3.B LINEAR TRANSFORMATIONS

Note Title 7/7/2013

A linear transformation T between two vector spaces V and W is a tranformation

$$T : V \to W$$

s.t.

1. $T(\underset{\text{in } V}{u+v}) = \underset{\text{in } W}{T(u) + T(v)}$
2. $T(cu) = c\,T(u)$

Linearity Conditions $\iff T(c_1u+c_2v) = c_1T(u)+c_2T(v)$

Special case: if $V=W$, T is called a linear operator.

Linear transformations "preserve" the vector space operations.
They take
- sums to sums
- scalar products to scalar products.

Examples: 1. Matrix transformation. Because if A is the matrix of the transformation, then

- $T(x_1 + x_2) = A(x_1 + x_2) = Ax_1 + Ax_2 = T(x_1) + T(x_2)$
- $T(cx) = cT(x)$

Linear

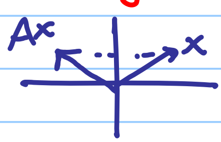
2. The special matrix transformations with matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

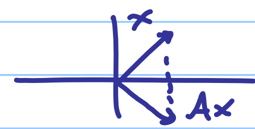
Linear

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = T(u)$$

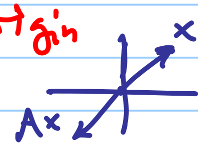
reflection about y-axis



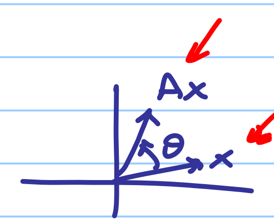
reflection about x-axis



reflection about the origin



rotation by θ radian about the origin



3. $T: M_{22} \to P_3 : T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = d + cx + (b-a)x^3$ is linear.

Linear

To see this check properties

1. $T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} = (d_1+d_2) + (c_1+c_2)x + ((b_1+b_2)-(a_1+a_2))x^3$

$T\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + T\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = (d_1 + c_1x + (b_1-a_1)x^3) + (d_2 + c_2x + (b_2-a_2)x^3)$ $/\!/$ property 1 ok

2. $T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = (kd) + (kc)x + (kb-ka)x^3$

$kT\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k(d + cx + (b-a)x^3)$ $=$ property 2 ok

4. $T: M_{22} \to P_3$, $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a^2 + bx^3$ is <u>nonlinear.</u>

Nonlinear

Verify!