3.D DIAGONALIZATION

Note Title 7/7/2013

Matrix arithmetic with diagonal matrices is easy. Look at:

$$\underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} a \\ b \end{bmatrix}} = \begin{bmatrix} 2a \\ 3b \end{bmatrix}$$ ← No component mixing of vectors in Dx.

$$\underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{A}{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \end{bmatrix}$$ ← No row mixing in DA.

$$\underset{D^k}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^k} = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$$ ← Easy to compute powers.

We want to take advantage of this!
We introduce matrices that can be "transformed" to diagonal

Let A, B be $n \times n$ matrices. We say that A is similar to B, if there is an invertible matrix P such that ($\overset{s}{\sim}$)

$$\boxed{B = P^{-1} A P}$$

Note: It's easy to check that
1. $A \overset{s}{\sim} A$
2. $A \overset{s}{\sim} B \Rightarrow B \overset{s}{\sim} A$
3. $A \overset{s}{\sim} B$ and $B \overset{s}{\sim} C \Rightarrow A \overset{s}{\sim} C$

} Properties of an equivalence relation

Def^n: $A_{n \times n}$ is diagonalizable, if it is similar to a diagonal matrix. So

$$\boxed{P^{-1} A P = D}$$

for some P invertible and D diagonal

Hard

Easy!

First nice application: If we know P, D, then $A^k = P D^k P^{-1}$

Pf $A = PDP^{-1} \Rightarrow A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$ (Then induction)

If A is diagonalizable, we say that A can be diagonilized.
The process of finding P and D is called a diagonalization of A.
We also say that P and D diagonalize A.

Theorem: Let A be $n \times n$ matrix.

1. A is diagonalizable $\Longleftrightarrow$ A has n L.I. EVEs.

2. If A is diagonalizable with $P^{-1}AP = D$, then the columns of P are EVEs of A and the diagonal entries of D are the corresponding EVAs of A.

3. If $\{v_1, \ldots, v_n\}$ are L.I. EVEs of A with corresponding EVAs $\lambda_1, \ldots, \lambda_n$, then A can be diagonalized by

$$P = [v_1 \; v_2 \; \cdots \; v_n] \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \ddots \; \lambda_n \end{bmatrix}$$

We also have the following useful theorem.

Theorem: Let $A_{n\times n}$. TFAE ($\leftarrow$ The following are equivalent)

1. A is diagonalizable.
2. $\mathbb{R}^n$ has a basis of EVEs of A.

Example: Diagonalize, if possible, $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol^n: We found before that $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$ and

$E_0 = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, $E_1 = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

A has 3 L.I. EVEs so it is diagolizable: $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

(You don't have to but can check: $\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}}_{P^{-1}} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D}$)

Example: Diagonalize, if possible, $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & -2 \end{bmatrix}$.

Sol^n: We leave it as exercise that

$E_{1} = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, $E_{-2} = \text{Span}\left\{\begin{bmatrix} 1/3 \\ 1 \\ 1 \end{bmatrix}\right\}$, $E_{-4} = \text{Span}\left\{\begin{bmatrix} 1/5 \\ 1 \\ 0 \end{bmatrix}\right\}$

A has 3 L.I. EVE so it's diagonalizable: $P = \begin{bmatrix} 1 & 1/3 & 1/5 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & & \\ & -2 & \\ & & -4 \end{bmatrix}$

Theorem: Let $\lambda_1, \dots, \lambda_k$ be any set of distinct EVAs of $A_{n \times n}$.

1. Then any corresponding EVEs $v_1, \dots, v_k$ are L.I.
2. If $B_1, \dots, B_k$ are bases for the corresp. EVEs then $B = B_1 \cup \dots \cup B_k$ is L.I.
3. Let k be the number of all distinct EVAs of A. Then A is diagonalizable $\iff$ B in part 2 has exactly n elements.
4. A is diagonalizable $\iff$ For each EVA λ of A we have

$$\boxed{\text{alg. mult. of } \lambda = \text{geom. mult. of } \lambda}$$

Example: Is $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$ diagonalizable?

Sol^n: It is easy to see that $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$.

Let's check the EVEs:

$$\lambda_1 = \lambda_2 = 0 : \quad [A - 0I \mid 0] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ -1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Sol}^n} \begin{bmatrix} -3r \\ -r \\ r \end{bmatrix}$$

So $E_0 = \text{Span}\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}$ STOP $\lambda_1 = \lambda_2 = 0$ has alg. mult. 2 but geom. mult 1

A is non-diagonalizable

(Engineers called them 'defective'!)