

Synthesising Recursive Functions for First-Order Model Counting

Paulius Dilkas

Joint work with Vaishak Belle (University of Edinburgh, UK)

18th April 2023

National University of Singapore, Singapore

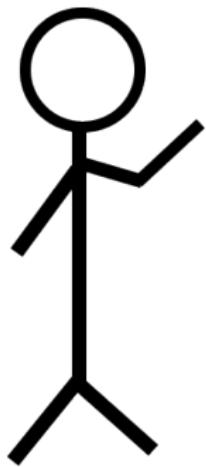


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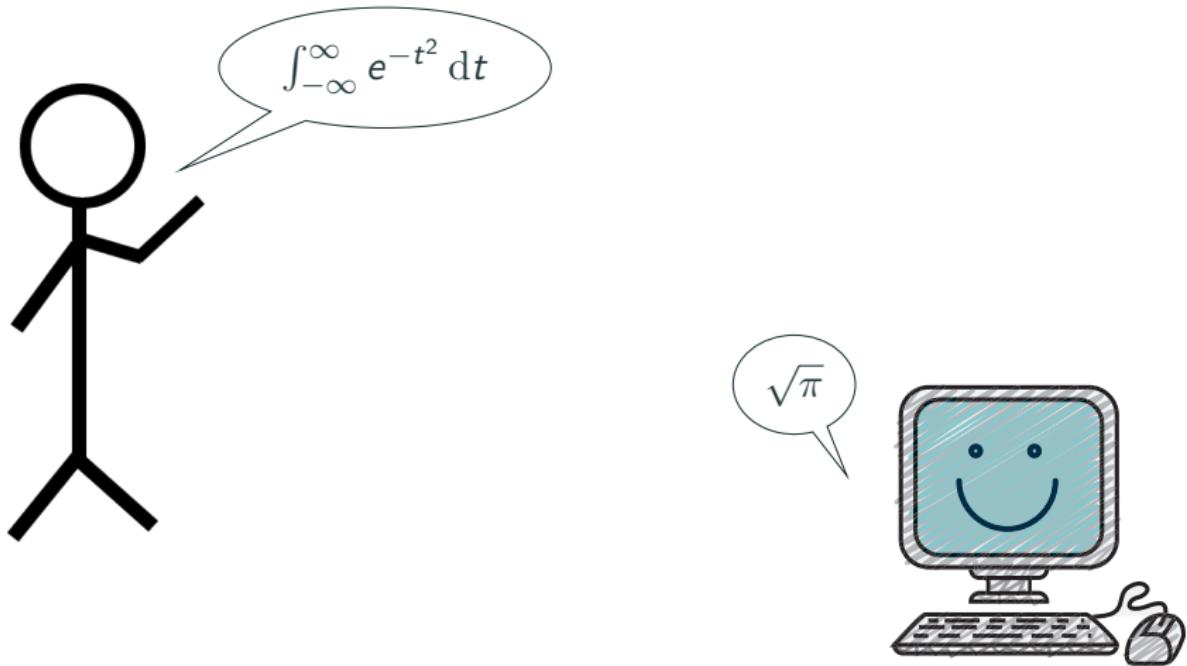


Engineering and
Physical Sciences
Research Council

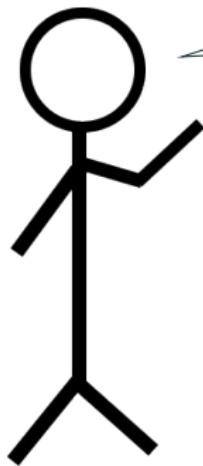
What Computers Can and Cannot Do



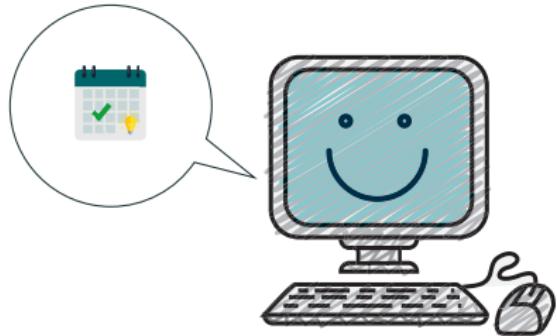
What Computers Can and Cannot Do



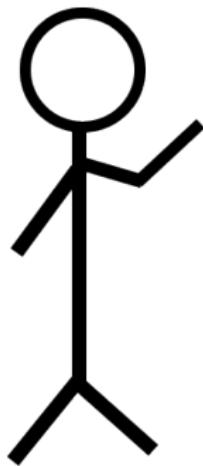
What Computers Can and Cannot Do



Produce a schedule for the
nurses at the local hospital.



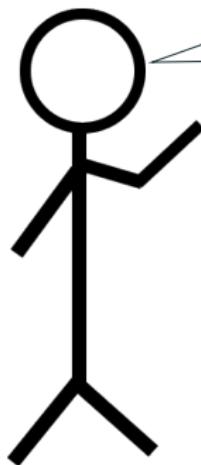
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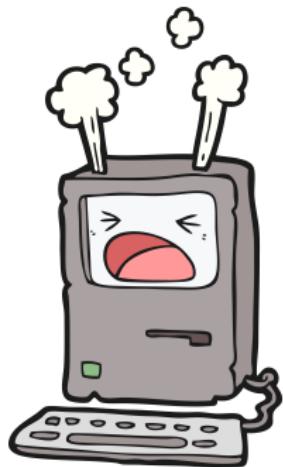
Paint a baroque oil painting of a
raccoon queen wearing a crown.



What Computers Can and Cannot Do



If I shuffle a deck of n cards,
how many possible outcomes
are there?



Terms and conditions apply.

Who Cares About Counting?

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Probabilistic Programming

*Inference and learning in probabilistic logic
programs using weighted Boolean formulas*

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS,
DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON,
GERDA JANSSENS and LUC DE RAEDT

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Neuro-symbolic AI

A Semantic Loss Function for Deep Learning with Symbolic Knowledge

Jingyi Xu¹ Ziliu Zhang² Tal Friedman¹ Yitao Liang¹ Guy Van den Broeck¹

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Natural Language Processing

Joint Inference for Knowledge Extraction from Biomedical Literature

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**Learning Relational Affordance Models for Robots
in Multi-Object Manipulation Tasks**

Bogdan Moldovan Plinio Moreno Martijn van Otterlo José Santos-Victor Luc De Raedt

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PheNetic: Network-based interpretation of unstructured gene lists in E. coli
Dries De Maeyer¹, Joris Renkens², Lore Cloots¹, Luc De Raedt^{1,2}, Kathleen Marchal^{1,3}

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Combinatorics

Automatic Conjecturing of P-Recursions
Using Lifted Inference

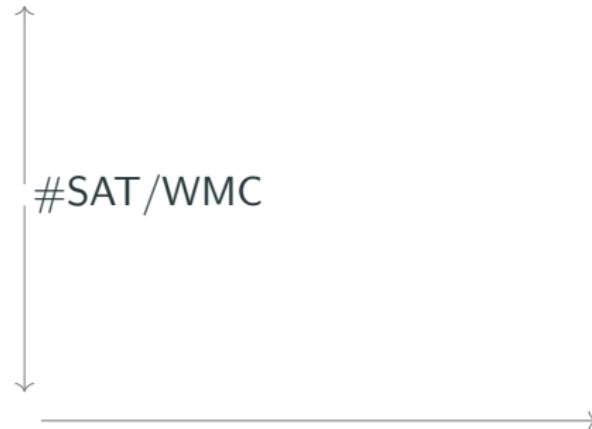
Jáchym Barvínek^{1(✉)}, Timothy van Bremen², Yuyi Wang³, Filip Železný¹,
and Ondřej Kuzelka¹

¹ Czech Technical University in Prague, Prague, Czech Republic
barvijac@fel.cvut.cz

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³ ETH Zurich, Zurich, Switzerland

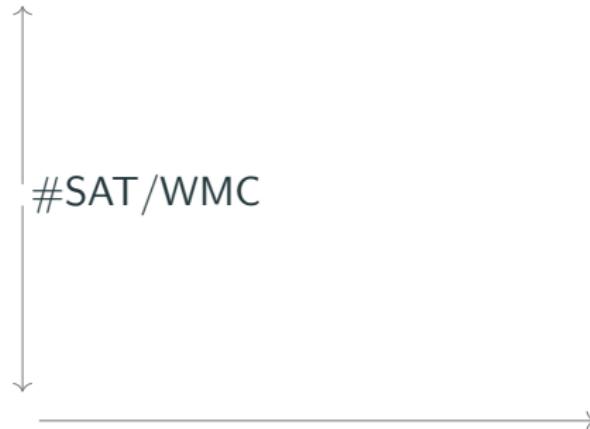
(Some of the) Many Ways to Count



#SAT (Valiant 1979)

- Input formula: $x \vee y$
- Interpretations: $\emptyset, \{x\}, \{y\}, \{x, y\}$
- Models: $\{x\}, \{y\}, \{x, y\}$

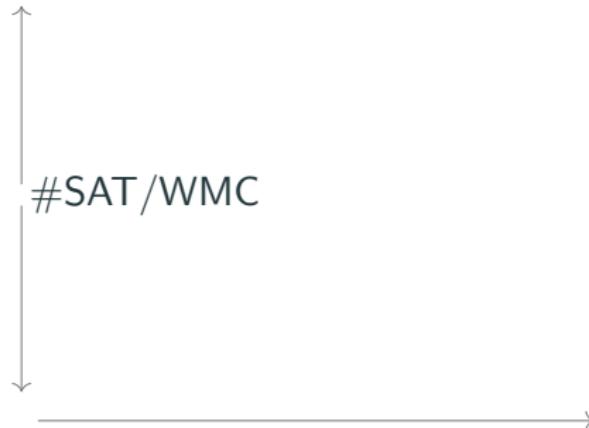
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- Answer (model count): 3

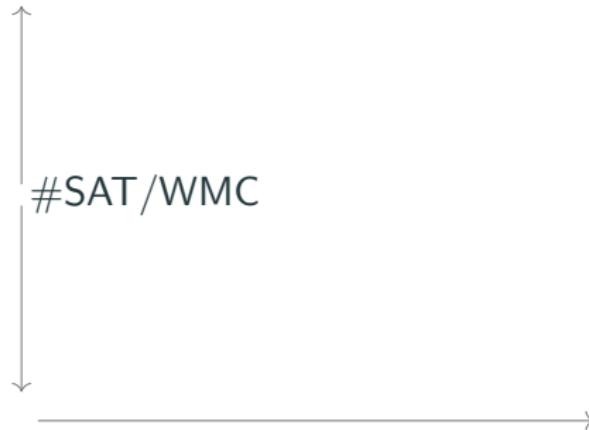
(Some of the) Many Ways to Count



Weighted Model Counting (Chavira and Darwiche 2008)

- Input formula: $x \vee y$
- Input weights: $w(x) = 0.3, w(\neg x) = 0.7,$
 $w(y) = 0.2, w(\neg y) = 0.8$

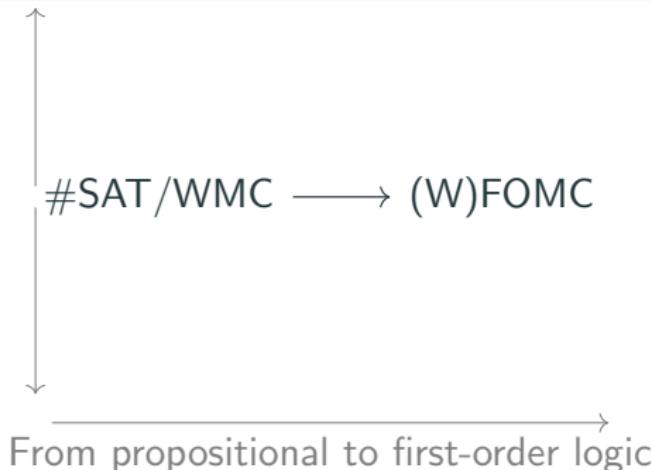
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 $w(y) = 0.2, w(\neg y) = 0.8$
- Answer (weighted model count):
 $w(x)w(y) + w(x)w(\neg y) + w(\neg x)w(y) = 0.44$

(Some of the) Many Ways to Count

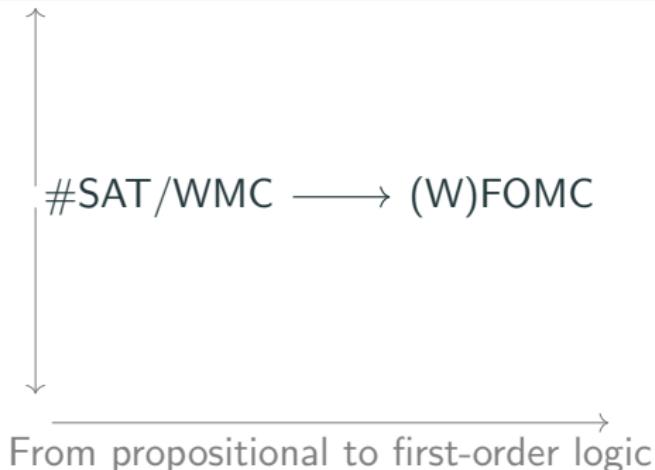


(Weighted) (Symmetric) First-Order Model Counting

(Van den Broeck et al. 2011)

- Input formula: $\forall \textcolor{teal}{x} \in \Delta. P(\textcolor{teal}{x})$
- Input weights: $w^+(P) = 0.3, w^-(P) = 0.7$
- Input domain size(s): $|\Delta| = 2$

(Some of the) Many Ways to Count

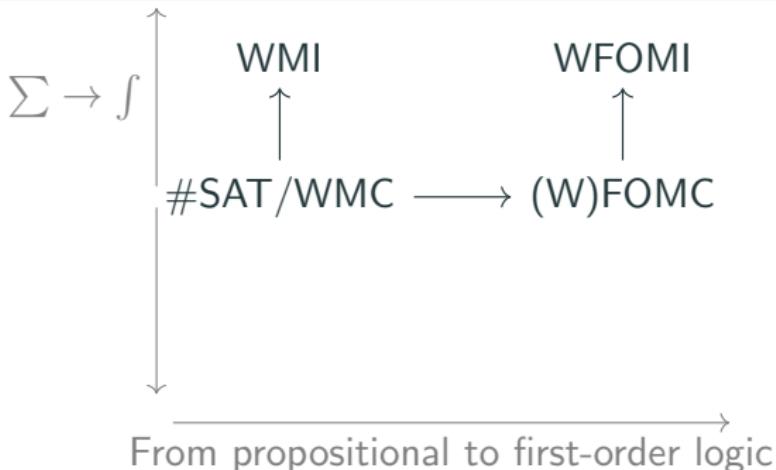


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- Answer: $(w^+(P))^{|\Delta|} = 0.09$

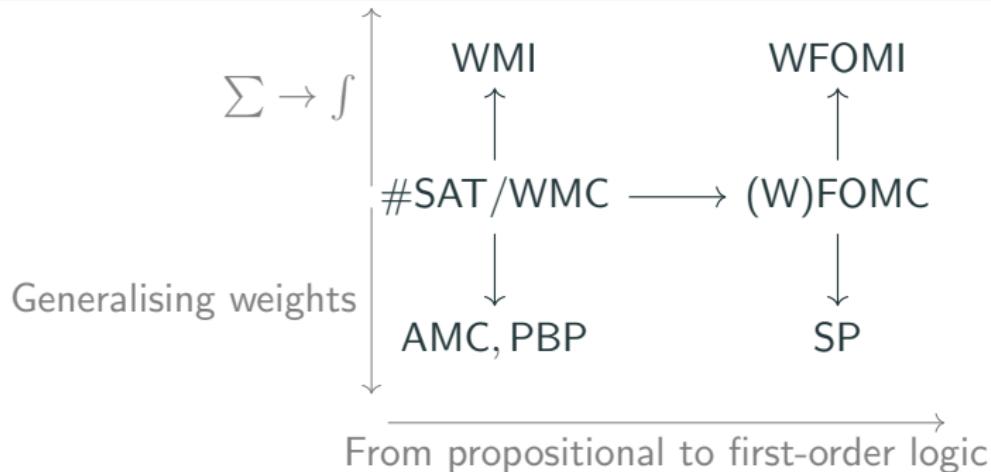
(Some of the) Many Ways to Count



Extensions to Continuous Domains

- Weighted model integration
 - (Belle, Passerini and Van den Broeck 2015)
- Weighted first-order model integration
 - (Feldstein and Belle 2021)

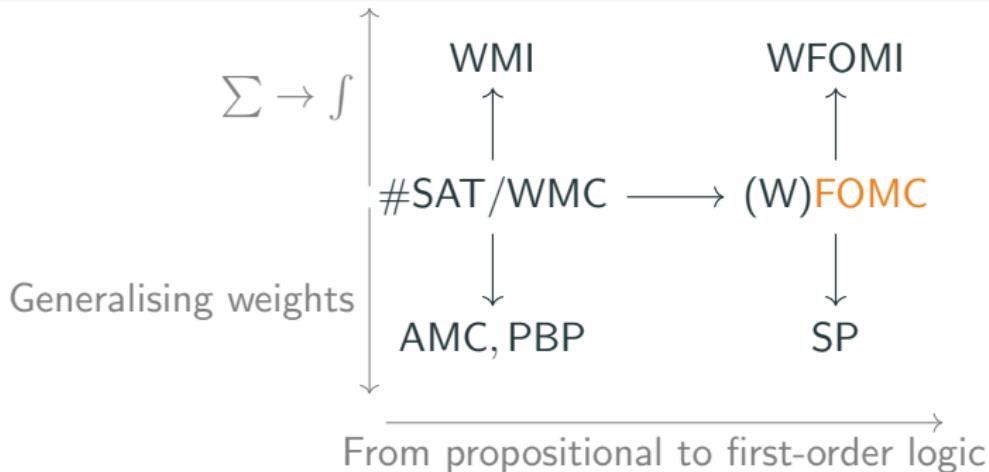
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Generalisations of the weight function

- Algebraic model counting
 - (Kimmig, Van den Broeck and De Raedt 2017)
 - From $\mathbb{R}_{\geq 0}$ to commutative semirings
 - Pseudo-Boolean projection (D. and Belle 2021)
 - Weights not necessarily on literals
 - Semiring programming (Belle and De Raedt 2020)

(Some of the) Many Ways to Count



(Unweighted) First-Order Model Counting

- Example formula:

$$\forall x \in \Delta. P(x) \vee Q(x)$$

- Let $\Delta := \{1, 2\}$

- Interpretations: all subsets of

$$\{P(1), Q(1), P(2), Q(2)\}$$

(Unweighted) First-Order Model Counting

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- Models:

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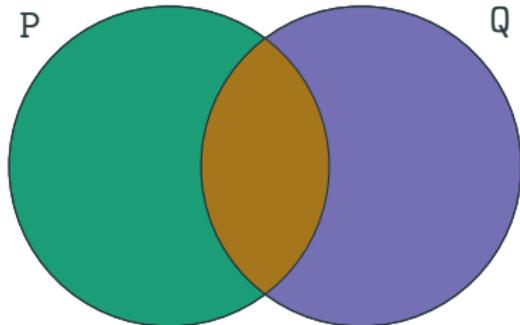
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$$\{P(1), Q(1), P(2)\}, \quad \{P(1), Q(1), Q(2)\}, \quad \{P(1), Q(1), P(2), Q(2)\}$$



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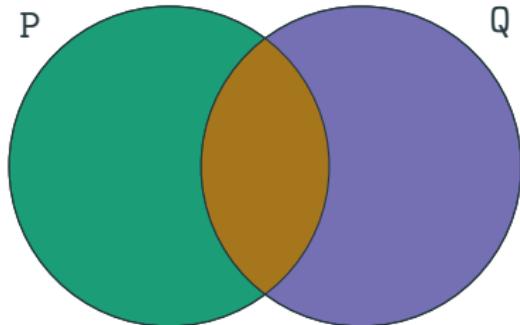
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Intuition

- Each 1-ary predicate is like a subset
- For $n > 1$, each n -ary predicate is like a relation
- FOMC counts combinations of relations

Approaches to FOMC

- **ForcLift** (Van den Broeck et al. 2011)
 - knowledge compilation to **FO d-DNNF**
- **L2C** (Kazemi and Poole 2016)
 - knowledge compilation to **C++ code**
- **Alchemy** (Gogate and Domingos 2016)
 - **DPLL**-style search
- **FastWFOMC** (van Bremen and Kuželka 2021)
 - knowledge compilation to **sd-DNNF**

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Our Contribution



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ForcLift

Recursion

Crane

ForcLift and First-Order Knowledge Compilation

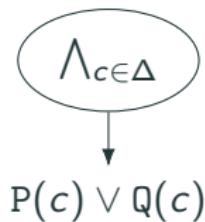
$$\forall x \in \Delta. \ P(x) \vee Q(x)$$

ForcLift and First-Order Knowledge Compilation

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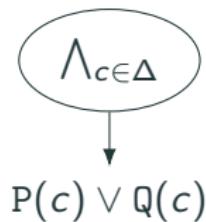
Independent partial grounding (introduces a constant $c \in \Delta$)

ForcLift and First-Order Knowledge Compilation



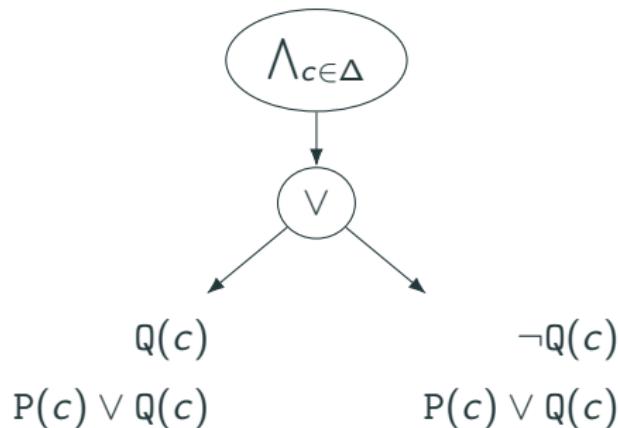
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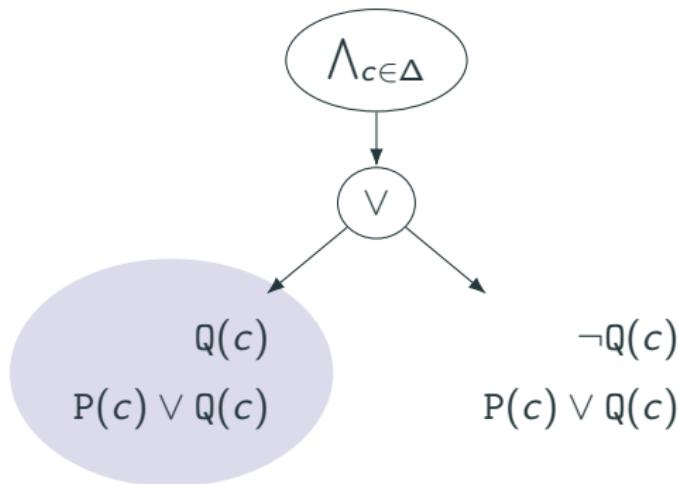
Shannon decomposition (a.k.a. Boole's expansion theorem) on $Q(c)$

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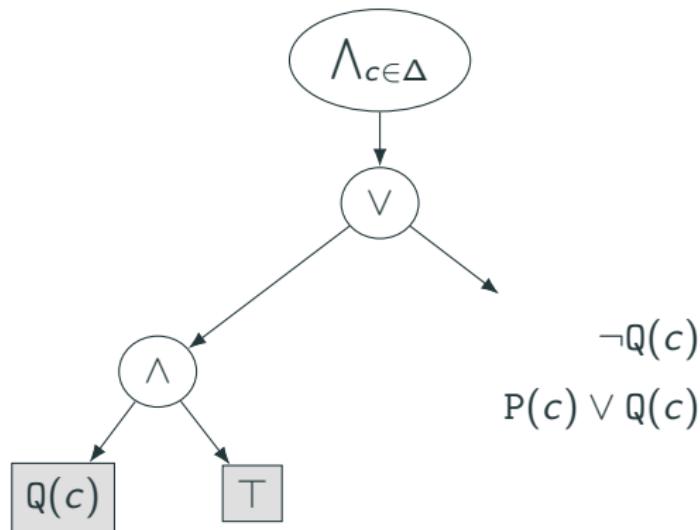
Shannon decomposition (a.k.a. Boole's expansion theorem) on $\text{Q}(c)$

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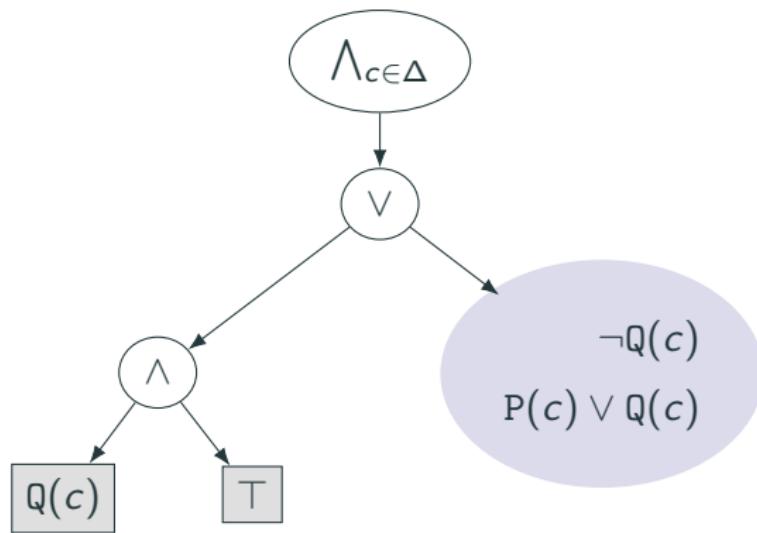
Positive unit propagation of $Q(c)$

ForcLift and First-Order Knowledge Compilation



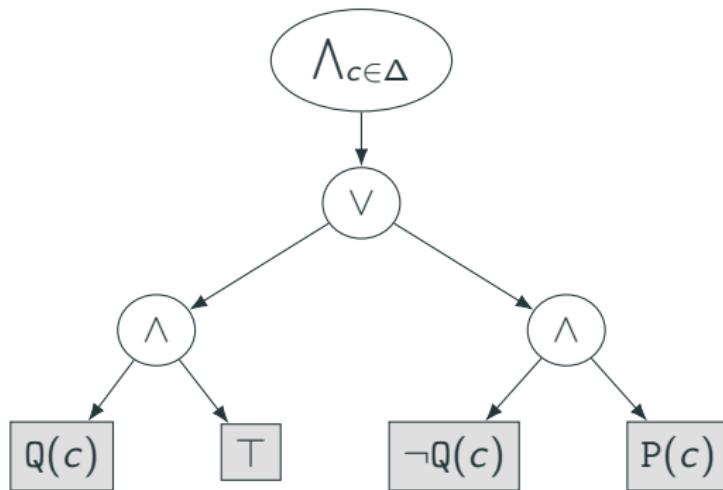
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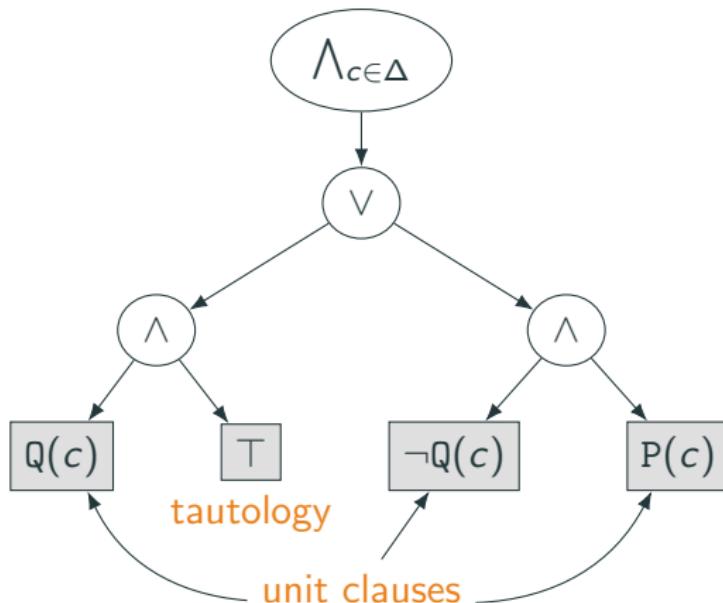
Negative unit propagation of $\neg Q(c)$

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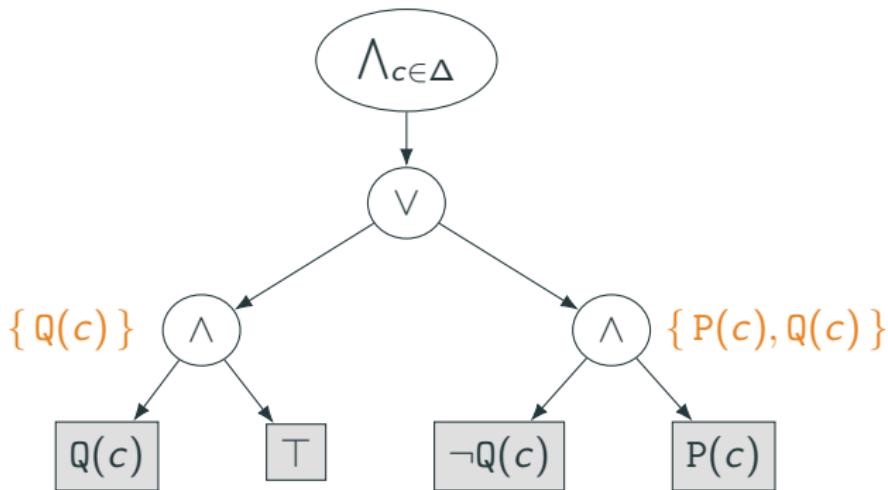
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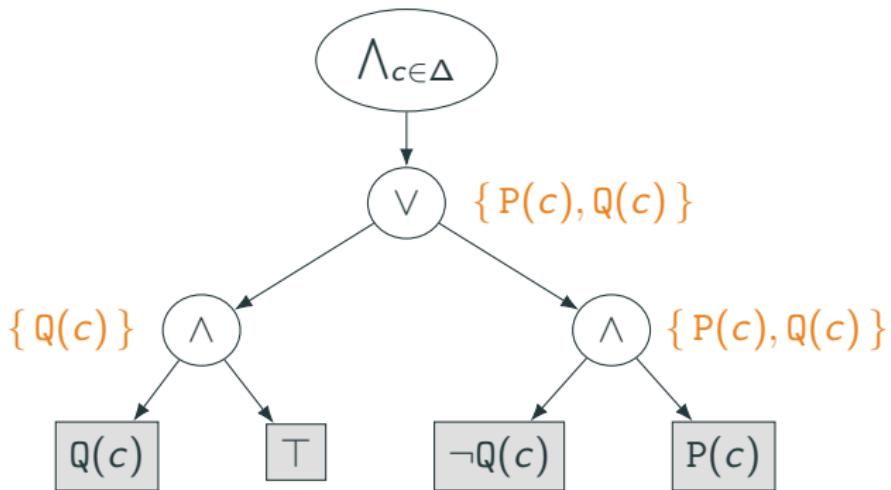
Compilation is complete ✓

ForcLift and First-Order Knowledge Compilation



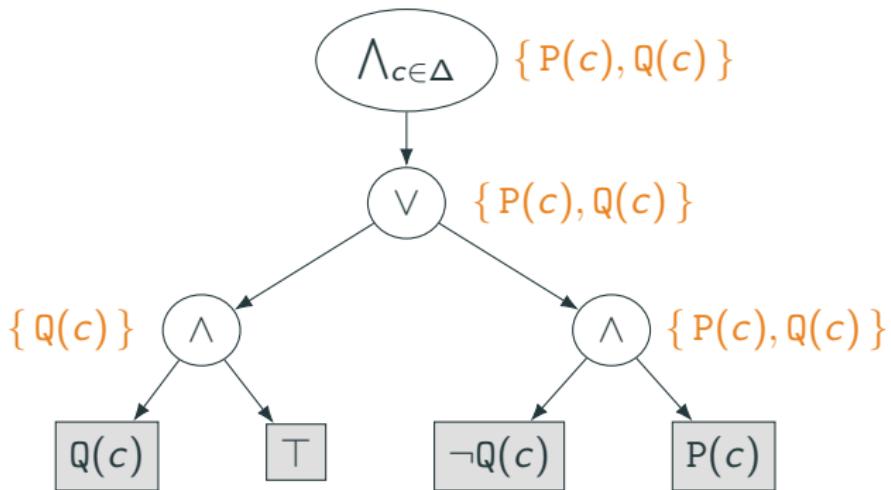
Smoothing: propagating atoms upwards

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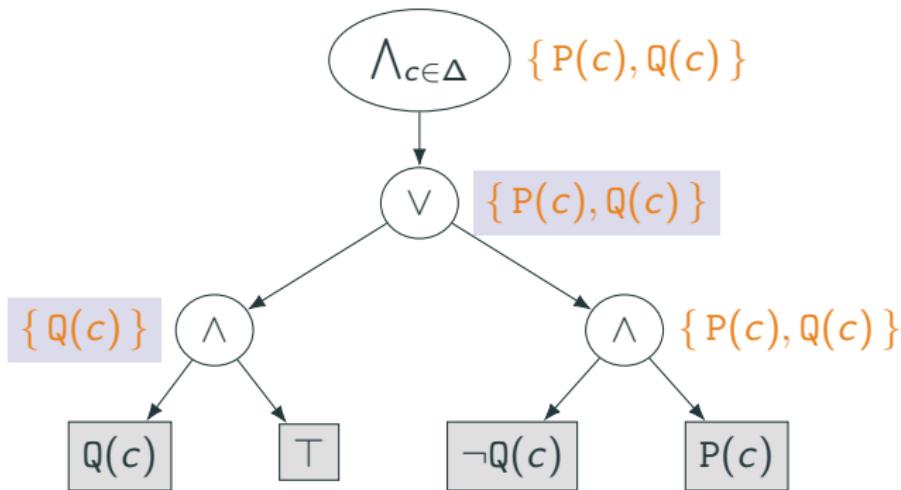
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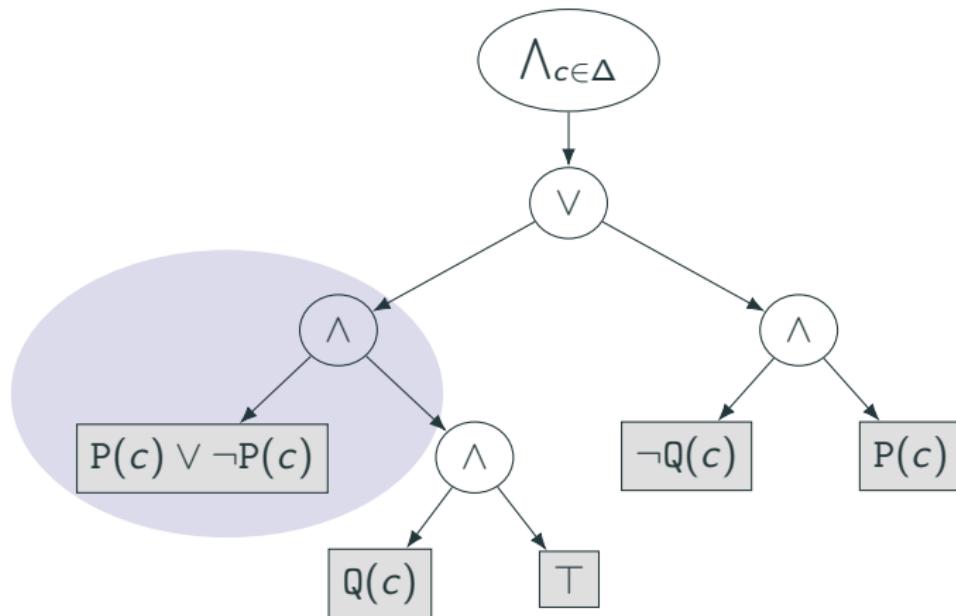
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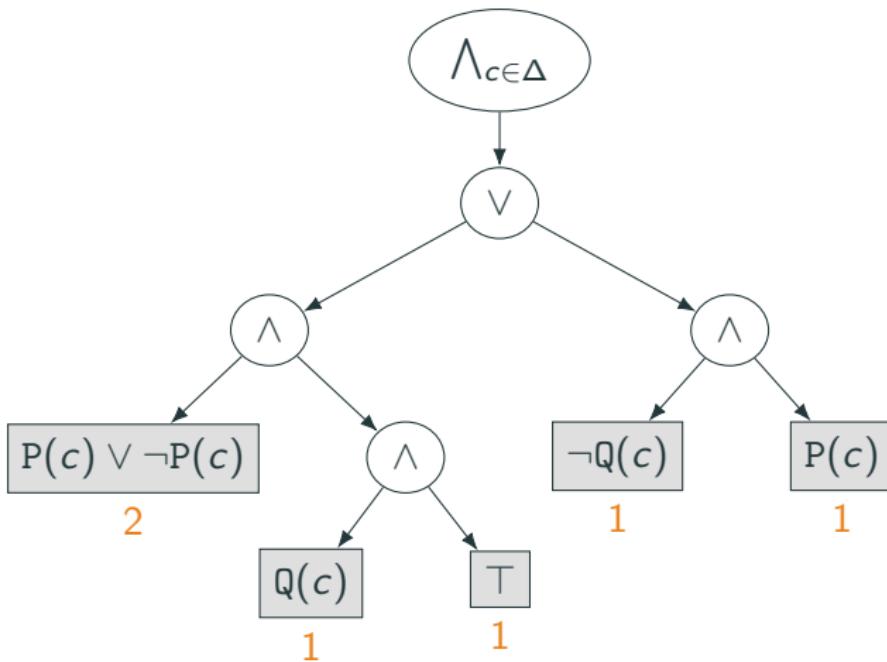
Smoothing: adding new atoms

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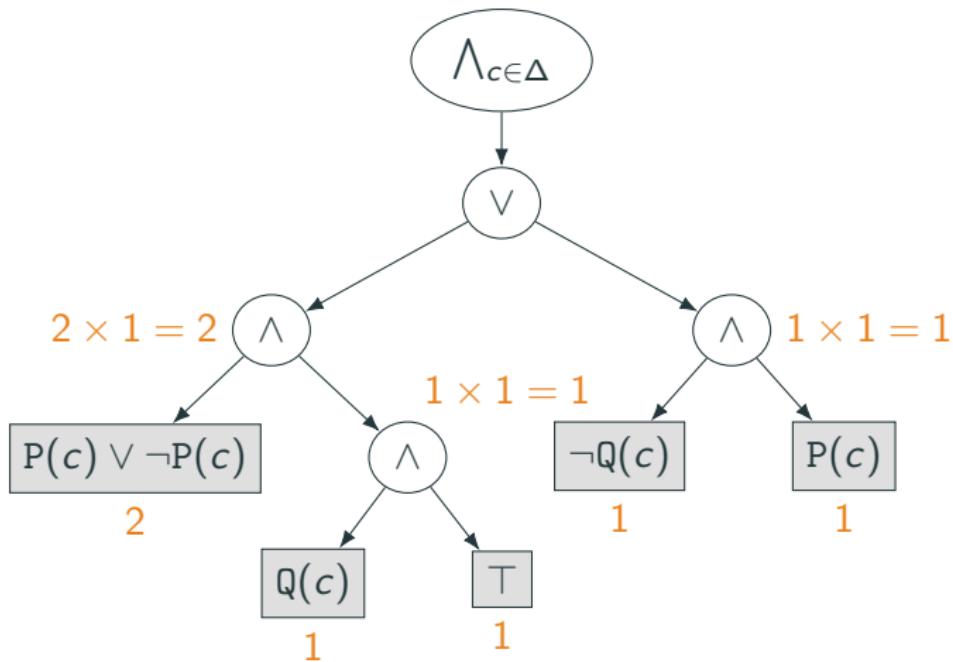
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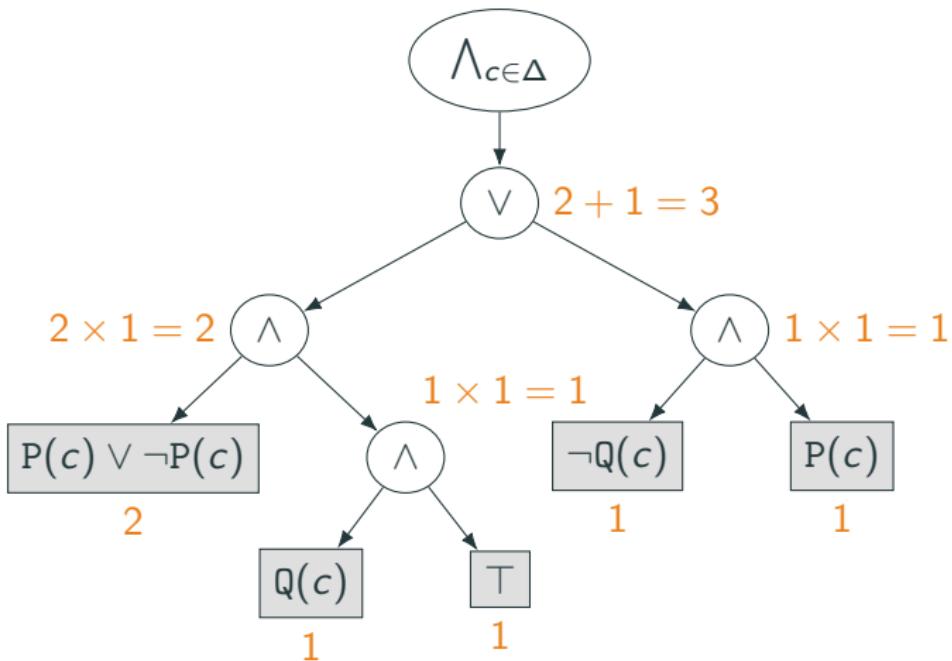
Propagating the model count

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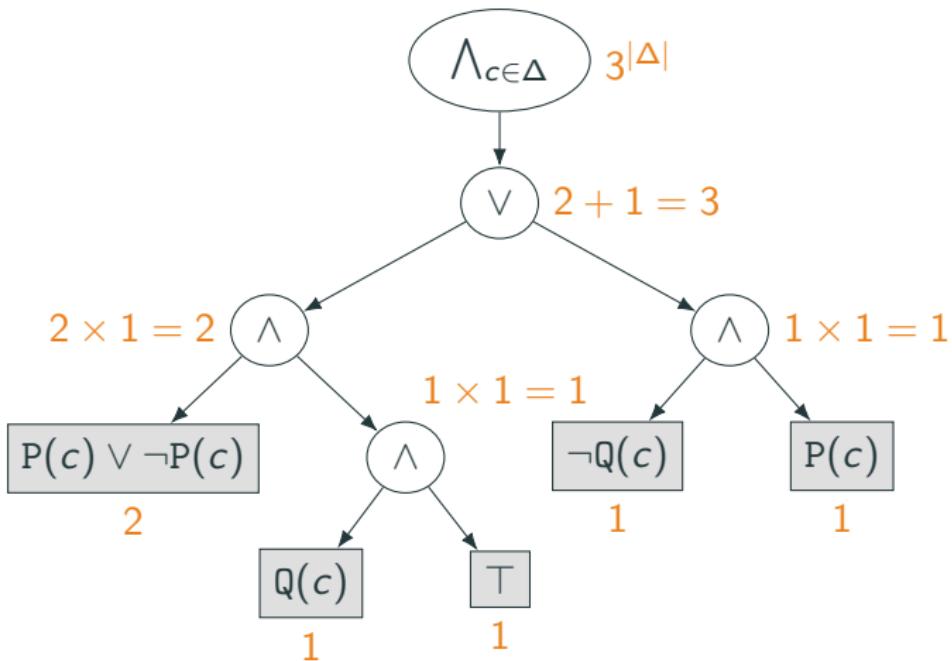
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ForcLift and First-Order Knowledge Compilation



Propagating the model count

ForcLift and First-Order Knowledge Compilation



Propagating the model count

A (Slightly) More Complicated Example

Suppose this room has n seats, and there are $m \leq n$ people in the audience. How many ways are there to seat everyone?

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More explicitly, we assume that:

- each attendee gets exactly one seat,
- and a seat can accommodate at most one person.

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More explicitly, we assume that:

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Answer: $n^m = n \cdot (n - 1) \cdots (n - m + 1)$.

Note: this problem is equivalent to counting $[m] \rightarrow [n]$ injections.

Let's Express This Problem in Logic!

- Let Γ and Δ be sets (i.e., domains)
 - such that $|\Gamma| = m$, and $|\Delta| = n$.
- Let $P \subseteq \Gamma \times \Delta$ be a relation (i.e., predicate) over Γ and Δ .
- We can describe all of the constraints in first-order logic:

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$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \tag{1}$$

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$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \quad (1)$$

- one person cannot occupy multiple seats

$$\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z \quad (2)$$

Let's Express This Problem in Logic!

- Let Γ and Δ be sets (i.e., domains)
 - such that $|\Gamma| = m$, and $|\Delta| = n$.
- Let $P \subseteq \Gamma \times \Delta$ be a relation (i.e., predicate) over Γ and Δ .
- We can describe all of the constraints in first-order logic:
 - each attendee gets a seat (i.e., at least one seat)

$$\forall x \in \Gamma. \exists y \in \Delta. P(x, y) \quad (1)$$

- one person cannot occupy multiple seats

$$\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z \quad (2)$$

- one seat cannot accommodate multiple attendees

$$\forall w, x \in \Gamma. \forall y \in \Delta. P(w, y) \wedge P(x, y) \Rightarrow w = x \quad (3)$$

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(1) and (2) constrain P to be a function, and (3) makes it injective.

Recursion



Back to Our Example

The following function counts injections:

$$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n - 1) + mf(m - 1, n - 1) & \text{otherwise.} \end{cases}$$

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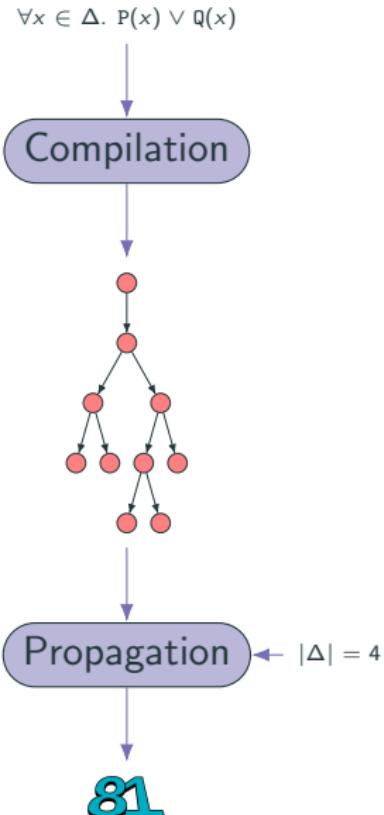
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- The rest of this talk is about how to find such functions automatically

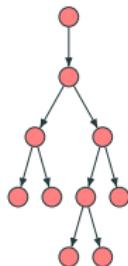
First-Order Knowledge Compilation: Before and After



First-Order Knowledge Compilation: Before and After

$$\forall x \in \Delta. P(x) \vee Q(x)$$

Compilation



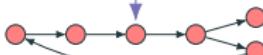
Propagation

81

$$|\Delta| = 4$$

$$\begin{aligned} & \forall x \in \Gamma. \exists y \in \Delta. P(x, y) \\ & \forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z \\ & \forall w, x \in \Gamma. \forall y \in \Delta. P(w, y) \wedge P(x, y) \Rightarrow w = x \end{aligned}$$

Compilation



Conversion

$$f(m, n) = \sum_{l=0}^m \binom{m}{l} [l < 2] \times f(m - l, n - 1)$$

Simplification

$$f(m, n) = f(m, n - 1) + mf(m - 1, n - 1)$$

Evaluation

$$\begin{aligned} & f(0, 0) = 1, f(m, 0) = 0 \\ & m = 2, n = 7 \end{aligned}$$

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Circuits vs Graphs

Circuits (Van den Broeck et al. 2011)...

- ... extend d-DNNF circuits (Darwiche 2001) for propositional knowledge compilation with **more node types**
- ... are **acyclic**.

Circuits vs Graphs

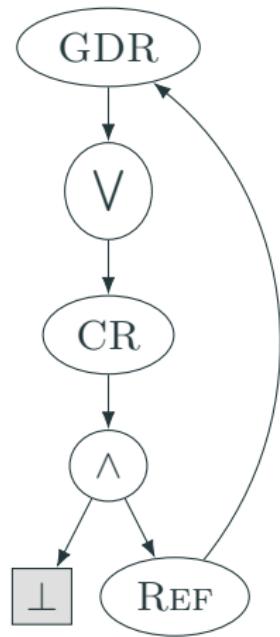
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- ... extend d-DNNF circuits (Darwiche 2001) for propositional knowledge compilation with **more node types**
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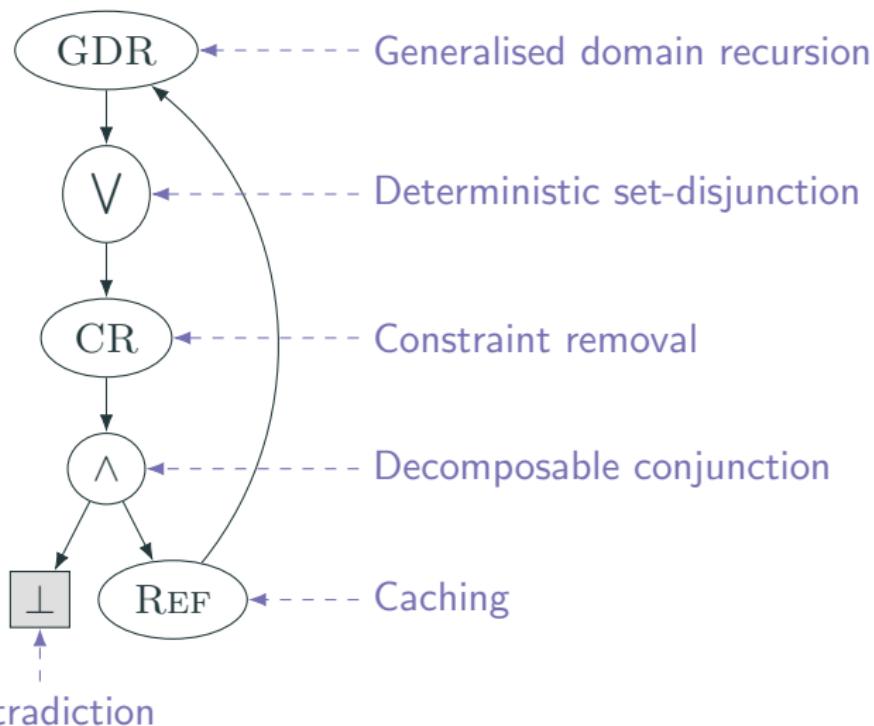
First-Order Computational Graphs (FCGs) are...
directed **acyclic** (weakly connected) graphs with:

- a single source,
- labelled nodes,
- and ordered outgoing edges.

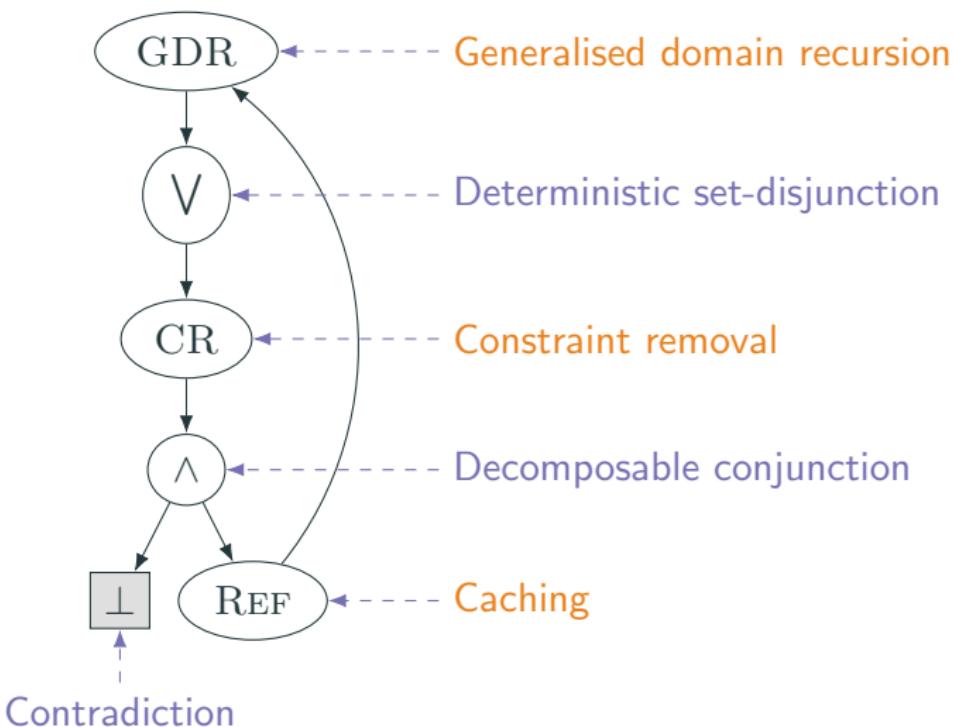
How to Interpret an FCG



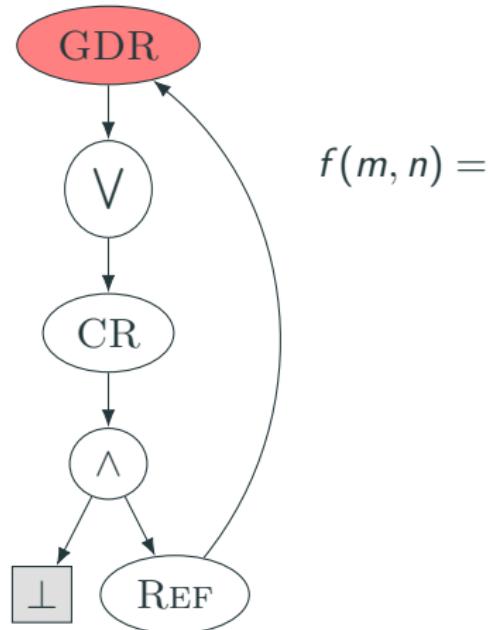
How to Interpret an FCG



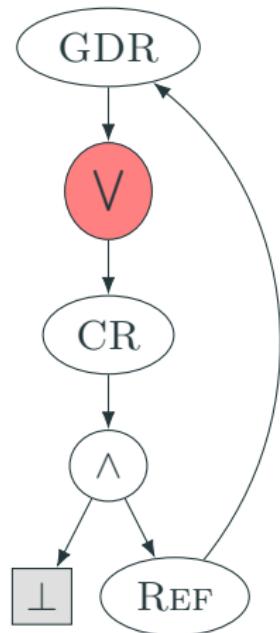
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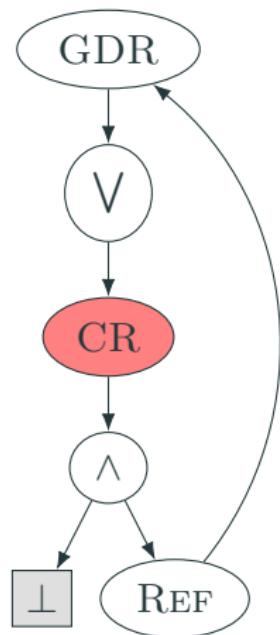


How to Interpret an FCG



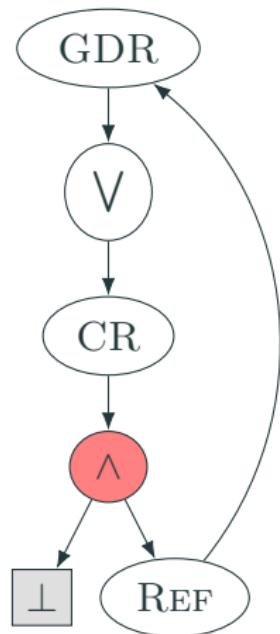
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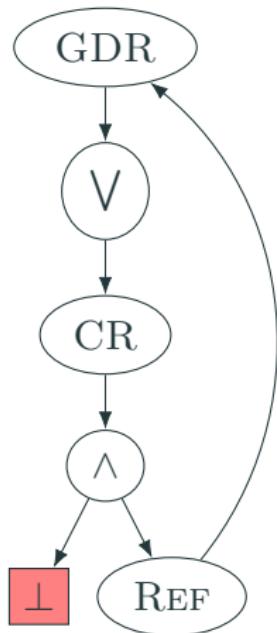
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$$f(m, n) = \sum_{l=0}^m \binom{m}{l} \quad \times$$

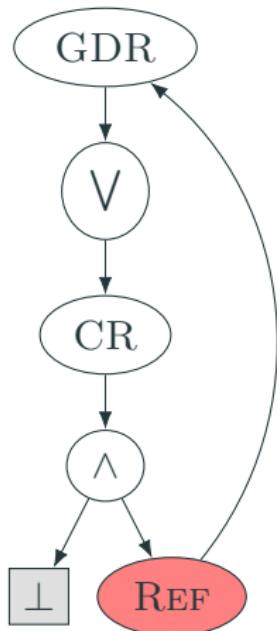
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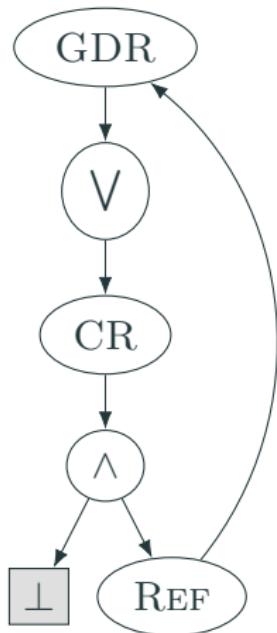
$$[\phi] = \begin{cases} 1 & \text{if } \phi \\ 0 & \text{if } \neg\phi \end{cases}$$

How to Interpret an FCG



$$f(m, n) = \sum_{l=0}^m \binom{m}{l} [l < 2] \times f(m - l, n - 1)$$

How to Interpret an FCG



$$\begin{aligned}f(m, n) &= \sum_{l=0}^m \binom{m}{l} [l < 2] \times f(m - l, n - 1) \\&= f(m, n - 1) + mf(m - 1, n - 1)\end{aligned}$$

Compilation: How FCGs Are Built

Definition

A **(compilation) rule** is a function that takes a **formula** and returns a set of **(G, L)** pairs, where

- G is an FCG,
- and L is a list of formulas.

The formulas in L are then **compiled**, and the resulting FCGs are **inserted** into G according to a **set order**.

Example Compilation Rule: Independence

Input formula:

$$(\forall x, y \in \Omega. x = y) \wedge \quad (1)$$

$$(\forall x \in \Gamma. \forall y, z \in \Delta. P(x, y) \wedge P(x, z) \Rightarrow y = z) \wedge \quad (2)$$

$$(\forall w, x \in \Gamma. \forall y \in \Delta. P(w, y) \wedge P(x, y) \Rightarrow w = x) \quad (3)$$

Example Compilation Rule: Independence

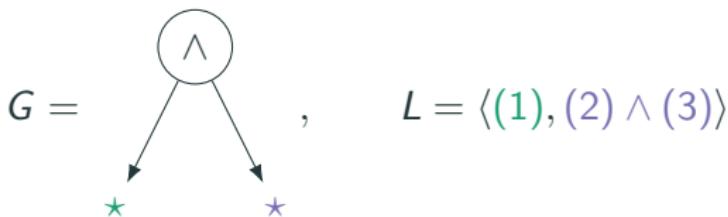
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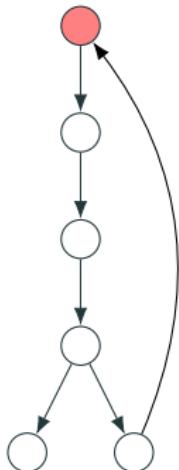
Only one (G, L) pair:



New Rule 1/3: Generalised Domain Recursion

Example

Input formula:



$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

Output formula (with a new constant $c \in \Gamma$):

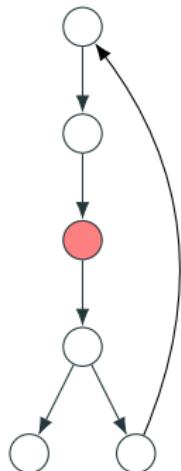
$$\forall y, z \in \Delta. y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$

$$\begin{aligned} \forall x \in \Gamma. \forall y, z \in \Delta. & x \neq c \wedge y \neq z \Rightarrow \\ & \neg P(x, y) \vee \neg P(x, z) \end{aligned}$$

New Rule 2/3: Constraint Removal

Example

Input formula (with a constant $c \in \Gamma$):



$$\forall x \in \Gamma. \forall y, z \in \Delta. x \neq c \wedge y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

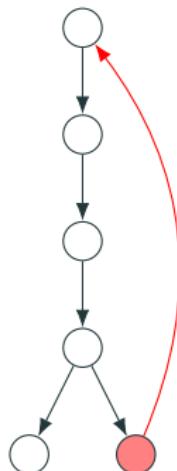
$$\forall w, x \in \Gamma. \forall y \in \Delta. w \neq c \wedge x \neq c \wedge w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

Output formula (with a new domain $\Gamma' := \Gamma \setminus \{c\}$):

$$\forall x \in \Gamma'. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

New Rule 3/3: Identifying Possibilities for Recursion



Goal

Check if the input formula is equivalent (up to domains) to a previously encountered formula.

Outline

1. Consider pairs of 'similar' clauses.
2. Consider bijections between their sets of variables.
3. Extend each such bijection to a map between sets of domains.
4. If the bijection makes the clauses equal, and the domain map is compatible with previous domain maps, move on to another pair of clauses.

How These Rules Fit Together (1/5)

$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma. \forall y \in \Delta. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Generalised domain recursion



$$\forall y, z \in \Delta. y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$

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How These Rules Fit Together (2/5)

$$\forall y, z \in \Delta. \ y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$

$$\forall x \in \Gamma. \ \forall y, z \in \Delta. \ y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall x \in \Gamma. \ \forall y \in \Delta. \ x \neq c \Rightarrow \neg P(c, y) \vee \neg P(x, y)$$

$$\forall w \in \Gamma. \ \forall y \in \Delta. \ w \neq c \Rightarrow \neg P(w, y) \vee \neg P(c, y)$$

$$\forall w, x \in \Gamma. \ \forall y \in \Delta. \ w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Atom counting and unit propagation



$$\forall y, z \in \Delta^{\top}. \ y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma. \ \forall y, z \in \Delta^{\perp}. \ y \neq z \wedge x \neq c \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma. \ \forall y \in \Delta^{\perp}. \ w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

How These Rules Fit Together (3/5)

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$$\forall w, x \in \Gamma. \forall y \in \Delta^\perp. w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$



Constraint removal



$$\forall y, z \in \Delta^\top. y \neq z \Rightarrow \perp$$

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

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How These Rules Fit Together (4/5)

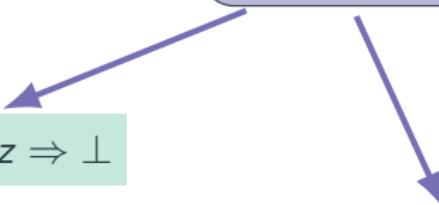
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Independence



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How These Rules Fit Together (5/5): Recursion

$$\forall x \in \Gamma'. \forall y, z \in \Delta^\perp. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

$$\forall w, x \in \Gamma'. \forall y \in \Delta^\perp. w \neq x \Rightarrow \neg P(w, y) \vee \neg P(x, y)$$

||

$$\forall x \in \Gamma. \forall y, z \in \Delta. y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

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+

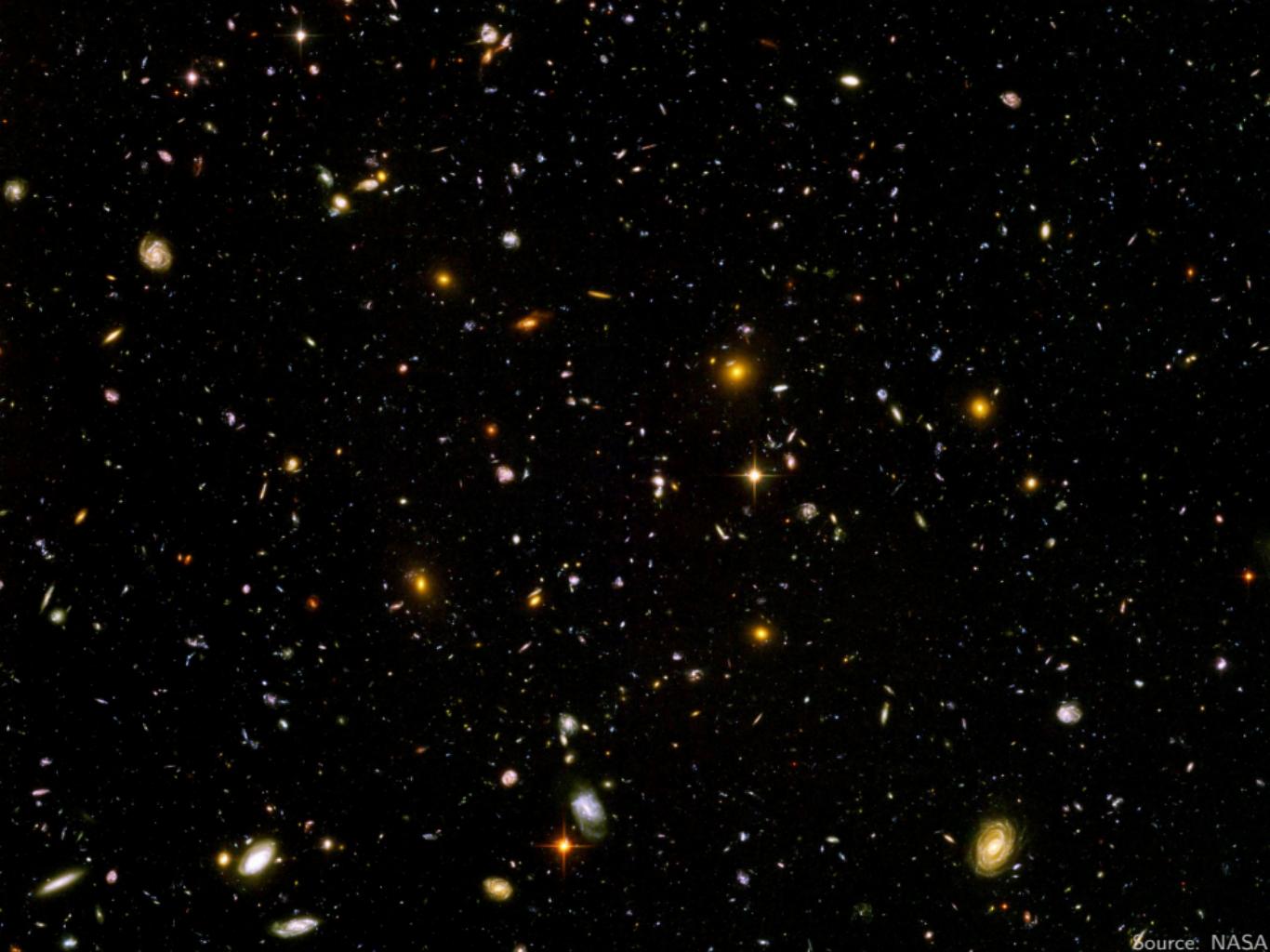
$$\{ \Gamma \mapsto \Gamma', \Delta^\perp \mapsto \Delta \}$$

Resulting Improvements to Counting Functions

Let Γ and Δ be two sets with cardinalities $|\Gamma| = m$ and $|\Delta| = n$.

Our new rules enable Crane to efficiently count $\Gamma \rightarrow \Delta$ functions such as:

- injections in $\Theta(mn)$ time
 - by hand: $\Theta(m)$
- partial injections in $\Theta(mn)$ time
 - by hand: $\Theta(\min\{m, n\}^2)$
- bijections in $\Theta(m)$ time
 - optimal!
 - FastWFOMC seems to be $\Omega(m^4)$



Summary & Future Work

Summary

The circuits hitherto used for FOMC become more powerful with:

- cycles,
- generalised domain recursion,
- and some more new compilation rules that support domain recursion.

Future Work

- Automate:
 - extracting and simplifying the definitions of functions,
 - finding all base cases.
- Open questions:
 - What kind of **sequences** are computable in this way?
 - Would using a **different logic** extend the capabilities of FOMC further?

Major Directions for Future Work

- An algebraic description of what kind of sequences and functions with domain \mathbb{N}_0^k are computable in this way
 - monotonicity, maximal growth rate, etc.
- A different input format or logic that allows the same approach to capture more computations
 - fixed-point logic with counting
 - ‘let domain $\Delta := \{1, 2, \dots, MC(\phi)\}$ ’
- parameterised Markov logic networks
 - ‘What equations do the domain sizes have to satisfy for the probability of event E to be at least 95%?’