Towards Practical First-Order Model Counting

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Motivation

Example Setting

- Let Δ be a set of cardinality n
- Suppose we want to count all $P \subseteq \Delta^2$ (as a function of n) that are:
  - functions,
  - bijections,
  - partial orders,
  - symmetric,
  - transitive,
  - etc.

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  - etc.

- Propositional model counting (#SAT) is #P-complete
- But many of these counting problems have efficient solutions
- And we can find them using first-order model counting
  - i.e., reasoning about sets, subsets, and arbitrary elements without grounding them

More Formally: What Is the Input?

Example Input Sentence

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \land P(x, z) \Rightarrow y = z$$

Many-Sorted Function-Free First-Order Logic with Equality

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- Any number of variables and constants
- $\exists$ and $\forall$ quantifiers can be nested arbitrarily deeply
- All domains are finite
  - Solutions are functions that take domain sizes as input

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First-Order Model Counting (FOMC)

- Each predicate acts like a subset
  - of a domain or product of domains
- Goal: count combinations of subsets that satisfy the sentence

Exact Algorithms for FOMC

Predecessors of This Work

- ForcLift (Van den Broeck et al. 2011)
  - knowledge compilation to FO d-DNNF
- Crane (Dilkas and Belle 2023)
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  - extends ForcLift with support for:
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    - recursive solutions

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Other Approaches

- L2C (Kazemi and Poole 2016)
  - knowledge compilation to C++ code
- Alchemy (Gogate and Domingos 2016)
  - DPLL-style search
- FastWFOMC (van Bremen and Kuželka 2021)
  - based on cell enumeration

Previous Work: Crane (Dilkas and Belle 2023)

- A knowledge compilation approach:
  - Sentences $\rightarrow$ labelled digraphs $\rightarrow$ function-defining equations
- Two variants: greedy search and breadth-first search (BFS)

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An Example Solution for Counting Bijections

$$f(m, n) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} g(l, m),$$
$$g(l, m) = g(l-1, m) + m g(l-1, m-1)$$

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Issues We Are Going to Address

Completeness: recursive functions (like g) have no base cases

Usability: how do I compute, e.g., $f(7, 7)$?

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$$g(l, m) = g(l-1, m) + m g(l-1, m-1)$$

3. ($\Rightarrow$) Identify a sufficient set of base cases
   - e.g., $\{ g(0, m), g(l, 0) \}$

4. For each base case:

$g(0, m)$

$g(l, 0)$

The Workflow of CRANE2 (2/2)

4. For each base case:

$$g(0, m) \xrightarrow{} \forall y \in \Delta.\ S(y) \vee \neg S(y) \qquad\qquad g(l, 0) \xrightarrow{} \top$$

4.1. ($\Rightarrow$) Construct the corr. sentence

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$g(0, m)$ $\quad\quad$ $g(l, 0)$

4.1. ($\Rightarrow$) Construct the corr. sentence

$$g(0, m) \longrightarrow \forall y \in \Delta.\ S(y) \lor \neg S(y) \qquad\qquad g(l, 0) \longrightarrow \top$$

4.2. Recurse

$$\forall y \in \Delta.\ S(y) \lor \neg S(y) \longrightarrow g(0, m) = ??? \qquad\qquad \top \longrightarrow g(l, 0) = ???$$

The Workflow of Crane2 (2/2)

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4.1. ($\Rightarrow$) Construct the corr. sentence

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$$
\begin{tikzcd}
g(0, m) \arrow[d] & & g(l, 0) \arrow[d] \\
\forall y \in \Delta.\ S(y) \lor \neg S(y) \arrow[d] & & \top \arrow[d] \\
g(0, m) = 0^m & & g(l, 0) = 1
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$$

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The Workflow of Crane2 (2/2)

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\forall y \in \Delta.\ S(y) \lor \neg S(y) \arrow[d] & \text{4.2. Recurse} & \top \arrow[d] \\
g(0, m) = 0^m \arrow[dr] & \text{4.3. Add to } \mathcal{E} & g(l, 0) = 1 \arrow[dl] \\
& \mathcal{E} \arrow[d, "\text{5. } (\Rightarrow) \text{ Compile to C++}"] & \\
& \boxed{\text{C++ code}} &
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The Workflow of CRANE2 (2/2)

4. For each base case:

$g(0, m)$ $\qquad$ $g(l, 0)$

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$\forall y \in \Delta.\ S(y) \vee \neg S(y)$ $\qquad$ $\top$

4.2. Recurse

$g(0, m) = 0^m$ $\qquad$ 4.3. Add to $\mathcal{E}$ $\qquad$ $g(l, 0) = 1$

$\mathcal{E}$

5. ($\Rightarrow$) Compile to C++

Domain sizes $\longrightarrow$ C++ code $\longrightarrow$ Model count

Finding (a Sufficient Set of) Base Cases

Outline

1. For every function call:
   1.1 For every argument of the form $var - const$:
      1.1.1 Replace the signature parameter with $0, 1, \ldots, const - 1$
   1.2 For every argument of the form $const$:
      1.2.1 Replace the corresponding signature parameter with $const$

Example

The signature of g is $g(l, m)$.

$$
\begin{tikzcd}
\text{Function calls:} & g(l-1, m) \arrow[d] & g(l-1, m-1) \arrow[dl] \arrow[d] \\
\text{Base cases:} & g(0, m) & g(l, 0)
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Theoretical Guarantees

Theorem

The maximum recursion depth of the compilation algorithm is upper bounded by the number of domains.

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Theorem

The evaluation of a recursive function always terminates.

Proof (hints).

- There exists a topological ordering of functions
- All function calls follow the structure from the previous slide
- Some common-sense assumptions about the evaluation order and previous work

□

From a Base Case to a Sentence

From Previous Work (Dilkas and Belle 2023)

- Crane associates each function f with a sentence ϕ such that Crane(ϕ) produces the definition of f
- There is a bijection between the parameters of f and the domains of ϕ

Example

- Base case: $g(0, m)$

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- Base case: $g(\overset{\Gamma}{\overset{\updownarrow}{0}}, \overset{\Delta}{\overset{\updownarrow}{m}})$

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- Base case: $g(\overset{\Gamma}{\overset{\updownarrow}{0}}, \overset{\Delta}{\overset{\updownarrow}{m}})$
- Part of the sentence of g:

$$\forall x \in \Gamma.\ \forall y \in \Delta.\ S(y) \lor \neg P(x, y) \tag{1}$$

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- $g(0, \ldots)$ means we need to simplify (1) by assuming $|\Gamma| = 0$
- Result: $\forall y \in \Delta.\ S(y) \lor \neg S(y)$ (Smoothing)

An Outline of the Resulting C++ Program

initialise $\text{Cache}_{g(0,m)}$, $\text{Cache}_{g(l,0)}$, Cache_{g}, and Cache_{f};

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Function $g_{l,0}(l)$**:** …
Function $g(l, m)$**:**
    if $(l, m) \in \text{Cache}_g$ **then return** $\text{Cache}_g(l, m)$;
    if $l = 0$ **then return** $g_{0,m}(m)$;
    if $m = 0$ **then return** $g_{l,0}(l)$;
    $r \leftarrow g(l-1, m) + mg(l-1, m-1)$;
    $\text{Cache}_g(l, m) \leftarrow r$;
    return r;
Function $f(m, n)$**:** …
Function `Main`**:**
    $(m, n) \leftarrow$ `ParseCommandLineArguments()`;
    return $f(m, n)$;

Benchmarks

- Friends & Smokers

$$(\forall x, y \in \Delta.\ S(x) \land F(x, y) \to S(y)) \land (\forall x \in \Delta.\ S(x) \to C(x))$$

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$$(\forall x, y \in \Delta.\ S(x) \wedge F(x, y) \rightarrow S(y)) \wedge (\forall x \in \Delta.\ S(x) \rightarrow C(x))$$

- Functions

$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge$$
$$(\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \rightarrow y = z)$$

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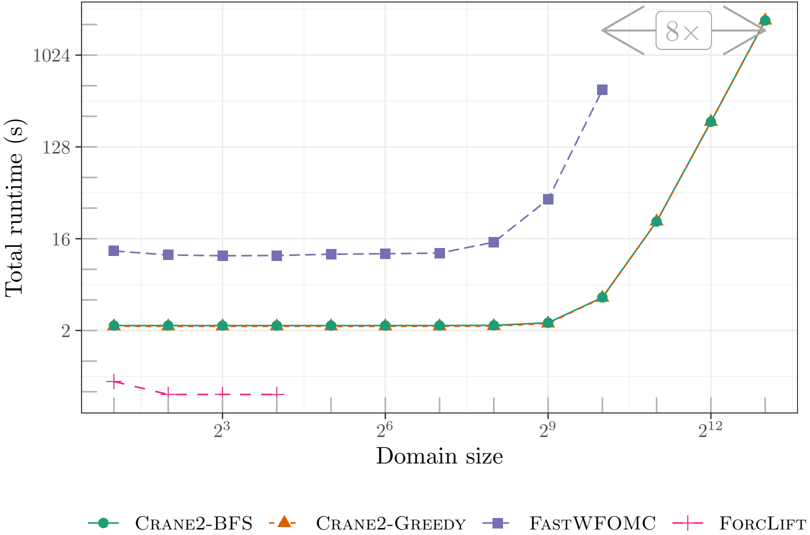
- Functions

$$\begin{gathered}(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge \\ (\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \rightarrow y = z)\end{gathered}$$

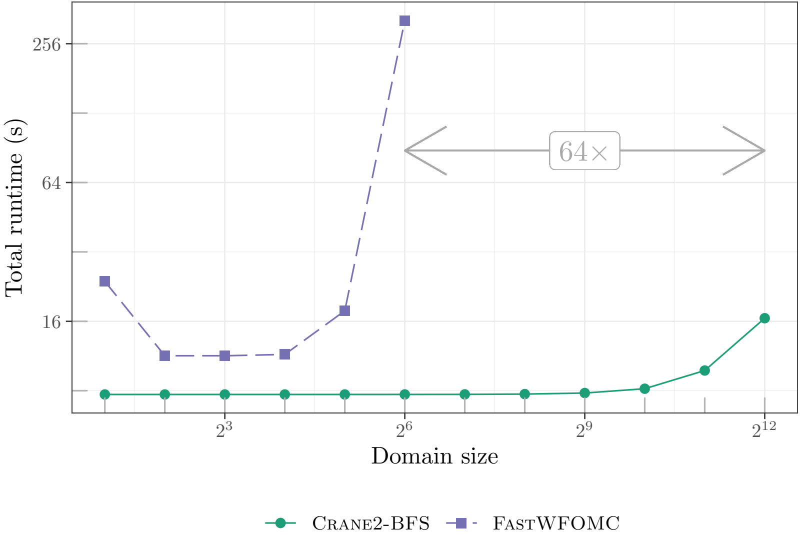
- Bijections

$$\begin{gathered}(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge \\ (\forall y \in \Delta.\ \exists x \in \Gamma.\ P(x, y)) \wedge \\ (\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \rightarrow y = z) \wedge \\ (\forall x, z \in \Gamma.\ \forall y \in \Delta.\ P(x, y) \wedge P(z, y) \rightarrow x = z)\end{gathered}$$

Friends & Smokers



Bijections



Total runtime (s)
256
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16
64×
2^3
2^6
2^9
2^{12}
Domain size
Crane2-BFS
FastWFOMC

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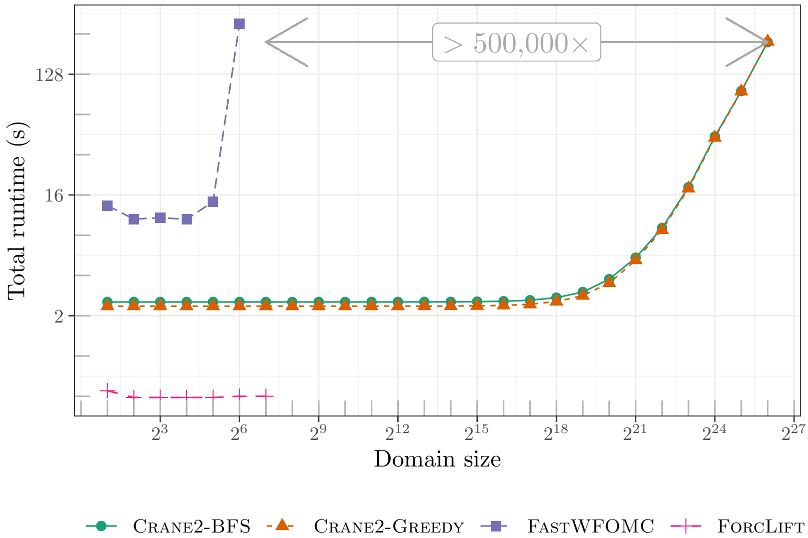

> 500,000×
128
16
2
Total runtime (s)
2^3 2^6 2^9 2^{12} 2^{15} 2^{18} 2^{21} 2^{24} 2^{27}
Domain size
Crane2-BFS
Crane2-Greedy
FastWFOMC
ForcLift

Summary & Future Work

Contributions

Completeness: recursive solutions now come with base cases

Usability: compilation to C++ programs

Scalability compared to other FOMC algorithms

- 8 to 500,000 times higher domain sizes

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Future Work

- Support for weighted counting (trivial)
- Experiments on a large set of benchmarks
- Completeness for fragments of first-order logic
- Fine-grained complexity