Note Title 8/21/2013

# REVIEW 3 PARTIAL FRACTIONS

$f(x) = \dfrac{P(x)}{Q(x)}$ , $P, Q$ polynomials

$f$ rational function

If $\deg(P) \geq \deg(Q)$ we may use long division :

$P(x) = Q(x)\,S(x) + R(x)$ , $R(x)$ = remainder , $\deg(R) < \deg(Q)$

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

integration of poly easy

So if we want to integrate $\int f = \int S + \int \dfrac{R}{Q}$

May assume that $f(x) = \dfrac{P(x)}{Q(x)}$ where $\deg(P) < \deg(Q)$

$f = \frac{P}{Q}$ , $\deg(P) < \deg(Q)$

Want to split $f$ into simpler fractions.

Case 1 : $Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)$ (Product Distinct Linear Factors)

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$

<u>Example 1</u>: $\int \frac{2x^2 + 2x - 1}{x^3 + x^2 - 2x}\, dx$

<u>Sol^n</u>: $x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x-1)(x+2)$

$$\frac{2x^2 + 2x - 1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$ (Multiply by $x(x-1)(x+2)$)

$\rightarrow$ $2x^2 + 2x - 1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$

$$2x^2+2x-1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

$$2x^2+2x-1 = (A+B+C)x^2 + (A+2B-C)x - 2A$$

Hence $-1 = -2A \Rightarrow A = \frac{1}{2}$

$$\left.\begin{aligned} 2 &= A+2B-C \\ 2 &= A+B+C \end{aligned}\right\} \text{add : } \quad 4 = \underbrace{2A}_{\frac{1}{2}} + 3B \Rightarrow B = 1$$

$$2 = \underbrace{A}_{\frac{1}{2}} + 2\underbrace{B}_{1} - C \Rightarrow C = \frac{1}{2}$$

Hence, $\displaystyle\int \frac{2x^2+2x-1}{x^3+x^2-2x}\,dx = \int \frac{\overset{=A}{1/2}}{x}\,dx + \int \frac{\overset{=B}{1}}{x-1}\,dx + \int \frac{\overset{=C}{1/2}}{x+2}\,dx$

$$= \frac{1}{2}\ln|x| + \ln|x-1| + \frac{1}{2}\ln|x+2| + \cancel{C}\ K$$

**Note**

$x=0$

$-1 = -2A \Rightarrow A = \frac{1}{2}$

$x=1$

$3 = 3B \Rightarrow B = 1$

$x=-2$

$3 = C\cdot(-2)(-3) \Rightarrow C = \frac{1}{2}$

## Case 2: Some linear Factors are repeated.

If $a_1x+b_1$ is repeated $r$ times, so we have $(a_1x+b_1)^r$ in the factorization, then for this factor we need

$$\rightarrow \quad \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \dots + \frac{A_r}{(a_1x+b_1)^r}$$

For example, $\dfrac{x^3-x+5}{x^2(x-2)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} + \dfrac{D}{(x-2)^2} + \dfrac{E}{(x-2)^3}$

$x^2$

**Example 2:** $\displaystyle\int \frac{8x}{(x-1)^2(x+1)}\,dx$

Sol$^n$:

$$\frac{8x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \Rightarrow 8x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$8x = (A+C)x^2 + (B-2C)x + (-A+B+C)$$

$$8x = (A+C)x^2 + (B-2C)x + (-A+B+C)$$

$$\begin{aligned} A + C &= 0 \\ B - 2C &= 8 \\ -A + B + C &= 0 \end{aligned} \quad \text{Solve to get} \quad \left.\begin{aligned} A &= 2 \\ B &= 4 \\ C &= -2 \end{aligned}\right\}$$

So

$$\int \frac{8x}{(x-1)^2(x+1)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{4}{(x-1)^2}\,dx + \int \frac{-2}{x+1}\,dx$$

$$= 2\ln|x-1| - \frac{4}{x-1} - 2\ln|x+1| + \cancel{C}\,K$$

Case 3 : $Q(x)$ contains irreducible quadratic factors (unrepeated)
(complex roots)

For each such factor $ax^2+bx+c$ we need a fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$

For example,

$$\cancel{\frac{B}{x^2+1}}$$

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

We would compute just as before and use $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

**Exercise** Use partial fractions to show that

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + K$$

$x^3+4x = x(x^2+4)$

**Case 4**: Repeated irreducible quadratic: For $(ax^2+bx+c)^r$ we need

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$