1 Given:

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x \in (0,1) \\ 3-x/4 & x \in (1,3) \\ 0 & x > 3 \end{cases}$$

1a. Median ?

Let's Calculate C.D.F first.

$F(y<0) = 0$

$F(y)$ for $y \in (0,1) = \int_0^y x\,dx \Rightarrow \frac{x^2}{2}\Big|_0^y$

$\therefore y^2/2$.

$F(y)$ for $y \in (1,3) = y^2/2\Big|_0^1 + \int_1^y \frac{3-x}{4}\,dx$

$\Rightarrow \frac{1}{2} + \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_1^y$

$\Rightarrow \frac{1}{2} + \frac{1}{4}\left[3y - \frac{y^2}{2} - \left[3 - \frac{1}{2}\right]\right]$

$\Rightarrow \frac{1}{2} + \frac{1}{4}\left[3y - \frac{y^2}{2} - \left[\frac{5}{2}\right]\right]$

$\Rightarrow \frac{1}{2} + \frac{3y}{4} - \frac{y^2}{8} - \frac{5}{8} \Rightarrow \frac{3y}{4} - \frac{y^2}{8} - \frac{1}{8}$

$F(y)$ for $y > 3 = 1$

$\therefore$ C.D.F, $F(y)$

$$= \begin{cases} 0 & y < 0 \\ y^2/2 & 0 \le y < 1 \\ \frac{3y}{4} - y^2/8 - \frac{1}{8} & 1 \le y < 3 \\ 1 & y \geq 3. \end{cases}$$

$\therefore$ Median, $F(0.5) = 2$

① $0 \le y < 1$

$0.5 = y^2/2 \Rightarrow y^2 = 1$

$y = \pm 1.$

$1 \le y < 3$

② $0.5 = \frac{3y}{4} - \frac{y^2}{8} - \frac{1}{8}$

$8 \times 0.5 = 6y - y^2 - 1$

$4 = 6y - y^2 - 1 \Rightarrow y^2 - 6y + 5 = 0.$

$y^2 - 5y - y + 5 = 0$

$y(y-5) - 1(y-5) = 0$

$\therefore y = 1$ or 5

Because, '1', falls in the range of $1 \leq y < 3$, the median is $\underline{\underline{1}}$.

11 b. $EX = 2$

$$EX = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$\Rightarrow \int_0^1 x \cdot x \cdot dx + \int_1^3 x \cdot \frac{(3-x)}{4} dx$$

$$\Rightarrow \int_0^1 x^2 dx + \int_1^3 \frac{3x - x^2}{4} dx.$$

$$= \left.\frac{x^3}{3}\right|_0^1 + \left.\frac{3x^2}{2\times4} - \frac{x^3}{3\times4}\right|_1^3$$

$$= \frac{1}{3} + \left[\frac{3\times9}{8} - \frac{27}{12} - \left(\frac{3}{8} - \frac{1}{12}\right)\right]$$

$$= \frac{1}{3} + \left[\frac{27}{8} - \frac{27}{12} - \frac{3}{8} + \frac{1}{12}\right]$$

$$\Rightarrow \underline{\underline{1.1667}}.$$

Hence, $E(x)$ $\underline{\underline{1.1667}} >$ Median, which is 1

1d. IQR (x).

IQR = $q_3 - q_1$

q_3 = F(0.75).

① $0 \leq y < 1$

$\Rightarrow 0.75 = \frac{y^2}{2}$

$y^2 = 1.5$

$y = \pm 1.22$.

② $1 \leq y < 3$.

$\frac{3y}{4} - \frac{y^2}{8} - \frac{1}{8} = 0.75$

$6y - y^2 - 1 = 6$

$y^2 - 6y + 7 = 0$. [from quadratic].

$\therefore y = 3 \pm \sqrt{2} \Rightarrow 4.41$ or 1.59.

Because 1.59 falls within the range of $1 \leq y < 3$,
$q^3 = 1.59$.

IIIly, for q^1

$q1 = F(0.25)$

① $0 \leq y < 1$

$$0.25 = \frac{y^2}{2}$$

$$y = \pm 0.7071.$$

(2) $1 \le y < 3.$

$$\frac{3y}{4} - \frac{y^2}{8} - \frac{1}{8} = 0.25$$

$$6y - y^2 - 1 = 2.$$

$$y^2 - 6y + 3 = 0.$$

$$y = 5.449 \text{ or } 0.55$$

Hence, $q^1 = 0.7071$

$$IQR = q^3 - q^1$$

$$= 1.59 - 0.7071$$

$$= \underline{\underline{0.8829}}.$$

4. Given:

a. $$f(x) = \begin{cases} 0.3 & 0 \le x < 1 \\ 0.7 & 1 \le x < 2 \\ 0 & \text{Otherwise.} \end{cases}$$

4a. Value 'a', that minimizes $E|x-a|$.

From lecture, the value that minimizes $E|x-a|$ is Median.

Let's calculate C.D.F first.

* for $y < 0$, $F(y) = 0$
* $0 \le x < 1$, $F(y) = \int_0^y 0.3\,dx \Rightarrow 0.3y$.
* $1 \le x < 2$, $F(y) = 0.3 + \int_1^y 0.7\,dx$

$$\Rightarrow 0.3 + 0.7y - 0.7 \Rightarrow 0.7y - 0.4.$$

* $F(y \ge 2) = 0.3 + [0.7y - 0.4]_1^2$

$$= 0.3 + 1.4 - 0.4 - (0.7 - 0.4).$$

$$= 1$$

$\therefore$ $$F(y) = \begin{cases} 0 & y < 0 \\ 0.3y & 0 \le y < 1 \\ 0.7y - 0.4 & 1 \le y < 2 \\ 1 & y \ge 2. \end{cases}$$

Median $F(0.5) = 2$

Substituting in C.D.F, we get

[1] $0 \le y < 1$

$0.5 = 0.3y$

$\therefore y = \underline{1.67}$, As 1.67 is not in the range, we move to next one.

[2] $1 \le y < 2$.

$\therefore 0.7y - 0.4 = 0.5$

$0.7y = 0.9$

$y = 9/7 = \underline{\underline{1.285}}$

Because, 1.285 is the range $1 \le y < 2$, the value 'a' that minimizes $E|X-a| = \underline{\underline{1.285}}$.

4b. Constant value 'b', that minimizes $E[(X-b)^2]$ is the expected value EX.

we know that

$$EX = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$\Rightarrow \int_0^1 0.3x\,dx + \int_1^2 0.7x\,dx$$

$$\Rightarrow \frac{0.3x^2}{2}\Big|_0^1 + \frac{0.7x^2}{2}\Big|_1^2$$

$$\frac{0.3}{2} + \left[\frac{0.7 \times 4}{2} - \frac{0.7}{2}\right]$$

$$= 0.15 + \left[1.4 - 0.35\right] \Rightarrow \underline{1.2}$$

So the constant value 'b', that minimizes $E((x-b)^2) = \underline{1.2}$.

1c. $P(0.5 < x < 1.5)$.

$\Rightarrow F(1.5) - F(0.5)$

$F(1.5) = \int_{-\infty}^{\infty} f(x)\,dx$

$\Rightarrow \int_{-\infty}^{0} 0\,f(x) + \int_{0}^{1} x\,dx + \int_{1}^{1.5} \frac{3-x}{4}\,dx$.

$= 0 + \frac{x^2}{2}\Big|_0^1 + \frac{3x}{4} - \frac{x^2}{8}\Big|_1^{1.5}$

$= \frac{1}{2} + \frac{3\times1.5}{4} - \frac{1.5^2}{8} - \left[\frac{3}{4} - \frac{1}{8}\right]$

$= \frac{1}{2} + \frac{4.5}{4} - \frac{2.25}{8} - \left[\frac{5}{8}\right]$

$= \underline{\underline{0.71875}}$

$F(0.5)$

$$= \int_{-\infty}^{0.5} f(x)\,dx.$$

$$= 0 + \int_{0}^{0.5} x\,dx$$

$$= \frac{x^2}{2}\Big|_0^{0.5} = \frac{0.5^2}{2} = 0.125$$

$$\therefore P(0.5 < X < 1.5) = 0.71875 - 0.125$$

$$= 0.59375$$