Towards Practical First-Order Model Counting

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SAT 2025

Motivation

Example Setting

- Let Δ be a set of cardinality n
- Suppose we want to count all $P \subseteq \Delta^2$ (as a function of n) that are:
  - functions,
  - bijections,
  - partial orders,
  - symmetric,
  - transitive,
  - etc.

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👎 Propositional model counting (#SAT) is #P-complete

👍 But many of these counting problems have efficient solutions

- And we can find them using first-order model counting
  - i.e., reasoning about sets, subsets, and arbitrary elements without grounding them

First-Order Model Counting

The Problem with Crane

A Solution Produced for the Bijection-Counting Problem

$$f(m,n) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} g(l,m),$$

$$g(l,m) = g(l-1,m) + mg(l-1,m-1)$$

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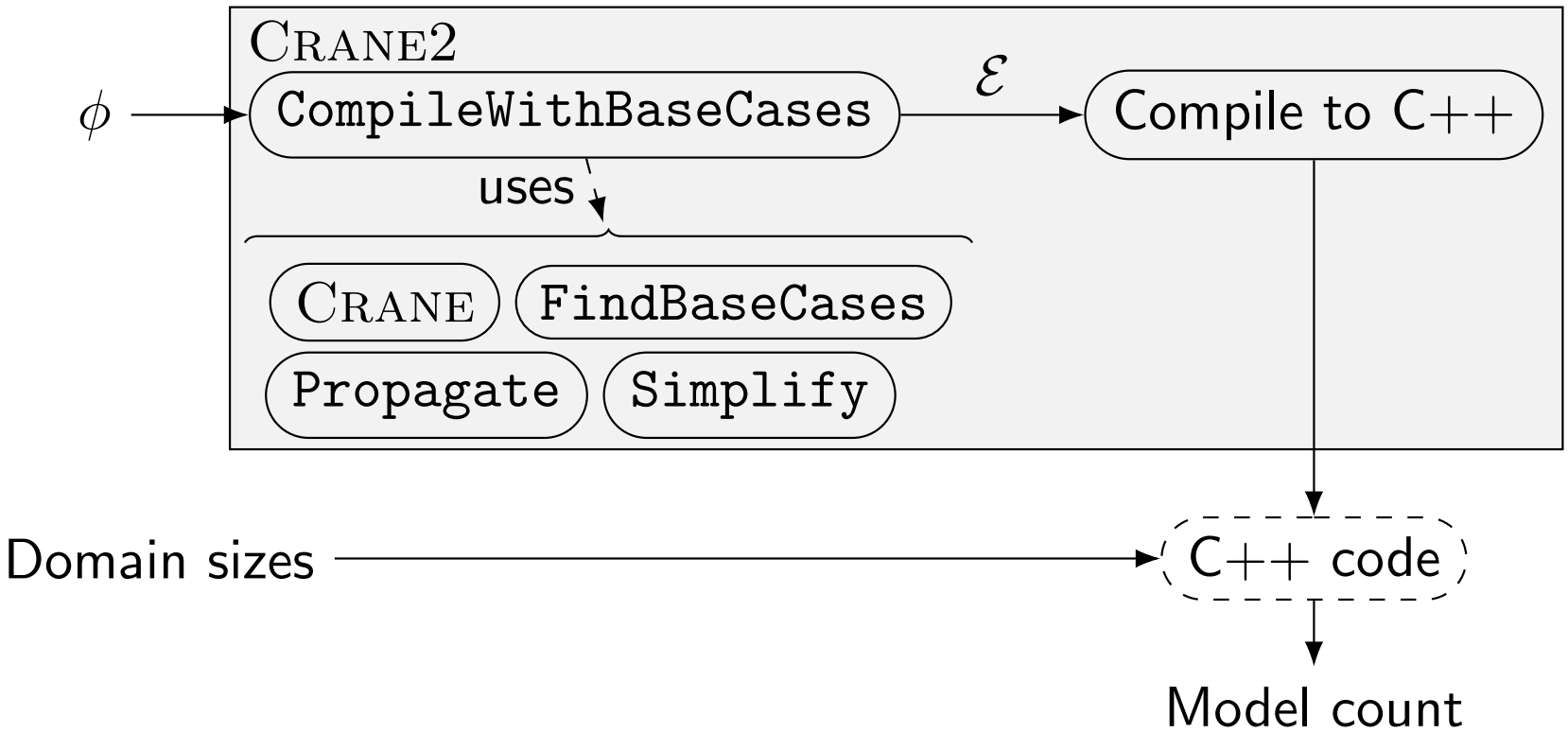
Issues

Completeness: what are the base cases of g?

Usability: how do I compute, e.g., $f(7,7)$?

Knowledge Compilation Workflow



Knowledge Compilation Workflow (1/2)

1. Use CRANE to compile ϕ into a set of equations $\mathcal{E}$
2. Simplify them, e.g.,

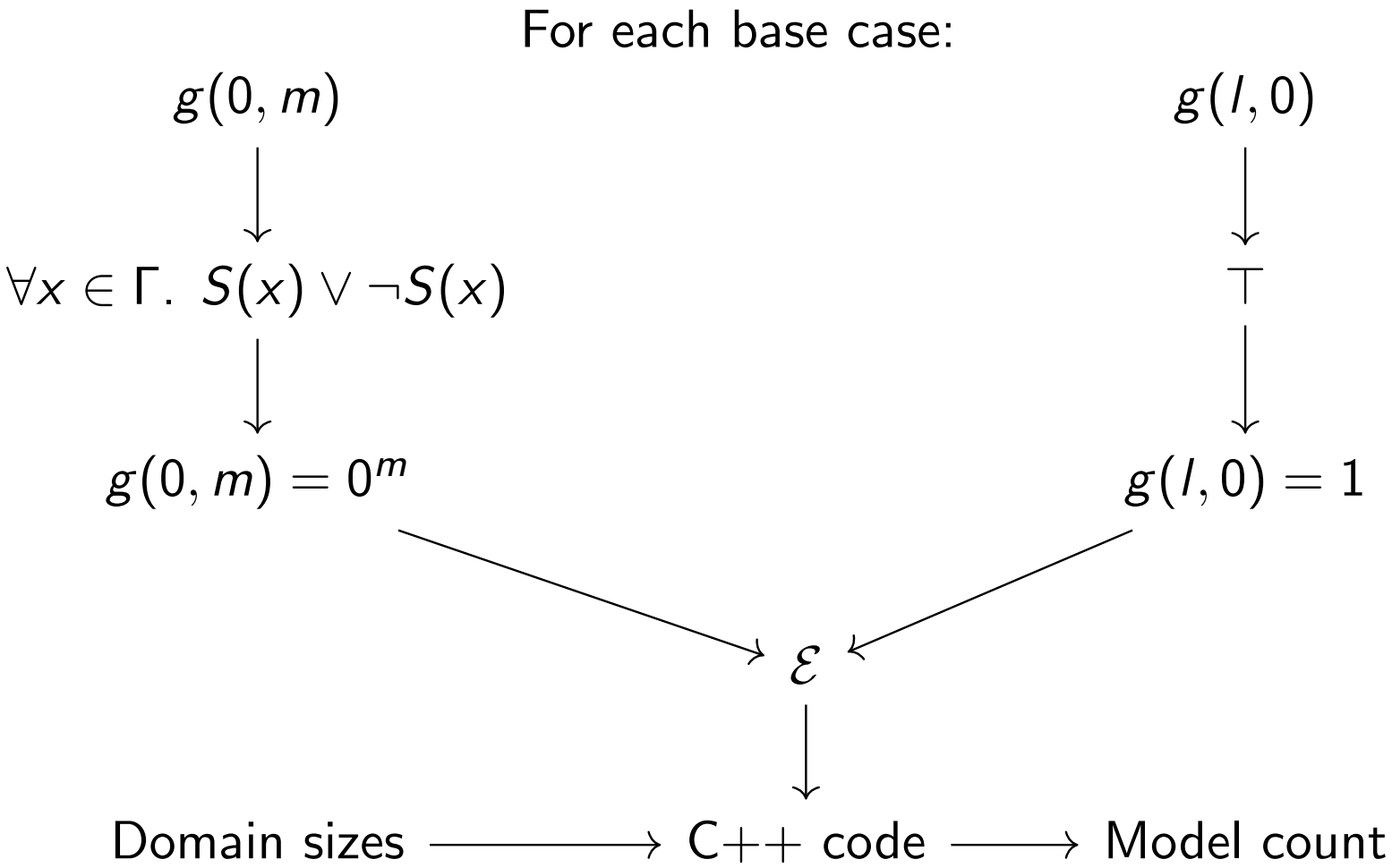
$$g(l,m) = \sum_{k=0}^{m} [0 \leq k \leq 1] \binom{m}{k} g(l-1, m-k)$$

   becomes

$$g(l,m) = g(l-1,m) + mg(l-1,m-1)$$

3. Identify a sufficient set of base cases
   - e.g., $\{ g(0,m), g(l,0) \}$

Knowledge Compilation Workflow (2/2)



Benchmarks

- Friends & Smokers

$$(\forall x, y \in \Delta.\ S(x) \wedge F(x, y) \rightarrow S(y)) \wedge (\forall x \in \Delta.\ S(x) \rightarrow C(x))$$

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$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge$$
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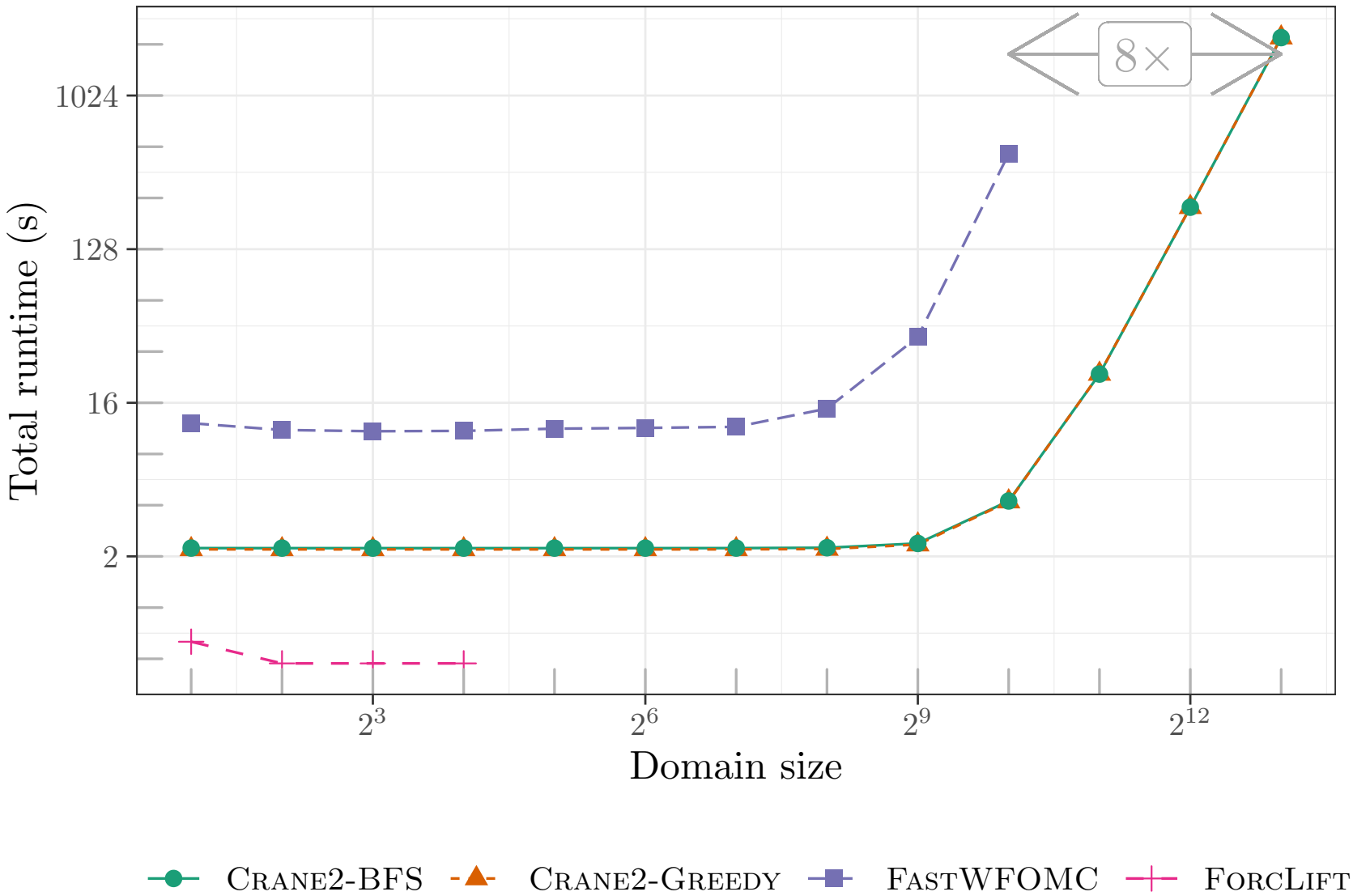
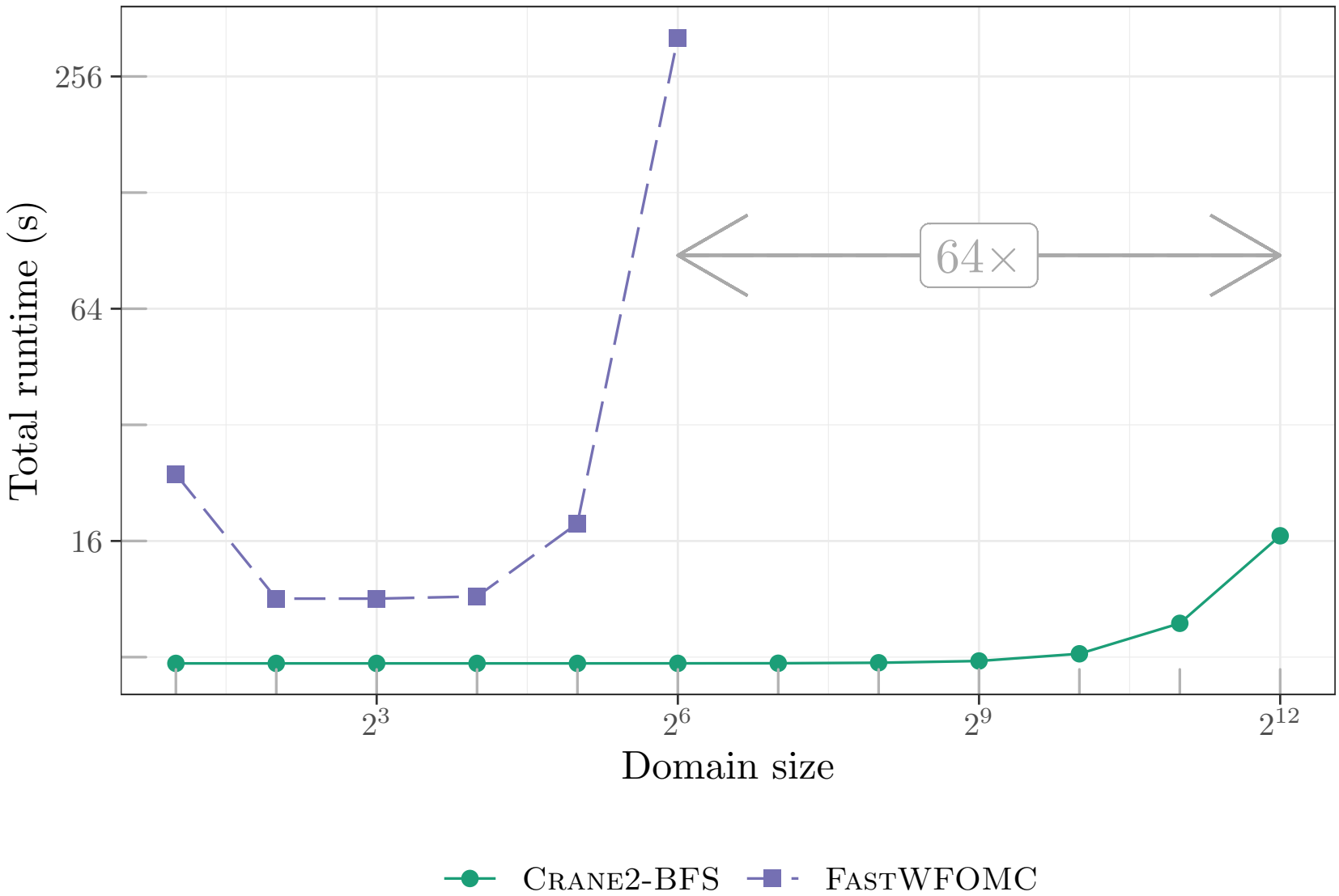
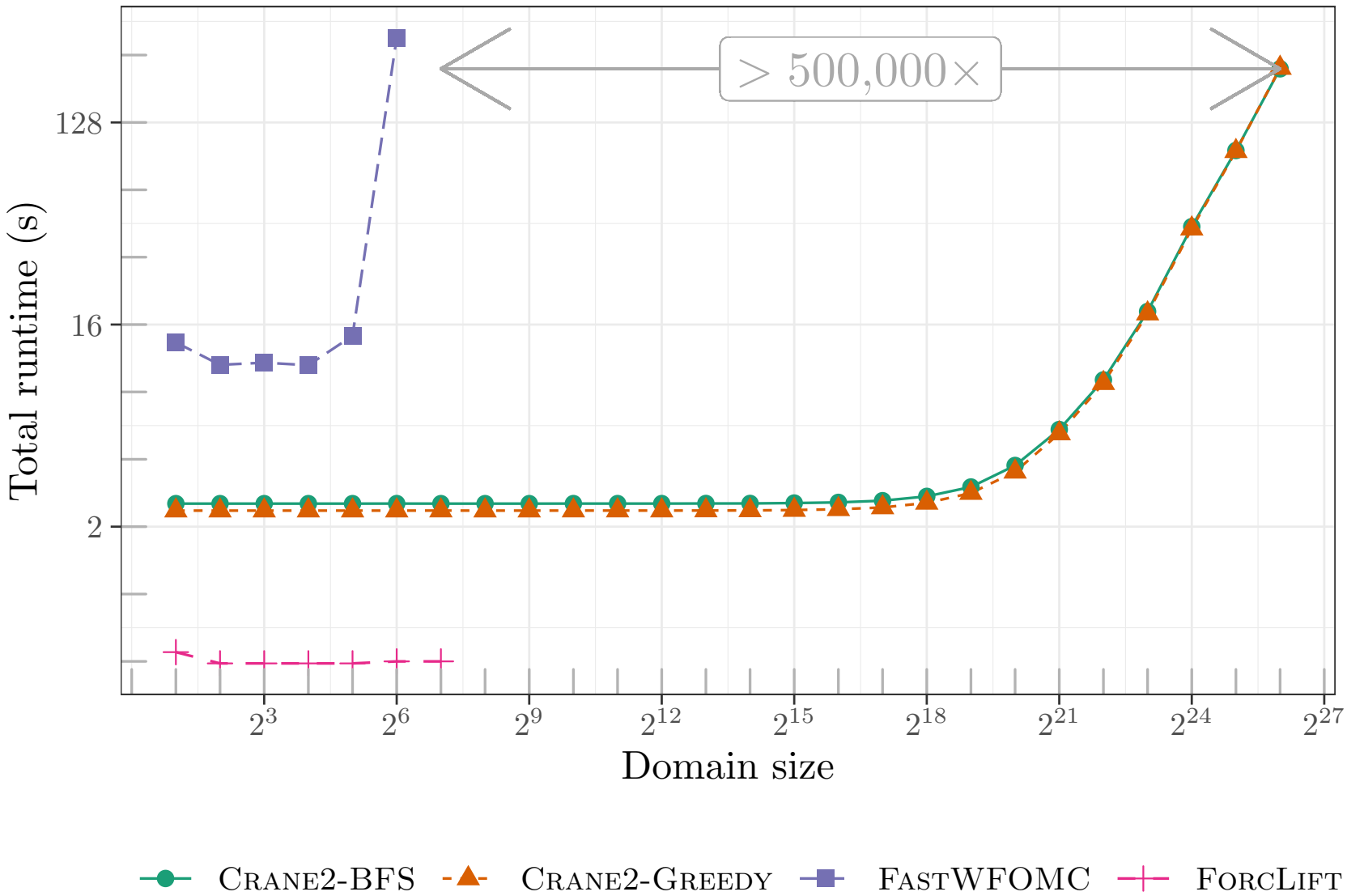
Friends & Smokers



Bijections


256
64
16
Total runtime (s)
64×
2^3
2^6
2^9
2^{12}
Domain size
Crane2-BFS
FastWFOMC

Functions


> 500,000×
128
16
2
Total runtime (s)
2^3
2^6
2^9
2^{12}
2^{15}
2^{18}
2^{21}
2^{24}
2^{27}
Domain size
Crane2-BFS
Crane2-Greedy
FastWFOMC
ForcLift

Summary

TODO: and future work