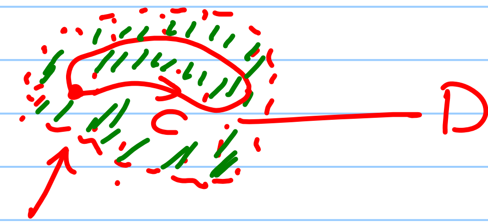
12.B CAUCHY'S INTEGRAL THEOREM

Theorem 1 (Cauchy's Integral Theorem)

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed curve C in D,

contour

$$\boxed{\oint_C f(z)\,dz = 0}$$



Example 1: Entire functions (i.e. analytic in all of $\mathbb{C}$).

e^z

$\sin z$

$\cos z$

polynomials

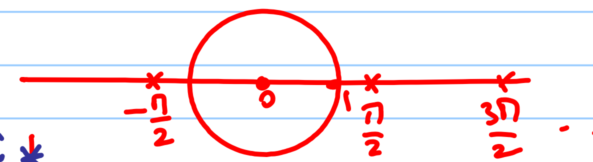
$$\oint_C e^z\,dz = 0, \quad \oint_C \cos z\,dz = 0, \quad \oint_C z^n\,dz = 0$$

for any closed path since the functions are entire

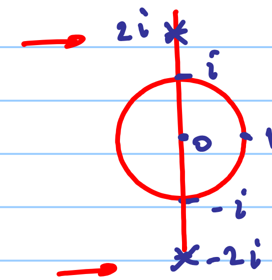
Example 2: Singularities outside contour (a simple closed curve)

$$\oint_C \sec z \, dz = 0, \qquad \oint_C \frac{dz}{z^2+4} = 0 \qquad \text{where } C \text{ is the unit circle}$$

$\sec z = \dfrac{1}{\cos z}$ is not analytic at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$
but these points are outside the unit circle.



Similarly for the second $z^2+4=0 \Leftrightarrow z = \pm 2i$.
These points too are outside the unit circle.



Example 3: Non analytic function

Note $\overline{e^{it}} = e^{-it}$

$$\oint_{C} \bar{z} \, dz = \int_0^{2\pi} e^{-it} \, i e^{it} \, dt = 2\pi i \quad \text{(try it!)}$$

C: unit circle

This does not contradict Cauchy's Theorem because $f(z) = \bar{z}$ is not analytic.

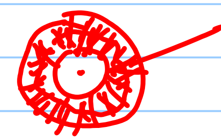
Example 4: Analyticity sufficient but not necessary.

$$\oint_C \frac{dz}{z^2} = 0 \qquad \text{(Try it!)}$$

C: $\odot$ 1

This result does not follow from Cauchy's Theorem because $f(z) = \frac{1}{z^2}$ is not analytic at $z=0$. So, the condition that f is analytic in D is sufficient rather than necessary for $\oint_C f(z)\,dz = 0$ to be true.

Example 5: Simple connectedness essential.



$$\oint_C \frac{dz}{z} = 2\pi i$$

C: $\odot$ 1

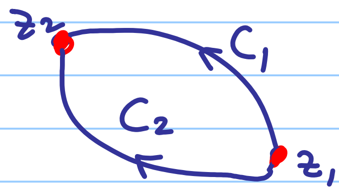
C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $f(z) = \frac{1}{z}$ is analytic, but this domain is not simply connected so that Cauchy's Theorem cannot be applied.

Independence of Path

Theorem 2 (Independence of Path)

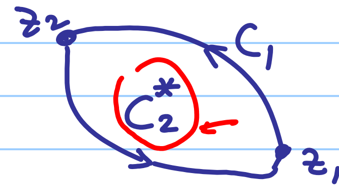
If $f(z)$ is analytic in a simply connected domain D, then the integral $\int_C f(z)\,dz$ is <u>independent of path</u> (i.e. it only depends on the endpoints of C and not on C itself).

Proof:



We'll prove $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

Consider



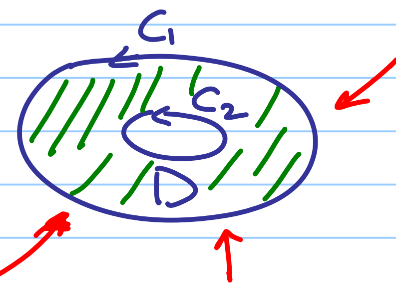
By Cauchy's Integral Theorem

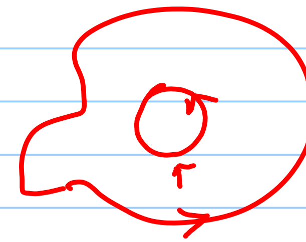
$$\int_{C_1, C_2^*} f(z)\,dz = 0 = \int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz$$

$$\Rightarrow \quad 0 = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

Principle of Deformation of Path

A doubly connected domain is



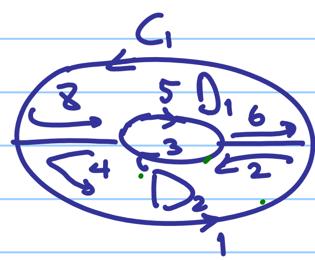


Theorem If $f(z)$ is analytic in any domain D^* and contains D and its boundary curves, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

(both taken counterclockwise or clockwise)

Proof:



$$\int_{C_1} = \int_1 + \int_7 = \int_1 + \underbrace{\int_2 + \int_6}_{=0} + \int_7 + \underbrace{\int_4 + \int_8}_{=0}$$

$$= \left(\int_6 + \int_7 + \int_8\right) + \left(\int_1 + \int_2 + \int_4\right) \underset{\text{Cauchy's Th}}{=} -\int_5 - \int_3 = \int_{C_2}$$

Example 6 $\displaystyle\int_C \frac{1}{z}\,dz = 2\pi i$

C any simple closed curve $\circlearrowleft$ that contains 0