Towards Practical First-Order Model Counting

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Motivation

Example Setting

- Let Δ be a set of cardinality n
- Suppose we want to count all $P \subseteq \Delta^2$ (as a function of n) that are:
  - functions,
  - bijections,
  - partial orders,
  - symmetric,
  - transitive,
  - etc.

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  - etc.

- Propositional model counting (#SAT) is #P-complete
- But many of these counting problems have efficient solutions
- And we can find them using first-order model counting
  - i.e., reasoning about sets, subsets, and arbitrary elements without grounding them

More Formally: What Is the Input?

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \Rightarrow y = z$$

Many-Sorted Function-Free First-Order Logic with Equality

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- Any number of variables and constants
- $\exists$ and $\forall$ quantifiers can be nested arbitrarily deeply
- All domains are finite
  - Solutions are functions that take domain sizes as input

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$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \land P(x, z) \Rightarrow y = z$$

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First-Order Model Counting (FOMC)

- Each predicate acts like a subset
  - of a domain or
  - of a Cartesian product of domains
- Goal: count combinations of subsets that satisfy the sentence

Previous Work: Crane (Dilkas and Belle 2023)

- A knowledge compilation approach:
  - Sentences $\rightarrow$ labelled digraphs $\rightarrow$ function-defining equations
- Capable of constructing recursive solutions

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An Example Solution for Counting Bijections

$$f(m,n) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} g(l,m),$$

$$g(l,m) = g(l-1,m) + mg(l-1,m-1)$$

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$$g(l, m) = g(l-1, m) + m g(l-1, m-1)$$

Issues

Completeness: what are the base cases of g?

Usability: how do I compute, e.g., $f(7, 7)$?

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$$g(l, m) = g(l-1, m) + mg(l-1, m-1)$$

3. ($\Rightarrow$) Identify a sufficient set of base cases
   - e.g., $\{\, g(0, m), g(l, 0) \,\}$

Knowledge Compilation Workflow (2/2)

4. For each base case:

$g(0, m)$

$g(l, 0)$

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$$\begin{array}{ccc} g(0, m) & & g(l, 0) \\ \downarrow & \text{4.1. } (\Rightarrow) \text{ Construct the corr. sentence} & \downarrow \\ \forall y \in \Delta.\ S(y) \lor \neg S(y) & & \top \end{array}$$

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4.1. ($\Rightarrow$) Construct the corr. sentence

$$\forall y \in \Delta.\ S(y) \lor \neg S(y) \qquad\qquad \top$$

4.2. Recurse

$$g(0, m) = ??? \qquad\qquad g(l, 0) = ???$$

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$$\begin{tikzcd}
g(0, m) \arrow[d] & & g(l, 0) \arrow[d] \\
\forall y \in \Delta.\ S(y) \lor \neg S(y) \arrow[d] & & \top \arrow[d] \\
g(0, m) = 0^m & & g(l, 0) = 1
\end{tikzcd}$$

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 & \text{4.2. Recurse} & \\
g(0, m) = 0^m \arrow[dr] & \text{4.3. Add to } \mathcal{E} & g(l, 0) = 1 \arrow[dl] \\
 & \mathcal{E} &
\end{tikzcd}
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Knowledge Compilation Workflow (2/2)

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$g(0, m)$ | $g(l, 0)$

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4.3. Add to $\mathcal{E}$

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5. Compile to C++

C++ code

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$\mathcal{E}$

$\mathcal{E} \xrightarrow{\text{5. Compile to C++}} \boxed{\text{C++ code}}$

Domain sizes $\longrightarrow$ $\boxed{\text{C++ code}}$ $\longrightarrow$ Model count

Finding (a Sufficient Set of) Base Cases

Outline

1. For every function call:
   1.1 For every argument of the form $var - const$:
      1.1.1 Replace the signature parameter with $0, 1, \ldots, const - 1$
   1.2 For every argument of the form $const$:
      1.2.1 Replace the corresponding signature parameter with $const$

Example

The signature of g is $g(l, m)$.

$$
\begin{tikzcd}
\text{Function calls:} & g(l-1, m) \arrow[d] & & g(l-1, m-1) \arrow[dll] \arrow[d] \\
\text{Base cases:} & g(0, m) & & g(l, 0)
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Theoretical Results

Theorem

The maximum recursion depth of the compilation algorithm is upper bounded by the number of domains.

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Theorem

The evaluation of a recursive function always terminates.

Proof (hints).

- There exists a topological ordering of functions
- All function calls follow the structure from the previous slide
- Some common-sense assumptions about the evaluation order and previous work

□

From a Base Case to a Sentence

From Previous Work (Dilkas and Belle 2023)

- $\textsc{Crane}$ associates each function f with a sentence ϕ such that $\textsc{Crane}(\phi)$ produces the definition of f
- There is a bijection between the parameters of f and the domains of ϕ

Example

- Base case: $g(0, m)$

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- Base case: $g(\overset{\Gamma}{\overset{\updownarrow}{0}}, \overset{\Delta}{\overset{\updownarrow}{m}})$

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- Part of the sentence of g:

$$\forall x \in \Gamma.\ \forall y \in \Delta.\ S(y) \lor \neg P(x, y) \tag{1}$$

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- $g(0, \dots)$ means we need to simplify (1) by assuming $|\Gamma| = 0$
- Result: $\forall y \in \Delta.\ S(y) \lor \neg S(y)$ (Smoothing)

Benchmarks

- Friends & Smokers

  $$(\forall x, y \in \Delta.\ S(x) \wedge F(x,y) \rightarrow S(y)) \wedge (\forall x \in \Delta.\ S(x) \rightarrow C(x))$$

Benchmarks

- Friends & Smokers

$$(\forall x, y \in \Delta.\ S(x) \wedge F(x, y) \rightarrow S(y)) \wedge (\forall x \in \Delta.\ S(x) \rightarrow C(x))$$

- Functions

$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge$$
$$(\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \rightarrow y = z)$$

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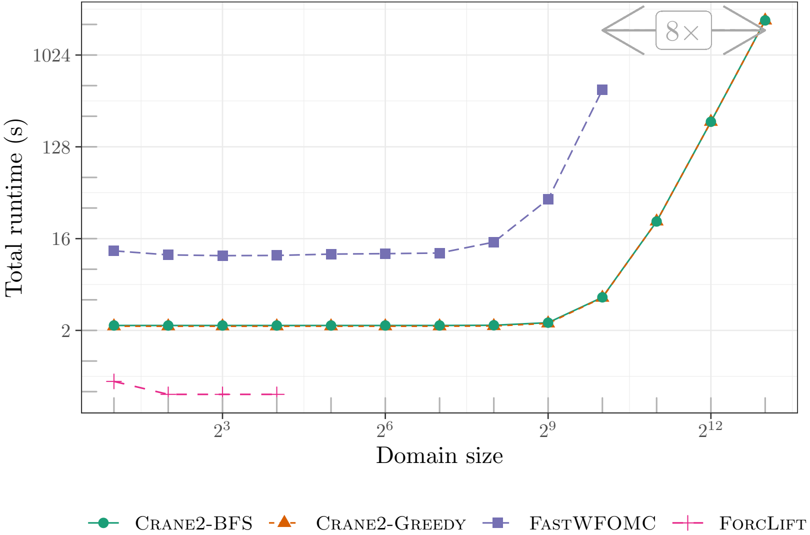
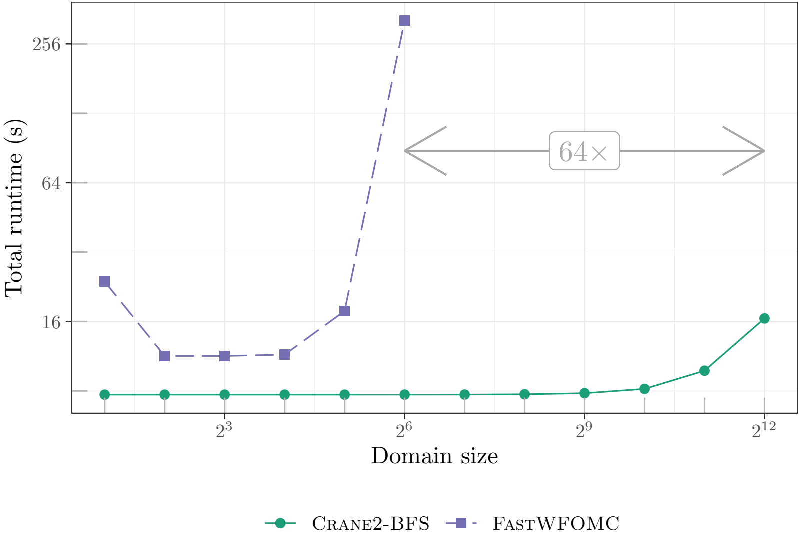
- Bijections

$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \wedge$$
$$(\forall y \in \Delta.\ \exists x \in \Gamma.\ P(x, y)) \wedge$$
$$(\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \to y = z) \wedge$$
$$(\forall x, z \in \Gamma.\ \forall y \in \Delta.\ P(x, y) \wedge P(z, y) \to x = z)$$

Friends & Smokers

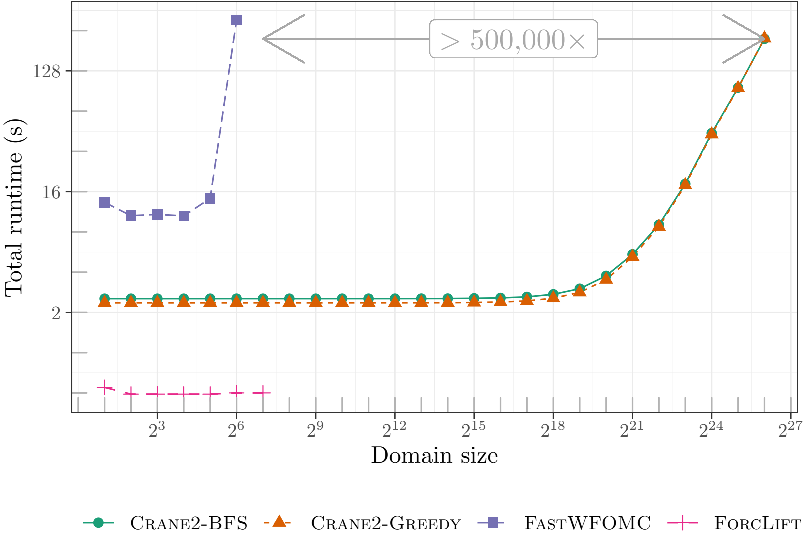


Bijections



256
64
16
Total runtime (s)
64×
2^3
2^6
2^9
2^{12}
Domain size
Crane2-BFS
FastWFOMC

Functions


> 500,000×
128
16
2
Total runtime (s)
2^3
2^6
2^9
2^{12}
2^{15}
2^{18}
2^{21}
2^{24}
2^{27}
Domain size
Crane2-BFS
Crane2-Greedy
FastWFOMC
ForcLift

Summary

TODO: and future work