

Generating Random WMC Instances

An Empirical Analysis with Varying Primal Treewidth

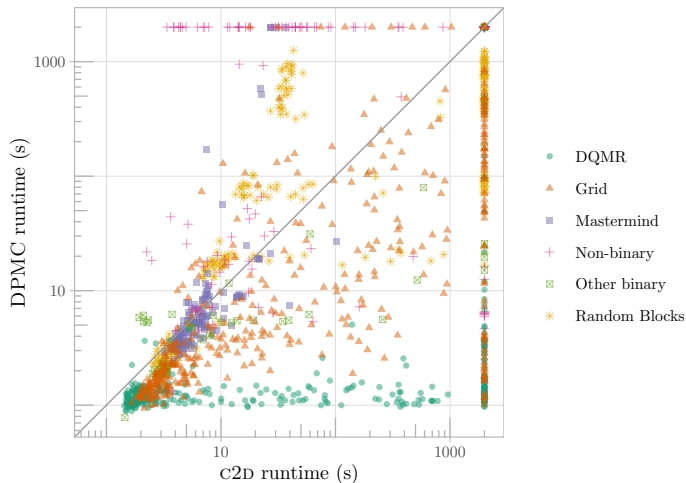
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30th May 2024



Which Algorithm Is Better? It Depends on the Data



The runtime data is from Dilkas and Belle (2021): various Bayesian networks encoded using the approach by Darwiche (2002)

The Problem: Weighted Model Counting (WMC)

- A generalisation of propositional model counting ($\#SAT$)
- Applications:
 - graphical models
 - probabilistic programming
 - neuro-symbolic AI
- WMC algorithms use:
 - dynamic programming
 - knowledge compilation
 - SAT solvers

Example

$$w(x) = 0.3, w(\neg x) = 0.7, \\ w(y) = 0.2, w(\neg y) = 0.8$$

$$WMC(x \vee y) = w(x)w(y) + \\ w(x)w(\neg y) + w(\neg x)w(y) = 0.44$$

(Some of the) WMC Algorithms

- CACHET (Sang et al. 2004)
 - a SAT solver with **clause learning** and **component caching**
- C2D (Darwiche 2004)
 - knowledge compilation to **d-DNNF**
- D4 (Lagniez and Marquis 2017)
 - knowledge compilation to **decision-DNNF**
- MINIC2D (Oztok and Darwiche 2015)
 - knowledge compilation to **decision sentential decision diagrams**
- DPMC (Dudek, Phan and Vardi 2020)
 - dynamic programming with **algebraic decision diagrams** and **tree decomposition** based planning

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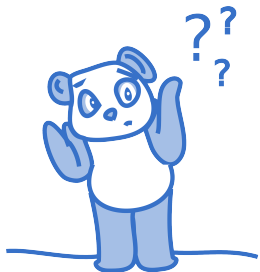
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Q: Why am I giving a talk about this **now**?



A:

Tree Decompositions and Primal Treewidth

A formula in CNF:

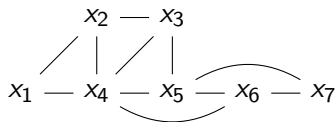
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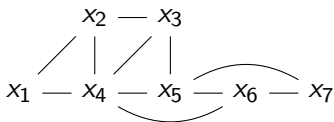


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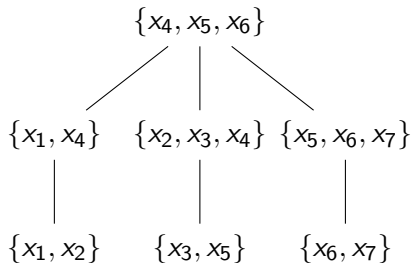
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Its minimum-width tree decomposition:

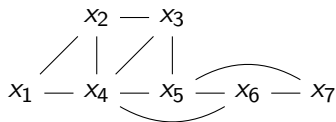


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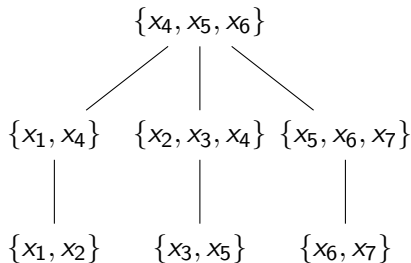
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The primal graph of ϕ is:



\therefore the primal treewidth of ϕ is 2

Its minimum-width tree decomposition:



Definition (Robertson and Seymour 1984)

A **tree decomposition** of a graph G is a pair (T, χ) , where T is a tree and $\chi: \mathcal{V}(T) \rightarrow 2^{\mathcal{V}(G)}$ is a labelling function, with the following properties:

- $\bigcup_{t \in \mathcal{V}(T)} \chi(t) = \mathcal{V}(G)$;
- for every edge $e \in \mathcal{E}(G)$, there is $t \in \mathcal{V}(T)$ s.t. e has both endpoints in $\chi(t)$;
- for all $t, t', t'' \in \mathcal{V}(T)$, if t' is on the path between t and t'' , then $\chi(t) \cap \chi(t'') \subseteq \chi(t')$.

The **width** of tree decomposition (T, χ) is $\max_{t \in \mathcal{V}(T)} |\chi(t)| - 1$. The **treewidth** of graph G is the smallest w such that G has a tree decomposition of width w .

The Parameterised Complexity of WMC Algorithms

Let n be the number of **variables** and m be the number of **clauses**.

- Component caching (used in CACHET) is $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$, where w is the **branchwidth** of the underlying hypergraph (Bacchus, Dalmao and Pitassi 2009)
 - Branchwidth is within a constant factor of primal treewidth
- C2D is based on an algorithm, which is $\mathcal{O}(2^w mw)$, where w is at most **primal treewidth** (Darwiche 2001; Darwiche 2004)
- DPMC can be shown to be $\mathcal{O}(4^w mn)$, where w is an upper bound on **primal treewidth**

Generating Random WMC Instances: The Algorithm

$\phi \leftarrow$ empty CNF formula;

$G \leftarrow$ empty graph;

for $i \leftarrow 1$ **to** m **do** \leftarrow

$X \leftarrow \emptyset$;

for $j \leftarrow 1$ **to** k **do** \leftarrow

$x \leftarrow \text{newVariable}(X, G)$; \leftarrow

$\mathcal{V}(G) \leftarrow \mathcal{V}(G) \cup \{x\}$;

$\mathcal{E}(G) \leftarrow \mathcal{E}(G) \cup \{\{x, y\} \mid y \in X\}$;

$X \leftarrow X \cup \{x\}$;

$\phi \leftarrow \phi \cup \{\{l \stackrel{?}{\sim} \mathcal{U}\{x, \neg x\} \mid x \in X\}\}$; \leftarrow

- the number of clauses
- clause width
- a function to pick a variable
- a (fair) coin flip

How to Pick a Variable

Parameter $\rho \in [0, 1]$ biases the probability distribution towards adding variables that would introduce fewer new edges.

Function `newVariable(set of variables X , primal graph G):`

```

$$\begin{aligned} & N \leftarrow \{ e \in \mathcal{E}(G) \mid |e \cap X| = 1 \}; \\ & \text{if } N = \emptyset \text{ then return } x \leftarrow \mathcal{U}(\{ x_1, x_2, \dots, x_n \} \setminus X); \\ & \text{return} \\ & \quad x \leftarrow \left( \{ x_1, x_2, \dots, x_n \} \setminus X, y \mapsto \frac{1-\rho}{n-|X|} + \rho \frac{|\{ z \in X \mid \{y, z\} \in \mathcal{E}(G) \}|}{|N|} \right); \end{aligned}$$

```

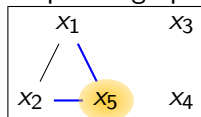

From Random SAT to Random WMC

We introduce parameter $\rho \in [0, 1]$ that biases the probability distribution towards adding variables that would introduce fewer new edges to the primal graph.

Example partially-filled formula:

$$(\neg x_5 \vee x_2 \vee x_1) \wedge (x_5 \vee ? \vee ?)$$

Its primal graph:



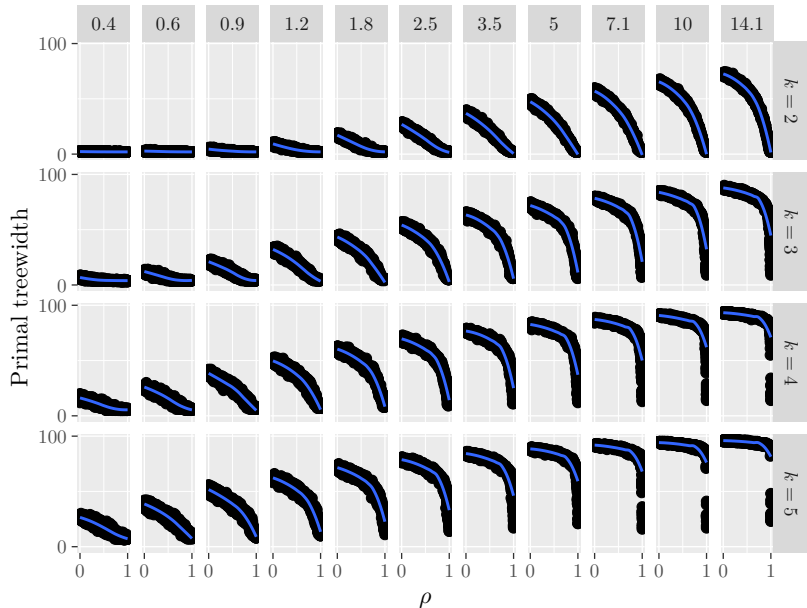
The probability distribution for the next variable

Base probability of each variable being chosen:

$$\frac{1 - \rho}{4}.$$

Both x_1 and x_2 get a bonus probability of $\rho/2$ for each being the endpoint of **one** out of the **two** neighbourhood edges.

The Relationship Between ρ and Primal Treewidth



Peak Hardness w.r.t. Density

Let μ denote the **density**, i.e., the number of clauses divided by the number of variables.

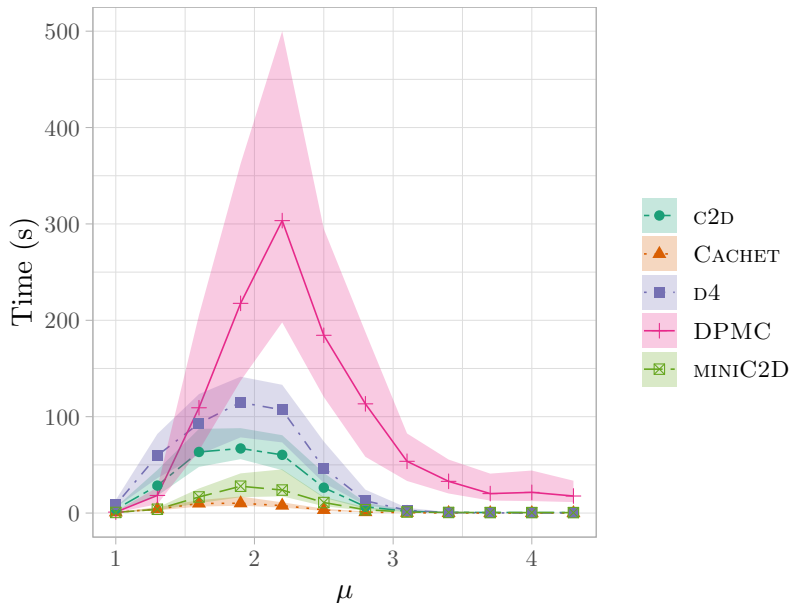
- CACHET is known to peak at $\mu = 1.8$ (Sang et al. 2004)
- Bayardo Jr. and Pehoushek (2000) show some #SAT algorithms to peak at $\mu = 1.2$ and $\mu = 1.9$

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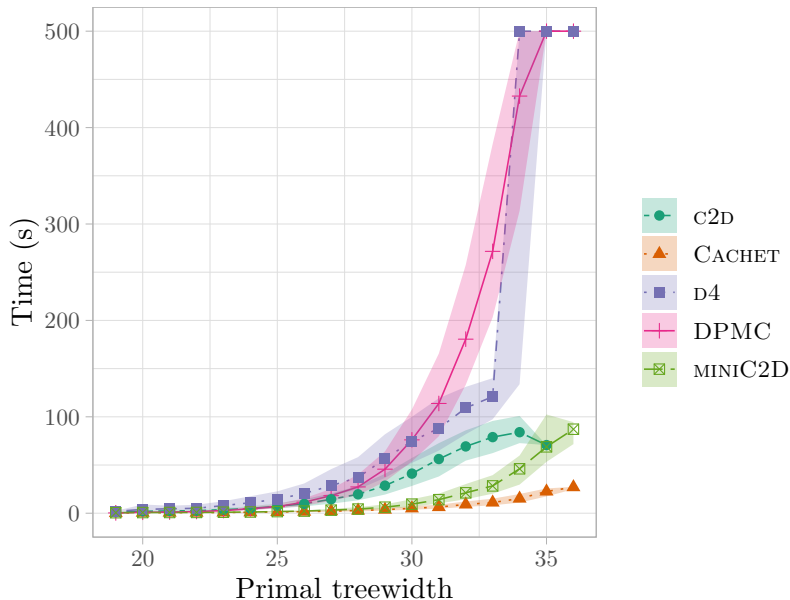
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- In our experiments:
 - DPMC peaks at $\mu = 2.2$
 - all other algorithms peak at $\mu = 1.9$

Peak Hardness w.r.t. Density (when $\rho = 0$)



Hardness w.r.t. Primal Treewidth (when $\mu = 1.9$)



Is The Relationship Exponential: Two Approaches

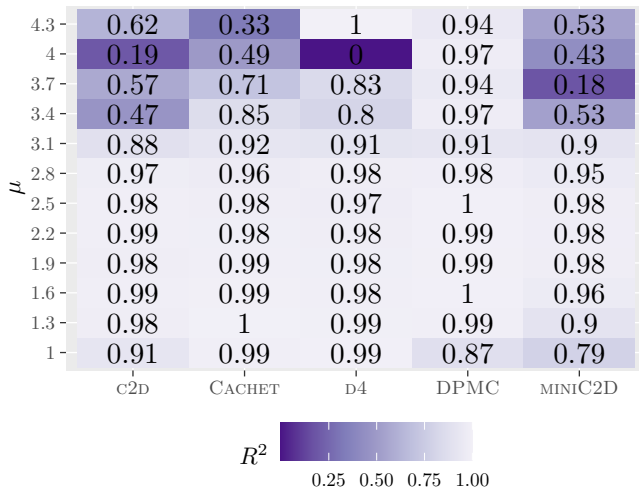
Linear Regression

We fit the model $\ln t \sim \alpha w + \beta$, i.e., $t \sim e^\beta (e^\alpha)^w$, where t is runtime, and w is primal treewidth

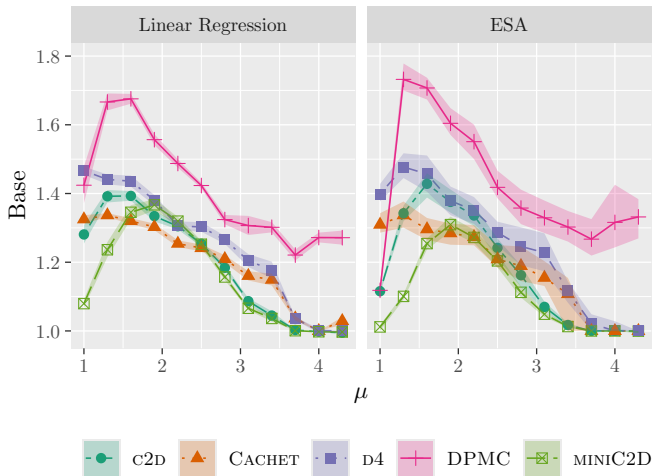
Empirical Scaling Analyzer (ESA) v2 (Pushak and Hoos 2020)

- 1 Prepare a list of hypotheses about scalability, e.g.:
exponential: $t \sim \alpha \beta^w$,
polynomial: $t \sim \alpha w^\beta$
- 2 Use 30 % of the data with the largest values of w for testing
- 3 For each hypothesis, ESA produces:
 - estimates of parameter values,
 - support loss, and
 - challenge loss

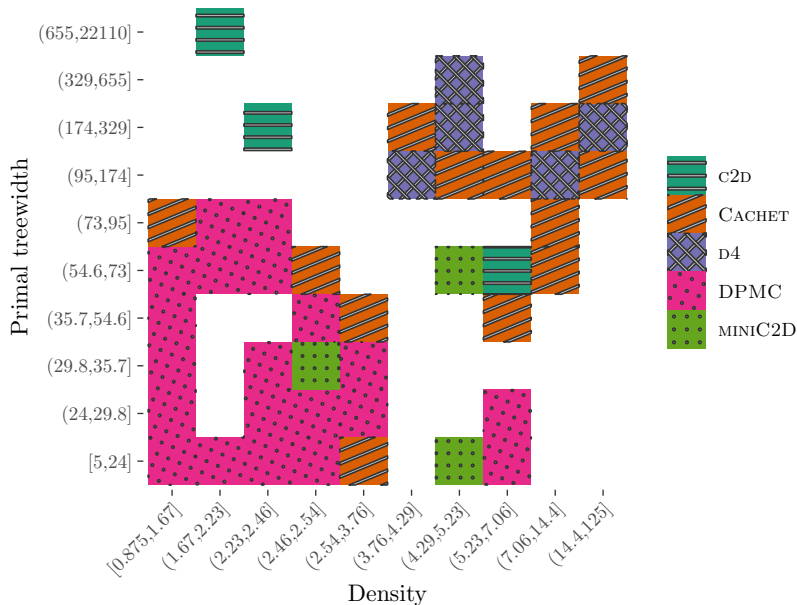
How Well Does Linear Regression Explain the Data?



The Base of the Exponential

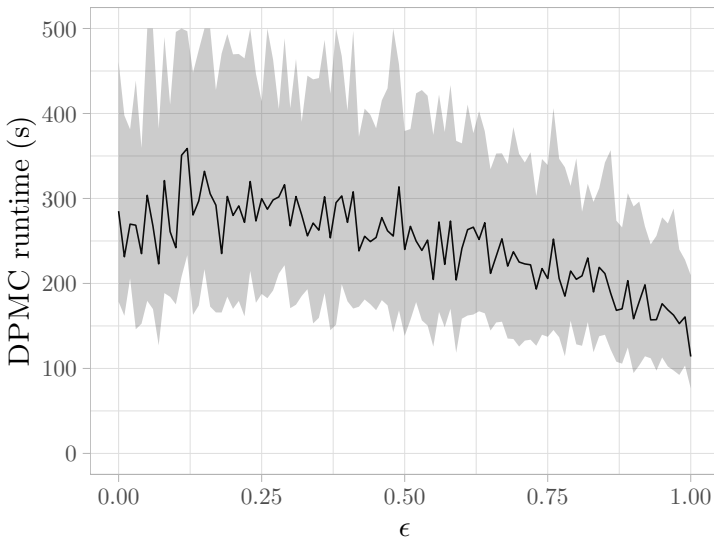


Does Real Data Confirm Our Observations?

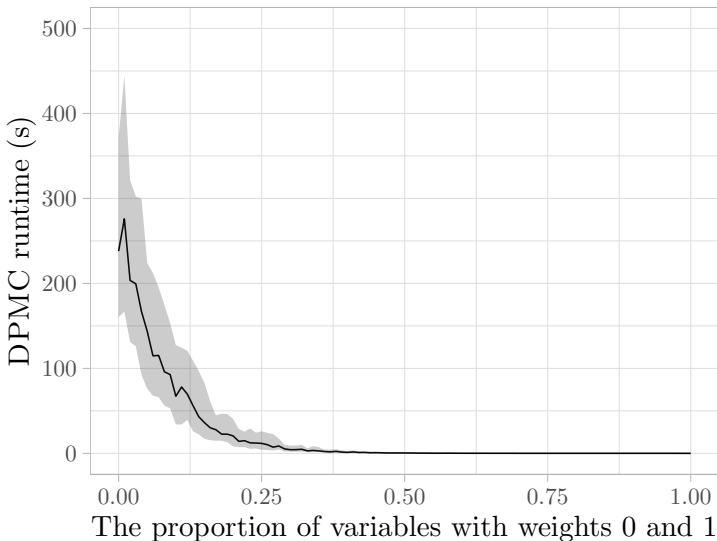


Bonus 1: How DPMC Reacts to Redundancy in Weights

Let ϵ be the proportion of variables x s.t. $w(x) = w(\neg x) = 0.5$



Bonus 2: 0/1 Weights Make Counting Easy



Summary

- This work introduced a **random model** for WMC instances with a parameter that indirectly controls **primal treewidth**
- Observations:
 - All algorithms **scale exponentially** w.r.t. primal treewidth
 - The running time of DPMC:
 - peaks at a higher density
 - and scales worse w.r.t. primal treewidth
- Future work:
 - A theoretical relationship between ρ and primal treewidth
 - Non- k -CNF instances
 - Algorithm portfolios for WMC

Future Work: (Per-Instance) Algorithm Selection

Definition (Bischl et al. 2016)

Given a set \mathcal{I} of problem instances, a space of algorithms \mathcal{A} , and a performance measure $m: \mathcal{I} \times \mathcal{A} \rightarrow \mathbb{R}$, the **algorithm selection problem** is to find a mapping $s: \mathcal{I} \rightarrow \mathcal{A}$ that optimises $\mathbb{E}[m(i, s(i))]$.

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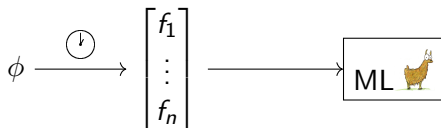
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$$\phi \xrightarrow{\text{clock}} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

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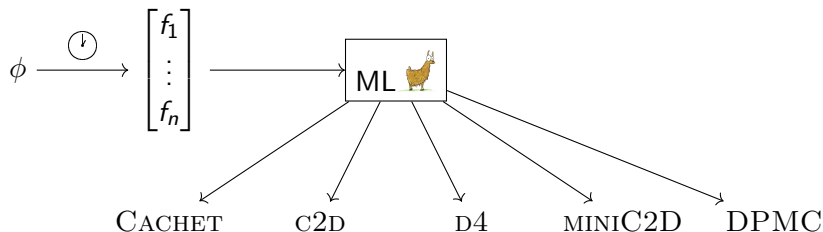
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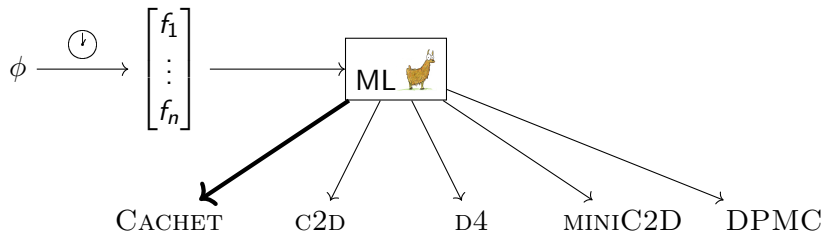
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