

# Synthesising Recursive Functions for First-Order Model Counting

---

Paulius Dilkas

Joint work with Vaishak Belle (University of Edinburgh, UK)

18th April 2023

National University of Singapore, Singapore

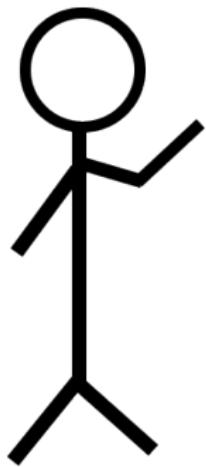


National University  
of Singapore

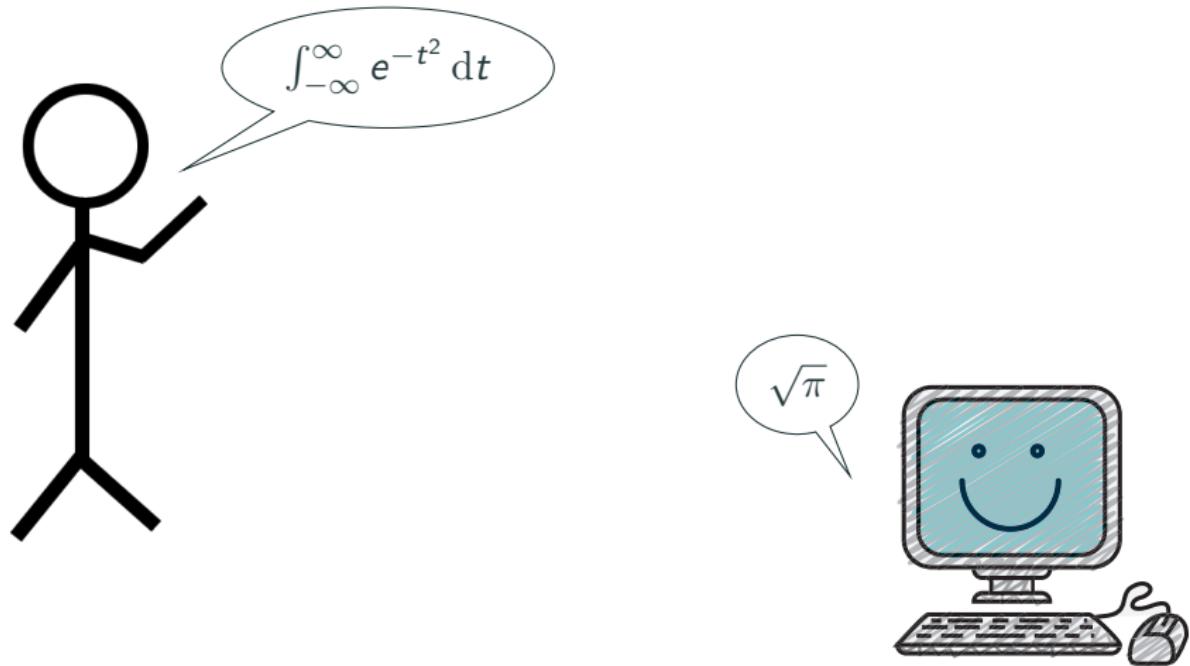


Engineering and  
Physical Sciences  
Research Council

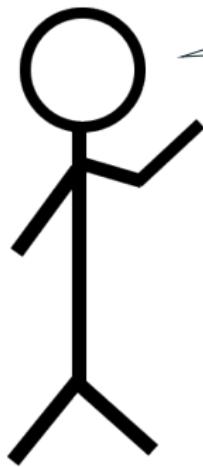
# What Computers Can and Cannot Do



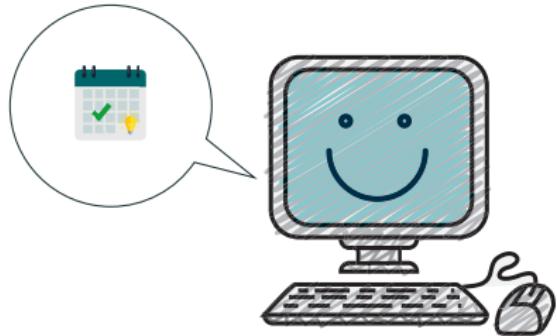
# What Computers Can and Cannot Do



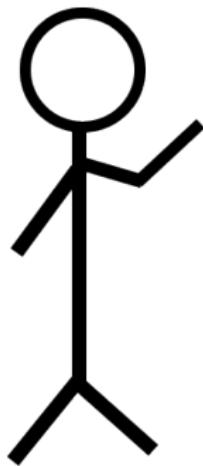
# What Computers Can and Cannot Do



Produce a schedule for the  
nurses at the local hospital.



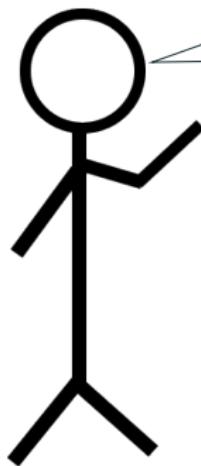
# What Computers Can and Cannot Do



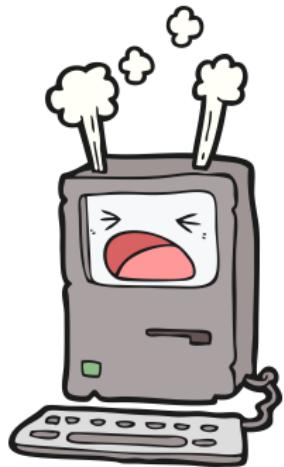
Paint a baroque oil painting of a  
raccoon queen wearing a crown.



# What Computers Can and Cannot Do



If I shuffle a deck of  $n$  cards,  
how many possible outcomes  
are there?



---

Terms and conditions apply.

# Who Cares About Counting?

# Who Cares About Counting?

## Probabilistic Programming

*Inference and learning in probabilistic logic  
programs using weighted Boolean formulas*

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS,  
DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON,  
GERDA JANSSENS and LUC DE RAEDT

*Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium  
(e-mail: FirstName.LastName@cs.kuleuven.be)*

# Who Cares About Counting?

## Probabilistic Programming

*Inference and learning in probabilistic logic programs using weighted Boolean formulas*

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS,  
DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON,  
GERDA JANSSENS and LUC DE RAEDT

*Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium  
(e-mail: FirstName.LastName@cs.kuleuven.be)*

## Neuro-symbolic AI

---

A Semantic Loss Function for Deep Learning with Symbolic Knowledge

---

Jingyi Xu<sup>1</sup> Ziliu Zhang<sup>2</sup> Tal Friedman<sup>1</sup> Yitao Liang<sup>1</sup> Guy Van den Broeck<sup>1</sup>

# Who Cares About Counting?

## Probabilistic Programming

*Inference and learning in probabilistic logic programs using weighted Boolean formulas*

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS,  
DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON,  
GERDA JANSSENS and LUC DE RAEDT

*Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium  
(e-mail: FirstName.LastName@cs.kuleuven.be)*

## Neuro-symbolic AI

A Semantic Loss Function for Deep Learning with Symbolic Knowledge

Jingyi Xu<sup>1</sup> Ziliu Zhang<sup>2</sup> Tal Friedman<sup>1</sup> Yitao Liang<sup>1</sup> Guy Van den Broeck<sup>1</sup>

## Natural Language Processing

Joint Inference for Knowledge Extraction from Biomedical Literature

**Hoifung Poon\***

Dept. of Computer Sci. & Eng.  
University of Washington  
Seattle, WA 98195  
hoifung@cs.washington.edu

**Lucy Vanderwende**

Microsoft Research  
Redmond, WA 98052  
Lucy.Vanderwende@microsoft.com

# Who Cares About Counting?

## Probabilistic Programming

*Inference and learning in probabilistic logic programs using weighted Boolean formulas*

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS,  
DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON,  
GERDA JANSSENS and LUC DE RAEDT

*Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium  
(e-mail: FirstName.LastName@cs.kuleuven.be)*

## Robotics

**Learning Relational Affordance Models for Robots  
in Multi-Object Manipulation Tasks**

Bogdan Moldovan   Plinio Moreno   Martijn van Otterlo   José Santos-Victor   Luc De Raedt

## Neuro-symbolic AI

A Semantic Loss Function for Deep Learning with Symbolic Knowledge

Jingyi Xu<sup>1</sup> Ziliu Zhang<sup>2</sup> Tal Friedman<sup>1</sup> Yitao Liang<sup>1</sup> Guy Van den Broeck<sup>1</sup>

## Natural Language Processing

Joint Inference for Knowledge Extraction from Biomedical Literature

**Hoifung Poon\***

Dept. of Computer Sci. & Eng.  
University of Washington  
Seattle, WA 98195  
hoifung@cs.washington.edu

**Lucy Vanderwende**

Microsoft Research  
Redmond, WA 98052  
Lucy.Vanderwende@microsoft.com

# Who Cares About Counting?

## Probabilistic Programming

*Inference and learning in probabilistic logic programs using weighted Boolean formulas*

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS,  
DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON,  
GERDA JANSSENS and LUC DE RAEDT

*Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium  
(e-mail: FirstName.LastName@cs.kuleuven.be)*

## Neuro-symbolic AI

A Semantic Loss Function for Deep Learning with Symbolic Knowledge

Jingyi Xu<sup>1</sup> Zili Zhang<sup>2</sup> Tal Friedman<sup>1</sup> Yitao Liang<sup>1</sup> Guy Van den Broeck<sup>1</sup>

## Natural Language Processing

Joint Inference for Knowledge Extraction from Biomedical Literature

Hoifung Poon\*

Dept. of Computer Sci. & Eng.  
University of Washington  
Seattle, WA 98195  
hoifung@cs.washington.edu

Lucy Vanderwende

Microsoft Research  
Redmond, WA 98052  
Lucy.Vanderwende@microsoft.com

## Robotics

**Learning Relational Affordance Models for Robots  
in Multi-Object Manipulation Tasks**

Bogdan Moldovan Plinio Moreno Martijn van Otterlo José Santos-Victor Luc De Raedt

## Bioinformatics

*PheNetic: Network-based interpretation of unstructured gene lists in E. coli*  
Dries De Maeyer<sup>1</sup>, Joris Renkens<sup>2</sup>, Lore Cloots<sup>1</sup>, Luc De Raedt<sup>1,2</sup>, Kathleen Marchal<sup>1,3</sup>

<sup>1</sup>Center of Microbial and Plant Genetics, Kasteelpark Arenberg 20, B-3001, Leuven, Belgium

<sup>2</sup>Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3001 Heverlee, Belgium

<sup>3</sup>Department of Plant Biotechnology and Bioinformatics, Ghent University, Technologiepark 927, 9052 Gent, Belgium

# Who Cares About Counting?

## Probabilistic Programming

*Inference and learning in probabilistic logic programs using weighted Boolean formulas*

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS,  
DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON,  
GERDA JANSSENS and LUC DE RAEDT

Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium  
(e-mail: FirstName.LastName@cs.kuleuven.be)

## Neuro-symbolic AI

A Semantic Loss Function for Deep Learning with Symbolic Knowledge

Jingyi Xu<sup>1</sup> Zili Zhang<sup>2</sup> Tal Friedman<sup>1</sup> Yitao Liang<sup>1</sup> Guy Van den Broeck<sup>1</sup>

## Natural Language Processing

Joint Inference for Knowledge Extraction from Biomedical Literature

Hoifung Poon\*

Dept. of Computer Sci. & Eng.  
University of Washington  
Seattle, WA 98195  
hoifung@cs.washington.edu

Lucy Vanderwende

Microsoft Research  
Redmond, WA 98052  
Lucy.Vanderwende@microsoft.com

## Robotics

Learning Relational Affordance Models for Robots  
in Multi-Object Manipulation Tasks

Bogdan Moldovan Plinio Moreno Martijn van Otterlo José Santos-Victor Luc De Raedt

## Bioinformatics

PheNetic: Network-based interpretation of unstructured gene lists in *E. coli*  
Dries De Maeyer<sup>1</sup>, Joris Renkens<sup>2</sup>, Lore Cloots<sup>1</sup>, Luc De Raedt<sup>1</sup>, Kathleen Marchal<sup>1,3</sup>

<sup>1</sup>Center of Microbial and Plant Genetics, Kasteelpark Arenberg 20, B-3001, Leuven, Belgium

<sup>2</sup>Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3001 Heverlee, Belgium

<sup>3</sup>Department of Plant Biotechnology and Bioinformatics, Ghent University, Technologiepark 927, 9052 Gent, Belgium

## Combinatorics

Automatic Conjecturing of P-Recursions  
Using Lifted Inference

Jáchym Barvínek<sup>1(✉)</sup>, Timothy van Bremen<sup>2</sup>, Yuyi Wang<sup>3</sup>, Filip Železný<sup>1</sup>,  
and Ondřej Kuzelka<sup>1</sup>

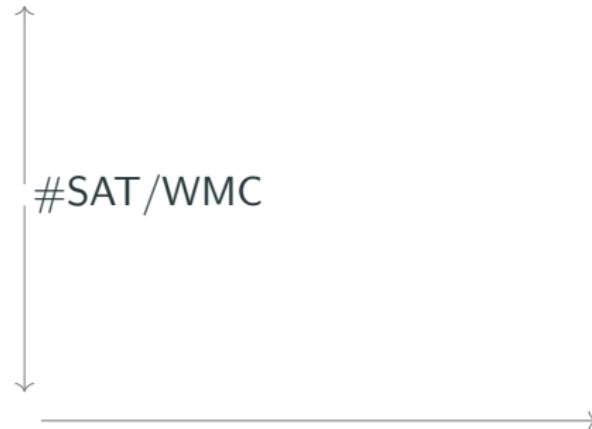
<sup>1</sup> Czech Technical University in Prague, Prague, Czech Republic

barvijac@fel.cvut.cz

<sup>2</sup> KU Leuven, Leuven, Belgium

<sup>3</sup> ETH Zurich, Zurich, Switzerland

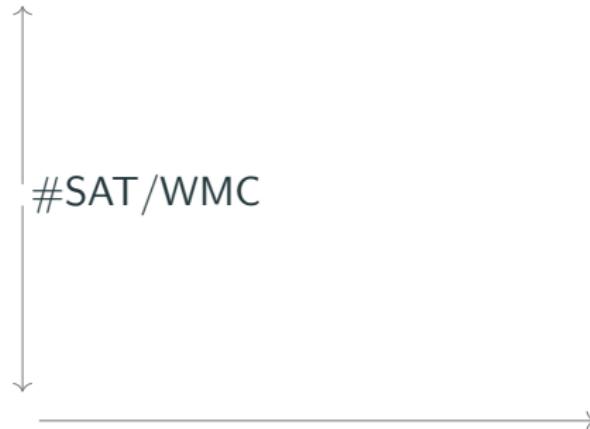
# (Some of the) Many Ways to Count



## #SAT (Valiant 1979)

- Input formula:  $x \vee y$
- Interpretations:  $\emptyset, \{x\}, \{y\}, \{x, y\}$
- Models:  $\{x\}, \{y\}, \{x, y\}$

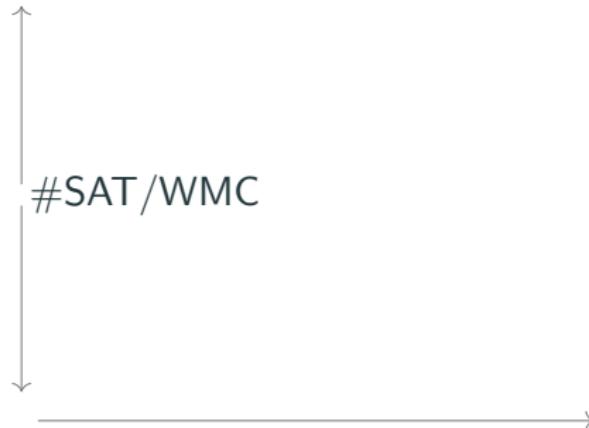
# (Some of the) Many Ways to Count



## #SAT (Valiant 1979)

- Input formula:  $x \vee y$
- Interpretations:  $\emptyset, \{x\}, \{y\}, \{x, y\}$
- Models:  $\{x\}, \{y\}, \{x, y\}$
- Answer (model count): 3

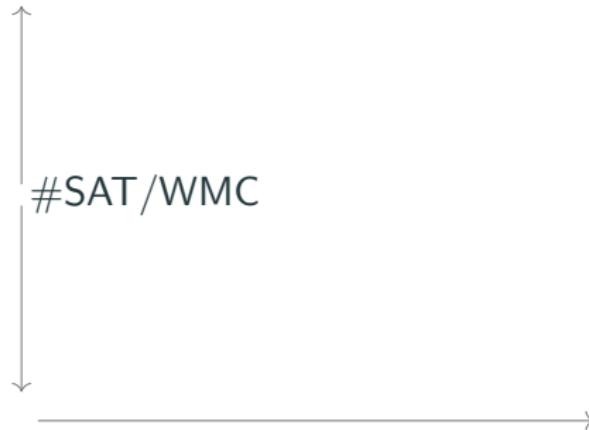
# (Some of the) Many Ways to Count



## Weighted Model Counting (Chavira and Darwiche 2008)

- Input formula:  $x \vee y$
- Input weights:  $w(x) = 0.3, w(\neg x) = 0.7,$   
 $w(y) = 0.2, w(\neg y) = 0.8$

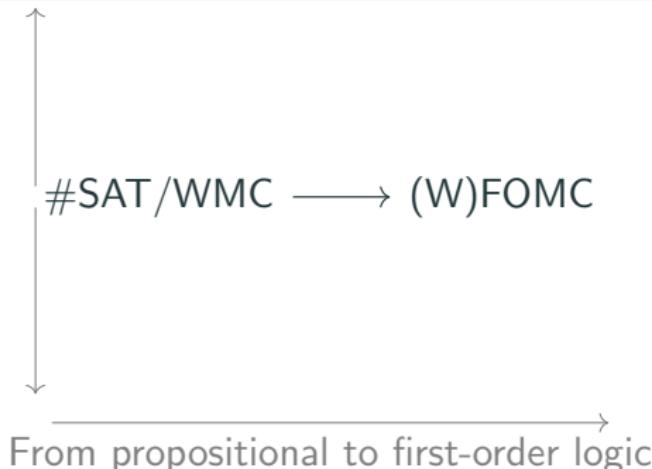
# (Some of the) Many Ways to Count



## Weighted Model Counting (Chavira and Darwiche 2008)

- Input formula:  $x \vee y$
- Input weights:  $w(x) = 0.3, w(\neg x) = 0.7,$   
 $w(y) = 0.2, w(\neg y) = 0.8$
- Answer (weighted model count):  
 $w(x)w(y) + w(x)w(\neg y) + w(\neg x)w(y) = 0.44$

# (Some of the) Many Ways to Count

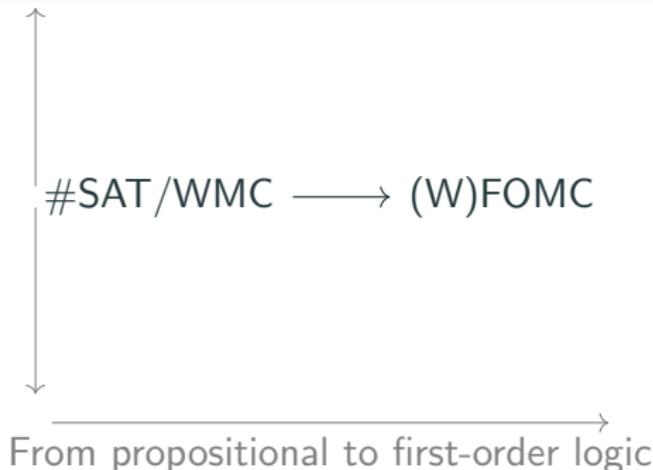


## (Weighted) (Symmetric) First-Order Model Counting

(Van den Broeck et al. 2011)

- Input formula:  $\forall \textcolor{teal}{x} \in \Delta. P(\textcolor{teal}{x})$
- Input weights:  $w^+(P) = 0.3, w^-(P) = 0.7$
- Input domain size(s):  $|\Delta| = 2$

# (Some of the) Many Ways to Count

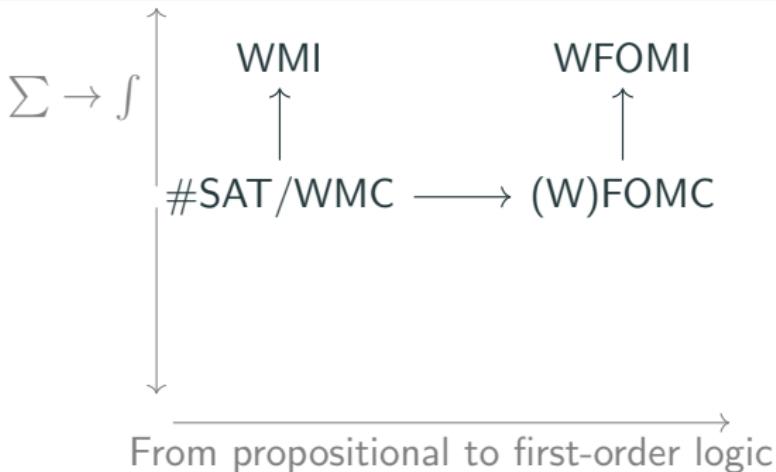


## (Weighted) (Symmetric) First-Order Model Counting

(Van den Broeck et al. 2011)

- Input formula:  $\forall \textcolor{teal}{x} \in \Delta. P(\textcolor{teal}{x})$
- Input weights:  $w^+(P) = 0.3, w^-(P) = 0.7$
- Input domain size(s):  $|\Delta| = 2$
- Answer:  $(w^+(P))^{|\Delta|} = 0.09$

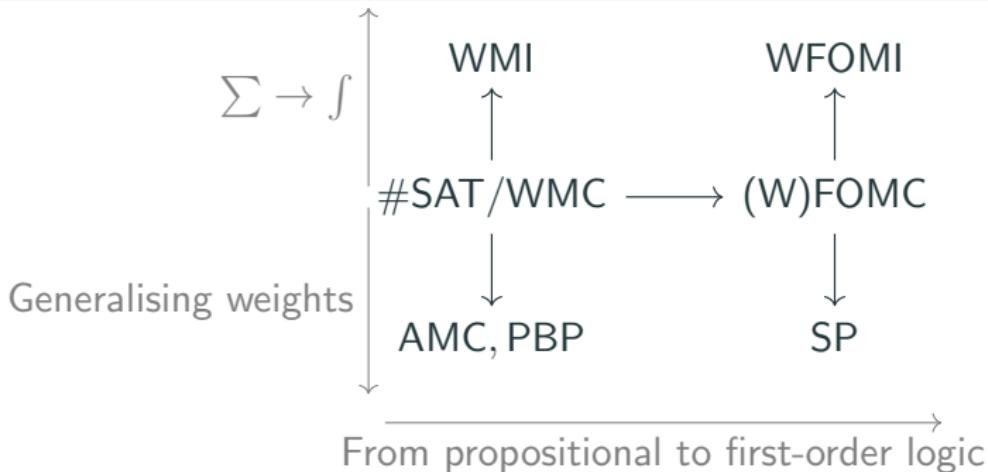
# (Some of the) Many Ways to Count



## Extensions to Continuous Domains

- Weighted model integration
  - (Belle, Passerini and Van den Broeck 2015)
- Weighted first-order model integration
  - (Feldstein and Belle 2021)

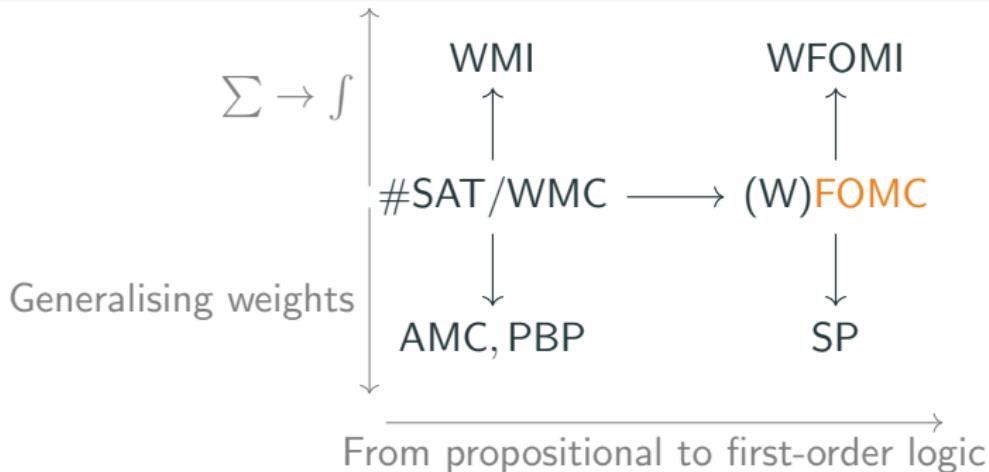
# (Some of the) Many Ways to Count



## Generalisations of the weight function

- Algebraic model counting
    - (Kimmig, Van den Broeck and De Raedt 2017)
    - From  $\mathbb{R}_{\geq 0}$  to commutative semirings
  - Pseudo-Boolean projection (D. and Belle 2021)
    - Weights not necessarily on literals
  - Semiring programming (Belle and De Raedt 2020)

# (Some of the) Many Ways to Count



## (Unweighted) First-Order Model Counting

- Example formula:

$$\forall x \in \Delta. P(x) \vee Q(x)$$

- Let  $\Delta := \{1, 2\}$

- Interpretations: all subsets of

$$\{P(1), Q(1), P(2), Q(2)\}$$

# (Unweighted) First-Order Model Counting

- Example formula:

$$\forall x \in \Delta. P(x) \vee Q(x)$$

- Let  $\Delta := \{1, 2\}$

- Interpretations: all subsets of

$$\{P(1), Q(1), P(2), Q(2)\}$$

- Models:

$$\{P(1), P(2)\},$$

$$\{P(1), Q(2)\},$$

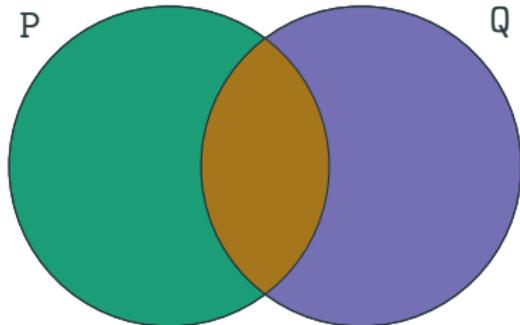
$$\{P(1), P(2), Q(2)\},$$

$$\{Q(1), P(2)\},$$

$$\{Q(1), Q(2)\},$$

$$\{Q(1), P(2), Q(2)\},$$

$$\{P(1), Q(1), P(2)\}, \quad \{P(1), Q(1), Q(2)\}, \quad \{P(1), Q(1), P(2), Q(2)\}$$



# (Unweighted) First-Order Model Counting

- Example formula:

$$\forall x \in \Delta. P(x) \vee Q(x)$$

- Let  $\Delta := \{1, 2\}$

- Interpretations: all subsets of

$$\{P(1), Q(1), P(2), Q(2)\}$$

- Models:

$$\{P(1), P(2)\},$$

$$\{P(1), Q(2)\},$$

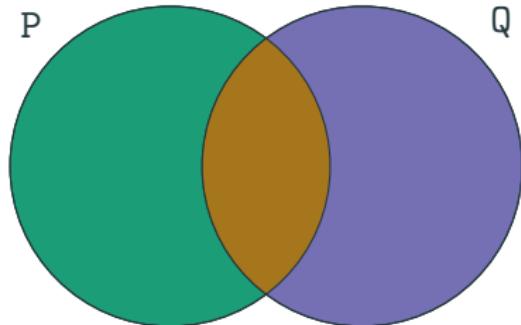
$$\{P(1), P(2), Q(2)\},$$

$$\{Q(1), P(2)\},$$

$$\{Q(1), Q(2)\},$$

$$\{Q(1), P(2), Q(2)\},$$

$$\{P(1), Q(1), P(2)\}, \quad \{P(1), Q(1), Q(2)\}, \quad \{P(1), Q(1), P(2), Q(2)\}$$



## Intuition

- Each 1-ary predicate is like a subset
- For  $n > 1$ , each  $n$ -ary predicate is like a relation
- FOMC counts combinations of relations

# Approaches to FOMC

- **ForcLift** (Van den Broeck et al. 2011)
  - knowledge compilation to **FO d-DNNF**
- **L2C** (Kazemi and Poole 2016)
  - knowledge compilation to **C++ code**
- **Alchemy** (Gogate and Domingos 2016)
  - **DPLL**-style search
- **FastWFOMC** (van Bremen and Kuželka 2021)
  - knowledge compilation to **sd-DNNF**

# Approaches to FOMC

- **ForcLift** (Van den Broeck et al. 2011)
  - knowledge compilation to **FO d-DNNF**
- **L2C** (Kazemi and Poole 2016)
  - knowledge compilation to **C++** code
- **Alchemy** (Gogate and Domingos 2016)
  - **DPLL**-style search
- **FastWFOMC** (van Bremen and Kuželka 2021)
  - knowledge compilation to **sd-DNNF**

## Our Contribution



+



=



ForcLift

Recursion

Crane

# ForcLift and First-Order Knowledge Compilation

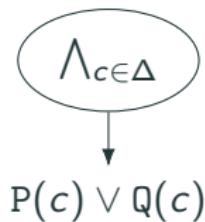
$$\forall x \in \Delta. \ P(x) \vee Q(x)$$

## ForcLift and First-Order Knowledge Compilation

$$\forall x \in \Delta. \ P(x) \vee Q(x)$$

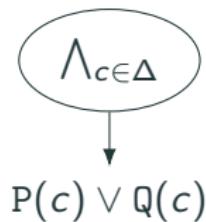
Independent partial grounding (introduces a constant  $c \in \Delta$ )

# ForcLift and First-Order Knowledge Compilation



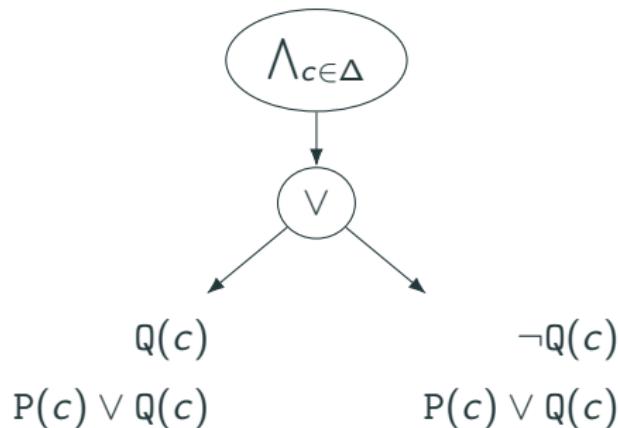
Independent partial grounding (introduces a constant  $c \in \Delta$ )

# ForcLift and First-Order Knowledge Compilation



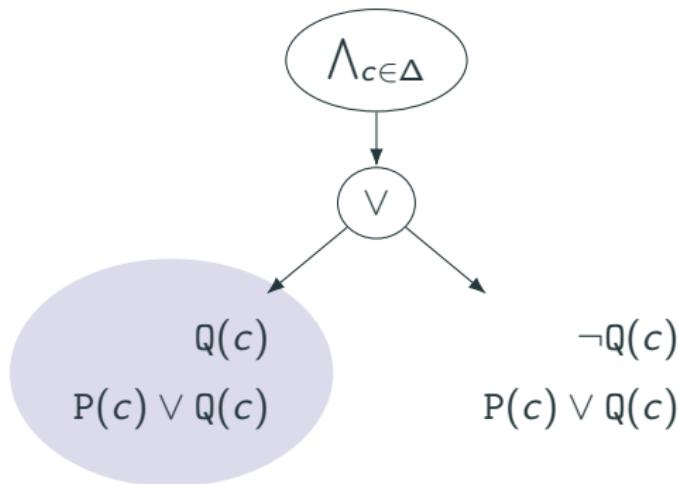
Shannon decomposition (a.k.a. Boole's expansion theorem) on  $Q(c)$

# ForcLift and First-Order Knowledge Compilation



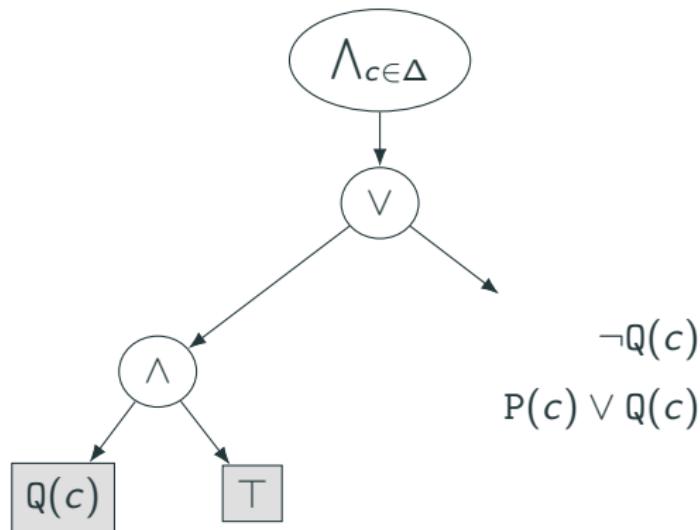
Shannon decomposition (a.k.a. Boole's expansion theorem) on  $\text{Q}(c)$

# ForcLift and First-Order Knowledge Compilation



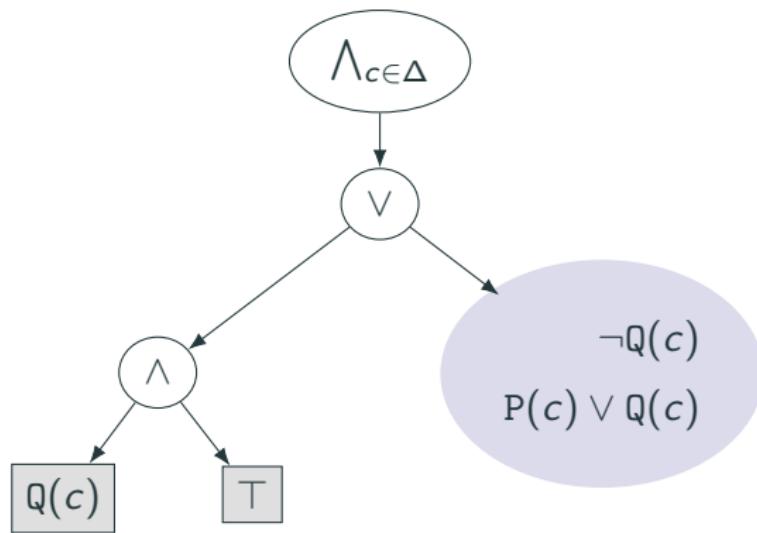
Positive unit propagation of  $Q(c)$

# ForcLift and First-Order Knowledge Compilation



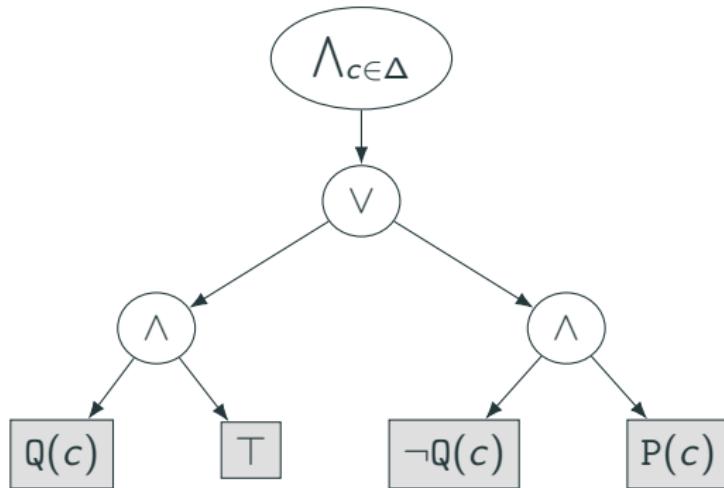
Positive unit propagation of  $Q(c)$

# ForcLift and First-Order Knowledge Compilation



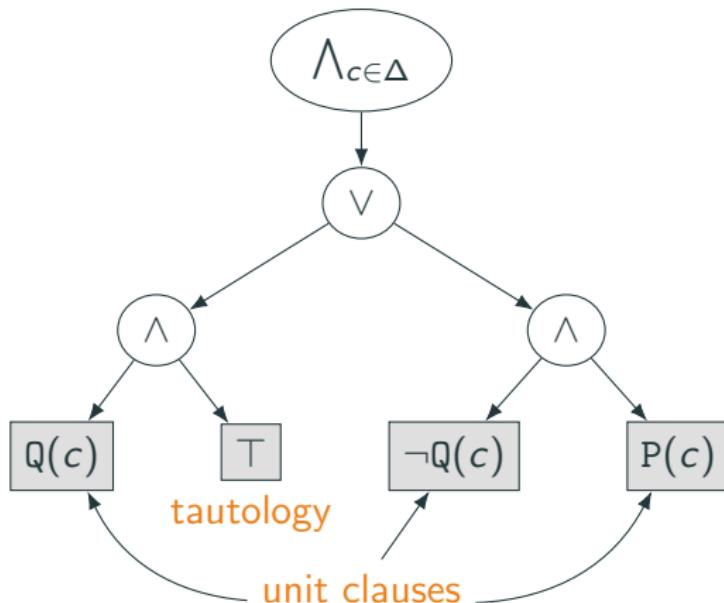
Negative unit propagation of  $\neg Q(c)$

# ForcLift and First-Order Knowledge Compilation



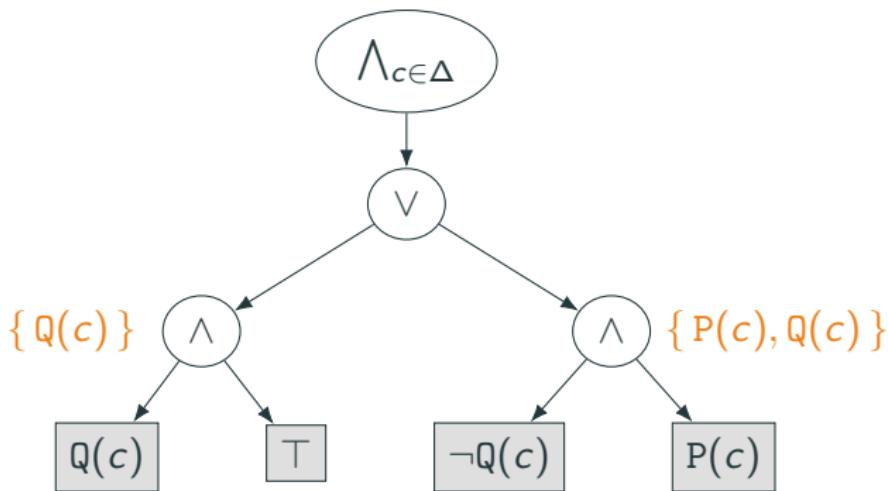
Negative unit propagation of  $\neg Q(c)$

# ForcLift and First-Order Knowledge Compilation



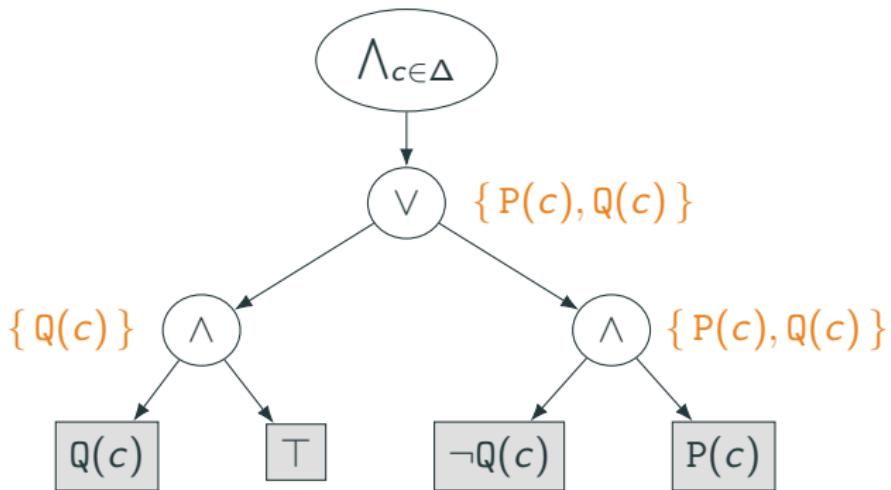
Compilation is complete ✓

# ForcLift and First-Order Knowledge Compilation



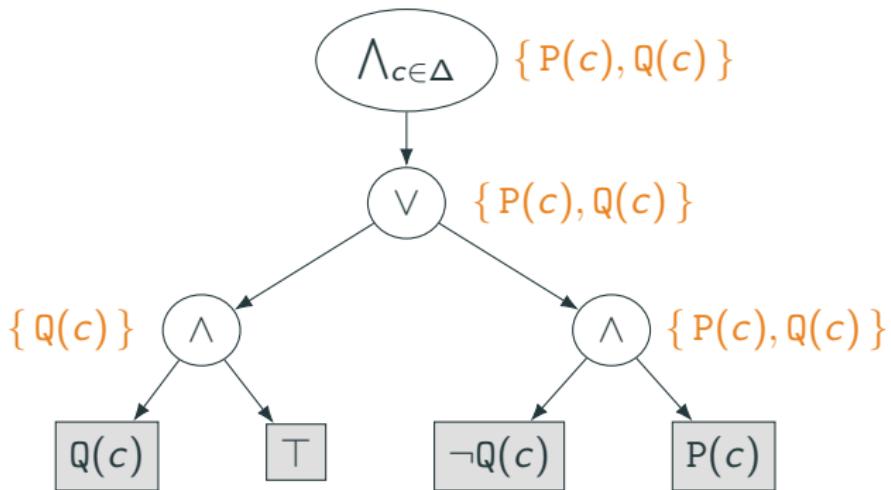
Smoothing: propagating atoms upwards

# ForcLift and First-Order Knowledge Compilation



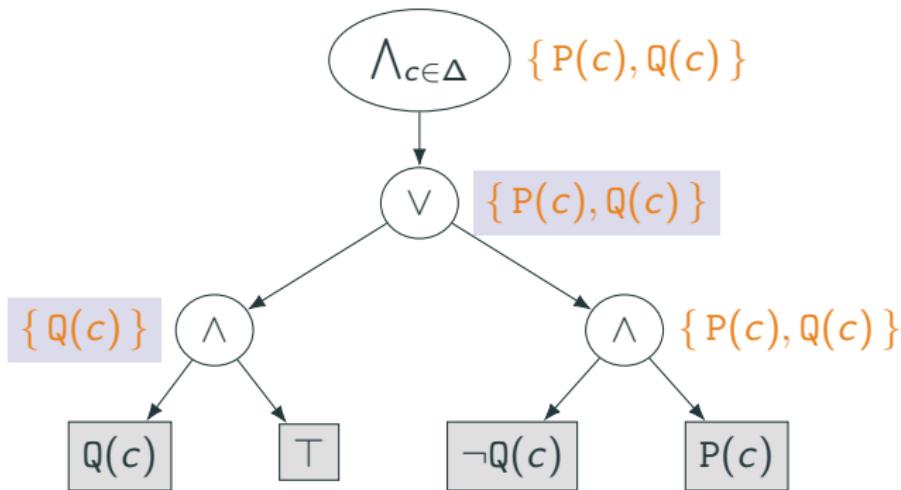
Smoothing: propagating atoms upwards

# ForcLift and First-Order Knowledge Compilation



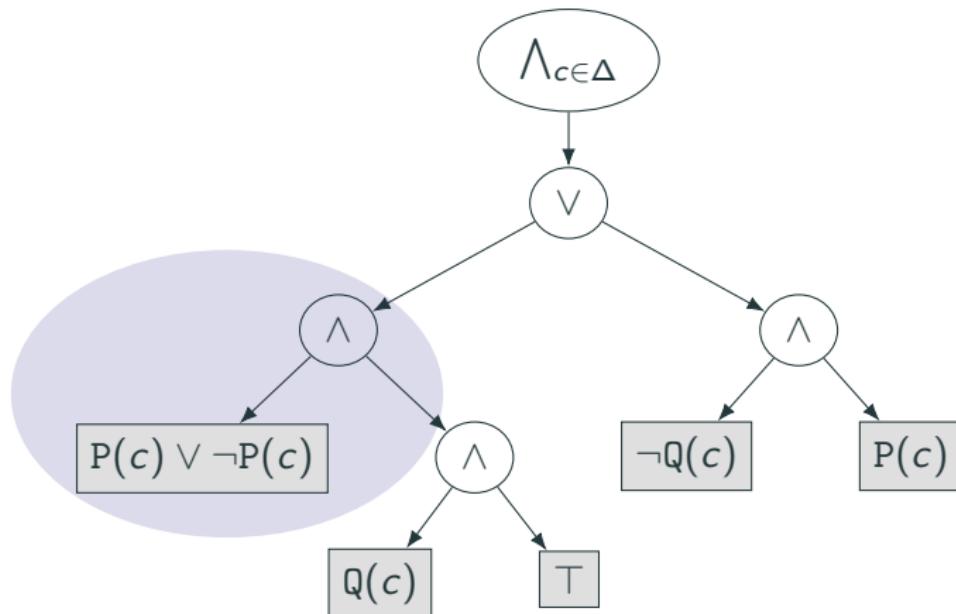
Smoothing: propagating atoms upwards

# ForcLift and First-Order Knowledge Compilation



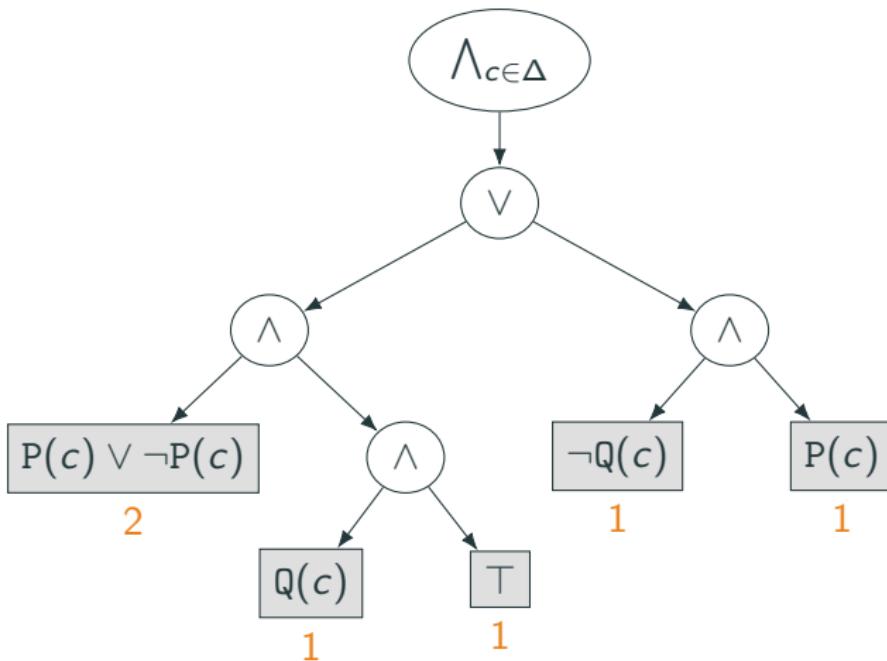
Smoothing: adding new atoms

# ForcLift and First-Order Knowledge Compilation



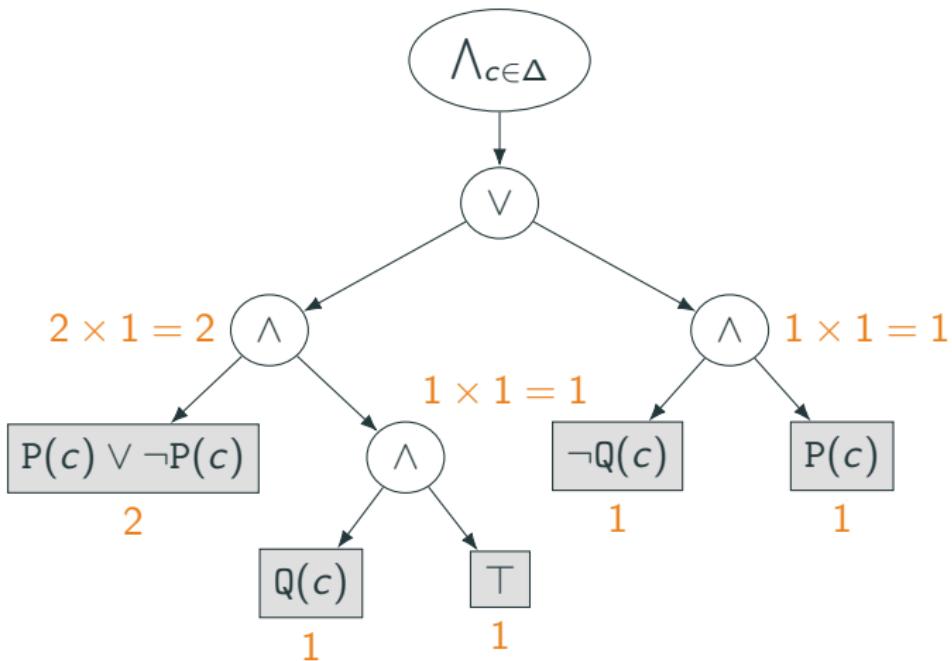
Smoothing: adding new atoms

# ForcLift and First-Order Knowledge Compilation



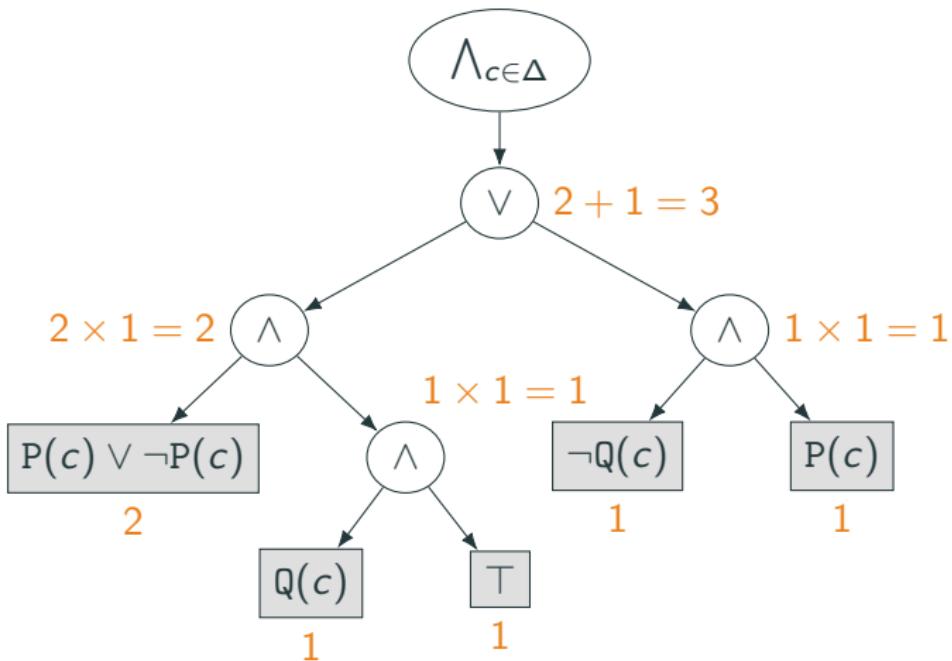
Propagating the model count

# ForcLift and First-Order Knowledge Compilation



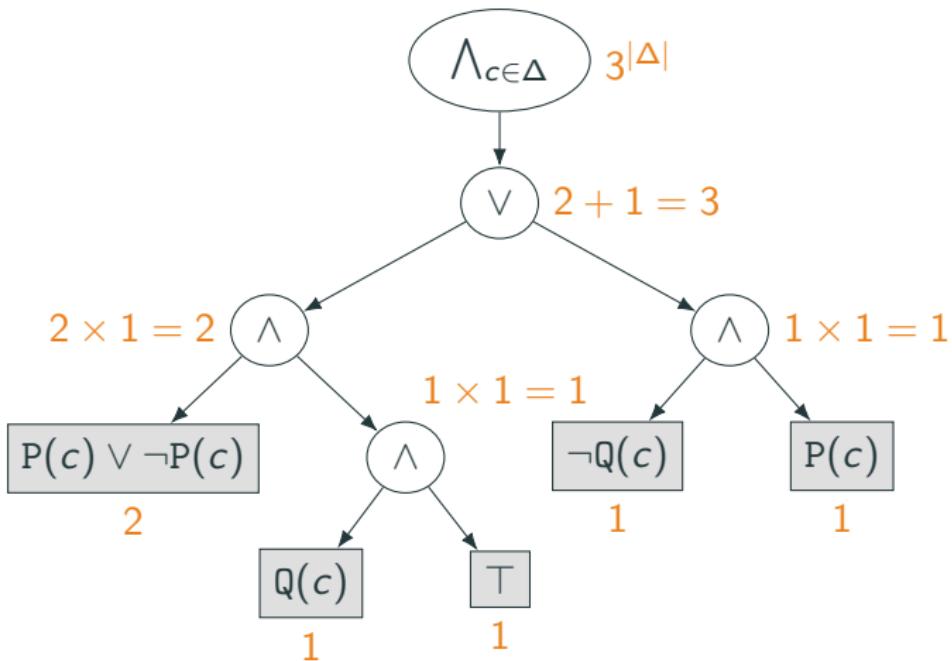
Propagating the model count

# ForcLift and First-Order Knowledge Compilation



Propagating the model count

# ForcLift and First-Order Knowledge Compilation



Propagating the model count

## A (Slightly) More Complicated Example

Suppose this room has  $n$  seats, and there are  $m \leq n$  people in the audience. How many ways are there to seat everyone?

## A (Slightly) More Complicated Example

Suppose this room has  $n$  seats, and there are  $m \leq n$  people in the audience. How many ways are there to seat everyone?

More explicitly, we assume that:

- each attendee gets exactly one seat,
- and a seat can accommodate at most one person.

## A (Slightly) More Complicated Example

Suppose this room has  $n$  seats, and there are  $m \leq n$  people in the audience. How many ways are there to seat everyone?

More explicitly, we assume that:

- each attendee gets exactly one seat,
- and a seat can accommodate at most one person.

**Answer:**  $n^m = n \cdot (n - 1) \cdots (n - m + 1)$ .

Note: this problem is equivalent to counting  $[m] \rightarrow [n]$  injections.

## Let's Express This Problem in Logic!

- Let  $\Gamma$  and  $\Delta$  be sets (i.e., domains)
  - such that  $|\Gamma| = m$ , and  $|\Delta| = n$ .
- Let  $P \subseteq \Gamma \times \Delta$  be a relation (i.e., predicate) over  $\Gamma$  and  $\Delta$ .
- We can describe all of the constraints in first-order logic:

## Let's Express This Problem in Logic!

- Let  $\Gamma$  and  $\Delta$  be sets (i.e., domains)
  - such that  $|\Gamma| = m$ , and  $|\Delta| = n$ .
- Let  $P \subseteq \Gamma \times \Delta$  be a relation (i.e., predicate) over  $\Gamma$  and  $\Delta$ .
- We can describe all of the constraints in first-order logic:
  - each attendee gets a seat (i.e., at least one seat)

$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \tag{1}$$

# Let's Express This Problem in Logic!

- Let  $\Gamma$  and  $\Delta$  be sets (i.e., domains)
  - such that  $|\Gamma| = m$ , and  $|\Delta| = n$ .
- Let  $P \subseteq \Gamma \times \Delta$  be a relation (i.e., predicate) over  $\Gamma$  and  $\Delta$ .
- We can describe all of the constraints in first-order logic:
  - each attendee gets a seat (i.e., at least one seat)

$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \quad (1)$$

- one person cannot occupy multiple seats

$$\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z \quad (2)$$

# Let's Express This Problem in Logic!

- Let  $\Gamma$  and  $\Delta$  be sets (i.e., domains)
  - such that  $|\Gamma| = m$ , and  $|\Delta| = n$ .
- Let  $P \subseteq \Gamma \times \Delta$  be a relation (i.e., predicate) over  $\Gamma$  and  $\Delta$ .
- We can describe all of the constraints in first-order logic:
  - each attendee gets a seat (i.e., at least one seat)

$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \quad (1)$$

- one person cannot occupy multiple seats

$$\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z \quad (2)$$

- one seat cannot accommodate multiple attendees

$$\forall w, x \in \Gamma. \forall y \in \Delta. P(w, y) \wedge P(x, y) \Rightarrow w = x \quad (3)$$

# Let's Express This Problem in Logic!

- Let  $\Gamma$  and  $\Delta$  be sets (i.e., domains)
  - such that  $|\Gamma| = m$ , and  $|\Delta| = n$ .
- Let  $P \subseteq \Gamma \times \Delta$  be a relation (i.e., predicate) over  $\Gamma$  and  $\Delta$ .
- We can describe all of the constraints in first-order logic:
  - each attendee gets a seat (i.e., at least one seat)

$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \quad (1)$$

- one person cannot occupy multiple seats

$$\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z \quad (2)$$

- one seat cannot accommodate multiple attendees

$$\forall w, x \in \Gamma. \forall y \in \Delta. P(w, y) \wedge P(x, y) \Rightarrow w = x \quad (3)$$

(1) and (2) constrain  $P$  to be a function, and (3) makes it injective.

# Recursion



## Back to Our Example

The following function counts injections:

$$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n - 1) + mf(m - 1, n - 1) & \text{otherwise.} \end{cases}$$

## Back to Our Example

The following function counts injections:

$$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n - 1) + mf(m - 1, n - 1) & \text{otherwise.} \end{cases}$$

- $f(m, n)$  can be computed in  $\Theta(mn)$  time
  - using dynamic programming

## Back to Our Example

The following function counts injections:

$$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n - 1) + mf(m - 1, n - 1) & \text{otherwise.} \end{cases}$$

- $f(m, n)$  can be computed in  $\Theta(mn)$  time
  - using dynamic programming
- Optimal time complexity to compute  $n^m$  is  $\Theta(m)$
- But  $\Theta(mn)$  is still much better than translating to propositional logic and solving a  $\#P$ -complete problem

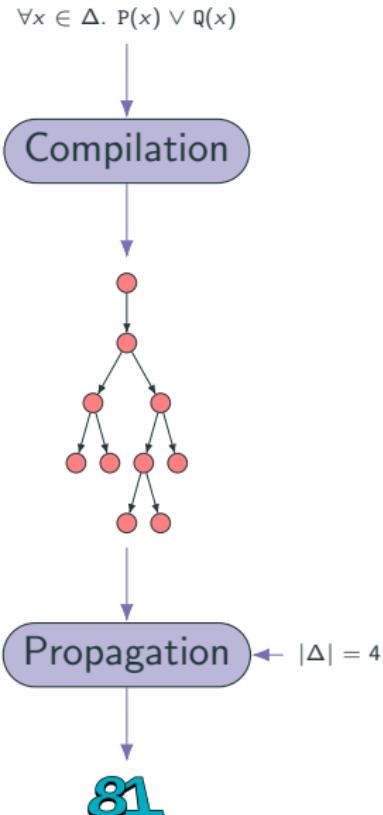
## Back to Our Example

The following function counts injections:

$$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n - 1) + mf(m - 1, n - 1) & \text{otherwise.} \end{cases}$$

- $f(m, n)$  can be computed in  $\Theta(mn)$  time
  - using dynamic programming
- Optimal time complexity to compute  $n^m$  is  $\Theta(m)$
- But  $\Theta(mn)$  is still much better than translating to propositional logic and solving a  $\#P$ -complete problem
- The rest of this talk is about how to find such functions automatically

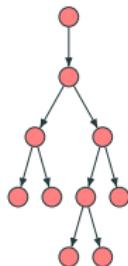
# First-Order Knowledge Compilation: Before and After



# First-Order Knowledge Compilation: Before and After

$$\forall x \in \Delta. P(x) \vee Q(x)$$

Compilation



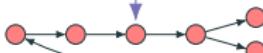
Propagation

81

$$|\Delta| = 4$$

$$\begin{aligned} & \forall x \in \Gamma. \exists y \in \Delta. P(x, y) \\ & \forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z \\ & \forall w, x \in \Gamma. \forall y \in \Delta. P(w, y) \wedge P(x, y) \Rightarrow w = x \end{aligned}$$

Compilation



Conversion

$$f(m, n) = \sum_{l=0}^m \binom{m}{l} [l < 2] \times f(m - l, n - 1)$$

Simplification

$$f(m, n) = f(m, n - 1) + mf(m - 1, n - 1)$$

Evaluation

$$\begin{aligned} & f(0, 0) = 1, f(m, 0) = 0 \\ & m = 2, n = 7 \end{aligned}$$

42

# Circuits vs Graphs

## Circuits (Van den Broeck et al. 2011)...

- ... extend d-DNNF circuits (Darwiche 2001) for propositional knowledge compilation with **more node types**
- ... are **acyclic**.

# Circuits vs Graphs

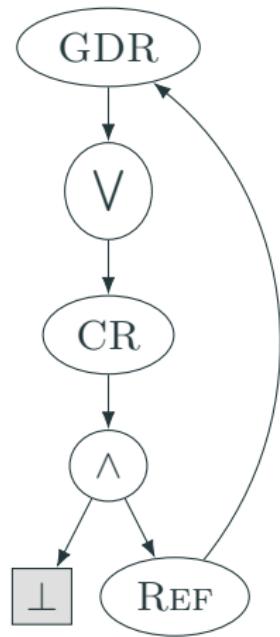
## Circuits (Van den Broeck et al. 2011)...

- ... extend d-DNNF circuits (Darwiche 2001) for propositional knowledge compilation with **more node types**
- ... are **acyclic**.

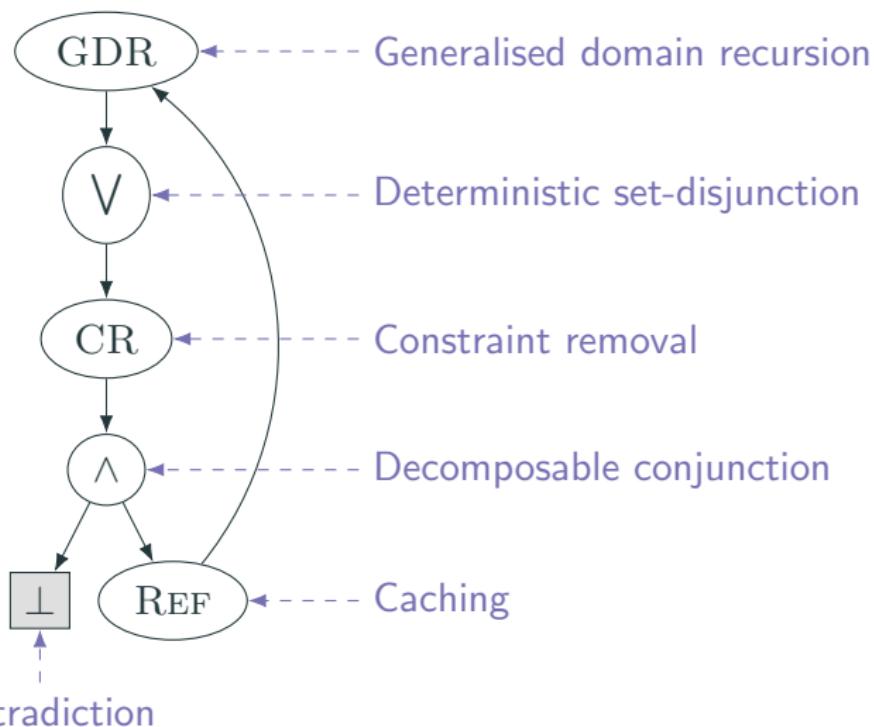
**First-Order Computational Graphs (FCGs) are...**  
directed **acyclic** (weakly connected) graphs with:

- a single source,
- labelled nodes,
- and ordered outgoing edges.

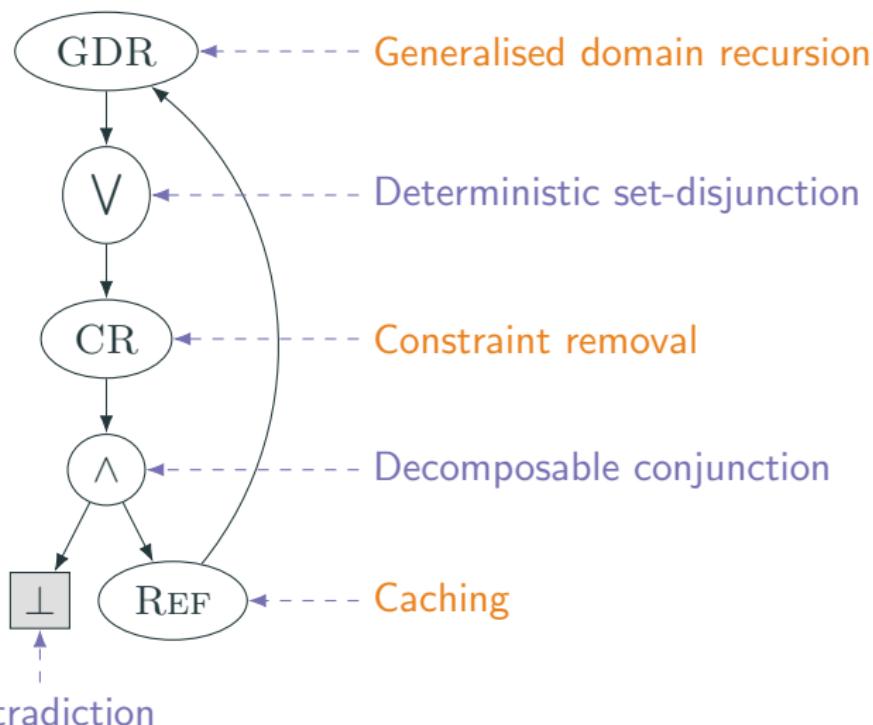
# How to Interpret an FCG



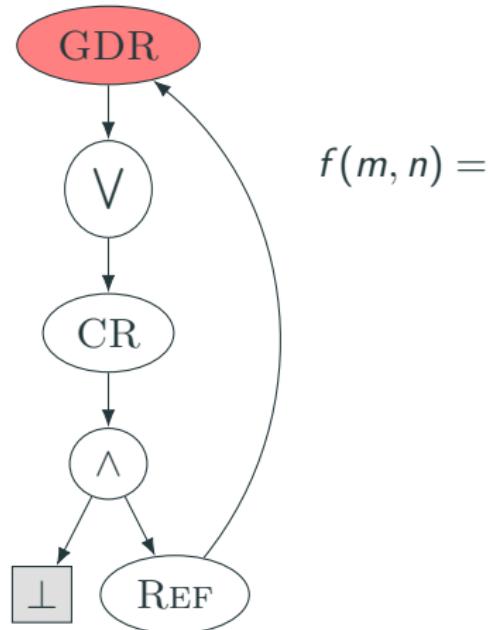
# How to Interpret an FCG



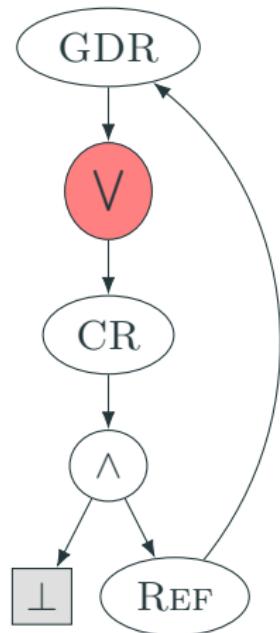
# How to Interpret an FCG



# How to Interpret an FCG

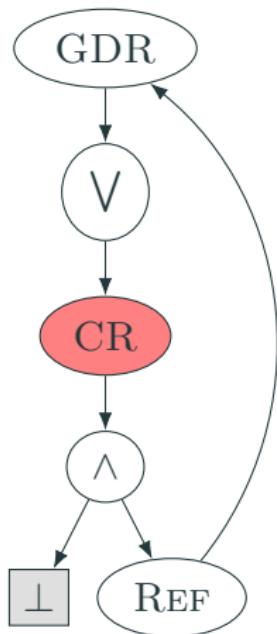


# How to Interpret an FCG



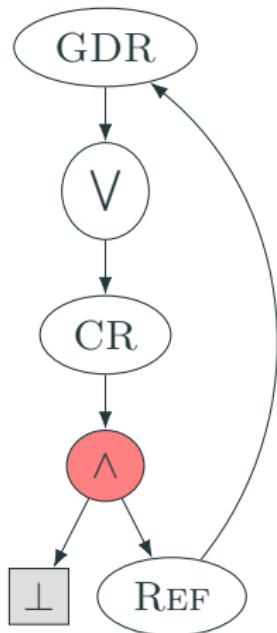
$$f(m, n) = \sum_{l=0}^m \binom{m}{l}$$

# How to Interpret an FCG



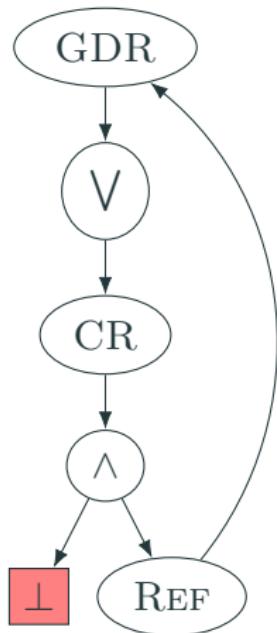
$$f(m, n) = \sum_{l=0}^m \binom{m}{l}$$

# How to Interpret an FCG



$$f(m, n) = \sum_{l=0}^m \binom{m}{l} \quad \times$$

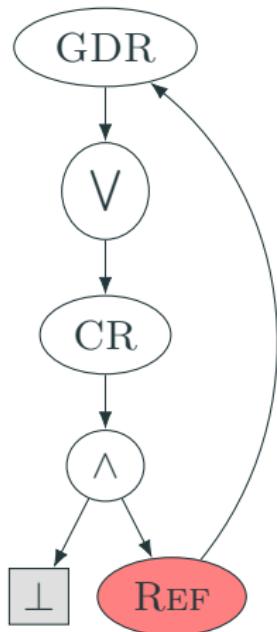
# How to Interpret an FCG



$$f(m, n) = \sum_{l=0}^m \binom{m}{l} [l < 2] \times$$

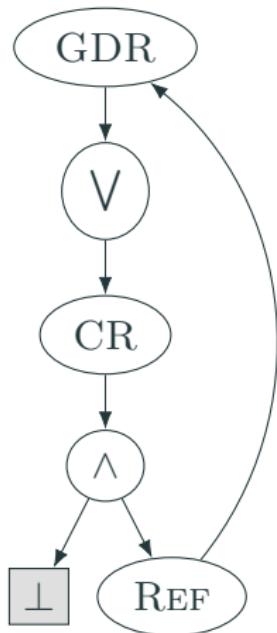
$$[\phi] = \begin{cases} 1 & \text{if } \phi \\ 0 & \text{if } \neg\phi \end{cases}$$

# How to Interpret an FCG



$$f(m, n) = \sum_{l=0}^m \binom{m}{l} [l < 2] \times f(m - l, n - 1)$$

## How to Interpret an FCG



$$\begin{aligned}f(m, n) &= \sum_{l=0}^m \binom{m}{l} [l < 2] \times f(m - l, n - 1) \\&= f(m, n - 1) + mf(m - 1, n - 1)\end{aligned}$$

# Compilation: How FCGs Are Built

## Definition

A **(compilation) rule** is a function that takes a **formula** and returns a set of  **$(G, L)$**  pairs, where

- $G$  is an FCG,
- and  $L$  is a list of formulas.

The formulas in  $L$  are then **compiled**, and the resulting FCGs are **inserted** into  $G$  according to a **set order**.

## Example Compilation Rule: Independence

Input formula:

$$(\forall x, y \in \Omega. x = y) \wedge \quad (1)$$

$$(\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z) \wedge \quad (2)$$

$$(\forall w, x \in \Gamma. \forall y \in \Delta. P(w, y) \wedge P(x, y) \Rightarrow w = x) \quad (3)$$

## Example Compilation Rule: Independence

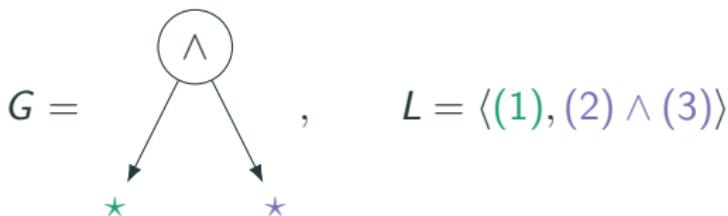
Input formula:

$$(\forall x, y \in \Omega. x = y) \wedge \quad (1)$$

$$(\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z) \wedge \quad (2)$$

$$(\forall w, x \in \Gamma. \forall y \in \Delta. P(w, y) \wedge P(x, y) \Rightarrow w = x) \quad (3)$$

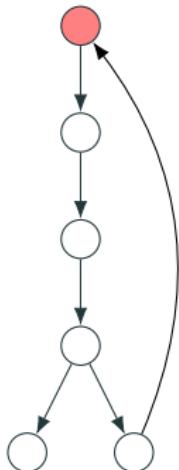
Only one (G, L) pair:



## New Rule 1/3: Generalised Domain Recursion

### Example

Input formula:



$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

Output formula (with a new constant  $c \in \Gamma$ ):

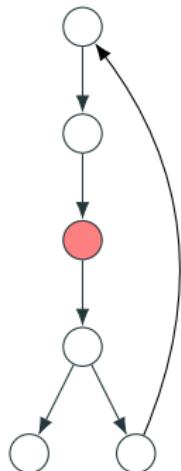
$$\forall y, z \in \Delta. y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$

$$\begin{aligned} \forall x \in \Gamma. \forall y, z \in \Delta. & \quad x \neq c \wedge y \neq z \Rightarrow \\ & \quad \neg P(x, y) \vee \neg P(x, z) \end{aligned}$$

## New Rule 2/3: Constraint Removal

### Example

Input formula (with a constant  $c \in \Gamma$ ):



$$\forall x \in \Gamma. \forall y, z \in \Delta. x \neq c \wedge y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

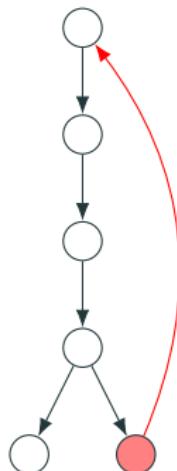
$$\forall w, x \in \Gamma. \forall y \in \Delta. w \neq c \wedge x \neq c \wedge w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

Output formula (with a new domain  $\Gamma' := \Gamma \setminus \{c\}$ ):

$$\forall x \in \Gamma'. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

## New Rule 3/3: Identifying Possibilities for Recursion



### Goal

Check if the input formula is equivalent (up to domains) to a previously encountered formula.

### Outline

1. Consider pairs of 'similar' clauses.
2. Consider bijections between their sets of variables.
3. Extend each such bijection to a map between sets of domains.
4. If the bijection makes the clauses equal, and the domain map is compatible with previous domain maps, move on to another pair of clauses.

## How These Rules Fit Together (1/5)

$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma. \forall y \in \Delta. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Generalised domain recursion



$$\forall y, z \in \Delta. y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$

$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall x \in \Gamma. \forall y \in \Delta. x \neq c \Rightarrow \neg P(c, y) \vee \neg P(x, y)$$

$$\forall w \in \Gamma. \forall y \in \Delta. w \neq c \Rightarrow \neg P(w, y) \vee \neg P(c, y)$$

$$\forall w, x \in \Gamma. \forall y \in \Delta. w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

## How These Rules Fit Together (2/5)

$$\forall y, z \in \Delta. \ y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$

$$\forall x \in \Gamma. \ \forall y, z \in \Delta. \ y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall x \in \Gamma. \ \forall y \in \Delta. \ x \neq c \Rightarrow \neg P(c, y) \vee \neg P(x, y)$$

$$\forall w \in \Gamma. \ \forall y \in \Delta. \ w \neq c \Rightarrow \neg P(w, y) \vee \neg P(c, y)$$

$$\forall w, x \in \Gamma. \ \forall y \in \Delta. \ w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Atom counting and unit propagation



$$\forall y, z \in \Delta^{\top}. \ y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma. \ \forall y, z \in \Delta^{\perp}. \ y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma. \ \forall y \in \Delta^{\perp}. \ w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

## How These Rules Fit Together (3/5)

$$\forall y, z \in \Delta^\top. y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma. \forall y, z \in \Delta^\perp. y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma. \forall y \in \Delta^\perp. w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Constraint removal



$$\forall y, z \in \Delta^\top. y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

## How These Rules Fit Together (4/5)

$$\forall y, z \in \Delta^T. y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Independence

$$\forall y, z \in \Delta^T. y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

## How These Rules Fit Together (5/5): Recursion

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

||

$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma. \forall y \in \Delta. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

+

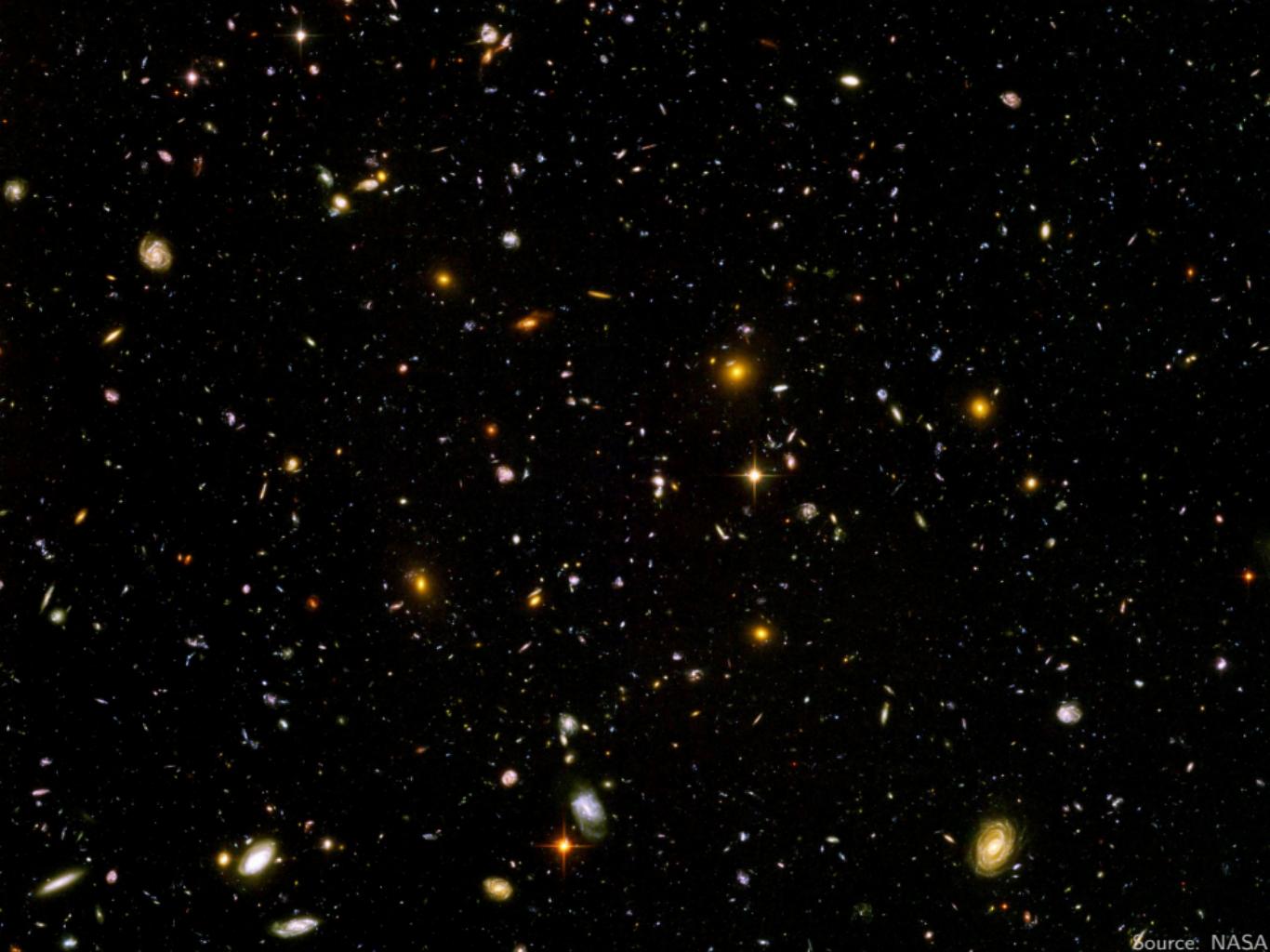
$$\{ \Gamma \mapsto \Gamma', \Delta^\perp \mapsto \Delta \}$$

## Resulting Improvements to Counting Functions

Let  $\Gamma$  and  $\Delta$  be two sets with cardinalities  $|\Gamma| = m$  and  $|\Delta| = n$ .

Our new rules enable Crane to efficiently count  $\Gamma \rightarrow \Delta$  functions such as:

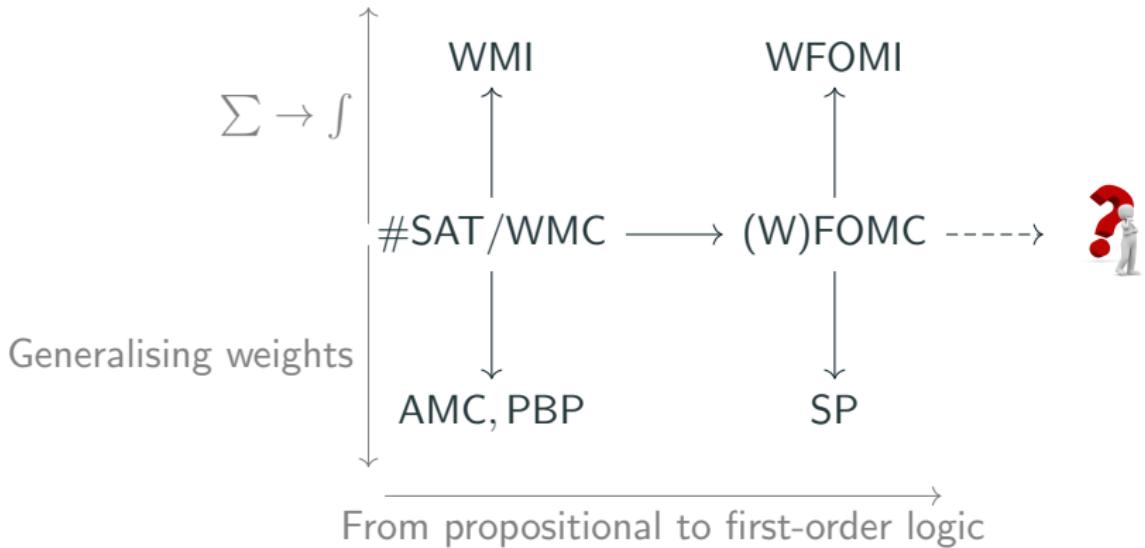
- injections in  $\Theta(mn)$  time
  - by hand:  $\Theta(m)$
- partial injections in  $\Theta(mn)$  time
  - by hand:  $\Theta(\min\{m, n\}^2)$
- bijections in  $\Theta(m)$  time
  - optimal!
  - FastWFOMC seems to be  $\Omega(m^4)$



## What Have We Learned?

- Knowledge compilation can build **graphs with cycles**.
- Graphs (as well as circuits) define **functions**.
- Cycles can represent **recursive calls**, including:
  - mutual recursion
  - and function calls as complex as  $f(n - k - 2)$ .
- Recursion helps us solve counting problems that were previously beyond the reach of FOMC.
- In some cases, even if a polynomial-time solution is already known, Crane is able to find **more efficient solutions**, with a lower degree polynomial.

# Beyond First-Order Logic



What kind of logic is needed to succinctly describe, e.g.,

- $f(n) = f(f(n - 1))$
- or the Fibonacci sequence?

## Algebraic Solutions to Parameterised Problems (1/2)

- Suppose we have a Markov logic network that models the probability  $P$  that some system will fail.
- Here:
  - domain sizes describe the numbers of various components,
  - and weights express probabilities that:
    - some component fails,
    - or some combination of failures leads to another failure.
- Crane can express  $P$  as a function of the domain sizes and weights.

## Algebraic Solutions to Parameterised Problems (2/2)

With the help of a computer algebra system, we can then:

- determine how  $P$  scales with the number of users,
- find combinations of domain sizes that keep  $P$  below some threshold,
- find ranges of weights that keep  $P$  sufficiently small across a range of domain size values.