LU Decomposition

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Problem

We have a matrix A, now we have decompose matrix A into two matrices L(Lower Triangular Matrix) and U(Upper Triangular Matrix) such that

$$A = L * U$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, U = \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}.$$

Motivation

So the first question come to mind that why we are
doing this?
so straight forward answer would be that it will simplify things.
How?
Most of the time in mathematics modeling we came up with
system of linear equations in the form of

$$\mathbf{Ax = b}$$

so finding A^{-1} is quite difficult so we will use LU decomposition

How?

Example

Lets we have system of eqations

$$[A]\{x\} = \{b\}$$
$$[L][U]\{x\} = \{b\}$$
$$(\therefore [A] = [L][U])$$
$$\{y\} = [U]\{x\}$$
$$[L]\{y\} = \{b\}$$
$$\because [] = matrix, \{\} = vector$$

This will make our system so simple to solve...

Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}.$$

$$A = LU \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} * \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+i \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+i \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

Now compare the values and get the values of elements of L and U.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

The next step is to solve $[L]\{y\}=\{b\}$ for the vector $\{y\}$ that we consider

$$Ly = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} = b$$

which can be solved by forward substitution $\{y\} = [3\ 4\ \text{-}6]^T$ now that we have found y we finish the procedure by solving

$$Ux = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} = y$$

by using backward substitution we will get $\{x\}$.

Methods

There are many methods of LU decompositions like:

Gaussian Elimination
Doolittle's method
Crout's method
and many more...

so what are implementing here is: `Doolittle's method`

Algorithm

```
1.    procedure COL_LU (A)
2.    begin
3.        for k := 1 to n do
4.            for j := k to n do
5.                A[j, k]:= A[j, k]/A[k, k];
6.            endfor;
7.            for j := k + 1 to n do
8.                for i := k + 1 to n do
9.                    A[i, j] := A[i, j] - A[i, k] x A[k, j];
10.               endfor;
11.           endfor;
    /*
After this iteration, column A[k + 1 : n, k] is logically the kth
column of L and row A[k, k : n] is logically the kth row of U.
    */
12.       endfor;
13.   end COL_LU
```
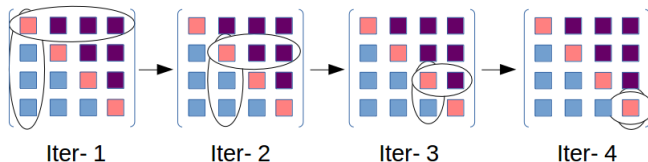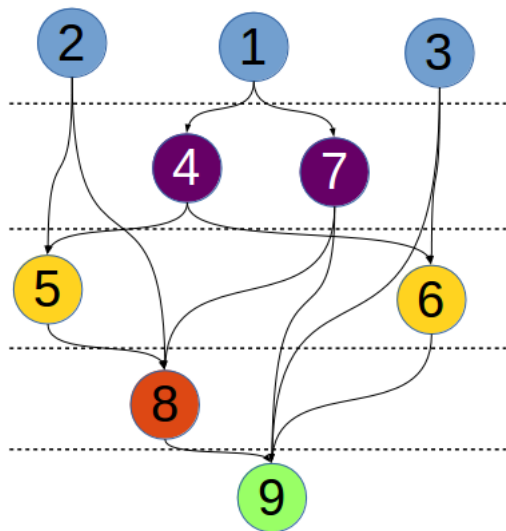

Iter- 1   Iter- 2   Iter- 3   Iter- 4
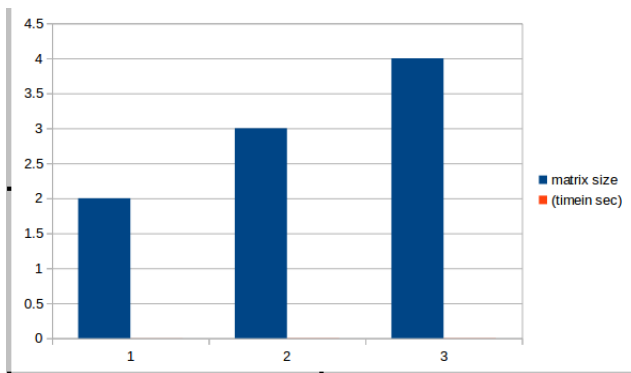
Task Generation and Dependency Graph

1. $L_{11} = a_{11}$
2. $L_{21} = a_{21}$
3. $L_{31} = a_{31}$
4. $U_{12} = \frac{a_{12}}{L_{11}}$
5. $L_{22} = a_{22} - L_{21} U_{12}$
6. $L_{32} = a_{32} - L_{31} U_{12}$
7. $U_{13} = \frac{a_{13}}{L_{11}}$
8. $U_{23} = \frac{a_{23} - L_{23} U_{13}}{L_{22}}$
9. $L_{33} = a_{33} - L_{31} U_{13} - L_{32} U_{23}$

Time Scalability

Time Scalability


```
==8863==
==8863== I   refs:      137,915
==8863== I1  misses:        930
==8863== LLi misses:        922
==8863== I1  miss rate:    0.67%
==8863== LLi miss rate:    0.66%
==8863==
==8863== D   refs:       53,622  (34,964 rd   + 18,658 wr)
==8863== D1  misses:      1,764  ( 1,255 rd   +    509 wr)
==8863== LLd misses:      1,544  ( 1,083 rd   +    461 wr)
==8863== D1  miss rate:    3.2% (   3.5%     +   2.7%  )
==8863== LLd miss rate:    2.8% (   3.0%     +   2.4%  )
==8863==
==8863== LL refs:         2,694  ( 2,185 rd   +    509 wr)
==8863== LL misses:       2,466  ( 2,005 rd   +    461 wr)
==8863== LL miss rate:     1.2% (   1.1%     +   2.4%  )
```

Complexity

Time $= O(n^3)$
Space $=$ Two nxn matrices space

What is Next ??

Make Algorithm Parallel

Thank You![1]

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[1]Resources: wikipedia.orgadf
http://www.engr.colostate.edu/ thompson/hPage/CourseMat/Tutorials/CompMethods/doolittle.pdf