535.641 Midterm

Name Dilip kumar

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1. (a) Let $S_1 = \{(-3,0,3),(6,-4,7),(2,1,-3)\}$ and $S_2 = \{(3,0,3),(3,-4,-1),(1,1,2)\}$. Test each set for linear independence. **Note:** The vectors in the sets are column vectors. They are written in row form in order to save space.

(b) Use step-by-step **Gauss Elimination** to solve the system.

$$\begin{bmatrix} -1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ -3 & 2 & 2 & 0 \\ -2 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 0 \\ 3 \end{bmatrix}$$

Ans:

(a) Find determinant of S_1 after converting vector form into matrix form

$$\det\left(\begin{bmatrix} -3 & 6 & 2 \\ 0 & -4 & 1 \\ 3 & 7 & -3 \end{bmatrix}\right) = -3\begin{vmatrix} -4 & 1 \\ 7 & -3 \end{vmatrix} - 6\begin{vmatrix} 0 & 1 \\ 3 & -3 \end{vmatrix} + 2\begin{vmatrix} 0 & -4 \\ 3 & 7 \end{vmatrix}$$

$$= -3(12-7) - 6(0-3) + 2(0+12)$$
$$= -15 + 18 + 24$$
$$= -15 + 42$$
$$= 27$$

Since $\det(|S_1|)$ is not zero, therefore S_1 are linearly independent.

Let's do similar for S_2

$$\det\left(\begin{bmatrix} 3 & 3 & 1 \\ 0 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix}\right) = 3\begin{vmatrix} -4 & 1 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & -4 \\ 3 & -1 \end{vmatrix}$$

$$= 3(-8+1) - 3(-3) + (+12)$$
$$= -21 + 9 + 12$$
$$= 0$$

Since $\det(|S_2|)$ is zero therefore S_2 are linearly dependent.

(b) We need to solve

$$Ax = b$$

where

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ -3 & 2 & 2 & 0 \\ -2 & 6 & 4 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 19 \\ 0 \\ 3 \end{bmatrix}$$

We can write augmented matrix as below

$$\left[\begin{array}{cccc|c} -1 & 3 & 2 & 0 & 3 \\ 2 & 1 & 4 & 0 & 19 \\ -3 & 2 & 2 & 0 & 0 \\ -2 & 6 & 4 & 1 & 3 \end{array}\right]$$

Let's now perform following operation to get the row echelon form.

$$R_1 \to -R_1$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & -3 \\ 2 & 1 & 4 & 0 & 19 \\ -3 & 2 & 2 & 0 & 0 \\ -2 & 6 & 4 & 1 & 3 \end{array}\right]$$

Let's apply following operatio

$$R_2 \to R_2 - 2R_1$$

$$R_3 \to R_3 + 3R_1$$

$$R_1 \to R_4 + 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & -3 \\ 0 & \cancel{4}7 & 8 & 0 & 25 \\ 0 & -7 & -4 & 0 & -9 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$$

More operations

$$R_3 \to R_3 + R_2$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & -3 \\ 0 & 7 & 8 & 0 & 25 \\ 0 & 0 & 4 & 0 & 16 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$$

Now matrix is in row echelon form. Let's solve equation using back substitution

$x_1 - 3x_2 - 2x_3 = -3$ — ①

$7x_2 + 8x_3 = 25$ — ②

$4x_3 = 16$ — ③

$x_4 = -3$ — ④

This gives

$x_4 = -3$

$x_3 = 4$

$7x_2 + 8\times 4 = 25 \Rightarrow 7x_2 = -7 \Rightarrow x_2 = -1$

$x_1 = -3 + 3\times -1 + 2\times 4 = -3 - 3 + 8 = +2$

Solution is: $x_1 = 2,\ x_2 = -1,\ x_3 = 4,\ x_4 = -3$

2. (a) Compute the inverse of $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}$. Show all steps.

(b) Find matrices P and D that diagonalize $A = \begin{bmatrix} 0 & 5 & -9 \\ 0 & 5 & -9 \\ 0 & 5 & -9 \end{bmatrix}$. Show all steps.

Ans:

(a) $M = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

To find the inverse of M, let's use Gauss-Jordan elimination method. Following is augmented matrix [M | I]

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

Now apply operation to convert left side into identity matrix.

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & -2 & 2 & 1 & 0 \\ 0 & 0 & 4 & -2 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2/4$
$R_3 \to R_3/4$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & 1/4 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/4 \end{array}\right]$$

$R_1 \to R_1 + R_3$

$R_2 \to R_2 + \frac{1}{2} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 1/4 \\ 0 & 1 & 0 & 1/4 & 1/4 & 1/8 \\ 0 & 0 & 1 & -1/2 & 0 & 1/4 \end{array}\right]$$

This gives inverse matrix of M as below

$$M^{-1} = \begin{bmatrix} 1/2 & 0 & 1/4 \\ 1/4 & 1/4 & 1/8 \\ -1/2 & 0 & 1/4 \end{bmatrix}$$

(b) To diagonalize A

$$A = P D P^{-1}$$

for $A = \begin{bmatrix} 0 & 5 & -9 \\ 0 & 5 & -9 \\ 0 & 5 & -9 \end{bmatrix}$

let's find eigenvalues of A using characteristic equation

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 0 & 5 & -9 \\ 0 & 5 & -9 \\ 0 & 5 & -9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 5 & -9 \\ 0 & 5-\lambda & -9 \\ 0 & 5 & -9-\lambda \end{bmatrix}\right) = 0$$

$$-\lambda \begin{vmatrix} 5-\lambda & -9 \\ 5 & -9-\lambda \end{vmatrix} - 5 \begin{vmatrix} 0 & -9 \\ 0 & -9-\lambda \end{vmatrix} - 9 \begin{vmatrix} 0 & 5-\lambda \\ 0 & 5 \end{vmatrix} = 0$$

$$-\lambda((5-\lambda)(-9-\lambda) - 5\times -9) - 0 - 0 = 0$$

$$-\lambda(-45 - 5\lambda + 9\lambda + \lambda^2 + 45) = 0$$

$$-\lambda(\lambda^2 + 4\lambda) = 0$$

$$\Rightarrow \lambda^2(\lambda + 4) = 0$$

The eigenvalues are

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -4$$

Since we don't have unique eigenvalues therefore let's find eigenvectors as well

For $\lambda_1 = 0$:

$$(A - \lambda I)v = 0$$

$$(A - 0I)v = 0$$

$$Av = 0$$

$$\begin{bmatrix} 0 & 5 & -9 \\ 0 & 5 & -9 \\ 0 & 5 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x_2 - 9x_3 = 0$ Variable x_1 is free.

$\Rightarrow 5x_2 = 9x_3$

The solution space has two linear independent vectors.

Let $x_1 = 1,\ x_2 = 0,\ x_3 = 0.$

This gives eigenvector

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_1 = 0,\ x_2 = 9,\ x_3 = 5$, this gives

$$V_2 = \begin{bmatrix} 0 \\ 9 \\ 5 \end{bmatrix}$$

For $\lambda_2 = -4$

Solve $(A \cancel{+} \lambda I)V = 0$

$$(A + 4\cancel{\lambda}I)V = 0$$

$$\left(\begin{bmatrix} 0 & 5 & -9 \\ 0 & 5 & -9 \\ 0 & 5 & -9 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) V = 0$$

$$\begin{bmatrix} 4 & 5 & -9 \\ 0 & 9 & -9 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Second row gives $9x_2 \cancel{+} -9x_3 = 0 \Rightarrow x_2 = x_3$

From row one: $4x_1 + 5x_2 - 9x_3 = 0$

$\Rightarrow 4x_1 + 5x_3 - 9x_3 = 0 \Rightarrow 4x_1 = 4x_3 \Rightarrow x_1 = x_3$

This gives simplest eigenvector as

$$V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now construct P and D

D: Is a diagonal matrix containing the eigenvalues

P: Corresponding eigenvectors as its column

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 9 & 1 \\ 0 & 5 & 1 \end{bmatrix}$$

3. (a) Check the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ -3 \end{bmatrix} \right\}$$

for orthogonality.

(b) Determine if S is a subspace of $\mathbf{R}^4$. If it is, find a basis and the dimension.

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, 2x_2 - 3x_4 = 0 \right\} \subseteq \mathbf{R}^4$$

Ans (a) We have

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad V_2 = \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix} \quad V_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -3 \end{bmatrix}$$

For these vectors to be orthogonal, their pair of dot product should be zero.

Let's find for each pair

$$V_1 \cdot V_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix} = 4 - 6 + 6 - 4 = 0$$

$$V_1 \cdot V_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \\ -3 \end{bmatrix} = -1 + 2 + 6 - 12 = -5$$

Since $V_1 \cdot V_3$ is not zero therefore set of V_1, V_2 and V_3 is not orthogonal.

(b) S to be a subspace of R^4, S must satisfy three conditions.

Condition 1: Contains the zero vector

The given condition, let's let's apply zero vector

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_2 - 3x_4 = 0$$
$$2 \cdot 0 - 3 \cdot 0 = 0$$
$$0 - 0 = 0$$
$$0 = 0$$

which is true, it means S contains zero vector.

Condition 2: Closed under addition

Let's use two vector

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

For u & v vectors in S, it should satisfy the condition:

$$2x_2 - 3x_4 = 0$$

let's try both vector

$$2u_2 - 3u_4 = 0 \quad -① $$
$$2v_2 - 3v_4 = 0 \quad -②$$

Now find sum

$$u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 \end{bmatrix}$$

Now check if $u+v$ exist in S testing the given condition

$$2(u_2+v_2)-3(u_4+v_4) = 2u_2-3u_4+2v_2-3v_4$$
$$= 0+0$$
$$= 0$$

which is true, it means S contains v_1+v_2

Condition 3: closed under scalar multiplication

Let u be a vector and c be any scalar

$$c\cdot u = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \\ cu_4 \end{bmatrix} \quad \text{and} \quad 2u_2-3u_4=0$$

Let's check if $c\cdot u$ exist in S

$$2(cu_2)-3(cu_4) = c(2u_2-3u_4)$$
$$= c(0)$$
$$= 0$$

This condition is also satisfied.

Since all three conditions are satisfied therefore S is a subspace of R^4.

Basis and Dimension:

The given equation for subspace

$$2x_2-3x_4=0$$
$$\Rightarrow x_2 = \frac{3}{2}x_4$$

This means variable x_1, x_3 and x_4 are free.
Let's set free variable as

$$x_1 = r$$
$$x_3 = s$$
$$x_4 = t$$

This gives
$$x_2 = \frac{3}{2}t$$

A general vector in S can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ \frac{3}{2}t \\ s \\ t \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 3/2 \\ 0 \\ 1 \end{bmatrix}$$

Three vectors on the right span the subspace S and are linearly independent. Therefore they form the basis.

Note: To avoid the fraction, we can multiply third vector by 2.

Basis of S can be defined as

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Dimension of subspace = No. of vectors in basis
$= 3$

4. For the linear differential system $\dot{\mathbf{x}} = A\mathbf{x}$, complete the following steps (a)-(c) for each A matrix given below.

   (a) Find the general solution of real functions.

   (b) Sketch **by hand** an approximate phase portrait indicating the direction of the trajectories.

   (c) Discuss why the trajectories are shaped as you drew them. What happens as $t \to +\infty$ and $t \to -\infty$?

$$A = \begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 4 \\ -4 & 4 \end{bmatrix}$$

Ans: First system:

$$A = \begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix}$$

(a) General solution

Solve eigenvalues using the characteristic equation

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 4-\lambda & 4 \\ 4 & -2-\lambda \end{bmatrix}\right) = 0$$

$$(4-\lambda)(-2-\lambda) - 4\cdot 4 = 0$$

$$-8 - 4\lambda + 2\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 2\lambda - 24 = 0$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\cdot 1\cdot -24}}{2\cdot 1}$$

$$= \frac{2 \pm \sqrt{4 + 96}}{2}$$

$$= \frac{2 \pm \sqrt{100}}{2}$$

$$= \frac{2 \pm 10}{2}$$

$$= 1 \pm 5$$

$$\lambda_1 = 6, \lambda_2 = -4$$

The eigenvalues are real and have opposite signs.

This indicates the origin is a saddle point.

Now, find the eigenvectors:

<u>For $\lambda_1 = 6$</u> let's solve

$$(A - \lambda I)v = 0$$

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) v = 0$$

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x + 4y = 0 \Rightarrow x - 2y = 0 \quad — ①$$

$$4x - 8y = 0 \Rightarrow x - 2y = 0 \quad — ②$$

This gives $x = 2y$

so eigenvector for simplified value for x & y

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

<u>For $\lambda_2 = -4$</u>

$$(A - \lambda I)v = 0$$

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) v = 0$$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$