4.C DERIVATIVES and ODES

Transforms of Derivatives

The following theorem that computes the tranform of the derivative is key to solving initial value problems of differential equations by using Laplace transforms.

Theorem: Let $f(t)$ be continuous with continuous derivatives $f', \dots, f^{(n-1)}$ s.t. $f, f', \dots, f^{(n-1)}$ are of exponential order and $f^{(n)}$ is piecewise continuous on all finite subintervals of $[0, \infty)$, then

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\,\mathcal{L}(f) - s f(0) - f'(0)$$

$$\vdots$$

$s^{n-3} f''(0)$

$$\mathcal{L}(f^{(n)}) = s^n\,\mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Example: Solve the IVP (initial value problem)

$$y'' + y = 2t, \quad y(0) = 1, \quad y'(0) = -1$$

Sol^n: Let $Y = L(y)$. We apply Laplace transform to both sides of the differential equation to get

$$L(y'') + L(y) = \frac{2}{s^2}$$

$$L(y'') = s^2 L(y) - s\,y(0) - y'(0)$$

$$L(y) = Y, \quad y(0) = 1, \quad y'(0) = (-1)$$

$$s^2 Y - s\,y(0) - y'(0) + Y = \frac{2}{s^2}$$

$$s^2 Y - s + 1 + Y = \frac{2}{s^2}$$

Algebraic!

$$(s^2+1)Y = s + \frac{2}{s^2} - 1$$

$$Y = \frac{s}{s^2+1} + \frac{2}{s^2(s^2+1)} - \frac{1}{s^2+1}$$

$$Y = \frac{s}{s^2+1} + \frac{2}{s^2(s^2+1)} - \frac{1}{s^2+1}$$

partial fraction

$$Y = \frac{s}{s^2+1} + \left[\frac{2}{s^2} - \frac{2}{s^2+1}\right] - \frac{1}{s^2+1}$$

$$Y = \frac{s}{s^2+1} - \frac{3}{s^2+1} + \frac{2}{s^2}$$

Then apply L^{-1} to get

$$y(t) = \cos(t) - 3\sin(t) + 2t$$