Recursive Solutions to First-Order Model Counting

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Workshop on Counting and Sampling 2022

Some Elementary Counting

A Counting Problem

Suppose this room has n seats, and there are $m \leq n$ people in the audience. How many ways are there to seat everyone?

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Answer: $n^{\underline{m}} = n \cdot (n-1) \cdots (n-m+1)$.
Note: this problem is equivalent to counting $[m] \to [n]$ injections.

Let's Express This Problem in Logic!

- Let M and N be sets (i.e., domains) such that $|M| = m$, and $|N| = n$
- Let $P \subseteq M \times N$ be a relation (i.e., predicate) over sets M and N
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The first two sentences constrain P to be a function, and the last one makes it injective.

Overview of the Problem

- First-order model counting (FOMC) is the problem of counting the models of a sentence in first-order logic.
- The (symmetric) weighted variation of the problem adds weights (e.g., probabilities) to predicates.
  - It is used for efficient probabilistic inference in relational models such as Markov logic networks.
- None of the (implemented) (W)FOMC algorithms are able to count, e.g., injective and bijective functions.

Claim

This shortcoming can be addressed via support for (almost arbitrary) recursive functions.

Back to Our Example

For instance, the following function counts injections

$$f(m,n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n-1) + mf(m-1, n-1) & \text{otherwise.} \end{cases}$$

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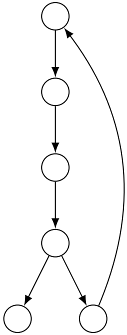
- f can be computed in $\Theta(mn)$ time (via dynamic programming).
- Optimal time complexity to compute $n^{\underline{m}}$ is $\Theta(\log m)$.
- But $\Theta(mn)$ is still much better than solving an equivalent #P-complete problem in propositional logic.
- The rest of this talk is about how such functions can be found automatically.

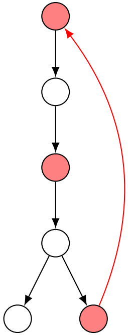
First-Order Knowledge Compilation with ForcLift



Workflow Before

1. Compile the formula to a circuit
2. Evaluate to get the answer

First-Order Knowledge Compilation with FORCLIFT



Workflow Before

1. Compile the formula to a circuit
2. Evaluate to get the answer

Workflow After

1. Compile the formula to a graph
2. Extract the definitions of functions
3. Simplify
4. Supplement with base cases
5. Evaluate to get the answer

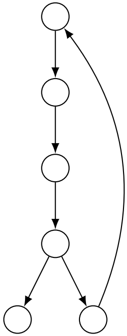
More Formally. . .

Definition

A first-order deterministic decomposable negation normal form computational graph (FCG) is a

- directed graph
- (which is weakly connected)
- with a single source,
- labelled vertices,
- and ordered outgoing edges.

How to Interpret an FCG



$$f(m, n) =$$

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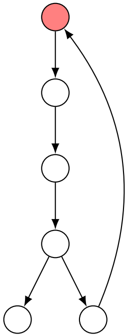

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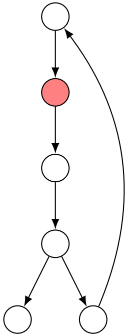

$$f(m,n) = \sum_{l=0}^{m} \binom{m}{l}$$

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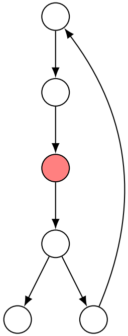

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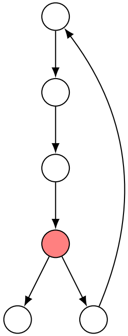
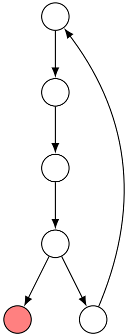


$$f(m,n) = \sum_{l=0}^{m} \binom{m}{l} \qquad \times$$

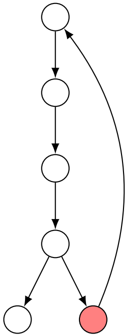
How to Interpret an FCG



$$f(m, n) = \sum_{l=0}^{m} \binom{m}{l} [l < 2] \times$$

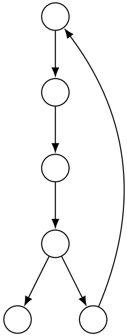
$$[\phi] = \begin{cases} 1 & \text{if } \phi \\ 0 & \text{if } \neg\phi \end{cases}$$

How to Interpret an FCG



$$f(m,n) = \sum_{l=0}^{m} \binom{m}{l} [l < 2] \times f(m-l, n-1)$$

How to Interpret an FCG



$$
\begin{aligned}
f(m, n) &= \sum_{l=0}^{m} \binom{m}{l} [l < 2] \times f(m - l, n - 1) \\
&= f(m, n - 1) + mf(m - 1, n - 1)
\end{aligned}
$$

Compilation Rules

Definition

A (compilation) rule is a function that takes a formula and returns a set of (G, L) pairs, where

- G is an FCG,
- and L is a list of formulas.

Example Rule: Independence

Input formula:

$$
\begin{aligned}
(\forall x, y \in L.\ x = y) \wedge \qquad (1) \\
(\forall x \in M.\ \forall y, z \in N.\ P(x, y) \wedge P(x, z) \Rightarrow y = z) \wedge \qquad (2) \\
(\forall w, x \in M.\ \forall y \in N.\ P(w, y) \wedge P(x, y) \Rightarrow w = x) \qquad (3)
\end{aligned}
$$

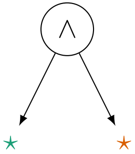
Example Rule: Independence

Input formula:

$$(\forall x, y \in L.\ x = y) \wedge \tag{1}$$
$$(\forall x \in M.\ \forall y, z \in N.\ P(x, y) \wedge P(x, z) \Rightarrow y = z) \wedge \tag{2}$$
$$(\forall w, x \in M.\ \forall y \in N.\ P(w, y) \wedge P(x, y) \Rightarrow w = x) \tag{3}$$

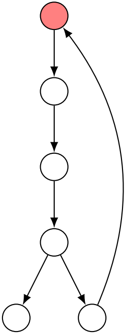
Only one (G, L) pair:

$G =$



, $\quad L = \langle (1), (2) \wedge (3) \rangle$

New Rule 1: Generalised Domain Recursion



Example

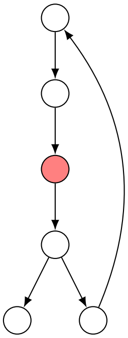
Input formula:

$$\forall x \in M.\ \forall y, z \in N.\ y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

Output formula (with a new constant $c \in M$):

$$\forall y, z \in N.\ y \neq z \Rightarrow \neg P(c, y) \vee \neg P(c, z)$$
$$\forall x \in M.\ \forall y, z \in N.\ x \neq c \wedge y \neq z \Rightarrow \neg P(x, y) \vee \neg P(x, z)$$

New Rule 2: Constraint Removal



Example

Input formula (with a constant $c \in M$):

$$\forall x \in M.\ \forall y, z \in N.\ x \neq c \land y \neq z \Rightarrow \neg P(x, y) \lor \neg P(x, z)$$

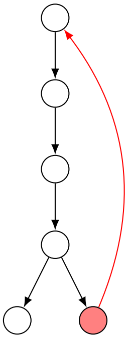
$$\forall w, x \in M.\ \forall y \in N.\ w \neq c \land x \neq c \land w \neq x \Rightarrow \neg P(w, y) \lor \neg P(x, y)$$

Output formula (with a new domain $M' := M \setminus \{ c \}$):

$$\forall x \in M'.\ \forall y, z \in N.\ y \neq z \Rightarrow \neg P(x, y) \lor \neg P(x, z)$$

$$\forall w, x \in M'.\ \forall y \in N.\ w \neq x \Rightarrow \neg P(w, y) \lor \neg P(x, y)$$

New Rule 3: Identifying Possibilities for Recursion



Goal

Check if the input formula is isomorphic (up to domains) to a previously encountered formula.

Rough Outline

1. Consider pairs of 'similar' clauses.
2. Consider bijections between their sets of variables.
3. Extend each such bijection to a map between sets of domains.
4. If the bijection makes the clauses equal, and the domain map is compatible with previous domain maps, move on to another pair of clauses.

Resulting Improvements to Counting Functions

Let M and N be two sets with cardinalities $|M| = m$ and $|N| = n$. Our new rules enable ForcLift to efficiently count $M \to N$ functions such as:

- injections in $\Theta(mn)$ time
  - best: $\Theta(m)$
- partial injections in $\Theta(mn)$ time
  - best: $\Theta(\min\{m, n\}^2)$
- bijections in $\Theta(m)$ time
  - optimal!

Summary & Future Work

Summary

The circuits hitherto used for FOMC become more powerful with:

- cycles,
- generalised domain recursion,
- and some more new compilation rules that support domain recursion.

Future Work

- Automate:
  - extracting and simplifying the definitions of functions,
  - finding all base cases.
- Open questions:
  - What kind of sequences are computable in this way?
  - Would using a different logic extend the capabilities of FOMC further?