

Generating Random WMC Instances

An Empirical Analysis with Varying Primal Treewidth

Paulius Dilkas

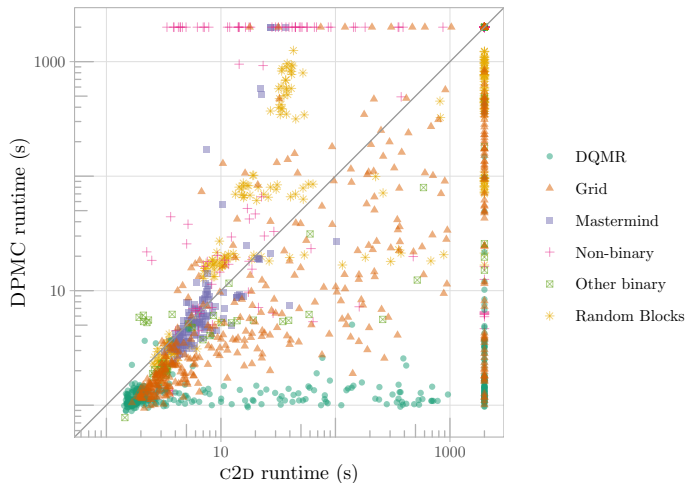
National University of Singapore, Singapore

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Engineering and
Physical Sciences
Research Council

Which Algorithm Is Better? It Depends on the Data



The runtime data is from Dilkas and Belle (2021): various Bayesian networks encoded using the approach by Darwiche (2002)

The Problem: Weighted Model Counting (WMC)

- A generalisation of propositional model counting ($\#SAT$)
- Applications:
 - graphical models
 - probabilistic programming
 - neuro-symbolic AI
- WMC algorithms use:
 - dynamic programming
 - knowledge compilation
 - SAT solvers

Example

$$w(x) = 0.3, w(\neg x) = 0.7, \\ w(y) = 0.2, w(\neg y) = 0.8$$

$$WMC(x \vee y) = w(x)w(y) + \\ w(x)w(\neg y) + w(\neg x)w(y) = 0.44$$

(Some of the) WMC Algorithms

- CACHET (Sang et al. 2004)
 - a SAT solver with **clause learning** and **component caching**
- C2D (Darwiche 2004)
 - knowledge compilation to **d-DNNF**
- D4 (Lagniez and Marquis 2017)
 - knowledge compilation to **decision-DNNF**
- MINIC2D (Oztok and Darwiche 2015)
 - knowledge compilation to **decision sentential decision diagrams**
- DPMC (Dudek, Phan and Vardi 2020)
 - dynamic programming with **algebraic decision diagrams** and **tree decomposition** based planning

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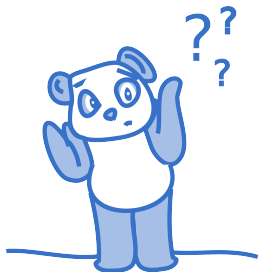
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Q: Why am I giving a talk about this **now**?



A:

Tree Decompositions and Primal Treewidth

Formula in CNF:

$$\phi = (x_4 \vee \neg x_3 \vee x_1) \wedge (\neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_4)$$

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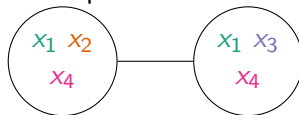
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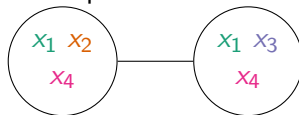
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Its primal graph:



Its minimum-width tree decomposition:



\therefore the primal treewidth of ϕ is 2

The Parameterised Complexity of WMC Algorithms

Let n be the number of **variables** and m be the number of **clauses**.

- Component caching (used in CACHET) is $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$, where w is the **branchwidth** of the underlying hypergraph (Bacchus, Dalmao and Pitassi 2009)
 - Branchwidth is within a constant factor of primal treewidth
- C2D is based on an algorithm, which is $\mathcal{O}(2^w mw)$, where w is at most **primal treewidth** (Darwiche 2001; Darwiche 2004)
- DPMC can be shown to be $\mathcal{O}(4^w mn)$, where w is an upper bound on **primal treewidth**

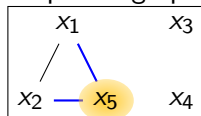
From Random SAT to Random WMC

We introduce parameter $\rho \in [0, 1]$ that biases the probability distribution towards adding variables that would introduce fewer new edges to the primal graph.

Example partially-filled formula:

$$(\neg x_5 \vee x_2 \vee x_1) \wedge (x_5 \vee ? \vee ?)$$

Its primal graph:



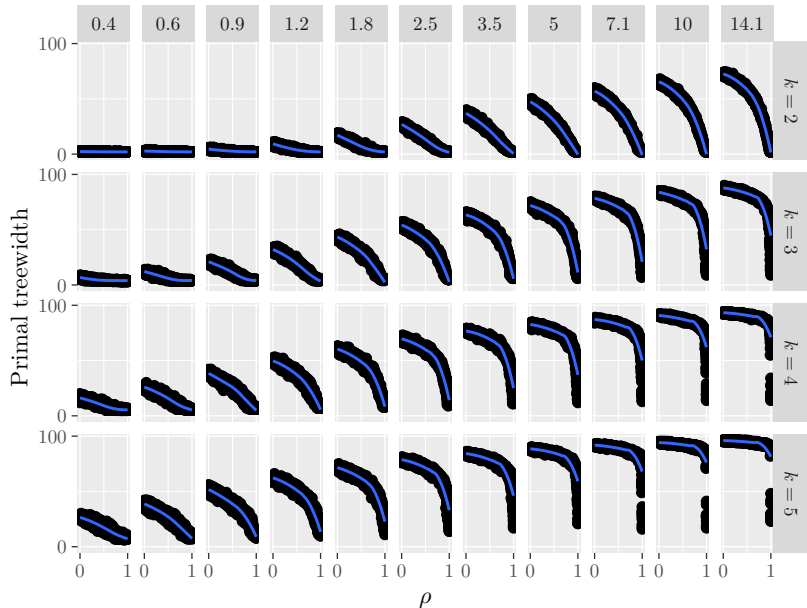
The probability distribution for the next variable

Base probability of each variable being chosen:

$$\frac{1 - \rho}{4}.$$

Both x_1 and x_2 get a bonus probability of $\rho/2$ for each being the endpoint of **one** out of the **two** neighbourhood edges.

The Relationship Between ρ and Primal Treewidth



Peak Hardness w.r.t. Density

Let μ denote the **density**, i.e., the number of clauses divided by the number of variables.

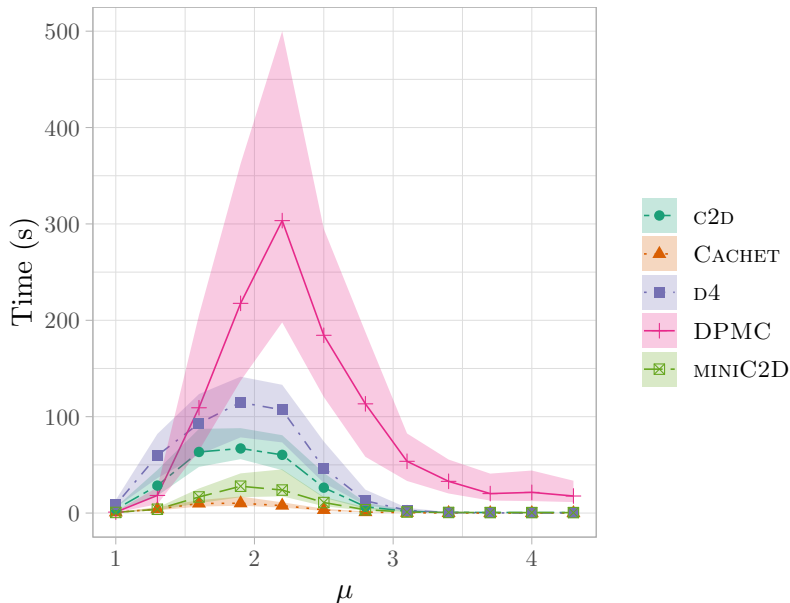
- CACHET is known to peak at $\mu = 1.8$ (Sang et al. 2004)
- Bayardo Jr. and Pehoushek (2000) show some #SAT algorithms to peak at $\mu = 1.2$ and $\mu = 1.9$

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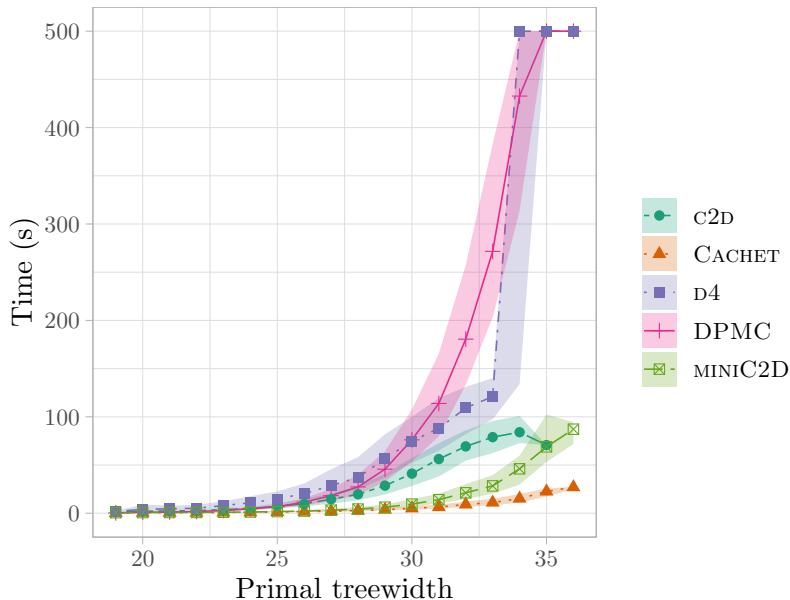
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- In our experiments:
 - DPMC peaks at $\mu = 2.2$
 - all other algorithms peak at $\mu = 1.9$

Peak Hardness w.r.t. Density (when $\rho = 0$)



Hardness w.r.t. Primal Treewidth (when $\mu = 1.9$)

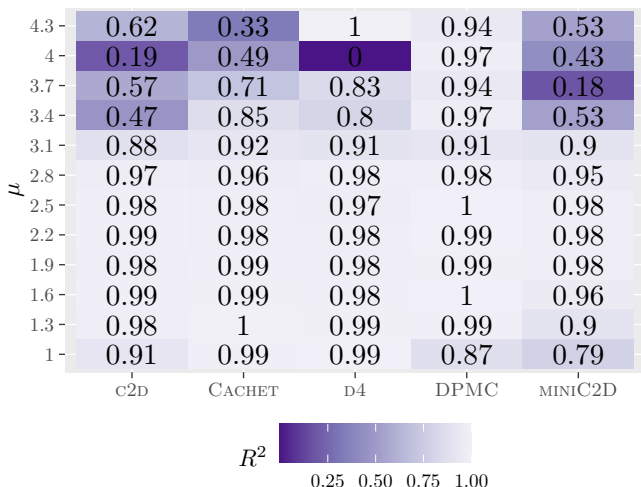


Is The Relationship Exponential?

Let us fit the model $\ln t \sim \alpha w + \beta$, i.e., $t \sim e^\beta (e^\alpha)^w$, where t is runtime, and w is primal treewidth

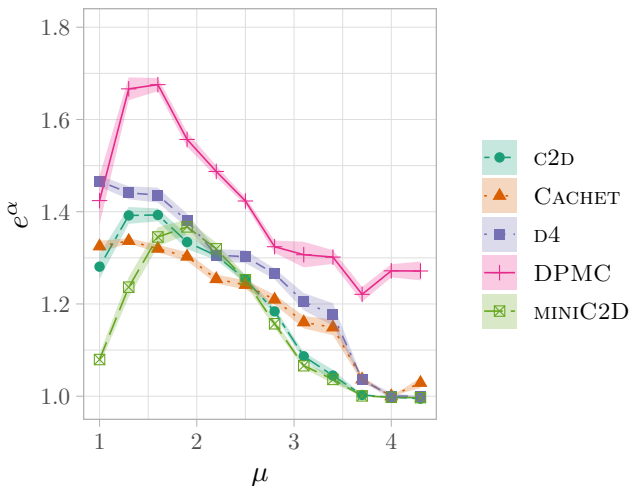
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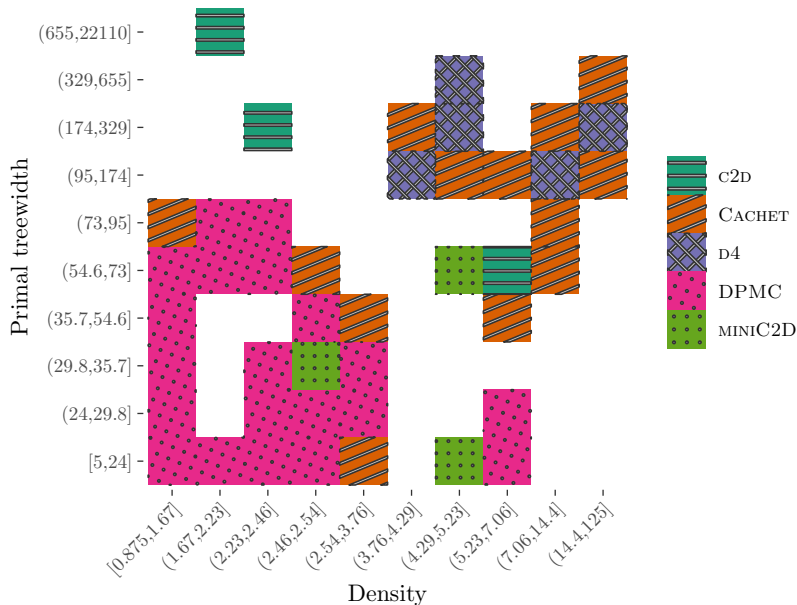


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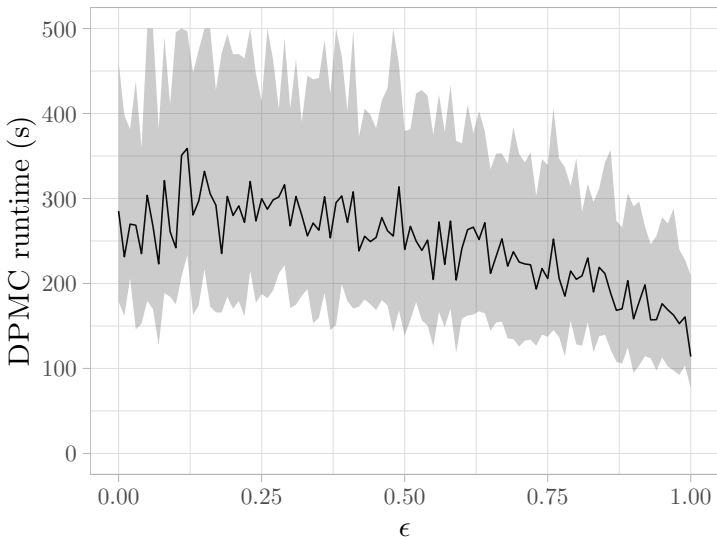


Does Real Data Confirm Our Observations?



Bonus: How DPMC Reacts to Redundancy in Weights

Let ϵ be the proportion of variables x s.t. $w(x) = w(\neg x) = 0.5$



Summary

- This work introduced a **random model** for WMC instances with a parameter that indirectly controls **primal treewidth**
- Observations:
 - All algorithms **scale exponentially** w.r.t. primal treewidth
 - The running time of DPMC:
 - peaks at a higher density
 - and scales worse w.r.t. primal treewidth
- Future work:
 - A theoretical relationship between ρ and primal treewidth
 - Non- k -CNF instances
 - Algorithm portfolios for WMC

Generating Random WMC Instances: The Algorithm

$\phi \leftarrow$ empty CNF formula;

$G \leftarrow$ empty graph;

for $i \leftarrow 1$ **to** m **do** \leftarrow

$X \leftarrow \emptyset$;

for $j \leftarrow 1$ **to** k **do** \leftarrow

$x \leftarrow \text{newVariable}(X, G)$;

$\mathcal{V}(G) \leftarrow \mathcal{V}(G) \cup \{x\}$;

$\mathcal{E}(G) \leftarrow \mathcal{E}(G) \cup \{\{x, y\} \mid y \in X\}$;

$X \leftarrow X \cup \{x\}$;

$\phi \leftarrow \phi \cup \{\{l \stackrel{!}{\leftarrow} \mathcal{U}\{x, \neg x\} \mid x \in X\}\}$;

- the number of clauses
- clause width
- a function to pick a variable
- a (fair) coin flip

How to Pick a Variable

Parameter $\rho \in [0, 1]$ biases the probability distribution towards adding variables that would introduce fewer new edges.

Function `newVariable`(*set of variables* X , *primal graph* G):

```

$$\begin{aligned} & N \leftarrow \{ e \in \mathcal{E}(G) \mid |e \cap X| = 1 \}; \\ & \text{if } N = \emptyset \text{ then return } x \leftarrow \mathcal{U}(\{x_1, x_2, \dots, x_n\} \setminus X); \\ & \text{return} \\ & \quad x \leftarrow \left( \{x_1, x_2, \dots, x_n\} \setminus X, y \mapsto \frac{1-\rho}{n-|X|} + \rho \frac{|\{z \in X \mid \{y, z\} \in \mathcal{E}(G)\}|}{|N|} \right); \end{aligned}$$

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