JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

Algorithms for Data Science

Optimization: Linear Programming

Introduction to Linear Programming

Linear programming (LP) is the process of optimizing a linear objective function subject to a set of linear constraints.

Standard Form:

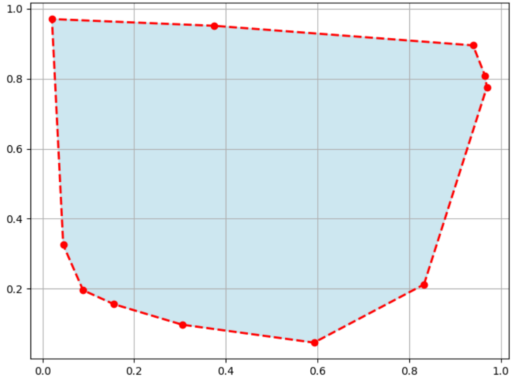
$$\text{Maximize or Minimize } c^T x \text{ s.t. } Ax \leq b, x \geq 0$$

Components:

$c^T x$ Linear objective function to be maximized or minimized.

$Ax \leq b$ Linear constraints representing limits on resources or conditions

$x \geq 0$ Non-negativity constraints on decision variables.



LP problems have a feasible region, and the optimal solution lies at one of its vertices.

Graphical Method (Two Variables)

The graphical method solves linear programming problems with two variables by visualizing the feasible region and objective function.

Steps:

1. Plot the constraints to define the feasible region.

2. Overlay the objective function as contour lines.

3. Identify the optimal solution at a vertex of the feasible region.

Example:

$$\text{Maximize } z = 3x_1 + 5x_2$$

S.t.

$$x_1 + 2x_2 \leq 8$$

$$3x_1 + 2x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

Graphical Method (Solved)

Example:

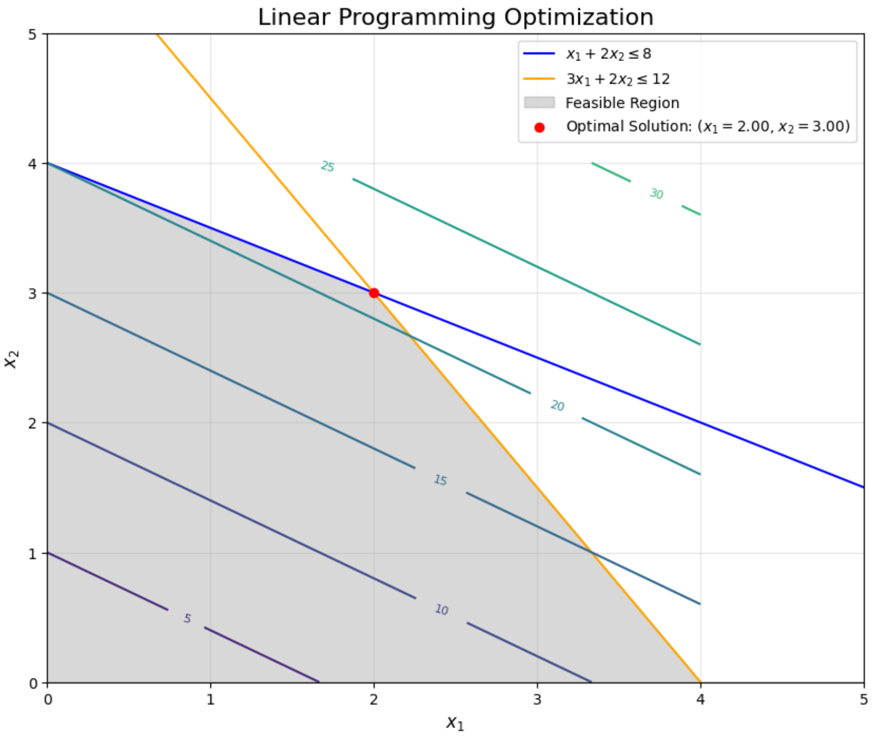
$$\text{Maximize } z = 3x_1 + 5x_2$$

S.t.

$$x_1 + 2x_2 \leq 8$$

$$3x_1 + 2x_2 \leq 12$$

$$x_1, x_2 \geq 0$$



Simplex Algorithm

A systematic method for solving LP problems by iterating over the vertices of the feasible region that guarantees an optimal solution for feasible LP problems.

Algorithm Steps

1. Initialization: Identify a starting vertex. $O(mn)$
2. Pivoting: Move to an adjacent vertex to improve solution.
   1. Evaluate reduced cost for each variable. $O(mn)$
   2. Select entering and leaving variables. $O(m)$
   3. Update the solution by solving a linear system. $O(m^2)$
3. Optimality Check $O(m)$

$$\left.\begin{matrix} O(mn) \\ O(m) \\ O\left(m^2\right) \\ O(m) \end{matrix}\right\} O\left(2^n\right) \text{ per pivot}$$

$$\text{Total Complexity } O\left(2^n \cdot m^2\right)$$

Simplex Algorithm Example

Example:

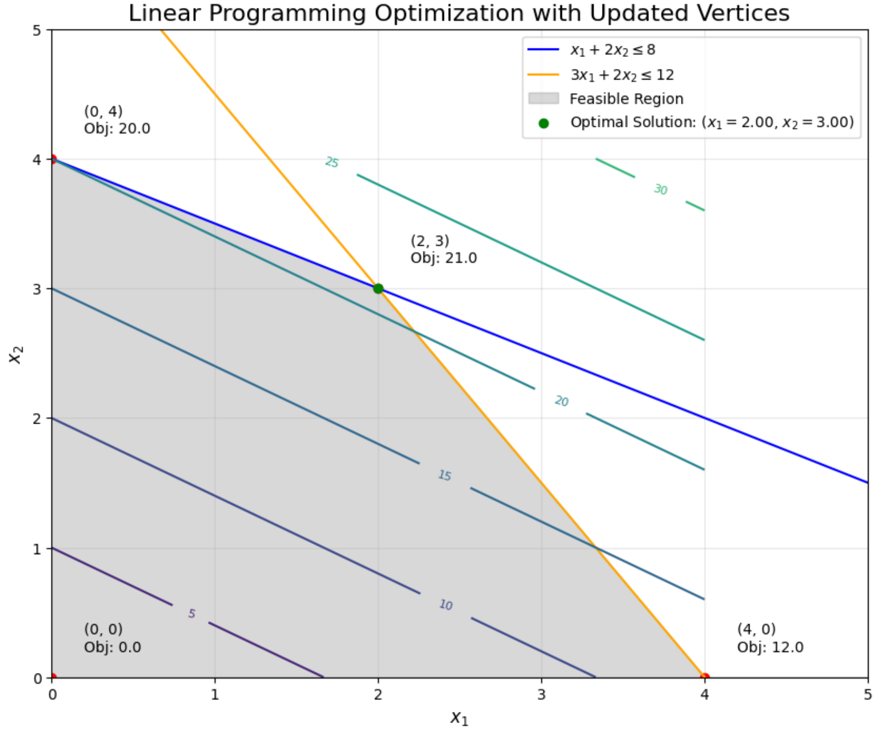
$$\text{Maximize } z = 3x_1 + 5x_2$$

$$\text{S.t.}$$

$$x_1 + 2x_2 \leq 8$$

$$3x_1 + 2x_2 \leq 12$$

$$x_1, x_2 \geq 0$$



LP in Python

Example:

$$\text{Maximize } z = 3x_1 + 5x_2$$

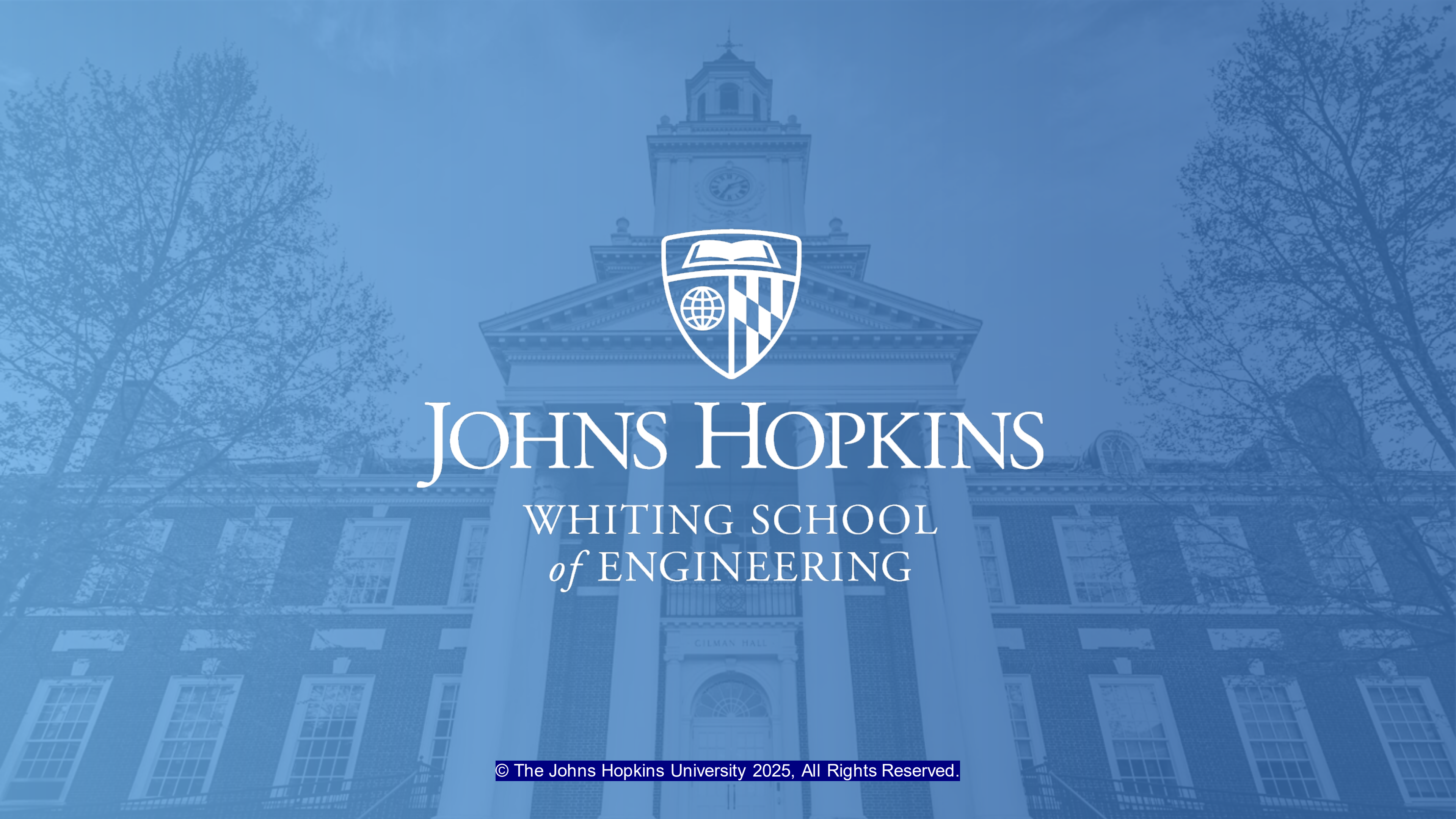
S.t.

$$x_1 + 2x_2 \leq 8$$

$$3x_1 + 2x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

```python
from scipy.optimize import linprog

# Coefficients of the objective function
c = [-3, -5]  # Negative for maximization

# Coefficients of the constraints
A = [[1, 2], [3, 2]]
b = [8, 12]

# Bounds for variables
x_bounds = [(0, None), (0, None)]

# Solve LP problem
result = linprog(c, A_ub=A, b_ub=b,
bounds=x_bounds, method='simplex')

# Print results
print("Optimal Value:", -result.fun)  # Flip the
sign back for maximization
print("Optimal Solution:", result.x)
```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