Constrained Optimization

Certain engineering problems can be converted to a formulation involving the quadratic form $Q(\underline{x}) = \underline{x}^T A \underline{x}$ subject to the constraint:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

$$\text{or} \quad \|\underline{x}\| = \|\underline{x}\|^2 = 1$$

Note: The max/min of $Q(\underline{x})$ with no cross-terms is relatively trivial to determine given the above constraint.

Example: $Q(x_1, x_2, x_3) = a x_1^2 + b x_2^2 + c x_3^2$
where $a \geq b \geq c$.
The max. can be readily found for $\|\underline{x}\| = 1$ following these steps

$$\begin{aligned} Q(\underline{x}) &= a x_1^2 + b x_2^2 + c x_3^2 \\ &\leq a x_1^2 + b x_2^2 + c x_3^2 \end{aligned}$$

$$= a\left(x_1^2 + x_2^2 + x_3^2\right)$$
$$= a$$

The min of $Q(\underline{x})$ can similarly be determined to be c for this example.

<u>Theorem</u>: If A is a symmetric matrix for the quadratic form $Q(\underline{x}) = \underline{x}^T A \underline{x}$, then

1. $\max\{Q(\underline{x})\}$ is the largest eigenvalue of A and $\max\{Q(\underline{x})\}$ is achieved when $\underline{x}$ is the corresponding unit eigenvector.

2. $\min\{Q(\underline{x})\}$ is the smallest eigenvalue of A and $\min\{Q(\underline{x})\}$ is achieved when $\underline{x}$ is the corresponding unit eigenvector.

<u>Proof</u>: Diagonalize $A \ni A = PDP^T$.

so then, $\underline{x}^T A \underline{x} = \underline{y}^T D \underline{y}$ when $\underline{x} = P\underline{y}$

also note:

$$
\begin{aligned}
\|\underline{x}\|^2 = \underline{x}^T\underline{x} &= (P\underline{y})^T(P\underline{y}) \\
&= \underline{y}^T P^T P \underline{y} \qquad \text{since } P^TP = I \\
&= \underline{y}^T \underline{y} \\
&= \|\underline{y}\|^2
\end{aligned}
$$

Without loss of generality, suppose that $A \in \mathbb{R}^{3\times 3}$ and is diagonalized $\ni$

$$
D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}
$$

where $a \geq b \geq c$, and all are eigenvalues of the matrix A.

It then follows from the logic applied in the above example that

1. $\max\{Q(\underline{x})\} = a$

2. $\min\{Q(\underline{x})\} = c$

We further note that $\underline{y}^T D \underline{y} = a$ is achieved for $\underline{y} = [1, 0, 0]^T$ by the construction of D.

The $\underline{x}$ that then corresponds to $\max\{Q(\underline{x})\} = a$ is then determined to be,

$$\underline{x} = P\underline{y} = [\underline{v}_1 \quad \underline{v}_2 \quad \underline{v}_3] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{v}_1$$

where $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$ are the eigenvectors corresponding to a, b, and c respectively.

We therefore conclude that the $\max\{Q(\underline{x})\}$ is achieved when $\underline{x}$ is the unit eigenvector corresponding to the largest eigenvalue of A.

Similar arguement can then be applied

to the smallest eigenvalue/eigenvector.

QED