2.D INVERSE and ADJOINT

Note Title 7/6/2013

INVERSE

Let A be a square matrix. A is invertible if there is a matrix B s.t.

$$AB = I \quad \text{and} \quad BA = I$$

Other names
invertible $\leftrightarrow$ nonsingular
noninvertible $\leftrightarrow$ singular

If no such B exists, A is non-invertible

If A is invertible, then B above is denoted by A^{-1} ('A inverse') and it is called the inverse of A.

Note: If A^{-1} exists, it is unique.

Idea for computation of A^{-1}. Let $B = [b_1\, b_2 \cdots b_n]$. Then
column

$$AB = I \Rightarrow A[b_1 \cdots b_n] = [e_1 \cdots e_n] \Rightarrow \begin{cases} Ab_1 = e_1 \\ Ab_2 = e_2 \\ \vdots \\ Ab_n = e_n \end{cases}$$

n linear systems with unknowns b_i

We can solve all these
$$\begin{matrix} Ab_1 = e_1 \\ \vdots \\ Ab_n = e_n \end{matrix}$$
simultaneously by reducing $[A \mid e_1 \dots e_n] = [A \mid I]$. REDUCE to RREF. If we get $[I \mid B]$ then $\boxed{A^{-1} = B}$

Example: Find A^{-1}, if it exists. $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & -2 \\ 3 & 5 & -2 \end{bmatrix}$

Sl^n Reducing

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 4 & -2 & 0 & 1 & 0 \\ 3 & 5 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & -3 & 1 & 0 \\ 0 & 5 & 1 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & -3 & 1 & 0 \\ 0 & 0 & -1/4 & 3/4 & -5/4 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 5 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & -4 \\ 0 & 4 & 0 & 0 & -4 & 4 \\ 0 & 0 & 1 & -3 & 5 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & -4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -3 & 5 & -4 \end{array}\right]$$

$[I \mid A^{-1}]$

So $A^{-1} = \begin{bmatrix} -2 & 5 & -4 \\ 0 & -1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$

Example: Find A^{-1}, if it exists. $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & -2 \\ -3 & -4 & 2 \end{bmatrix}$

Sol^n

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 4 & -2 & 0 & 1 & 0 \\ -3 & -4 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & -3 & 1 & 0 \\ 0 & -4 & -1 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & -3 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & 0 & 1 & 1 \end{array}\right]$$

STOP

Cannot get I

A is noninvertible (or singular)

Properties If A, B are invertible, then

- $(AB)^{-1} = B^{-1}A^{-1}$ (Reversed order!)
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^{-1})^{-1} = A$
- $AC = AD \Rightarrow C = D$. $KA = LA \Rightarrow K = L$ (Cancellation)

Pf

$AC = AD \Rightarrow A^{-1}(AC) = A^{-1}(AD)$

$(A^{-1}A)C = (A^{-1}A)D$

$IC = ID \Rightarrow C = D$

Determinants and Inversion

Cauchy's Theorem

- $\det(AB) = \det A \cdot \det B$ (For any A, B $n \times n$)
- A is invertible $\Longleftrightarrow$ $\det A \neq 0$ in this case

$$\boxed{\det A^{-1} = \frac{1}{\det A}}$$

Theorem Let A be invertible (so $\det A \neq 0$). Then

$Ax = b$
$A^{-1}(Ax) = A^{-1}b$
$(A^{-1}A)x = A^{-1}b$
$Ix = A^{-1}b$
$x = A^{-1}b$

- $Ax = b$ has the unique solution $x = A^{-1}b$
- $Ax = 0$ has only the trivial solution

$Ax = 0$
$A^{-1}Ax = A^{-1}0$
$x = 0$

Adjoint

A square. The adjoint of A, $\mathrm{Adj}(A)$ = Transpose of the matrix of cofactors of A, C_{ij}.

The (i,j) cofactor C_{ij} is:

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Checkboard

A_{ij} is the matrix obtained from A by deleting row i and column j.

Theorem If A is invertible, then

$$\boxed{A^{-1} = \frac{1}{\det A}\,\mathrm{Adj}(A)}$$

Example: Use Adj(A) to find A^{-1} for $A = \begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ 5 & 0 & 3 \end{bmatrix}$

Sol^n: Cofactors

$C_{11} = 9$ $\quad C_{12} = -2$ $\quad C_{13} = 15$ $\quad C_{11} = (-1)^{1+1} \det \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$

$C_{21} = -6$ $\quad C_{22} = 7$ $\quad C_{23} = 10$

$C_{31} = -10$ $\quad C_{32} = 6$ $\quad C_{33} = -11$ $\quad = (-1)^2 \cdot 9 = \boxed{9}$

$\rightarrow$ $\det A = 17$

So $\quad A^{-1} = \frac{1}{\det A} \operatorname{Adj} A = \frac{1}{17} \begin{bmatrix} 9 & -6 & -10 \\ -2 & 7 & 6 \\ 15 & 10 & -11 \end{bmatrix}$

Matrix Tranformations

Let $\underline{A_{m\times n}}$. We define a map (transformation) T by

$$T: \mathbb{R}^n \to \mathbb{R}^m \quad , \quad T(x) = \underset{(m\times n)\,(n\times 1)}{A\overset{n\text{-vector}}{x}} \leftarrow (m\times 1) \text{ in } \mathbb{R}^m$$

Such map is called a <u>matrix tranformation</u>. $T(x)$ is <u>the image of</u> x.

<u>Properties</u>

- $T(x_1 + x_2) = T(x_1) + T(x_2)$
- $T(cx) = c\,T(x)$

} <u>Linearity</u>

Example: Let $T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1+x_2+x_3\\x_1+2x_2+2x_3\\x_2+x_3\end{bmatrix}$

(a) Show that T is a matrix tranformation

(b) Which of $v_1=\begin{bmatrix}0\\-1\\-1\end{bmatrix}$ and $v_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$ are images of T?

Sol^n: (a) $T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\underline{\begin{bmatrix}1&1&1\\1&2&2\\0&1&1\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ So yes it's a mat. tranf. $A=\begin{bmatrix}1&1&1\\1&2&2\\0&1&1\end{bmatrix}$

(b) Need to find $\underline{u}$ (if it exists) s.t. $Tu=v_1$ or $\begin{bmatrix}1&1&1\\1&2&2\\0&1&1\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}0\\-1\\-1\end{bmatrix}$

Solve the system! $\left[\begin{array}{ccc|c}1&1&1&0\\1&2&2&-1\\0&1&1&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&1\\0&1&1&-1\\0&0&0&0\end{array}\right]\to$ solvable $\boxed{\text{YES}}$ v_1 is an image.

For: $\underline{v_2}$ need to solve $\left[\begin{array}{ccc|c}1&1&1&0\\1&2&2&0\\0&1&1&1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&0\\0&1&1&0\\0&0&0&1\end{array}\right]$ not solvable

No $\boxed{\text{NO}}$ v_2 is not an image.