5.D GENERALIZED FOURIER SERIES

Orthogonal sets of functions are important for the following reason:

If $S = \{g_1(x), \ldots, g_n(x), \ldots\}$ is orthogonal on $[a,b]$, and $f(x)$ is a function defined on $[a,b]$ s.t. $f(x)$ can be written as an infinite series in the g_n's:

$$f(x) = a_1 g_1(x) + a_2 g_2(x) + \cdots + a_n g_n(x) + \cdots \qquad (1)$$

$$= \sum_{n=1}^{\infty} a_n g_n(x) \quad \text{for all } x \text{ in } [a,b] \qquad (a_n \text{ constants})$$

Then it is <u>easy</u> to compute the a_n's:

$$a_n = \frac{\langle f, g_n \rangle}{\langle g_n, g_n \rangle} = \frac{\int_a^b f(x) g_n(x)\,dx}{\int_a^b (g_n(x))^2\,dx} \qquad (2)$$

$$\|g_n\|^2 = \langle g_n, g_n \rangle$$

Equation (1) is called a <u>Generalized Fourier Series</u> of $f(x)$ in terms of the orthogonal set S.

The constants a_n are computed by the simple formula (2) and they are called the <u>generalized Fourier coefficients</u> of (1).

To indicate how such formula (2) was obtained, we have:
(under strictly mathematical convergence conditions that we bypass here so that we focus on our purpose which is to use the series)

$$\underbrace{\langle f, g_n \rangle} = \langle \sum_{m=1}^{\infty} a_m g_m , g_n \rangle =$$

(by convergence conditions)

$$= \sum_{m=1}^{\infty} a_m \langle g_m , g_n \rangle$$

(by orthogonality $\langle g_m , g_n \rangle = 0$ if $m \neq n$)

$$= a_n \underbrace{\langle g_n , g_n \rangle}$$

Then divide both sides by $\langle g_n , g_n \rangle$ to get (2).

Examples

Example 1: (The Fourier Sine Series)

Recall that $\left\{ \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots \right\}$ is orthogonal on $[0, L]$

with norms $\quad \left\| \sin\left(\frac{n\pi x}{L}\right) \right\| = \sqrt{L/2} \quad , \quad n = 1, 2, \dots$

So for $f(x)$ defined on $[0, L]$ we have

$$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}}$$

where $b_n = \dfrac{\langle g_n, f \rangle}{\langle g_n, g_n \rangle} = \dfrac{\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\underbrace{\left\| \sin\left(\frac{n\pi x}{L}\right) \right\|^2}_{= \frac{L}{2}}}$

This is the Fourier Sine Series of $f(x)$ (FSS)

Example 2: (The Fourier Cosine Series)

Recall that $\left\{1, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \ldots\right\}$ is orthogonal on $[0, L]$

with norms

$$\|1\| = \sqrt{L}, \quad \left\|\cos\left(\frac{n\pi x}{L}\right)\right\| = \sqrt{L/2}, \quad n = 1, 2, \ldots$$

So for $f(x)$ defined on $[0, L]$ we have

$$\boxed{\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \\ a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

This is the Fourier Cosine Series of $f(x)$

(FCS)

To see this :

$\frac{a_0}{2}$ is the coefficient of the first function 1.

$$\frac{a_0}{2} = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_0^L f(x) \cdot 1 \, dx}{\int_0^L 1^2 \, dx} = \frac{1}{L} \int_0^L f(x) \, dx$$

Hence $\quad a_0 = \frac{2}{L} \int_0^L f(x) \, dx$

And $\quad a_n = \frac{\langle f, \cos(\frac{n\pi x}{L}) \rangle}{\langle \cos(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \rangle} = \frac{\int_0^L f(x) \cos(\frac{n\pi x}{L}) \, dx}{\|\cos(\frac{n\pi x}{L})\|^2 = \frac{L}{2}}$

$$= \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Note: we used $\frac{a_0}{2}$ instead of a_0 so that the integral formulas are more symmetric

__Example:__ (The (Classical) Fourier Series) We have seen before that

$$S = \{1, \cos(\tfrac{\pi x}{L}), \sin(\tfrac{\pi x}{L}), \ldots\}$$ is orthogonal on $[-L, L]$

(Classical) Fourier Series

Likewise we get

$$\boxed{\begin{gathered} f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \\ a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{gathered}}$$

Orthogonality with respect to Weight Function

Let $p(x)$ be a function defined on $[a,b]$ with all positive values:

$$p(x) > 0 \quad \text{for all} \quad x \text{ in } [a,b]$$

Such p is called a positive function

The assignment

$$\boxed{(f,g) \longrightarrow \langle f,g \rangle = \int_a^b p(x)\, f(x)\, g(x)\, dx}$$

defines an inner product on $C[a,b]$ (= all continuous $f: [a,b] \to \mathbb{R}$)

The norm defined by this inner product is

$$\boxed{\|f\| = \sqrt{\int_a^b p(x)\,(f(x))^2\, dx}}$$

Let $p(x)$ be a positive function defined on $[a,b]$. The sequence of functions $g_1(x), g_2(x), \ldots, g_n(x), \ldots$ is an orthogonal set on $[a,b]$ with respect to the weight function $p(x)$, if

$$\langle g_m, g_n \rangle = \int_a^b p(x)\, g_m(x)\, g_n(x)\, dx = 0 \quad \text{for } m \neq n$$

If each g_n has norm 1, then we have an orthonormal set with respect to the weight function $p(x)$.

Notes:

1. Orthogonality is the same as orthogonality with respect to weight function $p(x) = 1$ for all $x \in [a,b]$

2. If $g_1(x), g_2(x), \ldots$ is orthogonal w.r.t. weight $p(x)$ and we set $h_n(x) = \sqrt{p(x)}\, g(x)$, then $h_1(x), h_2(x), \ldots$ are orthogonal in the "usual" sense.