Towards Practical First-Order Model Counting

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Motivation

Example Setting

- Let Δ be a set of cardinality n
- Suppose we want to count all $P \subseteq \Delta^2$ (as a function of n) that are:
  - functions,
  - bijections,
  - partial orders,
  - symmetric,
  - transitive,
  - etc.

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  - etc.

- Propositional model counting (#SAT) is #P-complete
- But many of these counting problems have efficient solutions
- And we can find them using first-order model counting
  - i.e., reasoning about sets, subsets, and arbitrary elements without grounding them

More Formally: What Is the Input?

Example Input Sentence

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \land P(x, z) \Rightarrow y = z$$

Many-Sorted Function-Free First-Order Logic with Equality

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$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \Rightarrow y = z$$

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- Any number of variables and constants
- $\exists$ and $\forall$ quantifiers can be nested arbitrarily deeply
- All domains are finite
  - Solutions are functions that take domain sizes as input

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First-Order Model Counting (FOMC)

- Each predicate acts like a subset
  - of a domain or product of domains
- Goal: count combinations of subsets that satisfy the sentence

Previous Work: Crane (Dilkas and Belle 2023)

- A knowledge compilation approach:
  - Sentences $\rightarrow$ labelled digraphs $\rightarrow$ function-defining equations
- Capable of constructing recursive solutions
- Two variants: greedy search and breadth-first search (BFS)

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An Example Solution for Counting Bijections

$$f(m, n) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} g(l, m),$$
$$g(l, m) = g(l-1, m) + mg(l-1, m-1)$$

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Issues We Are Going to Address

Completeness: recursive functions (like g) have no base cases

Usability: how do I compute, e.g., $f(7,7)$?

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3. ($\Rightarrow$) Identify a sufficient set of base cases
   - e.g., $\{ g(0, m), g(l, 0) \}$

Workflow (2/2)

4. For each base case:

$g(0, m)$ $g(l, 0)$

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$$g(0, m) \quad\quad\quad\quad g(l, 0)$$

4.1. ($\Rightarrow$) Construct the corr. sentence

$$\forall y \in \Delta.\ S(y) \vee \neg S(y) \quad\quad\quad\quad \top$$

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4.2. Recurse

$$\begin{tikzcd}
g(0, m) \arrow[d] & g(l, 0) \arrow[d] \\
\forall y \in \Delta.\ S(y) \vee \neg S(y) \arrow[d] & \top \arrow[d] \\
g(0, m) = ??? & g(l, 0) = ???
\end{tikzcd}$$

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& \mathcal{E} \arrow[d, "\text{5. Compile to C++}"] & \\
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$$\mathcal{E}$$

$$\mathcal{E} \xrightarrow{\text{5. Compile to C++}} \text{C++ code}$$

$$\text{Domain sizes} \longrightarrow \boxed{\text{C++ code}} \longrightarrow \text{Model count}$$

Finding (a Sufficient Set of) Base Cases

Outline

1. For every function call:
   1.1 For every argument of the form $var - const$:
      1.1.1 Replace the signature parameter with $0, 1, \ldots, const - 1$
   1.2 For every argument of the form $const$:
      1.2.1 Replace the corresponding signature parameter with $const$

Example

The signature of g is $g(l, m)$.

$$\begin{tikzcd}
\text{Function calls:} & g(l-1, m) \arrow[d] & g(l-1, m-1) \arrow[dl] \arrow[d] \\
\text{Base cases:} & g(0, m) & g(l, 0)
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Theorem

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Proof (hints).

- There exists a topological ordering of functions
- All function calls follow the structure from the previous slide
- Some common-sense assumptions about the evaluation order and previous work

□

From a Base Case to a Sentence

From Previous Work (Dilkas and Belle 2023)

- CRANE associates each function f with a sentence ϕ such that CRANE(ϕ) produces the definition of f
- There is a bijection between the parameters of f and the domains of ϕ

Example

- Base case: $g(0, m)$

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Example

- Base case: $g(\overset{\Gamma}{\overset{\updownarrow}{0}}, \overset{\Delta}{\overset{\updownarrow}{m}})$

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- Base case: $g(\overset{\Gamma}{\overset{\updownarrow}{0}}, \overset{\Delta}{\overset{\updownarrow}{m}})$
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- $g(0, \dots)$ means we need to simplify (1) by assuming $|\Gamma| = 0$
- Result: $\forall y \in \Delta.\ S(y) \lor \neg S(y)$ (Smoothing)

An Outline of the Resulting C++ Program

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Function $g_{0,m}(m)$**:** …
Function $g_{l,0}(l)$**:** …
Function $g(l, m)$**:**
    if $(l, m) \in \text{Cache}_g$ **then return** $\text{Cache}_g(l, m)$;
    if $l = 0$ **then return** $g_{0,m}(m)$;
    if $m = 0$ **then return** $g_{l,0}(l)$;
    $r \leftarrow g(l-1, m) + mg(l-1, m-1)$;
    $\text{Cache}_g(l, m) \leftarrow r$;
    return r;
Function $f(m, n)$**:** …
Function `Main`**:**
    $(m, n) \leftarrow$ `ParseCommandLineArguments()`;
    return $f(m, n)$;

Benchmarks

- Friends & Smokers

$$(\forall x, y \in \Delta.\ S(x) \land F(x, y) \to S(y)) \land (\forall x \in \Delta.\ S(x) \to C(x))$$

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- Functions

$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \land \\ (\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \land P(x, z) \to y = z)$$

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- Bijections

$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \land$$
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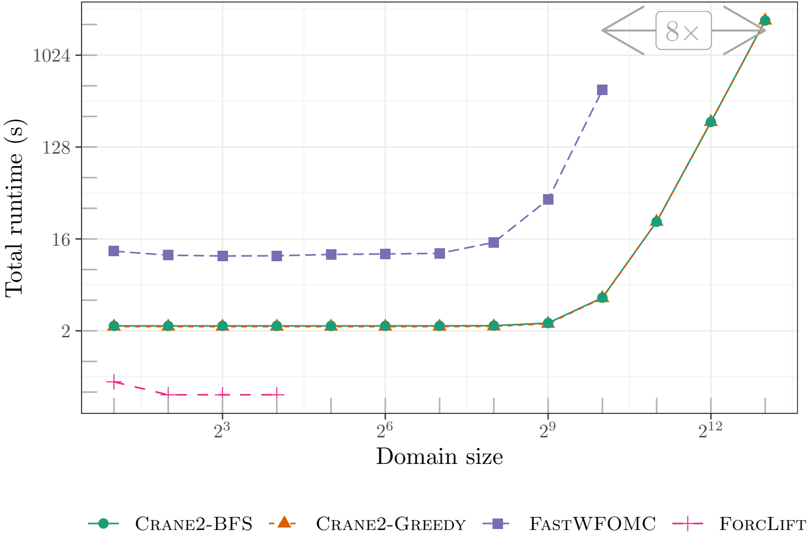
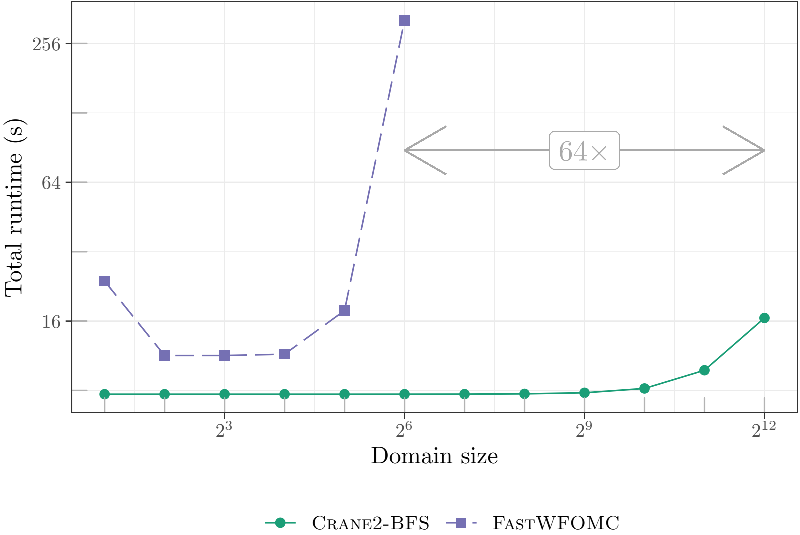
$$(\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \land P(x, z) \to y = z) \land$$
$$(\forall x, z \in \Gamma.\ \forall y \in \Delta.\ P(x, y) \land P(z, y) \to x = z)$$

Friends & Smokers

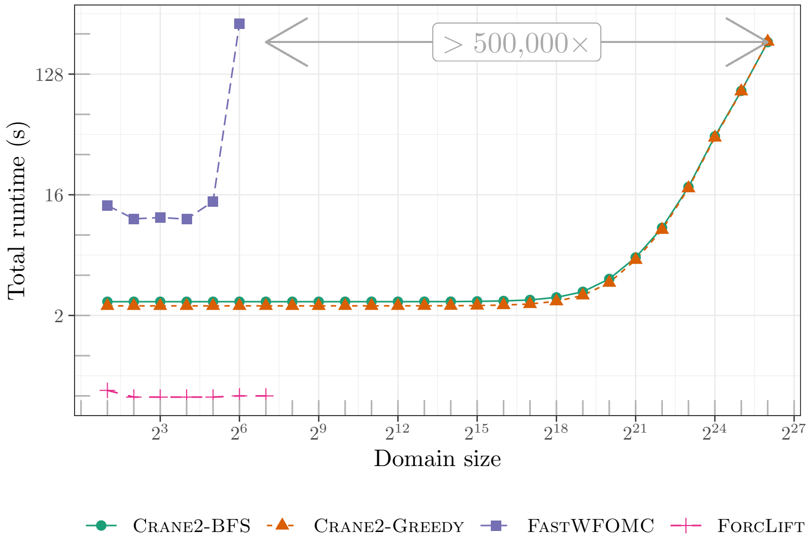


Bijections



256
64
16
Total runtime (s)
64×
2^3
2^6
2^9
2^{12}
Domain size
CRANE2-BFS
FASTWFOMC

Functions



> 500,000×
Total runtime (s)
128
16
2
2^3
2^6
2^9
2^12
2^15
2^18
2^21
2^24
2^27
Domain size
Crane2-BFS
Crane2-Greedy
FastWFOMC
ForcLift

Summary & Future Work

Contributions

Completeness: recursive solutions now come with base cases

Usability: compilation to C++ programs

Scalability compared to other FOMC algorithms

- 8 to 500,000 times higher domain sizes

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Future Work

- Support for weighted counting (trivial)
- Experiments on a large set of benchmarks
- Completeness for fragments of first-order logic
- Fine-grained complexity