5.C ORTHOGONAL SETS OF FUNCTIONS 2

Example 3: $S = \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots, \cos(nx), \sin(nx), \dots\}$

1. Show that S is orthogonal on $[-\pi, \pi]$

2. Find the norms.

Soln: 1.

(a) $\langle 1, \cos(mx) \rangle = \int_{-\pi}^{\pi} (1) \cos(mx)\, dx = \left. \dfrac{\sin(mx)}{m} \right|_{-\pi}^{\pi} = 0$

(b) $\langle 1, \sin(mx) \rangle = \int_{-\pi}^{\pi} (1) \sin(mx)\, dx = \left. -\dfrac{\cos(mx)}{m} \right|_{-\pi}^{\pi} = 0$

(c) If $m \neq n$, $\langle \sin(mx), \sin(nx) \rangle = 0$ (proved in Ex. 1)

(d) If $m \neq n$,

$$\langle \cos(mx), \sin(nx) \rangle = \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx \underset{\text{Trig. id.}}{=}$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \Big(\sin((m+n)x) - \sin((m-n)x)\Big)\,dx$$

$$= -\frac{1}{2}\,\frac{1}{(m+n)}\cos((m+n)x)\Big|_{-\pi}^{\pi} + \frac{1}{2}\,\frac{1}{\underset{\neq 0}{m-n}}\cos((m-n)x)\Big|_{-\pi}^{\pi}$$

$$= 0 + 0 = 0$$

(e) $$\langle \cos(mx), \sin(mx) \rangle = \int_{-\pi}^{\pi} \cos(mx)\sin(mx)\,dx \underset{\text{Trig. id.}}{=}$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2mx)\,dx$$

$$= -\frac{1}{2}\,\frac{\cos(2mx)}{2m}\Big|_{-\pi}^{\pi} = 0$$

f. If $m \neq n$,

$$\langle \cos(mx), \cos(nx) \rangle = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx \underset{\text{Trig. id.}}{=}$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left(\cos((m-n)x) + \cos((m+n)x)\right)dx$$

$$= \frac{1}{\underset{\neq 0}{2(m-n)}} \sin((m-n)x)\Big|_{-\pi}^{\pi} + \frac{1}{2(m+n)} \sin\frac{((m+n)x)}{m+n}\Big|_{-\pi}^{\pi}$$

$$= 0 + 0 = 0$$

2. Norms:

$$\|1\| = \left(\int_{-\pi}^{\pi} 1^2\,dx\right)^{1/2} = \sqrt{2\pi}$$

$$\|\cos(mx)\| = \left(\int_{-\pi}^{\pi} \cos^2(mx)\,dx\right)^{1/2} = \left(\frac{1}{2}\int_{-\pi}^{\pi}(1+\cos(2mx))\,dx\right)^{1/2} = \sqrt{\pi}$$

$$\|\sin(mx)\| = \int_{-\pi}^{\pi} \sin^2(mx)\,dx)^{1/2} = \left(\frac{1}{2}\int_{-\pi}^{\pi}(1-\cos(2mx))\,dx\right)^{1/2} = \sqrt{\pi}$$

Example 4: With an identical computation as in Ex. 3 we have

$$S = \left\{ 1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \ldots \right\}$$

is orthogonal on $[-L, L]$

with norms

$$\boxed{\begin{aligned} \|1\| &= \sqrt{2L} \checkmark \\ \left\|\cos\left(\frac{n\pi x}{L}\right)\right\| &= \sqrt{L} \checkmark \\ \left\|\sin\left(\frac{n\pi x}{L}\right)\right\| &= \sqrt{L} \checkmark \end{aligned}}$$

Special case $L = \pi$ yields Example 3.

Example 5: Show that the set

$$S = \{1, \cos(x), \cos(2x), \dots \cos(nx), \dots\}$$

is orthogonal on $[0,\pi]$ and find the norms of the functions.

Sol^n: Now that we have studied several examples, I leave it up to you to verify orthogonality on $[0,\pi]$ and compute the norms to get

$$\|1\| = \sqrt{\pi}, \quad \|\cos(nx)\| = \sqrt{\pi/2}$$

Example 6: $S=\{1, \cos(\frac{\pi x}{L}), \dots, \cos(\frac{n\pi x}{L}), \dots\}$ is orthogonal on $[0,L]$

with norms $\|1\| = \sqrt{L}$, $\|\cos(nx)\| = \sqrt{L/2}$

Sol^n: Exercise

Special case $L=\pi$ yields Example 5.