Recursive Solutions to First-Order Model Counting

Paulius Dilkas, Vaishak Belle

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Some Elementary Counting

A Counting Problem

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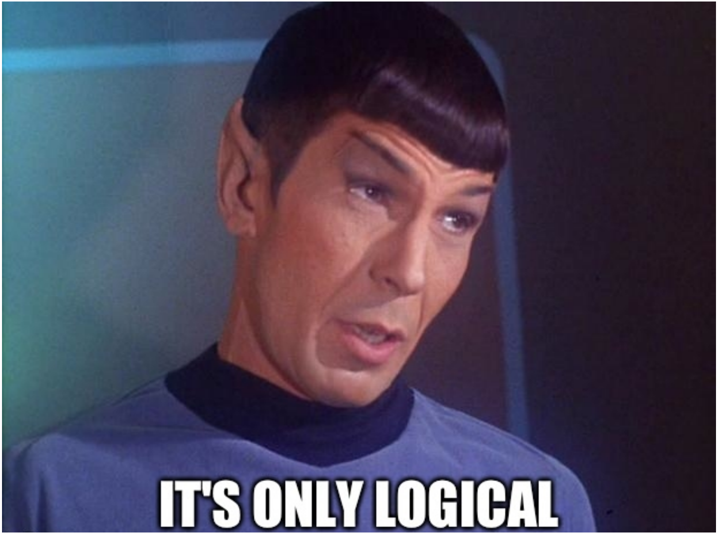
Answer: $n^{\underline{m}} = n \cdot (n-1) \cdots (n-m+1)$.

Note: this problem is equivalent to counting $[m] \to [n]$ injections.

Let's Express This Problem in Logic!



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- Let M and N be sets (i.e., domains) such that $|M| = m$, and $|N| = n$
- Let $P \subseteq M \times N$ be a relation (i.e., predicate) over sets M and N
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The first two sentences constrain P to be a function, and the last one makes it injective.

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Overview of the Problem

- First-order model counting is the problem of counting the models of a sentence in first-order logic.
- The (symmetric) weighted variation of the problem adds weights (e.g., probabilities) to predicates.
- Thus, SWFOMC can also be used for efficient probabilistic inference in relational models.
- None of the (implemented) (SW)FOMC algorithms are able to count, e.g., injective and bijective functions.

Claim

This shortcoming can be addressed via support for (almost arbitrary) recursive functions.

Back to Our Example

For instance, the following function counts injections

$$f(m,n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m,n-1) + mf(m-1,n-1) & \text{otherwise.} \end{cases}$$

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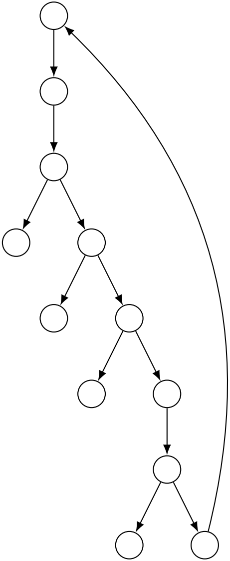
- f can be computed in $\Theta(mn)$ time (via dynamic programming).
- Optimal time complexity to compute $n^{\underline{m}}$ is $\Theta(\log m)$.
- But $\Theta(mn)$ is still much better than solving an equivalent $\#$P-complete problem in propositional logic.
- The rest of this talk is about how such functions can be found automatically.

First-Order Knowledge Compilation



Workflow Before

1. Compile the formula to a circuit
2. Evaluate to get the answer

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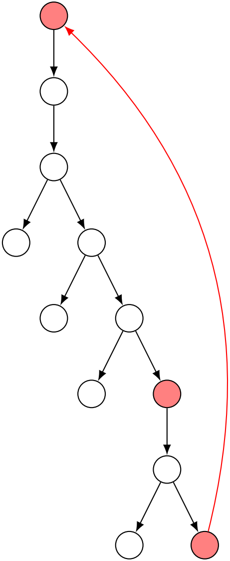

Workflow Before

1. Compile the formula to a circuit
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Workflow After

1. Compile the formula to a graph
2. Extract the definitions of functions
3. Simplify
4. Supplement with base cases
5. Evaluate to get the answer

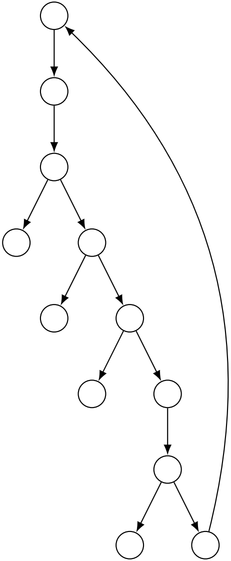
More Formally...

Definition

A first-order deterministic decomposable negation normal form computational graph (FCG) is a

- directed graph
- (which is weakly connected)
- with a single source,
- labelled vertices,
- and ordered outgoing edges.

How to Interpret an FCG



$f(m, n) =$

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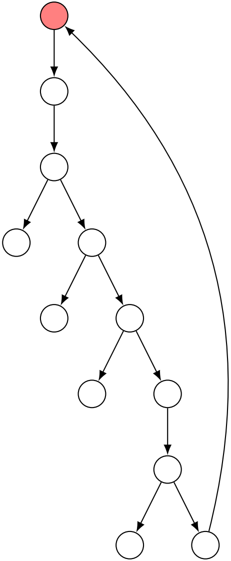

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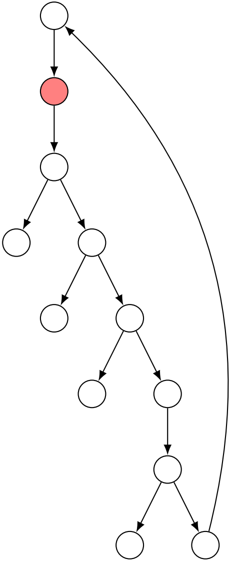

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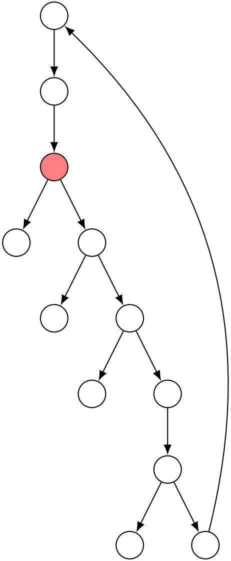

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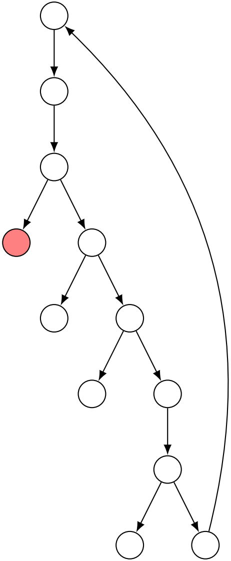

$$f(m,n) = \sum_{l=0}^{m} \binom{m}{l} 1 \times$$

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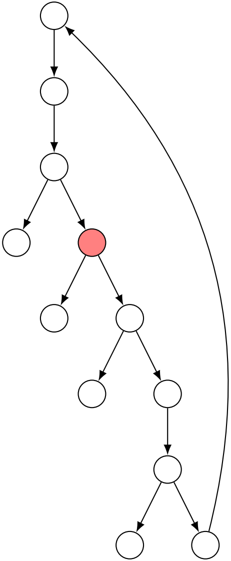

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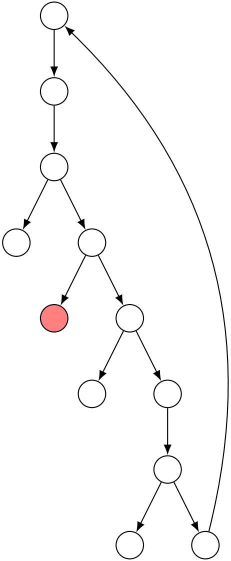

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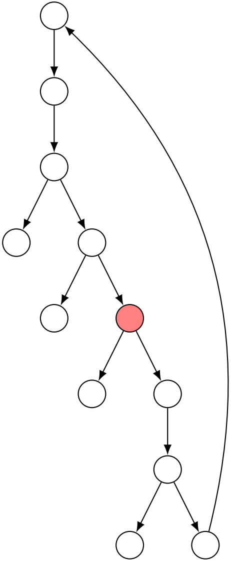

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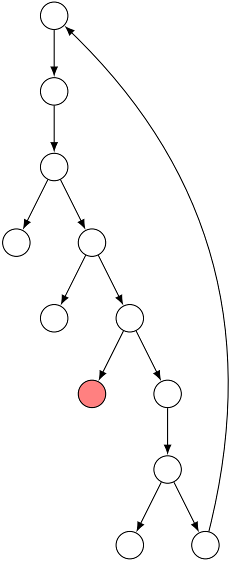

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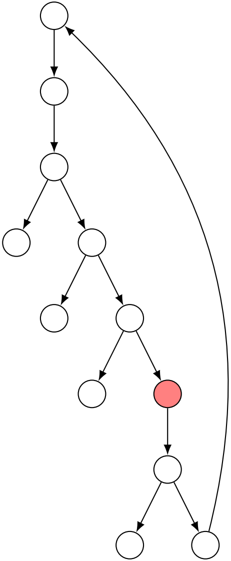

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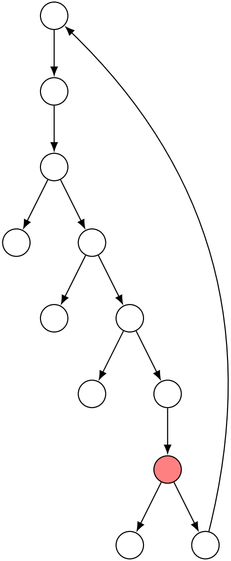

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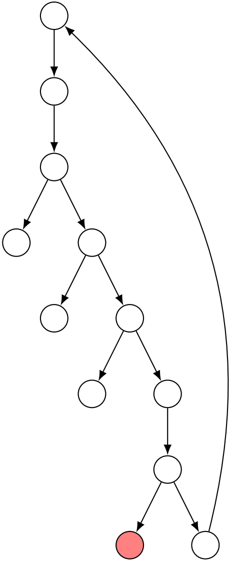
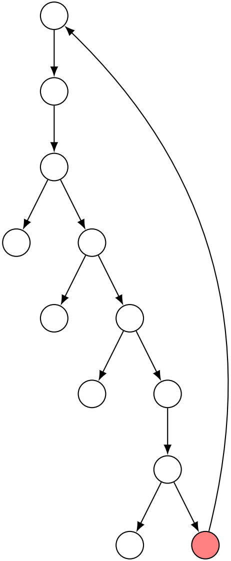


$$f(m, n) = \sum_{l=0}^{m} \binom{m}{l} 1 \times 1 \times 1 \times [l < 2] \times$$

$$[\phi] = \begin{cases} 1 & \text{if } \phi \\ 0 & \text{if } \neg\phi \end{cases}$$

How to Interpret an FCG



$$f(m,n) = \sum_{l=0}^{m} \binom{m}{l} 1 \times 1 \times 1 \times [l < 2] \times f(m-l, n-1)$$

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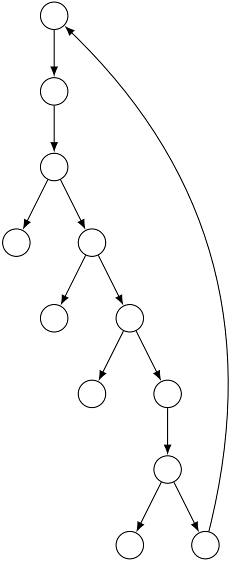

$$
\begin{aligned}
f(m, n) &= \sum_{l=0}^{m} \binom{m}{l} 1 \times 1 \times 1 \times [l < 2] \times f(m - l, n - 1) \\
&= f(m, n - 1) + mf(m - 1, n - 1)
\end{aligned}
$$

Compilation Rules

Definition

A (compilation) rule is a function that takes a formula and returns a set of (G, L) pairs, where

- G is a (potentially `null`) FCG,
- and L is a list of formulas.

Example Rule: Independence

Input formula:

$$(\forall x, y \in L.x = y) \land \tag{1}$$

$$(\forall x \in M.\forall y, z \in N.P(x, y) \land P(x, z) \Rightarrow y = z) \land \tag{2}$$

$$(\forall w, x \in M.\forall y \in N.P(w, y) \land P(x, y) \Rightarrow w = x) \tag{3}$$

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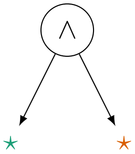
Only one (G, L) pair:

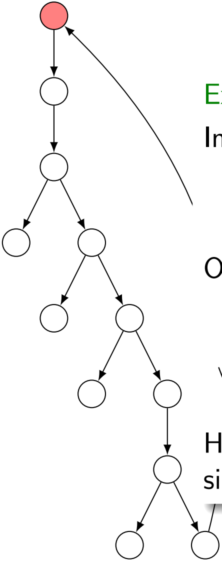
$G =$



$$, \qquad L = \langle (1), (2) \land (3) \rangle$$

Later, both G and L are incorporated into a larger search state.

New Rule 1: (Generalised) Domain Recursion



Example

Input formula:

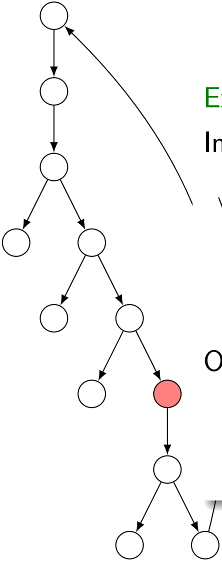
$$\forall x \in M. \forall y, z \in N. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

Output formula (with a new constant $c \in M$):

$$\forall y, z \in N. y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$
$$\forall x \in M. \forall y, z \in N. x \neq c \wedge y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

Here, domain recursion is applied to domain M. It could similarly be applied to N as well.

New Rule 2: Constraint Removal



Example

Input formula (with a constant $c \in M$):

$$\forall x \in M. \forall y, z \in N. x \neq c \land y \neq z \Rightarrow \neg P(x, y) \lor \neg P(x, z)$$

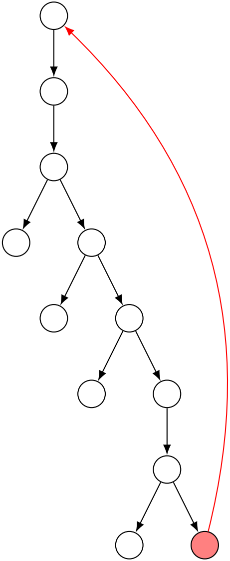
$$\forall w, x \in M. \forall y \in N. w \neq c \land x \neq c \land w \neq x \Rightarrow \neg P(w, y) \lor \neg P(x, y)$$

Output formula (with a new domain $M' := M \setminus \{ c \}$):

$$\forall x \in M'. \forall y, z \in N. y \neq z \Rightarrow \neg P(x, y) \lor \neg P(x, z)$$

$$\forall w, x \in M'. \forall y \in N. w \neq x \Rightarrow \neg P(w, y) \lor \neg P(x, y)$$

New Rule 3: Identifying Possibilities for Recursion



Goal

Check if the input formula is isomorphic (up to domains) to a previously encountered formula.

Rough Outline

1. Consider pairs of 'similar' clauses.
2. Consider bijections between their sets of variables.
3. Extend each such bijection to a map between sets of domains.
4. If the bijection makes the clauses equal, and the domain map is compatible with previous domain maps, move on to another pair of clauses.

Summary & Future Work

Summary

The circuits hitherto used for FOMC become more powerful with:

- cycles,
- generalised domain recursion,
- and some more new compilation rules that support domain recursion.

Future Work

- Automate:
  - extracting and simplifying the definitions of functions,
  - finding all base cases.
- Open questions:
  - What kind of sequences are computable in this way?
  - Would using a different logic extend the capabilities of FOMC further?