

Synthesising Recursive Functions for First-Order Model Counting

Paulius Dilkas

Joint work with Vaishak Belle (University of Edinburgh, UK)

CRIL Seminar, 25th May 2023

National University of Singapore, Singapore

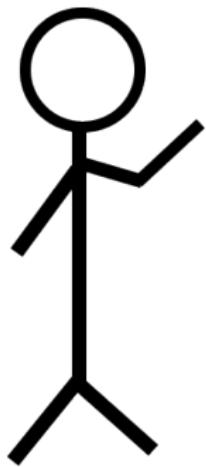


National University
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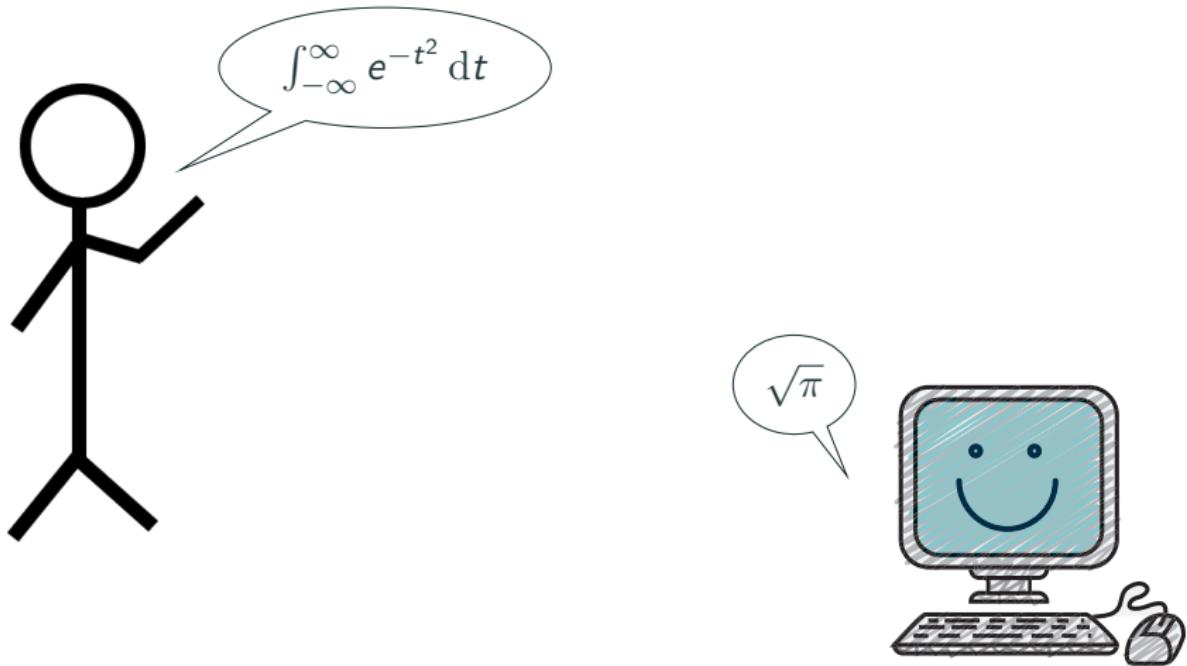


Engineering and
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Research Council

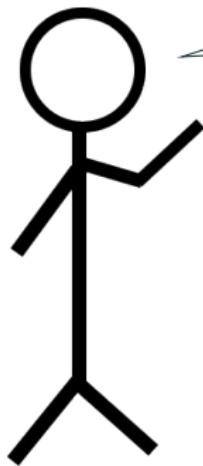
What Computers Can and Cannot Do



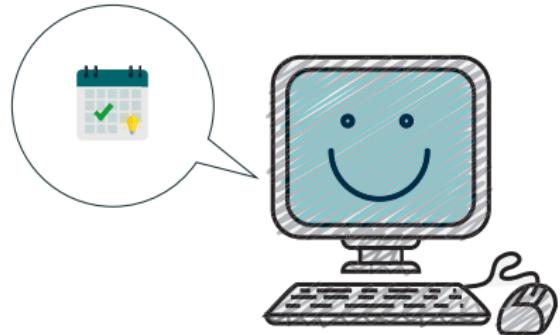
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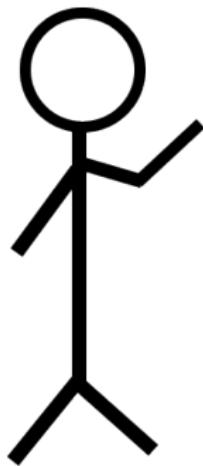
What Computers Can and Cannot Do



Produce a **schedule** for the
nurses at the local hospital.



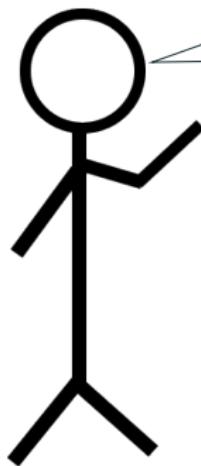
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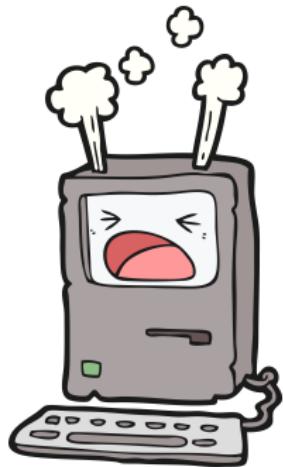
Paint a baroque oil painting of a
raccoon queen wearing a crown.



What Computers Can and Cannot Do



If I shuffle a deck of n cards,
how many possible outcomes
are there?



Terms and conditions apply.

Who Cares About Counting?

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Probabilistic Programming

*Inference and learning in probabilistic logic
programs using weighted Boolean formulas*

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS,
DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON,
GERDA JANSSENS and LUC DE RAEDT

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Neuro-symbolic AI

A Semantic Loss Function for Deep Learning with Symbolic Knowledge

Jingyi Xu¹ Ziliu Zhang² Tal Friedman¹ Yitao Liang¹ Guy Van den Broeck¹

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Natural Language Processing

Joint Inference for Knowledge Extraction from Biomedical Literature

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Robotics

**Learning Relational Affordance Models for Robots
in Multi-Object Manipulation Tasks**

Bogdan Moldovan Plinio Moreno Martijn van Otterlo José Santos-Victor Luc De Raedt

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PheNetic: Network-based interpretation of unstructured gene lists in E. coli
Dries De Maeyer¹, Joris Renkens², Lore Cloots¹, Luc De Raedt^{1,2}, Kathleen Marchal^{1,3}

¹Center of Microbial and Plant Genetics, Kasteelpark Arenberg 20, B-3001, Leuven, Belgium

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Combinatorics

Automatic Conjecturing of P-Recursions
Using Lifted Inference

Jáchym Barvínek^{1(✉)}, Timothy van Bremen², Yuyi Wang³, Filip Zelezny¹,
and Ondřej Kuzelka¹

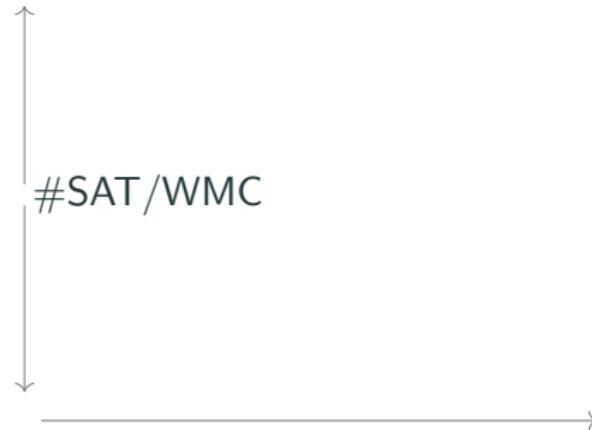
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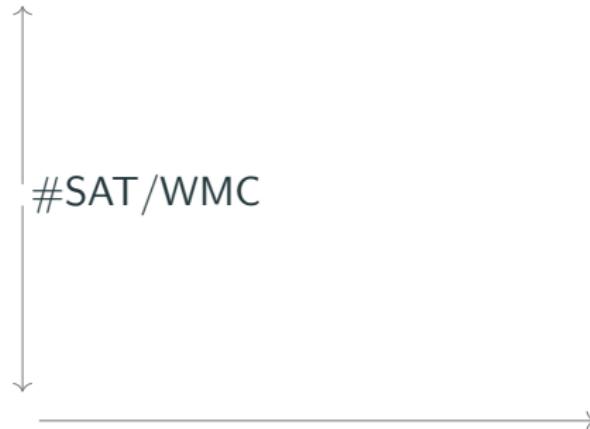
(Some of the) Many Ways to Count



#SAT (Valiant 1979)

- Input formula: $x \vee y$
- Interpretations: $\emptyset, \{x\}, \{y\}, \{x, y\}$
- Models: $\{x\}, \{y\}, \{x, y\}$

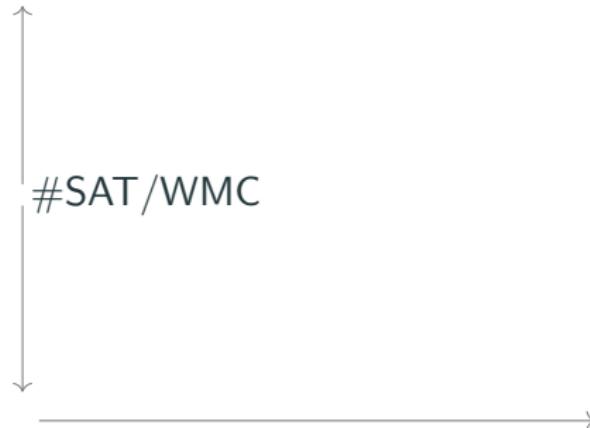
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- Answer (model count): 3

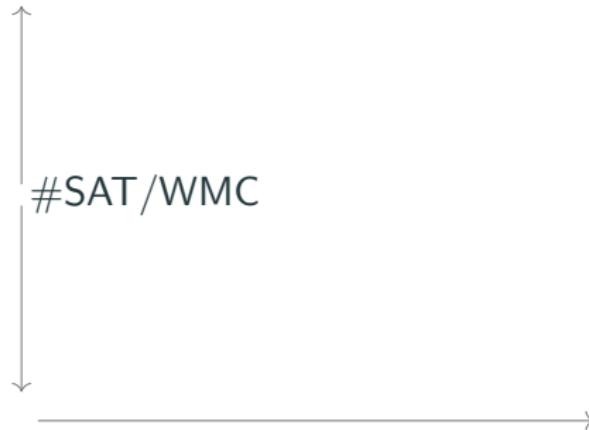
(Some of the) Many Ways to Count



Weighted Model Counting (Chavira and Darwiche 2008)

- Input formula: $x \vee y$
- Input weights: $w(x) = 0.3, w(\neg x) = 0.7,$
 $w(y) = 0.2, w(\neg y) = 0.8$

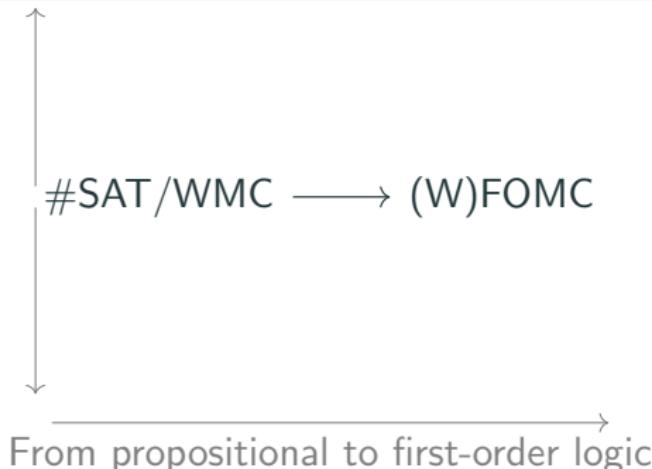
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 $w(y) = 0.2, w(\neg y) = 0.8$
- Answer (weighted model count):
 $w(x)w(y) + w(x)w(\neg y) + w(\neg x)w(y) = 0.44$

(Some of the) Many Ways to Count

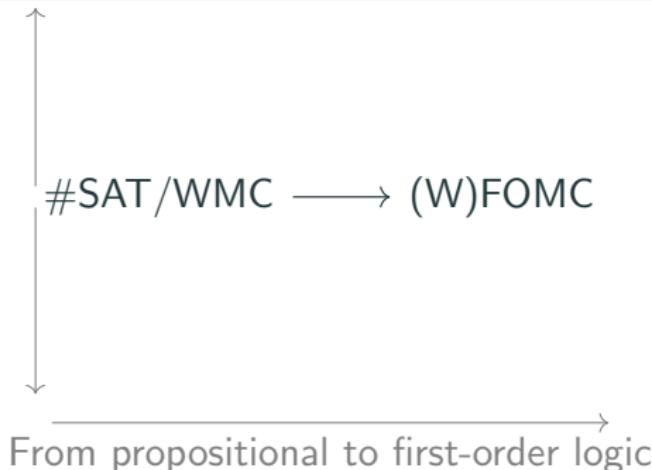


(Weighted) (Symmetric) First-Order Model Counting

(Van den Broeck et al. 2011)

- Input formula: $\forall \textcolor{teal}{x} \in \Delta. P(\textcolor{teal}{x})$
- Input weights: $w^+(P) = 0.3, w^-(P) = 0.7$
- Input domain size(s): $|\Delta| = 2$

(Some of the) Many Ways to Count

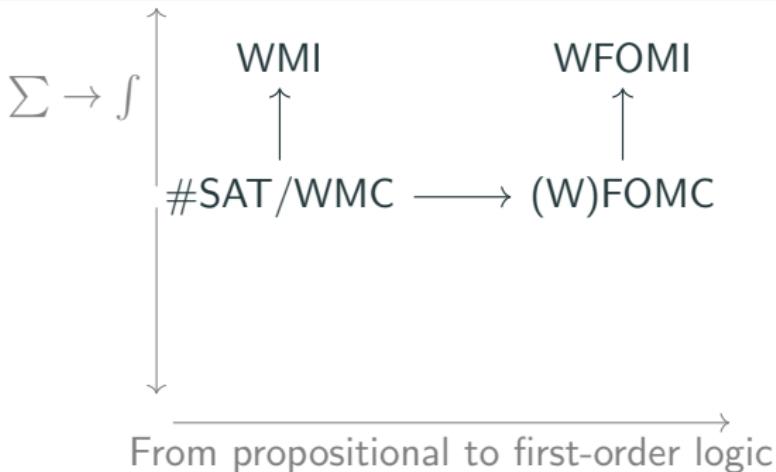


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- Answer: $(w^+(P))^{|\Delta|} = 0.09$

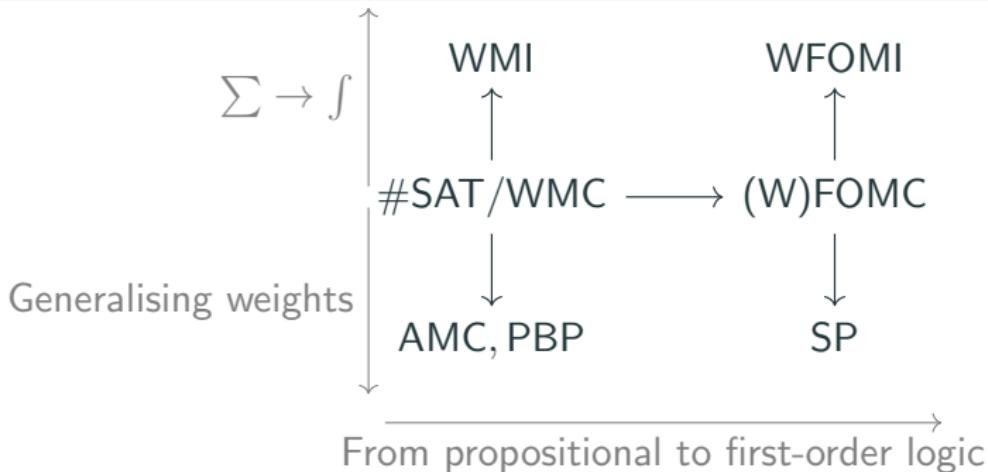
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Extensions to Continuous Domains

- Weighted model integration
 - (Belle, Passerini and Van den Broeck 2015)
- Weighted first-order model integration
 - (Feldstein and Belle 2021)

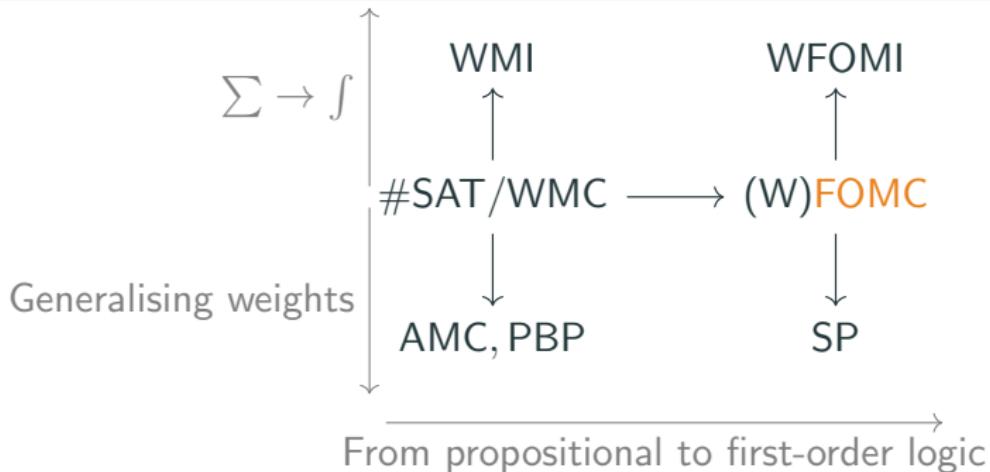
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Generalisations of the Weight Function

- Algebraic model counting
 - (Kimmig, Van den Broeck and De Raedt 2017)
 - From $\mathbb{R}_{\geq 0}$ to commutative semirings
 - Pseudo-Boolean projection (D. and Belle 2021)
 - Weights not necessarily on literals
 - Semiring programming (Belle and De Raedt 2020)

(Some of the) Many Ways to Count



(Unweighted) First-Order Model Counting

- Example formula:

$$\forall x \in \Delta. P(x) \vee Q(x).$$

- Let $\Delta := \{1, 2\}$.

- Interpretations: all subsets of

$$\{P(1), Q(1), P(2), Q(2)\}.$$

(Unweighted) First-Order Model Counting

- Example formula:

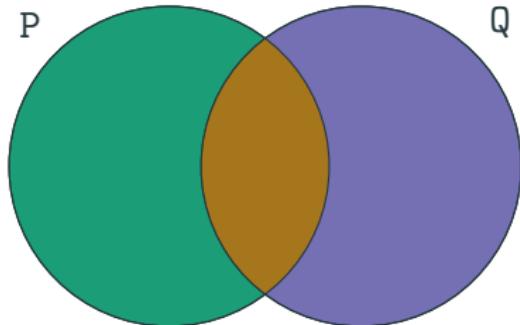
$$\forall x \in \Delta. P(x) \vee Q(x).$$

- Let $\Delta := \{1, 2\}$.

- Interpretations: all subsets of $\{P(1), Q(1), P(2), Q(2)\}$.

- Models:

$$\begin{array}{lll} \{P(1), P(2)\}, & \{P(1), Q(2)\}, & \{P(1), P(2), Q(2)\}, \\ \{Q(1), P(2)\}, & \{Q(1), Q(2)\}, & \{Q(1), P(2), Q(2)\}, \\ \{P(1), Q(1), P(2)\}, & \{P(1), Q(1), Q(2)\}, & \{P(1), Q(1), P(2), Q(2)\}. \end{array}$$



Intuition

- Each 1-ary predicate is like a subset.
- For $n > 1$, each n -ary predicate is like a relation.
- FOMC counts combinations of relations.

More Formally: What Is the Input?

$$\forall x \in \Gamma. \ \forall y, z \in \Delta. \ P(x, y) \wedge P(x, z) \Rightarrow y = z$$

Many-Sorted Function-Free First-Order Logic with Equality

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- Any number of variables
- All variables are bound

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Many-Sorted Function-Free First-Order Logic with Equality

- Any number of variables
- All variables are bound
- \forall and \exists quantifiers can be nested arbitrarily deep
- All domains are finite
- Predicates can have any arity

Exact Algorithms for FOMC

- **ForcLift** (Van den Broeck et al. 2011)
 - knowledge compilation to **FO d-DNNF**
- **L2C** (Kazemi and Poole 2016)
 - knowledge compilation to **C++ code**
- **Alchemy** (Gogate and Domingos 2016)
 - **DPLL**-style search
- **FastWFOMC** (van Bremen and Kuželka 2021)
 - knowledge compilation to **sd-DNNF**

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Our Contribution



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ForcLift

Recursion

Crane

ForcLift and First-Order Knowledge Compilation

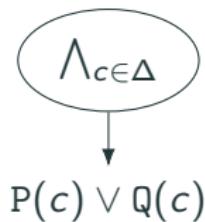
$$\forall x \in \Delta. \ P(x) \vee Q(x)$$

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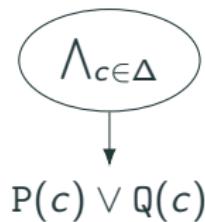
Independent partial grounding (introduces a constant $c \in \Delta$)

ForcLift and First-Order Knowledge Compilation



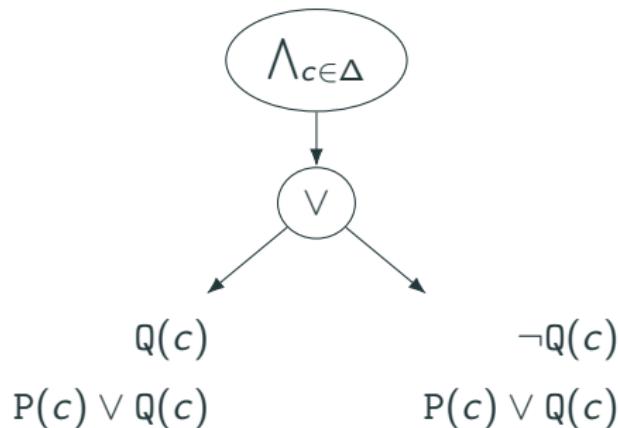
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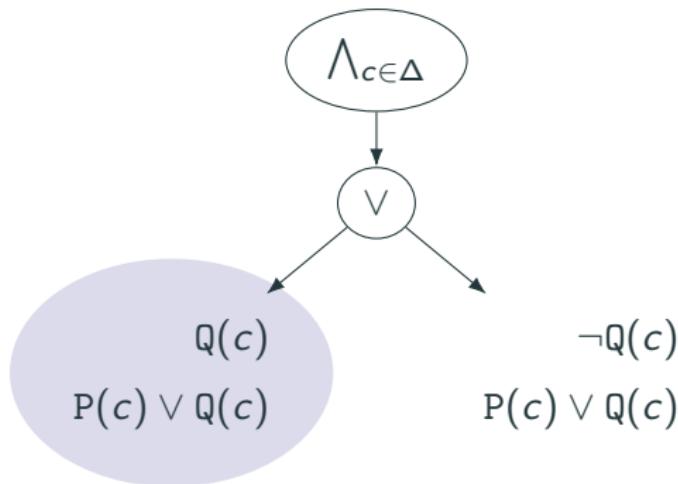
Shannon decomposition (a.k.a. Boole's expansion theorem) on $Q(c)$

ForcLift and First-Order Knowledge Compilation



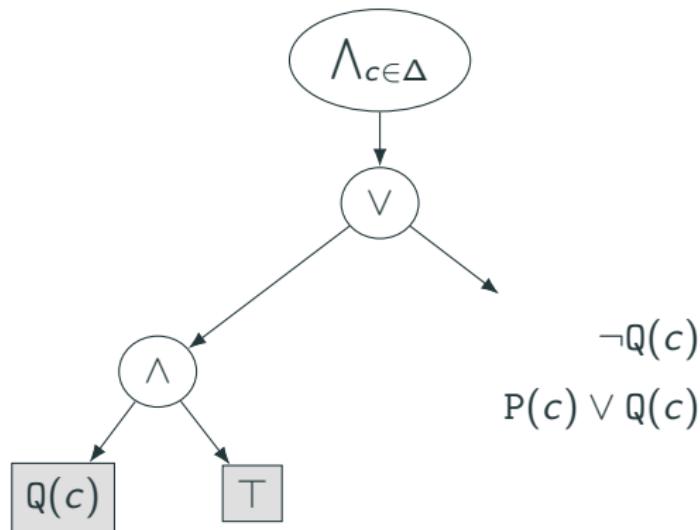
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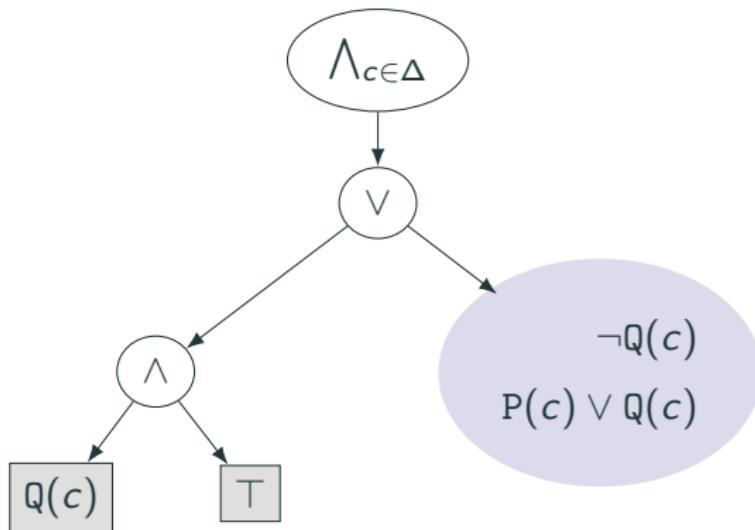
Positive unit propagation of $Q(c)$

ForcLift and First-Order Knowledge Compilation



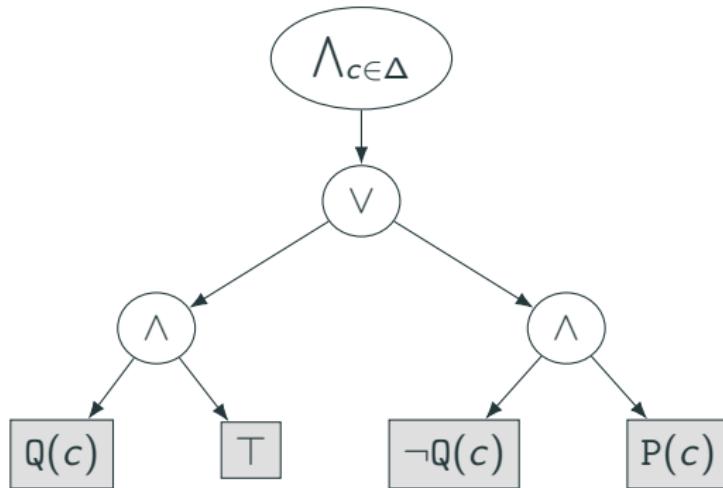
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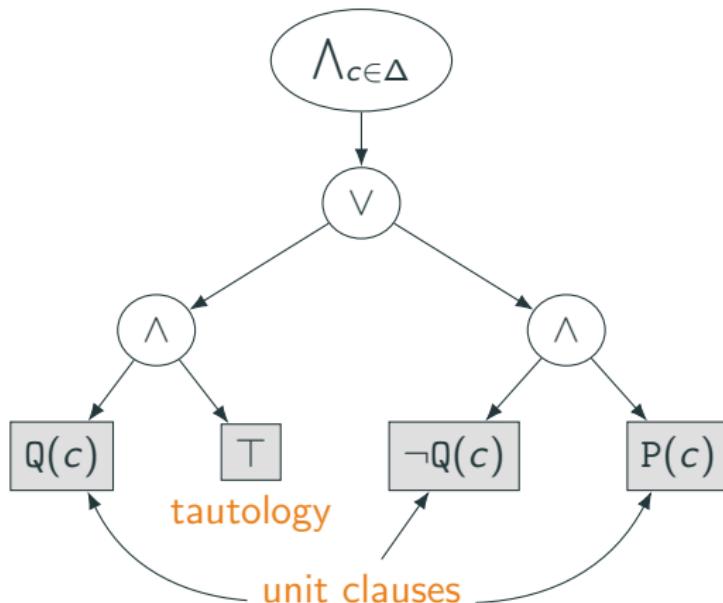
Negative unit propagation of $\neg Q(c)$

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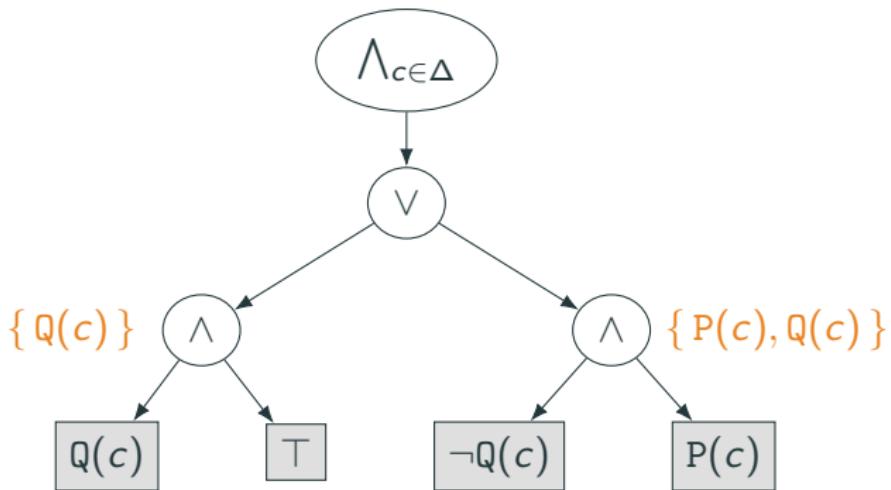
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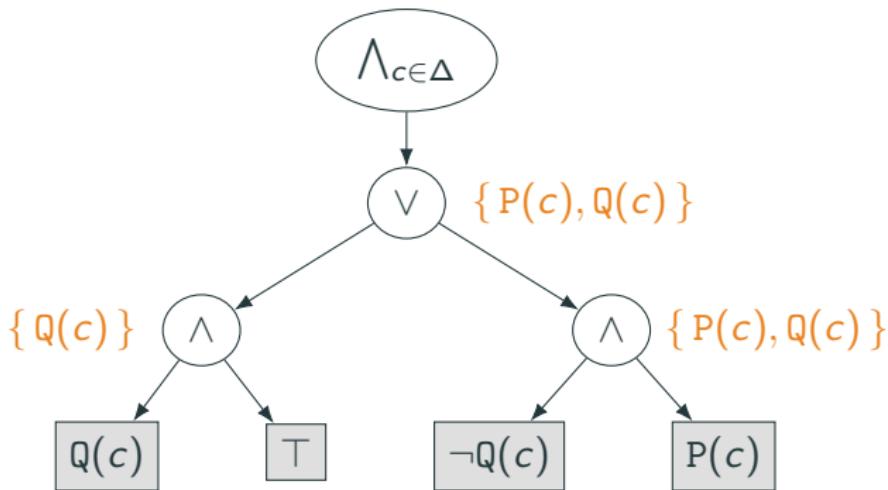
Compilation is complete ✓

ForcLift and First-Order Knowledge Compilation



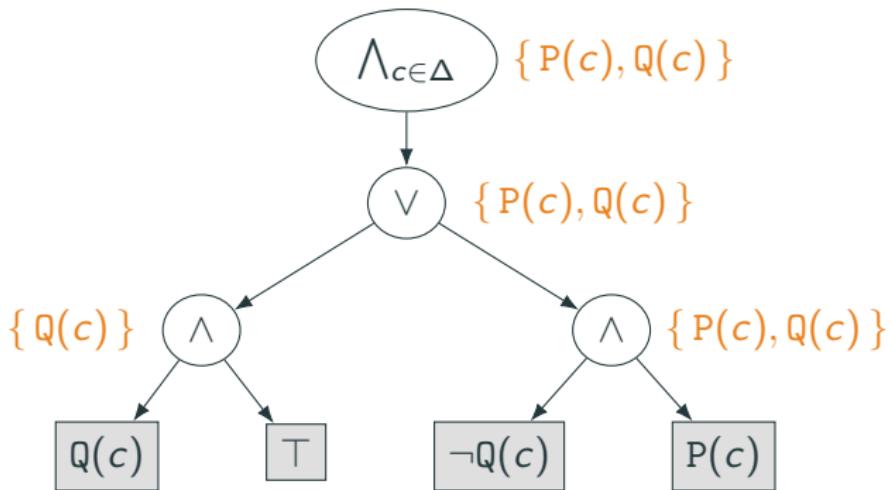
Smoothing: propagating atoms upwards

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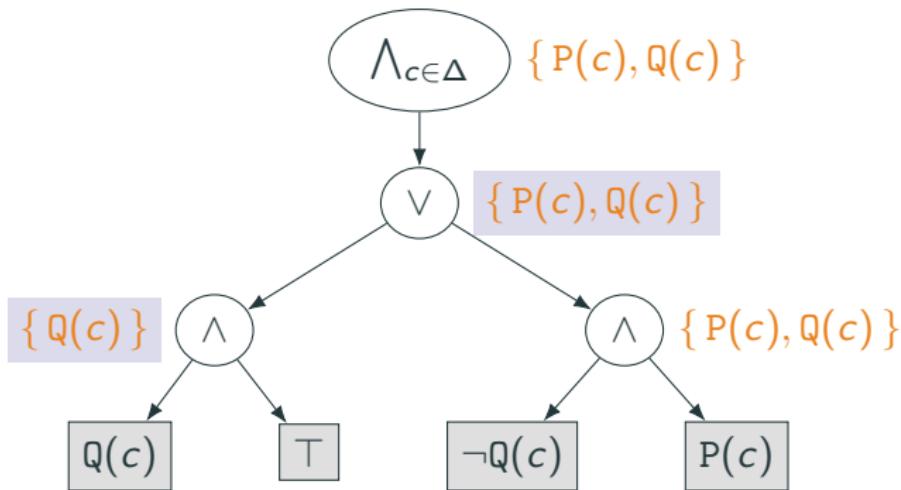
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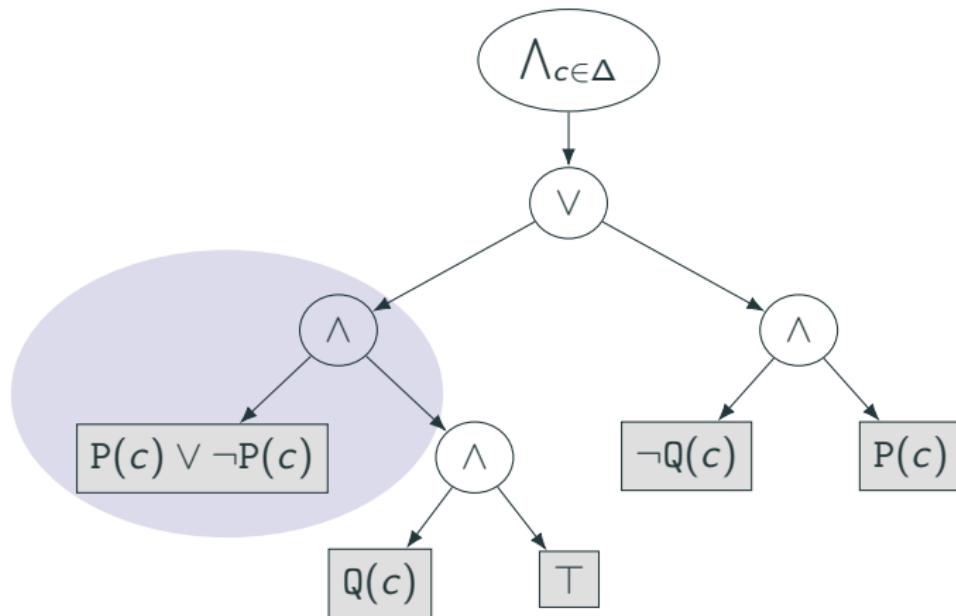
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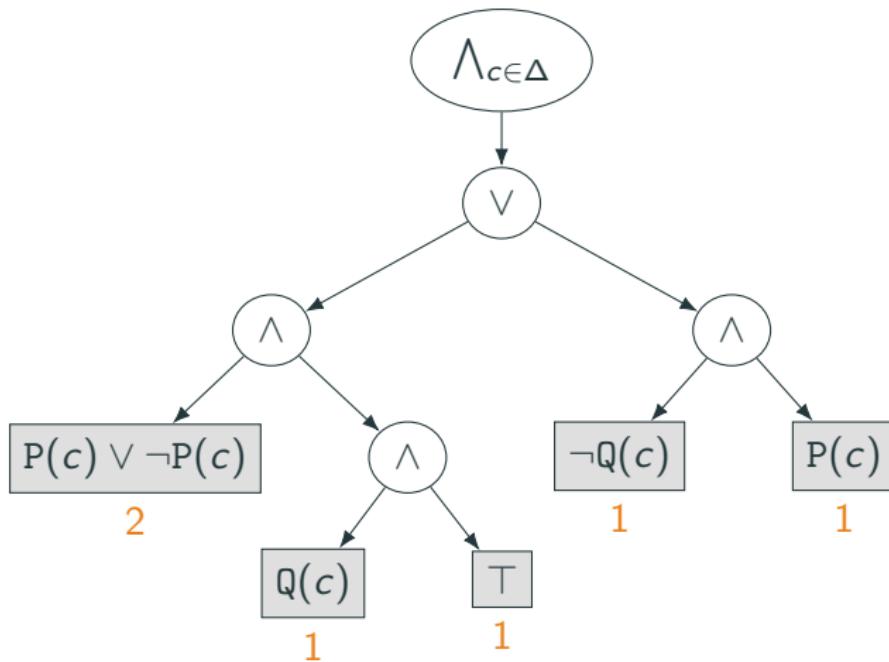
Smoothing: adding new atoms

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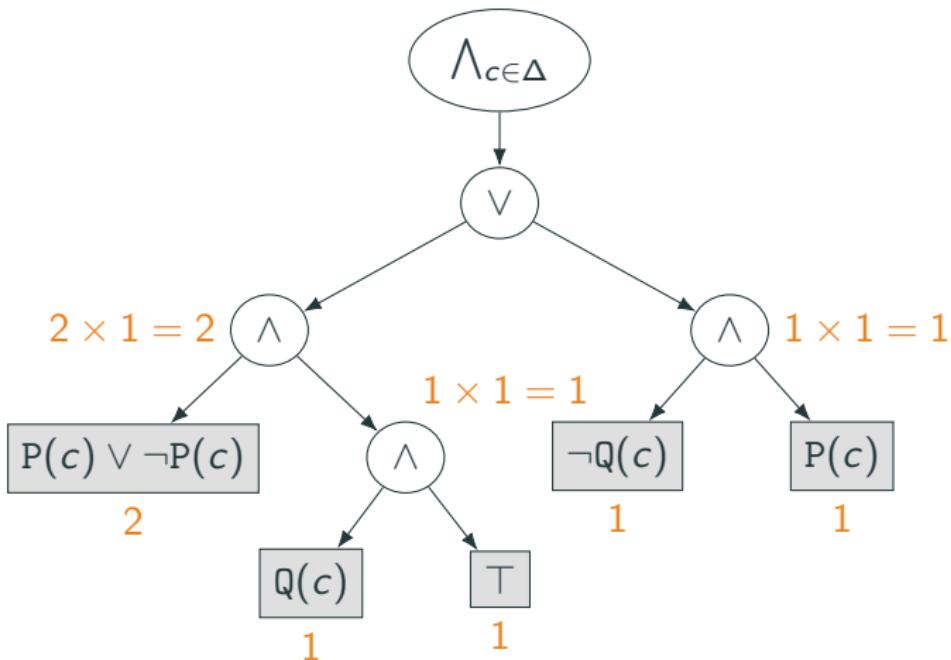
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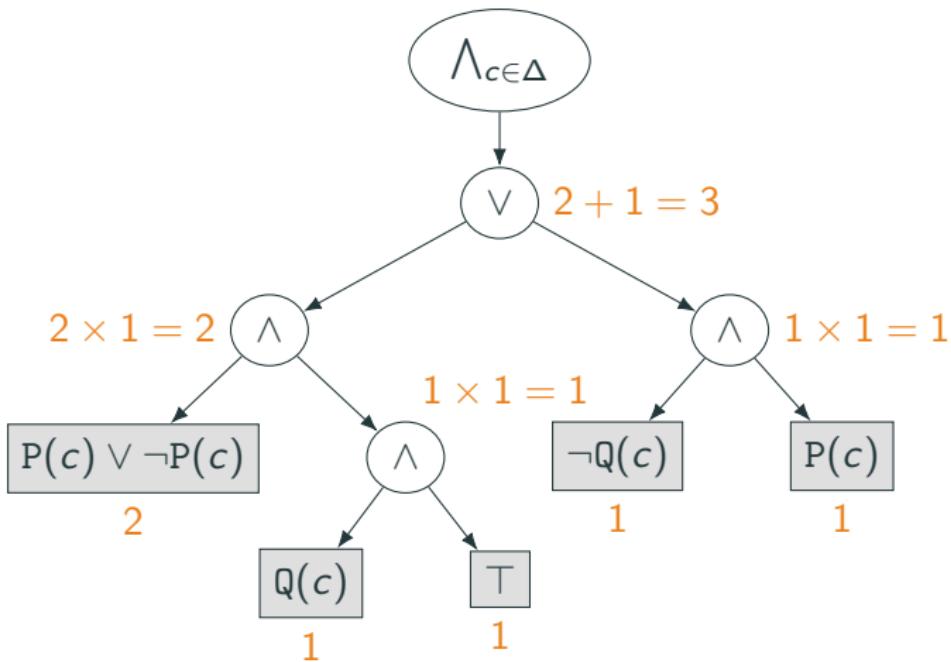
Propagating the model count

ForcLift and First-Order Knowledge Compilation



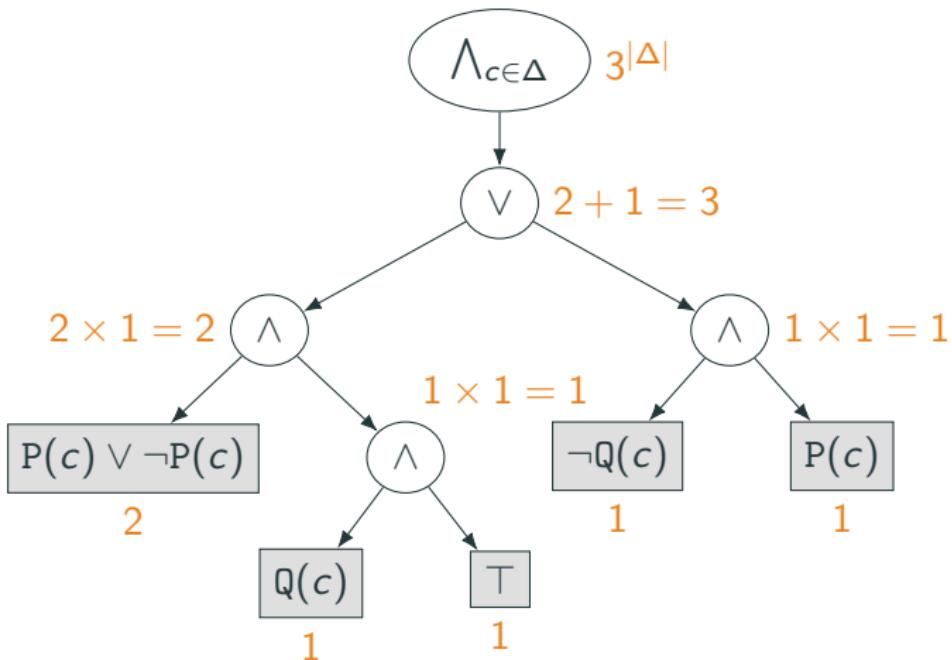
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Propagating the model count

A (Slightly) More Complicated Example

Suppose this room has n seats, and there are $m \leq n$ people in the audience. How many ways are there to seat everyone?

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More explicitly, we assume that:

- each attendee gets exactly one seat,
- and a seat can accommodate at most one person.

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More explicitly, we assume that:

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- and a seat can accommodate at most one person.

Answer: $n^m = n \cdot (n - 1) \cdots (n - m + 1)$.

Note: this problem is equivalent to counting $[m] \rightarrow [n]$ injections.

Let's Express This Problem in Logic!

- Let Γ and Δ be sets (i.e., domains)
 - such that $|\Gamma| = m$, and $|\Delta| = n$.
- Let $P \subseteq \Gamma \times \Delta$ be a relation (i.e., predicate) over Γ and Δ .
- We can describe all of the constraints in first-order logic:

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- We can describe all of the constraints in first-order logic:
 - each attendee gets a seat (i.e., at least one seat)

$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \tag{1}$$

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- Let $P \subseteq \Gamma \times \Delta$ be a relation (i.e., predicate) over Γ and Δ .
- We can describe all of the constraints in first-order logic:
 - each attendee gets a seat (i.e., at least one seat)

$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \quad (1)$$

- one person cannot occupy multiple seats

$$\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z \quad (2)$$

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(1) and (2) constrain P to be a function, and (3) makes it injective.

Recursion



Back to Our Example

The following function counts injections:

$$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n - 1) + m \times f(m - 1, n - 1) & \text{otherwise.} \end{cases}$$

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- But $\Theta(mn)$ is still much better than translating to propositional logic and solving a $\#P$ -complete problem.

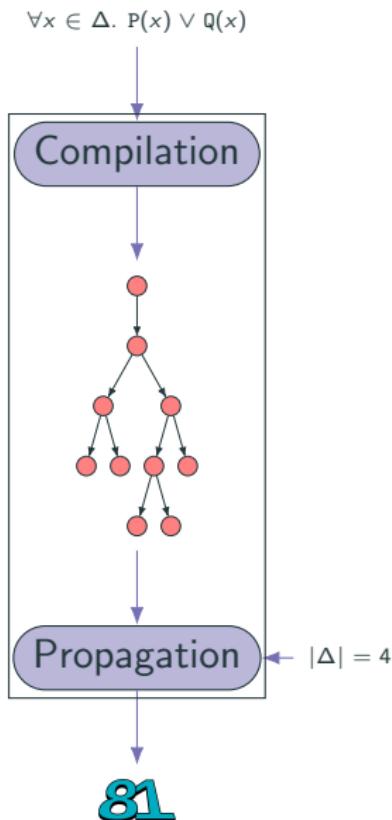
Back to Our Example

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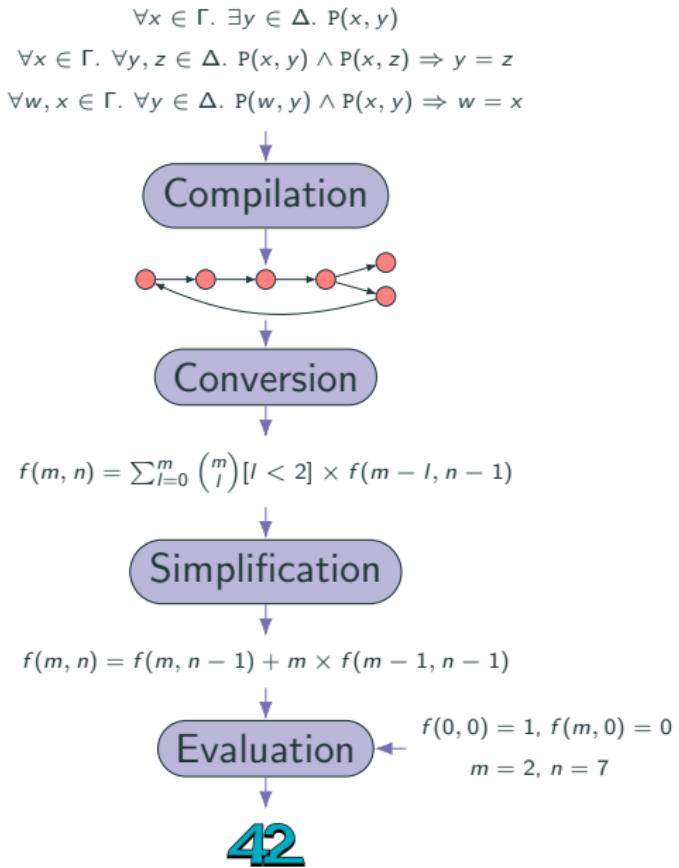
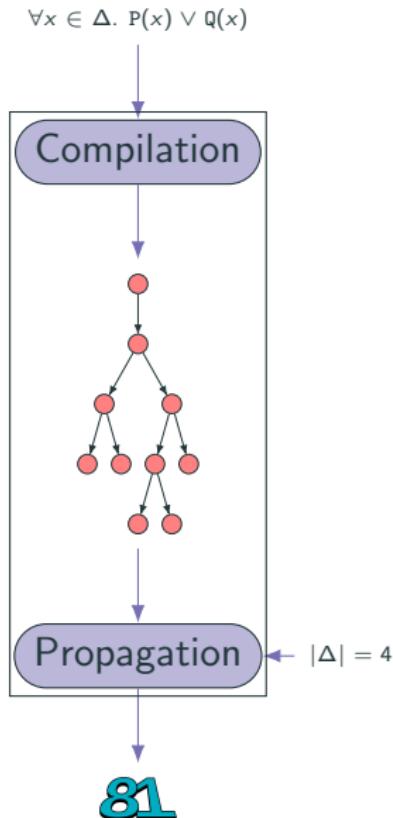
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- Optimal time complexity to compute n^m is $\Theta(m)$.
- But $\Theta(mn)$ is still much better than translating to propositional logic and solving a **#P-complete** problem.
- The rest of this talk is about how to construct such functions automatically.

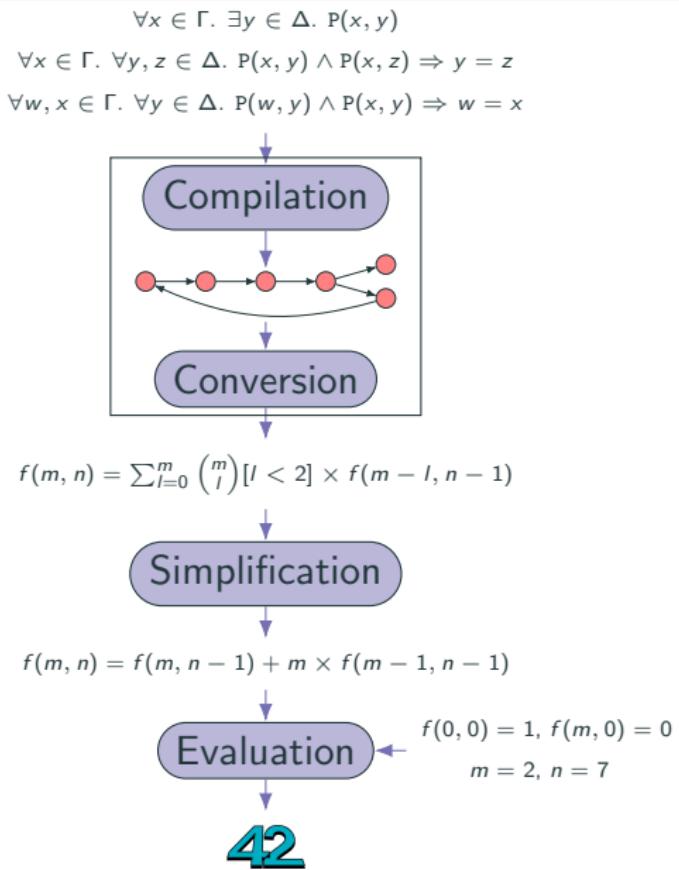
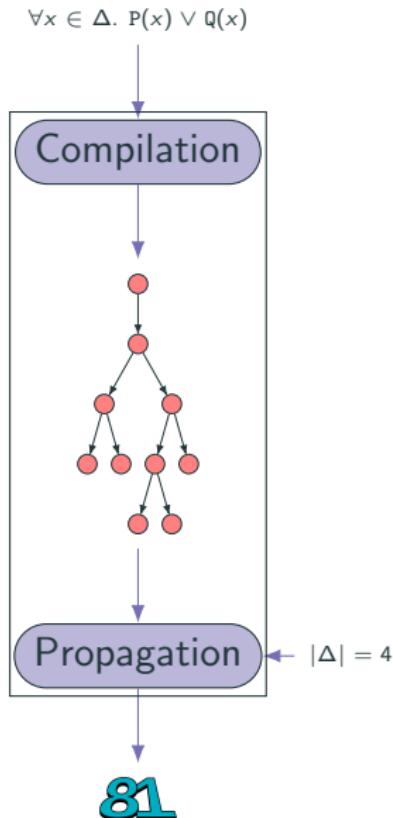
First-Order Knowledge Compilation: Before and After



First-Order Knowledge Compilation: Before and After



First-Order Knowledge Compilation: Before and After



Circuits vs Graphs

Circuits (Van den Broeck et al. 2011)...

- ... extend d-DNNF circuits (Darwiche 2001) for propositional knowledge compilation with **more node types**
- ... are **acyclic**.

Circuits vs Graphs

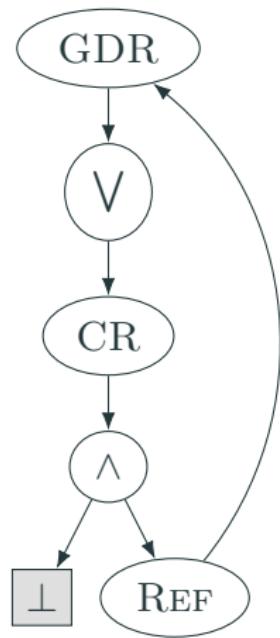
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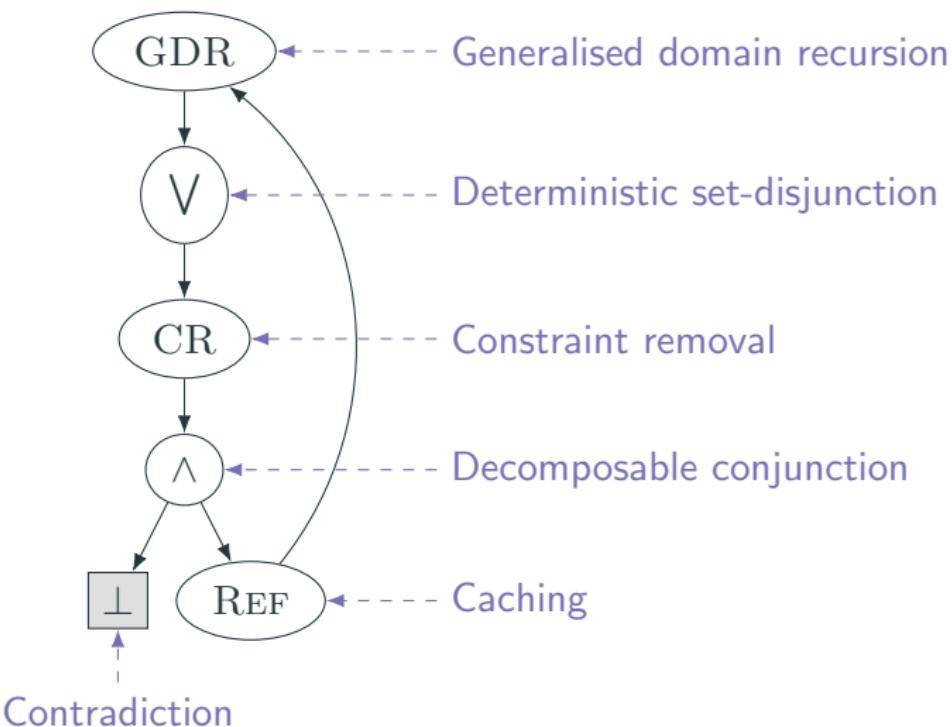
First-Order Computational Graphs (FCGs) are...
directed **acyclic** (weakly connected) graphs with:

- a single source,
- labelled nodes,
- and ordered outgoing edges.

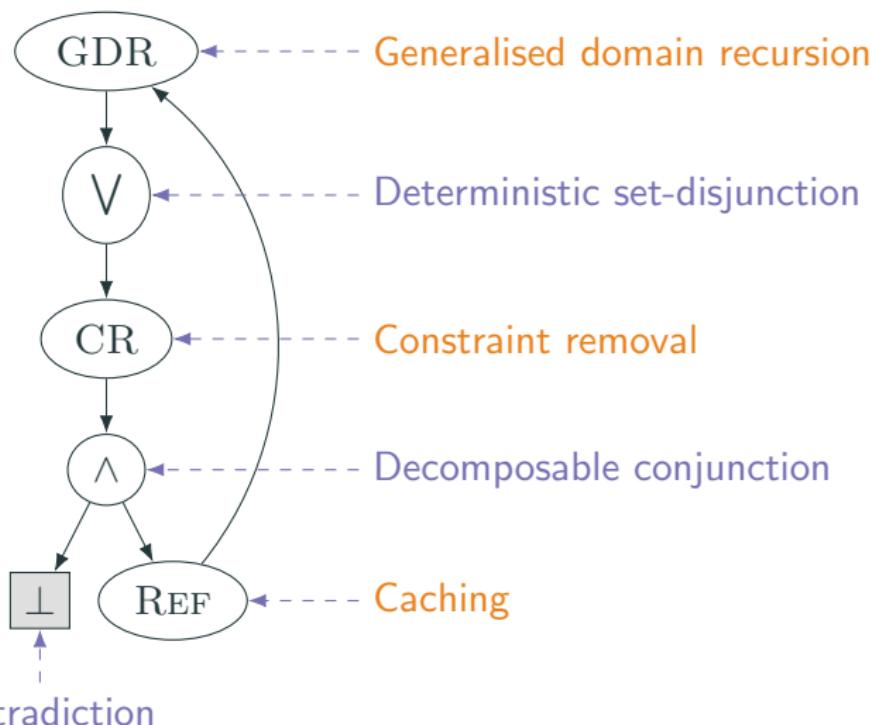
How to Interpret an FCG



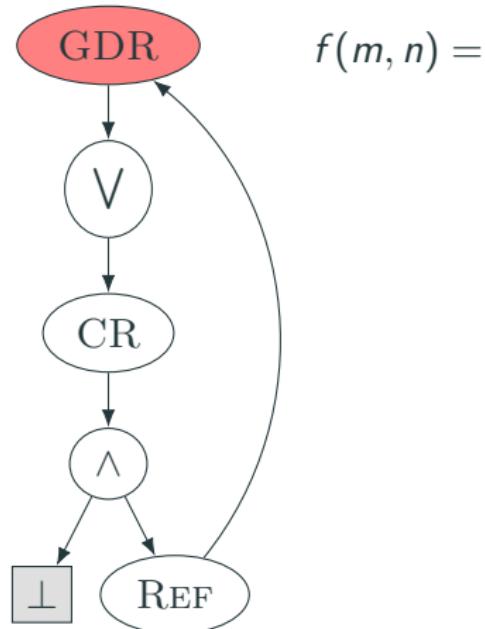
How to Interpret an FCG



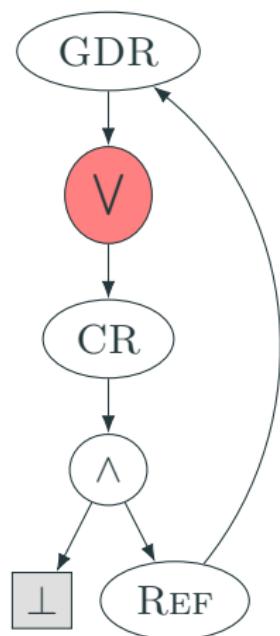
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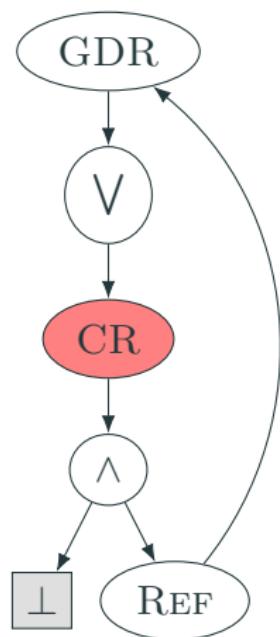


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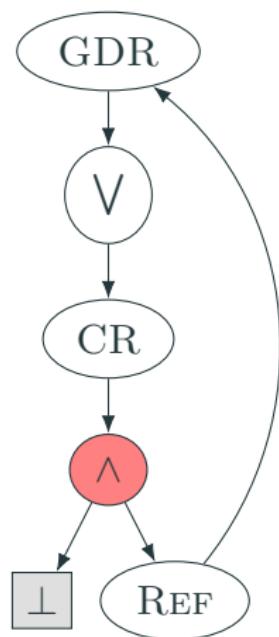
$$f(m, n) = \sum_{l=0}^m \binom{m}{l}$$

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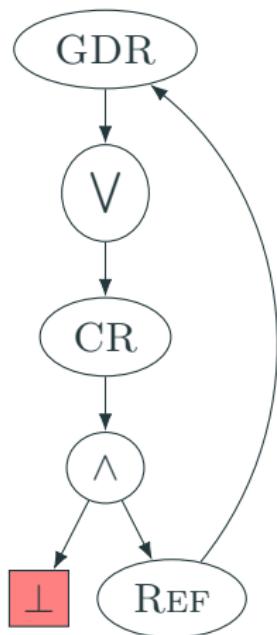
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$$f(m, n) = \sum_{l=0}^m \binom{m}{l} \quad \times$$

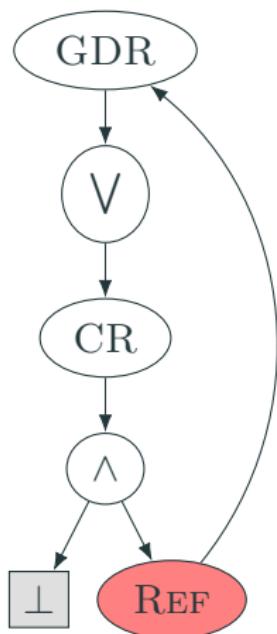
How to Interpret an FCG



$$f(m, n) = \sum_{l=0}^m \binom{m}{l} [l < 2] \times$$

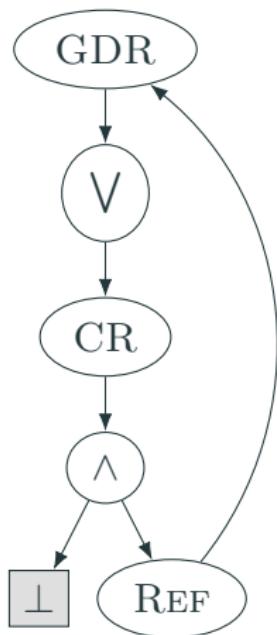
$$[\phi] = \begin{cases} 1 & \text{if } \phi \\ 0 & \text{if } \neg\phi \end{cases}$$

How to Interpret an FCG



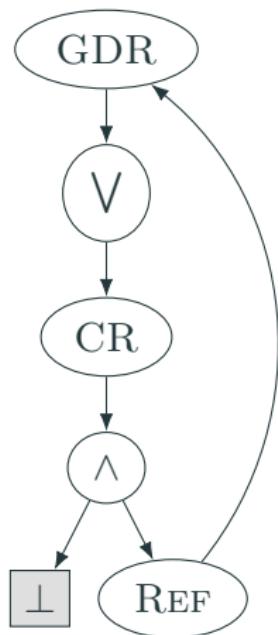
$$f(m, n) = \sum_{l=0}^m \binom{m}{l} [l < 2] \times f(m - l, n - 1)$$

How to Interpret an FCG



$$\begin{aligned}f(m, n) &= \sum_{l=0}^m \binom{m}{l} [l < 2] \times f(m - l, n - 1) \\&= \binom{m}{0} \times f(m - 0, n - 1) \\&\quad + \binom{m}{1} \times f(m - 1, n - 1)\end{aligned}$$

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Compilation: How FCGs Are Built

Definition

A **(compilation) rule** is a function that takes a **formula** and returns a set of **(G, L)** pairs, where

- G is a (possibly incomplete) FCG,
- and L is a list of formulas.

The formulas in L are then **compiled**, and the resulting FCGs are **inserted** into G according to a **set order**.

Example Compilation Rule: Independence

Input formula:

$$(\forall x, y \in \Omega. x = y) \wedge \quad (1)$$

$$(\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z) \wedge \quad (2)$$

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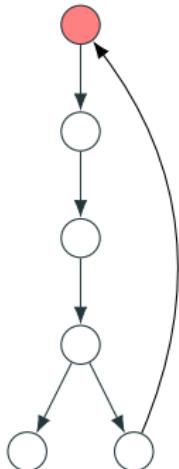
The independence compilation rule returns one (G, L) pair:



New Rule 1/3: Generalised Domain Recursion

Example

Input formula:



$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

Output formula (with a new constant $c \in \Gamma$):

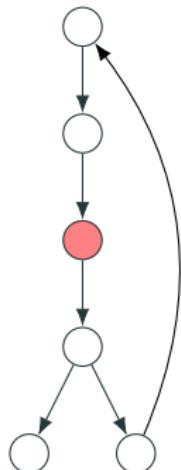
$$\forall y, z \in \Delta. y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$

$$\begin{aligned} \forall x \in \Gamma. \forall y, z \in \Delta. & x \neq c \wedge y \neq z \Rightarrow \\ & \neg P(x, y) \vee \neg P(x, z) \end{aligned}$$

New Rule 2/3: Constraint Removal

Example

Input formula (with a constant $c \in \Gamma$):



$$\forall x \in \Gamma. \forall y, z \in \Delta. x \neq c \wedge y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

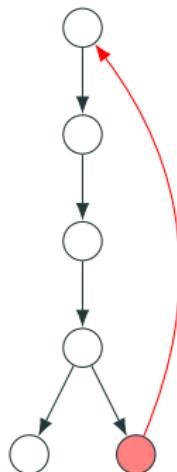
$$\forall w, x \in \Gamma. \forall y \in \Delta. w \neq c \wedge x \neq c \wedge w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

Output formula (with a new domain $\Gamma' := \Gamma \setminus \{c\}$):

$$\forall x \in \Gamma'. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

New Rule 3/3: Identifying Possibilities for Recursion



Goal

Check if the input formula is equivalent (up to domains) to a previously encountered formula.

Outline

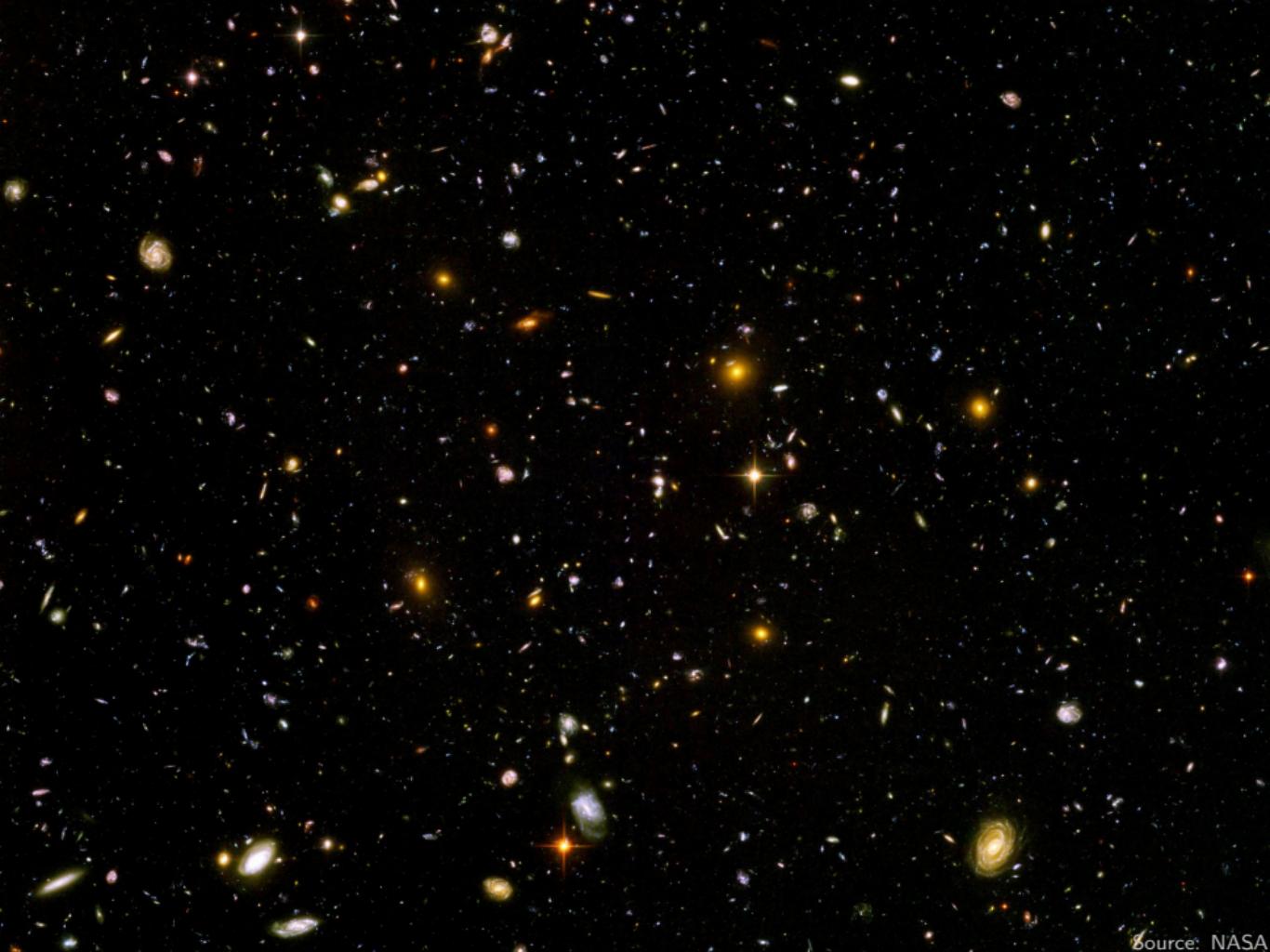
1. Consider pairs of 'similar' clauses.
2. Consider bijections between their sets of variables.
3. Extend each such bijection to a map between sets of domains.
4. If the bijection makes the clauses equivalent, and the domain map is compatible with previous domain maps, move on to another pair of clauses.

Resulting Improvements to Counting Functions

Let Γ and Δ be two sets with cardinalities $|\Gamma| = m$ and $|\Delta| = n$.

Our new rules enable Crane to efficiently count $\Gamma \rightarrow \Delta$ functions such as:

- injections in $\Theta(mn)$ time
 - by hand: $\Theta(m)$
- partial injections in $\Theta(mn)$ time
 - by hand: $\Theta(\min\{m, n\}^2)$
- bijections in $\Theta(m)$ time
 - optimal!



Source: NASA

What Have We Learned?

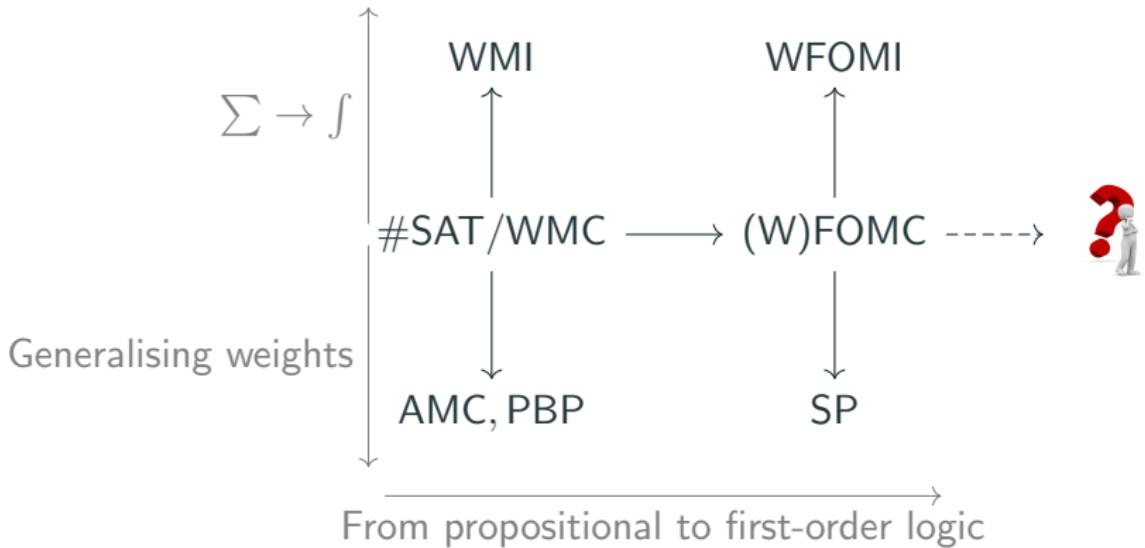
- Knowledge compilation can build **graphs with cycles**.
- Graphs (as well as circuits) define **functions**.
- Cycles can represent **recursive calls**, including:
 - mutual recursion
 - and function calls as complex as $f(n - k - 2)$.
- Recursion helps us solve counting problems that were previously beyond the reach of FOMC.
- In some cases, even if a polynomial-time solution is already known, Crane is able to find **more efficient solutions**, with a lower degree polynomial.

Future →

← Past



Beyond First-Order Logic



What kind of logic is needed to succinctly describe, e.g.,

- $f(n) = f(f(n - 1))$
- or the Fibonacci sequence?

Algebraic Solutions to Parameterised Problems (1/2)

- Suppose we have a Markov logic network that models the probability P that some system will fail.
- Here:
 - domain sizes describe the numbers of various components,
 - and weights express probabilities that:
 - some component fails,
 - or some combination of failures leads to another failure.
- Crane can express P as a function of the domain sizes and weights.

Algebraic Solutions to Parameterised Problems (2/2)

With the help of a computer algebra system, we can then:

- determine how P scales with the number of users,
- find combinations of domain sizes that keep P below some threshold,
- find ranges of weights that keep P sufficiently small across a range of domain size values.

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Reasoning with **functions**



Reasoning with **numbers**

How These Rules Fit Together (1/5)

$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$
$$\forall w, x \in \Gamma. \forall y \in \Delta. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$


Generalised domain recursion


$$\forall y, z \in \Delta. y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$
$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$
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Atom counting and unit propagation



$$\forall y, z \in \Delta^{\top}. \ y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma. \ \forall y, z \in \Delta^{\perp}. \ y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma. \ \forall y \in \Delta^{\perp}. \ w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

How These Rules Fit Together (3/5)

$$\forall y, z \in \Delta^\top. y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma. \forall y, z \in \Delta^\perp. y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma. \forall y \in \Delta^\perp. w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Constraint removal



$$\forall y, z \in \Delta^\top. y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

How These Rules Fit Together (4/5)

$$\forall y, z \in \Delta^T. y \neq z \Rightarrow \perp$$

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$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Independence

$$\forall y, z \in \Delta^T. y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

How These Rules Fit Together (5/5): Recursion

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

||

$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

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+

$$\{ \Gamma \mapsto \Gamma', \Delta \mapsto \Delta^\perp \}$$