535.641 Mathematical Methods Assignment 5

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Name DILIP KUMAR

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1. Rewrite the n^{th} order differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + ... + a_0 y(t) = b_0 u(t) + b_1 \dot{u}(t)$$

as a dimension-n linear state equation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$$

Hint: Let $x_n(t) = y^{(n-1)}(t) - b_1 u(t)$

<u>Ans:</u> Let's introduce unknown functions $x_i(t)$ as

$$
\begin{aligned}
x_1(t) &= y(t) \\
x_2(t) &= y'(t) = x_1'(t) \\
x_3(t) &= y''(t) = x_2'(t) \\
&\cdots \\
&\cdots \\
x_{n-1}(t) &= y^{(n-2)}(t) = x_{n-2}'(t)
\end{aligned}
$$

The hint provides the definition for the final state variables

$$x_n(t) = y^{(n-1)}(t) - b_1 u(t)$$

Now find the first derivative of each state variables $x_i'(t)$

$$x_1'(t) = y'(t) = x_2(t)$$
$$x_2'(t) = y''(t) = x_3(t)$$
$$\cdots \quad \cdots \quad \cdots$$
$$x_{n-2}'(t) = y^{n-2}(t) = x_{n-1}(t)$$
$$x_{n-1}'(t) = y^{n-1}(t)$$

From the hint definition

$$x_n(t) = y^{(n-1)}(t) - b_1 u(t)$$

$$y^{(n-1)}(t) = x_n(t) + b_1 u(t)$$

Therefore

$$x'_{n-1}(t) = x_n(t) + b_1 u(t)$$

Now for $x_n'(t)$, use the hint first

$$x_n(t) = y^{(n-1)}(t) - b_1 u(t)$$
$$x_n'(t) = y^n(t) - b_1 u'(t)$$

As per original differential equation

$$y^{(n)}(t) = -a_0 y(t) - a_1 y'(t) - \cdots - a_{n-1} y^{(n-1)}(t) + b_0 u(t) + b_1 u'(t)$$

Let's apply this to solve $x_n(t)$

$$x_n'(t) = [-a_0 y(t) - \cdots - a_{n-1} y^{(n-1)}(t) + b_0 u(t) + b_1 u'(t)] - b_1 u'(t)$$

$$= -a_0 y(t) - a_1 y'(t) - \cdots - a_{n-1} y^{(n-1)}(t) + b_0 u(t)$$

Now let's replace $y(t)$ and its derivative with state variable

$$x_n'(t) = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-2} x_{n-1} - a_{n-1}(x_n + b_1 u) + b_0 u$$

$$= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + (b_0 - a_{n-1} b_1) u(t)$$

Now assemble the derivative into matrix form

$$x'(t) = Ax(t) + Bu(t)$$

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \\ b_0 - a_{n-1} b_1 \end{bmatrix} u(t)$$

This gives the final state space representation with matrix A and B as below

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \\ b_0 - a_{n-1} b_1 \end{bmatrix}$$

2. Consider the system $\dot{\mathbf{x}} = A\mathbf{x}$. Sketch a phase portrait for each A matrix:

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Ans:

We first need to find the eigenvalue and eigenvectors of the matrix A.

Eigenvalues: Determin the stability and type of the fixed point at the origin.

Eigenvectors: Determin the orientation of the trajectories.

Matrix 1:

$$A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & 0.5 \\ 0.5 & 1-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)^2 - (0.5)^2 = 0$$

$$1-\lambda = \pm 0.5$$

$$\lambda = 1 \pm 0.5$$

$$\lambda_1 = 1.5, \quad \lambda_2 = 0.5$$

Since both eigenvalues are real, distinct and +ve, the origin is an unstable node.

Now find the eigenvectors using setup equation:

$$(A - \lambda I)V = 0$$

$$\begin{bmatrix} 1-\lambda & 0.5 \\ 0.5 & 1-\lambda \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (1-\lambda)V_1 + 0.5V_2 \\ 0.5V_1 + (1-\lambda)V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>At $\lambda_1 = 1.5$:</u>

$$-0.5V_1 + 0.5V_2 = 0 \Rightarrow V_1 = V_2$$

$$0.5V_1 - 0.5V_2 = 0 \Rightarrow V_1 = V_2$$

The simplest eigenvector $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ i.e $V_1 = V_2 = 1$

At $\lambda_2 = 0.5$

$$\begin{bmatrix} (1-0.5)v_1 + 0.5v_2 \\ 0.5v_1 + (1-0.5)v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

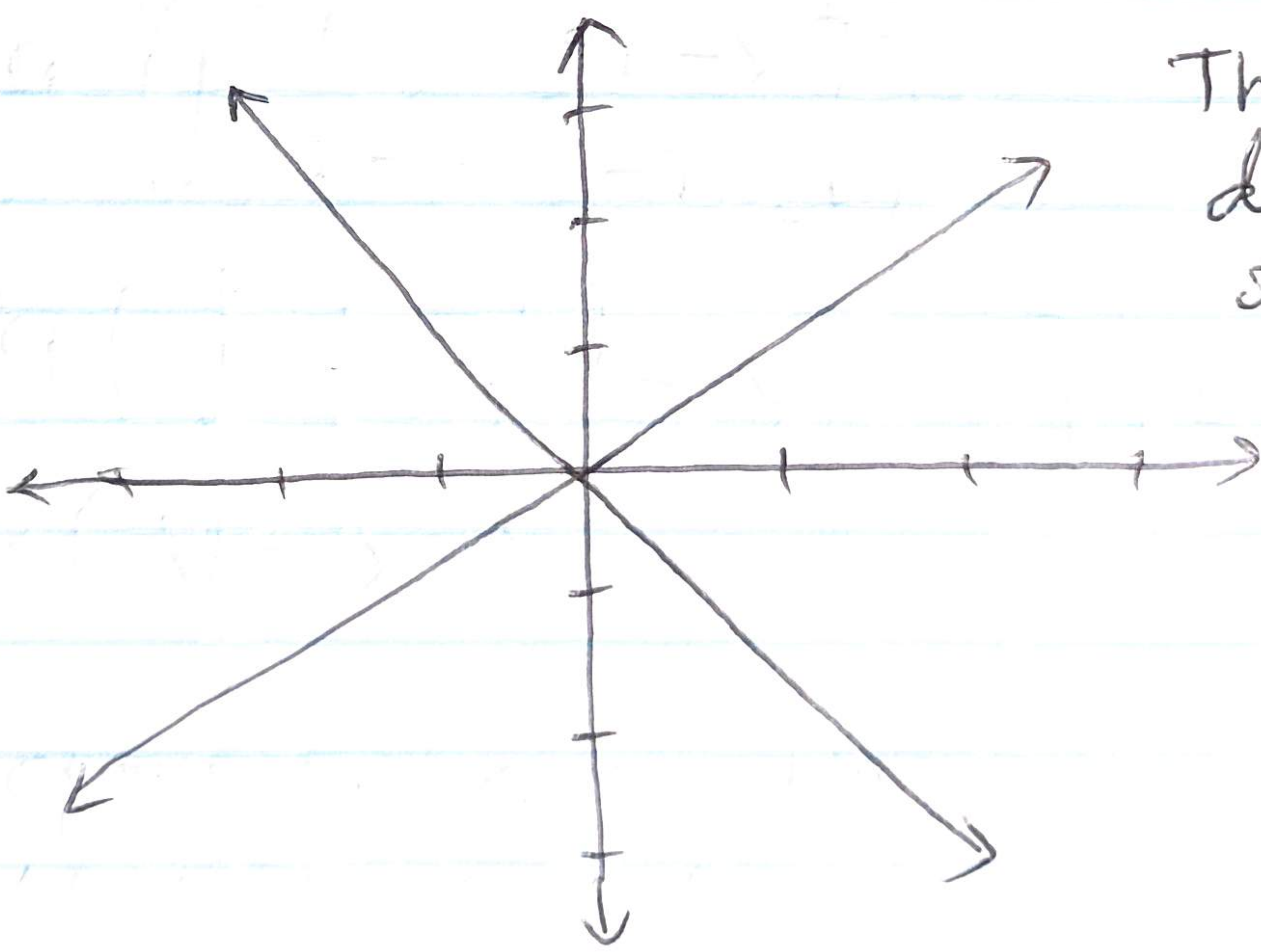
$0.5v_1 + 0.5v_2 = 0 \Rightarrow v_2 = -v_1$

$0.5v_1 + 0.5v_2 = 0 \Rightarrow v_2 = -v_1$

This gives simplest eigenvector

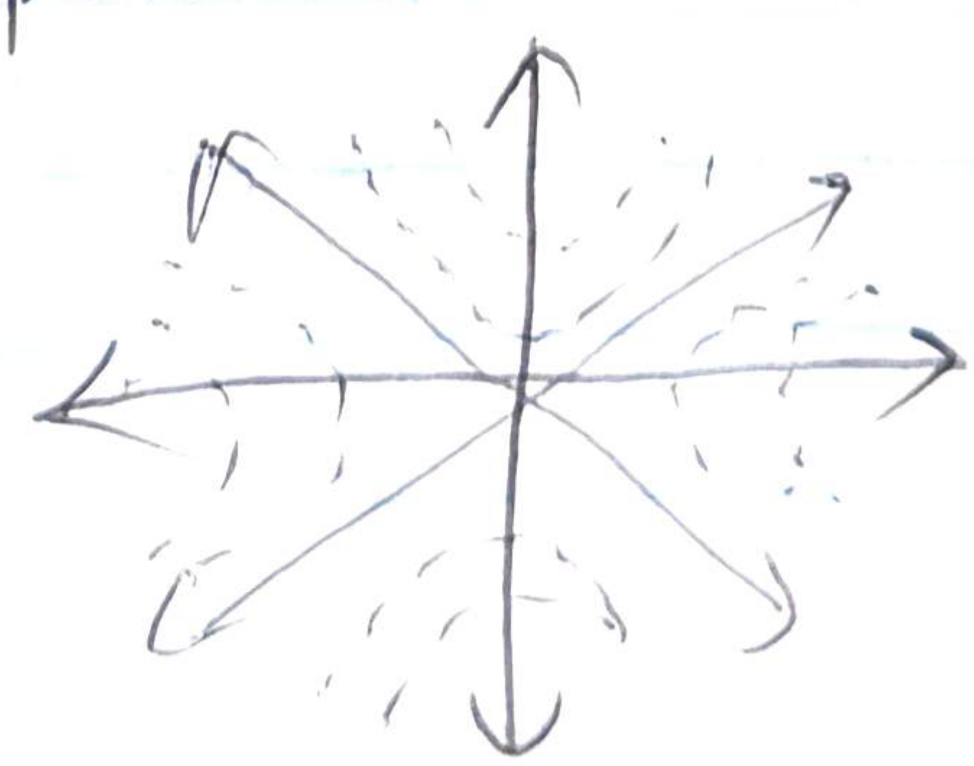
$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Sketch the phase portrait:



The trajectories defined by the eigenvectors.

The phase portrait of $Dx = Ax$



Matrix 2:

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

Let's find the eigenvalues & eigenvectors.
The characteristic equation

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -1-\lambda & -1 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

$$(-1-\lambda)^2 - (-1)(1) = 0$$

$$(\lambda+1)^2 = -1$$

$$\lambda + 1 = \pm i$$

$$\lambda = -1 \pm i$$

The eigenvalues are complex conjugates with a negative real part. The origin is a stable spiral. The trajectories spiral into the origin.

Direction of rotation:

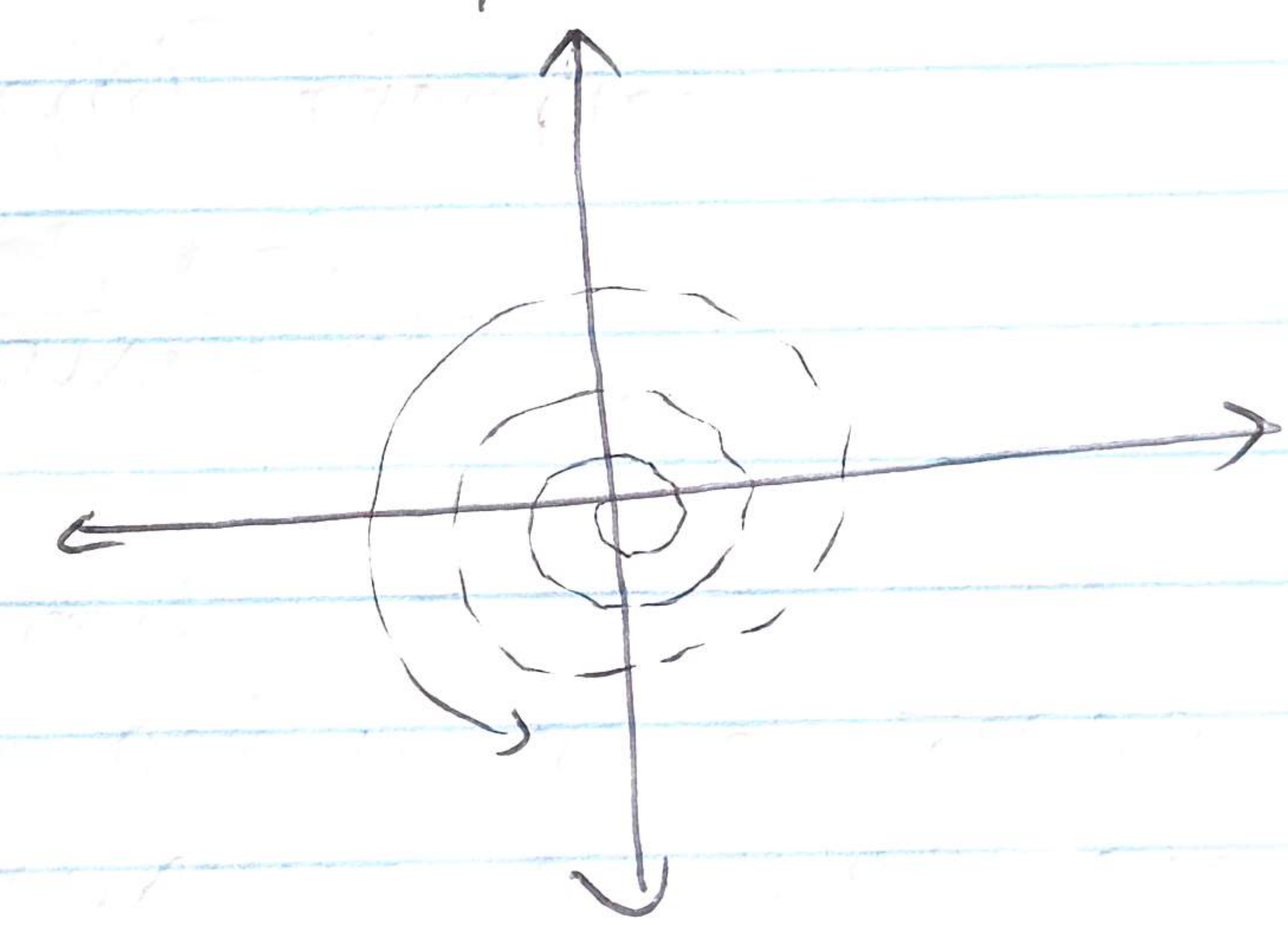
Let's use the test point $(1, 0)$

The velocity vector is

$$\dot{x} = Ax = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This vector points up and to the left, indicating a counter clockwise spiral.

Sketch the phase portrait:



Matrix 3:

$$A = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$$

To find eigenvalues, use characteristic equation

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right) = 0$$

$$(-2-\lambda)(1-\lambda) - 4 = 0$$
$$-2-\lambda+2\lambda+\lambda^2-4=0$$
$$\lambda^2+\lambda-6=0$$
$$(\lambda+3)(\lambda-2)=0$$
$$\lambda_1 = 2, \quad \lambda_2 = -3$$

Since eigenvalues are real and have opposite signs, the origin is a saddle point, which is unstable.

Find eigenvectors:

Using setup equation

$$(A-\lambda I)v = 0$$

$$\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-2-\lambda)v_1 - 2v_2 = 0 \Rightarrow v_2 = -\frac{1}{2}(2+\lambda)v_1$$
$$-2v_1 + (1-\lambda)v_2 = 0 \Rightarrow v_2 = \frac{2}{1-\lambda} v_1$$

At $\lambda_1 = 2$

$$V_2 = -\frac{1}{2}(2+2) V_1 = -2V_1$$

$$V_2 = \frac{2}{1-2} V_1 = -2V_1$$

the simplest eigenvector is

$$V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

This is unstable direction

At $\lambda_2 = -3$

$$V_2 = -\frac{1}{2}(2-3) V_1 = \frac{1}{2} V_1$$

$$V_2 = \frac{2}{1+3} V_1 = \frac{1}{2} V_1$$

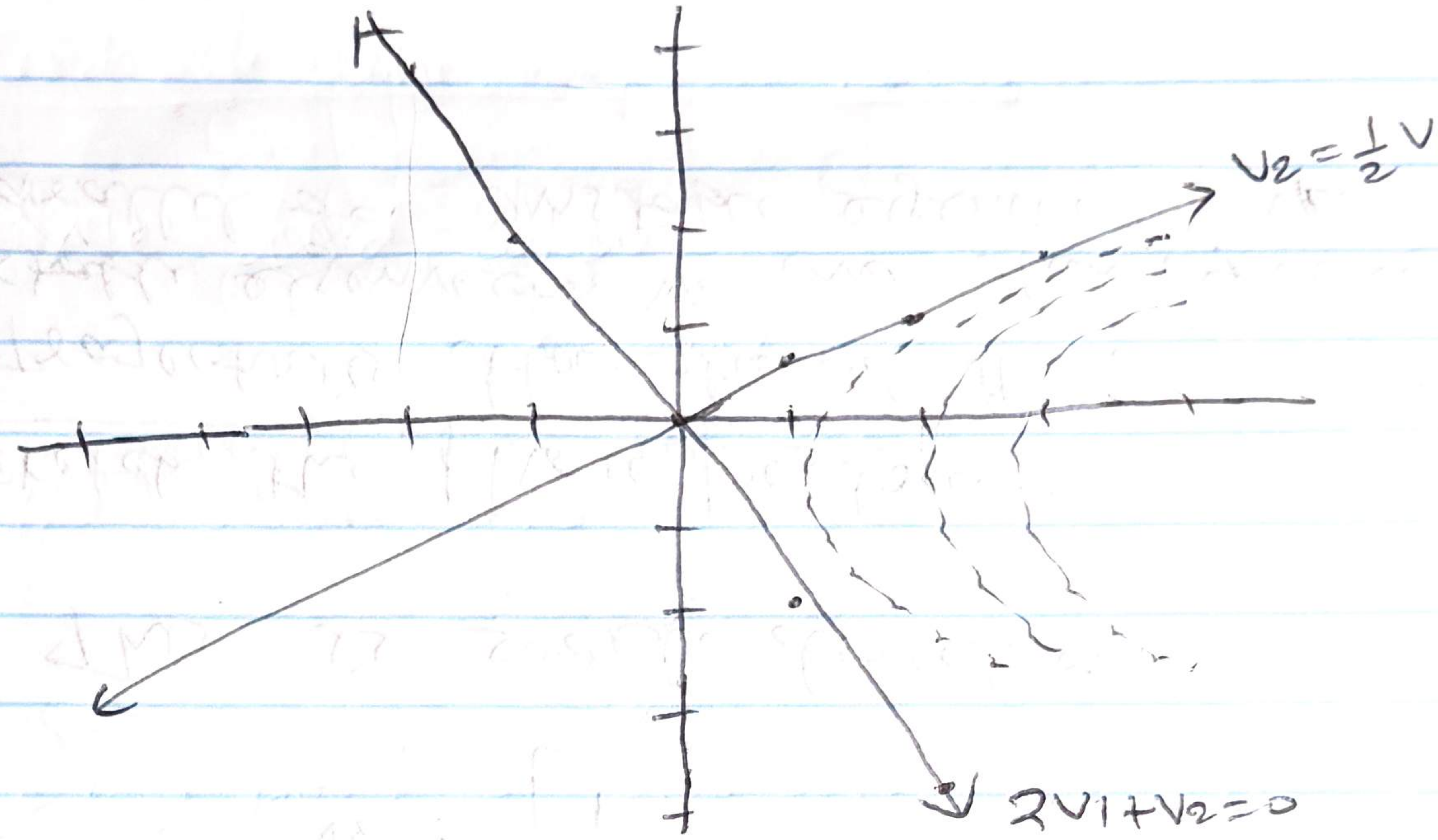
The simplest eigenvector is

$$V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

This is stable direction.

Sketch the phase portrait:

Trajectories flow toward the origin along the stable eigenvector V_2 and then repelled away parallel to unstable eigenvector V_1.



Matrix 4:

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

Find eigenvalues

$$(-1-\lambda)(1-\lambda) - (-2)(1) = 0$$
$$\lambda^2 - 1 + 2 = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

Since eigenvalues are purely imaginary with no real part, the origin is a center. Trajectories are stable, closed orbits.

Direction of rotation:

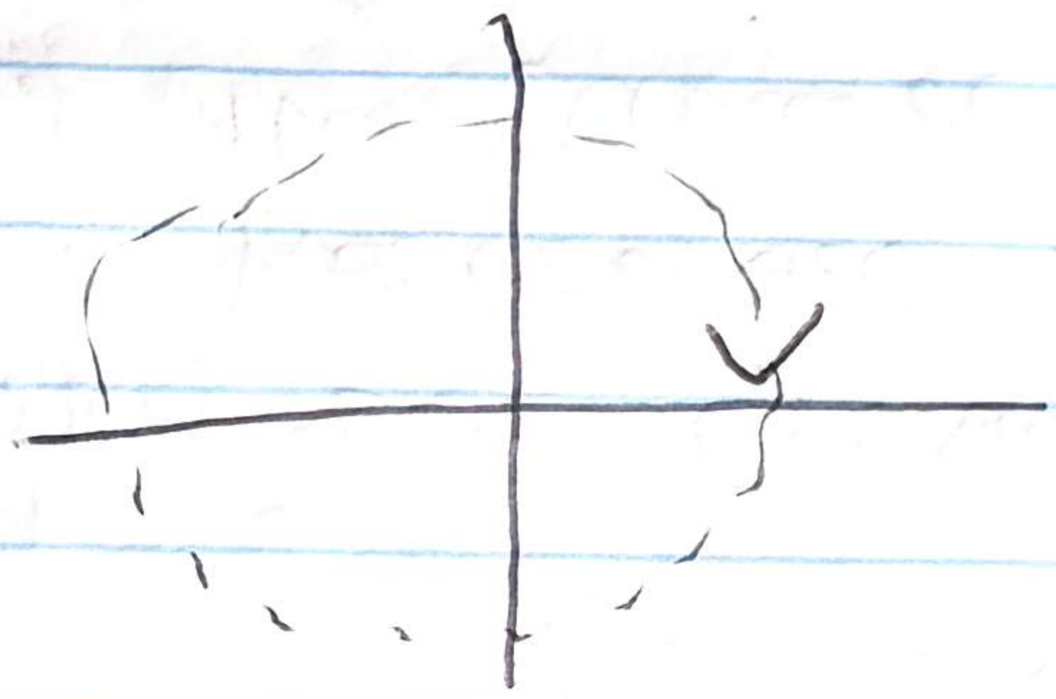
At the test point $(1,0)$, the velocity vector

$$x' = Ax = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

This vector points down and to the left. i.e clockwise direction.

Sketch the phase portrait:

Draw a series of concentric elliptical orbits around the origin with arrow indicating clockwise motion.



Matrix 5:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Find eigenvalues:

Since the matrix is triangular, the eigen-values are the diagonal entries

$$\lambda_1 = -1, \quad \lambda_2 = 0$$

The presence of zero eigenvalues mean the origin is not an isolated equilibrium point. There is line of fixed points.

Find the line of Fixed points:

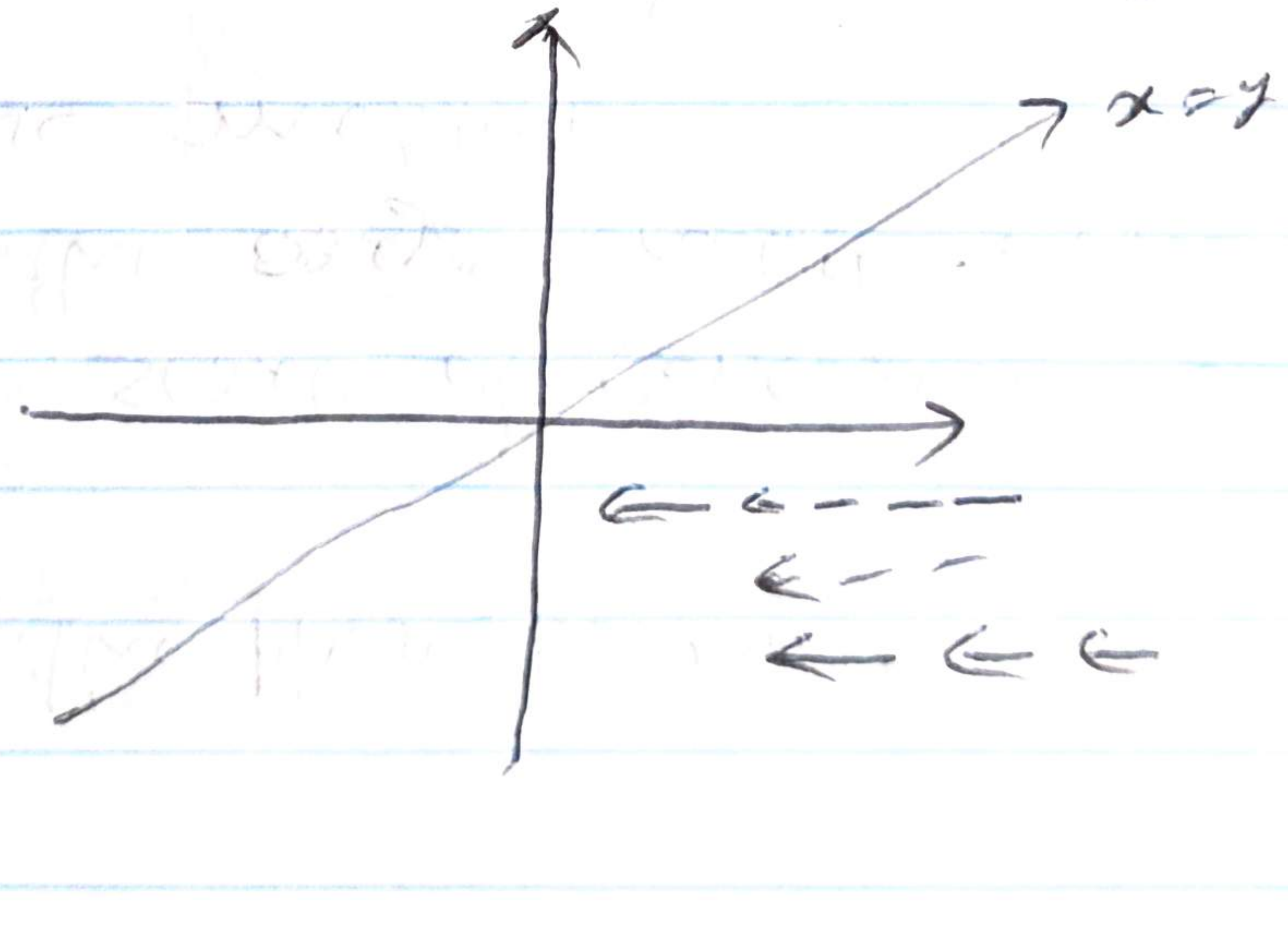
We find when
$$x' = Ax = 0$$

This occurs when
$$-x + y = 0$$
$$y = x$$
Every point on this line is a fixed point.

Sketch the phase portrait:

The system equations are
$$x' = -x + y$$
$$y' = 0$$ { it means all trajectories are horizontal lines }

Because the non zero eigenvalues is negative ($\lambda_1 = -1$), these horizontal trajectories all move toward the line of fixed points.



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3. Consider the linear system:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + Bu, \quad \mathbf{x} \in \mathbb{R}^2, \quad u \in \mathbb{R},$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ be a state feedback gain such that a forcing, $u(t)$, is prescribed according to,

$$u(t) = -K\mathbf{x}(t).$$

So that the closed-loop system is:

$$\frac{d\mathbf{x}}{dt} = (A - BK)\mathbf{x}.$$

(a) Show that the open-loop system (with $u = 0$) is unstable.

(b) Determine conditions on the feedback gain $K = [k_1 \ \ k_2]$ such that the closed-loop system is stable. *Hint: For a 2×2 matrix, all eigenvalues have negative real parts if and only if the trace is negative and the determinant is positive.*

(c) Among all such stabilizing gains K, find the one that minimizes the Euclidean norm

$$\min_K \|K\|_2 = \sqrt{k_1^2 + k_2^2}$$

(a) Show the open-loop system is unstable

The open-loop system is describe by the equation

at $u=0$

$$\frac{dx}{dt} = Ax + B\times 0 = Ax$$

The stability of this system is determined by the eigenvalues of the matrix A

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

Since A is an upper triangular matrix, its eigenvalues are simply the entries on its main diagonal

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

A linear system is unstable if any of its eigenvalues has a positive real part.
Since $\lambda_1 = 1$ is positive, the open-loop system is unstable.

(b) Conditions for closed-loop stability

The closed-loop system is given by

$$\frac{dx}{dt} = (A - BK)x$$

Find the closed-loop system matrix

$$A_{cl} = A - BK$$

$$BK = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix}$$

$$A_{cl} = A - Bk$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - k_1 & 2 - k_2 \\ -k_1 & -1 - k_2 \end{bmatrix}$$

For 2x2 matrix, the system is stable if and only if its trace is negative and its determinant is positive.

<u>Trace condition:</u>

$$tr(A_{cl}) = (1-k_1)+(-1-k_2) = -k_1 - k_2$$

For stability

$$tr(A_{cl}) < 0$$

$$-k_1 - k_2 < 0$$

$$k_1 + k_2 > 0$$

<u>Determinant condition:</u>

$$\det(A_{cl}) = \det\left(\begin{bmatrix} 1-k_1 & 2-k_2 \\ -k_1 & -1-k_2 \end{bmatrix}\right)$$

$$= (1-k_1)(-1-k_2) - (-k_1)(2-k_2)$$

$$= -(1-k_1)(1+k_2) + k_1(2-k_2)$$

$$= -(1-k_1+k_2-k_1k_2) + 2k_1 - k_1k_2$$

$$= -1+k_1-k_2+\cancel{k_1k_2}+2k_1-\cancel{k_1k_2}$$

$$= 3k_1 - k_2 - 1$$

For stability, we require

$$\det(A_{cl}) > 0$$

$$3k_1 - k_2 - 1 > 0$$

Therefore, the conditions on the feedback gain K for the closed-loop system to be stable are:

$$k_1 + k_2 > 0$$

$$3k_1 - k_2 > 1$$

(c) Stablizing gain that minimizes the norm

Minimizing the norm

$$\min_K \|K\|_2 = \sqrt{k_1^2 + k_2^2}$$

Minimizing the norm is equivalent to minimizing its square;

$$f(k_1, k_2) = k_1^2 + k_2^2$$

Find the point (k_1, k_2) in the stability region that is closest to the origin $(0, 0)$.

The stability region is an open, wedge-shaped area defined by inequalities from part (b)

$$k_2 > -k_1$$

$$k_2 < 3k_1 - 1$$

The point closest to origin will lie at the vertex where the boundary lines intersect

$$k_2 = -k_1$$

$$k_2 = 3k_1 - 1$$

This gives

$$-k_1 = 3k_1 - 1$$

$$\Rightarrow k_1 = \frac{1}{4}$$

$$k_2 = -\frac{1}{4}$$

The point $(1/4, -1/4)$ represents the theoretical minimum.

The gain that minimizes the Euclidean norm is

$$K = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

4. Alfvén waves describe the evolution of small amplitude perturbations in an electrically conducting fluid in the presence of a strong background magnetic field and can be modeled using the following equations

$$\frac{d\omega}{dt} = ik\ j,$$
$$\frac{dj}{dt} = ik\ \omega,$$

where ω is the vorticity, j is the induced current density, k is the wavenumber of the fluid disturbance, and $i = \sqrt{-1}$.

(a) Cast the set of equations in the form, $\dot{\mathbf{x}} = A\mathbf{x}$, and clearly identify $\mathbf{A}$ and $\mathbf{x}$.

(b) Classify the A matrix, (Hermitian/Skew-Hermitian/Unitary)?

(c) Diagonalize the matrix if possible.

(d) Obtain the second order ODEs for $\omega(t), j(t)$ from the given first order ODEs. Solve them as an initial value problem to find their solution. The initial conditions are $\omega(t=0) = \omega_0$, $j(t=0) = 0$.

(e) Confirm your solution in part (d) by expanding the solution vector $\mathbf{x}$ using the eigen solutions of matrix A and initial conditions given in (d).

Ans:

(a) State space form

To cast the equation in the form

$$x' = Ax$$

we first define the state vector x and x'.

Let:

$$x = \begin{bmatrix} \omega \\ j \end{bmatrix}, \quad x' = \begin{bmatrix} \omega' \\ j' \end{bmatrix}$$

Using given equation, we can write

$$\omega' = (0)\omega + (ik)j$$
$$j' = (ik)\omega + (0)j$$

This can be written in matrix form as

$$\begin{bmatrix} \omega' \\ j' \end{bmatrix} = \begin{bmatrix} 0 & ik \\ ik & 0 \end{bmatrix} \begin{bmatrix} \omega \\ j \end{bmatrix}$$

Thus, the state vector and matrix A are

$$x = \begin{bmatrix} \omega \\ j \end{bmatrix}, \quad A = \begin{bmatrix} 0 & ik \\ ik & 0 \end{bmatrix}$$

(b) Matrix classification

$$A = \begin{bmatrix} 0 & ik \\ ik & 0 \end{bmatrix}$$

<u>Transpose:</u>

$$A^T = \begin{bmatrix} 0 & ik \\ ik & 0 \end{bmatrix}$$

<u>Conjugate Transpose:</u>

$$A^{\dagger} = (A^T)^* = \begin{bmatrix} 0 & -ik \\ -ik & 0 \end{bmatrix}$$

Now check conditions

Hermitian:

$$\text{is } A = A^{t} \,?$$

It is not because $ik \neq -ik$

Skew-Hermitian:

$$\text{is } A = -A^{t} \,?$$

$$-A^{t} = -\begin{bmatrix} 0 & -ik \\ -ik & 0 \end{bmatrix} = \begin{bmatrix} 0 & ik \\ ik & 0 \end{bmatrix} = A$$

Yes, it is.

Unitary:

$$\text{is } A^{t}A = I \,?$$

$$A^{t}A = \begin{bmatrix} 0 & -ik \\ -ik & 0 \end{bmatrix} \begin{bmatrix} 0 & ik \\ ik & 0 \end{bmatrix} = \begin{bmatrix} (-ik)(ik) & 0 \\ 0 & (-ik)(ik) \end{bmatrix}$$

$$= \begin{bmatrix} k^2 & 0 \\ 0 & k^2 \end{bmatrix} \neq I$$

So it is not Unitary.

The matrix A is skew-Hermitian.

(c) Matrix Diagonalization

To diagonalize A, we need to find eigenvalue and eigenvectors.

Eigenvalues:

Use characterstic equation

$$\det(A-\lambda I)=0$$

$$\det\left(\begin{bmatrix}0 & ik\\ ik & 0\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)=0$$

$$\det\left(\begin{bmatrix}-\lambda & ik\\ ik & -\lambda\end{bmatrix}\right)=0$$

$$\lambda^2-(ik)^2=0$$

$$\Rightarrow \lambda^2+k^2=0$$

$$\Rightarrow \lambda^2=-k^2$$

$$\Rightarrow \lambda=\pm ik$$

$$\lambda_1=ik \text{ and } \lambda_2=-ik$$

Eigenvector: Using setup equation

$$(A-\lambda I)V=0$$

For $\lambda_1 = ik$

$$(A - \lambda_1 I) v_1 = 0$$

$$\begin{bmatrix} -ik & ik \\ ik & -ik \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-ikv_1 + ikv_2 = 0$$

$$\Rightarrow \quad v_1 = v_2$$

simplest eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -ik$

$$(A - \lambda_2 I) = 0$$

$$\begin{bmatrix} ik & ik \\ ik & ik \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$ikv_1 + ikv_2 = 0$$

$$\Rightarrow \quad v_1 = -v_2$$

simplest eigenvector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is diagonalizable as

$A = PDP^{-1}$, where

Eigenvector matrix P:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Eigenvalue matrix D:

$$D = \begin{bmatrix} ik & 0 \\ 0 & -ik \end{bmatrix}$$

(d) second-order ODEs and solutions

We can obtain second-order ODEs by differentiating and substitution

$\Rightarrow$ For $\omega(t)$

$$\omega' = ikj \Rightarrow \omega'' = ikj'$$

we have $j' = ik\omega$, this give

$$\omega'' = ik(ik\omega) = -k^2\omega$$

This gives the ODE

$$\omega'' + k^2\omega = 0$$

For $j(t)$:

$$j' = ik\omega \Rightarrow j'' = ik\omega'$$

we have $\omega' = ikj$, this gives

$$j'' = ik(ikj) = -k^2 j$$

This gives the ODE

$$j'' + k^2 j = 0$$

In the general solution format $C_1 \cos(kt) + C_2 \sin(kt)$

$\rightarrow \omega(t) = C_1 \cos(kt) + C_2 \sin(kt)$

$\rightarrow j(t) = C_3 \cos(kt) + C_4 \sin(kt)$

Applying the initial conditions

$$\omega(0) = 0, \; J(0) = 0$$

$\Rightarrow \omega(0) = C_1 = \omega_0$

$\rightarrow j(0) = C_3 = 0 \Rightarrow j(t) = C_4 \sin(kt)$

Now

$$\omega'(t) = -k\omega_0 \sin(kt) + kC_2 \cos(kt)$$

$$ikj(t) = ik(C_4 \sin(kt))$$

comparing these two equations

$$kC_2 = 0 \Rightarrow C_2 = 0 \text{ and}$$

$$-k\omega_0 = ikC_4 \Rightarrow C_4 = \frac{-k\omega_0}{ik} = i\omega_0$$

The final solutions are

$$\omega(t) = \omega_0 \cos(kt)$$
$$j(t) = i\omega_0 \sin(kt)$$

(e) Confirming using Eigen solution

The general solution of

$$x' = Ax$$

is

$$x(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$$

$$\begin{bmatrix} \omega(t) \\ j(t) \end{bmatrix} = c_1 e^{ikt} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-ikt} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Apply the initial condition $X(0) = \begin{pmatrix} \omega_0 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} \omega_0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This gives the system of equation

$$c_1 + c_2 = \omega_0 \quad \text{—(1)}$$
$$c_1 - c_2 = 0 \Rightarrow c_1 = c_2$$

Solving gives

$$c_1 = c_2 = \frac{\omega_0}{2}$$

which give

$$\begin{bmatrix} \omega(t) \\ j(t) \end{bmatrix} = \frac{\omega_0}{2} e^{ikt} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\omega_0}{2} e^{-ikt} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

On expanding the components

$$\omega(t) = \frac{\omega_0}{2}\left(e^{ikt} + e^{-ikt}\right) = \omega_0\left(\frac{e^{ikt} + e^{-ikt}}{2}\right)$$

$$j(t) = \frac{\omega_0}{2}\left(e^{ikt} - e^{-ikt}\right) = i\omega_0\left(\frac{e^{ikt} - e^{-ikt}}{2i}\right)$$

Using Euler's formula

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

we get

$$\omega(t) = \omega_0 \cos(kt)$$

$$j(t) = i\,\omega_0 \sin(kt)$$

This confirms the solution from part (d).