Nondeterministic Bigraphical Reactive Systems for Markov Decision Processes*

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Abstract. The abstract should briefly summarize the contents of the paper in 150–250 words.

Keywords: Bigraphs · Probabilistic semantics · Markov decision process.

1 Introduction

Definition 1 (Markov decision process). *For any finite set X, let $Dist(X)$ denote the set of discrete probability distributions over X. A* Markov Decision Process *is a tuple $(S, \overline{s}, A, P, L)$, where: S is a finite set of states and $\overline{s} \in S$ is the initial state; A is a finite set of* actions*; $P : S \times A \to Dist(S)$ is a (partial) probabilistic transition function, mapping state-action pairs to probability distributions over S; $L : S \to 2^P$ is a labelling with atomic propositions.*

Definition 2 (rewards). *A* reward structure *for an MDP $(S, \overline{s}, A, P, L)$ is a pair (ρ, ι), where $\rho : S \to \mathbb{R}$ is the* state reward function*, and $\iota : S \times A \to \mathbb{R}$ is the* transition reward function*.*

From [2].

Definition 3 (concrete place graph with sharing). *A* concrete place graph with sharing

$$F = (V_F, ctrl_F, prnt_F) : m \to n$$

is a triple having an inner interface m and an outer interface n. These index the sites and regions of the place graph, respectively. F has a finite set $V_F \subset \mathcal{V}$ of nodes, a control map $ctrl_F : V_F \to \mathcal{K}$, and a parent relation

$$prnt_F \subseteq (m \uplus V_F) \times (V_F \uplus n)$$

that is acyclic, i.e., $(v, v) \notin prnt_F^+$ for any $v \in V_F$.

Definition 4 (composition for place graphs with sharing). *If $F : k \to m$ and $G : m \to n$ are two concrete place graphs with sharing with $V_F \cap V_G = \emptyset$, their composite*

$$G \circ F = (V, ctrl, prnt) : k \to n$$

* Supported by organization x.

has nodes $V = V_F \uplus V_G$ *and control map* $ctrl = ctrl_F \uplus ctrl_G$. *Its parent relation* $prnt \subseteq (k \uplus V) \times (V \uplus n)$ *is given by:*

$$prnt \coloneqq prnt_G^{\triangleleft} \uplus prnt_{\circ} \uplus prnt_F^{\triangleright},$$

where

$$\begin{aligned} prnt_F^{\triangleright} &= prnt_F \triangleright V_F, \\ prnt_G^{\triangleleft} &= V_G \triangleleft prnt_G, \\ prnt_{\circ} &= (m \triangleleft prnt_G) \circ (prnt_F \triangleright m). \end{aligned}$$

Definition 5 (tensor product for place graphs). *If* $G_0 : m_0 \to n_0$ *and* $G_1 : m_1 \to n_1$ *are two concrete place graphs with sharing with* $V_F \cap V_G = \emptyset$, *their tensor product*

$$G_0 \otimes G_1 = (V, ctrl, prnt) : m_0 + m_1 \to n_0 + n_1$$

has nodes $V = V_{G_0} \uplus V_{G_1}$ *and control map* $ctrl \coloneqq ctrl_{G_0} \uplus ctrl_{G_1}$. *Its parent relation* $prnt \subseteq [(m_0 + m_1) \uplus V] \times [V \uplus (n_0 + n_1)]$ *is defined as*

$$prnt_{G_0} \uplus prnt_{G_1}^{(m_0,n_0)},$$

where

$$\begin{aligned} prnt_{G_1}^{(m_0,n_0)} = &\{(v,w) \mid (v,w) \in prnt_{G_1} \text{ and } v,w \in V_{G_1}\} \\ \uplus &\{(m_0+i,w) \mid (i,w) \in prnt_{G_1}, w \in V_{G_1} \text{ and } i \in m_1\} \\ \uplus &\{(v,n_0+i) \mid (v,i) \in prnt_{G_1}, v \in V_{G_1} \text{ and } i \in n_1\} \\ \uplus &\{(m_0+i,n_0+j) \mid (i,j) \in prnt_{G_1}, i \in m_1, \text{ and } j \in n_1\}. \end{aligned}$$

Definition 6 (concrete link graph). *A* concrete link graph

$$F = (V_F, E_F, ctrl_F, link_F) : X \to Y$$

is a quadruple having an inner face X *and an outer face* Y, *both finite subsets of* $\mathcal{X}$, *called respectivelly the inner and outer names of the link graph.* F *has finite sets* $V_F \subset \mathcal{V}$ *of nodes and* $E_F \subset \mathcal{E}$ *of edges, a control map* $ctrl_F : V_F \to \mathcal{K}$ *and a* link map

$$link_F : X \uplus P_F \to E_F \uplus Y,$$

where $P_F \coloneqq \{(v,i) \mid i \in ar(ctrl_F(v))\}$ *is the set of ports of* F. *Thus* (v,i) *is the* i*th port of node* v. *We shall call* $X \uplus P_F$ *the* points *of* F, *and* $E_F \uplus Y$ *its* links.

The sets of points and the set of ports of a link l are defined by

$$points_F(l) \coloneqq \{p \mid link_F(p) = l\}, \quad ports_F(l) \coloneqq points_F(l) \setminus X.$$

An edge is *idle* if it has no points. Identities over name sets are defined by $\mathsf{id}_X = (\emptyset, \emptyset, \emptyset, \mathsf{Id}_X) : X \to X$.

Definition 7 (concrete bigraph with sharing). *A concrete bigraph*

$$F = (V_F, E_F, ctrl_F, prnt_F, link_F) : \langle k, X\rangle \to \langle m, Y\rangle$$

consists of a concrete place graph with sharing $F^P = (V_F, ctrl_F, prnt_F) : k \to m$ and a concrete link graph $F^L = (V_F, E_F, ctrl_F, link_F) : X \to Y$. If $X = \epsilon$, then F is called ground. *We write the concrete bigraph with sharing as $F = (F^P, F^L)$.*

From PhD thesis [1]

Definition 8 (reaction rule). *A* reaction rule *is a pair*

$$\mathsf{R} = (R : m \to J, R' : m \to J),$$

sometimes written as $R \longrightarrow R'$, where R is the redex *and R' the* reactum*, and R is solid. The rule generates all the* ground reaction rules *(r, r'), where $r = (R \otimes \mathsf{id}_Y) \circ d$ and $r' = (R' \otimes \mathsf{id}_Y) \circ d$ for some discrete ground* parameter *$d : \epsilon \to \langle m, Y\rangle$. The* reaction relation *$\longrightarrow\!\!\!\triangleright_{\mathsf{R}}$ over ground bigraphs is defined by*

$$g \longrightarrow\!\!\!\triangleright_{\mathsf{R}} g' \text{ iff } g = Dr \text{ and } g' = Dr'$$

for some bigraph D and some ground reaction rule (r, r') generated from R.

Definition 9 (bigraphical reactive system (BRS)). *A* bigraphical reactive system *consists of a pair $(\mathcal{B}, \mathcal{R})$, where $\mathcal{B}$ is a set of bigraphs and $\mathcal{R}$ is a set of reaction rules defined over $\mathcal{B}$. It has a reaction relation*

$$\longrightarrow\!\!\!\triangleright_{\mathcal{R}} := \bigcup_{\mathsf{R} \in \mathcal{R}} \longrightarrow\!\!\!\triangleright_{\mathsf{R}},$$

which will be written $\longrightarrow\!\!\!\triangleright$ when $\mathcal{R}$ is understood.

1.1 Probabilistic bigraphs

Definition 10 (probabilistic reaction rule). *A* probabilistic reaction rule R *is a triple (R, R', p), sometimes written $R \xrightarrow{p} R'$, where (R, R') is a reaction rule and $p \in (0, 1]$ is a probability. Similarly to Definition 8, it generates a set of ground reaction rules of the form (r, r', p).*

Definition 11 (probabilistic bigraphical reactive system (PBRS)). *A* probabilistic bigraphical reactive system *consists of a pair $(\mathcal{B}, \mathcal{R})$, where $\mathcal{B}$ is a set of bigraphs and $\mathcal{R}$ is a set of probabilistic reaction rules defined over $\mathcal{B}$.*

Let g, g' be ground bigraphs, and $\{(r_i, r'_i, p_i)\}_{i=1}^n$ a set of ground probabilistic reaction rules, where for each r_i, there exists a bigraph D_i such that $g = D_i r_i$. Let $S = \{(r_i, r'_i, p_i) \mid g' = D_i r'_i\}$ (for the same D_i), and

$$s = \sum_{i=1}^{n} p_i.$$

Then the reaction relation is defined as

$$g \xrightarrow{p}\!\!\!\!\!\triangleright_{\mathcal{R}} g' \text{ iff } S \neq \emptyset,$$

where

$$p = \frac{1}{s} \sum_{(r,r',p') \in S} p'.$$

1.2 Nondeterministic bigraphs

Definition 12 (nondeterministic reaction rule). *Let A be a set of actions. A* nondeterministic reaction rule R *is a tuple (R, R', a, p), where (R, R', p) is a probabilistic reaction rule, and $a \in A$ is an action. We also define a* reaction reward function $r : A \to \mathbb{R}$ *that assigns a reward/cost to each action.*

Definition 13 (nondeterministic bigraphical reactive system (NBRS)). *A* nondeterministic bigraphical reactive system *consists of a pair $(\mathcal{B}, \mathcal{R})$, where $\mathcal{B}$ is a set of bigraphs and $\mathcal{R}$ is a set of nondeterministic reaction rules defined over $\mathcal{B}$.*

Let g, g' be ground bigraphs, $a \in A$ an action, and $\{(r_i, r_i', a, p_i)\}_{i=1}^n$ a set of ground nondeterministic reaction rules with action a, where for each r_i, there exists a bigraph D_i such that $g = D_i r_i$. Let $S = \{(r_i, r_i', a, p_i) \mid g' = D_i r_i'\}$ (for the same D_i), and

$$s = \sum_{i=1}^{n} p_i.$$

Then the reaction relation for action a is defined as

$$g \xrightarrow[r(a)]{p}\!\!\!\!\!\triangleright_{a} g' \text{ iff } S \neq \emptyset,$$

where

$$p = \frac{1}{s} \sum_{(r,r',a,p') \in S} p'.$$

BigraphER [3] supports defining predicates, i.e., bigraphs that are compared to every encountered state. We can associate a reward with each predicate, allowing us to assign rewards to states in a flexible and semantically meaningful way: the reward of a state is simply the sum of the rewards of all matching predicates (and 0 in case there are none).

Proposition 1. *Any MDP can be expressed as an NBRS.*

Proof. Obvious.

2 Implementation

Example 1. Consider an MDP $(S, \overline{s}, A, P, L)$, where $S = \{s_0, s_1, s_2, s_3\}$, $\overline{s} = s_0$, $A = \{a, b, c\}$, and P, L defined as follows:

$$\begin{aligned} P(s_0, a) &= [s_1 \mapsto 1], \\ P(s_1, b) &= [s_0 \mapsto 0.7, s_1 \mapsto 0.3], \\ P(s_1, c) &= [s_2 \mapsto 0.5, s_3 \mapsto 0.5], \\ P(s_2, a) &= [s_2 \mapsto 1], \\ P(s_3, a) &= [s_3 \mapsto 1], \end{aligned} \qquad \begin{aligned} L(s_0) &= \{initial\}, \\ L(s_1) &= \emptyset, \\ L(s_2) &= \{heads\}, \\ L(s_3) &= \{tails\}. \end{aligned}$$

Furthermore, equip it with a reward structure (ρ, ι), where $\rho(s_2) = 3$, $\iota(s_1, b) = 1$, and both functions are zero everywhere else.

The MDP can be represented as an NBRS with BigraphER code in Listing 1.1. Furthermore, the BigraphER's visualisation of the (automatically generated) transition system is in Fig. 1.

More specifically, reaction rule probabilities are represented as floating-point numbers inside the arrows (e.g. `-[0.7]->`), each action encompasses its reaction rules with **`action`** `actionName` and **`end`**. Rewards can be added to actions simply by inserting an integer enclosed in squared brackets after the action name (e.g. **`action`** `b[1]`). Lastly, predicate rewards have the same format, but are listed on the **`preds`** line of the **`begin nbrs`**—**`end`** block (e.g. **`preds`** `= {heads[3]};`).

Visually, each state is represented by a white ellipse (the starting state is highlighted with a thicker border), and each action (per state) is a grey ellipse. Text inside state ellipses lists all predicates that are satisfied by that state. Edges from actions to states are labelled with both the probability and the name of the reaction rule. Transitions from either a fully generated transition system or a simulation, labels from predicates, and state/transition rewards can be exported to the corresponding PRISM plain text formats.

3 Case study in autonomous agents

References

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3. Sevegnani, M., Calder, M.: BigraphER: Rewriting and analysis engine for bigraphs. In: Chaudhuri, S., Farzan, A. (eds.) Computer Aided Verification - 28th International Conference, CAV 2016, Toronto, ON, Canada, July 17-23, 2016, Proceedings, Part II. Lecture Notes in Computer Science, vol. 9780, pp. 494–501. Springer (2016). https://doi.org/10.1007/978-3-319-41540-6_27, https://doi.org/10.1007/978-3-319-41540-6_27

```
atomic ctrl S0 = 0;
atomic ctrl S1 = 0;
atomic ctrl S2 = 0;
atomic ctrl S3 = 0;

big initial = S0;
big heads = S2;
big tails = S3;

action a
  react toTemp = S0 -[1.]-> S1;
  react loopH = S2 -[1.]-> S2;
  react loopT = S3 -[1.]-> S3;
end

action b[1]
  react toInit = S1 -[0.7]-> S0
  react loop = S1 -[0.3]-> S1;
end

action c
  react toH = S1 -[0.5]-> S2;
  react toT = S1 -[0.5]-> S3;
end

begin nbrs
  init initial;
  rules = [ {toTemp, toInit,
    loop, toH, toT,
    loopH, loopT} ];
  preds = { initial, heads[3],
    tails };
end
```

Listing 1.1. BigraphER code.

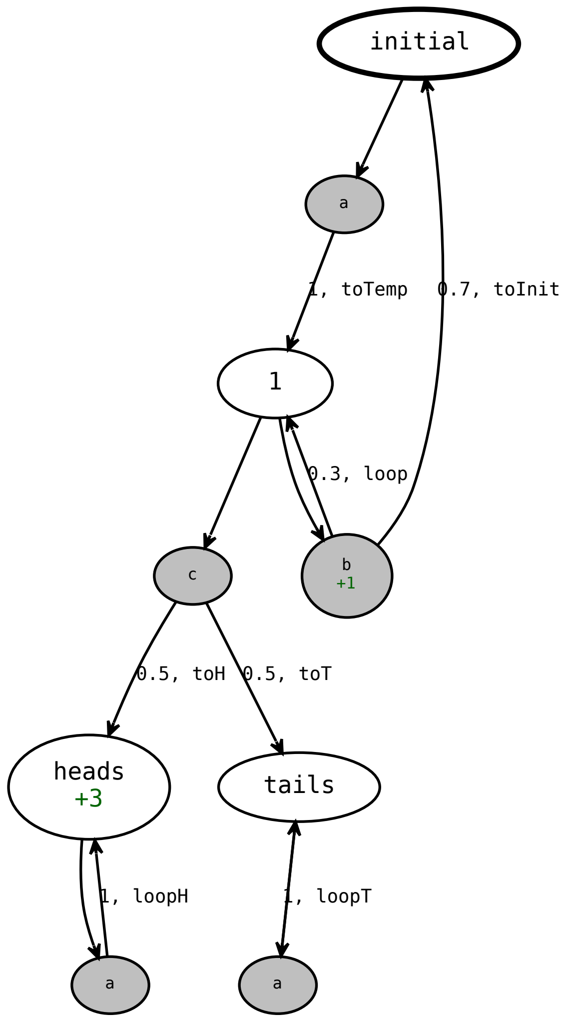

Fig. 1. The full transition system.

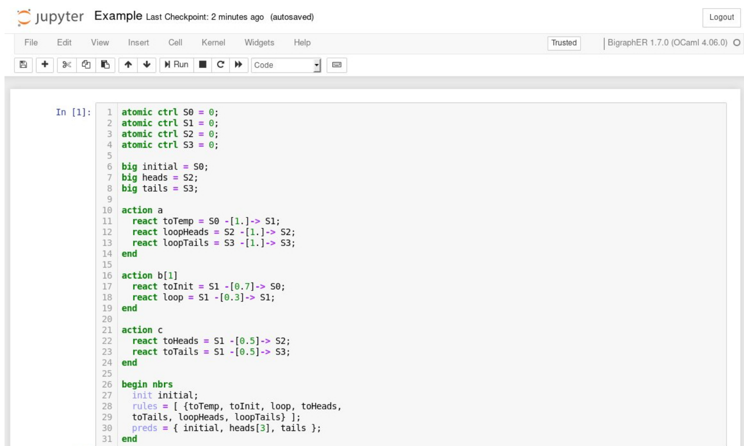

Fig. 2. The BigraphER Jupyter interface with syntax highlighting.

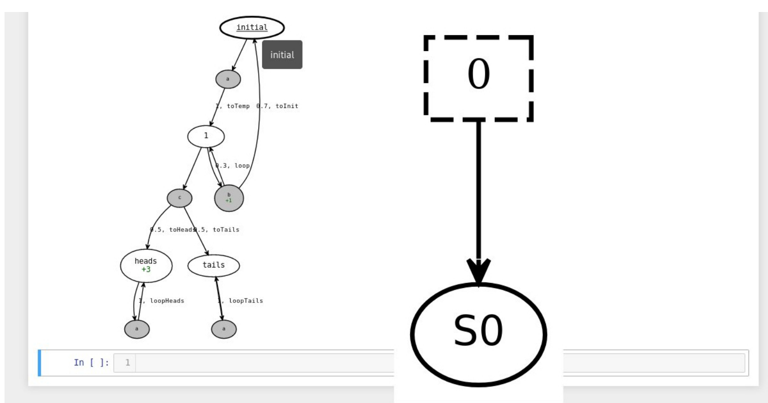

Fig. 3. ...

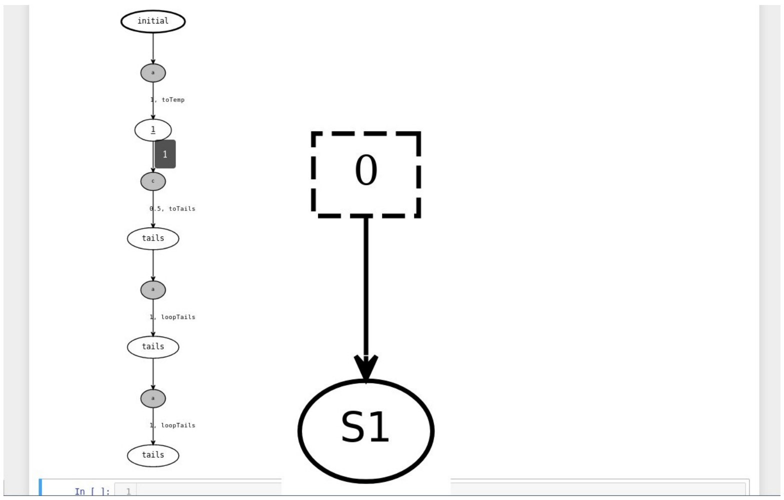

Fig. 4. ...

```
BigraphER: Bigraph Evaluator & Rewriting
========================================
Version:       1.7.0
Date:          Thu Aug 02 11:28:14 2018
Hostname:      tux
OS type:       Unix
Command line:  bigrapher validate -d jupyter-images/4 -f svg [4].big
Parsing model file [4].big ...
Type:          Nondeterministic BRS
Bindings:      14
# of rules:    7
Exporting declarations to jupyter-images/4 ...
Model file parsed correctly
```

Fig. 5. ...