

Generating Random WMC Instances

An Empirical Analysis with Varying Primal Treewidth

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Weighted Model Counting (WMC)

- ▶ A generalisation of propositional model counting (#SAT)

Example

$$w(x) = 0.3, w(\neg x) = 0.7, \\ w(y) = 0.2, w(\neg y) = 0.8$$

$$\text{WMC}(x \vee y) = w(x)w(y) + \\ w(x)w(\neg y) + w(\neg x)w(y) = 0.44$$

Weighted Model Counting (WMC)

- ▶ A generalisation of propositional model counting (#SAT)
- ▶ Applications:
 - ▶ graphical models
 - ▶ probabilistic programming
 - ▶ neural-symbolic artificial intelligence

Example

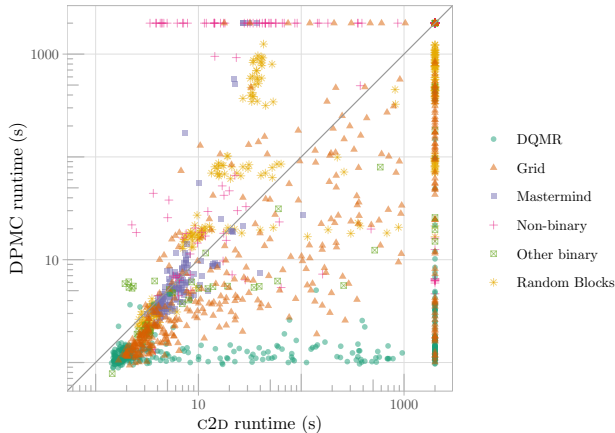
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(Some of the) WMC Algorithms

- ▶ CACHET (Sang et al. 2004)
 - ▶ component caching
- ▶ C2D (Darwiche 2004)
 - ▶ knowledge compilation to d-DNNF
- ▶ D4 (Lagniez and Marquis 2017)
 - ▶ knowledge compilation to decision-DNNF
- ▶ MINIC2D (Oztok and Darwiche 2015)
 - ▶ knowledge compilation to decision sentential decision diagrams
- ▶ DPMC (Dudek, Phan and Vardi 2020)
 - ▶ dynamic programming with algebraic decision diagrams (ADDs) and tree decomposition based planning

The Performance Characteristics of WMC Algorithms



Data from Dilkas and Belle (2021): various Bayesian networks encoded using the method by Darwiche (2002)

A Note on Primal Treewidth

Formula in CNF:

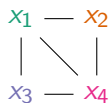
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Its primal graph:

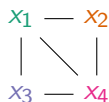


A Note on Primal Treewidth

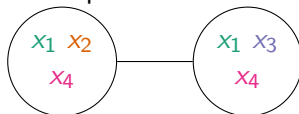
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Its primal graph:



Its minimum-width tree decomposition:

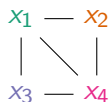


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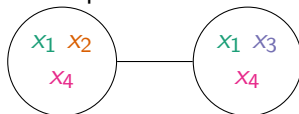
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Its primal graph:



Its minimum-width tree decomposition:



\therefore the primal treewidth of ϕ is 2

The Parameterised Complexity of WMC Algorithms

Let n be the number of variables and m be the number of clauses.

- ▶ Component caching (used in CACHET) is $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$, where w is the branchwidth of the underlying hypergraph (Bacchus, Dalmao and Pitassi 2009)
 - ▶ Branchwidth is within a constant factor of primal treewidth
- ▶ C2D is based on an algorithm, which is $\mathcal{O}(2^w mw)$, where w is at most primal treewidth (Darwiche 2001; Darwiche 2004)
- ▶ DPMC can be shown to be $\mathcal{O}(4^w mn)$, where w is an upper bound on primal treewidth

Generating Random WMC Instances: The Algorithm

$\phi \leftarrow$ empty CNF formula;

$G \leftarrow$ empty graph;

for $i \leftarrow 1$ **to** m **do** \leftarrow the number of clauses
 $X \leftarrow \emptyset$;
 for $j \leftarrow 1$ **to** k **do** \leftarrow clause width
 $x \leftarrow \text{newVariable}(X, G)$; \leftarrow a function to pick a variable
 $\mathcal{V}(G) \leftarrow \mathcal{V}(G) \cup \{x\}$;
 $\mathcal{E}(G) \leftarrow \mathcal{E}(G) \cup \{\{x, y\} \mid y \in X\}$;
 $X \leftarrow X \cup \{x\}$; \leftarrow a (fair) coin flip
 $\phi \leftarrow \phi \cup \{\{l \leftarrow \mathcal{U}\{x, \neg x\} \mid x \in X\}\}$;

How to Pick a Variable

Parameter $\rho \in [0, 1]$ biases the probability distribution towards adding variables that would introduce fewer new edges.

Function newVariable(*set of variables* X , *primal graph* G):

$N \leftarrow \{ e \in \mathcal{E}(G) \mid |e \cap X| = 1 \};$

if $N = \emptyset$ **then return** $x \leftarrow \mathcal{U}(\{x_1, x_2, \dots, x_n\} \setminus X);$

return

$x \leftarrow \left(\{x_1, x_2, \dots, x_n\} \setminus X, y \mapsto \frac{1-\rho}{n-|X|} + \rho \frac{|\{z \in X \mid \{y, z\} \in \mathcal{E}(G)\}|}{|N|} \right);$

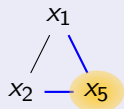
An Example

Setup

Let $n = 5$, $k = 3$, and $\rho = 0.3$.

Partial formula: $(\neg x_5 \vee x_2 \vee x_1) \wedge (x_5 \vee ? \vee ?)$.

Primal graph:



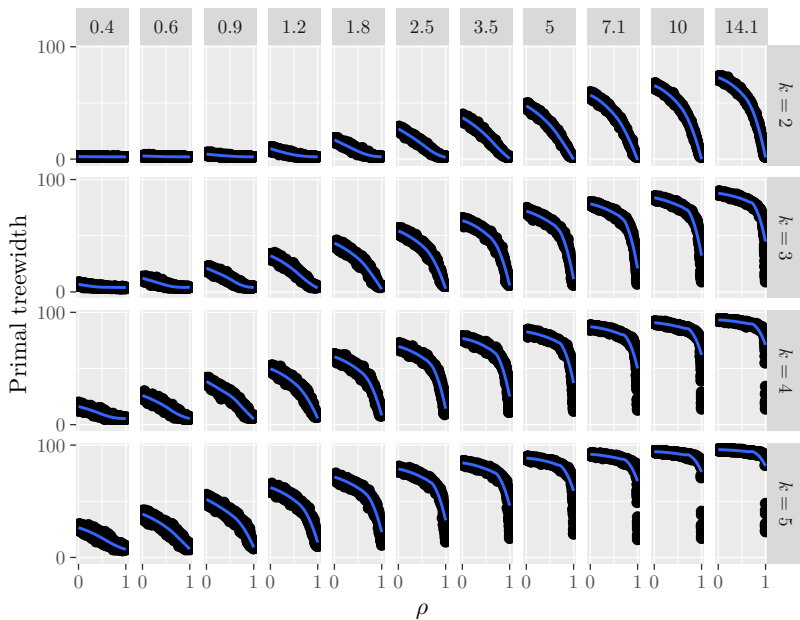
The probability distribution for the next variable

Base probability of each variable being chosen:

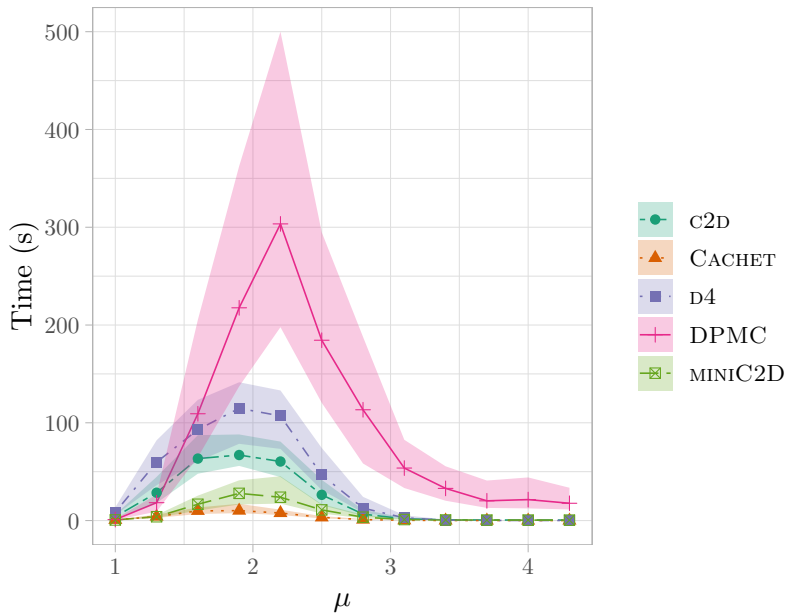
$$\frac{1 - \rho}{n - |X|} = \frac{1 - 0.3}{5 - 1} = 0.175.$$

Both x_1 and x_2 get a bonus probability of $\rho/2$ for each being the endpoint of **one** out of the **two** neighbourhood edges.

The Relationship Between ρ and Primal Treewidth



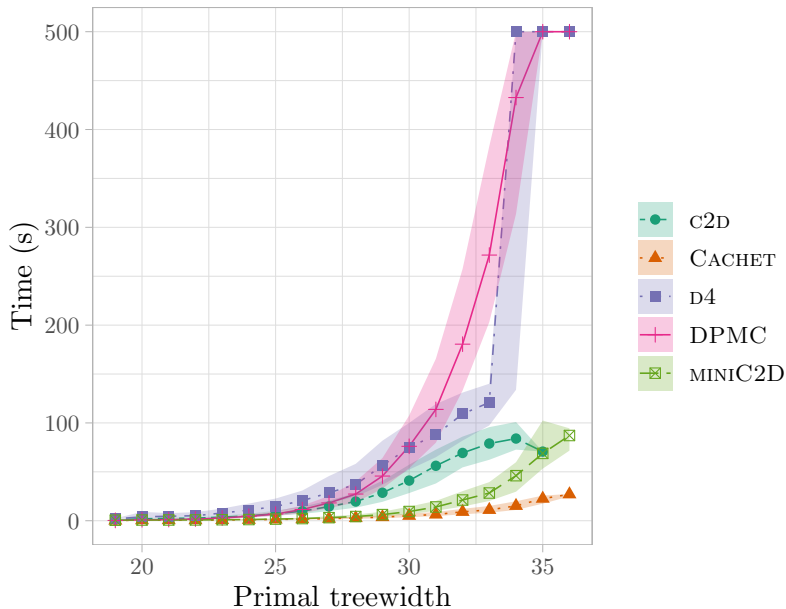
Hardness w.r.t. Density (when $\rho = 0$)



Peak Hardness w.r.t. Density

- ▶ In our experiments:
 - ▶ DPMC peaks at $\mu = 2.2$
 - ▶ all other algorithms peak at $\mu = 1.9$
- ▶ CACHET is known to peak at $\mu = 1.8$ (Sang et al. 2004)
- ▶ Bayardo Jr. and Pehoushek (2000) show some #SAT algorithms to peak at $\mu = 1.2$ and $\mu = 1.9$

Hardness w.r.t. Primal Treewidth (when $\mu = 1.9$)

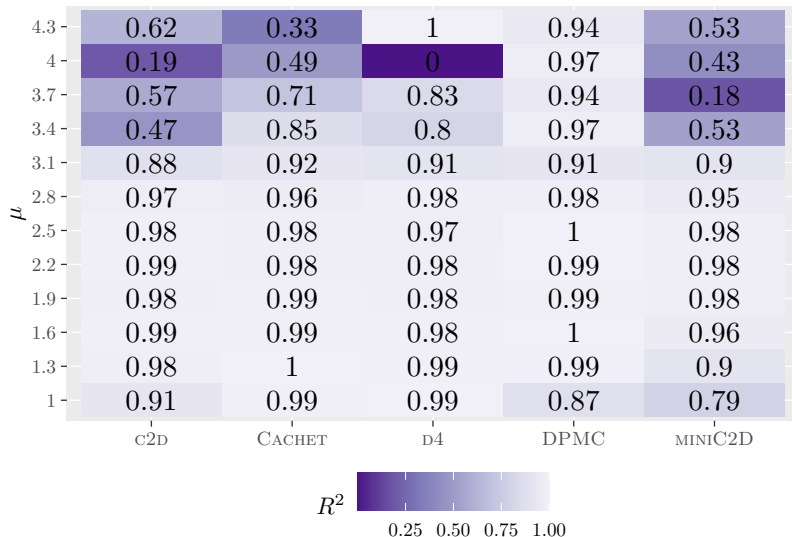


Is The Relationship Exponential?

Let us fit the model $\ln t \sim \alpha w + \beta$, i.e., $t \sim e^\beta (e^\alpha)^w$, where t is runtime, and w is primal treewidth

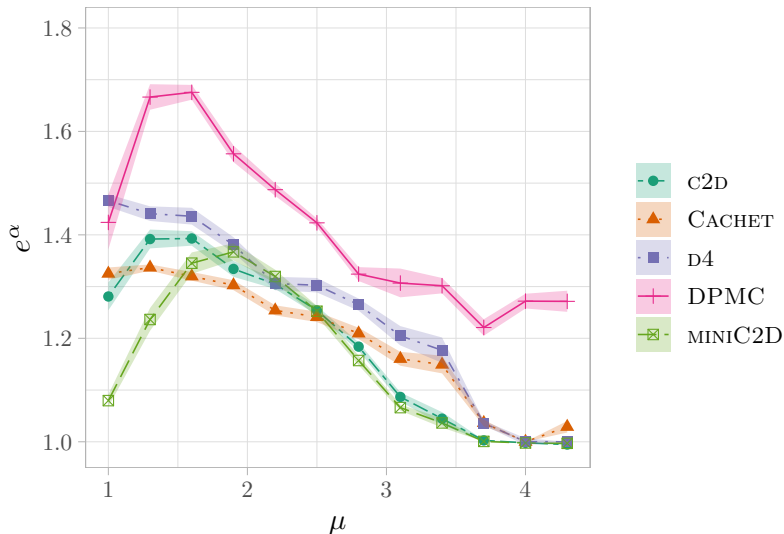
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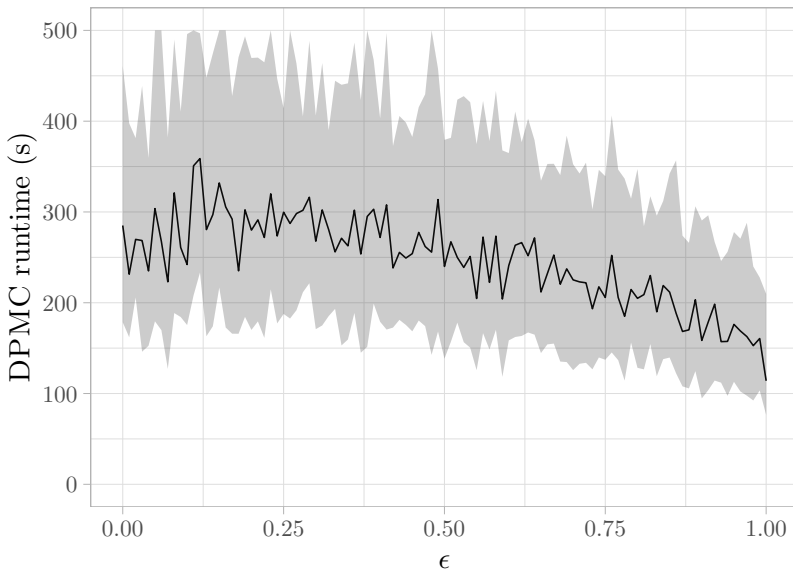
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Bonus: How DPMC Reacts to Redundancy in Weights

Let ϵ be the proportion of variables x s.t. $w(x) = w(\neg x) = 0.5$



Summary

Observations

- ▶ All algorithms scale exponentially w.r.t. primal treewidth
- ▶ The running time of DPMC peaks at a higher density and scales worse w.r.t. primal treewidth
- ▶ MINIC2D seems to be best on low density high primal treewidth instances

Future work

- ▶ A theoretical relationship between ρ and primal treewidth
- ▶ Non- k -CNF instances
- ▶ Algorithm portfolios for WMC