Generalising Weighted Model Counting

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Weighted Model Counting

Example

We have a biased coin that has a probability $p \in [0, 1]$ of landing heads. What is the probability that it lands heads at least once if we toss it three times?

In Propositional Logic...

- Formula: $x_1 \lor x_2 \lor x_3$
- Weights: $w(x_i) = p$, $w(\neg x_i) = 1 - p$ for $i = 1, 2, 3$
- Models: $\mathcal{P}(\{ x_1, x_2, x_3 \}) \setminus \{ \emptyset \}$

In First-Order Logic...

- Formula: $\exists x \in \{ 1, 2, 3 \}.\ P(x)$
- Weights: $w(P) = p$, $w(\neg P) = 1 - p$
- Models: $\mathcal{P}(\{ P(1), P(2), P(3) \}) \setminus \{ \emptyset \}$

Significance

Applications

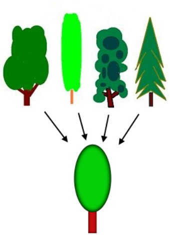
- Probabilistic inference: graphical models, statistical relational models, probabilistic programming
- Neural-symbolic artificial intelligence
- Bioinformatics
- Robotics
- Natural language processing
- Enumerative combinatorics

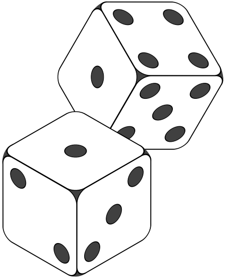
Impact

- Suitable WMC algorithm
- Appropriate input format
- Lifted reasoning
- Expressive data structures

- $> 100\times$ speedup
- provable tractability

Contributions





Generalising Representations

- Beyond weights on literals
- Circuits for recursion

Random-Instance Experiments

- Application-specific parameters
  - ProbLog predicates, arities
- Parameters of hardness
  - density, primal treewidth

Generalising Representations

WMC and Measures on Boolean Algebras

Definition

A measure is a function $\mu\colon \mathcal{P}(\mathcal{P}(X)) \to \mathbb{R}_{\geq 0}$ such that:

- $\mu(\bot) = 0$;
- $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x \wedge y = \bot$.

Observation

WMC corresponds to the process of calculating the value of $\mu(x)$ for some $x \in \mathcal{P}(\mathcal{P}(X))$.

Observation

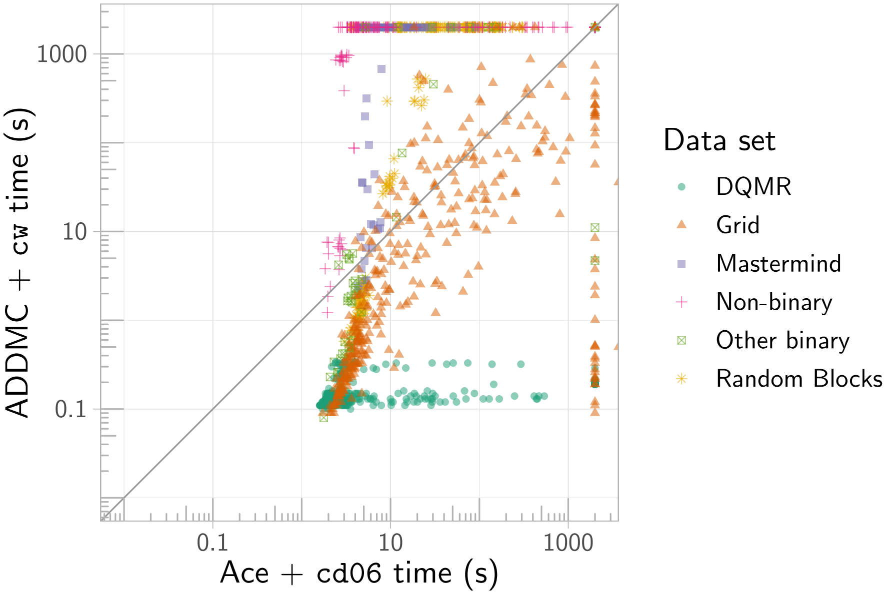
Classical WMC is only able to evaluate factorable measures (c.f., a collection of mutually independent random variables).

Theorem (Informal Version)

It is always possible to add more variables to turn a non-factorable measure into a factorable measure.

However, that is not necessarily a good idea!

Experiments with Bayesian Networks



Transforming Known WMC Encodings into PBP

For any propositional formula ϕ over a set of variables X and $p, q \in \mathbb{R}$, let $[\phi]^p_q \colon 2^X \to \mathbb{R}$ be the pseudo-Boolean function defined as

$$[\phi]^p_q(Y) \coloneqq \begin{cases} p & \text{if } Y \models \phi \\ q & \text{otherwise} \end{cases}$$

for any $Y \subseteq X$.

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Example

Clauses
$\neg x \Rightarrow p$
$p \Rightarrow \neg x$
$x \Rightarrow q$
$q \Rightarrow x$
$\neg x$

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Example

Clauses	In CNF
$\neg x \Rightarrow p$	$x \vee p$
$p \Rightarrow \neg x$	$\neg x \vee \neg p$
$x \Rightarrow q$	$\neg x \vee q$
$q \Rightarrow x$	$x \vee \neg q$
$\neg x$	$\neg x$

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Example

Clauses	In CNF	Pseudo-Boolean Functions
$\neg x \Rightarrow p$	$x \vee p$	$[\neg x]^{0.2}_1$
$p \Rightarrow \neg x$	$\neg x \vee \neg p$	
$x \Rightarrow q$	$\neg x \vee q$	$[x]^{0.8}_1$
$q \Rightarrow x$	$x \vee \neg q$	
$\neg x$	$\neg x$	$[\neg x]^1_0$

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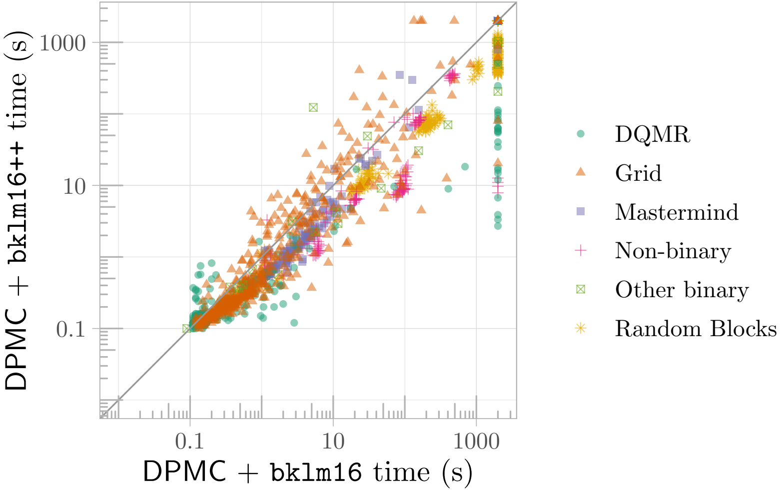
Example

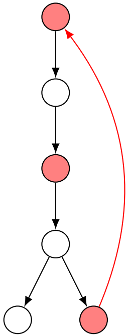
Clauses	In CNF	Pseudo-Boolean Functions	
$\neg x \Rightarrow p$	$x \vee p$	$[\neg x]^{0.2}_{1}$	
$p \Rightarrow \neg x$	$\neg x \vee \neg p$		$[x]^{0.8}_{0.2}$
$x \Rightarrow q$	$\neg x \vee q$	$[x]^{0.8}_{1}$	
$q \Rightarrow x$	$x \vee \neg q$		
$\neg x$	$\neg x$	$[\neg x]^{1}_{0}$	$[\neg x]^{1}_{0}$

Some Instances Become Tractable as a Result



First-Order Logic and Recursive Computations

Example (Counting $P\colon M \to N$ Injections)



Input Formula

$$\forall x \in M.\ \exists y \in N.\ P(x, y)$$
$$\forall x \in M.\ \forall y, z \in N.\ P(x, y) \land P(x, z) \Rightarrow y = z$$
$$\forall w, x \in M.\ \forall y \in N.\ P(w, y) \land P(x, y) \Rightarrow w = x$$

Recursive Solution

$$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n-1) + m \cdot f(m-1, n-1) & \text{otherwise.} \end{cases}$$

Resulting Improvements to Counting Functions

Let M and N be two sets with cardinalities $|M| = m$ and $|N| = n$. The new compilation rules enable FORCLIFT to efficiently count $M \to N$ functions such as:

- injections in $\Theta(mn)$ time
  - best: $\Theta(m)$
- partial injections in $\Theta(mn)$ time
  - best: $\Theta(\min\{ m, n \}^2)$
- bijections in $\Theta(m)$ time
  - optimal!

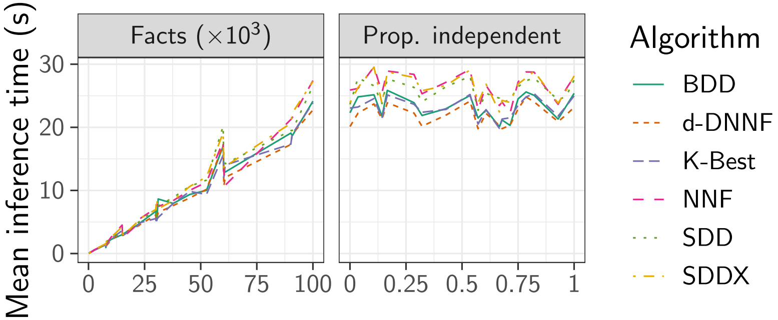
Random-Instance Experiments

A Constraint Model for (Probabilistic) Logic Programs

```
0.2::stress(P) :- person(P).
0.3::influences(P1, P2) :- friend(P1, P2).
0.1::cancer_spont(P) :- person(P).
0.3::cancer_smoke(P) :- person(P).
     smokes(X) :- stress(X).
     smokes(X) :- smokes(Y), influences(Y, X).
     cancer(P) :- cancer_spont(P).
     cancer(P) :- smokes(P), cancer_smoke(P).
     person(mary).
     person(albert).
     friend(albert, mary).
```

- predicates, arities
- variables
- constants
- probabilities
- length
- complexity

ProbLog Inference Algorithms on Random Instances



Random WMC Instances

Key Idea

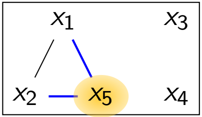
Parameter $\rho \in [0, 1]$ biases the probability distribution towards adding variables that would introduce fewer new edges in the primal graph.

Example

Partially-constructed formula:

$$(\neg x_5 \vee x_2 \vee x_1) \wedge (x_5 \vee ? \vee ?).$$

Its primal graph:



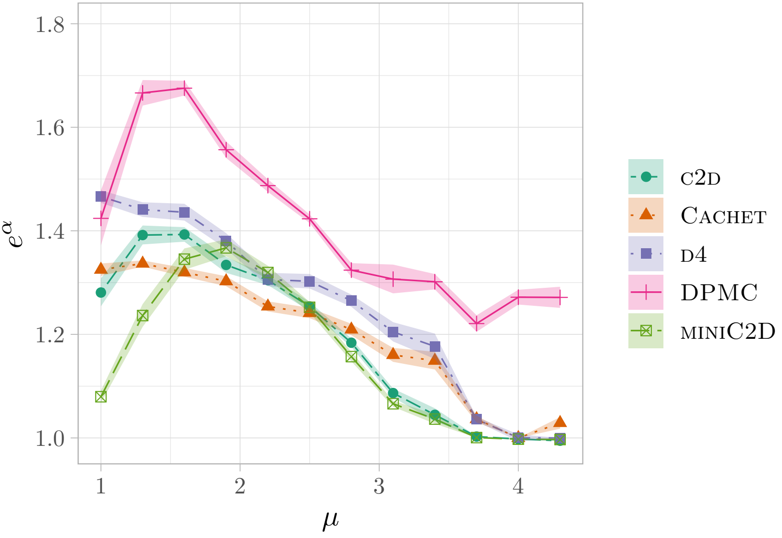
Base probability of each variable being chosen:

$$\frac{1-\rho}{4}.$$

Both x_1 and x_2 get a bonus probability of $\rho/2$ for each being the endpoint of one out of the two neighbourhood edges.

How WMC Algorithms Scale w.r.t. Primal Treewidth

We fit the model $\ln t \sim \alpha w + \beta$, i.e., $t \sim e^{\beta}(e^{\alpha})^{w}$, where t is runtime, and w is primal treewidth.



Summary

What Have We Learned?

- Pseudo-Boolean functions as an alternative to literal weights
- Cycles in graphs that encode recursive calls
- WMC is not always the bottleneck in probabilistic inference
- WMC algorithms based on algebraic decision diagrams are fundamentally different:
  - they can supports non-literal weights
  - their running time depends on the numerical values of weights
  - they peak at higher density
  - they scale worse w.r.t. primal treewidth

Future Directions

- PBP: new encodings, kernelization, pseudo-Boolean solvers
- WFOMC: full automation and new applications (e.g., in AI)