A Proofs

TODO: finish arranging the definitions

We begin with a few definitions that formalise the notion of a model.

For any set S, let 2^S denote its power set and $S^* \coloneqq \bigcup_{i=0}^{\infty} S^i$ the set of tuples of any finite length of elements of S. For example, $\{ a, b \}^* = \{ (), (a), (b), (a, a), \dots \}$.

Let Preds be the function that maps any clause or formula to the set of predicates used within.

For any $n \in \mathbb{N}_0$, let $[n] \coloneqq \{ 1, \dots, n \}$, e.g., $[0] = \emptyset$, and $[2] = \{ 1, 2 \}$.

Each formula ϕ has a map $\pi_\phi \colon \mathrm{Preds}(\phi) \to \mathrm{Doms}(\phi)^*$ s.t., for each predicate $\mathtt{p}/n \in \mathrm{Preds}(\phi)$, we have that $\pi_\phi(\mathtt{p}) \in \mathrm{Doms}(\phi)^n$.

Definition 1. Let ϕ be a formula and σ a domain size function. A *model* of (ϕ, σ) is a map $\mathfrak{M} \colon \mathrm{Preds}(\phi) \to 2^{\mathbb{N}_0^*}$ s.t. the following two conditions are satisfied.

1. Let $\mathtt{p}/n \in \mathrm{Preds}(\phi)$ be a predicate and let $\pi_\phi(\mathtt{p}) = (d_i)_{i=1}^n$ for some domains $d_i \in \mathrm{Doms}(\phi)$. Then
$$\mathfrak{M}(\mathtt{p}) \subseteq \prod_{i=1}^{n} [\sigma(d_i)].^{1} \tag{1}$$
2. As a collection of relations, $\mathfrak{M}$ satisfies ϕ.
   - TODO: introduce the substitution for formulas, moving the definition from elsewhere
   - define inductively as...
   - (not exhaustive)
   - $\mathfrak{M} \models \phi \land \psi$ iff $\mathfrak{M} \models \phi$ and $\mathfrak{M} \models \psi$
   - $\mathfrak{M} \models \phi \lor \psi$ iff $\mathfrak{M} \models \phi$ or $\mathfrak{M} \models \psi$
   - $\mathfrak{M} \models \neg\phi$ iff $\mathfrak{M} \not\models \phi$
   - $\mathfrak{M} \models (\forall X \in \Delta.\ \phi)$ iff $\mathfrak{M} \models \phi[x/\{ X \}]$ for all $x \in \Delta$
   - One can then add inequality constraints to the semantics as follows
   - $\mathfrak{M} \models (\forall X \in \Delta.\ X \neq c \Rightarrow \phi)$ iff $\mathfrak{M} \models \phi[x/\{ X \}]$ for all $x \in \Delta \setminus \{ c \}$
   - $\mathfrak{M} \models (\forall X, Y \in \Delta.\ X \neq Y \Rightarrow \phi)$ iff $\mathfrak{M} \models \phi[x/\{ X \}][y/\{ Y \}]$ for all $x, y \in \Delta$ s.t. $x \neq y$

Theorem 1 (Correctness of GDR)**.** *Let ϕ be the formula used as input to Algorithm 1, $\Omega \in \mathcal{D}$ the domain selected on line 2, and ϕ' the formula constructed by the algorithm for Ω. Suppose that $\Omega \neq \emptyset$. Then $\phi \equiv \phi'$.*

Proof. Let x be the constant introduced on line 4 and c a clause of ϕ selected on line 5 of the algorithm. We will show that c is equivalent to the conjunction of clauses added to ϕ' on lines 7 and 8.

If c has no variables with domain Ω, then lines 7 and 8 simply add a copy of c to ϕ'. If c has one variable with domain Ω, say, X, then $c \equiv \forall X \in \Omega.\ \psi$ for some formula ψ with one free variable X. In this case, lines 7 and 8 create two clauses: $\psi[x/\{ X \}]$ and $\forall X \in \Omega.\ X \neq x \Rightarrow \psi$. It is easy to see that their conjunction is equivalent to c provided that $\Omega \neq \emptyset$ and so x is a well-defined constant.

Let V be the set of variables introduced on line 6. It remains to show that repeating the transformation
$$\forall X \in \Omega.\ \psi \mapsto \{ \psi[x/\{ X \}], \forall X \in \Omega.\ X \neq x \Rightarrow \psi \}$$
for all variables $X \in V$ produces the same clauses as lines 7 and 8 (except for some tautologies that the latter method skips). Note that the order in which these operations are applied is immaterial. In other words, if we add inequality constraints for variables $\{ X_i \}_{i=1}^n$, and apply substitution for variables $V \setminus \{ X_i \}_{i=1}^n$, then—regardless of the order of operations—we get
$$\forall X_1, \dots, X_n \in \Omega.\ \bigwedge_{i=1}^{n} X_i \neq x \Rightarrow \psi\left[x/\left(V \setminus \{ X_i \}_{i=1}^n\right)\right].$$
This formula is equivalent to the clause generated on line 8 with $W = V \setminus \{ X_i \}_{i=1}^n$. Thus, for every clause c of ϕ, the new formula ϕ' gets a set of clauses whose conjunction is equivalent to c. Hence, the two formulas are equivalent. ☐

Theorem 2 (Correctness of CR)**.** *Let ϕ be the input formula of Algorithm 2, (Ω, x) a replaceable pair, and ϕ' the formula constructed by the algorithm, when (Ω, x) is selected on line 2. Then $\phi \equiv \phi'$, where the domain Ω' introduced on line 3 is interpreted as $\Omega \setminus \{ x \}$.*

Proof. Since there is a natural bijection between the clauses of ϕ and ϕ', we shall argue about the equivalence of each pair of clauses. Let c be an arbitrary clause of ϕ and c' its corresponding clause of ϕ'.

If c has no variables with domain Ω, then it cannot have any constraints involving x, so $c' = c$. Otherwise, for notational simplicity, let us assume that X is the only variable in c with domain Ω (the proof for an arbitrary number of variables is virtually the same). By Definition 4, we can rewrite c as $\forall X \in \Omega.\ X \neq x \Rightarrow \psi$, where ψ is a formula with X as the only free variable and with no mention of either x or Ω. Then $c' \equiv \forall X \in \Omega'.\ \psi$. Since $\Omega' \coloneqq \Omega \setminus \{ x \}$, we have that $c \equiv c'$. Since c was an arbitrary clause of ϕ, this completes the proof that $\phi \equiv \phi'$. ☐

Theorem 3 (Correctness of REF)**.** *Let ϕ be the formula used as input to Algorithm 3. Let ψ be any formula selected on line 1 of the algorithm s.t. $\rho \neq$ `null` on line 3. Let σ be a domain size function. Then the set of models of $(\psi, \sigma \circ \rho)$ is equal to the set of models of (ϕ, σ).*

Proof. We first show that the right-hand side of Eq. (1) is the same for models of both (ϕ, σ) and $(\psi, \sigma \circ \rho)$. Algorithm 3 ensures that $\psi \equiv \phi$ up to domains. In particular, this means that $\mathrm{Preds}(\psi) = \mathrm{Preds}(\phi)$. The square in
$$\begin{tikzcd}
\mathrm{Preds}(\psi) \arrow[r, equal] \arrow[d, "\pi_\psi"'] & \mathrm{Preds}(\phi) \arrow[d, "\pi_\phi"] & \\
\mathrm{Doms}(\psi)^* \arrow[r, "\rho"] & \mathrm{Doms}(\phi)^* \arrow[r, "\sigma"] & \mathbb{N}_0^*
\end{tikzcd} \tag{2}$$
commutes by Definition 3 and the definition of ρ. In other words, for any predicate $\mathtt{p}/n \in \mathrm{Preds}(\psi) = \mathrm{Preds}(\phi)$,

[1] For simplicity, Definition 1 ignores constants. To include constants, one would replace $[\sigma(d_i)]$ with a set that contains all constants associated with domain d_i, extended with enough new elements to make its cardinality $\sigma(d_i)$.

function ρ translates the domains associated with p in ψ to the domains associated with p in ϕ. Let $\pi_\phi(\mathrm{p}) = (d_i)_{i=1}^n$ and $\pi_\psi(\mathrm{p}) = (e_i)_{i=1}^n$ for some domains $d_i \in \mathrm{Doms}(\phi)$ and $e_i \in \mathrm{Doms}(\psi)$. Since $\rho(e_i) = d_i$ for all $i = 1, \ldots, n$ by the commutativity in diagram (2), we get that

$$\prod_{i=1}^{n}[\sigma(d_i)] = \prod_{i=1}^{n}[\sigma \circ \rho(e_i)] \tag{3}$$

as required. Certainly, any subset of the left-hand side of Eq. (3) (e.g., $\mathfrak{M}(\mathrm{p})$, where $\mathfrak{M}$ is a model of (ϕ, σ)) is a subset of the right-hand side and vice versa. Since ϕ and ψ semantically only differ in what domains they refer to, every model of ϕ satisfies ψ and vice versa, completing the proof. $\square$

B Solutions Found by CRANE

Here we list the exact function definitions produced by CRANE for all of the problem instances in Section 6 both before and after algebraic simplification (excluding multiplications by one). The correctness of all of them has been checked by identifying suitable base cases and verifying the numerical answers across a range of domain sizes.

1. A $\Theta(m)$ solution for counting $\Gamma \to \Gamma$ functions:

$$f(m) = \left(-1 + \sum_{l=0}^{m} \binom{m}{l} [l < 2]\right)^{m} = m^m.$$

2. A $\Theta(m^3 + n^3)$ solution for counting $\Gamma \to \Delta$ surjections:

$$\begin{aligned} f(m,n) = &\sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \\ &\left(\sum_{j=0}^{k} \binom{k}{j} [j < 2]\right)^{l} \\ = &\sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (k+1)^l. \end{aligned}$$

3. A $\Theta(m^3)$ solution for counting $\Gamma \to \Gamma$ surjections:

$$\begin{aligned} f(m) = &\sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \\ &\left(\sum_{j=0}^{k} \binom{k}{j} [j < 2]\right)^{l} \\ = &\sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (k+1)^l. \end{aligned}$$

4. A $\Theta(mn)$ solution for counting $\Gamma \to \Delta$ injections and partial injections (with different base cases):

$$\begin{aligned} f(m,n) &= \sum_{l=0}^{m} \binom{m}{l} [l < 2] f(m-l, n-1) \\ &= f(m, n-1) + m f(m-1, n-1). \end{aligned}$$

5. A $\Theta(m^3)$ solution for counting $\Gamma \to \Gamma$ injections:

$$\begin{aligned} f(m) &= \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} g(m,l); \\ g(m,l) &= \sum_{k=0}^{l} \binom{l}{k} [k < 2] g(m-1, l-k) \\ &= g(m-1, l) + l g(m-1, l-1). \end{aligned}$$

6. A $\Theta(m)$ solution for counting $\Gamma \to \Delta$ bijections:

$$f(m,n) = m f(m-1, n-1).$$

7. A $\Theta(l + mn)$ solution for counting $|\Lambda|$ partial injections $\Gamma \to \Delta$:

$$\begin{aligned} f(l,m,n) &= g(m,n)^l \\ g(m,n) &= \sum_{k=0}^{n} \binom{n}{k} [k < 2] g(m-1, n-k) \\ &= g(m-1, n) + n g(m-1, n-1). \end{aligned}$$