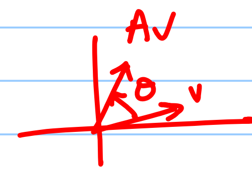
Note Title 7/7/2013

3.E ORTHOGONAL MATRICES

A square matrix A is orthogonal, if its columns form an orthonormal set. This means that each pair of columns is orthogonal and each column is unit.

Theorem: Let A be a square matrix. TFAE

1. A is orthogonal
2. $A^T A = I$ $\quad (\Leftrightarrow AA^T = I)$
3. $A^{-1} = A^T$



Example: Show that the rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal. Find A^{-1}.

Sol^n $\quad AA^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so A is orthogonal.

$A^{-1} = A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

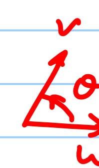
Example: It's easy to see that $B = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$ is orthogonal.

Theorem: For A $n \times n$ TFAE:

$u \cdot v = Au \cdot Av$

1. A is orthogonal.
2. $Au \cdot Av = u \cdot v$ for all $u, v \in \mathbb{R}^n$. (Preservation of dot products)
3. $\|Av\| = \|v\|$ for all $v \in \mathbb{R}^n$. (Preservation of Lengths)

Remark The matrix transformation $T(x) = Ax$ with A orthogonal is also called orthogonal.



The theorem says that orthogonal tranformations preserve dot products and lengths and angles. (since $u \cdot v = \|u\| \|v\| \cos\theta$)

Theorem: If λ is an EVA of an orthogonal A, then $|\lambda|=1$.

(Pf $\underbrace{\|v\|}_{\text{EVE}} = \|Av\| = \|\lambda v\| = |\lambda|\,\|v\|$ $\Rightarrow$ $|\lambda|=1$ (since $\|v\|\neq 0$))

$\|v\| = |\lambda|\,\|v\|$

This theorem also holds for complex EVAs.

Eigenvalues of Symmetric Matrices

Theorem: 1. A real symmetric matrix has only real eigenvalues.

2. A real skew-symmetric matrix has only eigenvalues that either pure imaginary or zero.

Example: $A=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ has EVAs: $1, 4, -1$. $B=\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ has EVAs: $0, \sqrt{6}\,i, -\sqrt{6}\,i$

symmetric   real   skew-symmetric   0 or pure imaginary

Hermitian and Unitary Matrices

Let A be a square complex matrix. Then

$\bar{A} =$ 'A conjugate'

$$\overline{\begin{bmatrix} 3i & 1+i \\ 0 & -5+2i \end{bmatrix}} = \begin{bmatrix} -3i & 1-i \\ 0 & -5-2i \end{bmatrix}$$

A is Hermitian, if $\bar{A}^T = A$

A is skew-Hermitian, if $\bar{A}^T = -A$

A is unitary, if $\bar{A}^T = A^{-1}$ $\quad (\Leftrightarrow A^T \bar{A} = I)$

Example: Show that $A = \begin{bmatrix} 4 & 2+i \\ 2-i & 0 \end{bmatrix}$ is Hermitian, $B = \begin{bmatrix} 0 & 2-i \\ -2-i & -4i \end{bmatrix}$ is skew-Hermitian, $C = \begin{bmatrix} 1/2 & -\sqrt{3}/2\, i \\ -\sqrt{3}/2\, i & 1/2 \end{bmatrix}$ is unitary

Sol^n: $\bar{A}^T = \overline{\begin{bmatrix} 4 & 2+i \\ 2-i & 0 \end{bmatrix}}^T = \begin{bmatrix} 4 & 2-i \\ 2+i & 0 \end{bmatrix}^T = \begin{bmatrix} 4 & 2+i \\ 2-i & 0 \end{bmatrix} = A$. Likewise, $\bar{B}^T = -B$

$$\bar{C}^T C = \overline{\begin{bmatrix} 1/2 & -\sqrt{3}/2\, i \\ -\sqrt{3}/2\, i & 1/2 \end{bmatrix}}^T \begin{bmatrix} 1/2 & -\sqrt{3}/2\, i \\ -\sqrt{3}/2\, i & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2\, i \\ \sqrt{3}/2\, i & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2\, i \\ -\sqrt{3}/2\, i & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Remarks

1. A real Hermitian matrix is just a symmetric matrix.
   (skew-) (skew-)

2. A real unitary matrix is just an orthogonal matrix.

3. The main diagonal of a Hermitian matrix consists of real numbers.

4. The main diagonal of a skew-Hermitian matrix consists of 0s or pure imaginary numbers.

Theorem: Let $A_{n\times n}$ be complex matrix.
1. If A is Hermitian, then its EVAs are real.
2. If A is skew-Hermitian, then its EVAs are 0 or pure imaginary.
3. If A is unitary, then its EVAs have absolute value 1.