Generating Random WMC Instances

An Empirical Analysis with Varying Primal Treewidth

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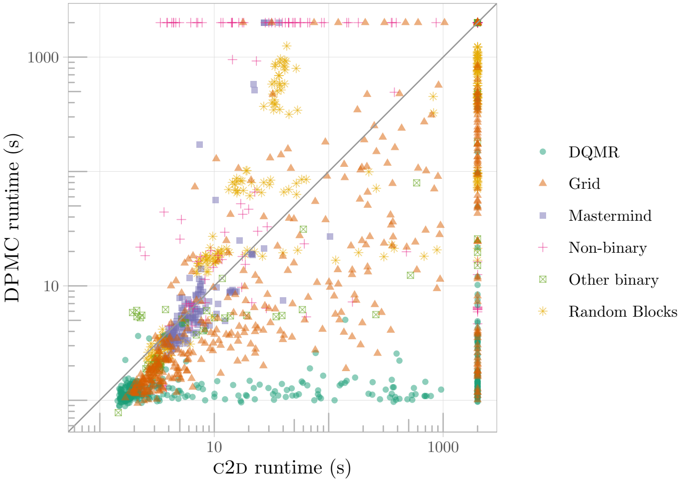
30th May 2024

Which Algorithm Is Better? It Depends on the Data



The runtime data is from Dilkas and Belle (2021): various Bayesian networks encoded using the approach by Darwiche (2002)

The Problem: Weighted Model Counting (WMC)

- A generalisation of propositional model counting (#SAT)
- Applications:
  - graphical models
  - probabilistic programming
  - neuro-symbolic AI
- WMC algorithms use:
  - dynamic programming
  - knowledge compilation
  - SAT solvers

Example

$w(x) = 0.3,\ w(\neg x) = 0.7,$
$w(y) = 0.2,\ w(\neg y) = 0.8$

$\mathrm{WMC}(x \lor y) = w(x)w(y) + w(x)w(\neg y) + w(\neg x)w(y) = 0.44$

(Some of the) WMC Algorithms

- Cachet (Sang et al. 2004)
  - a SAT solver with clause learning and component caching
- c2d (Darwiche 2004)
  - knowledge compilation to d-DNNF
- d4 (Lagniez and Marquis 2017)
  - knowledge compilation to decision-DNNF
- miniC2D (Oztok and Darwiche 2015)
  - knowledge compilation to decision-SDDs
- DPMC (Dudek, Phan and Vardi 2020)
  - dynamic programming with ADDs and tree decomposition based planning

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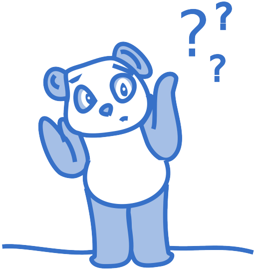
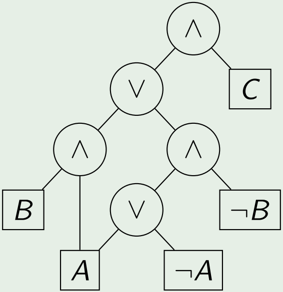
Q: Why am I giving a talk about this now?

A: 

Knowledge Compilation: d-DNNF and Decision-DNNF

Example

$C \land (A \lor \neg B)$ in d-DNNF/decision-DNNF:



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Negation normal form: $\neg$ is only applied to variables

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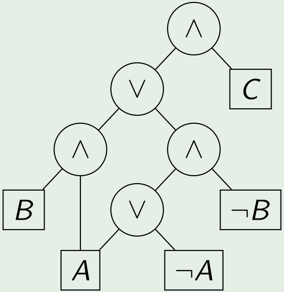

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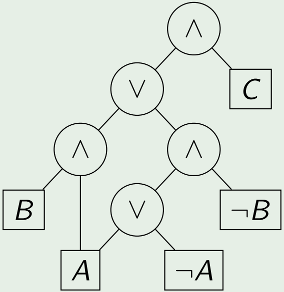

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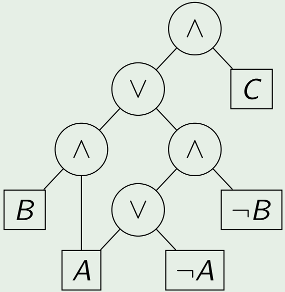

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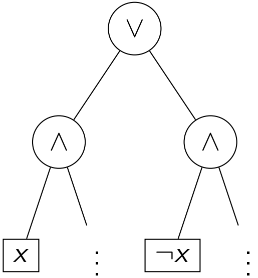
Decision: all disjunctions are of the form



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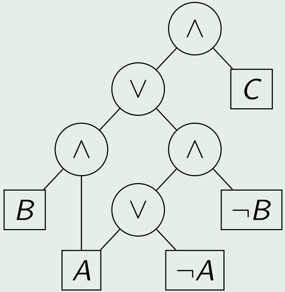

Tree Decompositions and Primal Treewidth

A formula in CNF:

$$\phi = \begin{aligned}[t] &(x_1 \lor x_2) \land (x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_4) \land (x_3 \lor x_5) \land (x_4 \lor x_5 \lor x_6) \\ &\land (x_5 \lor x_6 \lor x_7) \land (x_6 \lor x_7) \end{aligned}$$

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The primal graph of ϕ is:

$$\begin{tikzcd}
 & x_2 \arrow[r, dash] \arrow[dl, dash] \arrow[d, dash] & x_3 \arrow[dl, dash] \arrow[d, dash] & & \\
x_1 \arrow[r, dash] & x_4 \arrow[r, dash] \arrow[rr, dash, bend right] & x_5 \arrow[r, dash] \arrow[rr, dash, bend left] & x_6 \arrow[r, dash] & x_7
\end{tikzcd}$$

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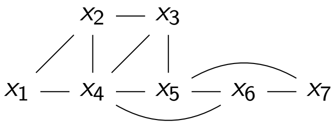
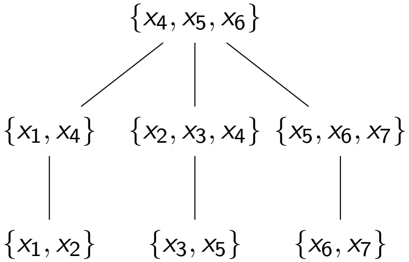


Its minimum-width tree decomposition:



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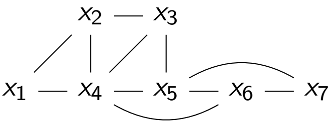

$\therefore$ the primal treewidth of ϕ is 2

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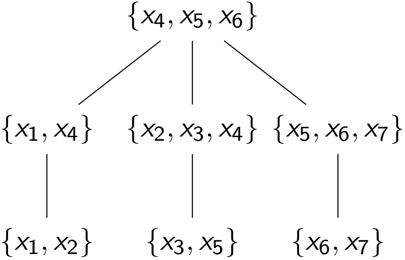

Formally...

Definition (Robertson and Seymour 1984)

A tree decomposition of a graph G is a pair (T, χ), where T is a tree and $\chi\colon \mathcal{V}(T) \to 2^{\mathcal{V}(G)}$ is a labelling function, with the following properties:

- $\bigcup_{t \in \mathcal{V}(T)} \chi(t) = \mathcal{V}(G)$;
- for every edge $e \in \mathcal{E}(G)$, there is $t \in \mathcal{V}(T)$ s.t. e has both endpoints in $\chi(t)$;
- for all $t, t', t'' \in \mathcal{V}(T)$, if t' is on the path between t and t'', then $\chi(t) \cap \chi(t'') \subseteq \chi(t')$.

The width of tree decomposition (T, χ) is $\max_{t \in \mathcal{V}(T)} |\chi(t)| - 1$. The treewidth of graph G is the smallest w such that G has a tree decomposition of width w.

The Parameterised Complexity of WMC Algorithms

Let n be the number of variables and m be the number of clauses.

- Component caching (used in CACHET) is $2^{\mathcal{O}(w)}n^{\mathcal{O}(1)}$, where w is the branchwidth of the underlying hypergraph (Bacchus, Dalmao and Pitassi 2009)
  - Branchwidth is within a constant factor of primal treewidth
- C2D is based on an algorithm, which is $\mathcal{O}(2^w mw)$, where w is at most primal treewidth (Darwiche 2001; Darwiche 2004)
- DPMC can be shown to be $\mathcal{O}(4^w mn)$, where w is an upper bound on primal treewidth

Early History of Random SAT

- Goldberg, Purdom Jr. and Brown (1982) show that (simplified) Davis-Putnam procedures run in polynomial time on average on the following model:
  - Fix the numbers of variables and clauses
  - Let $p \in (0, 0.5)$
  - For each clause C and variable x:
    - Add x to C w.p. p
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- Franco and Paull (1983) were the first to propose a 'reasonable' random k-CNF model:
  1. Fix the numbers of variables, clauses, and clause width
  2. Sample each clause independently as a subset of all possible literals
  3. Reject clauses that have x and $\neg x$ for some variable x
  - They show that the Davis-Putnam procedure requires exponential time w.p. one

Phase Transitions

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  - i.e., the number of clauses divided by the number of variables
  - (usually parameterised by clause width)
- The most comprehensive (experimental) study of phase transitions for SAT algorithms is by Coarfa et al. (2003)
  - They show that the transition from polynomial to exponential time depends on the solver

Generating Random WMC Instances: The Algorithm

$\phi \leftarrow$ empty CNF formula;
$G \leftarrow$ empty graph;
for $i \leftarrow 1$ **to** m **do**
    $X \leftarrow \emptyset$;
    for $j \leftarrow 1$ **to** k **do**
        $x \leftarrow$ `newVariable(`X`, `G`)`;
        $\mathcal{V}(G) \leftarrow \mathcal{V}(G) \cup \{ x \}$;
        $\mathcal{E}(G) \leftarrow \mathcal{E}(G) \cup \{ \{ x, y \} \mid y \in X \}$;
        $X \leftarrow X \cup \{ x \}$;
    $\phi \leftarrow \phi \cup \{ \{ l \leftsquigarrow \mathcal{U}\{ x, \neg x \} \mid x \in X \} \}$;

- the number of clauses
- clause width
- a function to pick a variable
- a (fair) coin flip

How to Pick a Variable

Parameter $\rho \in [0, 1]$ biases the probability distribution towards adding variables that would introduce fewer new edges.

Function `newVariable`(*set of variables* X, *primal graph* G)**:**

$N \leftarrow \{ e \in \mathcal{E}(G) \mid |e \cap X| = 1 \}$;

if $N = \emptyset$ **then return** $x \leftarrow\!\!\sim \mathcal{U}(\{ x_1, x_2, \ldots, x_n \} \setminus X)$;

return

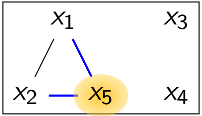
$$x \leftarrow\!\!\sim \left(\{ x_1, x_2, \ldots, x_n \} \setminus X, y \mapsto \frac{1-\rho}{n-|X|} + \rho \frac{|\{ z \in X \mid \{ y, z \} \in \mathcal{E}(G) \}|}{|N|} \right);$$

From Random SAT to Random WMC

We introduce parameter $\rho \in [0, 1]$ that biases the probability distribution towards adding variables that would introduce fewer new edges to the primal graph.

Example partially-filled formula:
$(\neg x_5 \lor x_2 \lor x_1) \land (x_5 \lor ? \lor ?)$

Its primal graph:



The probability distribution for the next variable

Base probability of each variable being chosen:

$$\frac{1-\rho}{4}.$$

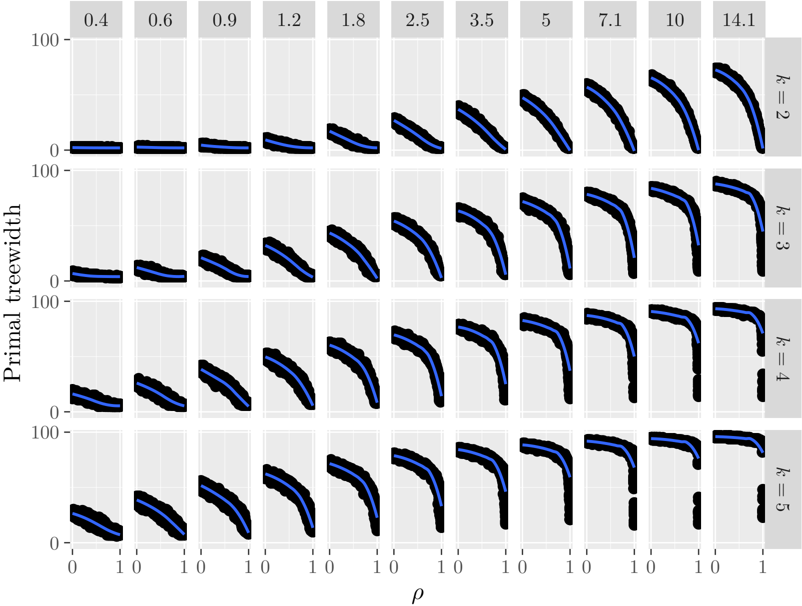
Both x_1 and x_2 get a bonus probability of $\rho/2$ for each being the endpoint of one out of the two neighbourhood edges.

The Relationship Between ρ and Primal Treewidth

Experiment 1 (Validation)

- Set the number of variables to 100
- Consider a geometric sequence of 11 densities from 0.4 to 14.1
- Let ρ range from zero to one in steps of 0.01
- Generate ten 2-, 3-, 4-, and 5-CNF formulas

The Relationship Between ρ and Primal Treewidth



Density, Primal Treewidth, and Runtime

Experiment 2 (The Main One)

- Set the number of variables to 70
- Consider densities ranging from 1 to 4.3 in steps of 0.3
- Let ρ range from 0 to 0.5 in steps of 0.01
- Generate one 3-CNF formula

Peak Hardness w.r.t. Density

Let μ denote the density, i.e., the number of clauses divided by the number of variables.

- CACHET is known to peak at $\mu = 1.8$ (Sang et al. 2004)
- Bayardo Jr. and Pehoushek (2000) show some #SAT algorithms to peak at $\mu = 1.2$ and $\mu = 1.9$

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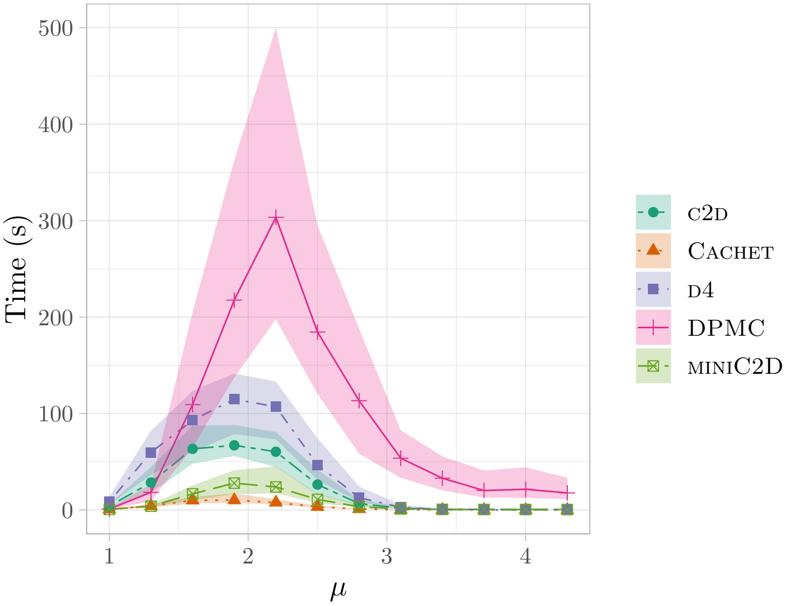
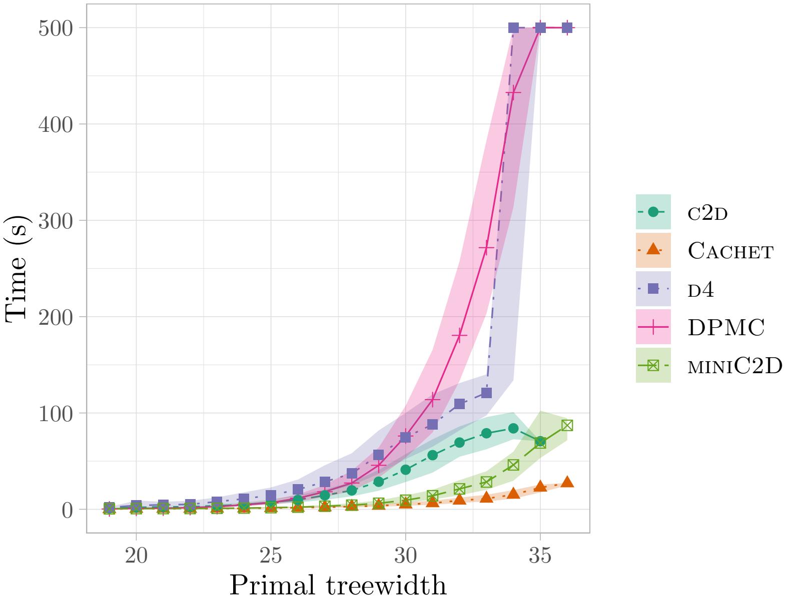
- Bayardo Jr. and Pehoushek (2000) show some #SAT algorithms to peak at $\mu = 1.2$ and $\mu = 1.9$
- In our experiments:
  - DPMC peaks at $\mu = 2.2$
  - all other algorithms peak at $\mu = 1.9$

Peak Hardness w.r.t. Density (when $\rho = 0$)



Hardness w.r.t. Primal Treewidth (when $\mu = 1.9$)



Is The Relationship Exponential: Two Approaches

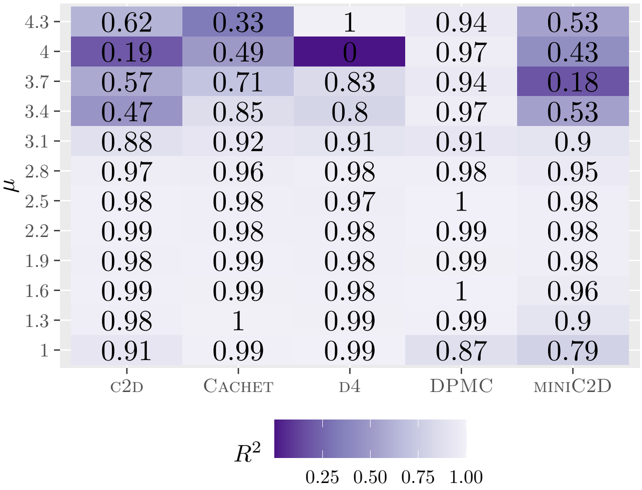
Linear Regression

We fit the model $\ln t \sim \alpha w + \beta$, i.e., $t \sim e^{\beta}(e^{\alpha})^{w}$, where t is runtime, and w is primal treewidth

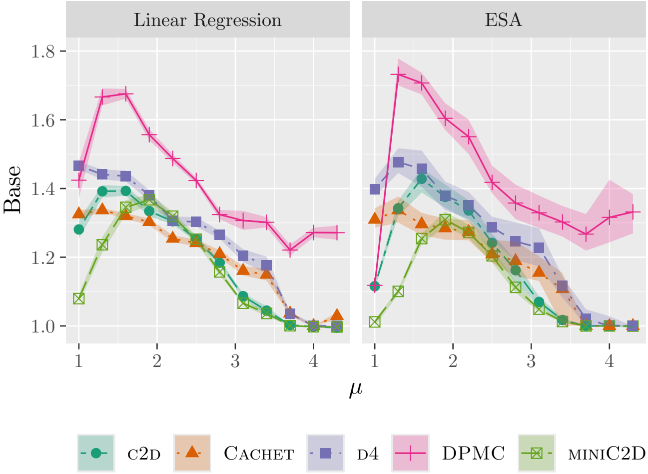
Empirical Scaling Analyzer (ESA) v2 (Pushak and Hoos 2020)

1. Prepare a list of hypotheses about scalability, e.g.:
   exponential: $t \sim \alpha\beta^{w}$,
   polynomial: $t \sim \alpha w^{\beta}$
2. Use $30\,\%$ of the data with the largest values of w for testing
3. For each hypothesis, ESA produces:
   - estimates of parameter values,
   - support loss, and
   - challenge loss

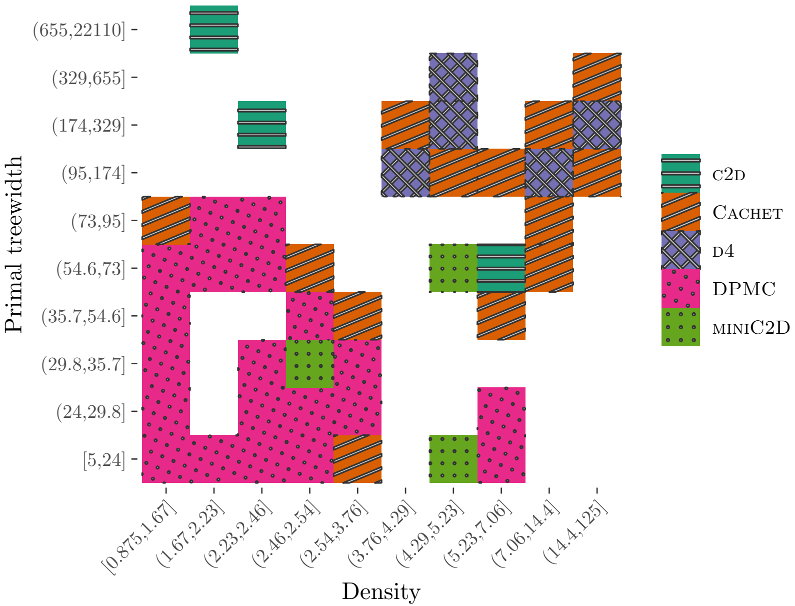
How Well Does Linear Regression Explain the Data?



The Base of the Exponential

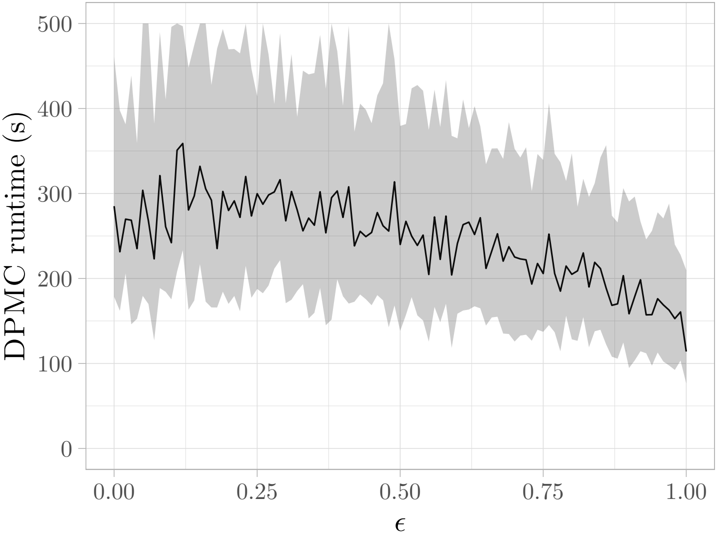


Does Real Data Confirm Our Observations?

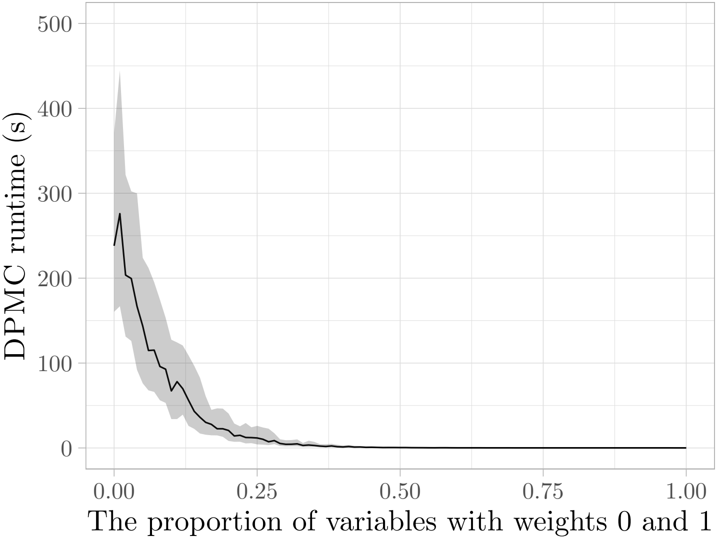


Bonus 1: How DPMC Reacts to Redundancy in Weights

Let ϵ be the proportion of variables x s.t. $w(x) = w(\neg x) = 0.5$



Bonus 2: 0/1 Weights Make Counting Easy



Summary

- This work introduced a random model for WMC instances with a parameter that indirectly controls primal treewidth
- Observations:
  - All algorithms scale exponentially w.r.t. primal treewidth
  - The running time of DPMC:
    - peaks at a higher density
    - and scales worse w.r.t. primal treewidth
- Future work:
  - A theoretical relationship between ρ and primal treewidth
  - Non-k-CNF instances
  - Algorithm portfolios for WMC

Future Work: (Per-Instance) Algorithm Selection

Definition (Bischl et al. 2016)

Given a set $\mathcal{I}$ of problem instances, a space of algorithms $\mathcal{A}$, and a performance measure $m\colon \mathcal{I} \times \mathcal{A} \to \mathbb{R}$, the algorithm selection problem is to find a mapping $s\colon \mathcal{I} \to \mathcal{A}$ that optimises $\mathbb{E}[m(i, s(i))]$.

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$$\phi \longrightarrow \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

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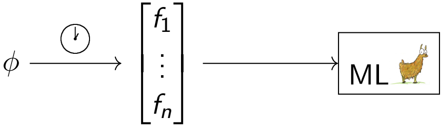

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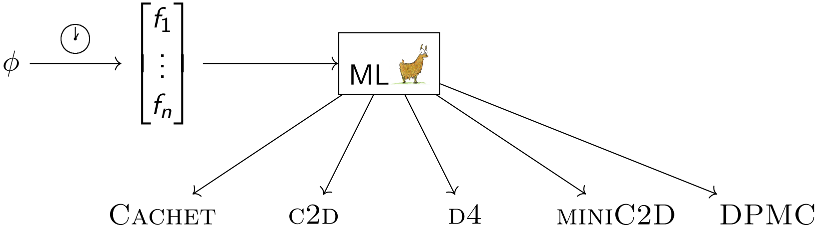

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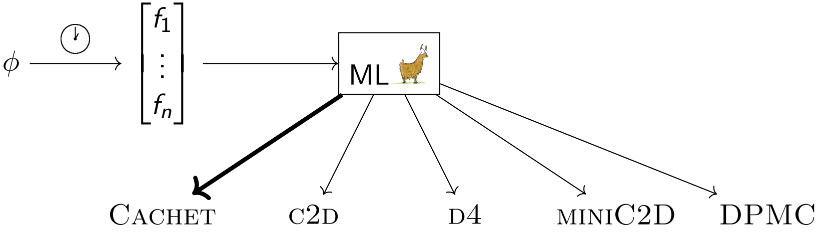

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