Problem Set - 4

1. Given:

$$f_x = \begin{cases} \frac{1}{30} & 0 \le x < 20 \\ 1/60 & 20 \le x < 40 \\ 0 & \text{otherwise} \end{cases}$$

1a. C.D.F of $X=2$.

we know that $F(y) = \int_0^y f(x)\,dx$.

* $F(0)$ for $y<0 = 0$ — [1].

* For $(0 \le x < 20)$, $F(y) = \int_0^y \left(\frac{1}{30}\right) dx$.

$= \frac{1}{30}[x]_0^y = \frac{y}{30} - 0 \Rightarrow \frac{y}{30}$ — [2].

* For $(20 \le x < 40)$, $F(y) = \int_0^{20} \frac{1}{30} dx + \int_{20}^{y} \frac{1}{60} dx$

$\Rightarrow \left[\frac{x}{30}\right]_0^{20} + \left[\frac{x}{60}\right]_{20}^{y}$

$\Rightarrow \frac{20}{30} + \left[\frac{y}{60} - \frac{20}{60}\right] \Rightarrow \frac{2}{3} + \frac{y}{60} - \frac{1}{3} \Rightarrow \frac{y}{60} + \frac{1}{3}$ — [3]

* For $(x \ge 40)$ $F(y) = \int_0^{20} \frac{1}{30} dx + \int_{20}^{40} \frac{1}{60} dx + \int_{40}^{y} 0\, dx$.

$\Rightarrow \left[\frac{x}{30}\right]_0^{20} + \left[\frac{x}{60}\right]_{20}^{40} + 0 \Rightarrow \frac{20}{30} + \left[\frac{40}{60} - \frac{20}{60}\right]$

$= \frac{2}{3} + \frac{2}{3} - \frac{1}{3} \Rightarrow 1$ //

Therefore

C.D.F of 'X', F(y) for all y is

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{30} & 0 \le y < 20 \\ y/60 + \frac{1}{3} & 20 \le y < 40 \\ 1 & y \ge 40 \end{cases}$$

1(b). Given :-

Buses move past at 9:20, 9:40.

i.e. only three an hour.

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$$

$$\Rightarrow \int_0^{20} \frac{x}{30}\, dx + \int_{20}^{40} \frac{x}{60}\, dx$$

$\left\{ \int x\,dx = \frac{x^2}{2} \right\}$

$$= \left[\frac{x^2}{60}\right]_0^{20} + \left[\frac{x^2}{120}\right]_{20}^{40}$$

$$= \frac{20^2}{60} - 0 + \left[\frac{40^2}{120} - \frac{20^2}{120}\right]$$

$$= 6.66 + 13.33 - 3.33$$

$$= \underline{16.66} \text{ minutes.}$$

10. To find the value of 'y', such that $F(y)=0.5$, lets substitute the value of $F(y)$ in the respective piece function

ie. $F(y) = 0.5 = \frac{y}{30} \Rightarrow y = 30 \times \frac{1}{2} = \underline{\underline{15}}$.

$\therefore F(15) = 0.5$, which is smaller than the expected time of 16.6 minutes.

Q2. Given :
$$\begin{cases} 0 & x < 1 \\ 2(x-1) & 1 \le x < 2 \\ 0 & x > 2 \end{cases}$$

b. Verify that 'y' is a pdf.

① For any given interval of 'x', the probability 'p' is ≥ 0, ~~and~~ ~~as~~

ie. for $x < 1$, $y(x) = 0$ — (1)

for $1 \le x < 2$, ⇒ non-negative — (2).

for $x \geq 2 \Rightarrow y(x) = 0$ non-negative (3)

② Let's integrate $y(x)$ over the entire range.

$$= \int_1^2 2(x-1)\,dx \Rightarrow \int_1^2 2x - 2\,dx.$$

$$= \left[\frac{2x^2}{2}\right]_1^2 - \left[2x\right]_1^2 \Rightarrow \left[x^2\right]_1^2 - 2\left[x\right]_1^2$$

$$= \left[4-1\right] - \left[2(2-1)\right] \Rightarrow 1 //$$

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Therefore from [1] and [2], we can conclude that $\int y'$ is a P.D.F.

2c. $P(1.5 < x < 1.75)$

$$\Rightarrow \int_{1.5}^{1.75} 2(x-1)\,dx. \qquad \left[F(1.75) - F(1.5)\right.$$

$$\Rightarrow \int_{1.5}^{1.75} (2x-2)\,dx \Rightarrow \left[\frac{2x^2}{2} - 2x\right]_{1.5}^{1.75}$$

$$\Rightarrow \left[x^2\right]_{1.5}^{1.75} - 2\left[x\right]_{1.5}^{1.75}$$

$$= (1.75)^2 - (1.5)^2 - \left[2\left(1.75 - 1.5\right)\right]$$

$$= 3.0625 - 2.25 - \left[2(0.25)\right]$$

$$= \underline{\underline{0.3125}}.$$

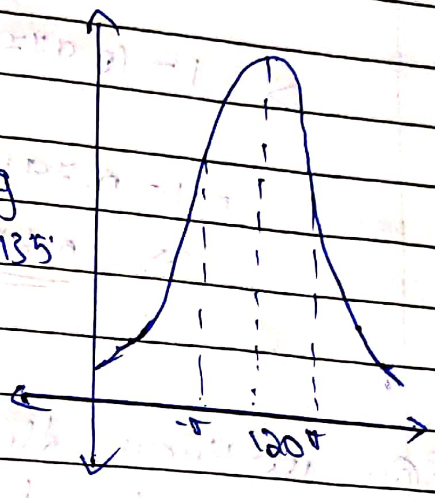
3.

(a) What is the probability that BPM is between (115,135).

$\Rightarrow$ given, mean $= 120$

$\sigma = 20$.

from 'R', we can calculate the probability of B.P.M being less than or greater than 115,135.



ie ① $\text{pnorm}(115, 120, 20) = 0.40129$.

ie. $P(BPM < 115) = 0.40129$.

② $\text{pnorm}(135, 120, 20) = 0.7734$.

ie. $P(BPN < 135) = 0.7734$.

$\therefore P(115 < x < 135) = 0.7734 - 0.40129 \Rightarrow \underline{\underline{0.3721}}$

(b) Probability of song with BPM over 160

$P(X > 160) = 1 - \text{pnorm}(160, 120, 20)$

$= 1 - 0.97725$ (T)

$= \underline{\underline{0.0227}}$.

Continuation of 3b.

⇒ Now the probability of BPM of atleast one song being more than 160 out of 10

⇒ 1 − P(No songs being over 160 BPM).

= ~~P(0.9772)~~

⇒ $1-(0.97725)^{10}$

$= 1-0.79443.$

⇒ $\underline{0.2056}$ or 20.56%.

3c. We know that, for two independent normal distribution,

$\mu(\text{sum}) = \mu_1 + \mu_2.$

$\mu_1 = 120, \quad \mu_2 = 120.$

$\therefore \mu(\text{sum}) = 240.$

$|||^{ly}$. $\text{variance}(\text{sum}) = \sigma_1^2 + \sigma_2^2$

$= 20^2 + 20^2$

$\sigma^2_{\text{sum}} = 800.$

$\therefore \sigma_{\text{sum}} = 28.284.$

To find the $\sigma_{\text{average}} = \frac{\sigma_{\text{sum}}}{\sqrt{n}} = \frac{28.284}{1.414} \Rightarrow 19.99 \approx 20.$

n = number of songs

Now, for the average for the 160, the sum is $(x_1 + x_2 = 320)$.

$\therefore P(x_1 + x_2 > 320) = 1 - \text{pnorm}(320, 240, 28.284)$

$= 0.0023$ or $.23\%$

4. Given:

$X_1 \sim Normal(1, 9)$ - $\mu_{x_1} = 1$
$\sigma_{x_1} = 3$.

$X_2 \sim Normal(3, 16)$ $\mu_{x_2} = 3$
$\sigma_{x_2} = 4$.

(a). $X_1 + X_2$? μ? σ^2 ?.

$\mu_x = \mu_{x_1} + \mu_{x_2}$

$\mu_x = 1 + 3 = \underline{\underline{4}}$

$\sigma_x^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2$

$\sigma_x^2 \Rightarrow \underline{\underline{9 + 16 = 25}}$

(b). $-X_2$ = ?

$\mu_{-x_2} = -\mu_{x_2} \Rightarrow \underline{\underline{-3}}$.

$\sigma_{-x_2}^2 = \sigma_{x_2}^2 = \underline{\underline{16}}$.

(c) $X_1 - X_2$.

$\mu_{x_1 - x_2} = \mu_{x_1} - \mu_{x_2} \Rightarrow 1 - 3 = \underline{\underline{-2}}$.

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$\sigma^2_{x_1-x_2} = \sigma_{x_1} + \sigma_2$.

$= 9 + 16 = \underline{\underline{25}}$.

(d). $2x_1 \Rightarrow ?$

$\mu_{2x_1} = 2 \times \mu_{x_1}$

$= 2 \times 1 \Rightarrow 2$ //

$\sigma^2_{2x_1} = (2)^2 \cdot \sigma^2_{2x_1} \Rightarrow 4 \times 9 = \underline{\underline{36}}$.

(e) $2x_1 - 2x_2 = ?$

$\mu_{2x_1-2x_2} = 2\mu_1 - 2\mu_2$.

$= 2\times 1 - 2\times 3$

$= 2 - 6 = -4$ ///

Variance $\sigma^2_{2x_1-2x_2} = 2^2 \times \sigma^2_{x_1} + 2^2 \sigma^2_{x_2}$

$= 4\times 9 + 4 \times 16$.

$= 36 + 64 = \underline{\underline{100}}$

Reference :- [Math.stackexchange.com].

Q5 Given:

$$f(x) = \begin{cases} 0.1 & 0 \le x < 1 \\ 0.2 & 1 \le x < 2 \\ 0.4 & 2 \le x < 3 \\ 0.3 & 3 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Complete C.D.F

① for $y < 0$, $F(0) = 0$ [as P.D.F = 0, at 0]

② for $0 \le y < 1$

$$F(y) = \int_0^y 0.1\,dx \Rightarrow [0.1x]_0^y \rightarrow 0.1y$$

③ for $1 \le y < 2$

$$F(y) = \int_1^y 0.2dx + \int_0^1 0.1dx \Rightarrow [0.2x]_1^y + [0.1x]_0^1$$

$$= 0.1 + 0.2y - 0.2 \Rightarrow 0.2y - 0.1$$

④ for $2 \le y < 3$

$$F(y) = \int_0^1 0.1dx + \int_1^2 0.2dx + \int_2^y 0.4dx.$$

$$\Rightarrow [0.1x]_0^1 + [0.2x]_1^2 + [0.4x]_2^y$$

$$= 0.4y + 0.1 + 0.4 - (0.4 \times 0.2)$$

$$\Rightarrow 0.4y - 0.5$$

(5) for $3 \le y < 4$.

$$F(y) = \int_3^y 0.3\,dx + \int_2^3 0.4\,dx + \int_1^2 0.2\,dx + \int_0^1 0.1\,dx$$

$$\Rightarrow 0.3[y-3] + 0.4(3-2) + 0.2[2-1] + 0.1[1]$$

$$= 0.3y - 0.9 + 0.4 + 0.2 + 0.1$$

$$= 0.3y - 0.2$$

(6) for $y \geq 4$ $F(y) = 1$ As the P.D.F is equal to zero for $x \geq 4$

$\therefore$ C.D.F is given by

$$F(y) = \begin{cases} 0 & y < 0 \\ 0.1y & 0 \le y < 1 \\ 0.2y - 0.1 & 1 \le y < 2 \\ 0.4y - 0.5 & 2 \le y < 3 \\ 0.3y - 0.2 & 3 \le y < 4 \\ 1 & y \geq 4 \end{cases}$$

5b) Median ?

The median is the value for which $F(y) = 0.5$.

for $x=2$, from the C.D.F obtained, we have

(1) $0.1y = 0.5$ from C.D.F at different intervals

$y = 5$

(2) $0.2(y-1) = 0.5 - 0.1$

$= 0.2(y-1) = 0.4$

$y - 1 = 2$

$y = 3$

(3) $0.4(y-2) = 0.5 - 0.3$

$0.4(y-2) = 0.2$

$y-2 = 0.5$

$y = 2.5$

(4). $0.3(y-3) = -0.5$

$y-3 = \frac{-0.5}{0.3}$

$y = 3 - 5/3 = 4/3$

from [1], [2], [3], [4], we have $F(3) = 0.5$

Hence Median = 3//.

5(c). Expected Value $E(X) = ?$

$$E(x) = \int_0^{\infty} x(y(x)) \cdot dx$$

$$\Rightarrow \int_0^1 x \cdot 0.1\, dx + \int_1^2 x \cdot 0.2\, dx + \int_2^3 x \cdot 0.4\, dx$$

$$+ \int_3^4 x \cdot 0.3\, dx$$

$$\Rightarrow 0.1\left[\frac{x^2}{2}\right]_0^1 + 0.2\left[\frac{x^2}{2}\right]_1^2 + 0.4\left(\frac{x^2}{2}\right)_2^3 + 0.3\left(\frac{x^2}{2}\right)_3^4$$

$$\Rightarrow \left(\frac{1}{2}\right)0.1 + \frac{0.2}{2}\left[\cancel{0} \cdot 2^2 - 1\right] + \frac{0.4}{2}\left[3^2 - 2^2\right]$$

$$+ \frac{0.3}{2}\left(4^2 - 3^2\right)$$

$$= 0.05 + 0.2 \times \frac{3}{2} + 0.2(5) + \frac{0.3}{2}(7)$$

$$= 0.05 + 0.3 + 1 + 1.05$$

$$\Rightarrow \underline{\underline{2.4}}.$$

Q6. Given:

$$g(x) = \begin{cases} 0 & x < 0 \\ x & x \in (0,1) \\ 1 & x \in [1,2] \\ 3-x & x \in [2,3] \\ 0 & x > 3 \end{cases}$$

let $f(x) = c.g(x)$

6 a. for what value of 'c', is f a P.D.F?

① Condition 1: for a P.D.F, the f^n must be non-negative for all values of x i.e. $f(x) \geq 0$.

② Condition 2: $\int_{-\infty}^{\infty} f(x)\,dx = 1$: , hence Equating we get:

$$\therefore \int_0^1 c.g(x)dx + \int_1^2 c.g(x)dx + \int_2^3 cg(x)dx = 1.$$

$$\Rightarrow c\left\{\int_0^1 x\,dx + \int_1^2 1\,dx + \int_2^3 (3-x)\,dx\right\} = 1$$

$$= c\left[\left[\frac{x^2}{2}\right]_0^1 + [x]_1^2 + \left[3x - \frac{x^2}{2}\right]_2^3\right] = 1$$

$$= c\left(\frac{1}{2} + (2-1) + 9 - \frac{9}{2} - [6-2]\right) = 1$$

$$= c\left(\frac{1}{2} + 1 + \frac{9}{2} - 4\right) = 1$$

$= c\left(6-4\right) = 1$

$c(2) = 1$

$\therefore c = \frac{1}{2}$

$\therefore f(x)$ can be written as

$$f_X = \begin{cases} 0 & x < 0 \\ x/2 & x \in [0,1] \\ 1/2 & x \in [1,2] \\ (3-x)/2 & x \in [2,3] \\ 0 & x > 3 \end{cases}$$

$$F_Y = \begin{cases} 0 & y < 4 \\ y^2/4 & 0 \le y < 1 \\ \frac{2y-1}{4} & 1 \le y < 2 \\ \frac{6y-y^2-5}{4} & 2 \le y < 3 \\ 1 & y \ge 3 \end{cases}$$

6(b). $P(1.5 < X < 2.5) = ?$

from the P.D.F computed ~~above~~, above, we have

$$P(1.5 < X < 2.5) = \int_{1.5}^{2.5} f(x)\,dx$$

$$\Rightarrow \int_{1.5}^{2.0} 0.5\,dx + \int_{2.0}^{2.5} \frac{(3-x)}{2}\,dx$$

$$= \left[0.5x\right]_{1.5}^{2.0} + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_{2.0}^{2.5}$$

$$= 0.5\times2 - 0.5\times1.5 + \left[\frac{1}{2}\left\{3\times2.5 - \frac{2.5^2}{2}\right\} - \frac{1}{2}\left[3\times2 - \frac{2^2}{2}\right]\right]$$

$$1 - 0.75 + \frac{1}{2}\left[3(0.5) - \frac{6.25}{2} - 2\right]$$

$$= 0.25 + \frac{1}{2}\left[1.5 - \frac{2.25}{2}\right]$$

$$= 0.25 + \frac{1}{2}\left[0.375\right]$$

$$= 0.25 + 0.1875$$

$$= \underline{\underline{0.4375}}.$$

Q6(c) $E(x) = 2$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx.$$

$$E(x) = \int_0^1 \frac{1}{2}x^2\,dx + \int_1^2 \left(\frac{x}{2}\right) dx + \int_2^3 \frac{1}{2}x(3-x)\,dx$$

$$\Rightarrow 0.5\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{4}\right]_1^2 + \frac{1}{2}\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_2^3$$

$$\Rightarrow 0.5\left[\frac{1}{3}\right] + \left(\frac{4}{4} - \frac{1}{4}\right) + \frac{1}{2}\left[3\times\frac{9}{2} - \frac{9}{3} - \left[\frac{3\times 4}{2} - \frac{8}{3}\right]\right]$$

$= \frac{0.5}{3} + \frac{3}{4} +$

$\frac{0.5}{3} + \frac{3}{4} + \frac{1}{2}\left[\frac{27}{2} - \frac{9}{3} - \left[6 - \frac{8}{3}\right]\right]$

$\frac{0.5}{3} + \frac{3}{4} + \frac{1}{2}\left[13.5 - 3 - \left[6 - \frac{8}{3}\right]\right]$

$\Rightarrow$ Simplifying we get

$E(x) = 1.5$

Q6(d). $F(1)$?

$$F(1) = \int_0^1 f(x)\,dx$$

$$= \int_0^1 0.5\cdot x\cdot dx.$$

$$\Rightarrow \left[0.5\cdot\frac{x^2}{2}\right]_0^1 \Rightarrow \left(0.5\times\frac{1}{2}\right) = \underline{\underline{0.25}}.$$

Q6(e). 0.9 quantile of (y)?

To compute 0.9 quantile we calculate 'y' such that

$F(y) = 0.90$. for different intervals from C.D.F

Case 1: when $y < 0$, $F(y) = 0$

Case 2 :- $0 \leq y < 1$.

$$F(y) = \frac{y^2}{4}$$

$$\frac{y^2}{4} = 0.9 \Rightarrow y = \sqrt{3.6} \Rightarrow 1.897$$

Case 3 :- $1 \leq y < 2 \Rightarrow F(y) = (2y-1)/4$.

$$\frac{2y-1}{4} = 0.9 \Rightarrow y = \frac{3.6+1}{2} \Rightarrow 2.3$$

Case 4 :- $2 \leq y < 3 \Rightarrow F(y) = \left(\frac{6y - y^2 - 5}{4}\right)$

$$0.9 = \frac{6y - y^2 - 5}{4}$$

$$3.6 + 5 = 6y - y^2$$

$$8.6 = 6y - y^2$$

$$y^2 - 6y + 8.6 = 0$$

through quardatic eq^n $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$y = \frac{6 \pm \sqrt{1.6}}{2}$$

$y = 3.632$ or, 2.367.

Though 3.632 does not fall in the interval $[2,3]$, 2.367 does, hence $F(2.367) = 0.90$