What Boolean Algebras Can Teach Us About Weighted Model Counting

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1 Introduction

Contributions.

- WMC defines a measure over a BA.
- WMC with weights on literals imposes an independence assumption. (Measures are 'slightly' more expressive than WMC with weights on models because they apply to non-atomic BAs.)
- A BA can be augmented with new literals in order to support any measure.
- (Maybe) a lower bound on the number of new literals needed in order to support any measure.
- Alternatively, one can use coproducts and pushouts to define a BA with precisely the right independence and conditional independence conditions. (This requires a relaxed version of WMC.)
- This results in a smaller problem for WMC algorithms (w.r.t. both the number of literals and the length of the theory) and is optimal for, e.g., Bayesian networks.
- (Maybe) this results in faster inference (?)

Notable previous/related work.

- Hailperin's approach to probability logic [9]
- Nilsson's (somewhat successful) probabilistic logic [14]
- Logical induction: a big paper with a good overview of previous attempts to assign probabilities to logical sentences in a sensible way [7]
- Measures on Boolean algebras
  - On possibility and probability measures in finite Boolean algebras [2]
  - Representation of conditional probability measures [12]

2 Preliminaries

Definition 1. A *Boolean algebra* (BA) is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ consisting of a set $\mathbf{B}$ with binary operations *meet* $\wedge$ and *join* $\vee$, unary operation $\neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;

- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;
- $\vee$ distributes over $\wedge$ and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

For clarity and succinctness, we will occasionally use three other operations that can be defined using the original three[1]:

$$
\begin{aligned}
a \to b &= \neg a \vee b, \\
a \leftrightarrow b &= (a \wedge b) \vee (\neg a \wedge \neg b), \\
a + b &= (a \wedge \neg b) \vee (\neg a \wedge b).
\end{aligned}
$$

We can also define a partial order $\leq$ on $\mathbf{B}$ as $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$) for all $a, b \in \mathbf{B}$. Furthermore, let $a < b$ denote $a \leq b$ and $a \neq b$. For the rest of this paper, let $\mathbf{B}$ refer to the BA $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$. For any $S \subseteq \mathbf{B}$, we write $\bigvee S$ for $\bigvee_{x \in S} x$ and call it the *supremum* of S. Similarly, $\bigwedge S = \bigwedge_{x \in S} x$ is the *infimum*. By convention, $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$. For any $a, b \in \mathbf{B}$, we say that a and b are *disjoint* if $a \wedge b = 0$.

Definition 2 ([10, 13])**.** An element $a \neq 0$ of $\mathbf{B}$ is an *atom* if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. Equivalently, $a \neq 0$ is an atom if there is no $x \in \mathbf{B}$ such that $0 < x < a$. We say that $\mathbf{B}$ is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 3 ([6])**.** *For any two distinct atoms a, $b \in \mathbf{B}$, $a \wedge b = 0$.*

Lemma 4 ([8])**.** *The following are equivalent:*

- $\mathbf{B}$ *is atomic.*
- *For any $x \in \mathbf{B}$, $x = \bigvee_{atoms\ a \leq x} a$.*
- 1 *is the supremum of all atoms.*

Lemma 5 ([8])**.** *All finite BAs are atomic.*

Definition 6 ([5, 10])**.** A *measure* on $\mathbf{B}$ is a function $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ such that:

- $m(0) = 0$;
- $m(a \vee b) = m(a) + m(b)$ for all $a, b \in \mathbf{B}$ whenever $a \wedge b = 0$.

If $m(1) = 1$, we call m a *probability measure*. Also, if $m(x) > 0$ for all $x \neq 0$, then m is *strictly positive*.

Definition 7 ([8])**.** Let $\mathbf{A}$ and $\mathbf{B}$ be BAs. A *(Boolean) homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $f \colon \mathbf{A} \to \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,
- $f(x \vee y) = f(x) \vee f(y)$,
- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Lemma 8 (Homomorphisms preserve order [8])**.** *Let $f \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism between two BAs $\mathbf{A}$ and $\mathbf{B}$. Then, for any $x, y \in \mathbf{A}$, if $x \leq y$, then $f(x) \leq f(y)$.*

Lemma 9 ([16])**.** *For any $a, b \in \mathbf{B}$, $a \leq b$ if and only if $a \wedge \neg b = 0$.*

Lemma 10 ([8])**.** *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure. Then for all $a, b \in \mathbf{B}$, if $a \leq b$, then $m(a) \leq m(b)$.*

[1]We use $+$ to denote symmetric difference because it is the additive operation of a Boolean ring.

3 WMC as a Measure

Definition 11. Let $\mathcal{L}$ be a propositional (or first-order) logic, and let Δ be a theory in $\mathcal{L}$. We can define an equivalence relation on formulas in $\mathcal{L}$ as

$$\alpha \sim \beta \quad \text{if and only if} \quad \Delta \vdash \alpha \leftrightarrow \beta$$

for all $\alpha, \beta \in \mathcal{L}$. Let $[\alpha]$ denote the equivalence class of $\alpha \in \mathcal{L}$ with respect to $\sim$. We can then let $B(\Delta) = \{[\alpha] \mid \alpha \in \mathcal{L}\}$ and define the structure of a BA on $B(\Delta)$ as

$$\begin{aligned}
[\alpha] \vee [\beta] &= [\alpha \vee \beta], \\
[\alpha] \wedge [\beta] &= [\alpha \wedge \beta], \\
\neg[\alpha] &= [\neg\alpha], \\
1 &= [\alpha \to \alpha], \\
0 &= [\alpha \wedge \neg\alpha]
\end{aligned}$$

for all $\alpha, \beta \in \mathcal{L}$. Then $B(\Delta)$ is the *Lindenbaum-Tarski algebra* of Δ [11, 17].

Example 12. Let $\mathcal{L}$ be a propositional logic with p and q as its only atoms. Then $L = \{p, q, \neg p, \neg q\}$ is its set of literals. Let $w : L \to \mathbb{R}_{\geq 0}$ be the *weight function* defined by

$$\begin{aligned}
w(p) &= 0.3, \\
w(\neg p) &= 0.7, \\
w(q) &= 0.2, \\
w(\neg q) &= 0.8.
\end{aligned}$$

Let Δ be a theory in $\mathcal{L}$ with a sole axiom p. Then Δ has two models, i.e., $\{p, q\}$ and $\{p, \neg q\}$. The *weighted model count* (WMC) [3] of Δ is then

$$\sum_{\omega \models \Delta} \prod_{\omega \models l} w(l) = w(p)w(q) + w(p)w(\neg q) = 0.3.$$

The corresponding BA $B(\Delta)$ can then be constructed using Definition 11. Alternatively, one can first construct the free BA generated by the set $\{p, q\}$ and then take a quotient with respect to either the filter generated by p or the ideal[2] generated by $\neg p$.

Each element of $B(\mathcal{L})$ can also be seen as a subset of the set of all models of $\mathcal{L}$, with 0 representing $\emptyset$, 1 representing the set of all (four) models, each atom representing a single model, and each edge going upward representing a subset relation. Thus, the Boolean-algebraic way of calculating the WMC of Δ consists of:

1. Identifying an element $a \in B(\mathcal{L})$ that corresponds to Δ.
2. Finding all atoms of $B(\mathcal{L})$ that are 'dominated' by a according to the partial order.
3. Using w to calculate the weight of each such atom.
4. Adding the weights of these atoms.

This motivates the following definition of WMC generalised to BAs.

Definition 13. Let $\mathbf{B}$ be an atomic BA, and let $M \subset \mathbf{B}$ be its set of atoms. Let $L \subset \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge L'$ for some $L' \subseteq L$, and let $w \colon L \to \mathbb{R}_{\geq 0}$ be arbitrary. The *weighted model count* $\mathrm{WMC}_w \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}_w(x) = \begin{cases} 0 & \text{if } x = 0 \\ \prod_{l \in L'} w(l) & \text{if } M \ni x = \bigwedge L' \\ \sum_{\text{atoms } a \leq x} \mathrm{WMC}_w(a) & \text{otherwise} \end{cases}$$

[2] More details on these concepts can be found in many books on BAs [8, 11].

for any $x \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{NWMC}_w \colon \mathbf{B} \to [0,1]$ as $\mathrm{NWMC}_w(x) = \frac{\mathrm{WMC}_w(x)}{\mathrm{WMC}_w(1)}$ for all $x \in \mathbf{B}$. For both WMC_w and NWMC_w, we will drop the subscript when doing so results in no potential confusion. Finally, we say that a measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is a *WMC measure* (or is *WMC-definable*) if there exists a subset $L \subset \mathbf{B}$ and a weight function $w \colon L \to \mathbb{R}_{\geq 0}$ such that $m = \mathrm{WMC}_w$.

Theorem 14. WMC *is a measure, and* NWMC *is a probability measure.*

Proof. First, note that WMC is non-negative and $\mathrm{WMC}(0) = 0$ by definition. Next, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Lemma 4 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned} \mathrm{WMC}(x \vee y) &= \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i \right) \vee \left(\bigvee_{j \in J} y_j \right) \right) = \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j) \\ &= \mathrm{WMC}(x) + \mathrm{WMC}(y), \end{aligned}$$

finishing the proof that WMC is a measure. This immediately shows that NWMC is a probability measure since, by definition, $\mathrm{NWMC}(1) = 1$. ☐

Given a theory Δ in a logic $\mathcal{L}$, the usual way of using WMC to compute the probability of a query q is [1, 15]

$$\Pr_{\Delta, w}(q) = \frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}.$$

In our algebraic formulation, this can be computed in two different ways:

- as $\frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}$ in $B(\mathcal{L})$,
- and as $\mathrm{NWMC}_w([q])$ in $B(\Delta)$.

But how does the measure defined on $B(\mathcal{L})$ transfer to $B(\Delta)$?

4 What Measures Are WMC-Definable?

4.1 WMC Requires Independent Literals

Lemma 15. *For any measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ and elements $a, b \in \mathbf{B}$,*

$$m(a \wedge b) = m(a)m(b) \tag{2}$$

if and only if

$$m(a \wedge b) \cdot m(\neg a \wedge \neg b) = m(a \wedge \neg b) \cdot m(\neg a \wedge b). \tag{3}$$

Proof. First, note that $a = (a \wedge b) \vee (a \wedge \neg b)$ and $(a \wedge b) \wedge (a \wedge \neg b) = 0$, so, by properties of a measure,

$$m(a) = m(a \wedge b) + m(a \wedge \neg b). \tag{4}$$

Applying Eq. (4) and the equivalent expression for $m(b)$ allows us to rewrite Eq. (2) as

$$m(a \wedge b) = [m(a \wedge b) + m(a \wedge \neg b)][m(a \wedge b) + m(\neg a \wedge b)]$$

which is equivalent to

$$m(a \wedge b)[1 - m(a \wedge b) - m(a \wedge \neg b) - m(\neg a \wedge b)] = m(a \wedge \neg b)m(\neg a \wedge b). \tag{5}$$

Since $a \wedge b$, $a \wedge \neg b$, $\neg a \wedge b$, $\neg a \wedge \neg b$ are pairwise disjoint and their supremum is 1,

$$m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b) + m(\neg a \wedge \neg b) = 1,$$

and this allows us to rewrite Eq. (5) into Eq. (3). As all transformations are invertible, the two expressions are equivalent. ☐

Theorem 16. *Let* $\mathbf{B}$ *be a free BA on* n *generators* $\{l_i\}_{i=1}^n$ *(for some* $n \in \mathbb{N}$*) with measure* $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$*, and let* $L = \{l_i\}_{i=1}^n \cup \{\neg l_i\}_{i=1}^n$*. Then there exists a weight function* $w\colon L \to \mathbb{R}_{\geq 0}$ *such that* $m = \mathrm{WMC}_w$ *if and only if*

$$m(l \wedge l') = m(l)m(l') \tag{6}$$

for all distinct $l, l' \in L$ *such that* $l \neq \neg l'$.

Remark. Note that if $n = 1$, then Eq. (6) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. ($\Leftarrow$) Let $w\colon L \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = m(l) \tag{7}$$

for all $l \in L$. We are going to show that $\mathrm{WMC}_w = m$. First, note that $\mathrm{WMC}_w(0) = 0 = m(0)$ by the definitions of both WMC_w and m. Second, let

$$a = \bigwedge_{i=1}^n a_i \tag{8}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\mathrm{WMC}(a) = \prod_{i=1}^n w(a_i) = \prod_{i=1}^n m(a_i) = m\left(\bigwedge_{i=1}^n a_i\right) = m(a)$$

by Definition 13 and Eqs. (6) to (8). Finally, note that if WMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

($\Rightarrow$) For the other direction, we are given a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ that induces a measure $m = \mathrm{WMC}_w\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (6) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$ for some $i, j \in [n]$. We then want to show that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \tag{9}$$

which is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j) \tag{10}$$

by Lemma 15. Then

$$\begin{aligned}
\mathrm{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \leq k_i \wedge k_j} \mathrm{WMC}(a) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\
&= \sum_{\text{atoms } a \leq k_i \wedge k_j} w(a_i) w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \leq k_i \wedge k_j} w(k_i) w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) \\
&= w(k_i) w(k_j) \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i) w(k_j) C,
\end{aligned}$$

where C denotes the part of $\mathrm{WMC}(k_i \wedge k_j)$ that will be the same for $\mathrm{WMC}(\neg k_i \wedge k_j)$, $\mathrm{WMC}(k_i \wedge \neg k_j)$, and $\mathrm{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (10) becomes

$$w(k_i) w(k_j) w(\neg k_i) w(\neg k_j) C^2 = w(k_i) w(\neg k_j) w(\neg k_i) w(k_j) C^2$$

which is trivially true. $\square$

4.2 Extending the Algebra

A well-known way to overcome this limitation of independence is by adding more literals [3], i.e., extending the set L covered by the WMC weight function $w \colon L \to \mathbb{R}_{\geq 0}$. Let us translate this idea to the language of Boolean algebras.

Theorem 17. *Let $\mathbf{B}$ be a finite Boolean algebra freely generated by some set of 'literals' L, and let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be an arbitrary measure. We know that $\mathbf{B}$ has $n = 2^{|L|}$ atoms. Let $(a_i)_{i=1}^n$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i \mid i \in [n]\} \cup \{\neg\phi_i \mid i \in [n]\}$ be the set L extended with $2n$ new literals. Let $\mathbf{B}'$ be the unique Boolean algebra with*

$$\{\phi_i \wedge a_i \mid i \in [n]\} \cup \{\neg\phi_i \wedge a_i \mid i \in [n]\}$$

as its set of atoms. Let $\iota \colon \mathbf{B} \to \mathbf{B}'$ be the inclusion homomorphism (i.e., $\iota(a) = a$ for all $a \in \mathbf{B}$). Let $w \colon L' \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = \begin{cases} \frac{m(a_i)}{2} & \text{if } l = \phi_i \text{ or } l = \neg\phi_i \text{ for some } i \in [n] \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m' \colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In simpler terms, any measure can be computed using WMC by extending the BA with more literals. More precisely, we are given the red part in

$$\begin{tikzcd}
\mathbb{R}_{\geq 0} & & \\
\mathbf{B} \arrow[u, "m"] \arrow[r, hook, "\iota"] & \mathbf{B}' \arrow[ul, "m'"'] & \\
L \arrow[u, hook] \arrow[r, hook] & L' \arrow[u, hook] \arrow[r, "w"] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

and construct the black part in such a way that the triangle commutes.

Proof. Since $\mathbf{B}$ is freely generated by L, each atom $a_i \in \mathbf{B}$ is an infimum of elements in L, i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either $b = \phi_i \wedge a_i$ or $b = \neg\phi_i \wedge a_i$ for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L'. Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies $\phi_i \wedge a_i \leq \iota(b)$ if and only if it satisfies $\neg\phi_i \wedge a_i \leq \iota(b)$. Then, according to the definition of w,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \leq \iota(b) \quad \text{if and only if} \quad a_i \leq b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. □

Now we can show that the construction in Theorem 17 is smallest possible.

Conjecture 18. *Let $\mathbf{B}$ and $\mathbf{B}'$ be Boolean algebras, and $\iota\colon \mathbf{B} \hookrightarrow \mathbf{B}'$ be the inclusion map such that $\mathbf{B}$ is freely generated by L, all atoms of $\mathbf{B}'$ can be expressed as meets of elements of L', and the following subset relations are satisfied:*

$$\begin{array}{ccc} \mathbf{B} & \xhookrightarrow{\iota} & \mathbf{B}' \\ \cup & & \cup \\ L & \subset & L' \end{array}$$

If, for any measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, one can construct a weight function $w\colon L' \to \mathbb{R}_{\geq 0}$ such that the WMC measure $\mathrm{WMC}\colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$ *with respect to w satisfies*

$$m = \mathrm{WMC} \circ \iota,$$

then $|L' \setminus L| \geq 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [4] and Sang et al. [15]. Suppose we have a discrete probability distribution with n variables, and the ith variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^{n} v_i$ models. My expansion would then use

$$\sum_{i=1}^{n} v_i + 2 \prod_{i=1}^{n} v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [4] would use

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \prod_{j=1}^{i} v_j$$

variables, while for the encoding by Sang et al. [15],

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n} (v_i - 1) \prod_{j=1}^{i-1} v_j$$

variables would suffice.

5 Representing Independence and Conditional Independence

5.1 Independence (all known results)

Definition 19. Given a BA $\mathbf{A}$, a *subalgebra* is a subset $\mathbf{B} \subseteq \mathbf{A}$ that, together with the operations, zero, and one of $\mathbf{A}$, is a BA.

Definition 20 ([8])**.** Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be BAs such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Let $f\colon \mathbf{A} \to \mathbf{C}$ and $g\colon \mathbf{B} \to \mathbf{C}$ be homomorphisms. Then f is an *extension* of g if $f(x) = g(x)$ for all $x \in \mathbf{B}$. If f is an extension of each member of a family $\{g_i\}_{i \in I}$ of homomorphisms, then f is called a *common extension* of $\{g_i\}_{i \in I}$.

Definition 21 ([8])**.** Let $\{\mathbf{A}_i\}_{i \in I}$ be a family of subalgebras of a BA $\mathbf{A}$. If for any BA $\mathbf{B}$ with a family of homomorphisms $\{f_i\colon \mathbf{A}_i \to \mathbf{B}\}_{i \in I}$ there exists a unique common extension of $\{f_i\colon \mathbf{A}_i \to \mathbf{B}\}_{i \in I}$ ($f\colon \mathbf{A} \to \mathbf{B}$ in the diagram),

$$\begin{tikzcd} \mathbf{A}_i \arrow[r, hook] \arrow[dr, "f_i"'] & \mathbf{A} \arrow[d, dashed, "f"] \\ & \mathbf{B} \end{tikzcd}$$

then $\mathbf{A}$ is the *internal sum*[3] of $\{\mathbf{A}_i\}_{i \in I}$. We will denote it as $\bigoplus_{i \in I} \mathbf{A}_i$.

Proposition 22 ([16])**.** *Let $\mathbf{A}$ be the internal sum of a family of BAs $\{\mathbf{A}_i\}_{i \in I}$, and let $\{m_i\colon \mathbf{A}_i \to \mathbb{R}_{\geq 0}\}_{i \in I}$ be a family of measures. Then there exists a unique measure $m\colon \mathbf{A} \to \mathbb{R}_{\geq 0}$ such that, for any finite subset $J \subseteq I$ and family of elements $\{x_j \in \mathbf{A}_j\}_{j \in J}$,*

$$m\left(\bigwedge_{j \in J} x_j\right) = \prod_{j \in J} m_j(x_j).$$

5.2 Conditional Independence

Definition 23 ([11])**.** Let $\mathbf{A}$ be a BA. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, and let $\{\mathbf{A}_i\}_{i \in I}$ be a family of subalgebras of $\mathbf{A}$ such that $\mathbf{A}_i \cap \mathbf{A}_j = \mathbf{B}$ for all $i \neq j$ in I. Let $\{\iota_i\colon \mathbf{B} \to \mathbf{A}_i\}$ be a family of inclusion homomorphisms. Then $\mathbf{A}$ is the *amalgamated free product*[4] *of* $\{\mathbf{A}_i\}_{i \in I}$ *over* $\mathbf{B}$ if, for any Boolean algebra $\mathbf{C}$ with a family of homomorphisms $\{f_i\colon \mathbf{A}_i \to \mathbf{C}\}_{i \in I}$ such that $f_i \circ \iota_i = f_j \circ \iota_j$ for all $i, j \in I$, there is a unique homomorphism $f\colon \mathbf{A} \to \mathbf{C}$ such that the triangle in

$$\begin{tikzcd} \mathbf{B} \arrow[r, hook, "\iota_i"] & \mathbf{A}_i \arrow[r, hook] \arrow[d, "f_i"'] & \mathbf{A} \arrow[dl, dashed, "f"] \\ & \mathbf{C} & \end{tikzcd}$$

commutes for all $i \in I$. We will denote this product as

$$\mathbf{A} = \bigoplus_{\substack{\mathbf{B} \\ i \in I}} \mathbf{A}_i.$$

[3] A slightly more general version of this definition is also known as the free product, the Boolean product, and the coproduct in the category of BAs [8, 11, 16].

[4] Also known as a (wide) pushout in the category of BAs.

5.2.1 New Results

Theorem 24. *Let $\{\mathbf{A}_i\}_{i \in I}$ be a family of BAs with measures $\{m_i \colon \mathbf{A}_i \to \mathbb{R}_{\geq 0}\}_{i \in I}$, let $\mathbf{B}$ be a BA, and let $\mathbf{A}$ be the amalgamated free product of $\{\mathbf{A}_i\}_{i \in I}$ over $\mathbf{B}$. Then there is a unique measure $m \colon \mathbf{A} \to \mathbb{R}_{\geq 0}$ such that, for any element $b \in \mathbf{B}$, finite subset $J \subseteq I$, and elements $\{a_j \in \mathbf{A}_j\}_{j \in J}$,*

$$m\left(b \wedge \bigwedge_{j \in J} a_j\right) = \prod_{j \in J} m_j(b \wedge a_j).$$

Theorem 25. *The number of weights needed to encode a Bayesian network using coproducts and pushouts is equal to the number of entries in the tables of the network (and the resulting theory is shorter).*

Theorem 26 (Pushouts of free BAs are free)**.** *Let*

$$\mathbf{A} = \bigoplus_{\substack{\mathbf{B} \\ i \in I}} \mathbf{A}_i$$

be an amalgamated free product such that $\{\mathbf{A}_i\}_{i \in I}$ are free BAs with $\{S_i\}_{i \in I}$ as their respective sets of generators. Let $S = \bigcup_{i \in I} S_i$. Then $\mathbf{A}$ is a free BA with generating set S.

Proof. Suppose we have a map from S to an arbitrary BA $\mathbf{C}$, as in

$$\begin{tikzcd}
S_i \arrow[r, hook] \arrow[d, hook] & S \arrow[d, hook] \arrow[ddr, bend left] & \\
\mathbf{A}_i \arrow[r, hook] \arrow[drr, dashed, bend right] & \mathbf{A} \arrow[dr, dashed] & \\
& & \mathbf{C}.
\end{tikzcd}$$

We want to show that there exists a unique homomorphism $\mathbf{A} \to \mathbf{C}$. For all $i \in I$, from $S_i \hookrightarrow S$ and $S \to \mathbf{C}$ we get a map $S_i \to \mathbf{C}$, so—by the definition of a free BA—there is a unique homomorphism $\mathbf{A}_i \to \mathbf{C}$. Furthermore, a family of homomorphisms $\{\mathbf{A}_i \to \mathbf{C}\}_{i \in I}$ uniquely determine a homomorphism $\mathbf{A} \to \mathbf{C}$ by the universal mapping property of a (wide) pushout. Thus $\mathbf{A}$ is a free BA with generating set S. ☐

Corollary 27. *Similarly, coproducts of free BAs are free.*

References

[1] Vaishak Belle. Weighted model counting with function symbols. In Gal Elidan, Kristian Kersting, and Alexander T. Ihler, editors, *Proceedings of the Thirty-Third Conference on Uncertainty in Artificial Intelligence, UAI 2017, Sydney, Australia, August 11-15, 2017*. AUAI Press, 2017.

[2] Elena Castiñeira, Susana Cubillo, and Enric Trillas. On possibility and probability measures in finite Boolean algebras. *Soft Comput.*, 7(2):89–96, 2002.

[3] Mark Chavira and Adnan Darwiche. On probabilistic inference by weighted model counting. *Artif. Intell.*, 172(6-7):772–799, 2008.

[4] Adnan Darwiche. A logical approach to factoring belief networks. In Dieter Fensel, Fausto Giunchiglia, Deborah L. McGuinness, and Mary-Anne Williams, editors, *Proceedings of the Eights International Conference on Principles and Knowledge Representation and Reasoning (KR-02), Toulouse, France, April 22-25, 2002*, pages 409–420. Morgan Kaufmann, 2002.

[5] Haim Gaifman. Concerning measures on Boolean algebras. *Pacific Journal of Mathematics*, 14(1):61–73, 1964.

[6] M. Ganesh. *Introduction to fuzzy sets and fuzzy logic.* PHI Learning Pvt. Ltd., 2006.

[7] Scott Garrabrant, Tsvi Benson-Tilsen, Andrew Critch, Nate Soares, and Jessica Taylor. Logical induction. *Electronic Colloquium on Computational Complexity (ECCC)*, 23:154, 2016.

[8] Steven Givant and Paul R. Halmos. *Introduction to Boolean algebras.* Springer Science & Business Media, 2008.

[9] Theodore Hailperin. Probability logic. *Notre Dame Journal of Formal Logic*, 25(3):198–212, 1984.

[10] Thomas Jech. *Set theory, Second Edition.* Perspectives in Mathematical Logic. Springer, 1997.

[11] Sabine Koppelberg, Robert Bonnet, and James Donald Monk. *Handbook of Boolean algebras*, volume 384. North-Holland Amsterdam, 1989.

[12] Peter H. Krauss. Representation of conditional probability measures on Boolean algebras. *Acta Mathematica Hungarica*, 19(3-4):229–241, 1968.

[13] Ken Levasseur and Al Doerr. *Applied Discrete Structures.* Lulu.com, 2012.

[14] Nils J. Nilsson. Probabilistic logic. *Artif. Intell.*, 28(1):71–87, 1986.

[15] Tian Sang, Paul Beame, and Henry A. Kautz. Performing Bayesian inference by weighted model counting. In Manuela M. Veloso and Subbarao Kambhampati, editors, *Proceedings, The Twentieth National Conference on Artificial Intelligence and the Seventeenth Innovative Applications of Artificial Intelligence Conference, July 9-13, 2005, Pittsburgh, Pennsylvania, USA*, pages 475–482. AAAI Press / The MIT Press, 2005.

[16] Roman Sikorski. *Boolean algebras.* Springer, third edition, 1969.

[17] Alfred Tarski. *Logic, semantics, metamathematics: papers from 1923 to 1938.* Hackett Publishing, 1983.