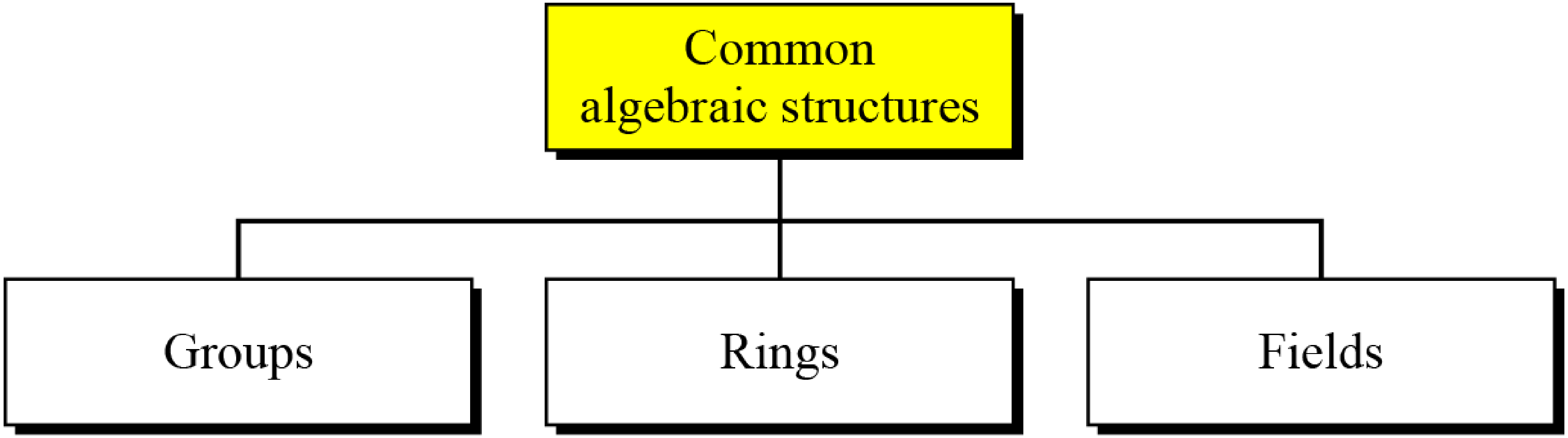
MATHEMATICS OF CRYPTOGRAPHY PART II ALGEBRAIC STRUCTURES

ALGEBRAIC STRUCTURES

- Cryptography requires sets of integers and specific operations that are defined for those sets.
- The combination of the set and the operations that are applied to the elements of the set is called an algebraic structure.
- Three common algebraic structures: groups, rings, and fields.

ALGEBRAIC STRUCTURES(cont.)



Common algebraic structure

Groups

- A group (G) is a set of elements with a binary operation (•) that satisfies four properties (or axioms).
  - Closure
  - Associativity
  - Existence of identity
  - Existence of inverse

Groups(cont.)

- Closure
  - If a and b are elements of G, then c = a•b is also an element of G.
- Associativity
  - If a, b and c are elements of G, then (a•b) •c=a•(b•c)
- Existence of identity
  - For all a in G, there exist an element e, called the identity element, such that e•a=a•e=a
- Existence of inverse
  - For each a in G, there exists an element a’, called the inverse of a, such that a•a’=a’•a=e

Groups(cont.)

- A Commutative group (Abelian group) is group in which the operator satisfies four properties plus an extra property that is commutativity.
  - For all a and b in G, we have a • b = b • a

Groups(cont.)

- Application
  - Although a group involves a single operation, the properties imposed on the operation allow the use of a pair of operations!!!!
  - How???

Groups(cont.)

- Example

The set of residue integers with the addition operator,

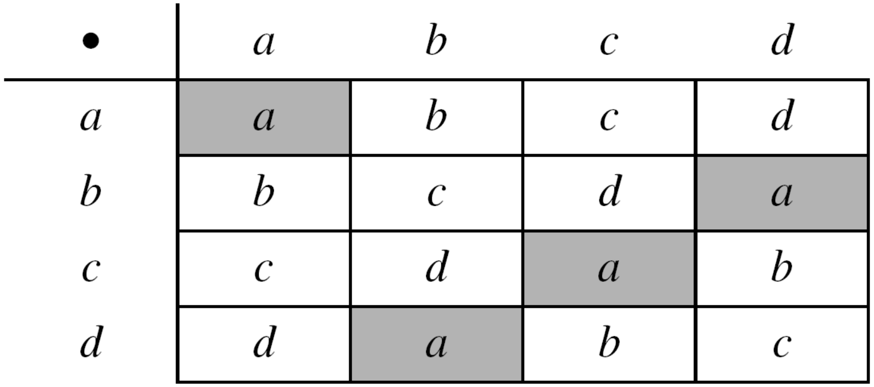
$$G = < Z_n , + >,$$

is a commutative group.

Check the properties…..

Groups(cont.)

- Example:
  - The set Zn* with the multiplication operator, G = <Zn*, ×>, is also an abelian group.
- Example:
  - Let us define a set G = < {a, b, c, d}, •> and the operation as shown in Table.



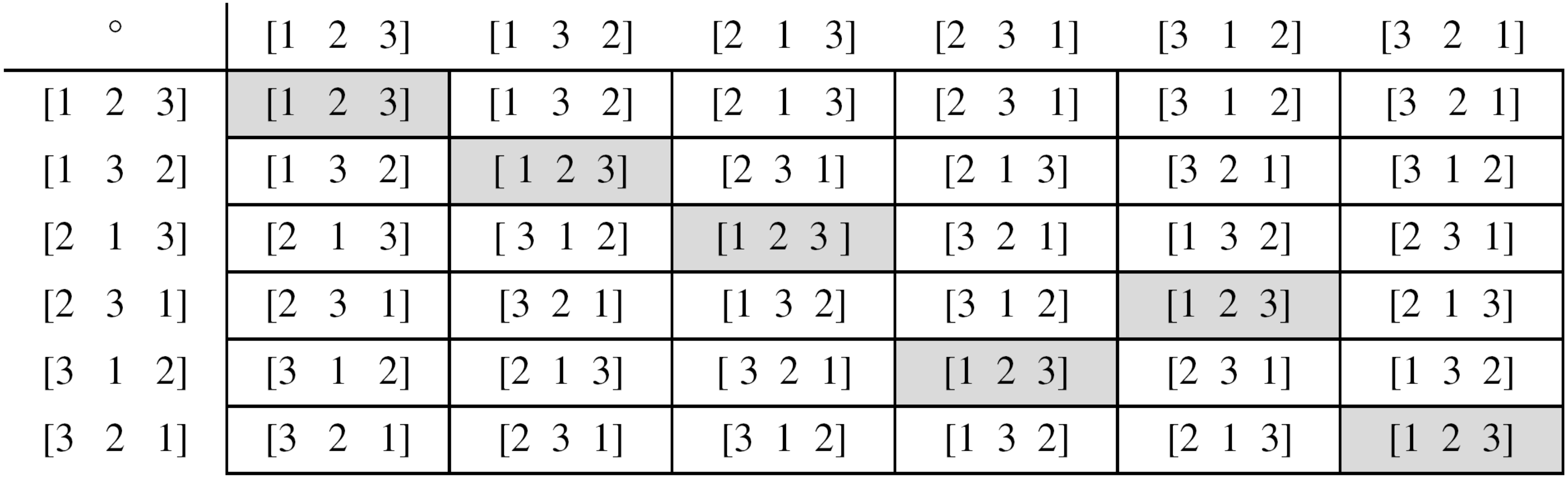
•	*a*	*b*	*c*	*d*
a	*a*	*b*	*c*	*d*
b	*b*	*c*	*d*	*a*
c	*c*	*d*	*a*	*b*
d	*d*	*a*	*b*	*c*

Groups(cont.)

- Example:
  - A very interesting group is the permutation group.
  - The set is the set of all permutations, and the operation is composition: applying one permutation after another.
  - Check for properties….
    - Is the group abelian????

Groups(cont.)

- Example(cont.):



∘	[1 2 3]	[1 3 2]	[2 1 3]	[2 3 1]	[3 1 2]	[3 2 1]
[1 2 3]	[1 2 3]	[1 3 2]	[2 1 3]	[2 3 1]	[3 1 2]	[3 2 1]
[1 3 2]	[1 3 2]	[1 2 3]	[2 3 1]	[2 1 3]	[3 2 1]	[3 1 2]
[2 1 3]	[2 1 3]	[3 1 2]	[1 2 3]	[3 2 1]	[1 3 2]	[2 3 1]
[2 3 1]	[2 3 1]	[3 2 1]	[1 3 2]	[3 1 2]	[1 2 3]	[2 1 3]
[3 1 2]	[3 1 2]	[2 1 3]	[3 2 1]	[1 2 3]	[2 3 1]	[1 3 2]
[3 2 1]	[3 2 1]	[2 3 1]	[3 1 2]	[1 3 2]	[2 1 3]	[1 2 3]

Operation table for permutation group

Groups(cont.)

- In the previous example, we showed that a set of permutations with the composition operation is a group.
- This implies that using two permutations one after another cannot strengthen the security of a cipher.
- Because we can always find a permutation that can do the same job because of the closure property.

Groups(cont.)

- Finite Group
  - If the set has a finite number of elements; otherwise, it is an infinite group.
- Order of a Group |G|
  - The number of elements in the group.
  - If the group is finite, its order is finite
- Subgroups
  - A subset H of a group G is a subgroup of G if H itself is a group with respect to the operation on G

Groups(cont.)

- Subgroups(cont.)
  - If G=<S, •> is a group, H=<T, •> is a group under the same operation, and T is a nonempty subset of S, then H is a subgroup of G
    - If a and b are members of both groups, then c=a•b is also member of both groups
    - The group share the same identity element
    - If a is a member of both groups, the inverse of a is also a member of both groups
    - The group made of the identity element of G, H=<{e}, •>, is a subgroup of G
    - Each group is a subgroup of itself

Groups(cont.)

- Exercise:
  - Is the group $H = <Z_{10}, +>$ a subgroup of the group $G = <Z_{12}, +>$?

Groups(cont.)

- Exercise:
  - Is the group H = <Z_{10}, +> a subgroup of the group G = <Z_{12}, +>?
- Solution:
  - The answer is no. Although H is a subset of G, the operations defined for these two groups are different. The operation in H is addition modulo 10; the operation in G is addition modulo 12.

Groups(cont.)

- Cyclic subgroups
  - If a subgroup of a group can be generated using the power of an element, the subgroup is called the cyclic subgroup.

$$\boxed{a^n \rightarrow a \bullet a \bullet \ldots \bullet a \quad (n \text{ times})}$$

Groups(cont.)

- Four cyclic subgroups can be made from the group G = <Z_6, +>.
- They are H_1 = <{0}, +>, H_2 = <{0, 2, 4}, +>, H_3 = <{0, 3}, +>, and H_4 = G.

$$0^0 \bmod 6 = 0$$

$$\begin{aligned}
1^0 &\bmod 6 = 0\\
1^1 &\bmod 6 = 1\\
1^2 &\bmod 6 = (1 + 1) \bmod 6 = 2\\
1^3 &\bmod 6 = (1 + 1 +1) \bmod 6 = 3\\
1^4 &\bmod 6 = (1 + 1 + 1 + 1) \bmod 6 = 4\\
1^5 &\bmod 6 = (1 + 1 + 1 + 1+ 1) \bmod 6 = 5
\end{aligned}$$

$$\begin{aligned}
2^0 &\bmod 6 = 0\\
2^1 &\bmod 6 = 2\\
2^2 &\bmod 6 = (2 + 2) \bmod 6 = 4
\end{aligned}$$

$$\begin{aligned}
3^0 &\bmod 6 = 0\\
3^1 &\bmod 6 = 3
\end{aligned}$$

$$\begin{aligned}
4^0 &\bmod 6 = 0\\
4^1 &\bmod 6 = 4\\
4^2 &\bmod 6 = (4 + 4) \bmod 6 = 2
\end{aligned}$$

$$\begin{aligned}
5^0 &\bmod 6 = 0\\
5^1 &\bmod 6 = 5\\
5^2 &\bmod 6 = 4\\
5^3 &\bmod 6 = 3\\
5^4 &\bmod 6 = 2\\
5^5 &\bmod 6 = 1
\end{aligned}$$

Groups(cont.)

- Exercise:
  - Find out the cyclic subgroups for group $G = \langle Z_{10}*, \times \rangle$.

Groups(cont.)

- Three cyclic subgroups can be made from the group $G = \langle Z_{10}*, \times \rangle$. G has only four elements: 1, 3, 7, and 9. The cyclic subgroups are $H_1 = \langle \{1\}, \times \rangle$, $H_2 = \langle \{1, 9\}, \times \rangle$, and $H_3 = G$.

$$1^0 \bmod 10 = 1$$

$$\begin{aligned} 3^0 \bmod 10 &= 1 \\ 3^1 \bmod 10 &= 3 \\ 3^2 \bmod 10 &= 9 \\ 3^3 \bmod 10 &= 7 \end{aligned}$$

$$\begin{aligned} 7^0 \bmod 10 &= 1 \\ 7^1 \bmod 10 &= 7 \\ 7^2 \bmod 10 &= 9 \\ 7^3 \bmod 10 &= 3 \end{aligned}$$

$$\begin{aligned} 9^0 \bmod 10 &= 1 \\ 9^1 \bmod 10 &= 9 \end{aligned}$$

Groups(cont.)

- Cyclic group
  - A cyclic group is a group that is its own cyclic subgroup.

$$\boxed{\{e, g, g^2, \ldots, g^{n-1}\}, \text{ where } g^n = e}$$

Groups(cont.)

- Cyclic group(cont.)
- Example:
  - Three cyclic subgroups can be made from the group G = < Z10∗, ×>.
  - The cyclic subgroups are H1 = <{1}, ×>, H2 = <{1, 9}, ×>, and H3 = G.
  - The group G = $<Z_{10}*, \times>$ is a cyclic group with two generators, $g = 3$ and $g = 7$.
  - The group G = $<Z_6, +>$ is a cyclic group with two generators, $g = 1$ and $g = 5$.

Groups(cont.)

- Lagrange's Theorem
  - Assume that G is a group, and H is a subgroup of G. If the order of G and H are |G| and |H|, respectively, then, based on this theorem, |H| divides |G|.
- Order of an Element
  - The order of an element is the order of the cyclic group it generates.

Groups(cont.)

- Example:
  - In the group G = <Z_6, +>, the orders of the elements are:

    ord(0) = 1, ord(1) = 6, ord(2) = 3, ord(3) = 2, ord(4) = 3, ord(5) = 6.

  - In the group G = <Z_{10}*, ×>, the orders of the elements are: ord(1) = 1, ord(3) = 4, ord(7) = 4, ord(9) = 2.

Ring

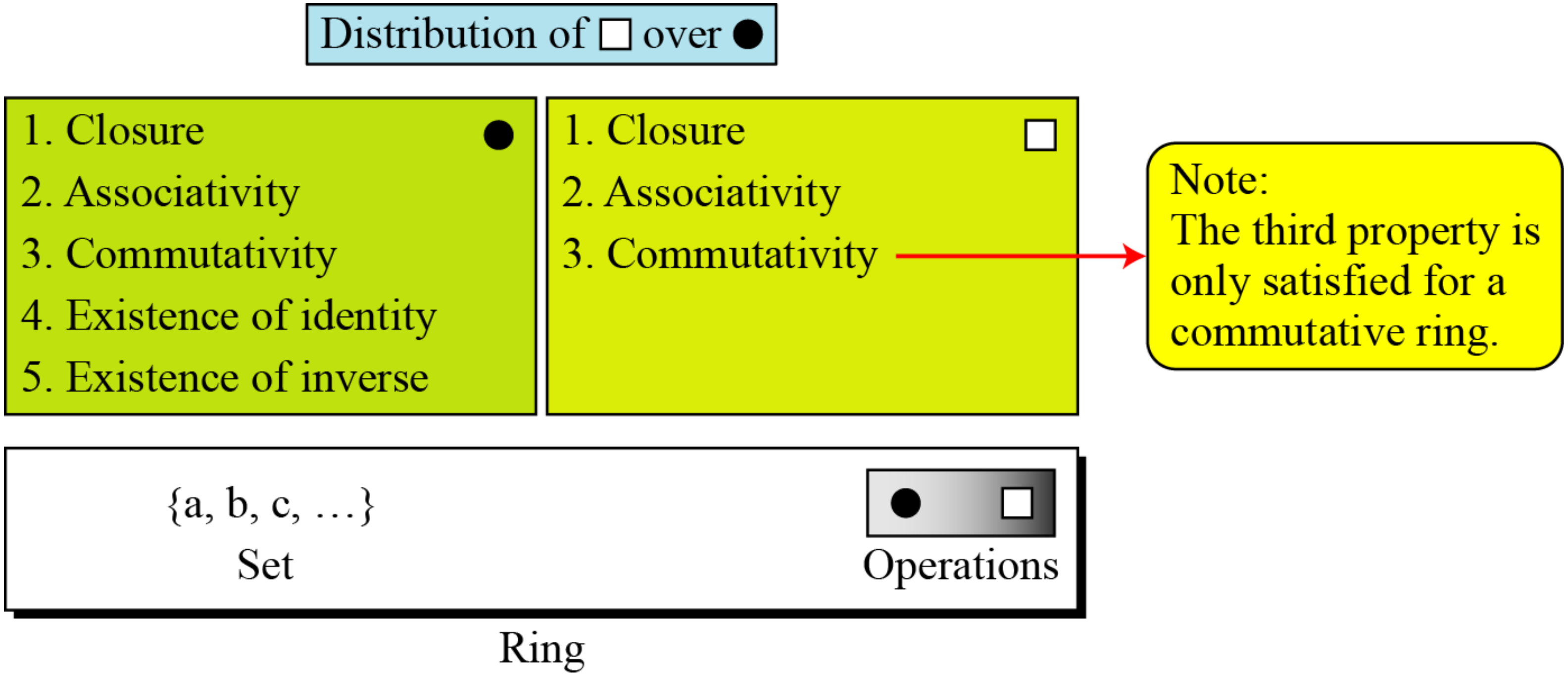
- A ring, R = <{…}, •,■ >, is an algebraic structure with two operations.
- First operation must satisfy all five properties
- Second operation must satisfy only the first two
- In addition, second operation must be distributed over first
  - i.e. for all a, b, and c elements of R, we have,

    $a \blacksquare (b \bullet c) = (a \blacksquare b) \bullet (a \blacksquare c)$ and

    $(a \bullet b) \blacksquare c = (a \blacksquare c) \bullet (a \blacksquare c)$

Ring(cont.)

- Commutative Ring



Distribution of □ over ●

1. Closure ●
2. Associativity
3. Commutativity
4. Existence of identity
5. Existence of inverse

1. Closure □
2. Associativity
3. Commutativity

Note:
The third property is only satisfied for a commutative ring.

{a, b, c, …}
Set

● □
Operations

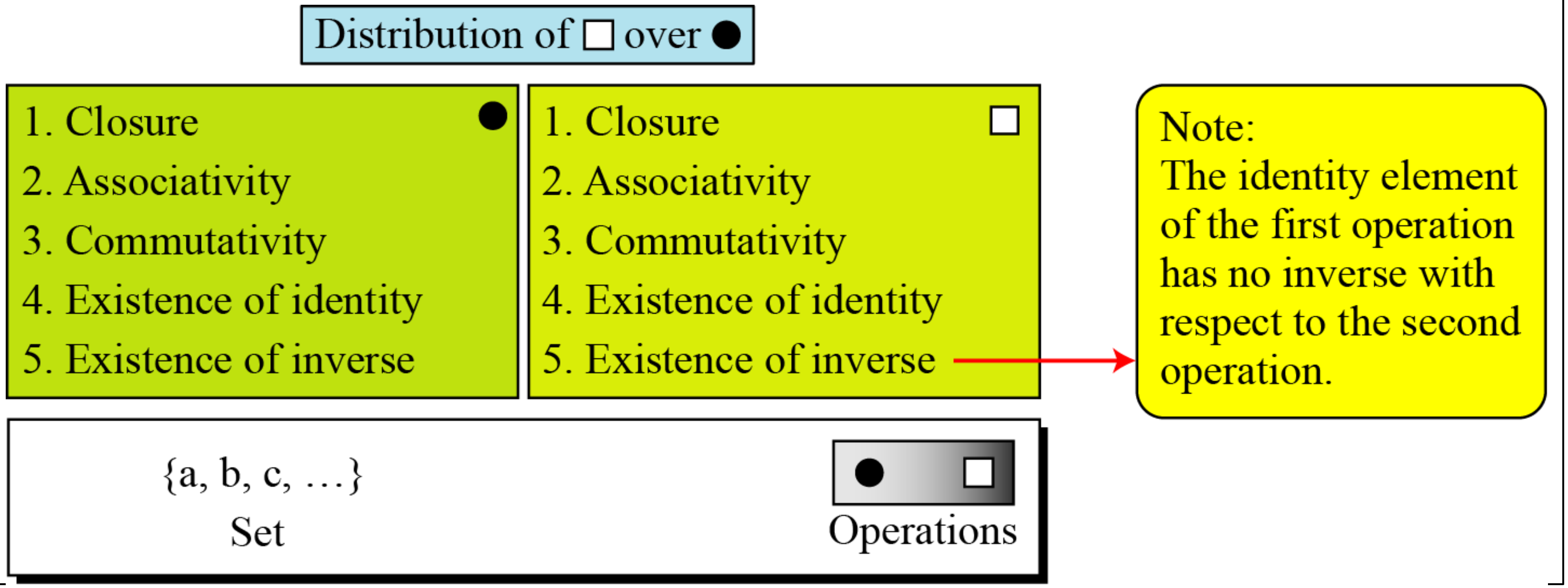
Ring

Ring(cont.)

- The set Z with two operations, addition and multiplication, is a commutative ring.
- We show it by R = <Z, +, ×>.
- Addition satisfies all of the five properties; multiplication satisfies only three properties.

Field

- A field, denoted by F = <{…}, •,■ > is a commutative ring in which the second operation satisfies all five properties defined for the first operation except that the identity of the first operation has no inverse.



Field

Field(cont.)

- Finite Fields
  - Galois showed that for a field to be finite, the number of elements should be p^n, where p is a prime and n is a positive integer.

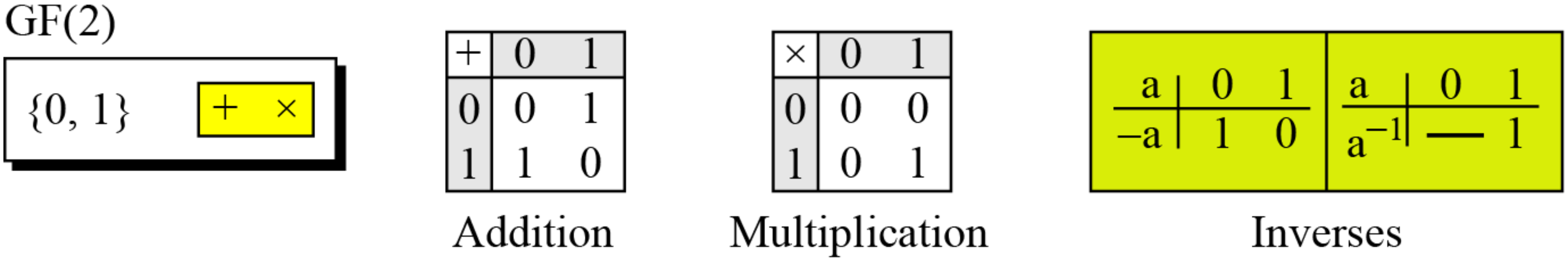
A Galois field, GF(p^n), is a finite field with p^n elements.

Field(cont.)

- GF(p) Fields
  - When $n = 1$, we have GF(p) field.
  - This field can be the set Z_p, {0, 1, …, p − 1}, with two arithmetic operations.

Field(cont.)

- A very common field in this category is GF(2) with the set {0, 1} and two operations, addition and multiplication.



GF(2)

{0, 1} + ×

+	0	1
0	0	1
1	1	0

Addition

×	0	1
0	0	0
1	0	1

Multiplication

a	0	1
−a	1	0

a	0	1
a^{-1}	—	1

Inverses

GF(2) field

Field(cont.)

- We can define GF(5) on the set Z_5 (5 is a prime) with addition and multiplication operators.

Field(cont.)

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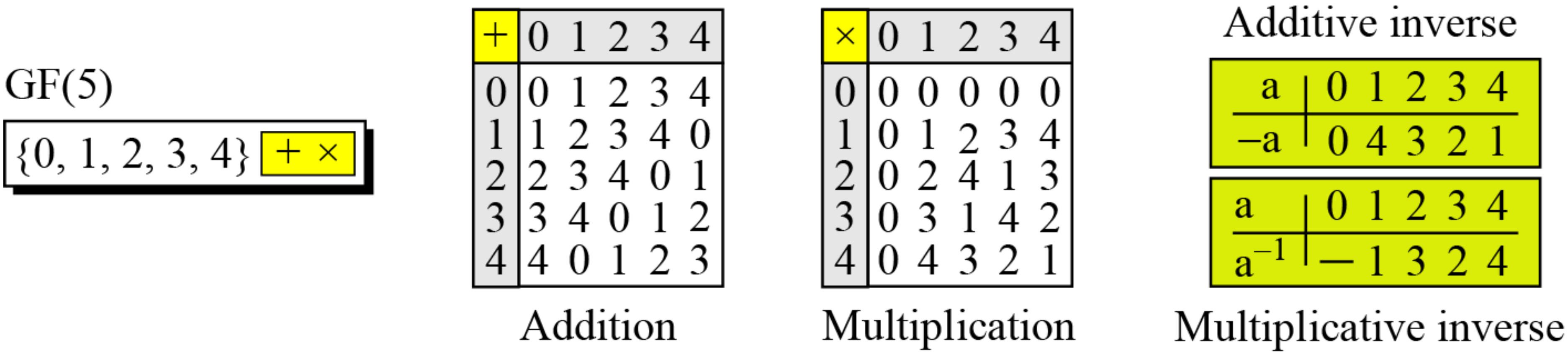


GF(5)

{0, 1, 2, 3, 4} + ×

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Addition

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Multiplication

Additive inverse

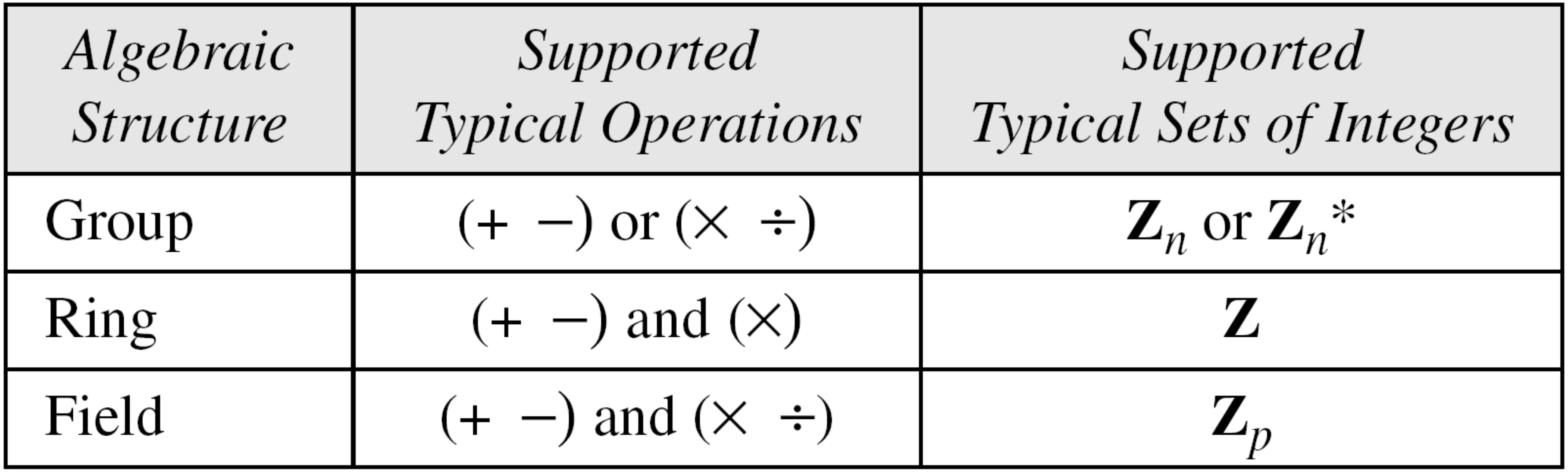
a	0	1	2	3	4
−a	0	4	3	2	1

a	0	1	2	3	4
a^{-1}	—	1	3	2	4

Multiplicative inverse

GF(5) field

- Summary:



Algebraic Structure	*Supported Typical Operations*	*Supported Typical Sets of Integers*
Group	(+ −) or (× ÷)	$\mathbf{Z}_n$ or $\mathbf{Z}_n{}^*$
Ring	(+ −) and (×)	$\mathbf{Z}$
Field	(+ −) and (× ÷)	$\mathbf{Z}_p$

GF(2^n) FIELDS

- In cryptography, we often need to use four operations(addition, subtraction, multiplication and division).
- In other words, we need to use fields.
- However, when we work with computers, the positive integers are stored in the computers as n-bit words in which n is usually 8,16,32 and so on.
- Range of integers is 0 to $2^n – 1$
- Hence modulus is ?????
- What if we want to use field????

GF(2^n) FIELDS (cont.)

- Solution 1
  - Use GF(p), with the set Zp, where p is the largest prime number less than 2^n
  - But the problem ???
- Solution 2
  - Use GF(2^n)
  - Use a set of 2^n words
  - The elements in this set are n-bit words
  - E.g. for n=3, the set is {000,001,010,011,100,101,110,111}

GF(2^n) FIELDS (cont.)

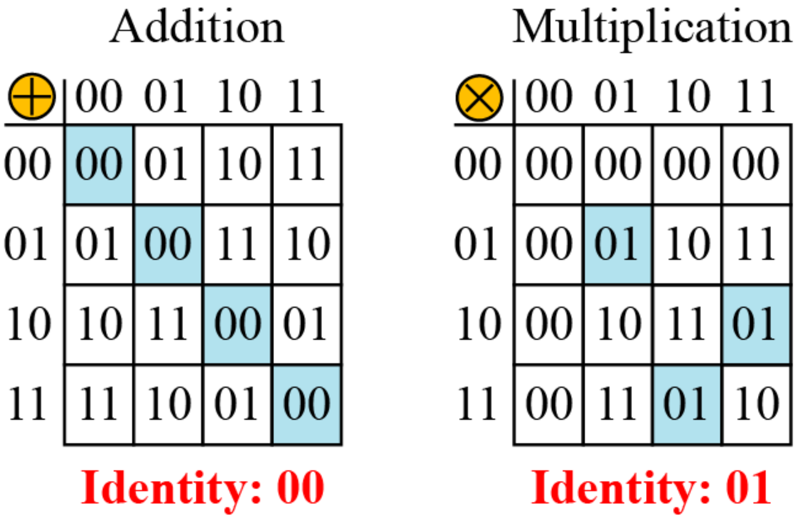
- Solution 2
  - But the problem???

GF(2^n) FIELDS (cont.)

- Solution 2
  - But the problem???
  - 2^n is not prime
  - Need to define operations on the set of elements in GF(2^n)

GF(2^n) FIELDS (cont.)

- Let us define a GF(2^2) field in which the set has four 2-bit words: {00, 01, 10, 11}.
- We can redefine addition and multiplication for this field in such a way that all properties of these operations are satisfied.



Addition

⊕	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

Identity: 00

Multiplication

⊗	00	01	10	11
00	00	00	00	00
01	00	01	10	11
10	00	10	11	01
11	00	11	01	10

Identity: 01

An example of GF(2^2) field

Polynomials

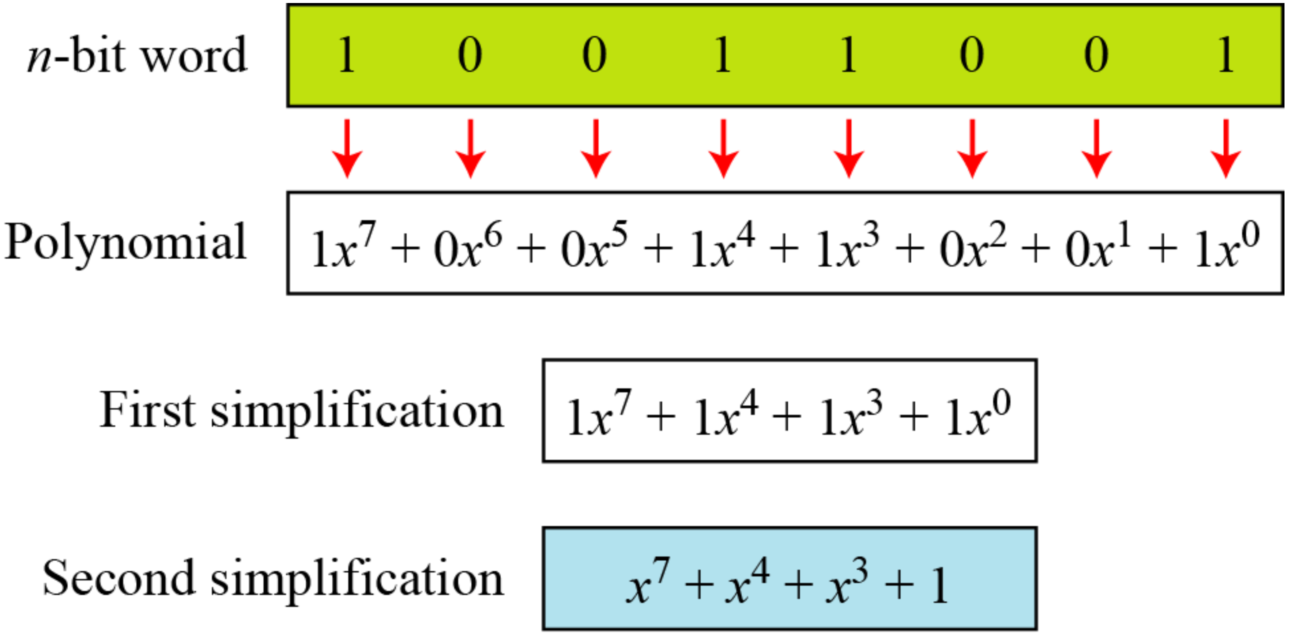
- A polynomial of degree *n* – 1 is an expression of the form

$$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x^1 + a_0x^0$$

- where x^i is called the ith term and a_i is called coefficient of the *i*th term.

Polynomials (cont.)

- We can represent the 8-bit word (10011001) using a polynomial.



n-bit word: 1 0 0 1 1 0 0 1

Polynomial: $1x^7 + 0x^6 + 0x^5 + 1x^4 + 1x^3 + 0x^2 + 0x^1 + 1x^0$

First simplification: $1x^7 + 1x^4 + 1x^3 + 1x^0$

Second simplification: $x^7 + x^4 + x^3 + 1$

Polynomials (cont.)

- Find the 8-bit word related to the polynomial $x^5 + x^2 + x$, we first supply the omitted terms.
- Since n = 8, it means the polynomial is of degree 7. The expanded polynomial is,

$$0x^7 + 0x^6 + 1x^5 + 0x^4 + 0x^3 + 1x^2 + 1x^1 + 0x^0$$

- This is related to the 8-bit word 00100110.

Polynomials (cont.)

- Operations on polynomials
  - Actually involves two operations
    - Operation on coefficients and operation on polynomials
  - Hence, need to define two fields
  - What for coefficient??
  - What for polynomials???

Polynomials (cont.)

- Operations on polynomials
  - Actually involves two operations
    - Operation on coefficients and operation on polynomials
  - Hence, need to define two fields
  - What for coefficient??
  - What for polynomials???

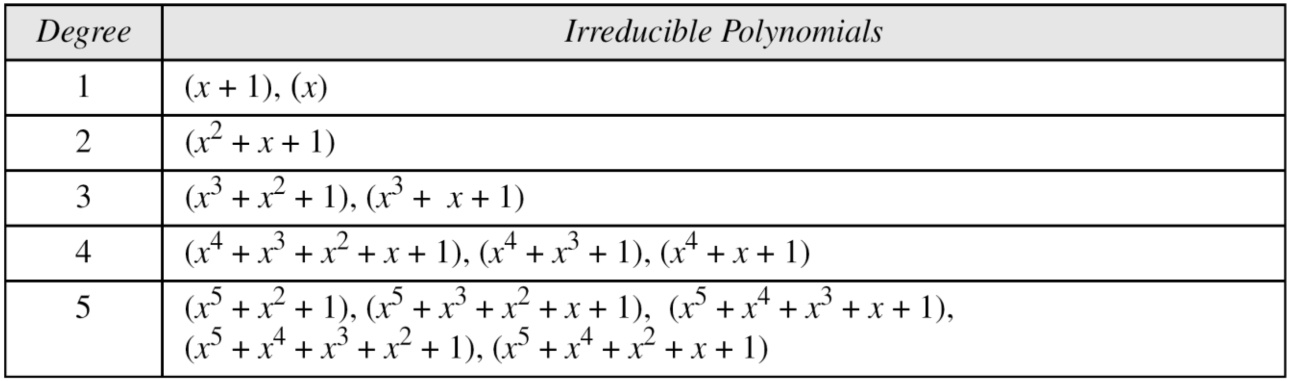
  - GF(2) and GF(2^n) is the answer….

Polynomials (cont.)

- Modulus
  - For the sets of polynomials in GF(2^n), a group of polynomials of degree n is defined as the modulus.
  - Such polynomials are referred to as irreducible polynomials.

Polynomials (cont.)

- irreducible polynomials.
  - No polynomial in the set can divide this polynomial
  - Can not be factored into a polynomial with degree of less than n



Degree	*Irreducible Polynomials*
1	$(x + 1), (x)$
2	$(x^2 + x + 1)$
3	$(x^3 + x^2 + 1), (x^3 + x + 1)$
4	$(x^4 + x^3 + x^2 + x + 1), (x^4 + x^3 + 1), (x^4 + x + 1)$
5	$(x^5 + x^2 + 1), (x^5 + x^3 + x^2 + x + 1), (x^5 + x^4 + x^3 + x + 1),$ $(x^5 + x^4 + x^3 + x^2 + 1), (x^5 + x^4 + x^2 + x + 1)$

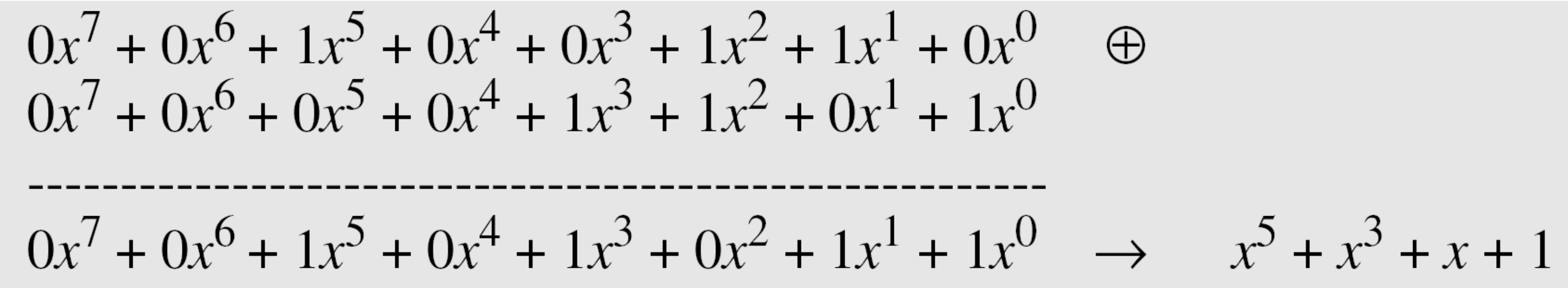
Polynomials (cont.)

- Polynomial addition

Addition and subtraction operations on polynomials are the same operation.

Polynomials (cont.)

- Example
- Let us do $(x^5 + x^2 + x) \oplus (x^3 + x^2 + 1)$ in $GF(2^8)$. We use the symbol $\oplus$ to show that we mean polynomial addition. The following shows the procedure:



$$\begin{array}{lcl}
0x^7 + 0x^6 + 1x^5 + 0x^4 + 0x^3 + 1x^2 + 1x^1 + 0x^0 & \oplus & \\
0x^7 + 0x^6 + 0x^5 + 0x^4 + 1x^3 + 1x^2 + 0x^1 + 1x^0 & & \\
\hdashline
0x^7 + 0x^6 + 1x^5 + 0x^4 + 1x^3 + 0x^2 + 1x^1 + 1x^0 & \rightarrow & x^5 + x^3 + x + 1
\end{array}$$

Polynomials (cont.)

- Short cut method
  - Addition in GF(2) means the exclusive-or (XOR) operation.
  - So we can exclusive-or the two words, bits by bits, to get the result.
  - In the previous example, $x^5 + x^2 + x$ is 00100110 and $x^3 + x^2 + 1$ is 00001101.
  - The result is 00101011 or in polynomial notation $x^5 + x^3 + x + 1$.

Polynomials (cont.)

- Multiplication
  - The coefficient multiplication is done in GF(2).
  - The multiplying x^i by x^j results in x^{i+j}.
  - The multiplication may create terms with degree more than $n - 1$, which means the result needs to be reduced using a modulus polynomial.

Polynomials (cont.)

- Example
  - Find the result of $(x^5 + x^2 + x) \otimes (x^7 + x^4 + x^3 + x^2 + x)$ in GF(2^8) with irreducible polynomial $(x^8 + x^4 + x^3 + x + 1)$.

$$P_1 \otimes P_2 = x^5(x^7 + x^4 + x^3 + x^2 + x) + x^2(x^7 + x^4 + x^3 + x^2 + x) + x(x^7 + x^4 + x^3 + x^2 + x)$$

$$P_1 \otimes P_2 = x^{12} + x^9 + x^8 + x^7 + x^6 + x^9 + x^6 + x^5 + x^4 + x^3 + x^8 + x^5 + x^4 + x^3 + x^2$$

$$P_1 \otimes P_2 = (x^{12} + x^7 + x^2) \bmod (x^8 + x^4 + x^3 + x + 1) = x^5 + x^3 + x^2 + x + 1$$

  - To find the final result, divide the polynomial of degree 12 by the polynomial of degree 8 (the modulus) and keep only the remainder.

Polynomials (cont.)

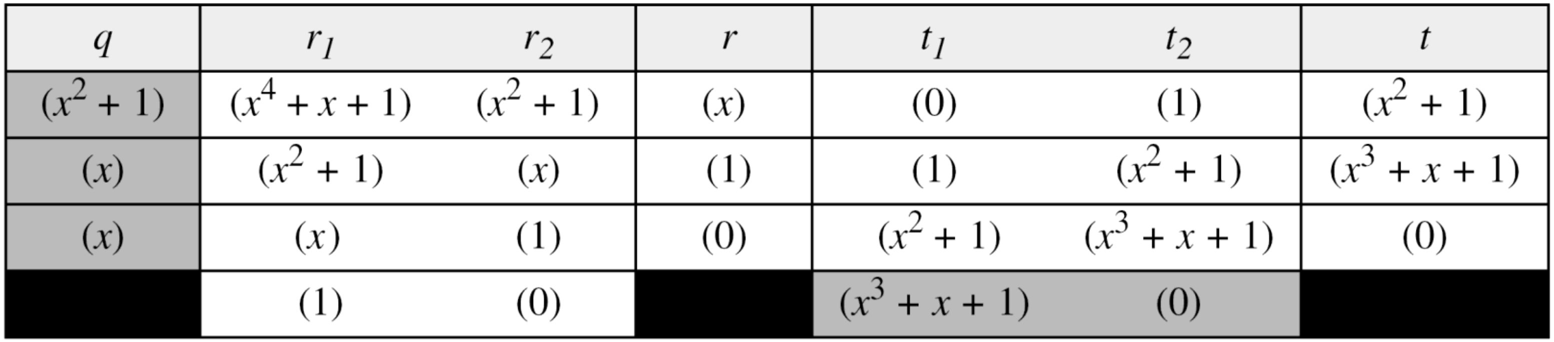
- Polynomial division with coefficients in GF(2)

$$\begin{array}{r|l}
 & x^4 + 1 \\
\cline{2-2}
x^8 + x^4 + x^3 + x + 1 & x^{12} + x^7 + x^2 \\
 & x^{12} + x^8 + x^7 + x^5 + x^4 \\
\cline{2-2}
 & x^8 + x^5 + x^4 + x^2 \\
 & x^8 + x^4 + x^3 + x + 1 \\
\cline{2-2}
\end{array}$$

$$\text{Remainder} \quad \boxed{x^5 + x^3 + x^2 + x + 1}$$

Polynomials (cont.)

- Example:
  - In GF (2^4), find the inverse of $(x^2 + 1)$ modulo $(x^4 + x + 1)$.
- Solution
  - The answer is $(x^3 + x + 1)$



q	r_1	r_2	r	t_1	t_2	t
$(x^2 + 1)$	$(x^4 + x + 1)$	$(x^2 + 1)$	(x)	(0)	(1)	$(x^2 + 1)$
(x)	$(x^2 + 1)$	(x)	(1)	(1)	$(x^2 + 1)$	$(x^3 + x + 1)$
(x)	(x)	(1)	(0)	$(x^2 + 1)$	$(x^3 + x + 1)$	(0)
	(1)	(0)		$(x^3 + x + 1)$	(0)	

Polynomials (cont.)

- Example:
  - In GF(2^8), find the inverse of (x^5) modulo ($x^8 + x^4 + x^3 + x + 1$).

Polynomials (cont.)

- Example:
  - In GF(2^8), find the inverse of (x^5) modulo ($x^8 + x^4 + x^3 + x + 1$).

- Solution

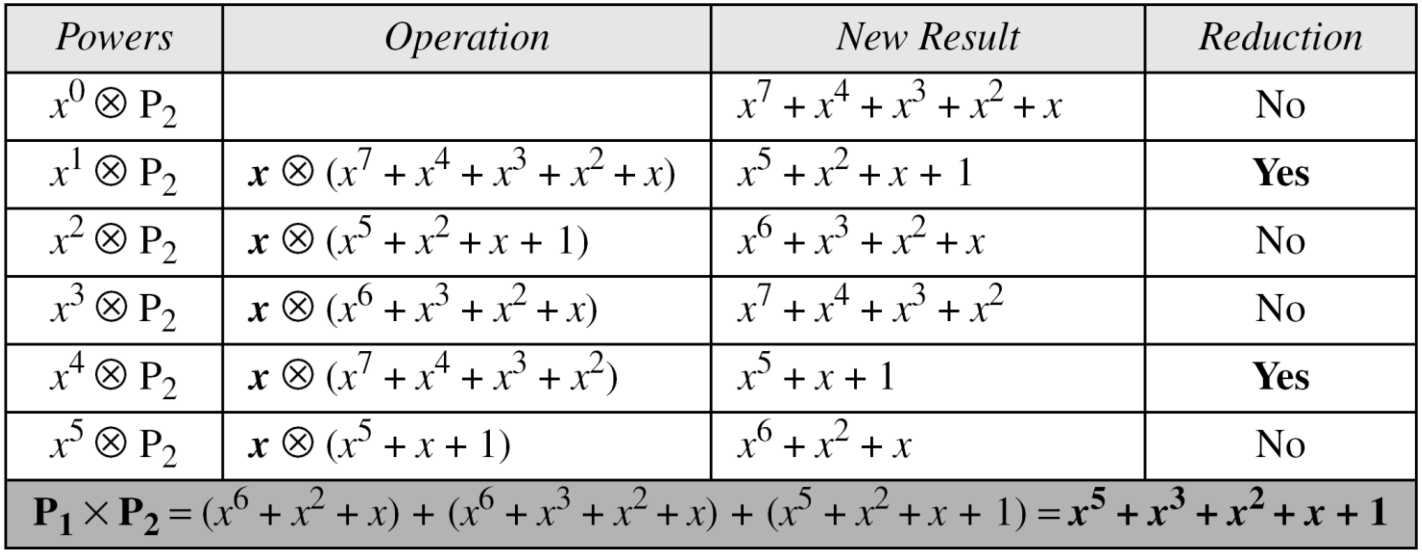
q	r_1	r_2	r	t_1	t_2	t
(x^3)	$(x^8 + x^4 + x^3 + x + 1)$	(x^5)	$(x^4 + x^3 + x + 1)$	(0)	(1)	(x^3)
$(x + 1)$	(x^5)	$(x^4 + x^3 + x + 1)$	$(x^3 + x^2 + 1)$	(1)	(x^3)	$(x^4 + x^3 + 1)$
(x)	$(x^4 + x^3 + x + 1)$	$(x^3 + x^2 + 1)$	(1)	(x^3)	$(x^4 + x^3 + 1)$	$(x^5 + x^4 + x^3 + x)$
$(x^3 + x^2 + 1)$	$(x^3 + x^2 + 1)$	(1)	(0)	$(x^4 + x^3 + 1)$	$(x^5 + x^4 + x^3 + x)$	(0)
	(1)	(0)		$(x^5 + x^4 + x^3 + x)$	(0)	

Polynomials (cont.)

- A better algorithm: Obtain the result by repeatedly multiplying a reduced polynomial by x.
- Example:
  - Find the result of multiplying $P_1 = (x^5 + x^2 + x)$ by $P_2 = (x^7 + x^4 + x^3 + x^2 + x)$ in GF(2^8) with irreducible polynomial $(x^8 + x^4 + x^3 + x + 1)$

Polynomials (cont.)

- Solution:
  - We first find the partial result of multiplying x^0, x^1, x^2, x^3, x^4, and x^5 by P_2. Note that although only three terms are needed, the product of $x^m \otimes P_2$ for m from 0 to 5 because each calculation depends on the previous result.



Powers	*Operation*	*New Result*	*Reduction*
$x^0 \otimes P_2$		$x^7 + x^4 + x^3 + x^2 + x$	No
$x^1 \otimes P_2$	$\boldsymbol{x} \otimes (x^7 + x^4 + x^3 + x^2 + x)$	$x^5 + x^2 + x + 1$	**Yes**
$x^2 \otimes P_2$	$\boldsymbol{x} \otimes (x^5 + x^2 + x + 1)$	$x^6 + x^3 + x^2 + x$	No
$x^3 \otimes P_2$	$\boldsymbol{x} \otimes (x^6 + x^3 + x^2 + x)$	$x^7 + x^4 + x^3 + x^2$	No
$x^4 \otimes P_2$	$\boldsymbol{x} \otimes (x^7 + x^4 + x^3 + x^2)$	$x^5 + x + 1$	**Yes**
$x^5 \otimes P_2$	$\boldsymbol{x} \otimes (x^5 + x + 1)$	$x^6 + x^2 + x$	No
$\mathbf{P_1 \times P_2} = (x^6 + x^2 + x) + (x^6 + x^3 + x^2 + x) + (x^5 + x^2 + x + 1) = \boldsymbol{x^5 + x^3 + x^2 + x + 1}$			

Polynomials (cont.)

- Exercise:

Find the result of multiplying $P_1 = (x^3 + x^2 + x + 1)$ by $P_2 = (x^2 + 1)$ in GF(2^4) with irreducible polynomial ($x^4 + x^3 + 1$)

Polynomials (cont.)

- Exercise:

Find the result of multiplying (10101) by (10000) in GF(2^5) using ($x^5 + x^2 + 1$) as modulus.