(3) critical point $(-2, -2)$

<u>Linearization:</u>

$$A = J(-2,-2) = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$$

<u>Eigenvalue & Eigenvector:</u>

→ Eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -5$

→ Eigenvectors at $\lambda_1 = -2$

$$V_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

at $\lambda_2 = -5$

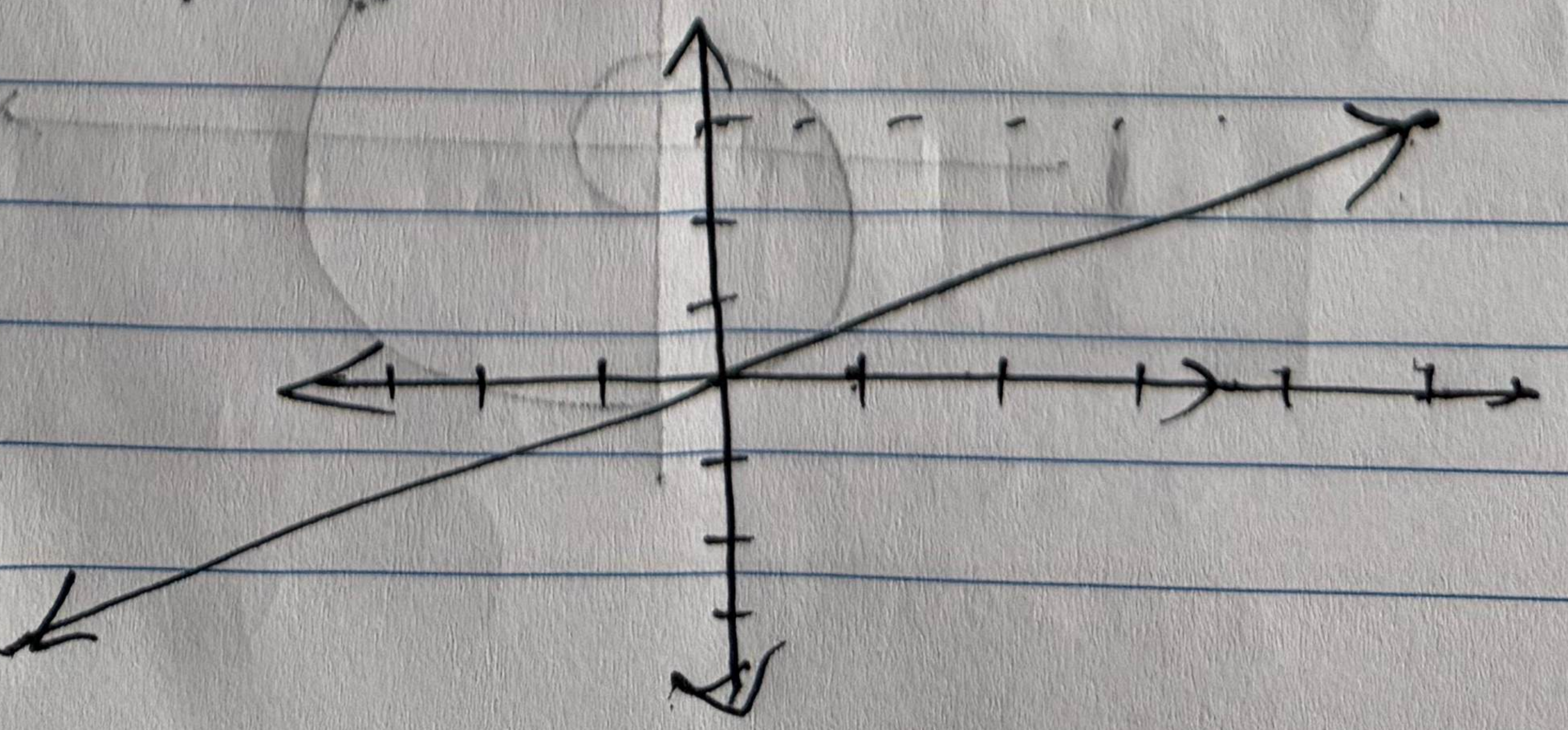
$$V_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

<u>Linearized Phase Portrait sketch:</u>

Since both eigenvalues are real and ~~stable~~ ~~and~~ negative, therefore this is a stable node.

system solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-5t}$$



Because $|\lambda_1| < |\lambda_2|$, trajectories will approach the origin tangent to the slower eigenvector V_1.

(4) Critical point $(3, -2)$

Linearization:

$$A = J(3,-2) = \begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$$

Eigenvalues & Eigenvectors:

→ Eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 3$

→ Eigenvector for $\lambda_1 = 5$

$$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

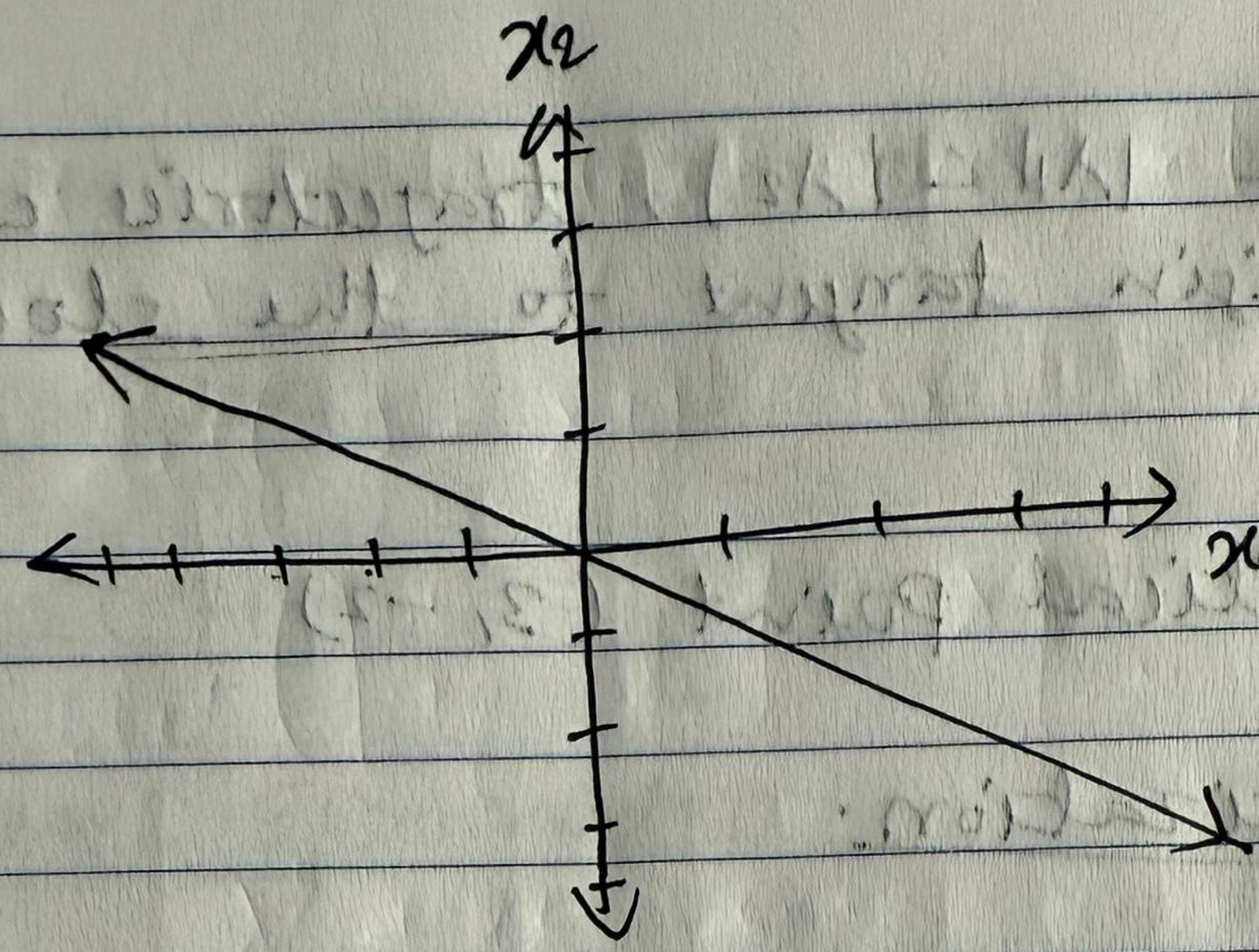
→ Eigenvector for $\lambda_2 = 3$

$$V_2 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

Linearized phase portrait sketch:

Since both eigenvalues are real and positive therefore this is an unstable node. system solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{3\lambda}$$



All trajectories moves away from the origin.
Because $|\lambda_2| < |\lambda_1|$, trajectories will leave the origin tangent to the slower eigenvector v_2.

Phase Portrait of the Nonlinear System:

We have given system as below

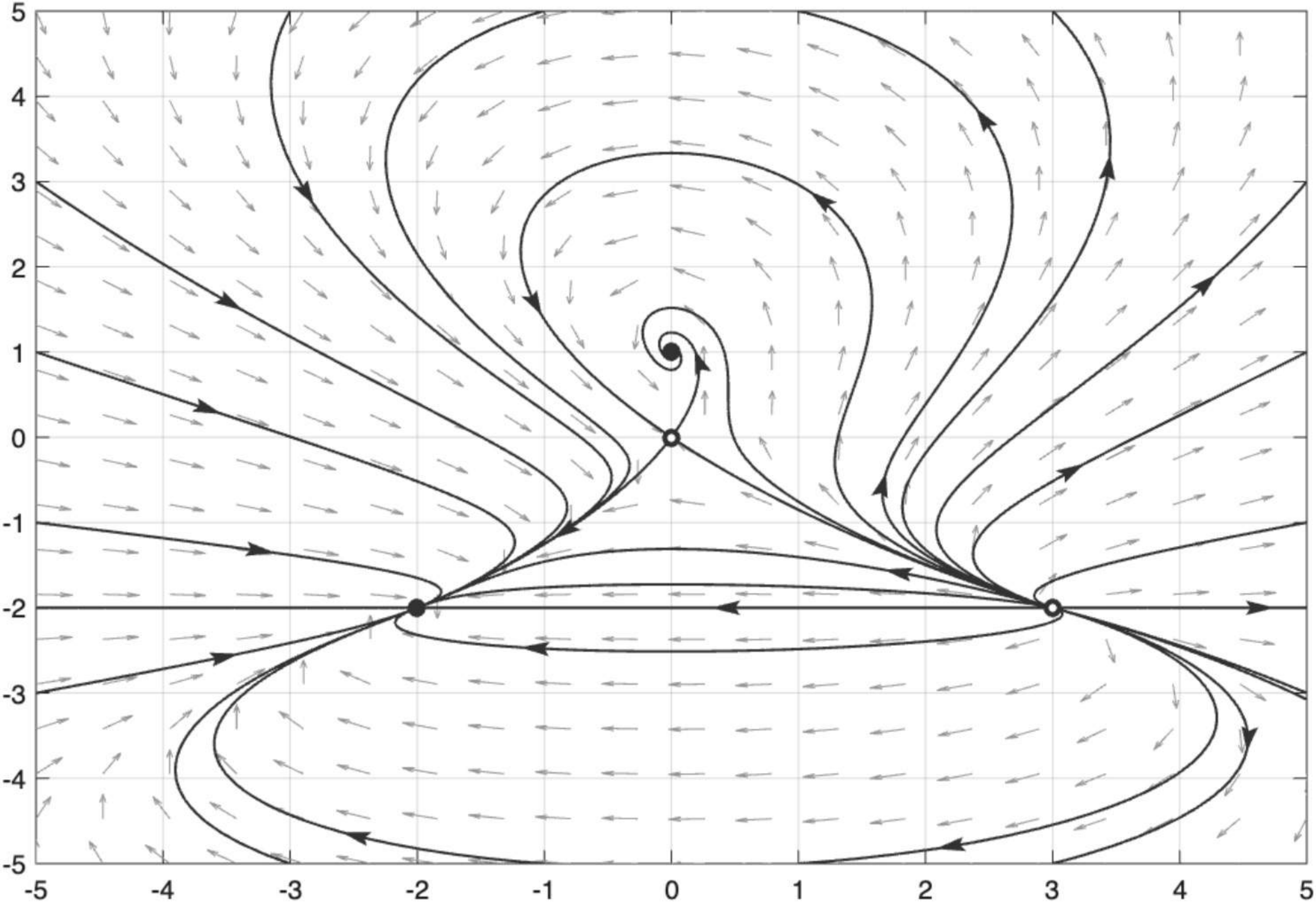
$$\frac{dx}{dt} = x^2 - x - y^2 + y$$

$$\frac{dy}{dt} = 2x + xy$$

We will use existing plotpp() method as below

```
plotpp( @ (t, x) [ x(1)^2 - x(1) - x(2)^2 + x(2);
                   2 x(1) + x(1)* x(2)])
```

This matlab code generates plot as given in next page.



5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

3. Find all critical points of the system and for each critical point state each classification that applies: (a) stable, (b) unstable, (c) attractor, (d) repeller, (e) saddle point.

$$\frac{dx}{dt} = 4 - 2y$$
$$\frac{dy}{dt} = 12 - 3x^2$$

Ans: To find critical points, we set both derivative to zero and solve for x and y

$\frac{dx}{dt} = 4 - 2y = 0 \Rightarrow 2y = 4 \Rightarrow y = 2$

$\frac{dy}{dt} = 12 - 3x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

This gives us two critical points
$(2, 2)$ and $(-2, 2)$

Linearization and classification:

Let's find Jacobian matrix to classify these points

$$J(x, y) = \begin{bmatrix} \frac{\partial}{\partial x}(4-2y) & \frac{\partial}{\partial y}(4-2y) \\ \frac{\partial}{\partial x}(12-3x^2) & \frac{\partial}{\partial y}(12-3x^2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \\ -6x & 0 \end{bmatrix}$$

Now let's evaluate Jacobian at each critical point.

Critical point (2,2):

$$J(2,2) = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$$

Let's find eigenvalues using characteristic equation

$$\det(J - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$(-\lambda)(-\lambda) - (-2)(-12) = 0$$

$$\lambda^2 = 24$$

$$\lambda = \pm 2\sqrt{6}$$

Since eigenvalues are real and have opposite signs, the critical point (2,2) is saddle point. It can classify as below

— (b) unstable

— (e) saddle point

Critical point (-2,2):

$$J(-2,2) = \begin{bmatrix} 0 & -2 \\ 12 & 0 \end{bmatrix}$$

Calculate eigenvalues using characteristic equation

$$\det(J - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 0 & -2 \\ 12 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\lambda^2 + 24 = 0$$

$$\lambda = \pm i 2\sqrt{6}$$

Since eigenvalues are purely imaginary, the linearized system has center. This is considered stable but not attractor.

It can be classified as below

— (a) Stable

4. Find the linearization of the following system at each critical point. For each linearization find the eigenvalues and eigenvectors (if the eigenvectors are complex do not compute the eigenvectors) and sketch each one of the linearized phase portraits. Then use software to draw a phase portrait of the system itself.

$$\frac{dx}{dt} = 4 - 2y$$
$$\frac{dy}{dt} = 12 - 3x^2$$

Ans: system equation given above is same as previous problem. We know that the critical points are (2,2) and (−2,2), and the Jacobian matrix is

$$J(x,y) = \begin{bmatrix} 0 & -2 \\ -6x & 0 \end{bmatrix}$$

Critical point (2,2):

⇒ Linearization:

The linearized system at (2,2) is

$$u' = Au$$

with matrix A given by

$$A = J(2,2) = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$$

Eigenvalues:

$\lambda_1 = 2\sqrt{6}$, $\lambda_2 = -2\sqrt{6}$

Eigenvector: Calculate using $\det(A-\lambda I)=0$

At $\lambda_1 = 2\sqrt{6}$, $V_1 = \begin{bmatrix} 1 \\ -\sqrt{6} \end{bmatrix}$

At $\lambda_2 = -2\sqrt{6}$, $V_2 = \begin{bmatrix} 1 \\ \sqrt{6} \end{bmatrix}$

Linearized Phase Portrait Sketch:

This is a saddle point. Trajectories move away from the point (2,2) parallel to eigenvector V_1 and move toward the point parallel to V_2.

2. Critical point (-2,2)

Linearization:

$$A = J(-2,2) = \begin{bmatrix} 0 & -2 \\ 12 & 0 \end{bmatrix}$$

Eigenvalues:

$$\lambda = \pm i 2\sqrt{6}$$

Linearized Phase Portrait Sketch:

Because eigenvalues are purely imaginary, this point is center.

The trajectories are closed ~~around~~ ellipse around $(-2,2)$. To find the direction of rotation, we can check the velocity at local $(1,0)$

~~Phase~~

$$u' = A\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \begin{bmatrix}0 & -2\\12 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \begin{bmatrix}0\\12\end{bmatrix}$$

which points upward. This indicates a counter clockwise rotation.

Phase Portrait of the Non linear System:

Using given matlab program, we can write code for following given system

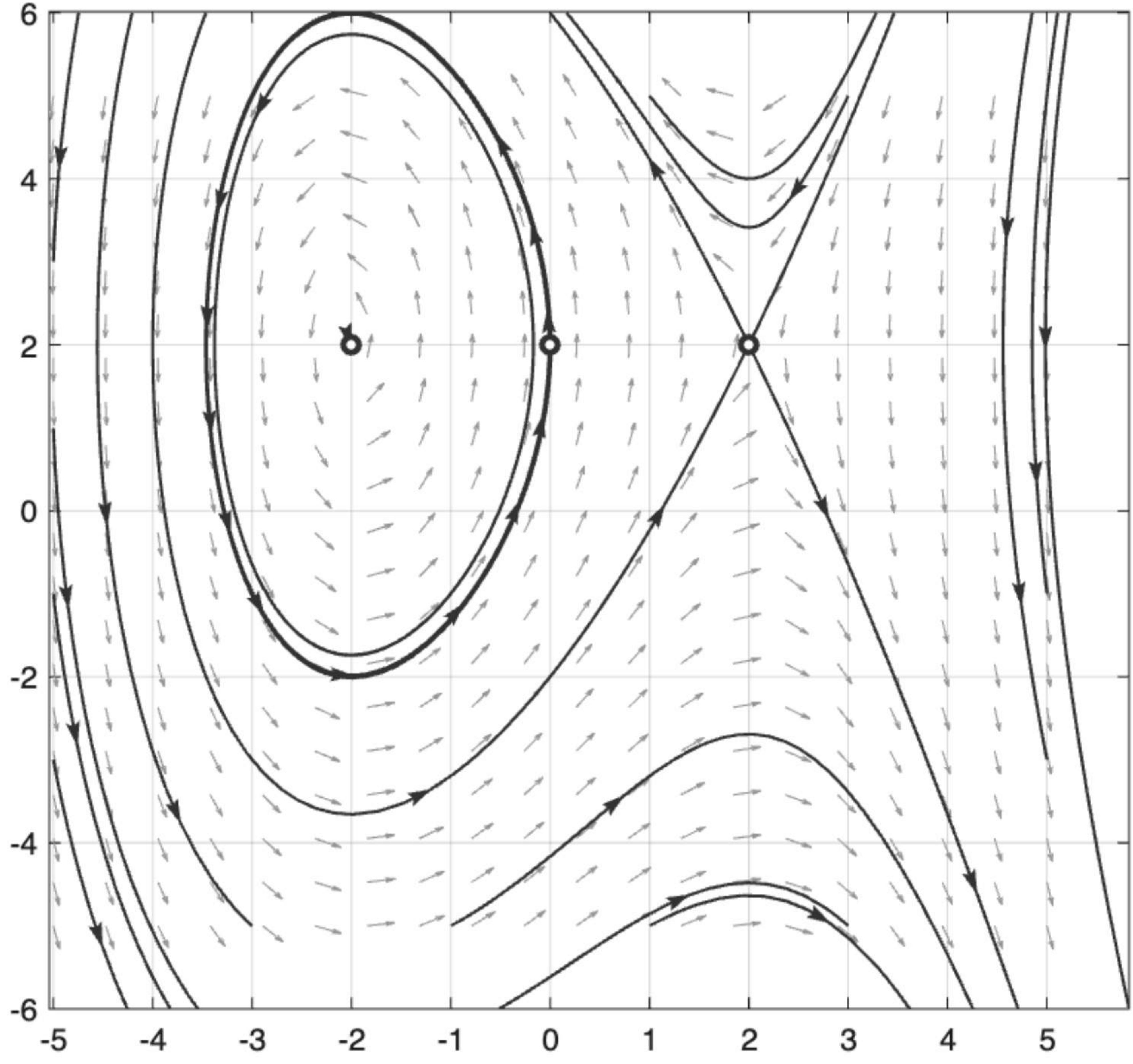
$$\frac{dx}{dt} = 4 - 2y$$

$$\frac{dy}{dt} = 12 - 3x^2$$

as below

```
plotpp(@(t,x) [4-2*x(2); 12-3*x(1)^2]
```

This generates plot as given on next page.



6
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5