Foundations for Inference in Probabilistic Relational Models

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Outline

1. Introduction
2. Equivalence
3. Random Programs
4. WMC
5. Future Work

Probabilistic Relational Models

Markov Logic Network (Richardson and Domingos 2006)

$$
\begin{array}{ll}
0.7 & \forall x \forall y \forall z\, \mathtt{Friends}(x, y) \land \mathtt{Friends}(y, z) \Rightarrow \mathtt{Friends}(x, z) \\
2.3 & \forall x\, \neg\exists y\, \mathtt{Friends}(x, y) \Rightarrow \mathtt{Smokes}(x) \\
1.5 & \forall x\, \mathtt{Smokes}(x) \Rightarrow \mathtt{Cancer}(x) \\
1.1 & \forall x \forall y\, \mathtt{Friends}(x, y) \Rightarrow (\mathtt{Smokes}(x) \Leftrightarrow \mathtt{Smokes}(y))
\end{array}
$$

Probabilistic Relational Models

ProbLog (De Raedt, Kimmig and Toivonen 2007)

$1.0 :: \texttt{likes}(X, Y) \texttt{:-} \texttt{friendOf}(X, Y).$

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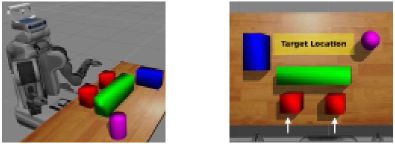
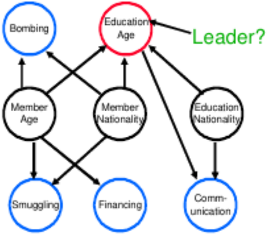
- What do these models have in common?
- When performing inference...
  - do we have to consider every detail?
  - what makes inference challenging?
  - can we do any better?
- How can we learn PRMs from data?

Applications



Moldovan and De Raedt 2014



Delaney et al. 2010

Predicate	Instance	Source(s)
ethnicGroup	Cubans	CSEAL
arthropod	spruce beetles	CPL, CSEAL
female	Kate Mara	CPL, CMC
sport	BMX bicycling	CSEAL, CMC
profession	legal assistants	CPL
magazine	Thrasher	CPL
bird	Buff-throated Warbler	CSEAL
river	Fording River	CPL, CMC
mediaType	chemistry books	CPL, CMC
cityInState	(troy, Michigan)	CSEAL
musicArtistGenre	(Nirvana, Grunge)	CPL
tvStationInCity	(WLS-TV, Chicago)	CPL, CSEAL
sportUsesEquip	(soccer, balls)	CPL
athleteInLeague	(Dan Fouts, NFL)	RL
starredIn	(Will Smith, Seven Pounds)	CPL
productType	(Acrobat Reader, FILE)	CPL
athletePlaysSport	(scott shields, baseball)	RL
cityInCountry	(Dublin Airport, Ireland)	CPL

Table 1: Example beliefs promoted by NELL.

Carlson et al. 2010

```
is_malignant(Case):-
        biopsyProcedure(Case,usCore),
        changes_Sizeinc(Case,missing),
        feature_shape(Case).
is_malignant(Case):-
        assoFinding(Case,asymmetry),
        breastDensity(Case,scatteredFDensities),
        vacuumAssisted(Case,yes).
is_malignant(Case):-
        needleGauge(Case,9),
        offset(Case,14),
        vacuumAssisted(Case,yes).
```

Côrte-Real, Dutra and Rocha 2017

Outline

$$\mathtt{Husband}(\mathit{joffrey}, \mathit{margaery})$$
$$\mathtt{Husband}(\mathit{tommen}, \mathit{margaery})$$
$$\mathtt{Husband}(\mathit{renly}, \mathit{margaery})$$
$$\mathtt{Parent}(\mathit{cersei}, \mathit{joffrey})$$
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$$\mathtt{Female}(X) \mathbin{:\!-} \mathtt{Husband}(\mathit{joffrey}, X).$$
$$\mathtt{Female}(X) \mathbin{:\!-} \mathtt{Parent}(X, \mathit{joffrey}).$$
$$\mathtt{Female}(X) \mathbin{:\!-} \mathtt{Parent}(\mathit{cersei}, X), \neg\mathtt{Husband}(X, \mathit{margaery}).$$

Main Results

Definition (Equivalence)

Two n-tuples of constants a and b are equivalent if

$$(\mathtt{P} \circ \rho)(a) = (\mathtt{P} \circ \rho)(b)$$

for all atoms $\mathtt{P} \circ \rho$ acting on n variables.

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Theorem

There is a logic program $\mathcal{L} \colon \mathcal{KB}(P_1, C) \to \mathcal{KB}(P_2, C)$ *such that* $\mathcal{L}(\Delta_1) = \Delta_2$ *if and only if* $\sim_{\Delta_2}$ *is coarser than* $\sim_{\Delta_1}$.

Outline

What Characterises a Program?

$1.0 :: \texttt{likes}(X, Y) \mathbin{:\!-} \texttt{friendOf}(X, Y).$

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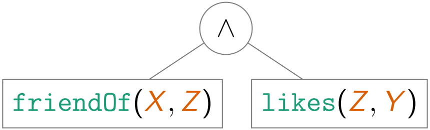

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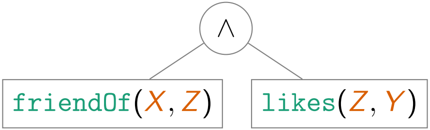
Also:

- cyclicity
- (conditional) independence
- required subformulas



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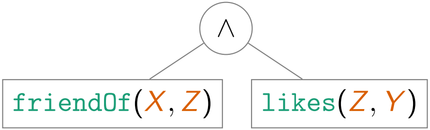
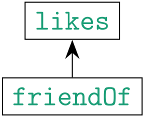
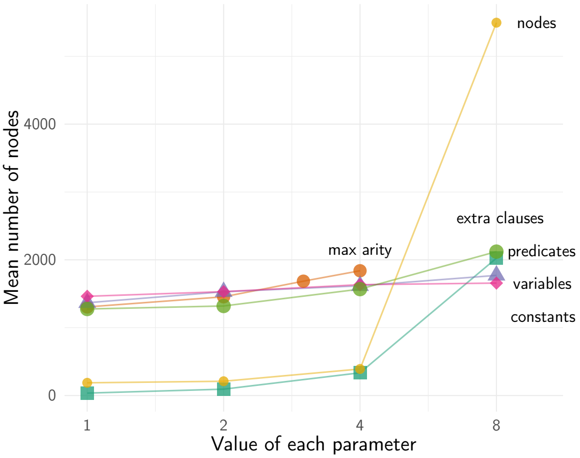




What Programs Are Hard to Generate?



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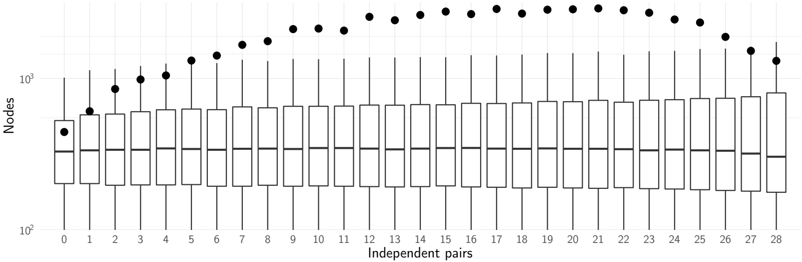

Outline

Defining WMC

Definition

Let $\mathbf{B}$ be an atomic Boolean algebra. Let $L \subset \mathbf{B}$ be such that every atom m can be uniquely expressed as $m = \bigwedge L'$ for some $L' \subseteq L$, and let $w \colon L \to \mathbb{R}_{\geq 0}$ be arbitrary. The weighted model count $\mathrm{WMC}_w \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}_w(x) = \begin{cases} 0 & \text{if } x = 0 \\ \prod_{l \in L'} w(l) & \text{if } x = \bigwedge L' \text{ is an atom} \\ \sum_{\text{atoms } a \leq x} \mathrm{WMC}_w(a) & \text{otherwise} \end{cases}$$

for any $x \in \mathbf{B}$.

WMC Requires Independent Literals

Theorem

Let $\mathbf{B}$ *be a free Boolean algebra over* $\{l_i\}_{i=1}^n$ *with measure*

$$m\colon \mathbf{B} \to \mathbb{R}_{\geq 0},$$

and let

$$L = \{l_i\}_{i=1}^n \cup \{\neg l_i\}_{i=1}^n.$$

Then there exists a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ *such that* $m = \mathrm{WMC}_w$ *if and only if*

$$m(l \wedge l') = m(l)m(l')$$

for all distinct $l, l' \in L$ *such that* $l \neq \neg l'$.

Extending the Algebra

$$
\begin{tikzcd}
\mathbb{R}_{\geq 0} & & \\
\mathbf{B} \arrow[u, "m"] \arrow[r, hook, "\iota"] & \mathbf{B}' \arrow[ul, "m'"'] & \\
L \arrow[u, hook] \arrow[r, hook] & L' \arrow[u, hook] \arrow[r, "w"] & \mathbb{R}_{\geq 0}
\end{tikzcd}
$$

How Can This Benefit Inference?

Theorem (Sikorski 1969)

If $\mathbf{B} = \mathcal{F}\{a\} + \mathcal{F}\{b\}$, *then* $\Pr(a \wedge b) = \Pr(a)\Pr(b)$.

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If $\mathbf{B} = \mathcal{F}\{a\} + \mathcal{F}\{b\}$, *then* $\Pr(a \land b) = \Pr(a)\Pr(b)$.

Conjecture

If $\mathbf{B} = \mathcal{F}\{a\} +_{\mathcal{F}\{c\}} \mathcal{F}\{b\}$, *then* $\Pr(a \land b \land c) = \Pr(a \land c)\Pr(b \land c)$.

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Using coproducts and pushouts, one can encode a Bayesian network into WMC with fewer literals and a shorter theory than before.

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Conjecture

A #SAT algorithm can be adapted without sacrificing efficiency.

Outline

Abstraction: Before

$$
\begin{tikzcd}
& \boxed{\text{Theory } \Delta_h} & \boxed{\text{Constants}} \arrow[d, "\boxed{\text{Predicates}}"'] & & \boxed{\text{Constants}} \arrow[d, "\boxed{\text{Predicates}}"] & \boxed{\text{Theory } \Delta_l} & \\
& & \text{Atoms} \arrow[d, hook] \arrow[ddrr, "\text{Refinement}"] & & \text{Atoms} \arrow[d, hook] & & \\
\mathbb{R}_{\geq 0} & & \text{Literals} \arrow[ll, "\boxed{w_h}"'] \arrow[d, hook] & & \text{Literals} \arrow[rr, "\boxed{w_l}"] \arrow[d, hook] & & \mathbb{R}_{\geq 0} \\
& & \text{Formulas} \arrow[loop left, "\wedge,\vee,\neg"] \arrow[dr, "\Pr"'] & & \text{Formulas} \arrow[loop right, "\wedge,\vee,\neg"] \arrow[dl, "\Pr"] & & \\
& & & {[0,1]} & & &
\end{tikzcd}
$$

Abstraction: After

$$
\begin{tikzcd}
S_h \arrow[d, hook] \arrow[dr] & S_l \arrow[d, hook] \\
\mathbf{B}_h \arrow[r, dashed] \arrow[d] & \mathbf{B}_l \arrow[d] \\
\mathbf{B}_h/(\neg\Delta_h) \arrow[r, dashed] & \mathbf{B}_l/(\neg\Delta_l)
\end{tikzcd}
$$

Reflections & Future Work

- The equivalence paper needs significant rework.
- The program generation paper could be improved by demonstrating the model's usefulness.
- The WMC paper needs successful experimental results.
- Understanding PRMs as algebras is promising—in one way or another.