LU - Decomposition

Motivation

It is common in engineering applications to solve a system of equations:

$$A \underline{x} = \underline{b}$$

where $A \in \mathbb{R}^{n \times n}$, $\underline{x}, \underline{b} \in \mathbb{R}^{n \times 1}$.

More so, it is common to solve a sequence of equations in which the right-hand side vector $\underline{b}$ changes and the coefficient matrix A remains unchanged.

For example, if spatially discretizing a problem, the coefficient matrix A depends on the mesh and the vector $\underline{b}$ depends on the forcing or some specific configuration. Typically you will want to change the forcing, but use the same mesh.

LU decomposition can be useful if solving a sequence of systems of equations

$$A\underline{x}_1 = \underline{b}_1 \quad , \quad A\underline{x}_2 = \underline{b}_2 \quad , \quad \dots , \quad A\underline{x}_k = \underline{b}_k$$

since $A = LU$ is unchanged for all systems.

Example

Consider the system of equations $A\underline{x} = \underline{b}$ where,

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 6 & 18 & -1 \end{bmatrix} \quad , \quad \underline{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

We want to decompose A such that

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{U}$$

To find these L and U matrices, we perform row-reduction on A. The L matrix is composed of the steps needed to row-reduce A into an upper triangular matrix U.

We first take row 2 and subtract row 1 multiplied by 2. We will denote this as $R_2 - 2R_1$. We do this to have a zero in the 1st column, 2nd row.

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 6 & 18 & -1 \end{bmatrix} \xrightarrow{\times(-2)} \sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 6 & 18 & -1 \end{bmatrix}$$

Similarly, we compute $R_3 - 3R_1$ to zero-out the 1st column, 3rd row.

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 0 \\ 6 & 18 & -1 \end{bmatrix} \times(-3) \sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 9 & -4 \end{bmatrix}$$

Next, we zero out the 2nd column, 3rd row, by computing $R_3 - (-9) R_2$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 9 & -4 \end{bmatrix} \times(-9) \sim \boxed{\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}}$$

This is the U matrix

To compose the L matrix, we consider the row reduction steps taken:

$$R_2 - 2 R_1 = \ell_{21}$$
$$R_3 - 3 R_1 = \ell_{31}$$
$$R_3 - 9 R_1 = \ell_{32}$$

The L matrix is then,

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix}}$$

It is recommended that you confirm by performing matrix multiplication

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -9 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 6 & 18 & -1 \end{bmatrix}}_{A}$$

Now we use this LU-decomposition to solve the system of equations $A\underline{x} = \underline{b}$. This is done according to the following steps:

$$A\underline{x} = \underline{b} \qquad \text{Substitute } A = LU$$

$$L(U\underline{x}) = \underline{b} \qquad \text{Let } \underline{y} = U\underline{x}$$

$$L\underline{y} = \underline{b} \qquad \text{Solve for } \underline{y}$$
$$\underline{y} = L^{-1}\underline{b}$$
$$U\underline{x} = \underline{y} \qquad \text{Solve for } \underline{x}$$
$$\underline{x} = U^{-1}\underline{y}$$

For the given example, we solve for y. This only involves forward elimination. Solve for y_1 first, then y_2, then y_3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ 3 & 9 & 1 & 5 \end{array}\right] \rightarrow \begin{array}{l} y_1 = 1 \\ 2y_1 + y_2 = 3 \rightarrow y_2 = 1 \\ 3y_1 + 9y_2 + y_3 = 5 \rightarrow y_3 = -7 \end{array}$$

Now solving for $\underline{x}$, using y. This only involves backwards elimination. Solve for x_3 first, then x_2, then x_1.

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -7 \end{array}\right] \rightarrow \begin{array}{l} 2x_1 + 3x_2 + x_3 = 1 \rightarrow x_1 = -15/8 \\ x_2 = 1 \rightarrow x_2 = 1 \\ -4x_3 = -7 \rightarrow x_3 = 7/4 \end{array}$$

Therefore, $\left[-15/8 \right.$

$$\boxed{\underline{x} = \begin{bmatrix} 1 \\ 7/4 \end{bmatrix}}$$