Towards Practical First-Order Model Counting

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SAT 2025

Motivation

Example Setting

- Let Δ be a set of cardinality n
- Suppose we want to count all $P \subseteq \Delta^2$ (as a function of n) that are:
  - functions,
  - bijections,
  - partial orders,
  - symmetric,
  - transitive,
  - etc.

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👎 Propositional model counting (#SAT) is #P-complete

👍 But many of these counting problems have efficient solutions

- And we can find them using first-order model counting
  - i.e., reasoning about sets, subsets, and arbitrary elements without grounding them

First-Order Model Counting

The Problem with Crane

A Solution Produced for the Bijection-Counting Problem

$$f(m,n) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} g(l,m),$$

$$g(l,m) = g(l-1,m) + mg(l-1,m-1)$$

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Issues

Completeness: what are the base cases of g?

Usability: how do I compute, e.g., $f(7,7)$?

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3. ($\Rightarrow$) Identify a sufficient set of base cases
   - e.g., $\{ g(0,m), g(l,0) \}$

Knowledge Compilation Workflow (2/2)

4. For each base case:

$g(0, m)$ $g(l, 0)$

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$g(0, m)$ $\qquad$ $g(l, 0)$

$\downarrow$ $\qquad$ 4.1. ($\Rightarrow$) Construct the corr. sentence $\qquad$ $\downarrow$

$\forall x \in \Gamma.\ S(x) \vee \neg S(x)$ $\qquad$ $\top$

Knowledge Compilation Workflow (2/2)

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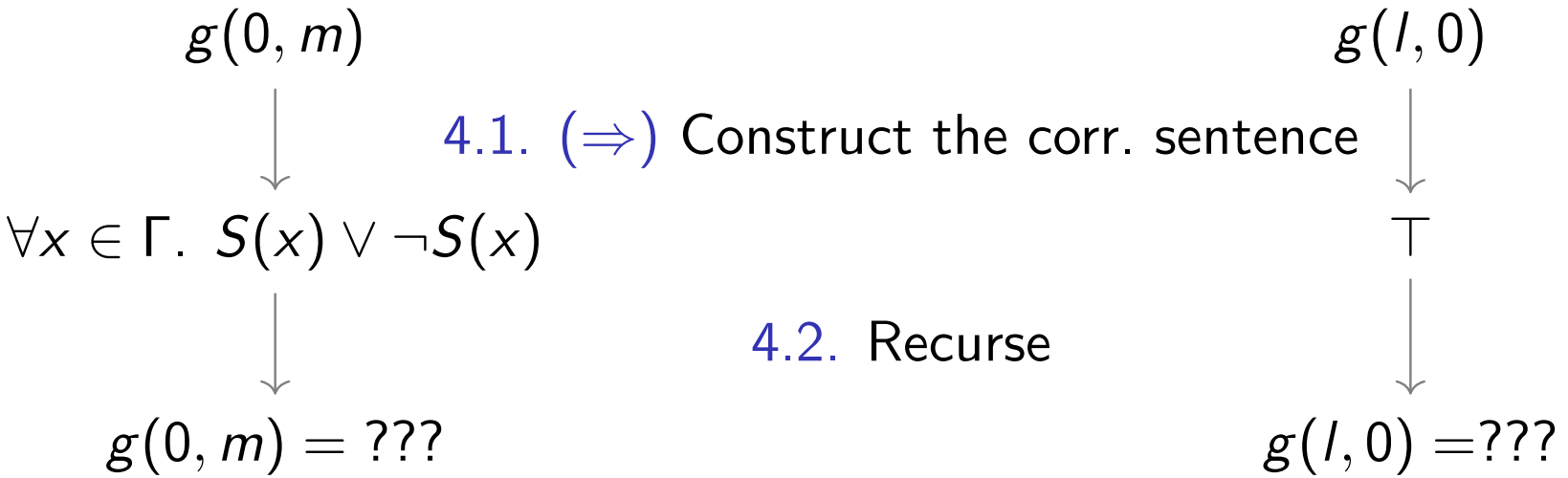

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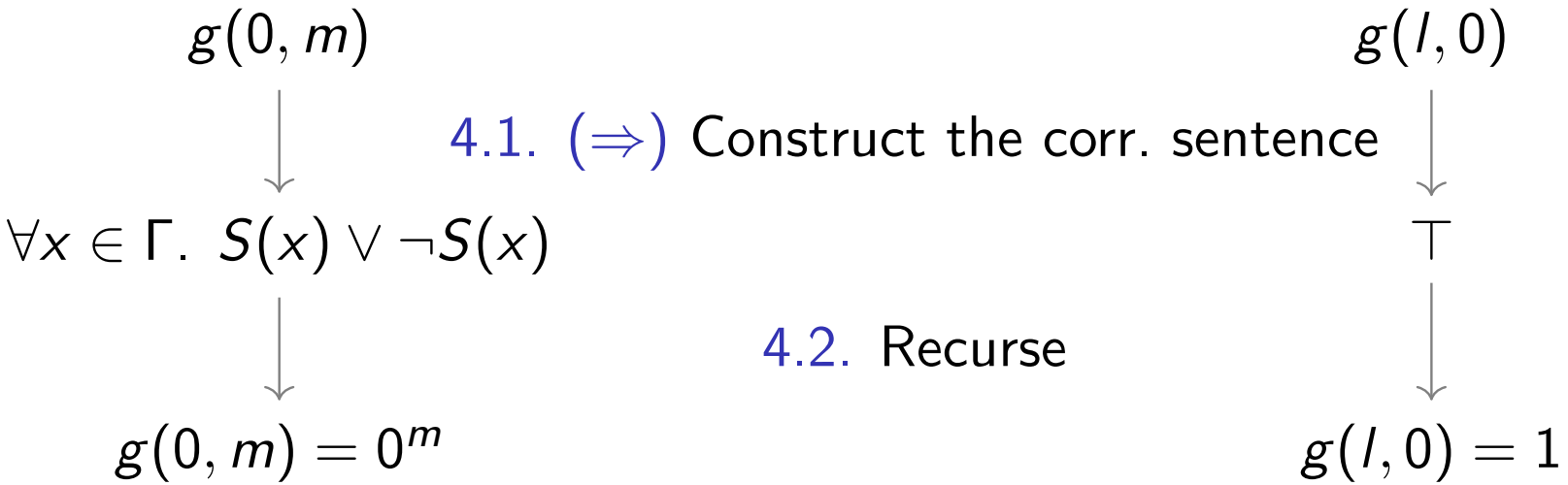

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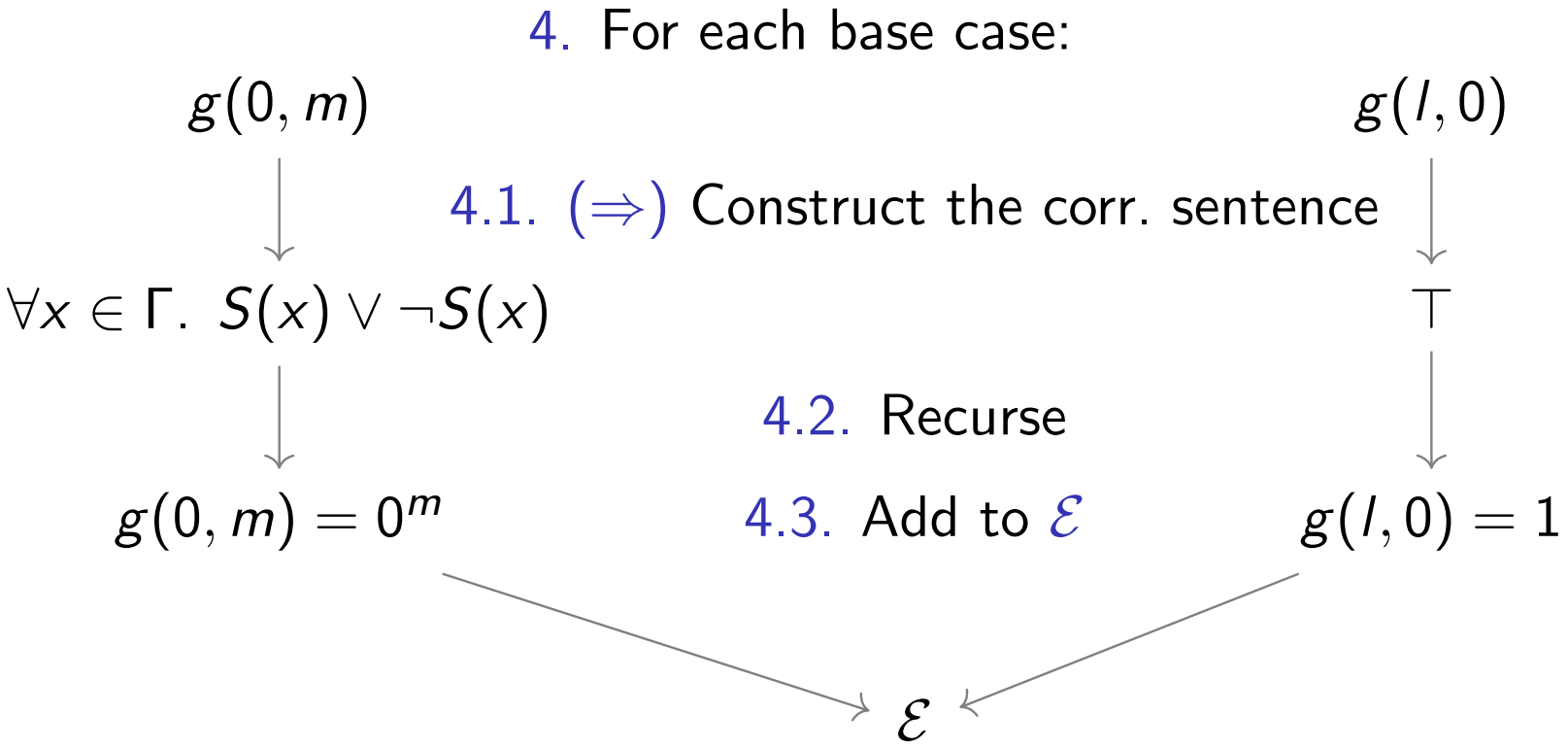

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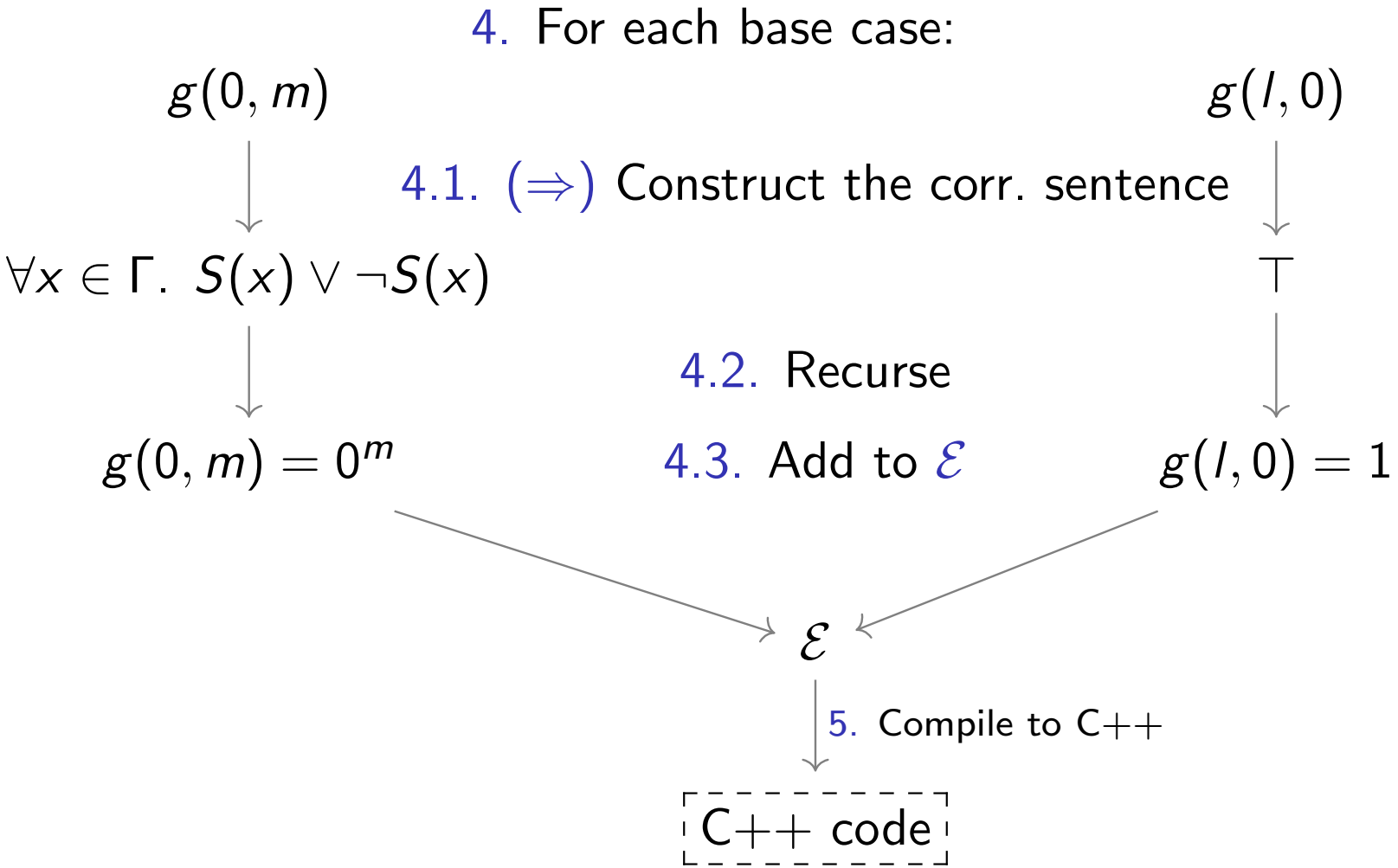

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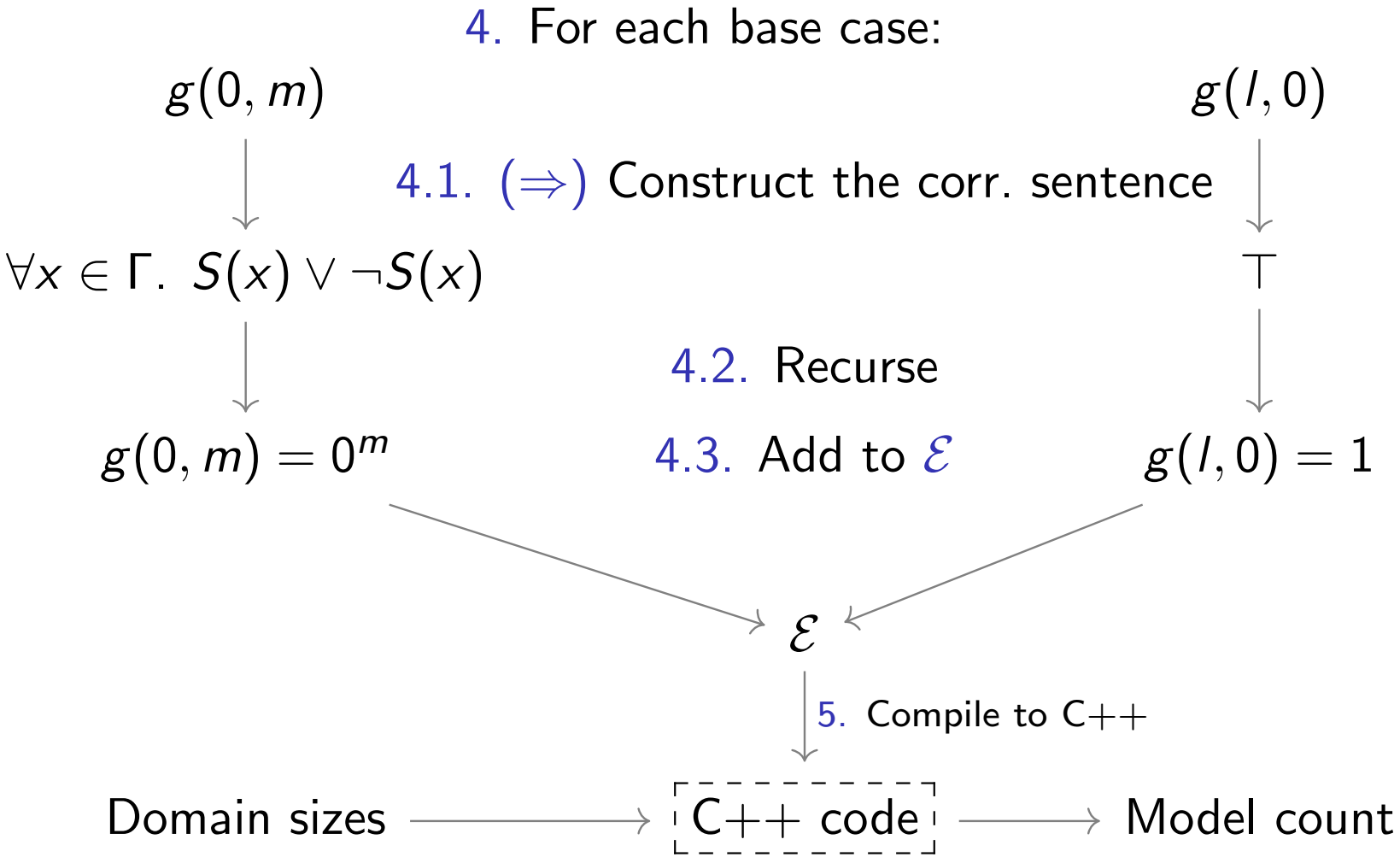


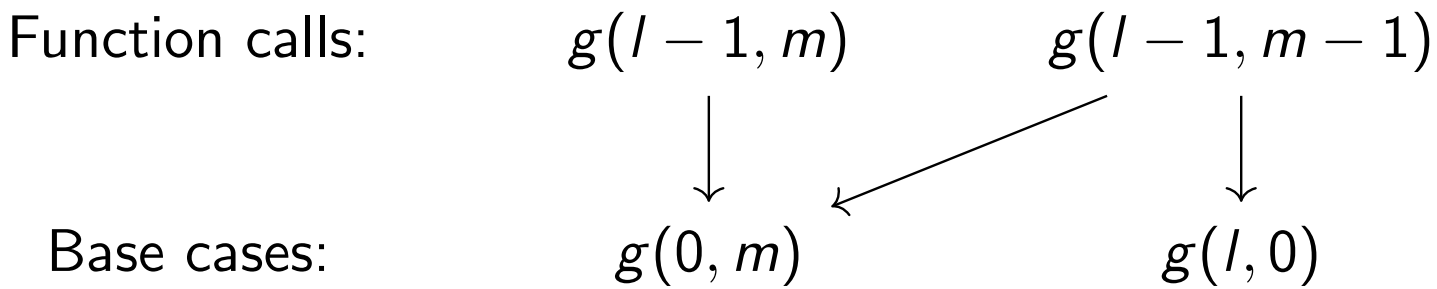
Finding (a Sufficient Set of) Base Cases

Outline

1. For every function call:
   - 1.1 For every argument of the form $var - const$:
     - 1.1.1 Replace the signature parameter with $0, 1, \ldots, const - 1$
   - 1.2 For every argument of the form $const$:
     - 1.2.1 Replace the corresponding signature parameter with $const$

Example

The signature of g is $g(l, m)$.



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The evaluation of a recursive function always terminates.

Proof (hints).

- There exists a topological ordering of functions
- All function calls follow the structure from the previous slide
- Some common-sense assumptions about the evaluation order and previous work

□

From a Base Case to a Sentence

Preliminaries

- CRANE associates each function f with a sentence ϕ such that CRANE(ϕ) produces the definition of f
- There is a bijection between the parameters of f and the domains of ϕ

Benchmarks

- Friends & Smokers

$$(\forall x, y \in \Delta.\ S(x) \wedge F(x, y) \rightarrow S(y)) \wedge (\forall x \in \Delta.\ S(x) \rightarrow C(x))$$

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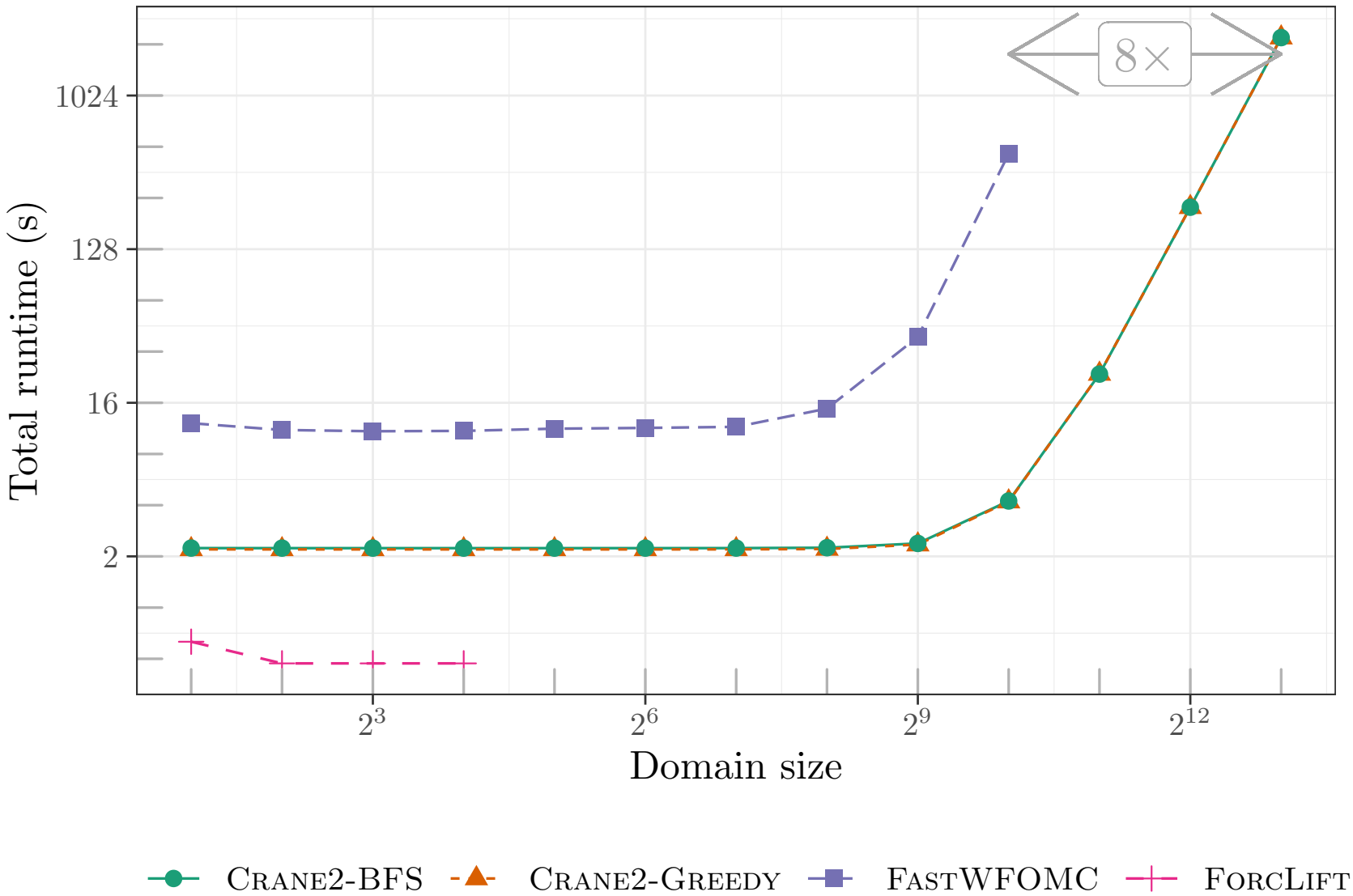
Friends & Smokers



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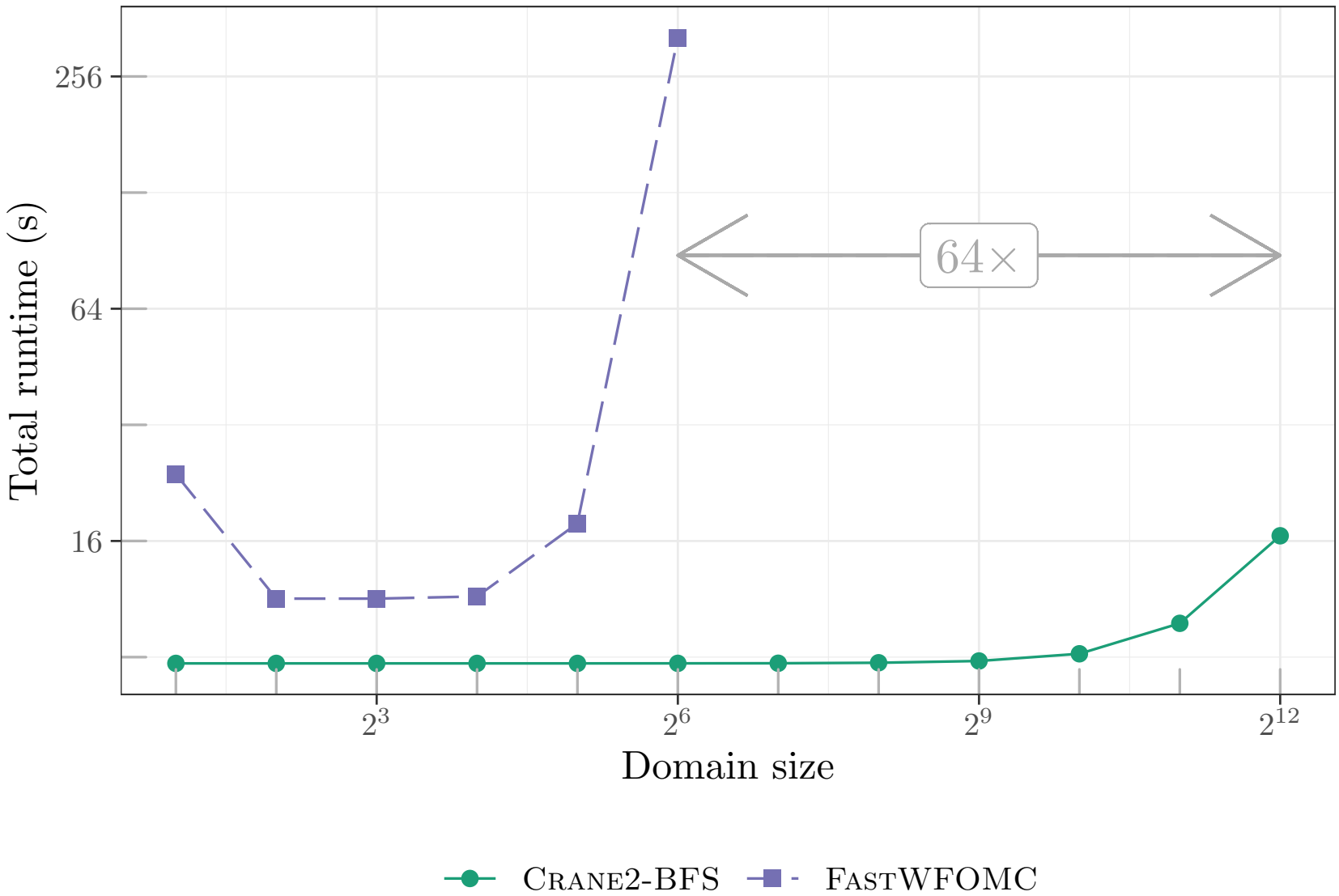
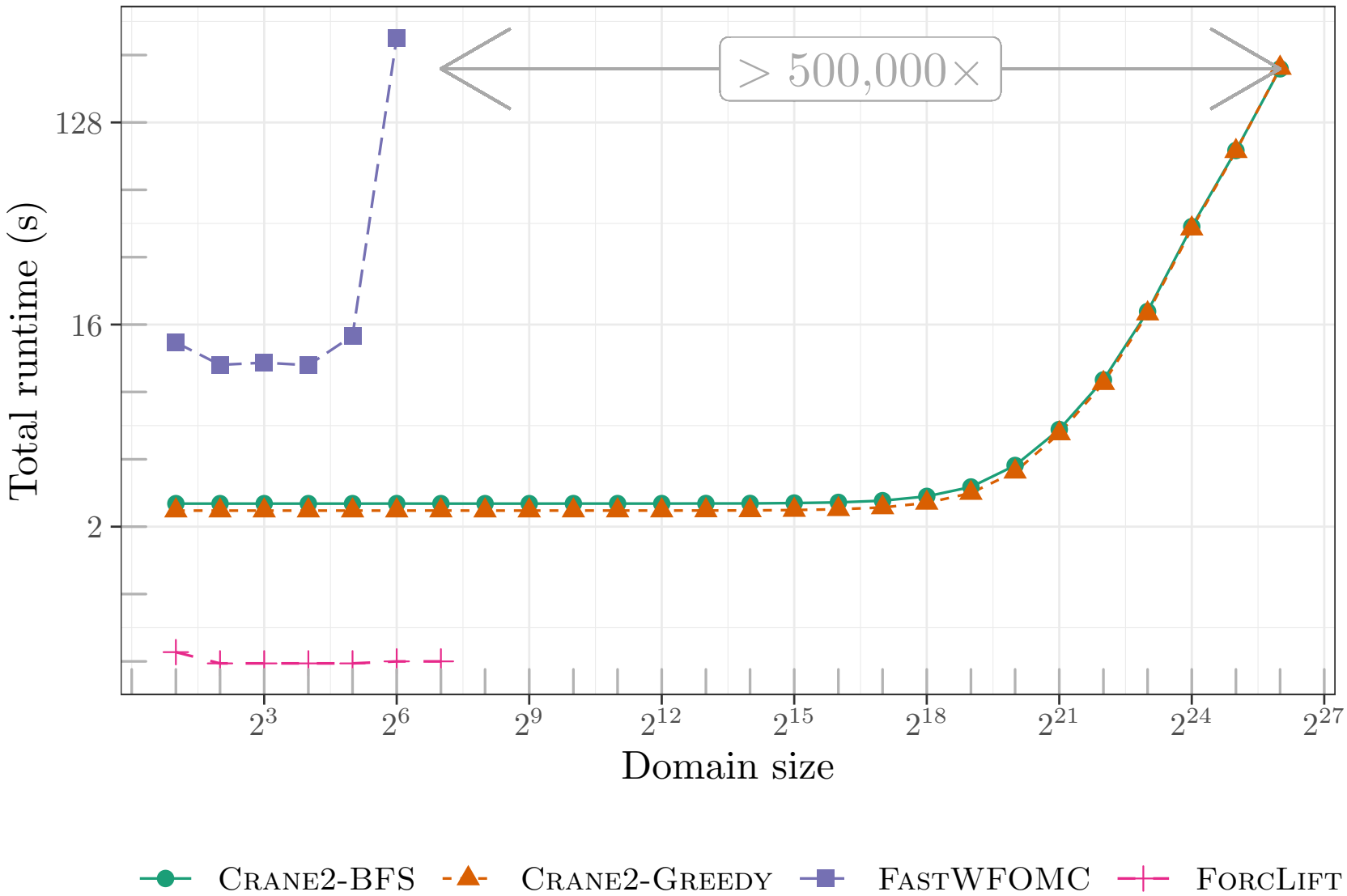

256
64
16
Total runtime (s)
64×
2^3
2^6
2^9
2^{12}
Domain size
Crane2-BFS
FastWFOMC

Functions


> 500,000×
128
16
2
Total runtime (s)
2^3
2^6
2^9
2^{12}
2^{15}
2^{18}
2^{21}
2^{24}
2^{27}
Domain size
Crane2-BFS
Crane2-Greedy
FastWFOMC
ForcLift

Summary

TODO: and future work