Algorithms for Data Science

Statistical Algorithms: Expectation Maximization (EM)

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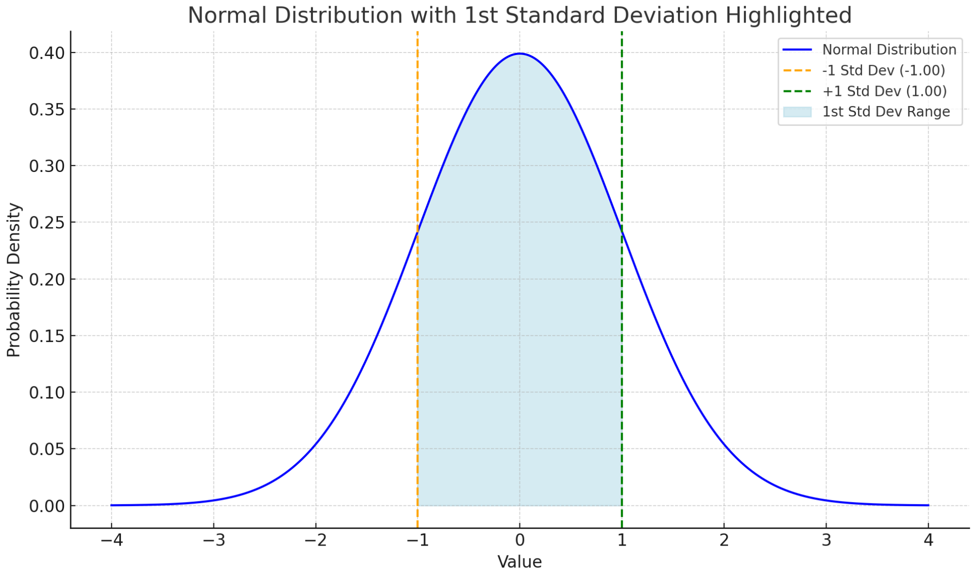
EM: Purpose

Even if data values of x_n are unknown, the distribution $f\left(x_n \mid p\right)$ can be used to determine an estimate for *p*.

Probability of the data point in the subpopulation.



EM: Algorithm

- Given an initial estimate of $p_k^{(0)}, m_k^{(0)}, \sigma_k^{(0)}$, EM iterates until convergence to a local maximum of the likelihood function:
  - **E-Step:**

$$p^i(k \mid n) = \frac{p_k^{(i)} g\left(x_n; m_k^{(i)}, \sigma_k^{(i)}\right)}{\sum_{k=1}^{K} p_k^{(i)} g\left(x_n; m_k^{(i)}, \sigma_k^{(i)}\right)}$$

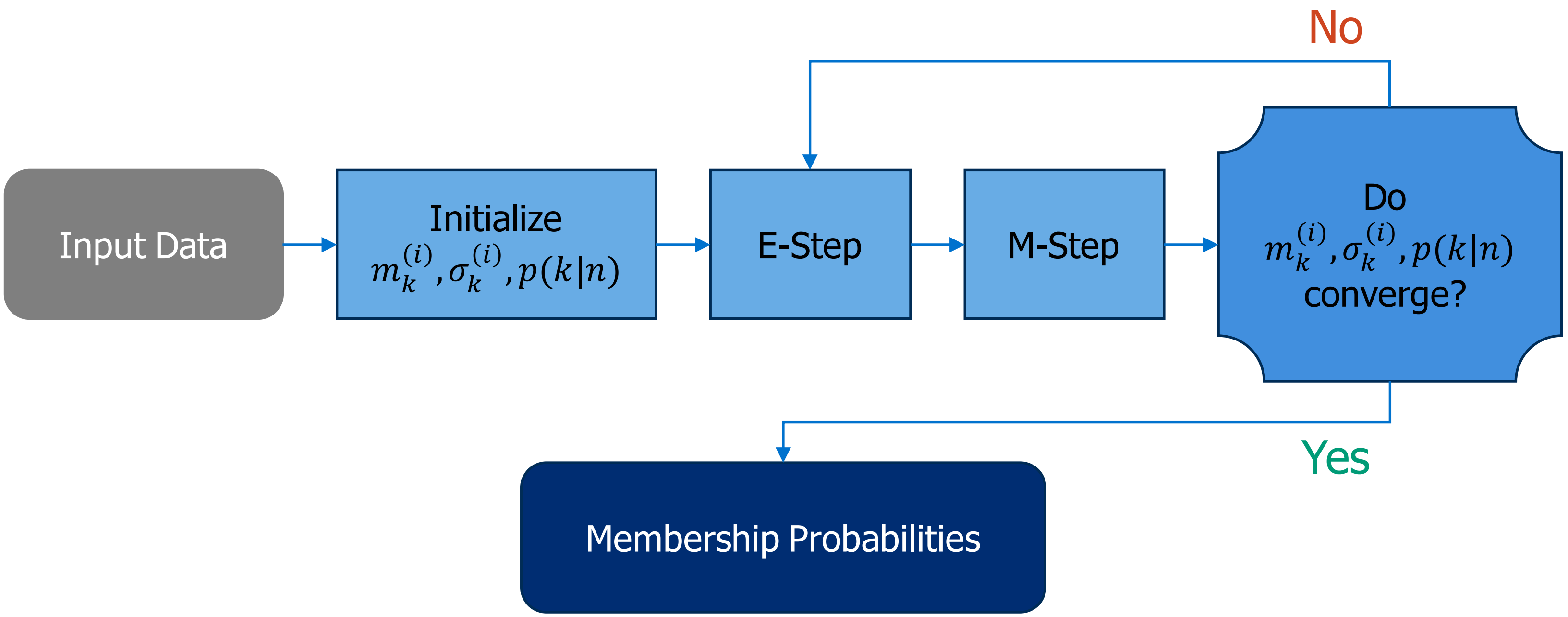
  - **M-Step:**

$$m_k^{(i+1)} = \frac{\sum_{n=1}^{N} p^i(k \mid n) x_n}{\sum_{i=1}^{N} p^i(k \mid n)}$$

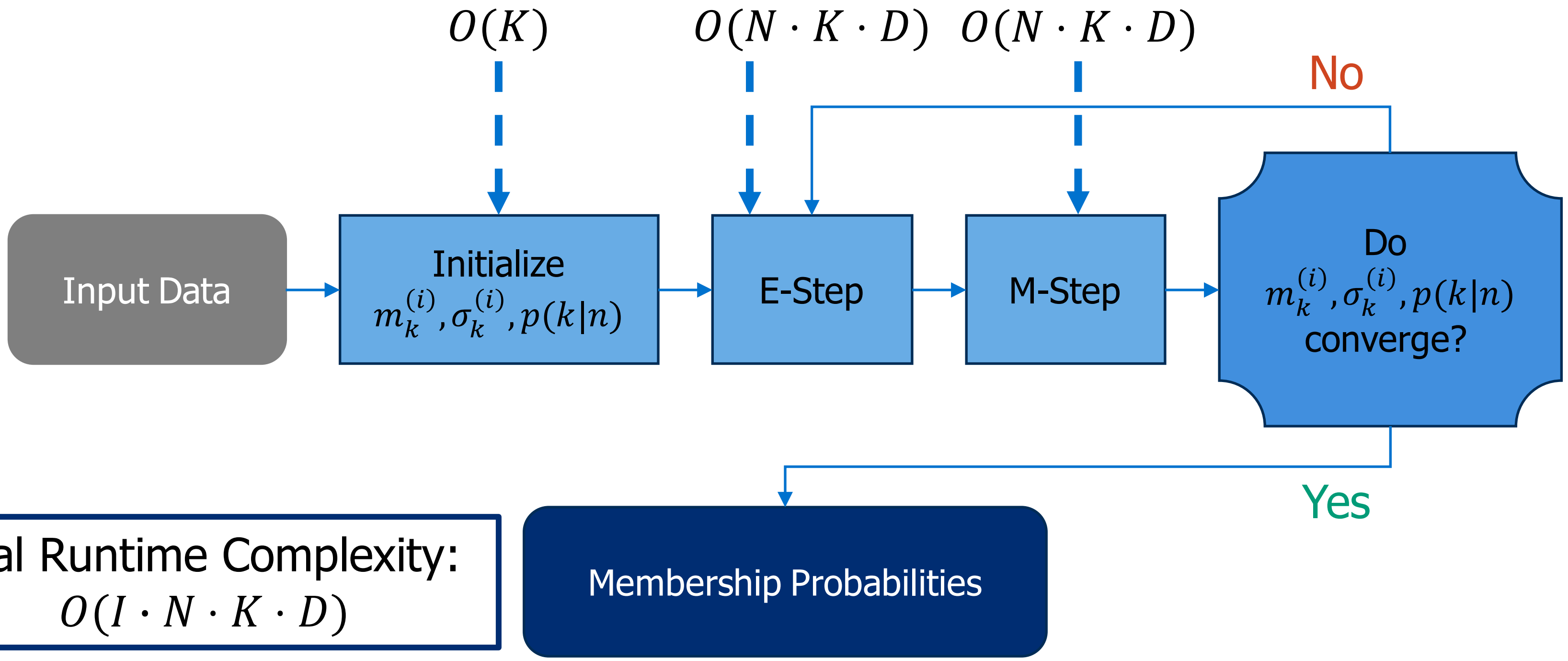
$$p_k^{(i+1)} = \frac{1}{N} \sum_{n=1}^{N} p^i(k \mid n)$$

$$\sigma_k^{(i+1)} = \sqrt{\frac{1}{D} \frac{\sum_{n=1}^{N} p^i(k \mid n) \left\| x_n - m_k^{(i+1)} \right\|^2}{\sum_{n=1}^{N} p^i(k \mid n)}}$$

EM: Algorithm



EM: Runtime Analysis

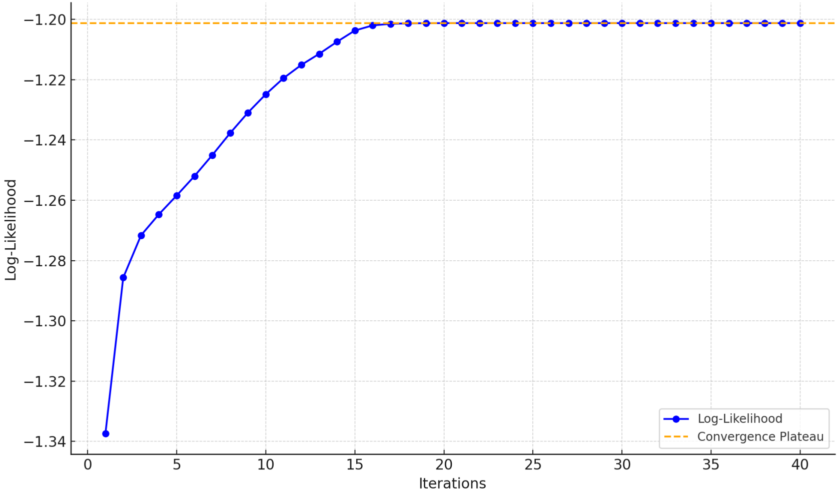


Total Runtime Complexity:
$O(I \cdot N \cdot K \cdot D)$

EM: Correctness

Theorem: The EM algorithm converges to a local maximum of the likelihood function for the observed data.

1. **E-Step:**
   - Computes the expected value of the complete data log-likelihood.
   - Ensures the expected likelihood does not decrease.
2. **M-Step:**
   - Updates parameters by maximizing the expected log-likelihood.
   - Guarantees monotonic increase in likelihood.
3. **Monotonic Convergence:**
   - Likelihood increases or remains constant at each iteration.
   - Stops at a local maximum when convergence criteria are met.



EM: Applications

Gaussian Mixture Models

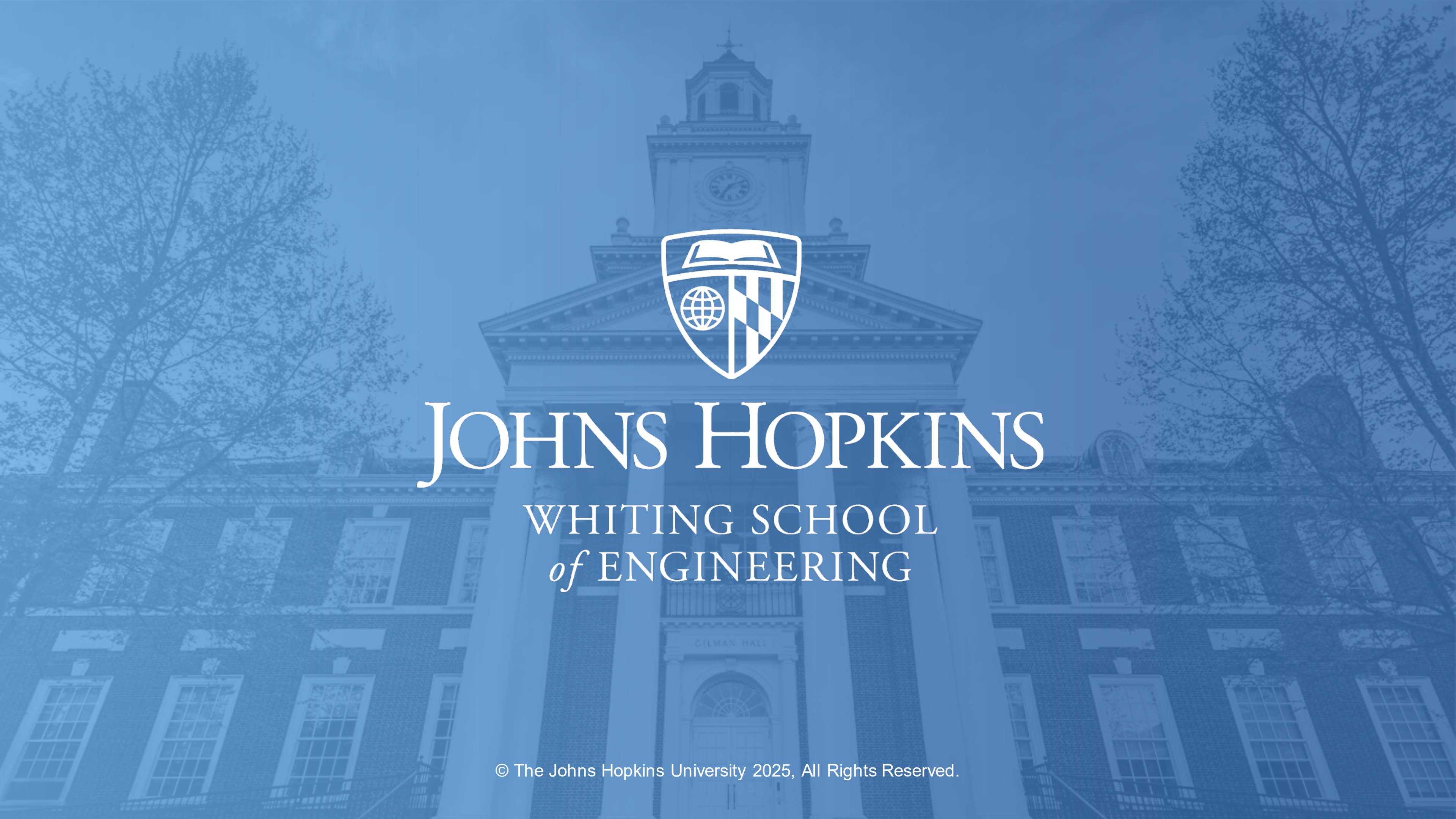
Clustering data into a predefined number of clusters, where each cluster is modeled as a Gaussian distribution.

Hidden Markov Models

Sequence analysis where states are hidden but observations are available.

Missing Data Imputation

Filling in incomplete datasets by estimating missing values iteratively.



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