Towards Practical First-Order Model Counting

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Motivation

Example Setting

- Let Δ be a set of cardinality n
- Suppose we want to count all $P \subseteq \Delta^2$ (as a function of n) that are:
  - functions,
  - bijections,
  - partial orders,
  - symmetric,
  - transitive,
  - etc.

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👎 Propositional model counting (#SAT) is #P-complete

👍 But many of these counting problems have efficient solutions

- And we can find them using first-order model counting
  - i.e., reasoning about sets, subsets, and arbitrary elements without grounding them

First-Order Model Counting

The Problem with CRANE

A Solution Produced for the Bijection-Counting Problem

$$f(m, n) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} g(l, m),$$
$$g(l, m) = g(l-1, m) + m g(l-1, m-1)$$

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Issues

Completeness: what are the base cases of g?

Usability: how do I compute, e.g., $f(7, 7)$?

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   becomes

$$g(l, m) = g(l-1, m) + m g(l-1, m-1)$$

3. $(\Rightarrow)$ Identify a sufficient set of base cases
   - e.g., $\{\, g(0, m), g(l, 0) \,\}$

Knowledge Compilation Workflow (2/2)

4. For each base case:

$g(0, m)$ $g(l, 0)$

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4. For each base case:

4.1. ($\Rightarrow$) Construct the corr. sentence

$$\begin{tikzcd} g(0, m) \arrow[d] & & g(l, 0) \arrow[d] \\ \forall y \in \Delta.\ S(y) \lor \neg S(y) & & \top \end{tikzcd}$$

Knowledge Compilation Workflow (2/2)

4. For each base case:

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4.2. Recurse

$$\begin{tikzcd}
g(0, m) \arrow[d] & g(l, 0) \arrow[d] \\
\forall y \in \Delta.\ S(y) \lor \neg S(y) \arrow[d] & \top \arrow[d] \\
g(0, m) = ??? & g(l, 0) = ???
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4.3. Add to $\mathcal{E}$

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g(0, m) = 0^m \arrow[dr] & & g(l, 0) = 1 \arrow[dl] \\
& \mathcal{E} &
\end{tikzcd}$$

Knowledge Compilation Workflow (2/2)

4. For each base case:

$$\begin{tikzcd}
g(0, m) \arrow[d] & \text{4.1. } (\Rightarrow) \text{ Construct the corr. sentence} & g(l, 0) \arrow[d] \\
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g(0, m) = 0^m \arrow[dr] & \text{4.2. Recurse} \quad \text{4.3. Add to } \mathcal{E} & g(l, 0) = 1 \arrow[dl] \\
& \mathcal{E} \arrow[d, "\text{5. Compile to C++}"] & \\
& \boxed{\text{C++ code}} &
\end{tikzcd}$$

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g(0, m) = 0^m \arrow[dr] & \text{4.3. Add to } \mathcal{E} & g(l, 0) = 1 \arrow[dl] \\
 & \mathcal{E} \arrow[d, "\text{5. Compile to C++}"] & \\
\text{Domain sizes} \arrow[r] & \boxed{\text{C++ code}} \arrow[r] & \text{Model count}
\end{tikzcd}$$

Finding (a Sufficient Set of) Base Cases

Outline

1. For every function call:
   1.1 For every argument of the form $var - const$:
      1.1.1 Replace the signature parameter with $0, 1, \ldots, const - 1$
   1.2 For every argument of the form $const$:
      1.2.1 Replace the corresponding signature parameter with $const$

Example

The signature of g is $g(l, m)$.

$$\begin{tikzcd}
\text{Function calls:} & g(l-1, m) \arrow[d] & g(l-1, m-1) \arrow[dl] \arrow[d] \\
\text{Base cases:} & g(0, m) & g(l, 0)
\end{tikzcd}$$

Theoretical Results

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The maximum recursion depth of the compilation algorithm is upper bounded by the number of domains.

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Theorem

The evaluation of a recursive function always terminates.

Proof (hints).

- There exists a topological ordering of functions
- All function calls follow the structure from the previous slide
- Some common-sense assumptions about the evaluation order and previous work

□

From a Base Case to a Sentence

From Previous Work (Dilkas and Belle 2023)

- Crane associates each function f with a sentence ϕ such that $\textsc{Crane}(\phi)$ produces the definition of f
- There is a bijection between the parameters of f and the domains of ϕ

Example

- Base case: $g(0, m)$

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Example

- Base case: $g(\overset{\overset{\Gamma}{\updownarrow}}{0}, \overset{\overset{\Delta}{\updownarrow}}{m})$

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$$\forall x \in \Gamma.\ \forall y \in \Delta.\ S(y) \lor \neg P(x, y) \tag{1}$$

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- $g(0, \dots)$ means we need to simplify (1) by assuming $|\Gamma| = 0$
- Result: $\forall y \in \Delta.\ S(y) \lor \neg S(y)$ (Smoothing)

Benchmarks

- Friends & Smokers

  $(\forall x, y \in \Delta.\ S(x) \wedge F(x, y) \to S(y)) \wedge (\forall x \in \Delta.\ S(x) \to C(x))$

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$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge$$
$$(\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \rightarrow y = z)$$

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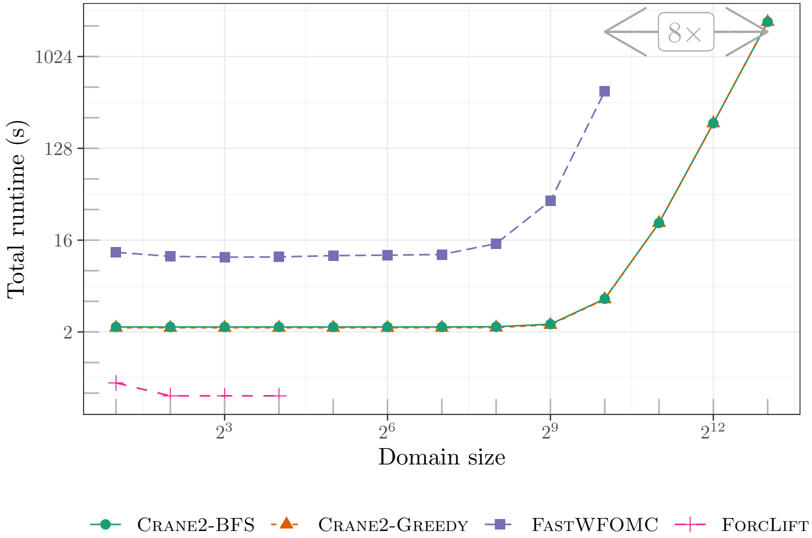
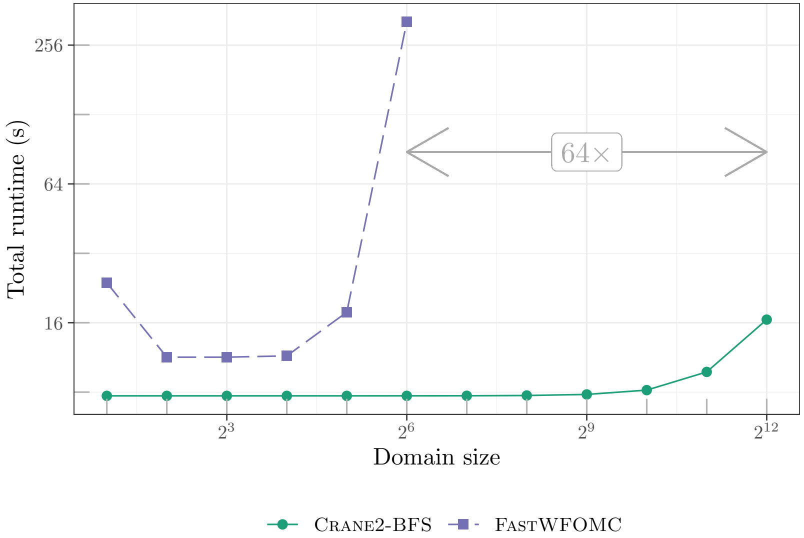
- Bijections

$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge \\ (\forall y \in \Delta.\ \exists x \in \Gamma.\ P(x, y)) \wedge \\ (\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \to y = z) \wedge \\ (\forall x, z \in \Gamma.\ \forall y \in \Delta.\ P(x, y) \wedge P(z, y) \to x = z)$$

Friends & Smokers

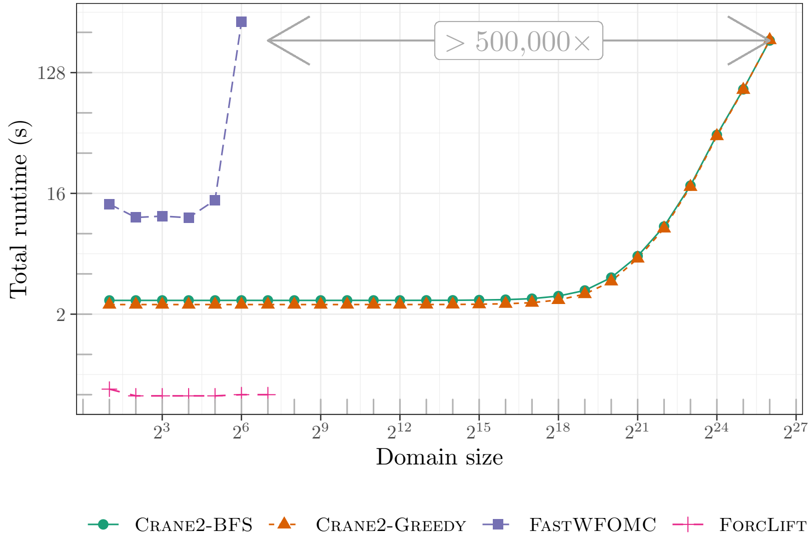


Bijections



Total runtime (s)
256
64
16
2^3
2^6
2^9
2^{12}
Domain size
64×
Crane2-BFS
FastWFOMC

Functions



> 500,000×
Total runtime (s)
128
16
2
2^3
2^6
2^9
2^12
2^15
2^18
2^21
2^24
2^27
Domain size
CRANE2-BFS
CRANE2-GREEDY
FASTWFOMC
FORCLIFT

Summary

TODO: and future work