Towards Practical First-Order Model Counting: Technical Appendix

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1 The Proof of Theorem 1

To demonstrate that the base cases identified by `FindBaseCases` are sufficient, we begin with a few observations that stem from previous (Van den Broeck et al. 2011; Dilkas and Belle 2023) and this work. Let $\mathcal{E}$ represent the equations returned by `CompileWithBaseCases`.

Observation 1. *For each function f, there is precisely one equation $e \in \mathcal{E}$ with $f(\mathbf{x})$ on the left-hand side where all x_i's are variables (i.e., e is not a base case). We refer to e as the* definition *of f.*

Observation 2. *There is a* topological ordering *of all functions $(f_i)_i$ in $\mathcal{E}$ such that equations in $\mathcal{E}$ with f_i on the left-hand side do not contain function calls to f_j with $j > i$. This condition prevents mutual recursion and other cyclic scenarios.*

Observation 3. *For every equation $(f(\mathbf{x}) = \texttt{expr}) \in \mathcal{E}$, the evaluation of* `expr` *terminates when provided with the values of all relevant function calls.*

Corollary 1. *If f is a non-recursive function with no function calls on the right-hand side of its definition, then the evaluation of any function call $f(\mathbf{x})$ terminates.*

Observation 4. *For any equation $f(\mathbf{x}) = \texttt{expr}$, if $\mathbf{x}$ contains only constants, then* `expr` *cannot include any function calls to f.*

Additionally, we introduce an assumption about the structure of recursion.

Assumption 1. *For every equation $(f(\mathbf{x}) = \texttt{expr}) \in \mathcal{E}$, every recursive function call $f(\mathbf{y}) \in \texttt{expr}$ satisfies the following:*

- *Each y_i is either $x_i - c_i$ or c_i for some constant c_i.*
- *There exists i such that $y_i = x_i - c_i$ for some $c_i > 0$.*

Finally, we assume a particular order of evaluation for function calls using the equations in $\mathcal{E}$. Specifically, we assume that base cases are considered before the recursive definition. The exact order in which base cases are considered is immaterial.

Assumption 2. *When multiple equations in $\mathcal{E}$ match a function call $f(\mathbf{x})$, preference is given to an equation with the most constants on its left-hand side.*

With the observations and assumptions mentioned above, we prove the following theorem.

Theorem 1 (Termination)**.** *Let f be an n-ary function in $\mathcal{E}$ and $\mathbf{x} \in \mathbb{N}_0^n$. Then the evaluation of $f(\mathbf{x})$ terminates.*

For readability, we divide the proof into several lemmas of increasing generality.

Lemma 1. *Assume that $\mathcal{E}$ consists of just* one unary *function f. Then the evaluation of a function call $f(x)$ terminates for any $x \in \mathbb{N}_0$.*

Proof. If $f(x)$ is captured by a base case, then its evaluation terminates by Corollary 1 and Observation 4. If f is not recursive, the evaluation of $f(x)$ terminates by Corollary 1.

Otherwise, let $f(y)$ be an arbitrary function call on the right-hand side of the definition of $f(x)$. If y is a constant, then there is a base case for $f(y)$. Otherwise, let $y = x - c$ for some $c > 0$. Then there exists $k \in \mathbb{N}_0$ such that $0 \leq x - kc \leq c - 1$. So, after k iterations, the sequence of function calls $f(x), f(x-c), f(x-2c), \ldots$ will be captured by the base case $f(x \mod c)$. ☐

Lemma 2. *Generalising Lemma 1, let $\mathcal{E}$ be a set of equations for* one *n-ary function f for some $n \geq 1$. Then the evaluation of $f(\mathbf{x})$ terminates for any $\mathbf{x} \in \mathbb{N}_0^n$.*

Proof. If f is non-recursive, the evaluation of $f(\mathbf{x})$ terminates by previous arguments. We proceed by induction on n, with the base case of $n = 1$ handled by Lemma 1. Assume that $n > 1$. Any base case of f can be seen as a function of arity $n - 1$, since one of the parameters is fixed. Thus, the evaluation of any base case terminates by the inductive hypothesis. It remains to show that the evaluation of the recursive equation for f terminates, but that follows from Observation 3. ☐

Proof of Theorem 1. We proceed by induction on the number of functions n. The base case of $n = 1$ is handled by Lemma 2. Let $(f_i)_{i=1}^n$ be some topological ordering of these $n > 1$ functions. If $f = f_j$ for $j < n$, then the evaluation of $f(\mathbf{x})$ terminates by the inductive hypothesis since f_j cannot call f_n by Observation 2. Using the inductive hypothesis that all function calls to f_j (with $j < n$) terminate, the proof proceeds similarly to the Proof of Lemma 2. ☐

References

Dilkas, P.; and Belle, V. 2023. Synthesising Recursive Functions for First-Order Model Counting: Challenges, Progress, and Conjectures. In *KR*, 198–207.

Van den Broeck, G.; Taghipour, N.; Meert, W.; Davis, J.; and De Raedt, L. 2011. Lifted Probabilistic Inference by First-Order Knowledge Compilation. In *IJCAI*, 2178–2185. IJCAI/AAAI.