Recursive Solutions to First-Order Model Counting

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Some Elementary Counting

A Counting Problem

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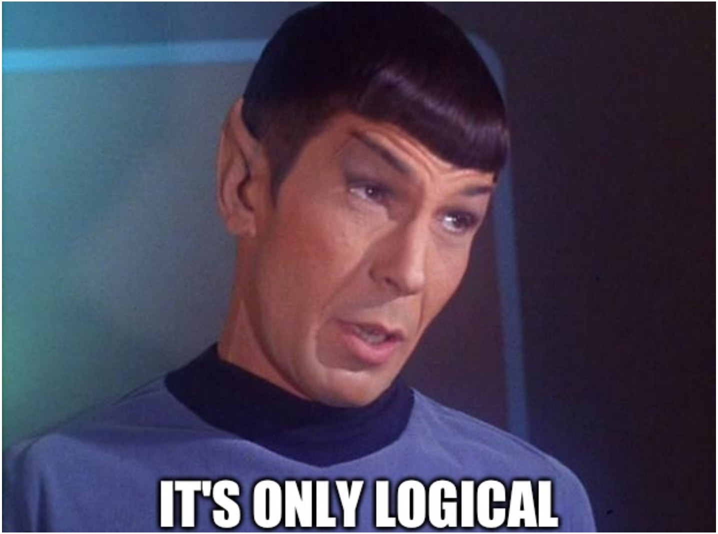
Answer: $n^{\underline{m}} = n \cdot (n-1) \cdots (n-m+1)$.
Note: this problem is equivalent to counting $[m] \to [n]$ injections.

Let's Express This Problem in Logic!



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- Let $P \subseteq M \times N$ be a relation (i.e., predicate) over sets M and N
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Overview of the Problem

- First-order model counting is the problem of counting the models of a sentence in first-order logic.
- The (symmetric) weighted variation of the problem adds weights (e.g., probabilities) to predicates.
- Thus, SWFOMC can also be used for efficient probabilistic inference in relational models.
- None of the (implemented) (SW)FOMC algorithms are able to count, e.g., injective and bijective functions.

Recursion to the Rescue!

- Optimal time complexity to compute $n^{\underline{m}}$ is $\Theta(\log m)$
- $$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n-1) + mf(m-1, n-1) & \text{otherwise.} \end{cases}$$