Recursive Solutions to First-Order Model Counting

Paulius Dilkas and Vaishak Belle

University of Edinburgh, Edinburgh, UK
p.dilkas@sms.ed.ac.uk, vbelle@ed.ac.uk

Abstract. First-order model counting (FOMC) is a computational problem that asks to count the models of a sentence in first-order logic. Despite being around for more than a decade, practical FOMC algorithms are still unable to compute functions as simple as a factorial. We argue that the capabilities of FOMC algorithms are severely limited by their inability to express arbitrary recursive computations. To enable arbitrary recursion, we relax the restrictions that typically accompany domain recursion and generalise circuits used to express a solution to an FOMC problem to graphs that may contain cycles. To this end, we enhance the most well-established (weighted) FOMC algorithm ForcLift with new compilation rules and an algorithm to check whether a recursive call is feasible. These improvements allow us to find efficient solutions to counting fundamental structures such as injections and bijections.

First-order model counting (FOMC) is the problem of computing the number of models of a sentence in first-order logic given the size(s) of its domain(s) [11]. The main application of FOMC (or, rather, its *weighted* variant WFOMC) is in probabilistic inference, particularly for statistical relational models such as Markov logic networks that define probabilities over sets of objects and relations thereof [2, 11].

Over slightly more than a decade, research on (W)FOMC advanced on both theoretical and empirical fronts. Several WFOMC algorithms such as ForcLift [11], probabilistic theorem proving [2], and L2C [4] were developed. More and more classes of formulas were shown to be *liftable*, i.e., solvable in polynomial time with respect to the size(s) of the domain(s) [3, 5, 6, 9]. Much of the researchers' attention was devoted to developing efficient solutions to formulas with up to two variables [7, 8].

However, none of the publicly available (W)FOMC algorithms can efficiently compute functions as simple as a factorial.[1] We claim that this shortcoming is due to the inability of these algorithms to construct recursive solutions.

The topic of recursion in the context of WFOMC has been studied before but in very limited ways. Barvínek et al. [1] use WFOMC to generate numerical

[1] The problem of computing the factorial can be described using two variables and counting quantifiers, so it belongs to a fragment of first-order logic which is known to be liftable [6].

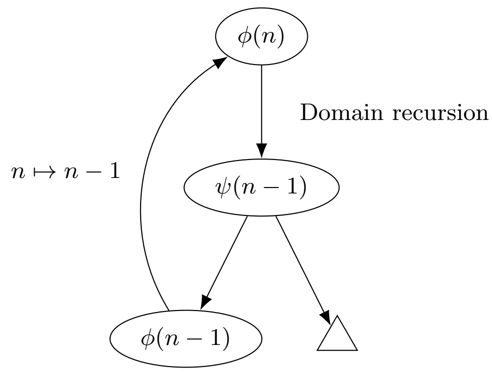

Fig. 1. A conceptual illustration of the main idea

data which is then used to conjecture recurrence relations that explain that data. Van den Broeck [10] introduced the idea of *domain recursion*. Intuitively, domain recursion partitions a domain of size n into a single explicitly named constant and the remaining domain of size $n-1$. However, many stringent conditions are enforced to ensure that the search for a tractable solution always terminates.

In this work, we show how new tractable solutions can be found by dispensing with these restrictions. With additional compilation rules and an algorithm for checking whether a 'recursive call' is possible, ForcLift [11] can construct recursive functions that efficiently solve counting problems that used to be beyond its reach. The main conceptual difference from the original algorithm is that the input formula is now compiled to a labelled directed graph rather than a circuit (i.e., cycles are allowed). This idea is illustrated in Fig. 1. Suppose the original formula ϕ depended on a domain of size $n \in \mathbb{N}$. Domain recursion transforms ϕ into a different formula ψ that depends on a domain of size $n-1$. After some number of subsequent transformations, the algorithm identifies that a solution to ψ can be constructed in part by finding a solution to a version of ϕ where the domain of size n is replaced by a domain of size $n-1$. Recognising ϕ from before, we can add a cycle-forming edge to the graph, which can be interpreted as function f relying on $f(n-1)$ to compute $f(n)$.

In our experiments, we consider variations of the function-counting problem. These functions vary in:

- whether they are full or partial,
- whether they are injective/surjective/bijective or not,
- and whether the domain and the codomain are the same.

Many versions of this problem were previously unsolvable by any available (W)FOMC algorithm, whereas we can find recursive solutions (that can be evaluated in polynomial time) to all except one of these problems.

Further work is necessary to fully automate this new way of computing the (W)FOMC of a formula. The new version of ForcLift produces a graph which then needs to be transformed to definitions of (potentially recursive) functions. In some cases, it is necessary to simplify the algebraic expressions in these definitions (e.g., reducing $x - x$ to zero for some expression x). Most importantly, the algorithm only gives us the recursive calls but not the base cases. What makes the problem of finding these base cases non-trivial is that the number of base cases is not constant for functions of arity greater than one (i.e., formulas that mention more than one domain). Nonetheless, these remaining challenges are minuscule compared to the broader goal of expanding the capabilities of (W)FOMC to new classes of instances.

References

1. Barvínek, J., van Bremen, T., Wang, Y., Zelezný, F., Kuzelka, O.: Automatic conjecturing of P-recursions using lifted inference. In: Katzouris, N., Artikis, A. (eds.) Inductive Logic Programming - 30th International Conference, ILP 2021, Virtual Event, October 25-27, 2021, Proceedings. Lecture Notes in Computer Science, vol. 13191, pp. 17–25. Springer (2021). https://doi.org/10.1007/978-3-030-97454-1_2
2. Gogate, V., Domingos, P.M.: Probabilistic theorem proving. Commun. ACM **59**(7), 107–115 (2016). https://doi.org/10.1145/2936726
3. Kazemi, S.M., Kimmig, A., Van den Broeck, G., Poole, D.: New liftable classes for first-order probabilistic inference. In: Lee, D.D., Sugiyama, M., von Luxburg, U., Guyon, I., Garnett, R. (eds.) Advances in Neural Information Processing Systems 29: Annual Conference on Neural Information Processing Systems 2016, December 5-10, 2016, Barcelona, Spain. pp. 3117–3125 (2016)
4. Kazemi, S.M., Poole, D.: Knowledge compilation for lifted probabilistic inference: Compiling to a low-level language. In: Baral, C., Delgrande, J.P., Wolter, F. (eds.) Principles of Knowledge Representation and Reasoning: Proceedings of the Fifteenth International Conference, KR 2016, Cape Town, South Africa, April 25-29, 2016. pp. 561–564. AAAI Press (2016), http://www.aaai.org/ocs/index.php/KR/KR16/paper/view/12861
5. Kuusisto, A., Lutz, C.: Weighted model counting beyond two-variable logic. In: Dawar, A., Grädel, E. (eds.) Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018. pp. 619–628. ACM (2018). https://doi.org/10.1145/3209108.3209168
6. Kuzelka, O.: Weighted first-order model counting in the two-variable fragment with counting quantifiers. J. Artif. Intell. Res. **70**, 1281–1307 (2021). https://doi.org/10.1613/jair.1.12320
7. Malhotra, S., Serafini, L.: Weighted model counting in FO2 with cardinality constraints and counting quantifiers: A closed form formula. CoRR **abs/2110.05992** (2021), https://arxiv.org/abs/2110.05992
8. van Bremen, T., Kuzelka, O.: Faster lifting for two-variable logic using cell graphs. In: de Campos, C.P., Maathuis, M.H., Quaeghebeur, E. (eds.) Proceedings of the Thirty-Seventh Conference on Uncertainty in Artificial Intelligence, UAI 2021, Virtual Event, 27-30 July 2021. Proceedings of Machine Learning Research, vol. 161, pp. 1393–1402. AUAI Press (2021), https://proceedings.mlr.press/v161/bremen21a.html

9. van Bremen, T., Kuzelka, O.: Lifted inference with tree axioms. In: Bienvenu, M., Lakemeyer, G., Erdem, E. (eds.) Proceedings of the 18th International Conference on Principles of Knowledge Representation and Reasoning, KR 2021, Online event, November 3-12, 2021. pp. 599–608 (2021). https://doi.org/10.24963/kr.2021/57
10. Van den Broeck, G.: On the completeness of first-order knowledge compilation for lifted probabilistic inference. In: Shawe-Taylor, J., Zemel, R.S., Bartlett, P.L., Pereira, F.C.N., Weinberger, K.Q. (eds.) Advances in Neural Information Processing Systems 24: 25th Annual Conference on Neural Information Processing Systems 2011. Proceedings of a meeting held 12-14 December 2011, Granada, Spain. pp. 1386–1394 (2011)
11. Van den Broeck, G., Taghipour, N., Meert, W., Davis, J., De Raedt, L.: Lifted probabilistic inference by first-order knowledge compilation. In: Walsh, T. (ed.) IJCAI 2011, Proceedings of the 22nd International Joint Conference on Artificial Intelligence, Barcelona, Catalonia, Spain, July 16-22, 2011. pp. 2178–2185. IJCAI/AAAI (2011). https://doi.org/10.5591/978-1-57735-516-8/IJCAI11-363