Note Title 7/24/2013

6.D THE PRINCIPLE OF SUPERPOSITION

When we compute the Fourier coefficients of a function $f(x)$ with respect to some orthogonal set $\{g_1(x), \dots, g_n(x), \dots\}$ we use the generic formula

$$a_n = \frac{\langle f, g_n \rangle}{\langle g_n, g_n \rangle} = \frac{\int_a^b f(x) g_n(x)\, dx}{\int_a^b (g_n(x))^2\, dx}$$

In the happy occassion when $f(x)$ is a linear combination in a few g_n's, then we do not have to integrate. In fact, we do not have to compute at all! We may use comparison or superposition to compute the a_n's.

Suppose $f(x) = \sum_{n=1}^{\infty} a_n g_n(x)$ is the Fourier series of $f(x)$ and suppose that $f(x)$ has the very special form $f(x) = \sum_{n=1}^{\infty} b_n g_n(x)$. Then we have

$$\langle f, g_m\rangle = \langle \sum_{n=1}^{\infty} a_n g_n, g_m\rangle = \sum_{n=1}^{\infty} a_n \underbrace{\overset{\overset{0 \text{ if } m\neq n}{\shortparallel}}{\langle g_n, g_m\rangle}}_{} = a_m \langle g_m, g_m\rangle$$

Likewise, $\langle f, g_m\rangle = b_m \langle g_m, g_m\rangle$

Therefore, $\boxed{a_m = b_m}$

This is a very simple observation with important implications.

Example: Find the deflection $u(x,t)$ of a string with $L=2$, $c^2=4$, initial velocity zero and initial deflection

$$\sum_{n=1}^{\infty} a_n \underbrace{\cos\left(\frac{cn\pi t}{L}\right)}_{t=0 \,\to\, 1} \sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,0) = 5\sin(2\pi x) - \frac{1}{2}\sin(3\pi x)$$

Sol^n: Because the initial velocity is zero, all $b_n = 0$. It remains to compute the a_n.

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right) = 5\sin(2\pi x) - \frac{1}{2}\sin(3\pi x)$$

$\sin(2\pi x)$ appears on the left with $n=4$ and $\sin(3\pi x)$ appears on the left with $n=6$. So by superposition,

Lucky break: the functions appear on the LHS so we are allowed to use SUPERPOSITION (comparison)

$$a_4 = 5, \quad a_6 = -\frac{1}{2}, \quad a_{rest} = 0$$

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So,

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{\overset{2}{c}\, n\pi}{\underset{2}{L}} t\right) \sin\left(\frac{n\pi x}{\underset{2}{L}}\right) = \sum_{n=1}^{\infty} a_n \cos(n\pi t) \sin\left(\frac{n\pi}{2} x\right)$$

reduces to:

$$u(x,t) = 5\cos(4\pi t)\sin(2\pi x) - \frac{1}{2}\cos(6\pi t)\sin(3\pi x)$$

We see that integration was avoided completely!

<u>Warning</u>: If we had something like

$$u(x,0) = 5\sin(2\pi x) - \frac{1}{2}\sin(3\pi x) + \overset{\text{new}}{\boxed{1}}$$

then we could not use Superposition, because the function 1 does <u>not</u> appear on LHS.

THE HARD WAY

In the last example if we had not noticed that superposition applies then we would have to 'integrate, as follows:

(Please read this to see a pitfall which confuses the students!)

By the formula:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \frac{2}{2}\int_0^2\left(5\sin(2\pi x) - \frac{1}{2}\sin(3\pi x)\right)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = 5\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx - \frac{1}{2}\int_0^2 \sin(3\pi x)\sin\left(\frac{n\pi x}{2}\right)dx$$

Let's look at the first integral.

$$\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx$$

Here we have to exercise caution! We actually have <u>two</u> cases:

Case 1. $n \neq 4$ This is because for integration we
Case 2. $n = 4$ require a different trig. identity!

1. $n \neq 4$: $\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_0^2 \cos\left(\left(2-\frac{n}{2}\right)\pi x\right)dx - \frac{1}{2}\int_0^2 \cos\left(\left(2+\frac{n}{2}\right)\pi x\right)dx$

$$= \frac{1}{2}\,\frac{\sin\left(\left(2-\frac{n}{2}\right)\pi x\right)}{n\pi\left(2-\frac{n}{2}\right)\neq 0!}\Bigg|_0^2 - \frac{1}{2}\,\frac{\sin\left((2+n/2)\pi x\right)}{n\pi(2+n/2)}\Bigg|_0^2 = 0 \ ✓$$

2. $n = 4$: $\int_0^2 \sin^2(2\pi x)\,dx = \frac{1}{2}\int_0^2 (1-\cos(4\pi x))dx = \frac{1}{2}\left(x - \frac{\sin(4\pi x)}{4\pi}\right)\Big|_0^2 = 1$ ✓

Pitfall: Usually people do not realize that there are two cases.
They only treat case 1 and get <u>zero</u>!

By a similar computation we get that the second integral

$$\int_0^2 \sin(3\pi x)\sin\left(\frac{n\pi x}{2}\right)dx = \begin{cases} 0, & \text{if } n \neq 6 \\ 1, & \text{if } n = 6 \end{cases}$$

Hence putting all together we get:

$$a_n = 5\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx - \frac{1}{2}\int_0^2 \sin(3\pi x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$\left.\begin{aligned} a_4 &= 5\cdot 1 - \frac{1}{2}\cdot 0 = 5 \\ a_6 &= 5\cdot 0 - \frac{1}{2}\cdot 1 = -\frac{1}{2} \\ a_{rest} &= 5\cdot 0 - \frac{1}{2}\cdot 0 = 0 \end{aligned}\right\} \text{So } a_4 = 5,\ a_6 = -\frac{1}{2},\ a_{rest} = 0$$

Same as before but with a lot more work.