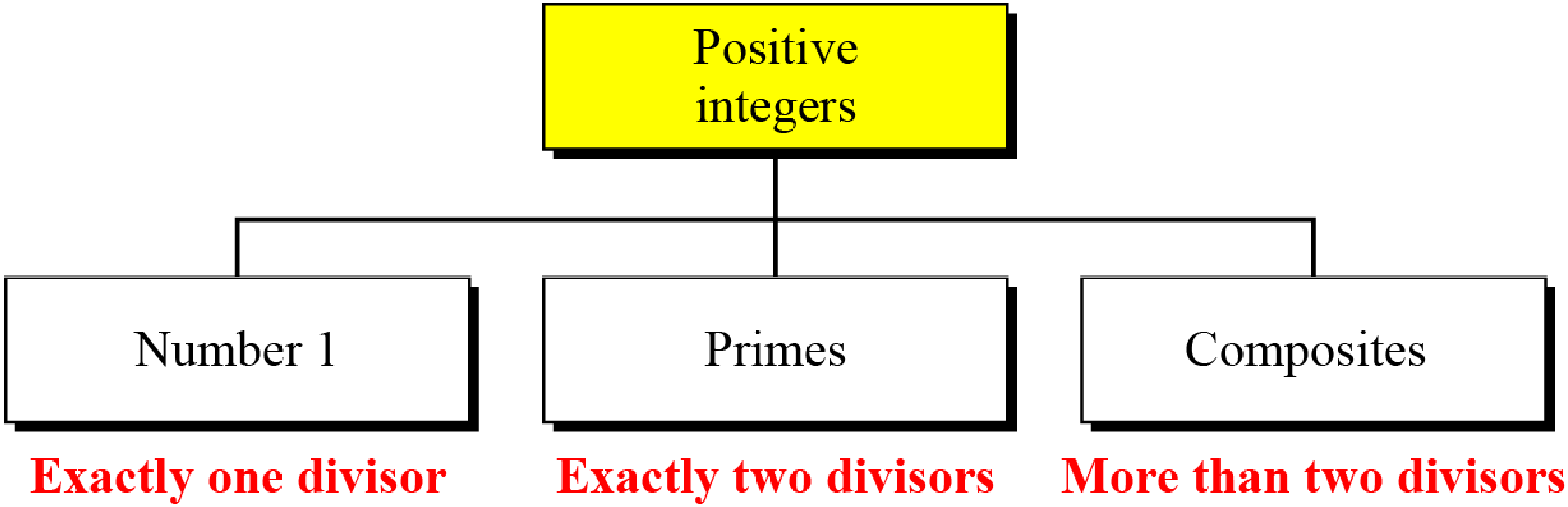
MATHEMATICS OF CRYPTOGRAPHY PART III

Primes and Related Congruence Equations

Primes

Three groups of positive integers



A prime is divisible only by itself and 1.

Primes(cont.)

- Cardinality of Primes

  There is an infinite number of primes.

- Number of Primes

$$[n / (\ln n)] \quad < \quad \pi(n) \quad < \quad [n/(\ln n - 1.08366)]$$

- E.g. Find the number of primes less than 1,000,000.
  - The approximation gives the range 72,383 to 78,543. The actual number of primes is 78,498

Checking for Primeness

- Given a number n, how can we determine if n is a prime?
  - The answer is that we need to see if the number is divisible by all primes less than $\sqrt{n}$
- Is 97 a prime?
  - The floor of $\sqrt{97} = 9$. The primes less than 9 are 2, 3, 5, and 7. We need to see if 97 is divisible by any of these numbers. It is not, so 97 is a prime.

Checking for Primeness(cont.)

- Is 301 a prime?
  - The floor of $\sqrt{301}$ = 17. We need to check 2, 3, 5, 7, 11, 13, and 17. The numbers 2, 3, and 5 do not divide 301, but 7 does. Therefore 301 is not a prime.

Euler's Phi-Function

- *Euler's phi-function*, ϕ (n), which is sometimes called the *Euler's totient function* plays a very important role in cryptography.
- The function finds the number of integers that are both smaller than n and relatively prime to n

1. $\phi(1) = 0$.
2. $\phi(p) = p - 1$ if p is a prime.
3. $\phi(m \times n) = \phi(m) \times \phi(n)$ if m and n are relatively prime.
4. $\phi(p^e) = p^e - p^{e-1}$ if p is a prime.

Euler's Phi-Function(cont.)

- We can combine the above four rules to find the value of $\phi(n)$. For example, if n can be factored as

$$n = p_1^{e1} \times p_2^{e2} \times \ldots \times p_k^{ek}$$

- Then we combine the third and the fourth rule to find

$$\phi(n) = (p_1^{e_1} - p_1^{e_1 - 1}) \times (p_2^{e_2} - p_2^{e_2 - 1}) \times \cdots \times (p_k^{e_k} - p_k^{e_k - 1})$$

The difficulty of finding $\phi(n)$ depends on the difficulty of finding the factorization of n.

Euler’s Phi-Function(cont.)

- Example 1
  - What is the value of $\phi(13)$?
- Solution
  - Because 13 is a prime, $\phi(13) = (13 - 1) = 12$.
- Example 2
  - What is the value of $\phi(10)$?
- Solution
  - We can use the third rule: $\phi(10) = \phi(2) \times \phi(5) = 1 \times 4 = 4$, because 2 and 5 are primes.

Euler's Phi-Function(cont.)

- Example 3
  - What is the value of $\phi(240)$?
- Solution
  - We can write $240 = 2^4 \times 3^1 \times 5^1$. Then

$$\phi\,(240) = (2^4 - 2^3) \times (3^1 - 3^0) \times (5^1 - 5^0) = 64$$

- Example 4
  - Can we say that $\phi\,(49) = \phi\,(7) \times \phi\,(7) = 6 \times 6 = 36$????

Euler's Phi-Function(cont.)

- Example 3
  - What is the value of $\phi(240)$?
- Solution
  - We can write $240 = 2^4 \times 3^1 \times 5^1$. Then

    $\phi(240) = (2^4 - 2^3) \times (3^1 - 3^0) \times (5^1 - 5^0) = 64$
- Example 4
  - Can we say that $\phi(49) = \phi(7) \times \phi(7) = 6 \times 6 = 36$????
- Solution
  - No. The third rule applies when *m* and *n* are relatively prime. Here $49 = 7^2$. We need to use the fourth rule: $\phi(49) = 7^2 - 7^1 = 42$.

Euler's Phi-Function(cont.)

- Example 5
  - What is the number of elements in Z_{14}^*?
- Solution
  - The answer is $\phi(14) = \phi(7) \times \phi(2) = 6 \times 1 = 6$. The members are 1, 3, 5, 9, 11, and 13.

Interesting point: If $n > 2$, the value of $\phi(n)$ is even.

Fermat’s Little Theorem

- First Version
  - If p is a prime and a is an integer such that p does not divide a,

$$a^{p-1} \equiv 1 \bmod p$$

- Second Version
  - Removes the condition on a
  - If p is prime and a is an integer,

$$a^{p} \equiv a \bmod p$$

Fermat's Little Theorem(cont.)

- Example 1
  - Find the result of 6^{10} mod 11.
- Solution
  - We have 6^{10} mod 11 = 1. This is the first version of Fermat's little theorem where p = 11.
- Example 2
  - Find the result of 3^{12} mod 11.
- Solution

Fermat's Little Theorem(cont.)

- Example 1
  - Find the result of $6^{10} \bmod 11$.
- Solution
  - We have $6^{10} \bmod 11 = 1$. This is the first version of Fermat's little theorem where $p = 11$.
- Example 2
  - Find the result of $3^{12} \bmod 11$.
- Solution
  - Here the exponent (12) and the modulus (11) are not the same. With substitution this can be solved using Fermat's little theorem.

$$3^{12} \bmod 11 = (3^{11} \times 3) \bmod 11 = (3^{11} \bmod 11)\,(3 \bmod 11) = (3 \times 3) \bmod 11 = 9$$

Fermat's Little Theorem(cont.)

- Multiplicative Inverses

$$a^{-1} \bmod p = a^{p-2} \bmod p$$

- The answers to multiplicative inverses modulo a prime can be found without using the extended Euclidean algorithm:

a. $8^{-1} \bmod 17 = 8^{17-2} \bmod 17 = 8^{15} \bmod 17 = 15 \bmod 17$

b. $5^{-1} \bmod 23 = 5^{23-2} \bmod 23 = 5^{21} \bmod 23 = 14 \bmod 23$

c. $60^{-1} \bmod 101 = 60^{101-2} \bmod 101 = 60^{99} \bmod 101 = 32 \bmod 101$

d. $22^{-1} \bmod 211 = 22^{211-2} \bmod 211 = 22^{209} \bmod 211 = 48 \bmod 211$

Euler's Theorem

- First Version
  - If a and n are coprime,

$$a^{\phi(n)} \equiv 1 \ (\mathrm{mod}\ n)$$

- Second Version
  - Removes the condition that a and n should be coprime

$$a^{k \times \phi(n) + 1} \equiv a \ (\mathrm{mod}\ n)$$

The second version of Euler's theorem is used in the RSA cryptosystem

Euler's Theorem(cont.)

- Example 1
  - Find the result of 6^{24} mod 35.
- Solution
  - We have 6^{24} mod 35 = $6^{\phi(35)}$ mod 35 = 1.
- Example 2
  - Find the result of 20^{62} mod 77???

Euler's Theorem(cont.)

- Example 1
  - Find the result of 6^{24} mod 35.
- Solution
  - We have 6^{24} mod 35 = $6^{\phi(35)}$ mod 35 = 1.
- Example 2
  - Find the result of 20^{62} mod 77.
- Solution

If we let k = 1 on the second version, we have

20^{62} mod 77 = (20 mod 77) ($20^{\phi(77)+1}$ mod 77) mod 77

= (20)(20) mod 77 = 15.

Euler's Theorem(cont.)

- Multiplicative Inverses
  - Euler's theorem can be used to find multiplicative inverses modulo a composite.

$$a^{-1} \bmod n = a^{\phi(n)-1} \bmod n$$

Euler's Theorem(cont.)

- Example
  - The answers to multiplicative inverses modulo a composite can be found without using the extended Euclidean algorithm if we know the factorization of the composite:

a. $8^{-1} \bmod 77 = 8^{\phi(77)-1} \bmod 77 = 8^{59} \bmod 77 = 29 \bmod 77$

b. $7^{-1} \bmod 15 = 7^{\phi(15)-1} \bmod 15 = 7^{7} \bmod 15 = 13 \bmod 15$

c. $60^{-1} \bmod 187 = 60^{\phi(187)-1} \bmod 187 = 60^{159} \bmod 187 = 53 \bmod 187$

d. $71^{-1} \bmod 100 = 71^{\phi(100)-1} \bmod 100 = 71^{39} \bmod 100 = 31 \bmod 100$

Generating Primes

- Mersenne Primes $\boxed{\mathbf{M_p = 2^p - 1}}$

$$
\begin{aligned}
M_2 &= 2^2 - 1 = 3 \\
M_3 &= 2^3 - 1 = 7 \\
M_5 &= 2^5 - 1 = 31 \\
M_7 &= 2^7 - 1 = 127 \\
M_{11} &= 2^{11} - 1 = 2047 && \textbf{Not a prime } (\mathbf{2047 = 23 \times 89}) \\
M_{13} &= 2^{13} - 1 = 8191 \\
M_{17} &= 2^{17} - 1 = 131071
\end{aligned}
$$

A number in the form $M_p = 2^p - 1$ is called a Mersenne number and may or may not be a prime.

Generating Primes(cont.)

- Fermat Primes $\boxed{F_n = 2^{2^n} + 1}$

$F_0 = 3$ $F_1 = 5$ $F_2 = 17$ $F_3 = 257$ $F_4 = 65537$

$F_5 = 4294967297 = 641 \times 6700417$ ***Not a prime***

Primality Testing

- Finding an algorithm to correctly and efficiently test a very large integer and output a prime or a composite has always been a challenge in number theory.
- Two types
  - Deterministic Algorithms
  - Probabilistic Algorithms

Deterministic Algorithms

- Divisibility Algorithm

Algorithm 9.1 *Pseudocode for the divisibility test*

```
Divisibility_Test (n)                    // n is the number to test for primality
{
  r ← 2
  while (r < √n )
  {
   if (r | n) return "a composite"
   r ← r + 1
  }
  return "a prime"
}
```

The bit-operation complexity of the divisibility test is O($2^{n_b/2}$) (exponential)

Deterministic Algorithms(cont.)

- Example
  - Assume n has 200 bits. What is the number of bit operations needed to run the divisibility-test algorithm?
- Solution
  - The bit-operation complexity of this algorithm is $2^{n_b/2}$. This means that the algorithm needs 2^{100} bit operations. On a computer capable of doing 2^{30} bit operations per second, the algorithm needs 2^{70} seconds to do the testing !!!!!

Deterministic Algorithms(cont.)

- AKS Algorithm

$$(x-a)^n \equiv (x^n - a) \pmod{n} \qquad (1) \qquad O((\log_2 n_b)^{12})$$

- Example
  - Assume n has 200 bits. What is the number of bit operations needed to run the AKS algorithm?
- Solution
  - This algorithm needs only $(\log_2 200)^{12}$ = 39,547,615,483 bit operations. On a computer capable of doing 1 billion bit operations per second, the algorithm needs only 40 seconds.

Probabilistic Algorithms

- Fermat Test $\boxed{\textbf{If } n \textbf{ is a prime, then } a^{n-1} \equiv 1 \bmod n.}$

If n is a prime, $a^{n-1} \equiv 1 \bmod n$
If n is a composite, it is possible that $a^{n-1} \equiv 1 \bmod n$

- Example
  - Does the number 561 pass the Fermat test?

Probabilistic Algorithms(cont.)

- Example
  - Does the number 561 pass the Fermat test?
- Solution
  - Use base 2

$$2^{561-1} = 1 \bmod 561$$

  - The number passes the Fermat test, but it is not a prime, because 561 = 33 × 17.

Probabilistic Algorithms(cont.)

- Square Root Test

If n is a prime, $\sqrt{1} \bmod n = \pm 1$.
If n is a composite, $\sqrt{1} \bmod n = \pm 1$ and possibly other values.

- Example
  - What are the square roots of 1 mod n if n is 7 (a prime)?
- Solution
  - The only square roots are 1 and −1. We can see that

$1^2 = 1 \bmod 7$	$(-1)^2 = 1 \bmod 7$
$2^2 = 4 \bmod 7$	$(-2)^2 = 4 \bmod 7$
$3^2 = 2 \bmod 7$	$(-3)^2 = 2 \bmod 7$

Probabilistic Algorithms(cont.)

– Note that we don't have to test 4, 5 and 6 because $4 = -3 \bmod 7$, $5 = -2 \bmod 7$ and $6 = -1 \bmod 7$.

Probabilistic Algorithms(cont.)

- Example
  - What are the square roots of 1 mod n if n is 8 (a composite)?
- Solution
  - There are four solutions: 1, 3, 5, and 7 (which is −1). We can see that

$$1^2 = 1 \bmod 8 \qquad (-1)^2 = 1 \bmod 8$$
$$3^2 = 1 \bmod 8 \qquad 5^2 = 1 \bmod 8$$

Probabilistic Algorithms(cont.)

- Example
  - What are the square roots of 1 mod n if n is 17 (a prime)?
- Solution
  - There are only two solutions: 1 and −1

$1^2 = 1 \bmod 17$	$(-1)^2 = 1 \bmod 17$
$2^2 = 4 \bmod 17$	$(-2)^2 = 4 \bmod 17$
$3^2 = 9 \bmod 17$	$(-3)^2 = 9 \bmod 17$
$4^2 = 16 \bmod 17$	$(-4)^2 = 16 \bmod 17$
$5^2 = 8 \bmod 17$	$(-5)^2 = 8 \bmod 17$
$6^2 = 2 \bmod 17$	$(-6)^2 = 2 \bmod 17$
$(7)^2 = 15 \bmod 17$	$(-7)^2 = 15 \bmod 17$
$(8)^2 = 13 \bmod 17$	$(-8)^2 = 13 \bmod 17$

Probabilistic Algorithms(cont.)

- Example
  - What are the square roots of 1 mod n if n is 22 (a composite)??????

Probabilistic Algorithms(cont.)

- Example
  - What are the square roots of 1 mod n if n is 22 (a composite)?
- Solution
  - Surprisingly, there are only two solutions, +1 and −1, although 22 is a composite.

$$
\begin{aligned}
1^2 &= 1 \bmod 22 \\
(-1)^2 &= 1 \bmod 22
\end{aligned}
$$

Probabilistic Algorithms(cont.)

- Miller-Rabin Test $\quad n - 1 = m \times 2^k$

Idea behind Fermat primality test

$$a^{n-1} = a^{m \times 2^k} = [a^m]^{2^k} = [a^m]^{\overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{k \text{ times}}}$$

The Miller-Rabin test needs from step 0 to step k − 1.

Probabilistic Algorithms(cont.)

- Pseudo code Miller-Rabin(n)

```
n-1= m 2^k, where m is odd(note that n-1 is even)
Choose a random integer a, 1<=a<=n-1
T=a^m mod n
if T ≡ 1(mod n)
    then return ("n is prime")
for i=1 to k
{   if T ≡ -1 (mod n)
    then return("n is prime")
    else T=T^2 mod n
}
Return("n is composite")
```

Probabilistic Algorithms(cont.)

- Example
  - Does the number 561 pass the Miller-Rabin test?
- Solution
  - Using base 2, let $561 - 1 = 35 \times 2^4$, which means $m = 35$, $k = 4$, and $a = 2$.

Initialization:	$T = 2^{35} \bmod 561 = 263 \bmod 561$	
$k = 1$:	$T = 263^2 \bmod 561 = 166 \bmod 561$	
$k = 2$:	$T = 166^2 \bmod 561 = 67 \bmod 561$	
$k = 3$:	$T = 67^2 \bmod 561 = +1 \bmod 561$	$\rightarrow$ **a composite**

Probabilistic Algorithms(cont.)

- Example
  - We already know that 27 is not a prime. Let us apply the Miller-Rabin test.
- Solution
  - With base 2, let $27 - 1 = 13 \times 2^1$, which means that $m = 13$, $k = 1$, and $a = 2$. The initialization step: $T = 2^{13} \bmod 27 = 11 \bmod 27$. However, because the algorithm enters the loop only once, it returns a composite.

Probabilistic Algorithms(cont.)

- Example
  - We know that 61 is a prime, let us see if it passes the Miller-Rabin test.
- Solution
  - We use base 2.

$$61 - 1 = 15 \times 2^2 \quad \rightarrow \quad m = 15 \quad k = 2 \quad a = 2$$

$$\textit{Initialization:} \quad T = 2^{15} \bmod 61 = 11 \bmod 61$$

$$k = 1 \qquad T = 11^2 \bmod 61 = -1 \bmod 61 \quad \rightarrow \quad \textbf{a prime}$$

Probabilistic Algorithms(cont.)

- Exercise
  - Check for the primality of the numbers 201 and 349 using Miller-Rabin test. Use base 2.

Recommended Primality test

- Combination of the divisibility test and the Miller-Rabin test.
- Example
  - The number 4033 is a composite (37 × 109). Does it pass the recommended primality test?
- Solution
  1. Perform the divisibility tests first. The numbers 2, 3, 5, 7, 11, 17, and 23 are not divisors of 4033.
  2. Perform the Miller-Rabin test with a base of 2, $4033 - 1 = 63 \times 64$, which means m is 63 and k is 6.

Initialization: $T \equiv 2^{63} \pmod{4033} \equiv 3521 \pmod{4033}$

$k = 1$ $\quad T \equiv T^2 \equiv 3521^2 \pmod{4033} \equiv -1 \pmod{4033}$ $\quad \rightarrow$ **Passes**

Recommended Primality test(cont.)

Cont…

3. But we are not satisfied. We continue with another base, 3.

Initialization: $T \equiv 3^{63} \pmod{4033} \equiv 3551 \pmod{4033}$

$k = 1$	$T \equiv T^2 \equiv 3551^2$ (mod 4033 $\equiv$ 2443 (mod 4033)
$k = 2$	$T \equiv T^2 \equiv 2443^2$ (mod 4033 $\equiv$ 3442 (mod 4033)
$k = 3$	$T \equiv T^2 \equiv 3442^2$ (mod 4033 $\equiv$ 2443 (mod 4033)
$k = 4$	$T \equiv T^2 \equiv 2443^2$ (mod 4033 $\equiv$ 3442 (mod 4033)
$k = 5$	$T \equiv T^2 \equiv 3442^2$ (mod 4033 $\equiv$ 2443 (mod 4033) $\rightarrow$ **Failed (composite)**

FACTORIZATION

Fundamental Theorem of Arithmetic

$$n = p_1^{e1} \times p_2^{e2} \times \cdots \times p_k^{ek}$$

- Greatest Common Divisor

$$a = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k} \qquad b = p_1^{b_1} \times p_2^{b_2} \times \cdots \times p_k^{b_k}$$

$$\gcd(a, b) = p_1^{\min(a_1, b_1)} \times p_2^{\min(a_2, b_2)} \times \cdots \times p_k^{\min(a_k, b_k)}$$

- Least Common Multiplier

$$a = p_1^{a1} \times p_2^{a2} \times \cdots \times p_k^{ak} \qquad b = p_1^{b1} \times p_2^{b2} \times \cdots \times p_k^{bk}$$

$$\mathrm{lcm}(a, b) = p_1^{\max(a_1, b_1)} \times p_2^{\max(a_2, b_2)} \times \cdots \times p_k^{\max(a_k, b_k)}$$

$$\mathrm{lcm}(a, b) \times \gcd(a, b) = a \times b$$

Factorization methods

- Trial Division Method

Algorithm 9.3 *Pseudocode for trial-division factorization*

```
Trial_Division_Factorization (n)                    // n is the number to be factored
{
    a ← 2
    while (a ≤ √n )
    {
        while (n mod a = 0)
        {
            output a                                // Factors are output one by one
            n = n / a
        }
        a ← a + 1
    }
    if (n > 1) output n                             // n has no more factors
}
```

Factorization methods(cont.)

- Example
  - Use the trial division algorithm to find the factors of 1233.
- Solution
  - We run a program based on the algorithm and get the following result.

$$1233 = 3^2 \times 137$$

Factorization methods(cont.)

- Example
  - Use the trial division algorithm to find the factors of 1523357784
- Solution
  - We run a program based on the algorithm and get the following result.

$$1523357784 = 2^3 \times 3^2 \times 13 \times 37 \times 43987$$

Fermat Method

$$n = x^2 - y^2 = a \times b \qquad \text{with } a = (x + y) \text{ and } b = (x - y)$$

Algorithm 9.4 *Pseudocode for Fermat factorization*

Feramat_Factorization (n) // n is the number to be factored

{

$x \leftarrow \sqrt{n}$ // smallest integer greater than $\sqrt{n}$

while ($x < n$)

{

$w \leftarrow x^2 - n$

if (w is perfect square) $y \leftarrow \sqrt{w}$; $a \leftarrow x + y$; $b \leftarrow x - y$; return a and b

$x \leftarrow x + 1$

}

}

Factorization methods(cont.)

- More methods
  - Pollard p-1
  - Pollard rho
  - Number Field Sieve
  - Quadratic Sieve

CHINESE REMAINDER THEOREM

- Used to solve a set of congruent equations with one variable but different moduli, which are relatively prime

$$
\begin{aligned}
x &\equiv a_1 \pmod{m_1} \\
x &\equiv a_2 \pmod{m_2} \\
&\cdots \\
x &\equiv a_k \pmod{m_k}
\end{aligned}
$$

Continued…

- Example
  - The following is an example of a set of equations with different moduli:

$$
\begin{aligned}
x &\equiv 2 \pmod{3} \\
x &\equiv 3 \pmod{5} \\
x &\equiv 2 \pmod{7}
\end{aligned}
$$

  - The solution to this set of equations is given in the next section; for the moment, note that the answer to this set of equations is $x = 23$. This value satisfies all equations: $23 \equiv 2 \pmod{3}$, $23 \equiv 3 \pmod{5}$, and $23 \equiv 2 \pmod{7}$.

Continued…

- Solution To Chinese Remainder Theorem
  - Find $M = m_1 \times m_2 \times \ldots \times m_k$. This is the common modulus.
  - Find $M_1 = M/m_1$, $M_2 = M/m_2$, …, $M_k = M/m_k$.
  - Find the multiplicative inverse of M_1, M_2, …, M_k using the corresponding moduli (m_1, m_2, …, m_k). Call the inverses M_1^{-1}, M_2^{-1}, …, M_k^{-1}.
  - The solution to the simultaneous equations is

$$\boxed{x = (a_1 \times M_1 \times M_1^{-1} + a_2 \times M_2 \times M_2^{-1} + \cdots + a_k \times M_k \times M_k^{-1}) \bmod M}$$

Continued…

- Example
  - Find the solution to the simultaneous equations:

$$
\begin{aligned}
x &\equiv 2 \pmod 3\\
x &\equiv 3 \pmod 5\\
x &\equiv 2 \pmod 7
\end{aligned}
$$

- Solution: We follow the four steps.

1. $M = 3 \times 5 \times 7 = 105$

2. $M_1 = 105 / 3 = 35$, $M_2 = 105 / 5 = 21$, $M_3 = 105 / 7 = 15$

3. The inverses are $M_1^{-1} = 2$, $M_2^{-1} = 1$, $M_3^{-1} = 1$

4. $x = (2 \times 35 \times 2 + 3 \times 21 \times 1 + 2 \times 15 \times 1) \bmod 105 = 23 \bmod 105$

Continued…

- Example
  - Find an integer that has a remainder of 3 when divided by 7 and 13, but is divisible by 12.
- Solution ????

Continued…

- Example
  - Find an integer that has a remainder of 3 when divided by 7 and 13, but is divisible by 12.
- Solution
  - This is a CRT problem. We can form three equations and solve them to find the value of x.

$$x = 3 \bmod 7$$
$$x = 3 \bmod 13$$
$$x = 0 \bmod 12$$

  - If we follow the four steps, we find x = 276. We can check that
    276 = 3 mod 7, 276 = 3 mod 13 and 276 is divisible by 12 (the quotient is 23 and the remainder is zero).

Continued…

- Assume we need to calculate z = x + y where x = 123 and y = 334, but our system accepts only numbers less than 100. These numbers can be represented as follows:

$$
\begin{array}{ll}
x \equiv 24 \pmod{99} & y \equiv 37 \pmod{99} \\
x \equiv 25 \pmod{98} & y \equiv 40 \pmod{98} \\
x \equiv 26 \pmod{97} & y \equiv 43 \pmod{97}
\end{array}
$$

- Adding each congruence in x with the corresponding congruence in y gives

$$
\begin{array}{lcl}
x + y \equiv 61 \pmod{99} & \rightarrow & z \equiv 61 \pmod{99} \\
x + y \equiv 65 \pmod{98} & \rightarrow & z \equiv 65 \pmod{98} \\
x + y \equiv 69 \pmod{97} & \rightarrow & z \equiv 69 \pmod{97}
\end{array}
$$

- Now three equations can be solved using the Chinese remainder theorem to find z. One of the acceptable answers is z = 457.

Continued…

Secret Sharing scheme in cryptography aims to distribute and later recover secret S among n parties. Secret S is distributed in form of shares which are generated from secret. Without cooperation of k no. of parties, the secret cannot be reconstructed from shares directly. Consider the following example:

Say our secret is S. The shares for n=4 no. of parties are generated taking modulus 11,13,17 and 19. They are respectively 1,12,2 and 3 and given by following equations:

Now, from four possible sets of k=3 shares (as k shares are necessary to reconstruct the secret), consider one possible set {1, 12, 2} and recover the secret S from it.

Continued…

Secret Sharing scheme in cryptography aims to distribute and later recover secret S among n parties. Secret S is distributed in form of shares which are generated from secret. Without cooperation of k no. of parties, the secret cannot be reconstructed from shares directly. Consider the following example:

Say our secret is S. The shares for n=4 no. of parties are generated taking modulus 11,13,17 and 19. They are respectively 1,12,2 and 3 and given by following equations:

S Ξ 1 mod 11,

S Ξ 12 mod 13,

S Ξ 2 mod 17,

S Ξ 3 mod 19.

Now, from four possible sets of k=3 shares (as k shares are necessary to reconstruct the secret), consider one possible set {1, 12, 2} and recover the secret S from it.

Continued…

Solution: The problem can be solved by Chinese remainder theorem.

For the set {1,12,2}, the equations available are,

$S \equiv 1 \bmod 11$,

$S \equiv 12 \bmod 13$,

$S \equiv 2 \bmod 17$,

Now solving this equation using CRT, M=11 *13*17 = 2431,

M1 = 2431/11=221,

M2 = 2431/13=187,

M3=2431/17=143

$M1^{-1}$, $M2^{-1}$ and $M3^{-1}$ can be calculated using Extended Euclidean Algorithm.

$M1^{-1} = 1$

$M2^{-1} = 8$

$M3^{-1} = 5$

Now, secret S= ((1*221*1) + (12*187*8) + (2*143*5)) mod 2431

S = 155 mod 2431

EXPONENTIATION AND LOGARITHM

EXPONENTIATION AND LOGARITHM

Exponentiation: $y = a^x$ $\rightarrow$ **Logarithm:** $x = \log_a y$

Exponentiation

- Fast Exponentiation
  - The idea behind the square-and-multiply method

$$y = a^{\boxed{x_{n_b-1} \times 2^{n_b-1} + x_{n_b-2} \times 2^{n_b-2} + \cdots + x_1 \times 2^1 + x_0 \times 2^0}}$$

in which x_i is 0 or 1

$$\downarrow$$

$$y = \boxed{\boxed{a^{2^{n_b-1}}} \text{ or } \boxed{1}} \times \boxed{\boxed{a^{2^{n_b-2}}} \text{ or } \boxed{1}} \times \cdots \times \boxed{\boxed{a^2} \text{ or } \boxed{1}} \times \boxed{\boxed{a} \text{ or } \boxed{1}}$$

Example:

$$\boxed{y = a^9 = a^{1001_2} = a^8 \times 1 \times 1 \times a}$$

Continued…

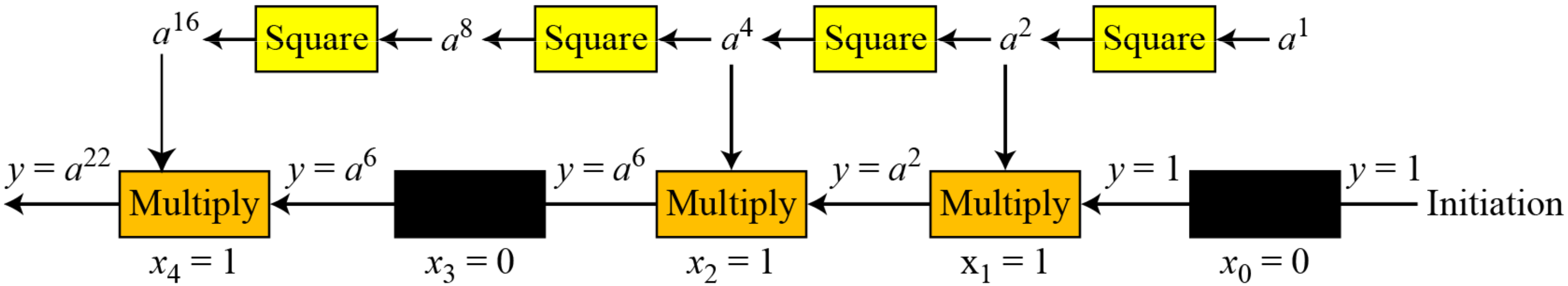
Algorithm 9.7 *Pseudocode for square-and-multiply algorithm*

```
Square_and_Multiply (a, x, n)
{
    y ← 1
    for (i ← 0 to n_b − 1)                  // n_b is the number of bits in x
    {
        if (x_i = 1)   y ← a × y  mod n     // multiply only if the bit is 1

        a ← a^2 mod n                       // squaring is not needed in the last iteration
    }
    return y
}
```

Continued…

- The process for calculating $y = a^x$
- In this case, $x = 22 = (10110)_2$ in binary.



Continued…

Table 9.3 *Calculation of 17^{22} mod 21*

i	x_i	*Multiplication (Initialization: y = 1)*		*Squaring (Initialization: a = 17)*
0	0		$\rightarrow$	$a = 17^2 \bmod 21 = 16$
1	1	$y = 1 \times 16 \bmod 21 = 16$	$\rightarrow$	$a = 16^2 \bmod 21 = 4$
2	1	$y = 16 \times 4 \bmod 21 = 1$	$\rightarrow$	$a = 4^2 \bmod 21 = 16$
3	0		$\rightarrow$	$a = 16^2 \bmod 21 = 4$
4	1	$y = 1 \times 4 \bmod 21 = 4$	$\rightarrow$	

Logarithm

- In cryptography we need to discuss modular logarithm

Algorithm 9.8 *Exhaustive search for modular logarithm*

```
Modular_Logarithm (a, y, n)
{
    for (x = 1 to n −1)                         // k is the number of bits in x
    {
        if (y ≡ a^x mod n) return x
    }
    return failure
}
```

Logarithm(cont.)

- Order of the Group.
- Example:
  – What is the order of group $G = \langle Z_{21}*, \times \rangle$?
    - $|G| = \phi\,(21) = \phi\,(3) \times \phi\,(7) = 2 \times 6 = 12$. There are 12 elements in this group: 1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, and 20. All are relatively prime with 21.

Logarithm(cont.)

- Order of an element
- Example:
  - Find the order of all elements in $G = \langle Z_{10}*, \times \rangle$.
  - This group has only $\phi(10) = 4$ elements: 1, 3, 7, 9.

a. $1^1 \equiv 1 \bmod (10) \rightarrow \text{ord}(1) = 1$.

b. $3^4 \equiv 1 \bmod (10) \rightarrow \text{ord}(3) = 4$.

c. $7^4 \equiv 1 \bmod (10) \rightarrow \text{ord}(7) = 4$.

d. $9^2 \equiv 1 \bmod (10) \rightarrow \text{ord}(9) = 2$.

Logarithm(cont.)

- Primitive roots
  - In the group G = <Z_n*, ×>, when the order of an element is the same as ϕ(n), that element is called the primitive root of the group.
  - Example
    - There are no primitive roots in G = <Z_8*, ×> because no element has the order equal to $\phi(8) = 4$.

Logarithm(cont.)

- Example
  - the result of $a^i \equiv x \pmod 7$ for the group $G = <Z_7*, \times>$. In this group, $\phi(7) = 6$.

Table 9.5 *Example 9.50*

		$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
	$a = 1$	$\boxed{x: 1}$	$x: 1$	$x: 1$	$x: 1$	$x: 1$	$x: 1$
	$a = 2$	$x: 2$	$x: 4$	$\boxed{x: 1}$	$x: 2$	$x: 4$	$x: 1$
Primitive root →	$a = 3$	$x: 3$	$x: 2$	$x: 6$	$x: 4$	$x: 5$	$\boxed{x: 1}$
	$a = 4$	$x: 4$	$x: 2$	$\boxed{x: 1}$	$x: 4$	$x: 2$	$x: 1$
Primitive root →	$a = 5$	$x: 5$	$x: 4$	$x: 6$	$x: 2$	$x: 3$	$\boxed{x: 1}$
	$a = 6$	$x: 6$	$\boxed{x: 1}$	$x: 6$	$x: 1$	$x: 6$	$x: 1$

Logarithm(cont.)

The group $G = \langle Z_n^*, \times\rangle$ has primitive roots only if n is 2, 4, p^t, or $2p^t$.

If the group $G = \langle Z_n^*, \times\rangle$ has any primitive root, the number of primitive roots is $\phi(\phi(n))$.

The group $G = \langle Z_n^*, \times\rangle$ is a cyclic group if it has primitive roots.
The group $G = \langle Z_p^*, \times\rangle$ is always cyclic.

Logarithm(cont.)

- The idea of Discrete Logarithm
- Properties of $G = \langle Z_p{}^*, \times \rangle$:
  1. Its elements include all integers from 1 to $p - 1$.
  2. It always has primitive roots.
  3. It is cyclic. The elements can be created using g^x where x is an integer from 1 to $\phi(n) = p - 1$.
  4. The primitive roots can be thought as the base of logarithm.

Logarithm(cont.)

- Solution to Modular Logarithm Using Discrete Logs
- Tabulation of Discrete Logarithms

Table 9.6 *Discrete logarithm for* $\mathbf{G} = \langle \mathbf{Z}_7{}^*, \times \rangle$

y	*1*	*2*	*3*	*4*	*5*	*6*
$x = L_3\, y$	6	2	1	4	5	3
$x = L_5\, y$	6	4	5	2	1	3

Logarithm(cont.)

- Find x in each of the following cases:

  a. $4 \equiv 3^x \pmod 7$

  b. $6 \equiv 5^x \pmod 7$

- Solution
  - Use the tabulation of the discrete logarithm

    a. $4 \equiv 3^x \bmod 7 \rightarrow x = L_3 4 \bmod 7 = 4 \bmod 7$

    b. $6 \equiv 5^x \bmod 7 \rightarrow x = L_5 6 \bmod 7 = 3 \bmod 7$

Logarithm(cont.)

Using Properties of Discrete Logarithms

Table 9.7 *Comparison of traditional and discrete logarithms*

Traditional Logarithm	*Discrete Logarithms*
$\log_a 1 = 0$	$L_g 1 \equiv 0 \pmod{\phi(n)}$
$\log_a (x \times y) = \log_a x + \log_a y$	$L_g(x \times y) \equiv (L_g x + L_g y) \pmod{\phi(n)}$
$\log_a x^k = k \times \log_a x$	$L_g x^k \equiv k \times L_g x \pmod{\phi(n)}$

The discrete logarithm problem has the same complexity as the factorization problem.