PROBLEM SET-3

1a. Toss a coin 7 times.

The coin is fair $\therefore P[H] = 0.5,\ P[T] = 0.5.$

$\Rightarrow X =$ Binomial Random variable representing the number of ~~toss~~ heads in 7 trials

$P(X=5) = ?$

From Binomial Probability Formula, we have,

$$P(X=k) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k}$$ where

$n =$ trials

$p = P(\text{success})\ (0.5)$

$k =$ successes (5).

$$\therefore P(X=5) = {}^{7}C_{5}\ (0.5)^{5}\ (1-0.5)^{2}$$

$$= \frac{7!}{2!\,5!}\ (0.5)^{7} \Rightarrow 21 \times 0.0078125$$

$$\therefore P(X=5) = \underline{0.164}$$

1b. $F(2) = P(X=0) + P(X=1) + P(X=2).$

From Binomial Probability formula,

$$P(X=0) = {}^{7}C_{0}\ (0.5)^{0} (0.5)^{7}$$

$$\Rightarrow 1 \times 1 \times 0.0078125$$

$$\Rightarrow \underline{\underline{0.0078125}}$$

$P(x=1) \Rightarrow {}^7C_1 \cdot (0.5)^1 \times (0.5)^6$

$\Rightarrow 7 \times 0.5^7$

$= \underline{\underline{0.05469}}.$

$P(x=2) = {}^7C_2 \, (0.5)^2 (0.5)^5$

$= \frac{7!}{5!\,2!} \times 0.0078125$

$= \underline{\underline{0.1640625}}$

$\therefore F(2) = 0.0078125 + 0.05469 + 0.164$

$\Rightarrow \underline{\underline{0.2265}}.$

1c. Expected Value $E(x)$ of a Binomial Random Variable is given by

$E(x) = n \cdot p$ n = Trials

$n = 7$, & $P(H) = 0.5$. p = 'P' of success $\boxed{\text{heads}}$

$\therefore E(x) = 7 \times 0.5$

$E(x) = \underline{\underline{3.5}}.$

Standard deviation $= \sqrt{\sigma^2}$.

$\sigma^2 = np(1-p)$ [for a B^n random Var]

$= 7 \times 0.5 (0.5)$

$= 1.75$

$\therefore \sigma = \underline{\underline{1.3228}}.$

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1. d. Let 'A' be the event of getting 4 heads in the first 4 tosses, & B be the event of getting 4 tails in the last four tosses.

Events A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

or

$P(A/B) = P(A)$

or

$P(B/A) = P(B)$.

From the question, if I have 4 heads in the first four tosses, its impossible for me to get 4 tails in the last four tosses out of 7 trials

i.e. A = H H H H H/T H/T H/T

B = H/T H/T H/T T T T T

Hence, the probability of event B happening depends on the probability of A, & vice-versa.

~~[illegible]~~

"~~[illegible]~~"

Hence, these two events are non-independent

2. Cat and Mouse:

(a) ~~Sol~~: Let's divide the given Problem into multiple scenarios.

①. Probability of mouse selecting outside exit = 60%

↳ Koko, Cat waits outside, it has 20% probability of catching it

↳ (100 - 20)% of not catching the mouse.

②. Probability of mouse choosing inside exit, 100 - 60 = 40

↳ Probability of cat falling asleep if it waits inside = 30%

↳ Probability of cat not falling asleep waiting inside = (100 - 30) = 70%

↳ If it stays awake, Probability of catching = 40%

↳ If it stays awake, Probability of not ~~ca~~ not catching = (100 - 40) = 60%.

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Hence, the probability of rat catching the mouse if it stays outside & if the mouse chooses outside exit

$= 60\% \times 20\%$

$= 0.6 \times 0.2 = \underline{0.12}$ or 12%

Probability of rat waiting ~~inside~~ inside, if awake & the mouse chooses inside exit

$= 40\% \times 70\% \times 40\%$

$= \underline{0.112}$ or 11.2%.

Hence, the rat has more probability of catching (12%) if it stays outside, than if it waits inside (11.2%)

2(b). Probability of rat catching the mouse waiting outside $= 0.12$ [from 2a].

P(not catching if rat stays outside) $= 1 - 0.12 = \underline{0.88}$.

To find the probability that it catches within 7 nights, we use complementary of probability that it does not catch in 7 nights

$\Rightarrow$ P[Not catching in first night] = 0.88.

P[No catching in first two nights] = 0.88 × 0.88.

Illy P[Not catching in 7 nights] = $(0.88)^7$.

$\Rightarrow$ 0.4086.

Therefore, Probability of the mouse being caught within 7 ~~days~~ nights

= 1 − P[not catching in 7 nights].

= 1 − 0.4086

= 59.132%

3. Given : 70% of Gen Z believe Government should solve Problems.

a. For a Binomial Random Variable, we know that, the probability of X is given by.

$$P(X=k) = {}^{n}C_{k} \cdot (p)^{k} \, (1-p)^{n-k}.$$

$$\therefore \; P(X=0) = {}^{2}C_{0} \cdot (0.7)^{0} \, (0.3)^{2}.$$

$$= 1 \times (0.3)^{2} \Rightarrow \underline{\underline{0.09}}.$$

$$P(X=1) = {}^{2}C_{1} \, (0.7) \times 0.3$$
$$= 2 \times 0.21 \Rightarrow \underline{\underline{0.42}}$$

$$P(X=2) = {}^{2}C_{2} \cdot (0.7)^{2} \times (0.3)^{0}$$
$$= \underline{\underline{0.49}}.$$

3.(b) The PMF for the random variable X is given by

$$f(x) = \begin{cases} 0.09 & x=0 \\ 0.42 & x=1 \\ 0.49 & x=2. \\ 0 & \text{otherwise.} \end{cases}$$

111ly the C.D.F is given by

$$F(y) = \begin{cases} 0 & y < 0 \\ 0.09 & 0 \leq y < 1 \\ 0.51 & 1 \leq y < 2 \\ 1 & y \geq 2 \end{cases}$$

3(c) Expected value $E(x)$ is given by: for a Binomial Random Variable.

$$E(x) = n \cdot p$$

$$\left\{ \begin{array}{l} n = 2 \\ p = 0.7 \end{array} \right\}$$

$$= 2 \times 0.7$$

$$= 1.4$$

4. Given :

5 Symbols " +, □, ★, O, ~ " illumin & the event of illuminating these symbols are independent.

4 (a) Symbols identified correctly by receiver = ?

Trials $n = 25$

probability of getting one right out of 5 symbols

$= \frac{1}{5} = 0.2$.

Hence, out of 25 trials, the number of identification to be correct

$= n \times p$

$= 25 \times 0.2 = \underline{\underline{5}}$.

4. (b). Let 'X' denote a random discrete variable denoting the number of correct identifications

$\therefore$ ~~P(X)~~ $X > 7$ = indication of ESP.

Therefore: $P(X>7) = 1 - P(X \leq 7)$

$= 1 - \text{pbinom}(7, 25, 1/5)$ [R]

$= 1 - 0.89091$

$\therefore P(X>7) = 0.10912$ or 10.9%

P.S: pbinom (x, n, p) gives the cumulative probability of an event with 'x' successes of n trials with 'p' as probability of success.

4(c) Let 'x' be a discrete random variable defining the number of people getting more than 7 matches in the test

$\therefore$ P[Atleast one will get score more than 7] is equal to

= 1 - P[No body getting more than 7]

from 4b, we know that probability of one person getting a score of > 7 is 0.10912.

$\therefore$ probability of not getting a score > 7

= 1 - 0.10912

= 0.89088

111^{ly}, probability of 20 receivers not getting a score $> 7 = (0.89088)^{20}$.

$\therefore \quad 1 - (0.89088)^{20}$

$= \quad \underline{\underline{0.90083}}$

Hence there is 90% probability that atleast one will score above 7 in the test, out of 20 receivers having 25 trials.

5. X = Uniform Random Variable with P.D.F

$$f(x) = \begin{cases} \frac{1}{20} & 20 \leq x < 40 \\ 0 & \text{otherwise} \end{cases}$$

5 (a) To prove $f(x)$ is always non-negative.

from the P.M.F of x, for all values of x, except for the interval $[20, 40)$, the probability is given as 0, and for the same interval $[20, 40)$, the probability is given as $\frac{1}{20}$, hence $f(x)$ is always non-negative for all $f(x)$.

To prove area under $f(x)$ is 1.

⇒ Let's integrate the area under $f(x)$ with the given interval

$$= \int_{20}^{40} \frac{1}{20} \, dx.$$

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Integration of a constant $\Rightarrow \int A\,dx = Ax + C$.

Hence, $\int_{20}^{40} \frac{1}{20}\,dx = \left[\frac{1}{20}x\right]_{20}^{40}$

$$\Rightarrow \frac{40}{20} - \frac{20}{20} \Rightarrow 2-1 = \underline{\underline{1}}.$$

As the result of integration over the interval of $y(x)$ is 1, the total area under $f(x) = 1$ as well.

5b Find CDF of X, F(y) for all y?

from C.D.F definition, we know that,

$$F(y) = \int_{-\infty}^{y} y(x)\,dx.$$

from the P.D.F given, if $y < a$, $y(x) = 0 \Rightarrow F(y) = 0$

when $a \le y \le b$, we have,

$$F(y) = \int_{a}^{y} y(x)\,dx = \left[\frac{1}{20}x\right]_{20}^{y} \Rightarrow \frac{y}{20} - \frac{20}{20}.$$

$$= \frac{y}{20} - 1.$$

for $y \ge 40$, $F(y) = 1$

Hence, CDF for the Uniform Distribution F^{n}

$$F_y = \begin{cases} 0 & y < 20 \\ \frac{y}{20} - 1 & 20 \le y < 40 \\ 1 & y \ge 40 \end{cases}$$

5(c) Find $P(30 \leq X < 50)$

$\Rightarrow P(30 \leq X < 50) = F(50) - F(30)$

$\Rightarrow 1 - \left[\frac{30-20}{20}\right]$ {from C.D.F}

$= 1 - \frac{1}{2}$

$= 0.5$

5(d) Expected Value of X.

we know that, for a Uniform Random Variable

$$EX = \int_a^b \frac{x}{b-a}\,dx \quad \text{OR} \quad \frac{a+b}{2}$$

given $a=20, b=40 \Rightarrow EX = \frac{40+20}{2} = 30$

5e. Variance is given by:

$$\sigma^2 = \frac{(b-a)^2}{12}$$

Given $a=20, b=40 \Rightarrow \frac{(40-20)^2}{12} \Rightarrow \frac{400}{12} \Rightarrow 33.33$

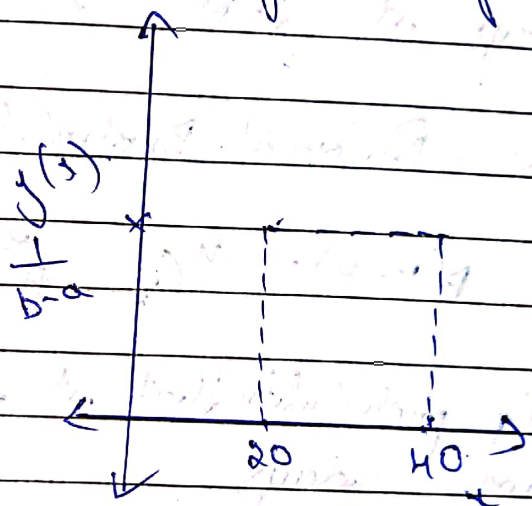
Standard deviation $\Rightarrow \sqrt{\sigma^2} \Rightarrow \sqrt{33.33} = 5.77$

6. Given : time

* Travelling time to office has a uniform distribution of $U[20, 40]$

* Assume the time taken one day is independent of other.

(a) $P(X < 25) = ?$



we know that, CDF of a uniform y^n is given by.

$$F(y) = \begin{cases} 0 & y < a \\ \frac{y-a}{b-a} & a \le y < b \\ 1 & y \ge b \end{cases} \Rightarrow \begin{cases} 0 & y < 20 \\ \frac{y-20}{20} & 20 \le y < 40 \\ 1 & \ge 40 \end{cases}$$

$\therefore P(X < 25) = F(25) \Rightarrow \frac{25-20}{20}$

$\Rightarrow \frac{5}{20} = \underline{\underline{0.25}}$

6(b). from previous question, we know that the probability of taking less than 25 minutes to work :

$$P(X<25) = 0.25.$$

So, the probability of taking more than or equal to 25 minutes :

$$P(X \geq 25) = 0.75.$$

Let 'x' be a random variable denoting the number of days it takes less than 25 minutes to travel.

From Binomial Probability formula we have.

$$P(X=k) = {}^nC_k \cdot p^k \cdot (1-p)^{n-k}$$

∴ the probability that it takes less than 25m on atleast two days

$$\Rightarrow P(X=2) + P(X=3) + P(X=4) + P(X=5).$$

$$\therefore P(X=2) = 5C_2 (0.25)^2 (0.75)^3 \Rightarrow 0.2636719$$

$$P(X=3) = 5C_3 (0.25)^3 (0.75)^2 \Rightarrow 0.08789.$$

$$P(X=4) = 5C_4 (0.25)^4 (0.75) \Rightarrow 0.014649$$

$$P(X=5) = 5C_5 (0.25)^5 (0.75)^0 = 0.00097$$

Adding All Probabilities, we get = 0.036718

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6(c) Let X_1, X_2, X_3, X_4, X_5 be independent discrete random variables ~~describing~~ ~~dice~~.

from Trosset chapter 4.3, Theorem 4.4, we have.

$$E(X_1+X_2] = EX_1 + EX_2.$$

$$\therefore E(X_1+X_2+X_3+X_4+X_5] = EX_1+EX_2+EX_3+EX_4+EX_5.$$

E(X) for a Uniform Random Variable is given by:

$$E(X) = \frac{a+b}{2}.$$

$$\therefore EX_1 = \frac{20+40}{2} = 30.$$

$|||^{ly}$ $EX_2 = \frac{20+40}{2} = 30 = EX_3, EX_4, EX_5.$

Summing up all the E(x), we get

$$E(Y) = 30\times5 \Rightarrow \underline{\underline{150}}.$$

$|||^{ly}$ from Theorem 4.8, we have.

$$Var(Y) = Var X_1 + Var X_2 \ldots Var X_5.$$

$$Var(X_1) = \frac{(b-a)^2}{12} \Rightarrow \frac{(40-20)^2}{12} = 33.33.$$

As individual distributions are independent

Therefor $Var(Y) = Var X_1 + Var X_2 \ldots Var Y_5$

$$= 33.33 + 33.33 + 33.33 + 33.33 + 33.33$$

$$Var(Y). \Rightarrow \underline{\underline{166.66}}.$$

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Standard deviation $\sigma = \sqrt{\text{Variance}}$.

$= \sqrt{166.6}$.

$\therefore \sigma(Y) = \underline{\underline{12.907}}$.