LU Decomposition

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Problem

We have a matrix A, now we have decompose matrix A into two matrices L(Lower Triangular Matrix) and U(Upper Triangular Matrix) such that

$$\mathbf{A = L * U}$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, U = \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}.$$

Motivation

`So the first question come to mind that why we are doing this?`

so straight forward answer would be that it will simplify things.
How?
Most of the time in mathematics modeling we came up with system of linear equations in the form of

$$\mathbf{Ax} = \mathbf{b}$$

so finding A^{-1} is quite difficult so we will use LU decomposition

How?

Example

Lets we have system of eqations

$$
\begin{aligned}
[A]\{x\} &= \{b\} \\
[L][U]\{x\} &= \{b\} \\
(\therefore [A] &= [L][U]) \\
\{y\} &= [U]\{x\} \\
[L]\{y\} &= \{b\} \\
\because [] = matrix, \{\} &= vector
\end{aligned}
$$

This will make our system so simple to solve...

Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}.$$

$$A = LU \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} * \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+i \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+i \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

Now compare the values and get the values of elements of L and U.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

The next step is to solve [L]{y}={b} for the vector {y} that we consider

$$Ly = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} = b$$

which can be solved by forward substitution {y} = $[3\ 4\ \text{-}6]^T$ now that we have found y we finish the procedure by solving

$$Ux = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} = y$$

by using backward substitution we will get {x}.

Methods

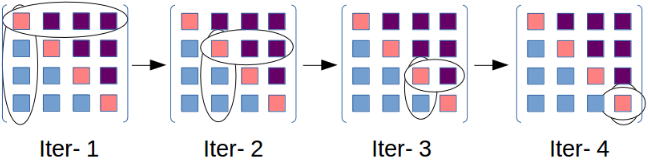
There are many methods of LU decompositions like:

Gaussian Elimination
Doolittle's method
Crout's method
and many more...

so what are implementing here is: `Doolittle's method`

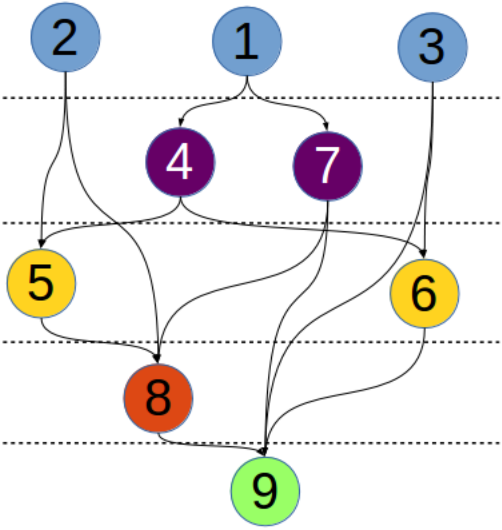
Algorithm

```
procedure COL_LU (A)
begin
    for k := 1 to n do
        for j := k to n do
            A[j, k]:= A[j, k]/A[k, k];
        endfor;
        for j := k + 1 to n do
            for i := k + 1 to n do
                A[i, j] := A[i, j] - A[i, k] x A[k, j];
            endfor;
        endfor;
  /*
After this iteration, column A[k + 1 : n, k] is logically the kth
column of L and row A[k, k : n] is logically the kth row of U.
  */
    endfor;
end COL_LU
```



Task Generation and Dependency Graph

1. $L_{11} = a_{11}$
2. $L_{21} = a_{21}$
3. $L_{31} = a_{31}$
4. $U_{12} = \frac{a_{12}}{L_{11}}$
5. $L_{22} = a_{22} - L_{21}U_{12}$
6. $L_{32} = a_{32} - L_{31}U_{12}$
7. $U_{13} = \frac{a_{13}}{L_{11}}$
8. $U_{23} = \frac{a_{23} - L_{23}U_{13}}{L_{22}}$
9. $L_{33} = a_{33} - L_{31}U_{13} - L_{32}U_{23}$



Complexity

Time = $O(n^3)$ Space = Two nxn matrices space

What is Next ??

Make Algorithm Parallel

Thank You![1]

[1] Resources: wikipedia.orgadf http://www.engr.colostate.edu/ thompson/hPage/CourseMat/Tutorials/CompMethods/doolittle.pdf