$

Let's compute helper components

$$A^{-1}u = \frac{1}{4}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$V^T A^{-1} = [2 \quad 1 \quad 0] \frac{1}{4} \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix} = [1 \quad 0 \quad 0]$$

Denominator:

$$1 + V^T A^{-1} u = 1 + [2 \quad 1 \quad 0] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1 + (2-1) = 2$$

Numerator:

$$A^{-1} u V^T A^{-1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} [1 \quad 0 \quad 0] = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Now use the formula

$$B^{-1} = (A + uV^T)^{-1}$$

$$= A^{-1} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$

(iii) Compute B^{-1} directly and verify

$$B = A + uv^T$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Perform row ~~oper~~ operation to find B^{-1}

Augmented matrix $[A \mid I]$

<u>step 1</u>: set up the Augmented matrix $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

<u>step 2</u>: Perform row operation to reach RREF

$\rightarrow R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 4 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow R_2 \leftarrow R_2 - 4R_1$

$R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -6 & -4 & 1 & -4 & 0 \\ 0 & -2 & 0 & 0 & -2 & 1 \end{bmatrix}$$

$\rightarrow R_2 \leftrightarrow R_3$

$R_2 \leftarrow -R_2/2$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & -6 & -4 & 1 & -4 & 0 \end{bmatrix}$$

$\rightarrow R_3 \leftarrow R_3 + 6R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & -4 & 1 & 2 & -3 \end{bmatrix}$$

$\rightarrow R_3 \leftarrow -R_3/4$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & -1/4 & -1/2 & 3/4 \end{bmatrix}$$

$\rightarrow R_1 \leftarrow R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/4 & 3/2 & -3/4 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & -1/4 & -1/2 & 3/4 \end{array}\right]$$

$\rightarrow R_1 \leftarrow R_1 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & -1/4 & -1/2 & 3/4 \end{array}\right]$$

This gives inverse of B

$$B^{-1} = \begin{bmatrix} 1/4 & -1/2 & 1/4 \\ 0 & 1 & -1/2 \\ -1/4 & -1/2 & 3/4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$

which is same as B^{-1} calculated in part (ii) using Sherman-Morrison identity to calculate B^{-1}.

3. Consider the matrix A and vector $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

(a) Decompose the matrix $A = LU$ such that L and U are respectively lower and upper triangular matrices.

(b) Use your decomposition $A = LU$ to solve the system $A\mathbf{x} = \mathbf{b}$.

Ans.

(a) LU Decomposition

U: Perform Gaussian elimination to find the upper triangular matrix U.

L: The multipliers used in U will form the lower triangular matrix L.

Given matrix

$$A = \begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix}$$

Step 1: Perform row operation to create zeros in the first column below the pivot.

$R_2 \leftarrow R_2 - 3R_1$ (multiplier is $l_{21} = 3$)

$R_3 \leftarrow R_3 + R_1$ (multiplier is $l_{31} = -1$)

$R_4 \leftarrow R_4 + 3R_1$ (multiplier is $l_{41} = -3$)

Result after row operations

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & -12 & 20 & -7 \end{bmatrix}$$

Step 2: Create a zero in the second column below the pivot.

$R_4 \leftarrow R_4 - 4R_2$ (multiplier is $l_{42} = 4$)

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -7 \end{bmatrix}$$

Step 3: Create a zero in the third column below the pivot

$R_4 \leftarrow R_4 + 2R_3$ (multiplier is $l_{43} = -2$)

$$U = \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The L matrix is constructed using multipliers from the eliminations step:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix}$$

(b) Solve the system $Ax = b$

$$A = LU$$

$$L\underbrace{Ux}_{y} = b$$

i.e. $Ux = y$

$\therefore$ $Ly = b$

We now solve in ~~two~~ two stages

Step 1: Solve $Ly = b$ using forward substitution

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

$y_1 = 1$

$3y_1 + y_2 = 6 \Rightarrow y_2 = 3$

$-y_1 + y_3 = 0 \Rightarrow y_3 = y_1 = 1$

$-3y_1 + 4y_2 - 2y_3 + y_4 = 3$

$\Rightarrow -3 + 12 - 2 + y_4 = 3$

$\Rightarrow y_4 = 3 - 7 = -4$

The solution vector y is

$$y = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -4 \end{bmatrix}$$

Step 2: Solve $Ux = y$

We do use backword substitution on the following system, using the vector y we calculated above.

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -4 \end{bmatrix}$$

$$x_1 - 2x_2 - 2x_3 - 3x_4 = 1$$
$$-3x_2 + 6x_3 = 3$$
$$2x_3 + 4x_4 = 1$$
$$x_4 = -4$$

This give following

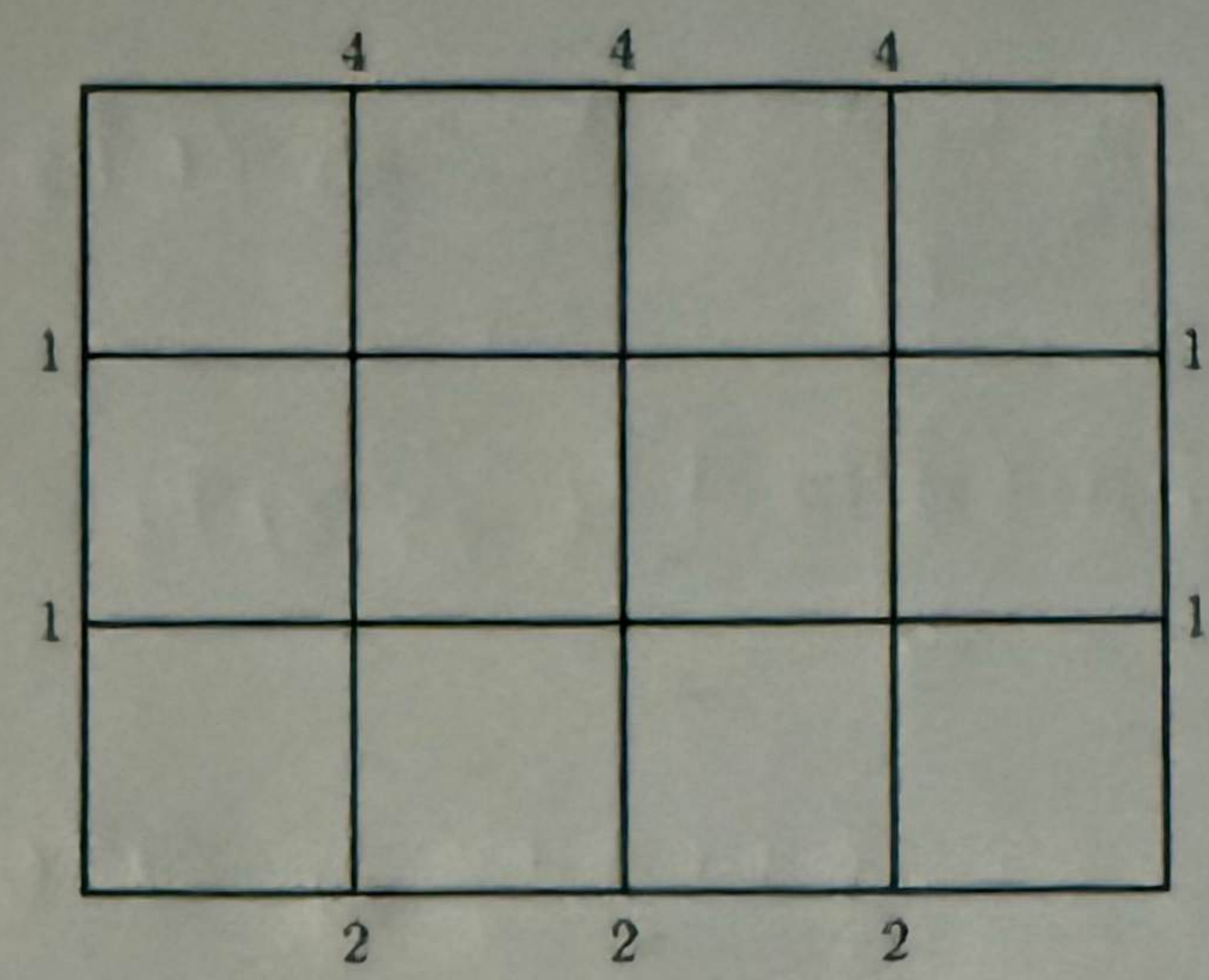
$$x_4 = -4$$
$$x_3 = (1 - 4 \times -4)/2$$
$$= 17/2$$
$$x_2 = \frac{3 - 6 \times 17/2}{-3}$$
$$= \frac{6 - 102}{-6}$$
$$= \frac{\cancel{96}\ 8\ 16}{\cancel{6}}$$
$$= 16$$
$$x_1 = 1 + 2 \times 16 + 2 \times \frac{17}{2} + 3 \times -4$$
$$= 1 + 32 + 17 - 12$$
$$= 50 - 12$$
$$= 38$$

The final solution is the vector x

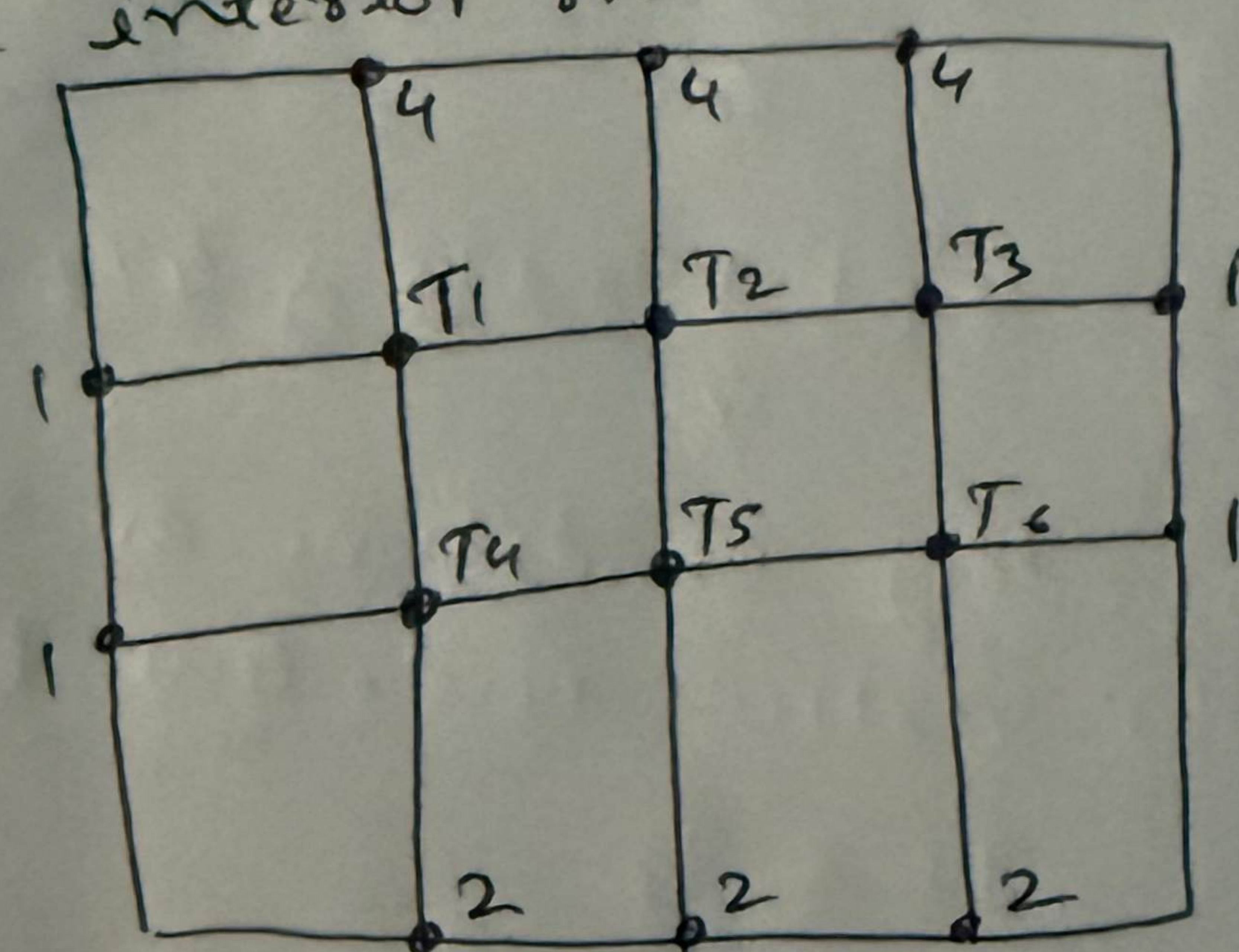
$$x = \begin{bmatrix} 38 \\ 16 \\ 17/2 \\ -4 \end{bmatrix}$$

4. Consider a rectangular metal plate with known boundary temperatures. Assume the material is isotropic and the mesh (shown below) is uniform.

Use the mean value property for heat conduction to determine the temperature at all six interior nodes. You may round your final answer.



Ans: We need to find temperature of following six interior nodes.



As per mean value property, the temperature at any interior node is simply the average of the temperatures of its immediate neighboring nodes (up, down, left and right)

This help us to create system of linear equations as below

$$T_1 = \frac{4+1+T_2+T_4}{4} \Rightarrow 4T_1 - T_2 - T_4 = 5$$

$$T_2 = \frac{4+T_1+T_3+T_5}{4} \Rightarrow 4T_2 - T_1 - T_3 - T_5 = 4$$

$$T_3 = \frac{4+1+T_2+T_6}{4} \Rightarrow 4T_3 - T_2 - T_6 = 5$$

$$T_4 = \frac{2+1+T_1+T_5}{4} \Rightarrow 4T_4 - T_1 - T_5 = 3$$

$$T_5 = \frac{2+T_2+T_4+T_6}{4} \Rightarrow 4T_5 - T_2 - T_4 - T_6 = 2$$

$$T_6 = \frac{2+1+T_3+T_5}{4} \Rightarrow 4T_6 - T_3 - T_5 = 3$$

Since boundary conditions are symmetrical across the vertical centerline therefore

$$T_1 = T_3$$
$$T_4 = T_6$$

It simplifies our system of equation as below

1. $4T_1 - T_2 - T_4 = 5$
2. $4T_2 - 2T_1 - T_5 = 4$
3. $4T_4 - T_1 - T_5 = 3$
4. $4T_5 - T_2 - 2T_4 = 2$

Let's solve this system of equation using matrix method. Following is initial matrix (T_1, T_2, T_4, T_5)

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 5 \\ -2 & 4 & 0 & -1 & 4 \\ -1 & 0 & 4 & -1 & 3 \\ 0 & -1 & -2 & 4 & 2 \end{array}\right]$$

It can be converted to RREF applying the row level operations.

step 1: $R_1 \leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} -1 & 0 & 4 & -1 & 3 \\ -2 & 4 & 0 & -1 & 4 \\ 4 & -1 & -1 & 0 & 5 \\ 0 & -1 & -2 & 4 & 2 \end{array}\right]$$

Step 2: $R_1 \leftarrow -1 \cdot R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -3 \\ -2 & 4 & 0 & -1 & 4 \\ 4 & -1 & -1 & 0 & 5 \\ 0 & -1 & -2 & 4 & 2 \end{array}\right]$$

Step 3: $R_2 \leftarrow R_2 + 2R_1$

$R_3 \leftarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & 0 & -4 & 1 & -3 \\ 0 & 4 & -8 & 1 & -2 \\ 0 & -1 & 15 & -4 & 17 \\ 0 & -1 & -2 & 4 & 2 \end{bmatrix}$$

step 4: $R_2 \leftrightarrow R_3$

$R_2 \leftarrow -1 \cdot R_2$

$$\begin{bmatrix} 1 & 0 & -4 & 1 & -3 \\ 0 & 1 & -15 & 4 & -17 \\ 0 & 0 & 52 & -15 & 66 \\ 0 & 0 & -17 & 8 & -15 \end{bmatrix}$$

step 5: $R_3 \leftarrow R_3 - 4R_2$

$R_4 \leftarrow R_4 + R_2$

$$\begin{bmatrix} 1 & 0 & -4 & 1 & -3 \\ 0 & 1 & -15 & 4 & -17 \\ 0 & 0 & 52 & -15 & 66 \\ 0 & 0 & -17 & 8 & -15 \end{bmatrix}$$

step 6: $R_4 \leftarrow 52R_4 + 17R_3$

$$\begin{bmatrix} 1 & 0 & -4 & 1 & -3 \\ 0 & 1 & -15 & 4 & -17 \\ 0 & 0 & 52 & -15 & 66 \\ 0 & 0 & 0 & 161 & 342 \end{bmatrix}$$

Step 7: $R_4 \leftarrow \frac{1}{161} R_4$ (Echelon form)

$R_3 \leftarrow \frac{1}{52} R_3$

$$\begin{bmatrix} 1 & 0 & -4 & 1 & -3 \\ 0 & 1 & -15 & 4 & -17 \\ 0 & 0 & 1 & -15/52 & 33/26 \\ 0 & 0 & 0 & 1 & 342/161 \end{bmatrix}$$

(RREF)

Step 8: $R_1 \leftarrow R_1 - R_4$

$R_2 \leftarrow R_2 - 4R_4$

$R_3 \leftarrow R_3 + \frac{15}{52} R_4$

$$\begin{bmatrix} 1 & 0 & -4 & 0 & -825/161 \\ 0 & 1 & -15 & 0 & -4097/161 \\ 0 & 0 & 1 & 0 & 303/161 \\ 0 & 0 & 0 & 1 & 342/161 \end{bmatrix}$$

Step 9: $R_1 \leftarrow R_1 + 4R_3$

$R_2 \leftarrow R_2 + 15R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 387/161 \\ 0 & 1 & 0 & 0 & 440/161 \\ 0 & 0 & 1 & 0 & 303/161 \\ 0 & 0 & 0 & 1 & 342/161 \end{bmatrix}$$

This gives temperature at six interior nodes as below

$$T_1 = \frac{387}{161} \approx 2.40$$

$$T_2 = \frac{440}{161} \approx 2.73$$

$$T_3 = T_1 \approx 2.40$$

$$T_4 = \frac{303}{161} \approx 1.88$$

$$T_5 = \frac{342}{161} \approx 2.12$$

$$T_6 = T_4 \approx 1.88$$