Synthesising Recursive Functions for First-Order Model Counting: Challenges, Progress, and Conjectures

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Abstract

First-order model counting (FOMC) is a computational problem that asks to count the models of a sentence in finite-domain first-order logic. Despite being around for more than a decade, practical FOMC algorithms are still unable to compute functions as simple as a factorial. We argue that the capabilities of FOMC algorithms to date are limited by their inability to express arbitrary recursive computations. To enable arbitrary recursion, we relax the restrictions that typically accompany domain recursion and generalise circuits used to express a solution to an FOMC problem to graphs that may contain cycles. To this end, we enhance the most well-established (weighted) FOMC algorithm FORCLIFT with new compilation rules and an algorithm to check whether a recursive call is feasible. These improvements allow the algorithm to construct efficient solutions to counting fundamental structures such as injections and bijections. We end with a few conjectures on what classes of instances could be liftable as a result.

1 Introduction

TODO: also cite 'WFOMC with DAG axioms' (Malhotra and Serafini 2023)

First-order model counting (FOMC) is the problem of computing the number of models of a sentence in first-order logic (FOL) given the size(s) of its domain(s) (Beame et al. 2015). *Symmetric weighted FOMC* (WFOMC) extends FOMC with (pairs of) weights on predicates and asks for a weighted sum across all models instead. By fixing the sizes of the domains, a WFOMC instance can be rewritten as an instance of (propositional) weighted model counting (Chavira and Darwiche 2008). WFOMC emerged as the dominant approach to *lifted (probabilistic) inference*. Lifted inference techniques exploit symmetries in probabilistic models by reasoning about sets rather than individuals (Kersting 2012). By doing so, many instances become solvable in polynomial time (Van den Broeck 2011). Lifted inference algorithms are typically used on probabilistic models such as probabilistic programming languages (De Raedt and Kimmig 2015; Riguzzi et al. 2017), Markov logic networks (Van den Broeck et al. 2011; Gogate and Domingos 2016; Richardson and Domingos 2006), and other lifted graphical (Kimmig, Mihalkova, and Getoor 2015) and statistical relational (De Raedt et al. 2016) models. Lifted inference techniques for probabilistic databases, while developed somewhat independently, have also been inspired by WFOMC (Gatterbauer and Suciu 2015; Gribkoff, Suciu, and Van den Broeck 2014). While WFOMC has received more attention in the literature, FOMC is an interesting problem in an of itself because of its connections to finite model theory (van Bremen and Kuželka 2021b) and applications in enumerative combinatorics (Barvínek et al. 2021). There is also a recent line of work that uses FOMC-based methods to count solutions to constraint satisfaction problems (Totis et al. 2023).

Traditionally in computational complexity theory, a problem is *tractable* if it can be solved in time polynomial in the instance size. The equivalent notion in (W)FOMC is liftability. A (W)FOMC instance is *(domain-)liftable* if it can be solved in time polynomial in the size(s) of the domain(s) (Jaeger and Van den Broeck 2012). Over more than a decade, many classes of instances were shown to be liftable (van Bremen and Kuželka 2021b; Kazemi et al. 2016; Kuusisto and Lutz 2018; Kuželka 2021). First, Van den Broeck (2011) showed that the class of all sentences of FOL with up to two variables (denoted FO^2) is liftable. Then Beame et al. (2015) proved that there exists a sentence with three variables for which FOMC is $\#\mathsf{P}_1$-complete (i.e., FO^3 is not liftable). Since these two seminal results, most of the research on (W)FOMC focused on developing faster solutions for the FO^2 fragment (van Bremen and Kuželka 2021a; Malhotra and Serafini 2022) and defining new liftable fragments. These fragments include $\mathsf{S}^2\mathsf{FO}^2$ and $\mathsf{S}^2\mathsf{RU}$ (Kazemi et al. 2016), U_1 (Kuusisto and Lutz 2018), C^2 (i.e., the two-variable fragment with counting quantifiers) (Kuželka 2021; Malhotra and Serafini 2022), and C^2 extended with tree axioms (van Bremen and Kuželka 2021b) and linear order axiom (Tóth and Kuželka 2022). On the empirical front, there are several implementations of exact WFOMC algorithms: ALCHEMY (Gogate and Domingos 2016), FASTWFOMC (van Bremen and Kuželka 2021a), FORCLIFT (Van den Broeck et al. 2011), and L2C (Kazemi and Poole 2016). Approximate counting is supported by ALCHEMY, APPROXWFOMC (van Bremen and Kuželka 2020), FORCLIFT (Van den Broeck, Choi, and Darwiche 2012), MAGICIAN (Venugopal, Sarkhel, and Gogate 2015), and TUFFY (Niu et al. 2011).

However, none of the publicly available exact (W)FOMC algorithms can efficiently compute functions as simple as

a factorial.[1] We claim that the capabilities of (W)FOMC algorithms can be significantly extended by empowering them with the ability to construct recursive solutions. The topic of recursion in the context of WFOMC has been studied before but in limited ways. Barvínek et al. (2021) use WFOMC to generate numerical data that is then used to conjecture recurrence relations that explain that data. Van den Broeck (2011) introduced the idea of *domain recursion*. Intuitively, domain recursion partitions a domain of size n into a single explicitly named constant and the remaining domain of size $n-1$. However, many stringent conditions are enforced to ensure that the search for a tractable solution always terminates.

In this work, we show how to relax these restrictions in a way that results in a stronger (W)FOMC algorithm, capable of handling more instances (e.g., counting various injective mappings) in a lifted manner. The ideas presented in this paper are implemented in CRANE—an extension of the arguably most well-known WFOMC algorithm FORCLIFT. FORCLIFT works in two stages: compilation and evaluation/propagation. In the first stage, various *(compilation) rules* are applied to the input (or some derivative) formula, gradually constructing a circuit. In the second stage, the weights of the instance are propagated through the circuit, computing the weighted model count. Along with new compilation rules, CRANE introduces changes to both stages of the process. First, while FORCLIFT applies compilation rules via greedy search, CRANE also supports a hybrid search algorithm that applies some rules greedily and some using breadth-first search.[2] This alternative was introduced because there is no reason to expect greedy search to be optimal. Second, the product of compilation is not directly evaluated but rather interpreted as a collection of functions on domain sizes. Hence, our approach is reminiscent of previous work on lifted inference via compilation to C++ programs (Kazemi and Poole 2016) and the broader area of functional synthesis (Golia, Roy, and Meel 2020; Kuncak et al. 2010; Sanathanan and Koerner 1963).

The main conceptual difference between CRANE and FORCLIFT is that we utilise labelled directed graphs instead of circuits, where cycles represent recursive calls. Suppose the input formula ϕ depends on a domain of size $n \in \mathbb{N}_0$. *Generalised domain recursion* (GDR)—one of the new compilation rules—transforms ϕ into a different formula ψ that has an additional constant and some new *constraints*. After some additional transformations, the constraints in ψ become 'uniform' and can be removed, replacing the domain of size n with a new domain of size $n-1$—this is the responsibility of the *constraint removal* (CR) compilation rule. Afterwards, another compilation rule recognizes that the resulting formula is just like the input formula ϕ but with a different domain. This observation allows us to add a cycle-forming edge to the graph, which can be interpreted as function f relying on $f(n-1)$ to compute $f(n)$.

We begin by defining the representation used for sentences in FOL in Section 2. Then, in Section 3, we define the graphs that replace circuits in representing a solution to a (W)FOMC problem. Section 4 introduces the new compilation rules. In Section 5, we discuss how a graph can be interpreted as a collection of (potentially recursive) functions. In Section 6, we compare FORCLIFT and CRANE on a range of function-counting problems. We show that CRANE performs as well as FORCLIFT on the instances that were already solvable by FORCLIFT but is also able to handle most of the instances that FORCLIFT fails on. Finally, in Section 7, we conclude by outlining some conjectures and directions for future work.

2 Preliminaries

Our representation of FOMC instances is largely based on the format used internally by FORCLIFT, some aspects of which are described by Van den Broeck et al. (2011). FORCLIFT can translate sentences in a variant of function-free many-sorted FOL with equality to this internal format. We use lowercase Latin letters for predicates (e.g., $\mathtt{p}$) and constants (e.g., x), uppercase Latin letters for variables (e.g., X), and uppercase Greek letters for domains (e.g., Δ). Sometimes we write predicate $\mathtt{p}$ as $\mathtt{p}/n$, where $n \in \mathbb{N}^+$ is the *arity* of $\mathtt{p}$. An *atom* is $\mathtt{p}(t_1, \ldots, t_n)$ for some predicate $\mathtt{p}/n$ and terms $t_1, \ldots, t_n$. A *term* is either a constant or a variable. A *literal* is either an atom or the negation of an atom (denoted by $\neg\mathtt{p}(t_1, \ldots, t_n)$). Let $\mathcal{D}$ be the set of all relevant (finite) domains. Initially, $\mathcal{D}$ contains all domains mentioned by the input formula. During compilation, new domains are added to $\mathcal{D}$ by some of the compilation rules. Each such new domain is interpreted as a subset of some other domain in $\mathcal{D}$.

Definition 1 (Constraint)**.** An *(inequality) constraint* is a pair (a, b), where a is a variable, and b is either a variable or a constant.

Definition 2 (Clause)**.** A *clause*[3] is a triple $c = (L, C, \delta_c)$, where L is a set of literals, C is a set of constraints, and δ_c is the domain map of c. Let Vars be the function that maps clauses and sets of either literals or constraints to the set of variables in them. In particular, $\mathrm{Vars}(c) \coloneqq \mathrm{Vars}(L) \cup \mathrm{Vars}(C)$. *Domain map* $\delta_c \colon \mathrm{Vars}(c) \to \mathcal{D}$ is a function that maps all variables in c to their domains such that (s.t.) if $(X, Y) \in C$ for some variables X and Y, then $\delta_c(X) = \delta_c(Y)$. For convenience, we sometimes write δ_c for the domain map of c without unpacking c into its three constituents.

[1] There is some previous work discussing FOMC for factorials (Kuželka 2021; van Bremen and Kuželka 2021b; Van den Broeck 2016). One of them (Van den Broeck 2016) is an example 'by hand' that does not directly correspond to any algorithm. Another (Kuželka 2021) proves that the problem is liftable but contains no experimental results. The third (van Bremen and Kuželka 2021b) contains experimental results performed with an algorithm that is not publicly available.

[2] In the current implementation, the rules that are applied in a non-greedy fashion are: atom counting, inclusion-exclusion, independent partial groundings, Shannon decomposition, shattering, and two of our new rules introduced in Sections 4.1 and 4.3—see previous work (Van den Broeck et al. 2011) for more information about the rules.

[3] Van den Broeck et al. (2011) refer to clauses as c-clauses.

Notation. For any set S, let 2^S denote its power set and $S^* \coloneqq \bigcup_{i=0}^{\infty} S^i$ the set of tuples of any finite length of elements of S. For any $n \in \mathbb{N}_0$, let $[n] \coloneqq \{1, \ldots, n\}$, e.g., $[0] = \emptyset$, and $[2] = \{1, 2\}$. We write $\langle\rangle$ for an empty list, $\langle x \rangle$ for a list with one element x, $|l|$ for the length of list l, $+\!\!+$ for list concatenation, and $\nrightarrow$ to denote partial functions. Let S be a set of constraints or literals, V a set of variables, and x a variable or a constant. We write $S[x/V]$ to denote S with all occurrences of all variables in V replaced with x. Let Doms (respectively, Preds) be the function that maps any clause or formula to the set of domains (respectively, predicates) used within, and let $\mathrm{dom}(f)$ denote the domain of function f.

Definition 3 (Formula)**.** A *formula* (called a c-theory by Van den Broeck et al. (2011)) is a set of clauses s.t. all constraints and atoms 'type check' with respect to domains. Each formula ϕ has a map $\pi_\phi \colon \mathrm{Preds}(\phi) \to \mathrm{Doms}(\phi)^*$ such that, for each predicate $\mathtt{p}/n \in \mathrm{Preds}(\phi)$, we have that $\pi_\phi(\mathtt{p}) \in \mathrm{Doms}(\phi)^n$.[4]

Example 1. Let $\phi \coloneqq \{c_1, c_2\}$ be a formula with clauses

$$c_1 \coloneqq (\{\neg \mathtt{p}(X, Y), \neg \mathtt{p}(X, Z)\}, \{(Y, Z)\}, \{X \mapsto \Gamma, Y \mapsto \Delta, Z \mapsto \Delta\}),$$
$$c_2 \coloneqq (\{\neg \mathtt{p}(X, Y), \neg \mathtt{p}(Z, Y)\}, \{(X, Z)\}, \{X \mapsto \Gamma, Y \mapsto \Delta, Z \mapsto \Gamma\})$$

for some predicate $\mathtt{p}/2$, variables X, Y, Z, and domains Γ and Δ. Based on δ_{c_1} and δ_{c_2}, we can infer that $\pi_\phi = \{\mathtt{p} \mapsto (\Gamma, \Delta)\}$. All variables that occur as the first argument to $\mathtt{p}$ are in Γ, and, likewise, all variables that occur as the second argument to $\mathtt{p}$ are in Δ. Therefore, ϕ 'type checks' as a valid formula.

There are two major differences between Definitions 1–3 and the corresponding concepts introduced by Van den Broeck et al. (2011). First, we decouple variable-to-domain assignments from constraints and move them to a separate function δ_c in Definition 2. Second, while Van den Broeck et al. (2011) allow for equality constraints and constraints of the form $X \notin \Delta$ for some variable X and domain Δ, we exclude such constraints simply because they are not needed.

One can read a formula in our format as a sentence in FOL. All variables in a clause are implicitly universally quantified (but note that variables are never shared among clauses), and all clauses in a formula are implicitly linked by a conjunction. Thus, formula ϕ from Example 1 reads as

$$(\forall X \in \Gamma.\ \forall Y, Z \in \Delta.\ Y \neq Z \Rightarrow \neg \mathtt{p}(X, Y) \vee \neg \mathtt{p}(X, Z)) \wedge$$
$$(\forall X, Z \in \Gamma.\ \forall Y \in \Delta.\ X \neq Z \Rightarrow \neg \mathtt{p}(X, Y) \vee \neg \mathtt{p}(Z, Y)).$$

Once domains are mapped to finite sets and constants to elements in those sets, a formula can be viewed as a set of conditions that the predicates (interpreted as relations) have to satisfy. Hence, FOMC is the problem of counting the number of combinations of relations that satisfy these conditions.

[4] Similarly to variables and predicates, constants are also mapped to domains, although we keep this mapping implicit.

Example 2. Let Δ be a domain of size $n \in \mathbb{N}_0$. The model count of $\forall X \in \Delta.\ \mathtt{p}(X) \vee \mathtt{q}(X)$ is then 3^n. Intuitively, since both predicates are of arity one, they can be interpreted as subsets of Δ. Thus, the formula says that each element of Δ has to be included in $\mathtt{p}$ or in $\mathtt{q}$ or in both, i.e., there are three possibilities.

Example 3. Consider a variant of the well-known 'friends and smokers' example $\forall X, Y \in \Delta.\ \mathtt{smokes}(X) \wedge \mathtt{friends}(X, Y) \Rightarrow \mathtt{smokes}(Y)$. Letting $n \coloneqq |\Delta|$ as in Example 2, the model count can be expressed as $\sum_{k=0}^{n} \binom{n}{k} 2^{n^2 - k(n-k)}$ (Van den Broeck, Meert, and Darwiche 2014).

To formalise the idea of a model, let $\sigma \colon \mathcal{D} \to \mathbb{N}_0$ be the *domain size function* that maps each domain to a non-negative integer.

Definition 4. Let ϕ be a formula and σ a domain size function. A *model* of (ϕ, σ) is a map $\mathfrak{M} \colon \mathrm{Preds}(\phi) \to 2^{\mathbb{N}_0^*}$ s.t. the following two conditions are satisfied.

1. Let $\mathtt{p}/n \in \mathrm{Preds}(\phi)$ be a predicate and let $\pi_\phi(\mathtt{p}) = (d_i)_{i=1}^n$ for some domains $d_i \in \mathrm{Doms}(\phi)$. Then
$$\mathfrak{M}(\mathtt{p}) \subseteq \prod_{i=1}^{n} [\sigma(d_i)].^5 \tag{1}$$
2. As a collection of relations, $\mathfrak{M}$ satisfies ϕ.

Example 4. Let ϕ be as in Example 1 and let $\sigma(\Gamma) = \sigma(\Delta) = 2$. Then the right-hand side of Eq. (1) becomes

$$[\sigma(\Gamma)] \times [\sigma(\Delta)] = [2] \times [2] = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

There are $2^4 = 16$ possible relations between Γ and Δ. Let us count how many of them satisfy the conditions imposed on predicate $\mathtt{p}$. The empty relation does. All four relations of cardinality one do too. Finally, there are two relations of cardinality two that satisfy the conditions as well. Thus, the FOMC of (ϕ, σ) is 7. Incidentally, it counts partial injections. We will continue to use the problem of counting partial injections (and the formula from Example 1 specifically) as the main running example throughout the paper.

3 From Circuits to Graphs

Darwiche (2001) introduced *deterministic decomposable negation normal form* (d-DNNF) circuits for propositional knowledge compilation and showed that the model count of a propositional formula can be computed in time linear in the size of the circuit. Van den Broeck et al. (2011) generalised them to FOL via *first-order d-DNNF (FO d-DNNF) circuits*. FO d-DNNF circuits (hereafter simply called *circuits*) are directed acyclic graphs with nodes corresponding to formulas in FOL—see Fig. 1 for an example. The following types of nodes are supported by FORCLIFT: caching (REF), contradiction ($\bot$), tautology ($\top$), decomposable conjunction ($\wedge$), decomposable set-conjunction ($\bigwedge$), deterministic disjunction ($\vee$), deterministic set-disjunction ($\bigvee$), domain recursion, grounding, inclusion-exclusion, smoothing,

[5] For simplicity, Definition 4 ignores constants. To include constants, one would replace $[\sigma(d_i)]$ with a set that contains all constants associated with domain d_i, extended with enough new elements to make its cardinality $\sigma(d_i)$.

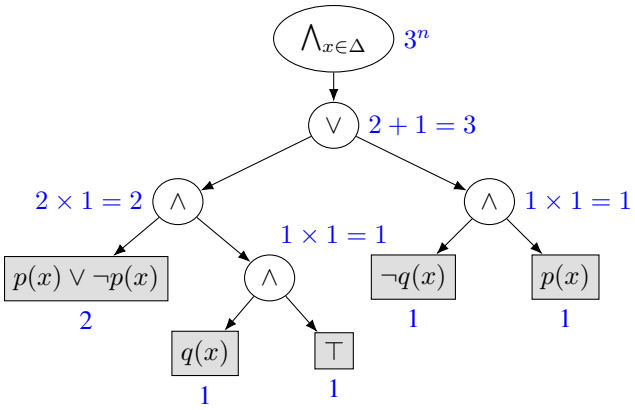

Figure 1: A circuit produced by FORCLIFT for Example 2. Values and computations in blue (on the outside of each node) show how a bottom-up evaluation of the circuit computes the model count.

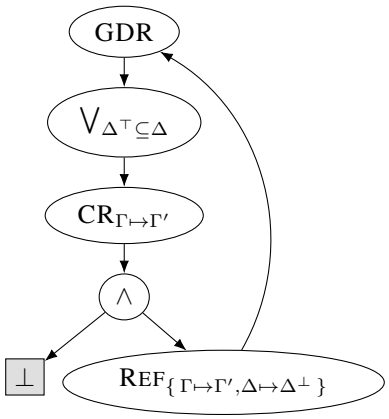

Figure 2: A simplified version of the FCG constructed by CRANE for the problem of counting partial injections from Example 1. Here we omit some parameters as well as nodes whose only arithmetic effect is multiplication by one.

and unit clause. We refer the reader to previous work (Van den Broeck 2011; Van den Broeck et al. 2011) for more information about node types and their interpretations for computing the (W)FOMC.

In this paper, we introduce *FO d-DNNF computational graphs* (FCGs) that generalise circuits by dispensing with acyclicity. Formally, an FCG is a (weakly connected) directed graph with a single source, node labels, and ordered outgoing edges. Node labels consist of two parts: the *type* and the *parameters*. The type of a node determines its outdegree. We make the following changes and additions to the node types already supported by FORCLIFT. First, we introduce a new node type for constraint removal (CR). Second, we replace domain recursion with generalised domain recursion (GDR). And third, for reasoning about partially-constructed FCGs, we write $\star$ for a placeholder type that is yet to be replaced. We write T_p for an FCG that has a node with label T_p (i.e., type T and parameter(s) p) and $\star$'s as all of its direct successors. We also write $T_p(v)$ for an FCG with one edge from a node labelled T_p to some other node v (and no other nodes or edges). See Fig. 2 for an example FCG.

Finally, we introduce a structure that represents a solution to a (W)FOMC problem while it is still being built. A *chip* is a pair (G, L), where G is an FCG, and L is a list of formulas, s.t. $|L|$ is equal to the number of $\star$'s in G. L contains formulas that still need to be compiled. Once a formula is compiled, it replaces one of the $\star$'s in G according to a set order. We say that an FCG is *complete* (i.e., it represents a *complete solution*) if it has no $\star$'s. Similarly, a chip is complete if its FCG is complete (or, equivalently, the list of formulas is empty).

4 New Compilation Rules

A *(compilation) rule* takes a formula and returns a set of chips. The cardinality of this set is the number of different ways in which the rule can be applied to the input formula. While FORCLIFT (Van den Broeck et al. 2011) heuristically chooses one of them, in an attempt to not miss a solution, CRANE returns them all. In particular, if a rule returns an empty set, then that rule does not apply to the formula.

4.1 Generalised Domain Recursion

The main idea behind domain recursion (both the original version by Van den Broeck (2011) and the one presented here) is as follows. Let $\Omega \in \mathcal{D}$ be a domain. Assuming that $\Omega \neq \emptyset$, pick some $x \in \Omega$. Then, for every variable $X \in \Omega$ that occurs in a literal, consider two possibilities: $X = x$ and $X \neq x$.

Example 5. Let ϕ be a formula with a single clause

$$(\{\,\neg\mathrm{p}(X,Y), \neg\mathrm{p}(X,Z)\,\}, \{\,(Y,Z)\,\}, \{\,X\mapsto\Gamma, Y\mapsto\Delta, Z\mapsto\Delta\,\}).$$

Then we can introduce constant $x \in \Gamma$ and rewrite ϕ as $\phi' = \{\,c_1, c_2\,\}$, where

$$c_1 = (\{\,\neg\mathrm{p}(x,Y), \neg\mathrm{p}(x,Z)\,\}, \{\,(Y,Z)\,\}, \{\,Y\mapsto\Delta, Z\mapsto\Delta\,\}),$$
$$c_2 = (\{\,\neg\mathrm{p}(X,Y), \neg\mathrm{p}(X,Z)\,\}, \{\,(X,x),(Y,Z)\,\}, \{\,X\mapsto\Gamma', Y\mapsto\Delta, Z\mapsto\Delta\,\}),$$

and $\Gamma' = \Gamma \setminus \{\,x\,\}$.

Van den Broeck (2011) imposes stringent conditions on the input formula to ensure that the expanded version of the formula (as in Example 5) can be handled efficiently. For instance, Example 1 cannot be handled by FORCLIFT because there is no root binding class, i.e., the two root variables belong to different equivalence classes with respect to the binding relationship. The clauses in this expanded formula are then partitioned into three parts based on whether the transformation introduced constants or constraints or both. The aforementioned conditions ensure that these parts can be treated independently.

In contrast, GDR has only one precondition: for GDR to be applicable on domain $\Omega \in \mathcal{D}$, there must be at least one variable with domain Ω that is featured in a literal (and not just in constraints). Without such variables, GDR would

Algorithm 1: The compilation rule for GDR nodes.

Input: formula ϕ, set of all relevant domains $\mathcal{D}$
Output: set of chips S

1 $S \leftarrow \emptyset$;
2 **foreach** *domain* $\Omega \in \mathcal{D}$ *s.t. there is* $c \in \phi$ *and* $X \in \mathrm{Vars}(L_c)$ *s.t.* $\delta_c(X) = \Omega$ **do**
3 $\phi' \leftarrow \emptyset$;
4 $x \leftarrow$ a new constant in domain Ω;
5 **foreach** *clause* $c = (L, C, \delta) \in \phi$ **do**
6 $V \leftarrow \{ X \in \mathrm{Vars}(L) \mid \delta(X) = \Omega \}$;
7 **foreach** $W \subseteq V$ *s.t.* $W^2 \cap C = \emptyset$ **and** $W \cap \{ X \in \mathrm{Vars}(C) \mid (X, y) \in C$ *for some constant* $y \} = \emptyset$ **do**
  /* δ' restricts δ to the new set of variables */
8 $\phi' \leftarrow \phi' \cup \{ (L[x/W], C[x/W] \cup \{ (X, x) \mid (X \in V \setminus W) \}, \delta') \}$;
9 $S \leftarrow S \cup \{ (\mathrm{GDR}, \langle \phi' \rangle) \}$;

have no effect on the formula. GDR is also simpler in that the expanded formula is left as-is to be handled by other compilation rules. Typically, after a few more rules are applied, a combination of CR and REF nodes introduces a cycle-inducing edge back to the GDR node, thus completing the definition of a recursive function. The GDR compilation rule is summarised as Algorithm 1 and explained in more detail using the example below.

Example 6. Let $\phi \coloneqq \{ c_1, c_2 \}$ be the formula from Example 1. While GDR is possible on both domains, here we illustrate how it works on Γ. Having chosen a domain, the algorithm iterates over the clauses of ϕ. Suppose line 5 picks $c = c_1$ as the first clause. Then, set V is constructed to contain all variables with domain $\Omega = \Gamma$ that occur in the literals of clause c. In this case, $V = \{ X \}$.

Line 7 iterates over all subsets $W \subseteq V$ of variables that can be replaced by a constant without resulting in evidently unsatisfiable formulas. We impose two restrictions on W. First, $W^2 \cap C = \emptyset$ ensures that there are no pairs of variables in W that are constrained to be distinct, since that would result in an $x \neq x$ constraint after substitution. Similarly, we want to avoid variables in W that have inequality constraints with constants: after the substitution, such constraints would transform into inequality constraints between two constants. In this case, both subsets of V satisfy these conditions, and line 8 generates two clauses for the output formula:

$$(\{ \neg\mathrm{p}(X, Y), \neg\mathrm{p}(X, Z) \}, \{ (Y, Z), (X, x) \}, \{ X \mapsto \Gamma, Y \mapsto \Delta, Z \mapsto \Delta \}),$$

from $W = \emptyset$ and

$$(\{ \neg\mathrm{p}(x, Y), \neg\mathrm{p}(x, Z) \}, \{ (Y, Z) \}, \{ Y \mapsto \Delta, Z \mapsto \Delta \})$$

from $W = V$.

When line 5 picks $c = c_2$, then $V = \{ X, Z \}$. The subset $W = V$ fails to satisfy the conditions on line 7 because of the $X \neq Z$ constraint. The other three subsets of V all generate clauses for ϕ'. Indeed, $W = \emptyset$ generates

$$(\{ \neg\mathrm{p}(X, Y), \neg\mathrm{p}(Z, Y) \}, \{ (X, Z), (X, x), (Z, x) \}, \{ X \mapsto \Gamma, Y \mapsto \Delta, Z \mapsto \Gamma \}),$$

$W = \{ X \}$ generates

$$(\{ \neg\mathrm{p}(x, Y), \neg\mathrm{p}(Z, Y) \}, \{ (Z, x) \}, \{ Y \mapsto \Delta, Z \mapsto \Gamma \}),$$

and $W = \{ Z \}$ generates

$$(\{ \neg\mathrm{p}(X, Y), \neg\mathrm{p}(x, Y) \}, \{ (X, x) \}, \{ X \mapsto \Gamma, Y \mapsto \Delta \}).$$

Theorem 1 (Correctness of GDR[6])**.** *Let ϕ be the formula used as input to Algorithm 1, $\Omega \in \mathcal{D}$ the domain selected on line 2, and ϕ' the formula constructed by the algorithm for Ω. Suppose that $\Omega \neq \emptyset$. Then $\phi \equiv \phi'$.*

Proof. Let x be the constant introduced on line 4 and c a clause of ϕ selected on line 5 of the algorithm. We will show that c is equivalent to the conjunction of clauses added to ϕ' on lines 7 and 8.

If c has no variables with domain Ω, then lines 7 and 8 simply add a copy of c to ϕ'. If c has one variable with domain Ω, say, X, then $c \equiv \forall X \in \Omega.\ \psi$ for some formula ψ with one free variable X. In this case, lines 7 and 8 create two clauses: $\psi[x/\{ X \}]$ and $\forall X \in \Omega.\ X \neq x \Rightarrow \psi$. It is easy to see that their conjunction is equivalent to c provided that $\Omega \neq \emptyset$ and so x is a well-defined constant.

Let V be the set of variables introduced on line 6. It remains to show that repeating the transformation

$$\forall X \in \Omega.\ \psi \mapsto \{ \psi[x/\{ X \}], \forall X \in \Omega.\ X \neq x \Rightarrow \psi \}$$

for all variables $X \in V$ produces the same clauses as lines 7 and 8 (except for some tautologies that the latter method skips). Note that the order in which these operations are applied is immaterial. In other words, if we add inequality constraints for variables $\{ X_i \}_{i=1}^n$, and apply substitution for variables $V \setminus \{ X_i \}_{i=1}^n$, then—regardless of the order of operations—we get

$$\forall X_1, \ldots, X_n \in \Omega.\ \bigwedge_{i=1}^{n} X_i \neq x \Rightarrow \psi\left[x/\left(V \setminus \{ X_i \}_{i=1}^n\right)\right].$$

This formula is equivalent to the clause generated on line 8 with $W = V \setminus \{ X_i \}_{i=1}^n$. Thus, for every clause c of ϕ, the new formula ϕ' gets a set of clauses whose conjunction is equivalent to c. Hence, the two formulas are equivalent. □

4.2 Constraint Removal

Recall that GDR on a domain Ω creates constraints of the form $X_i \neq x$ for some constant $x \in \Omega$ and family of variables $X_i \in \Omega$. Once certain conditions are satisfied, we can eliminate these constraints and replace Ω with a new domain Ω', which can be interpreted as $\Omega \setminus \{ x \}$. These conditions are that a constraint of the form $X \neq x$ exists for all variables $X \in \Omega$ across all clauses, and such constraints are the only place where x occurs. We formalise the conditions as Definition 5.

[6]Theorem 1 is equivalent to Proposition 3 by Van den Broeck (2011).

Algorithm 2: The compilation rule for CR nodes.

Input: formula ϕ, set of all relevant domains $\mathcal{D}$
Output: set of chips S

1 $S \leftarrow \emptyset$;
2 **foreach** *replaceable pair* $(\Omega \in \mathcal{D}, x \in \Omega)$ **do**
3 add a new domain Ω' to $\mathcal{D}$;
4 $\phi' \leftarrow \emptyset$;
5 **foreach** *clause* $(L, C, \delta) \in \phi$ **do**
6 $C' \leftarrow \{ (a, b) \in C \mid b \neq x \}$;
7 $\delta' \leftarrow X \mapsto \begin{cases} \Omega' & \text{if } \delta(X) = \Omega \\ \delta(X) & \text{otherwise;} \end{cases}$
8 $\phi' \leftarrow \phi' \cup \{ (L, C', \delta') \}$
9 $S \leftarrow S \cup \{ (\mathrm{CR}_{\Omega \mapsto \Omega'}, \langle \phi' \rangle) \}$;

Definition 5. With respect to a formula ϕ, a pair (Ω, x) of a domain $\Omega \in \mathcal{D}$ and its element $x \in \Omega$ is called *replaceable* if

- x does not occur in any literal of any clause of ϕ,
- and for each clause $c = (L, C, \delta_c) \in \phi$ and variable $X \in \mathrm{Vars}(c)$, either $\delta_c(X) \neq \Omega$ or $(X, x) \in C$.

Once a replaceable pair is found, Algorithm 2 proceeds to construct the new formula by removing constraints (on line 6) and constructing a new domain map δ' that replaces Ω with Ω' (on line 7).

Example 7. Let $\phi = \{ c_1, c_2 \}$ be a formula with clauses

$$c_1 = (\{ \neg \mathrm{p}(X, Y), \neg \mathrm{p}(X, Z) \}, \{ (X, x), (Y, Z) \}, \{ X \mapsto \Gamma, Y \mapsto \Delta, Z \mapsto \Delta \}),$$

$$c_2 = (\{ \neg \mathrm{p}(X, Y), \neg \mathrm{p}(Z, Y) \}, \{ (X, x), (Z, X), (Z, x) \}, \{ X \mapsto \Gamma, Y \mapsto \Delta, Z \mapsto \Gamma \}).$$

Domain Γ and its element $x \in \Gamma$ satisfy the preconditions for CR. The rule introduces a new domain Γ' and transforms ϕ to $\phi' = (c'_1, c'_2)$, where

$$c'_1 = (\{ \neg \mathrm{p}(X, Y), \neg \mathrm{p}(X, Z) \}, \{ (Y, Z) \}, \{ X \mapsto \Gamma', Y \mapsto \Delta, Z \mapsto \Delta \}),$$

$$c'_2 = (\{ \neg \mathrm{p}(X, Y), \neg \mathrm{p}(Z, Y) \}, \{ (Z, X) \}, \{ X \mapsto \Gamma', Y \mapsto \Delta, Z \mapsto \Gamma' \}).$$

Theorem 2 (Correctness of CR)**.** *Let ϕ be the input formula of Algorithm 2, (Ω, x) a replaceable pair, and ϕ' the formula constructed by the algorithm, when (Ω, x) is selected on line 2. Then $\phi \equiv \phi'$, where the domain Ω' introduced on line 3 is interpreted as $\Omega \setminus \{ x \}$.*

Proof. Since there is a natural bijection between the clauses of ϕ and ϕ', we shall argue about the equivalence of each pair of clauses. Let c be an arbitrary clause of ϕ and c' its corresponding clause of ϕ'.

If c has no variables with domain Ω, then it cannot have any constraints involving x, so $c' = c$. Otherwise, for notational simplicity, let us assume that X is the only variable in c with domain Ω (the proof for an arbitrary number of variables is virtually the same). By Definition 5, we can rewrite c as $\forall X \in \Omega.\ X \neq x \Rightarrow \psi$, where ψ is a formula with X as the only free variable and with no mention of either x or Ω. Then $c' \equiv \forall X \in \Omega'.\ \psi$. Since $\Omega' \coloneqq \Omega \setminus \{ x \}$, we have that $c \equiv c'$. Since c was an arbitrary clause of ϕ, this completes the proof that $\phi \equiv \phi'$. □

Algorithm 3: The compilation rule for REF nodes.

Input: formula ϕ, cache C
Output: a set of chips

1 **foreach** *formula and node* $(\psi, v) \in C(\#\phi)$ **do**
2 $\rho \leftarrow \mathtt{r}(\phi, \psi)$;
3 **if** $\rho \neq \mathtt{null}$ **then return** $\{ (\mathrm{REF}_\rho(v), \langle\rangle) \}$;
4 **return** $\emptyset$;
5 **Function** r (*formula ϕ, formula ψ, map $\rho = \emptyset$*) **:**
6 **if** $|\phi| \neq |\psi|$ **or** $\#\phi \neq \#\psi$ **then return** `null`;
7 **if** $\phi = \emptyset$ **then return** ρ;
8 **foreach** *clause* $c \in \psi$ **do**
9 **foreach** *clause* $d \in \phi$ *s.t.* $\#d = \#c$ **do**
10 **forall** $\gamma \in \mathtt{genMaps}(c, d, \rho)$ **do**
11 $\rho' \leftarrow \mathtt{r}(\phi \setminus \{ d \}, \psi \setminus \{ c \}, \rho \cup \gamma)$;
12 **if** $\rho' \neq \mathtt{null}$ **then return** ρ';
13 **return** `null`;

4.3 Identifying Opportunities for Recursion

Hashing. We use (integer-valued) hash functions to discard pairs of formulas that are too different for recursion to work. The hash code of a clause $c = (L, C, \delta_c)$ (denoted by $\#c$) combines the hash codes of the sets of constants and predicates in c, the numbers of positive and negative literals, the number of inequality constraints $|C|$, and the number of variables $|\mathrm{Vars}(c)|$. The hash code of a formula ϕ combines the hash codes of all its clauses and is denoted by $\#\phi$.

Caching. FORCLIFT (Van den Broeck et al. 2011) uses a cache to check if a formula is identical to one of the formulas that have already been fully compiled. To facilitate recursion, we extend the caching scheme to include formulas that have been encountered but not fully compiled yet. Formally, we define a *cache* to be a map from integers (e.g., hash codes) to sets of pairs of the form (ϕ, v), where ϕ is a formula, and v is an FCG node.

Algorithm 3 describes the compilation rule for creating REF nodes. For every formula ψ in the cache s.t. $\#\psi = \#\phi$, function `r` is called to check whether a recursive call is feasible. If it is, `r` returns a (total) map $\rho\colon \mathrm{Doms}(\psi) \to \mathrm{Doms}(\phi)$ that shows how ψ can be transformed into ϕ by replacing each domain $\Omega \in \mathrm{Doms}(\psi)$ with $\rho(\Omega) \in \mathrm{Doms}(\phi)$. Otherwise, `r` returns `null` to signify that ϕ and ψ are too different for recursion to work. This happens if ϕ and ψ (or their subformulas explored in recursive calls) are structurally different (i.e., the numbers of clauses or the hash codes fail to match) or if a clause of ψ cannot be paired with a sufficiently similar clause of ϕ. Function `r` works

by iterating over pairs of clauses of ϕ and ψ that have the same hash codes. For every pair of sufficiently similar clauses, `r` calls itself on the remaining clauses until the map $\rho\colon \mathrm{Doms}(\psi) \twoheadrightarrow \mathrm{Doms}(\phi)$ becomes total, and all clauses are successfully coupled.

Function `genMaps` checks the compatibility of a pair of clauses. It considers every possible bijection $\beta\colon \mathrm{Vars}(c) \to \mathrm{Vars}(d)$ and map $\gamma\colon \mathrm{Doms}(c) \to \mathrm{Doms}(d)$ s.t.

$$\begin{array}{ccc} \mathrm{Vars}(c) & \xrightarrow{\beta} & \mathrm{Vars}(d) \\ {\scriptstyle\delta_c}\downarrow & & \downarrow{\scriptstyle\delta_d} \\ \mathrm{Doms}(c) & \xrightarrow{\gamma} & \mathrm{Doms}(d) \\ \downarrow & & \downarrow \\ \mathrm{Doms}(\psi) & \xrightarrow[\rho]{} & \mathrm{Doms}(\phi). \end{array}$$

commutes, and c becomes equal to d when its variables are replaced according to β and its domains replaced according to γ. The function then returns each such γ as soon as possible. The commutativity check also ensures that $\rho \cup \gamma$ is possible on line 11 of Algorithm 3. The resulting (partial) function $\rho \cup \gamma$ is the unique function s.t. $\rho \cup \gamma|_{\mathrm{dom}(\rho)} = \rho$, and $\rho \cup \gamma|_{\mathrm{dom}(\gamma)} = \gamma$. The operation is defined when $\rho|_{\mathrm{dom}(\rho)\cap\mathrm{dom}(\gamma)} = \gamma|_{\mathrm{dom}(\rho)\cap\mathrm{dom}(\gamma)}$.

Theorem 3 (Correctness of REF)**.** *Let ϕ be the formula used as input to Algorithm 3. Let ψ be any formula selected on line 1 of the algorithm s.t. $\rho \neq$ `null` on line 3. Let σ be a domain size function. Then the set of models of $(\psi, \sigma \circ \rho)$ is equal to the set of models of (ϕ, σ).*

Proof. We first show that the right-hand side of Eq. (1) is the same for models of both (ϕ, σ) and $(\psi, \sigma \circ \rho)$. Algorithm 3 ensures that $\psi \equiv \phi$ up to domains. In particular, this means that $\mathrm{Preds}(\psi) = \mathrm{Preds}(\phi)$. The square in

$$\begin{array}{ccc} \mathrm{Preds}(\psi) & = & \mathrm{Preds}(\phi) \\ {\scriptstyle\pi_\psi}\downarrow & & \downarrow{\scriptstyle\pi_\phi} \\ \mathrm{Doms}(\psi)^* & \xrightarrow{\rho} & \mathrm{Doms}(\phi)^* \xrightarrow{\sigma} \mathbb{N}_0^* \end{array} \tag{2}$$

commutes by Definition 3 and the definition of ρ. In other words, for any predicate $\mathtt{p}/n \in \mathrm{Preds}(\psi) = \mathrm{Preds}(\phi)$, function ρ translates the domains associated with $\mathtt{p}$ in ψ to the domains associated with $\mathtt{p}$ in ϕ. Let $\pi_\phi(\mathtt{p}) = (d_i)_{i=1}^n$ and $\pi_\psi(\mathtt{p}) = (e_i)_{i=1}^n$ for some domains $d_i \in \mathrm{Doms}(\phi)$ and $e_i \in \mathrm{Doms}(\psi)$. Since $\rho(e_i) = d_i$ for all $i = 1, \ldots, n$ by the commutativity in Diagram (2), we get that

$$\prod_{i=1}^{n} [\sigma(d_i)] = \prod_{i=1}^{n} [\sigma \circ \rho(e_i)] \tag{3}$$

as required. Certainly, any subset of the left-hand side of Eq. (3) (e.g., $\mathfrak{M}(\mathtt{p})$, where $\mathfrak{M}$ is a model of (ϕ, σ)) is a subset of the right-hand side and vice versa. Since ϕ and ψ semantically only differ in what domains they refer to, every model of ϕ satisfies ψ and vice versa, completing the proof. □

Algorithm 4: Construct functions from an FCG

Input: $\mathcal{D}$—the set of domains of the input formula
Input: s—the source node of the FCG
Data: $\mathsf{F} = \emptyset$ (function names)
Output: a list of function definitions

```
1  V ← { Ω ↦ newVarName() | Ω ∈ 𝒟 };
2  if s is a direct successor of a REF node then
3  |  (e, funs) ← v(s, V);
4  |  return funs;
5  f ← newFunctionName(s);
6  (e, funs) ← v(s, V);
7  return ⟨f(⟨V(Ω) | Ω ∈ 𝒟⟩) = e⟩ +⧺ funs;
8  Function v (node v, variable names V):
9  |  if v is not a direct successor of a REF node then
10 |  |  return visit(v, V);
11 |  f ← newFunctionName(v);
12 |  V' ← V;
13 |  for Ω ∈ 𝔇(v) s.t. V'(Ω) is non-atomic do
14 |  |  V'(Ω) ← newVarName();
15 |  (e, funs) ← visit(v, V');
16 |  call ← f(⟨V(Ω) | Ω ∈ 𝔇(v)⟩);
17 |  signature ← f(⟨V'(Ω) | Ω ∈ 𝔇(v)⟩);
18 |  return (call, ⟨signature = e⟩ +⧺ funs);
19 Function visit(node v, variable names V):
20 |  switch label of v do
21 |  |  case GDR(v') do return v(v', V);
22 |  |  case CR_{Ω↦Ω'}(v') do
23 |  |  |  return v(v', V ∪ { Ω' ↦ (V(Ω) − 1) });
24 |  |  case REF_ρ(v') do
25 |  |  |  args ← ⟨V ∘ ρ(Ω) | Ω ∈ 𝔇(v')⟩;
26 |  |  |  return (F(v')(args), ⟨⟩);
27 |  |  ...
```

5 Converting FCGs to Function Definitions

- TODO: is maybe 'parameter name' (or something similar) would be more appropriate than 'variable name'?
- We call a variable name is *atomic* if it is free of subtraction (e.g., m, n); otherwise it is *non-atomic* (e.g., $m-1$, $n-l$).
- For each node v, let $\mathfrak{D}(v)$ denote the (precomputed) set of domains, the sizes of which we would use as parameters if we were to define a function that begins at v. This set is computed by iteratively setting $\mathfrak{D}(v) \leftarrow (\bigcup_u \mathfrak{D}(u) \setminus I_v) \cup U_v$ until convergence, where
  - u is any direct successor of v,
  - I_v is the set of domains *introduced* at node v, and
  - U_v is the set of domains *used* by v.
- We assume a fixed total ordering of all domains names. In particular, in our running example we assume that Γ goes before Δ.
- F is a partial map from the set of FCG nodes to function names (e.g., f, g).

- `newVarName` just generates a unique identifier for a variable
- `newFunctionName` does the same thing for function names and also records the new name in F
- The first function in the list is the main one.
- Note that a node is a direct successor of a REF node if and only if it has an in-degree greater than one.
- For simplicity, we refer to FCG nodes using their types.
- The algorithm consists of two main functions: `v` and `visit`. The former handles new function definitions, and the latter produces an algebraic interpretation of each node depending on its type. As there are many node types, Algorithm 4 only includes the ones pertinent to the contributions of this paper; see previous work (Van den Broeck et al. 2011) for information about the other types.
- Given some node v as input, both `v` and `visit` return a pair $(\mathsf{e}, \mathsf{funs})$. Here, e is the algebraic expression that should be used to evaluate v or use it in another expression, and funs is a list of auxiliary functions that were created while constructing e.
- TODO: $\mathfrak{D}$ must be a list. That's the ordering of domain names comes in, not in list comprehensions.

When FORCLIFT (Van den Broeck et al. 2011) compiles a WFOMC instance into a circuit, each node type encodes an arithmetic operation on its inputs and parameters. These operations are then immediately performed while traversing the circuit. With CRANE, the interpretation of an FCG is a list of functions. Each function has (some) domain sizes as parameters and may contain recursive calls to other functions, including itself. While there may be any number of subsidiary functions, there is always one main function that can be called with the sizes of the domains of the input formula as arguments. Henceforth, this function is always called f, and it is defined by the source node.

The interpretation of a node is decided by its type. Here we describe the interpretations of new (or significantly changed) types and refer the reader to previous work (Van den Broeck et al. 2011) for information on other types. The interpretation of a CR or a GDR node is simply the interpretation of its only direct successor. Obviously, $\star$ nodes have no interpretation as incomplete FCGs are not meant to be interpreted. The interpretation of a REF node is a function call. The direct successor of the REF node (say, v) then must introduce a function. The parameters of this function are the sizes of all domains used by nodes reachable from v.

Example 8. Let us see how Algorithm 4 handles the FCG from Fig. 2. Here, $\mathcal{D} = \{\Gamma, \Delta\}$. Let $\mathsf{V}_1 \coloneqq \{\Gamma \mapsto m, \Delta \mapsto n\}$, $\mathsf{V}_2 \coloneqq \mathsf{V}_1 \cup \{\Delta^\top \mapsto l, \Delta^\bot \mapsto (n-l)\}$, and $\mathsf{V}_3 \coloneqq \mathsf{V}_2 \cup \{\Gamma' \mapsto (m-1)\}$ denote versions of V at various points throughout the algorithm's execution. Here, l, m, and n are all arbitrary variable names generated by `newVarName`. Line 1 initialises V to V_1.

Figure 3 shows the outline of function calls and their return values. Once `v(`GDR, V_1`)` is called on line 3, `newFunctionName` updates F to $\{\text{GDR} \mapsto f\}$, where f is a new function name. There are no non-atomic

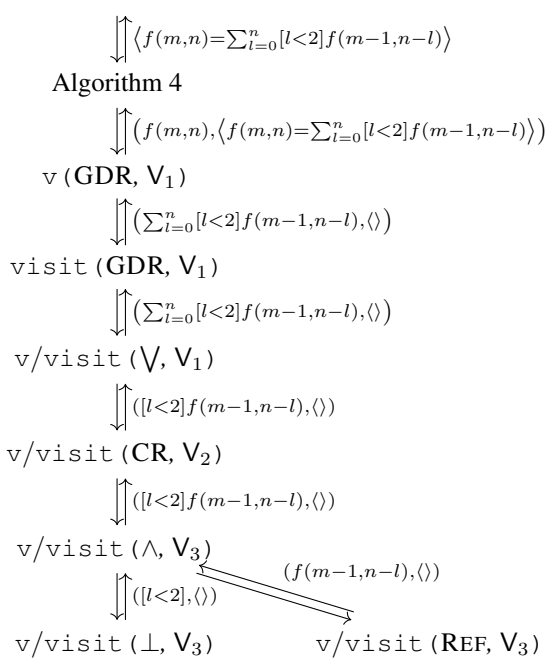

Figure 3: Function calls and their return values when Algorithm 4 is run on the FCG from Fig. 2. When `v` calls `visit` and returns its output unchanged, the two function calls are shortened as a single call to `v/visit`. TODO: this last sentence should be explained in text.

names in V_1, so lines 13 and 14 are skipped, and calls to `visit(`GDR, V_1`)` and `v/visit(`$\bigvee$, V_1`)` follow. Here, V_1 becomes V_2, i.e., the $\bigvee$ node introduces two domains $\Delta^\top$ and $\Delta^\bot$ that 'partition' Δ, i.e., $\mathsf{V}_2(\Delta^\top) + \mathsf{V}_2(\Delta^\bot) = l + (n-l) = n = \mathsf{V}_2(\Delta)$. Similarly, `visit(`CR, V_2`)` on line 23 adds Γ', replacing V_2 with V_3.

Eventually, `visit(`$\bot$, V_3`)` is called. A contradiction node with clause c as a parameter is interpreted as one if the clause has groundings and zero otherwise. In this case, the parameter is $c = (\emptyset, \{(X,Y)\}, \{X \mapsto \Delta^\top, Y \mapsto \Delta^\top\})$ (not shown in Fig. 2). Equivalently, $\forall X, Y \in \Delta^\top.\ X \neq Y \Rightarrow \bot$, i.e., $\forall X, Y \in \Delta^\top.\ X = Y$. This sentence is true if and only if $\mathsf{V}_3(\Delta^\top) < 2$. Thus, we can use the Iverson bracket notation to write

$$[l < 2] \coloneqq \begin{cases} 1 & \text{if } l < 2 \\ 0 & \text{otherwise.} \end{cases}$$

for the output expression of `visit(`$\bot$, V_3`)`.

Next, `visit(`$\wedge$, V_3`)` calls `v/visit(`REF, V_3`)`. Since $\mathfrak{D}(\text{GDR}) = \langle \Gamma, \Delta \rangle$, and

$$\langle \Gamma, \Delta \rangle \xrightarrow{\rho} \langle \Gamma', \Delta^\bot \rangle \xmapsto{\mathsf{V}_3} \langle m-1, n-l \rangle\ , \quad \mathsf{args}$$

is set to $\langle m-1, n-l \rangle$ on line 25, and the output expression becomes $f(m-1, n-l)$. The call to `visit(`$\wedge$, V_3`)` then returns the product of the two expressions $[l < 2] f(m-1, n-l)$. Next, the call to `visit(`$\bigvee$, V_1`)` prepends a summation and returns $\sum_{l=0}^{n} [l < 2] f(m-1, n-l)$. The same expression (and an empty list of functions) is then returned to line 15 of the

call to $\mathrm{v}(\mathrm{GDR}, \mathsf{V}_1)$. Here, both call and signature are set to $f(m,n)$. As the definition of f is the final answer and not part of some other algebraic expression, lines 3 and 4 discard the function call expression and return the definition of f.

Therefore,

$$f(m,n) = \sum_{l=0}^{n} \binom{n}{l} [l < 2] f(m-1, n-l)$$
$$= f(m-1,n) + nf(m-1,n-1). \quad (4)$$

To use this recursive function to compute the model count of the input formula for any domain sizes, one just needs to find the base cases $f(0,n)$ and $f(m,0)$ for all $m,n \in \mathbb{N}_0$.

6 Experimental Results

We compare CRANE and FORCLIFT[7] (Van den Broeck et al. 2011) on their ability to count various kinds of functions. This class of instances is chosen because of its simplicity and ubiquity and the inability of state-of-the-art WFOMC algorithms to solve many such instances. Note that other WFOMC algorithms mentioned in Section 1 are unable to solve any of the instances that FORCLIFT fails on. We begin by describing how such function-counting problems can be expressed in FOL. FORCLIFT then translates these sentences in FOL to formulas as defined in Definition 3.

Let p be a predicate that models relations between sets Γ and Δ. To restrict all such relations to just $\Gamma \to \Delta$ functions, in FOL one might write

$$\forall X \in \Gamma.\ \forall Y, Z \in \Delta.\ \mathrm{p}(X,Y) \wedge \mathrm{p}(X,Z) \Rightarrow Y = Z$$

and

$$\forall X \in \Gamma.\ \exists Y \in \Delta.\ \mathrm{p}(X,Y). \quad (5)$$

The former sentence says that one element of Γ can map to at *most* one element of Δ, and the latter sentence says that each element of Γ must map to at *least* one element of Δ. One can then add

$$\forall X, Z \in \Gamma.\ \forall Y \in \Delta.\ \mathrm{p}(X,Y) \wedge \mathrm{p}(Z,Y) \Rightarrow X = Z$$

to restrict p to injections or

$$\forall Y \in \Delta.\ \exists X \in \Gamma.\ \mathrm{p}(X,Y)$$

to ensure surjectivity or remove Eq. (5) to consider partial functions. Lastly, one can replace all occurrences of Δ with Γ to model endofunctions (i.e., functions with the same domain and codomain) instead.

In our experiments, we consider all sixteen combinations of these properties, i.e., injectivity, surjectivity, partiality, and endo-. We run each algorithm once on each of the sixteen instances.[8] CRANE is run in hybrid search mode until either five solutions are found or the search tree reaches height six. FORCLIFT is always run until it terminates. If successful, FORCLIFT generates a circuit, and CRANE generates one or more (complete) FCGs. In both cases, we manually convert the resulting graphs into definitions of functions as described in Section 5. We then assess the complexity of each solution by hand and pick the best in case CRANE returns several solutions of varying complexities.

Previous work often compares WFOMC algorithms by running them on a few instances with increasing domain sizes and measuring runtime (Van den Broeck 2011; Van den Broeck et al. 2011; Van den Broeck and Davis 2012). However, we can do much better and identify the exact worst-case asymptotic complexity of a solution produced by either CRANE or FORCLIFT. The asymptotic complexity of a solution can be determined by considering the total number of arithmetic operations needed to follow the definitions of constituent functions without recomputing the same quantity multiple times. In particular, we assume that each function call and binomial coefficient is computed at most once. We also assume that computing $\binom{n}{k}$ takes $\Theta(nk)$ time. For example, the complexity of Eq. (4) is $\Theta(mn)$ since $f(m,n)$ can be computed by a dynamic programming algorithm that computes $f(i,j)$ for all $i = 0,\ldots,m$ and $j = 0,\ldots,n$, taking a constant amount of time on the computation of each $f(i,j)$.

The experimental results are summarised in Table 1, where we compare the solutions found by both algorithms to the best known ways of computing the same quantities. The best-known asymptotic complexity for computing total surjections is by Earnest (2018). All other best-known complexity results are inferred from the formulas and programs on the on-line encyclopedia of integer sequences (OEIS Foundation Inc. 2023). On instances that could already be solved by FORCLIFT, the two algorithms perform equally well. However, CRANE can also solve all but one of the instances that FORCLIFT fails on in at most cubic time. See supplementary material for the exact solutions produced by CRANE that correspond to the complexity results in Table 1. We also note that standard (W)FOMC benchmarks such as Example 3 that are already supported by FORCLIFT remain feasible for CRANE as well. Moreover, our algorithm can also handle the linear order axiom, for which a bespoke algorithm was recently proposed by Tóth and Kuželka (2022).

7 Conclusion and Future Work

In this paper, we showed how a state-of-the-art (W)FOMC algorithm can be empowered by generalising domain recursion and adding support for cycles in the graph that encodes a solution. To construct such graphs, CRANE supplements FORCLIFT (Van den Broeck et al. 2011) with three new compilation rules. Our experiments revealed a range of counting problems that are liftable to CRANE but not to any other (W)FOMC algorithm. Although in this paper we focus on unweighted counting, FORCLIFT's support for weights trivially transfers to CRANE as well. The common thread across these newly liftable problems is (partial) injectivity. Thus, we can formulate the following conjecture.

Conjecture 1. *Let* IFO^2 *be the class of formulas in FOL that contain clauses with at most two variables as well as*

[7] https://dtaid.cs.kuleuven.be/wfomc

[8] While domain sizes have to be specified as part of the input, they are immaterial to the search procedure.

Function Class			Asymptotic Complexity of Counting		
Partial	Endo-	Class	Best Known	With FORCLIFT	With CRANE
✓/✗	✓/✗	Functions	$\log m$	m	m
✗	✗	Surjections	$n \log m$	$m^3 + n^3$	$m^3 + n^3$
✗	✓		$m \log m$	m^3	m^3
✓	✗		Same as injections from Δ to Γ		
✓	✓		Same as endo-injections		
✗	✗	Injections	m	—	mn
✗	✓		m	—	m^3
✓	✗		$\min\{m, n\}^2$	—	mn
✓	✓		m^2	—	—
✗	✗	Bijections	m	—	m
✗	✓		Same as (partial) (endo-)injections		
✓	✓/✗				

Table 1: The worst-case complexity of counting various types of functions, where m is the size of domain Γ, and n is the size of domain Δ. All asymptotic complexities are in $\Theta(\cdot)$. A dash means that the algorithm was not able to find a solution.

any number of copies of

$$
\begin{aligned}
&(\forall X \in \Gamma.\ \forall Y, Z \in \Delta. \\
&\quad Y \neq Z \Rightarrow \neg \mathrm{p}(X, Y) \vee \neg \mathrm{p}(X, Z)) \wedge \\
&(\forall X, Z \in \Gamma.\ \forall Y \in \Delta. \\
&\quad X \neq Z \Rightarrow \neg \mathrm{p}(X, Y) \vee \neg \mathrm{p}(Z, Y))
\end{aligned}
$$

for some predicate $\mathrm{p}/2$ *and domains* Γ *and* Δ*. Then* FO^2 *is liftable by* CRANE*.*

Recall that C^2 is the class of formulas with counting quantifiers and at most two variables. C^2 was recently shown to be liftable (Kuželka 2021) but without providing a usable (W)FOMC algorithm. Since the tasks of counting injections and bijections fall into the C^2 fragment, we can conjecture the following.

Conjecture 2. C^2 *is liftable by* CRANE *by either reformulating formulas in* C^2 *to avoid counting quantifiers or extending* CRANE *to support them.*

However, the most important direction for future work is to fully automate this new process. First, we need an algorithm that transforms FCGs into definitions of functions. Formalising this process would also allow us to prove the correctness of the new compilation rules in constructing FCGs that indeed compute the right (weighted) model counts. Second, these definitions must be simplified before they can be used, perhaps by a computer algebra system. Third, we need a way to find the base cases for the recursive definitions provided by CRANE. Fourth, since the first solution found by CRANE is not always optimal in terms of its complexity, an automated way to determine the asymptotic complexity of a solution would be helpful as well. Achieving these goals would enable CRANE to automatically constructing efficient ways to compute various functions and sequences. In addition to the potential impact on areas of artificial intelligence (AI) such as statistical relational AI (De Raedt et al. 2016), CRANE could be beneficial to research in combinatorics as well (Barvínek et al. 2021).

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