# 535.641 Mathematical Methods Assignment 3

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1. Consider a linear transformation $T : P_2 \to P_3$, where $P_2$ denotes the set of all 2nd order polynomials and $P_3$ denotes the set of all 3rd order polynomials. The transform is such that: $T(1) = 1-2x-x^2$, $T(x) = 3x+3x^2+2x^3$, and $T(x^2) = -2+x-x^2-2x^3$.

   (a) Find the standard matrix $A$ of this linear transform.

   (b) Find the image of $2-2x+3x^2$.

   (c) Is the polynomial $q = 1+x+2x^2+2x^3$ in the image of this transform? If it is, then find all $P_2$ polynomials, $p$ that would satisfy $T(p) = q$.

Ans: The basis for $P_2$ is

$$B = \{1, x, x^2\}$$

which is any polynomial $a+bx+cx^2$ correspond to the coordinate vector

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The basis for $P_3$ is

$$C = \{1, x, x^2, x^3\}$$

which is any polynomial $d+ex+fx^2+gx^3$ correspond to the coordinate vector

$$\begin{bmatrix} d \\ e \\ f \\ g \end{bmatrix}$$

(a) Find the standard matrix A of the linear transformation

The standard matrix A is constructed from the coordinate vectors of the transformation of the domain's basis vectors

The columns of A are $T(1)$, $T(x)$ and $T(x^2)$

⇒ Transform the basis vectors (given in the problem)

$$T(1) = 1 - 2x - 1x^2 + 0x^3$$
$$T(x) = 0 + 3x + 3x^2 + 2x^3$$
$$T(x^2) = -2 + 1x - 1x^2 - 2x^3$$

⇒ Find the coordinate vector for each transformed basis vector

$$[T(1)_c] = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix} \qquad [T(x^2)_c] = \begin{bmatrix} -2 \\ 1 \\ -1 \\ -2 \end{bmatrix}$$

$$[T(x)_c] = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$\Rightarrow$ Construct the standard matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 3 & 1 \\ -1 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

(b) Find the image of $2-2x+3x^2$

The input polynomial

$$p(x) = 2-2x+3x^2$$

corresponding co-ordinate vector

$$p = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

The resulting coordinate vector

$$q = Ap$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ -2 & 3 & 1 \\ -1 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2+0.-2+-2.-2 \\ -2.2+3.-2+1.-2 \\ -1.2+3.-2-1.3 \\ 0.2+2.2+-2.3 \end{bmatrix} = \begin{bmatrix} 2+0-6 \\ -4-6+3 \\ -2-6-3 \\ 0-4-6 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ -11 \\ -10 \end{bmatrix}$$

The resulting coordinate vector corresponds to a polynomial in the codomain $P_3$.

Image will be

$$-4-7x-11x^2-10x^3$$

(c) Given polynomial

$$q = 1+x+2x^2+2x^3$$

The coordinate vector

$$q = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

This question asks if following system is consistent.

$$Ap = q$$

The unknown vector

$$p = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Setup the Augmented matrix $[A|q]$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ -2 & 3 & 1 & 1 \\ -1 & 3 & -1 & 2 \\ 0 & 2 & -2 & 2 \end{bmatrix}$$

Apply row-reduce

$R_2 \leftarrow R_2 + 2R_1$
$R_3 \leftarrow R_3 + R_1$
$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 3 & -3 & 3 \\ 0 & 2 & -2 & 2 \end{bmatrix}$$

$R_2 \leftarrow R_2/3$
$R_4 \leftarrow R_4/2$
$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

system is consistent because there are no contradictory rows. Therefore, ~~yes~~ yes, the polynomial $q$ is in the ~~image~~ image of $T$.

$\Rightarrow$ Find all solutions:
Convert the reduce matrix back into equati
The variable $c$ has no pivot, so it is a free variable. let's say $c = t$

$a - 2c = 1 \Rightarrow a = 1 + 2c \Rightarrow a = 1 + 2t$
$b - c = 1 \Rightarrow b = 1 + c \Rightarrow b = 1 + t$

The general solution for given polynomial

$$p(x) = a + bx + cx^2$$

That maps to q is

$$p(x) = (1 + 2t) + (1 + t)x + t(x^2)$$

$$p(x) = (1 + x) + t(2 + x + x^2)$$

2. Consider the given vectors that form basis for a subspace of $\mathbb{R}^4$

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Construct an orthogonal basis for this subspace.

Ans: Let's say new orthogonal basis is

$\{v_1, v_2, v_3\}$

$\Rightarrow$ Find the $v_1$

$$v_1 = x_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$\Rightarrow$ Find the second vector $v_2$

$$v_2 = x_2 - \text{proj}_{v_1}(x_2)$$

$$= x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$x_2 . v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0.1 + 1.1 + 1.1 + 1.1 = 3$$

$$v_1 . v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1.1 + 1.1 + 1.1 + 1.1 = 4$$

This gives

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

To simplify, let's multiply by 4

$$v_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

⇒ Find the third vector $v_3$

$$v_3 = x_3 - \left( \frac{x_3 . v_1}{v_1 . v_1} \right) v_1 - \left( \frac{x_3 . v_2'}{v_2' . v_2'} \right) v_2'$$

$$x_3 \cdot v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2$$

$$x_3 \cdot v_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0 \cdot -3 + 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2$$

$$v_2' \cdot v_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -3 \cdot -3 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 12$$

This gives

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{4}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{12}\right) \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 0 - 1/2 + 1/2 \\ 0 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

To simplify, let's multiply by 3

$$v_3' = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

The orthogonal Basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

3. For the following matrix

$$A = \begin{bmatrix} 2 & 1 \\ c & 0 \end{bmatrix}$$

(a) Find all real values $c$ for which the matrix $\mathbf{A}$ has real eigenvalues

(b) Find the eigenvectors which correspond to the situation where all eigenvalues are real ($c \in \mathbb{R}$). *Hint: You should confirm the eigenvectors are reasonable for all $c$ being considered.*

Ans: Let's first solve characteristic equation

$$\det(A - \lambda I) = 0$$

(1) Setup matrix

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ c & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ c & -\lambda \end{bmatrix}$$

(2) Find the determinant and set to zero

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ c & -\lambda \end{bmatrix}\right)$$

$$0 = (2-\lambda)(-\lambda) - (1)(c)$$

$$0 = -2\lambda + \lambda^2 - c$$

(3) This gives polynomial

$$\lambda^2 - 2\lambda - c = 0$$

In generic form, we have

$a = 1$, $b = -2$, $c = -c$

For real roots

$$\text{discriminant } (b^2 - 4ac) \geq 0$$

$$(-2)^2 - 4(1)(-c) \geq 0$$

$$4 + 4c \geq 0$$

$$c \geq -1$$

The matrix A will have real eigenvalues for all real values of $c \geq -1$.

(b) Find the eigenvectors

Let's first solve eigenvalues

$$\lambda^2 - 2\lambda - c = 0 \quad \text{with} \quad c \geq -1$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-c)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 + 4c}}{2} = \frac{2 \pm 2\sqrt{1+c}}{2} = 1 \pm \sqrt{1+c}$$

The two real eigenvalues are

$$\lambda_1 = 1 + \sqrt{1+c}$$

$$\lambda_2 = 1 - \sqrt{1+c}$$

An eigenvector

$$V = \begin{bmatrix} x \\ y \end{bmatrix}$$

is a non zero vector that solves the system

$$(A - \lambda I)V = 0$$

Eigenvector for $\lambda_1 = 1 + \sqrt{1+c}$

$$(A - \lambda_1 I)V = 0$$

$$\begin{bmatrix} 2-\lambda_1 & 1 \\ c & -\lambda_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$(2-\lambda_1)x + y = 0 \quad \text{--- (1)}$$

$$cx - \lambda_1 \cdot y = 0 \quad \text{--- (2)}$$

Let's solve (1) for given $\lambda_1$

$$(2 - (1 + \sqrt{1+c}))x + y = 0$$

$$(1 - \sqrt{1+c})x + y = 0$$

$$y = -(1 - \sqrt{1+c})\,x = (\sqrt{1+c} - 1)x$$

To find eigenvector, we can let $x = 1$,

$$y = \sqrt{1+c} - 1$$

Eigen vector, $V_1 = \begin{bmatrix} 1 \\ \sqrt{1+c} - 1 \end{bmatrix}$

4. Consider the quadratic form,



$$Q(x_1, x_2) = 5x_1^2 + 5x_2^2 - 4x_1x_2$$

(a) find the maximum value of $Q(x_1, x_2)$ subject to the constraint $x_1^2 + x_2^2 = 1$

(b) find the location where this maximum is achieved

**Ans:** As per Principal Axes Theorem

$$Q(x) = x^T A x$$

The minimum value of $Q(x)$ is subject to the constraints that $x$ is a unit vector

$$||x||^2 = 1$$

which is the largest eigenvalue ($\lambda_{max}$) for symmetric matrix A.

The location when the maximum occurs is the unit eigenvector corresponding to the largest eigenvalue.

⇒ Find the symmetric matrix A

$$Q(x_1, x_2) = 5x_1^2 + 5x_2^2 \cancel{+} - 4x_1x_2$$

To write into $Q(x) = x^T A x$ form

→ coefficient of squared term $x_1^2, x_2^2$ goes on main diagonal

→ The cross term $(x_1x_2)$ split evenly between the off diagonal

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

⇒ Find the eigenvalue of A

$$A - \lambda I = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)\cdot(5-\lambda) - (-2)(-2)$$

$$= (5-\lambda)^2 - 4$$

To solve $\det(A - \lambda I) = 0$

$$(5-\lambda)^2 = 4$$

$$5-\lambda = \pm 2$$

$$\lambda = 5 \pm 2$$

$$\lambda_1 = 3, \quad \lambda_2 = 7$$

(a) Find the maximum values

The maximum value of the quadratic form under the constraints is the largest eigenvalue

$$\lambda_{max} = \cancel{9}7$$

i.e maximum value $Q(x_1, x_2) = 7$

(b) Find the location where this maximum is acheived

The maximum is acheived at the unit eigenvectors corresponding to the largest eigenvalue $\lambda = 7$

To find these eigenvectors

$$(A - 7I)v = 0$$

$$\begin{bmatrix} 5-7 & -2 \\ -2 & 5-7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 = 0 \quad — ①$$

$$-2x_1 - 2x_2 = 0 \quad — ②$$

i.e $\quad x_1 = -x_2$

The eigenspace for $\lambda = 7$ is spanned by any vector that satisfy this condition

$$V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

However, the location must satisfy the Constraints

$$x_1^2 + x_2^2 = 1$$

So we must use the unit eigenvectors.

Eigenvectors $\begin{bmatrix} t \\ -t \end{bmatrix}$

$$(t)^2 + (-t)^2 = 1$$

$$2t^2 = 1$$

$$t = \pm \frac{1}{\sqrt{2}}$$

This gives two locations where the maximum is acheived

(1) For $t = \frac{1}{\sqrt{2}}$, the location

$$(x_1, x_2) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

(2) For $t = -\frac{1}{\sqrt{2}}$

location $(x_1, x_2) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$