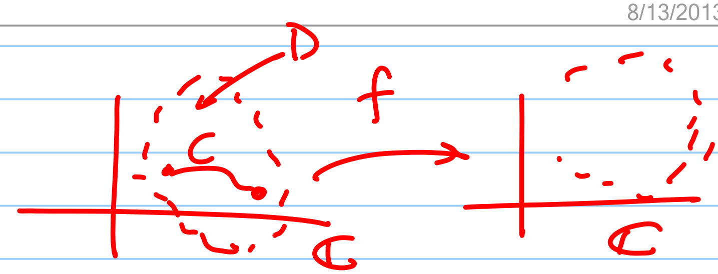
12.A COMPLEX LINE INTEGRALS

Line Integral in the Complex Plane

$$\int_C f(z)\,dz$$

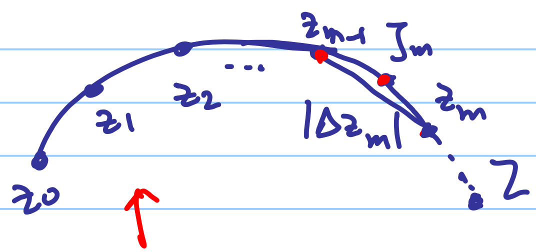
path of integration $\rightarrow C$, $\leftarrow$ integrand



C is a smooth curve $z(t)$ in a complex domain. Smooth means it is continuous and has nonzero derivative $\frac{dz}{dt} \neq 0$ at each point of the domain. Geometrically, C has a unique and continuously turning tangent. $\rightarrow$ $\boxed{C : z(t) = x(t) + i\,y(t)}$ $a \leq t \leq b$ curve parametrization

Definition of the integral: We partition $a \leq t \leq b$: $t_0 = a < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ and look at the corresponding points of the curve $z_0, z_1, \ldots, z_n \,(= Z)$

$z(t_0)$, $z(t_1)$, $z(t_n)$

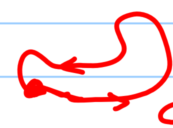


On each point of the subdivision we choose an arbitrary point, say, ζ_m between z_m and z_{m-1} and form the (Riemann) sum

$$S_n = \sum_{m=1}^{n} f(\zeta_m)\,\Delta z_m \qquad \Delta z_m = z_m - z_{m-1}$$

We take the limit of S_n as $n \to \infty$ and as the partitions become finer and finer

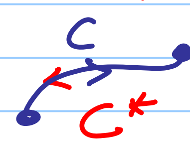
$$\underbrace{\lim S_n = \int_C f(z)\,dz}$$ the complex line integral of $f(z)$ over the path C.

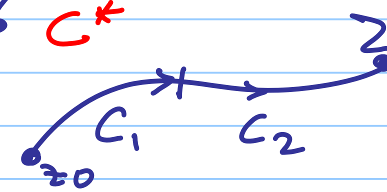
Special case : If C is a closed path 

we usually write $\oint_C f(z)\,dz$

Properties: 1. $\int_C \left(k_1 f_1(z) + k_2 f_2(z)\right) dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$

2. $\int_{z_0}^{Z} f(z)\,dz = -\int_{Z}^{z_0} f(z)\,dz$ This means that if C reverses orientation then the integral changes sign.

or $\int_C f(z)\,dz = -\int_{C^*} f(z)\,dz$ 

3. $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ 

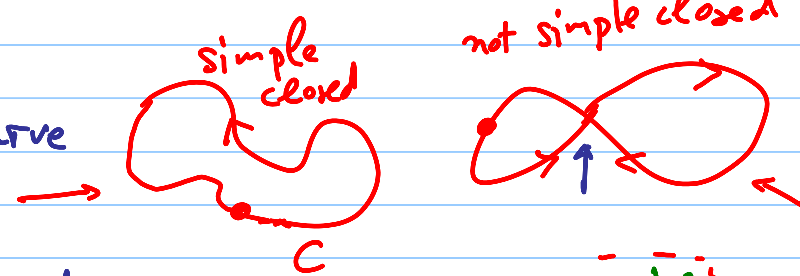
partitioning of path

Note: If $f(z)$ is a continuous function and C is a piecewise smooth curve (i.e., it consists of finitely many smooth curves joined end to end), then the integral always exists.

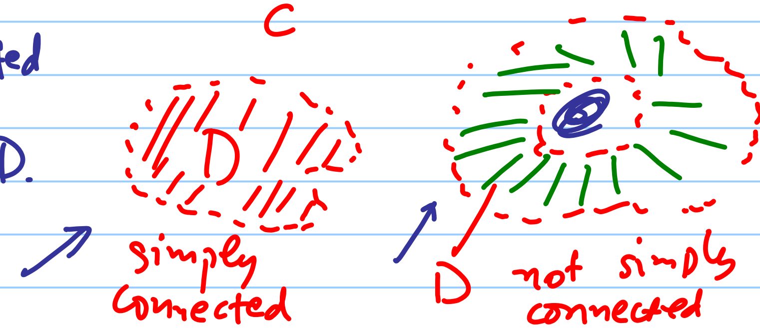
Two Methods of Integration

1. By Indefinite Integration

A simple closed curve is a smooth closed curve that does not cross itself.



A simply connected domain D is an open connected subset of the complex plane for which every simple closed curve entirely in D encloses only points of D.

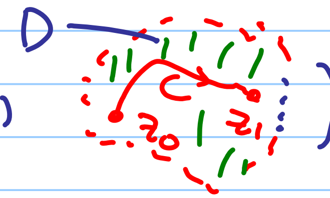


Theorem 1 (Indefinite Integration of analytic Functions)

Let $f(z)$ be analytic in a simply connected domain D. Then exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_C f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

$(F'(z) = f(z))$



Example 1: Find $\int_C z^2\,dz$ for any path C connecting 0 to $1+i$.

Sol^n: $\displaystyle\int_C z^2\,dz = \int_0^{1+i} z^2\,dz = \frac{z^3}{3}\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$

Example 2: Find $\displaystyle\int_{-\pi i}^{\pi i} \cos z\,dz$ $\left(\int_C \cos z\,dz\right)$

Sol^n: $\displaystyle\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z\Big|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i\sinh \pi = 23.097\,i$

Example 3: Find $\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz$.

Sol^n: $\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz = 2e^{z/2}\Big|_{8+\pi i}^{8-\pi i} = 2\left(e^{4-3\pi i/2} - e^{4+\pi i/2}\right) = 0$ (e^{z} is periodic with period $2\pi i$)

Example 4: Find $\int_{-i}^{i} \frac{dz}{z}$.

Sol^n: $\int_{-i}^{i} \frac{dz}{z} = \mathrm{Ln}(i) - \mathrm{Ln}(-i) = \frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right) = i\pi$

2. By Parametrization of Path (more useful)

This method is not restricted to analytic functions, whose antiderivative is known, but it applies to all continuous complex functions.

Theorem 2 (Integration by Using the Path)

Let C be a piecewise smooth path represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$\boxed{\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt}$$!

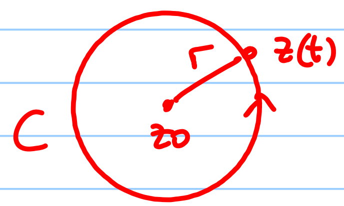
$\left(\dot{z} = \frac{dz}{dt}\right)$

$e^{it} = \cos t + i\sin t$

$(\cos t, \sin t)$

$0 \le t \le 2\pi$

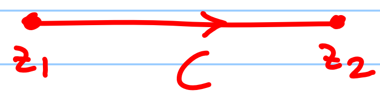
Parametrization of a Circle



$$\boxed{C: \quad z(t) = z_0 + re^{it}, \qquad 0 \le t \le 2\pi}$$

(or may use $\underline{\cos t + i\sin t}$ for $\underline{e}^{it}$)

Parametrization of a Straight Line Segment



$$\boxed{C: \quad z(t) = (1-t)z_1 + t z_2, \qquad 0 \le t \le 1}$$

Example 5: Show that $\int \frac{dz}{z} = 2\pi i$

C = unit circle counterclockwise

C $\circlearrowleft$ 0, 1

$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt$

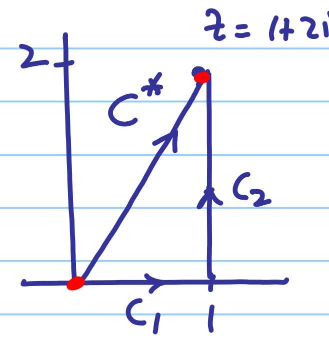
Proof: Parametrize the unit circle:

$\rightarrow$ $z(t) = e^{it}$, $0 \le t \le 2\pi$

$\rightarrow$ $f(z) = \frac{1}{z}$, $f(z(t)) = \frac{1}{e^{it}}$, $\dot{z}(t) = i e^{it}$

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}} \cdot i e^{it}\,dt = i \int_0^{2\pi} 1\,dt = 2\pi i$$

Example 6: Integrate $\mathrm{Re}(z)$ from 0 to $1+2i$
(a) along C^*
(b) along C consisting of C_1 and C_2.



Sol^n (a) C^* $z(t) = (1-t).0 + t(1+2i)$, $0 \le t \le 1$ $\quad (1-t)z_1 + t z_2$, $0 \le t \le 1$

$= t(1+2i)$

$f(z(t)) = \mathrm{Re}(t(1+2i)) = \mathrm{Re}(t + i(2t)) = t$

$\dot{z}(t) = 1+2i$

$$\int_{C^*} \mathrm{Re}(z)\,dz = \int_0^1 t(1+2i)\,dt = (1+2i)\int_0^1 t\,dt = (1+2i)\frac{t^2}{2}\Big|_0^1 = \frac{1}{2} + i$$

(b) C_1: $z(t) = (1-t).0 + t.1 = t$, $0 \le t \le 1$

$\dot{z}(t) = 1$, $f(z(t)) = \mathrm{Re}(t) = t$. Hence, $\int_{C_1} \mathrm{Re}(z)\,dz = \int_0^1 t\,dt = \frac{1}{2}$

C_2: $z(t) = (1-t)1 + t(1+2i) = 1 + 2it$, $0 \le t \le 1$

$\dot{z}(t) = 2i$, $\mathrm{Re}(z(t)) = 1$ $\quad$ Hence, $\int_{C_2} \mathrm{Re}(z)\,dz = \int_0^1 1 \cdot 2i\,dt = 2i$

TOTAL: $\frac{1}{2} + 2i$ $\quad$ INTEGRAL IS PATH DEPENDED.