Towards Practical First-Order Model Counting

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SAT 2025

Motivation

Example Setting

- Let Δ be a set of cardinality $n \in \mathbb{N}_0$
- Suppose we want to count all $P \subseteq \Delta^2$ (as a function of n) that are:
  - functions,
  - bijections,
  - partial orders,
  - symmetric,
  - transitive,
  - etc.

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  - etc.

- Propositional model counting (#SAT) is #P-complete
- But many of these counting problems have efficient solutions
- And we can find them using first-order model counting
  - i.e., reasoning about sets, subsets, and arbitrary elements without grounding them

More Formally: What Is the Input?

Example Input Sentence

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \land P(x, z) \Rightarrow y = z$$

Many-Sorted Function-Free First-Order Logic with Equality

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- Any number of variables and constants
- $\exists$ and $\forall$ quantifiers can be nested arbitrarily deeply
- All domains are finite
  - Solutions are functions that take domain sizes as input
- Of course, not all valid inputs have tractable solutions

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First-Order Model Counting (FOMC)

- Each predicate acts like a subset
  - of a domain or product of domains
- Goal: count combinations of subsets that satisfy the sentence

Exact Algorithms for FOMC

Predecessors of This Work

- ForcLift (Van den Broeck et al. 2011)
  - knowledge compilation to FO d-DNNF
- Crane (Dilkas and Belle 2023)
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  - extends ForcLift with support for:
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Other Approaches

- L2C (Kazemi and Poole 2016)
  - knowledge compilation to C++ code
- Alchemy (Gogate and Domingos 2016)
  - DPLL-style search
- FastWFOMC (van Bremen and Kuželka 2021)
  - based on cell enumeration

Previous Work: Crane (Dilkas and Belle 2023)

- A knowledge compilation approach:
  - Sentences $\rightarrow$ labelled digraphs $\rightarrow$ function-defining equations
- Two variants: greedy search and breadth-first search (BFS)

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An Example Solution for Counting Bijections

$$f(m, n) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} g(l, m),$$

$$g(l, m) = g(l-1, m) + mg(l-1, m-1)$$

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Issues We Are Going to Address

Completeness: recursive functions (like g) have no base cases

Usability: how do I compute, e.g., $f(7,7)$?

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3. ($\Rightarrow$) Identify a sufficient set of base cases
   - e.g., $\{\, g(0,m), g(l,0) \,\}$

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\big\downarrow & \text{4.2. Recurse} & \big\downarrow \\
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\begin{tikzcd}
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& \mathcal{E} &
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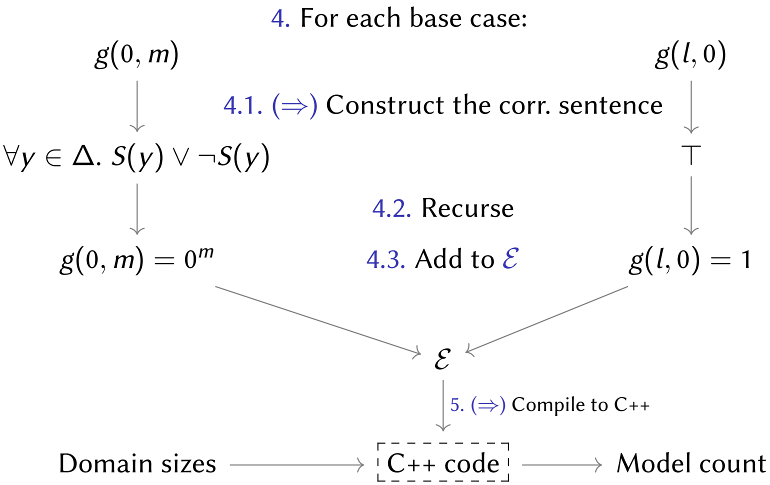
$g(0, m) = 0^m$ $\qquad$ 4.3. Add to $\mathcal{E}$ $\qquad$ $g(l, 0) = 1$

$\mathcal{E}$

5. ($\Rightarrow$) Compile to C++

C++ code

The Workflow of CRANE2 (2/2)



Outline

1. For every function call:
    - 1.1 For every argument of the form $var - const$:
        - 1.1.1 Replace the signature parameter with $0, 1, \ldots, const - 1$
    - 1.2 For every argument of the form $const$:
        - 1.2.1 Replace the corresponding signature parameter with $const$

Example

The signature of g is $g(l, m)$.

$$\begin{tikzcd}[column sep=large, row sep=large]
\text{Function calls:} & g(l - 1, m) \arrow[d] & g(l - 1, m - 1) \arrow[dl] \arrow[d] \\
\text{Base cases:} & g(0, m) & g(l, 0)
\end{tikzcd}$$

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Theorem

The maximum recursion depth of the compilation algorithm is upper bounded by the number of domains.

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Theorem

The evaluation of a recursive function always terminates.

Proof (hints).

- There exists a topological ordering of functions
- All function calls follow the structure from the previous slide
- Some common-sense assumptions about the evaluation order and previous work

□

From a Base Case to a Sentence

From Previous Work (Dilkas and Belle 2023)

- CRANE associates each function f with a sentence ϕ such that CRANE(ϕ) produces the definition of f
- There is a bijection between the parameters of the main function f and the domains of the input sentence ϕ

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- Base case: $g(0, m)$

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- Base case: $g(\overset{\Gamma}{\overset{\updownarrow}{0}}, \overset{\Delta}{\overset{\updownarrow}{m}})$

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- Base case: $g(\overset{\Gamma}{\updownarrow}{0}, \overset{\Delta}{\updownarrow}{m})$
- Part of the sentence of g:

$$\forall x \in \Gamma.\ \forall y \in \Delta.\ S(y) \lor \neg P(x, y) \tag{1}$$

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- $g(0, \dots)$ means we need to simplify (1) by assuming $|\Gamma| = 0$
- Result: $\forall y \in \Delta.\ S(y) \lor \neg S(y)$ (Smoothing)

An Outline of the Resulting C++ Program

initialise $\text{Cache}_{g(0,m)}$, $\text{Cache}_{g(l,0)}$, Cache_{g}, and Cache_{f};

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Function $g_{l,0}(l)$**:** …

Function $g(l, m)$**:**

- **if** $(l, m) \in \mathrm{Cache}_g$ **then return** $\mathrm{Cache}_g(l, m)$;
- **if** $l = 0$ **then return** $g_{0,m}(m)$;
- **if** $m = 0$ **then return** $g_{l,0}(l)$;
- $r \leftarrow g(l-1, m) + mg(l-1, m-1)$;
- $\mathrm{Cache}_g(l, m) \leftarrow r$;
- **return** r;

Function $f(m, n)$**:** …

Function `Main`**:**

- $(m, n) \leftarrow$ `ParseCommandLineArguments()`;
- **return** $f(m, n)$;

Benchmarks

- Friends & Smokers

$$(\forall x, y \in \Delta.\ S(x) \land F(x, y) \to S(y)) \land (\forall x \in \Delta.\ S(x) \to C(x))$$

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- Functions

$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \land$$
$$(\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \land P(x, z) \to y = z)$$

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- Friends & Smokers

$$(\forall x, y \in \Delta.\ S(x) \wedge F(x, y) \to S(y)) \wedge (\forall x \in \Delta.\ S(x) \to C(x))$$

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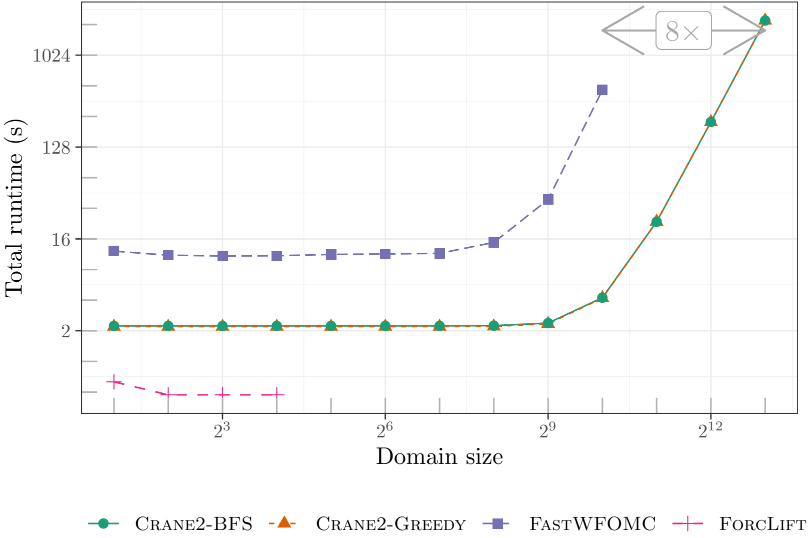
$$\begin{gathered}(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge \\ (\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \to y = z)\end{gathered}$$

- Bijections

$$\begin{gathered}(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge \\ (\forall y \in \Delta.\ \exists x \in \Gamma.\ P(x, y)) \wedge \\ (\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \to y = z) \wedge \\ (\forall x, z \in \Gamma.\ \forall y \in \Delta.\ P(x, y) \wedge P(z, y) \to x = z)\end{gathered}$$

Friends & Smokers



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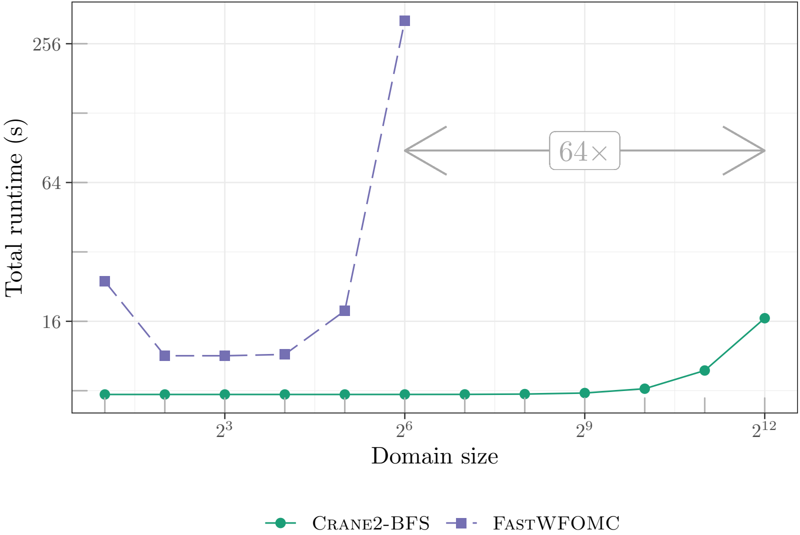
Total runtime (s)
256
64
16
64×
2^3
2^6
2^9
2^{12}
Domain size
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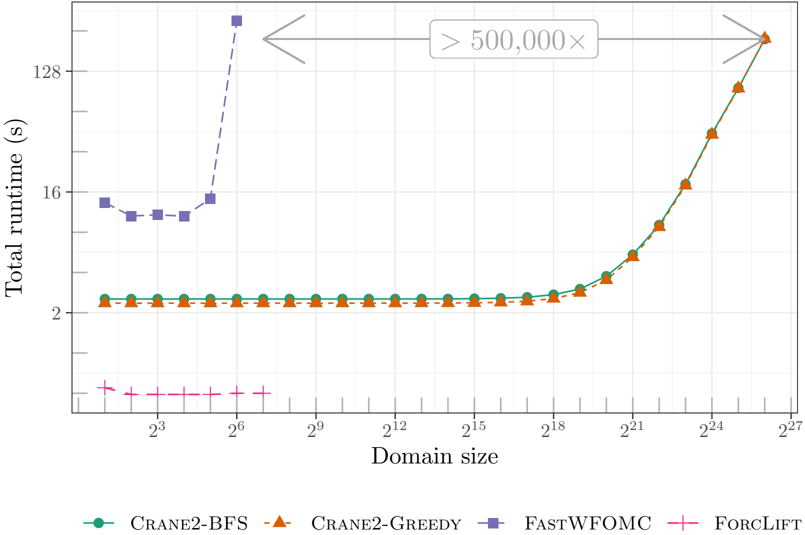

> 500,000×
Total runtime (s)
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2^6
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2^{15}
2^{18}
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2^{24}
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Summary & Future Work

Contributions

Completeness: recursive solutions now come with base cases

Usability: compilation to C++ programs

Scalability compared to other FOMC algorithms

- 8 to 500,000 times higher domain sizes

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Future Work

- Support for weighted counting (trivial)
- Experiments on a large set of benchmarks
- Completeness for fragments of first-order logic
- Fine-grained complexity