Weighted Model Counting/Integration from the Perspective of Boolean Algebras

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1 Introduction

Previous/related work:

- Hailperin's approach to probability logic [10]
- Nilsson's (somewhat successful) probabilistic logic [16]
- Semiring programming [3]
- WMC with functions [1]
- WMI [2]
- Measures on Boolean algebras: overview articles (from most cited to least cited)
  - Horn and Tarski [11]
  - Concerning measures on Boolean algebras [7]
  - Jech – Measures on Boolean algebras (arXiv) [13]
- Measures on Boolean algebras: more specific articles
  - On possibility and probability measures in finite Boolean algebras [4]
  - Representation of conditional probability measures [14]

2 WMC as a Measure

2.1 Preliminaries

Definition 1. A *Boolean algebra* (BA) is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ of a set $\mathbf{B}$ with operations $\wedge, \vee, \neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;
- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;
- $\vee$ distributes over $\wedge$ and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

We can also define a partial order $\leq$ on $\mathbf{B}$ as $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$) for $a, b \in \mathbf{B}$. Furthermore, let $a < b$ denote $a \leq b$ and $a \neq b$.

For the rest of this paper, let $\mathbf{B}$ refer to the BA $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$.

Definition 2 ([12, 15])**.** An element $a \neq 0$ of $\mathbf{B}$ is an *atom* if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. Equivalently, $a \neq 0$ is an atom if there is no $x \in \mathbf{B}$ such that $0 < x < a$. A BA $\mathbf{B}$ is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([8])**.** *For any two distinct atoms a, $b \in \mathbf{B}$, $a \wedge b = 0$.*

Lemma 2 ([9])**.** *All finite BAs are atomic.*

Theorem 1. *Let $\mathbf{B}$ be a finite Boolean algebra. Then every $x \in \mathbf{B} \setminus \{0\}$ can be uniquely expressed as*

$$x = \bigvee_{\textit{atoms } a \leq x} a.$$

Proof. A simple consequence of the theorem that every finite Boolean algebra is isomorphic to a field of subsets of a set, where the cardinality of the set is equal to the number of atoms in the Boolean algebra. ☐

Remove the requirement for being strictly positive

Definition 3 ([7])**.** A *(strictly positive) measure* on a Boolean algebra $\mathbf{B}$ is a function $m \colon \mathbf{B} \to [0, 1]$ such that:

1. $m(1) = 1$, and $m(x) > 0$ for $x \neq 0$;
2. $m(x \vee y) = m(x) + m(y)$ for all $x, y \in \mathbf{B}$ whenever $x \wedge y = 0$.

2.2 New Results

Allow weight to be zero

Definition 4. Let $\mathbf{B}$ be a finite Boolean algebra, and let $M \subseteq \mathbf{B}$ be its set of atoms. Let $L \subseteq \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge_{i \in I} l_i$ for some $\{l_i\}_{i \in I} \subseteq L$, and let $w \colon L \to \mathbb{R}_{>0}$ be arbitrary. The *weighted model count* $\mathrm{WMC} \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}(a) = \begin{cases} 0 & \text{if } a = 0 \\ \prod_{i \in I} w(l_i) & \text{if } M \ni a = \bigwedge_{i \in I} l_i \\ \sum_{i \in I} \mathrm{WMC}(m_i) & \text{if } \mathbf{B} \setminus (M \cup \{0\}) \ni a = \bigvee_{i \in I} m_i \end{cases}$$

for any $a \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{nWMC} \colon \mathbf{B} \to [0, 1]$ as $\mathrm{nWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)}$ for all $a \in \mathbf{B}$.

Proposition 1. nWMC *is a measure for any finite Boolean algebra* $\mathbf{B}$.

Proof. First, note that Property 1 of Definition 3 is satisfied by the definition of nWMC. Next, in order to prove Property 2, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{nWMC}(x \vee y) = \mathrm{nWMC}(x) + \mathrm{nWMC}(y)$$

which is equivalent to

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Theorem 1 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned} \mathrm{WMC}(x \vee y) &= \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i \right) \vee \left(\bigvee_{j \in J} y_j \right) \right) = \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j) \\ &= \mathrm{WMC}(x) + \mathrm{WMC}(y), \end{aligned}$$

finishing the proof. □

3 What Measures Are WMC-Computable?

3.1 WMC Requires Independent Literals

Proposition 2. *Let $\mathbf{B}$ be a finite measure algebra with measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$. Let $L \subset \mathbf{B}$ be defined as*

$$L = \{l_i \mid i \in [n]\} \cup \{\neg l_i \mid i \in [n]\}$$

for some $n \in \mathbb{N}$. Finally, assume that $\mathbf{B}$ has 2^n atoms, where each atom $a \in \mathbf{B}$ is an infimum

$$a = \bigwedge_{i=1}^{n} a_i$$

such that $a_i \in \{l_i, \neg l_i\}$ for $i \in [n]$. Then there exists a weight function $w \colon L \to \mathbb{R}_{>0}$ that makes m a WMC measure if and only if

$$m(l \wedge l') = m(l)m(l') \tag{2}$$

for all distinct $l, l' \in L$ such that $l \neq \neg l'$.

Remark. Note that if $n = 1$, then Eq. (2) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. Let us begin with the 'if' part of the statement. Let $w \colon L \to \mathbb{R}_{>0}$ be defined by

$$w(l) = m(l) \tag{3}$$

for all $l \in L$. We are going to show that $\mathrm{nWMC} = m$. First, note that $\mathrm{nWMC}(0) = 0 = m(0)$ by the definitions of both nWMC and m. Second, let

$$a = \bigwedge_{i=1}^{n} a_i \tag{4}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\mathrm{nWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)} = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} w(a_i) = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} m(a_i) = \frac{1}{\mathrm{WMC}(1)} m\left(\bigwedge_{i=1}^{n} a_i\right) = \frac{m(a)}{\mathrm{WMC}(1)}$$

by Definition 4 and Eqs. (2) to (4). Now we just need to show that $\mathrm{WMC}(1) = 1$. Indeed,

$$\begin{aligned} \mathrm{WMC}(1) &= \sum_{\text{atoms } a \in \mathbf{B}} \mathrm{WMC}(a) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} w(a_i) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} m(a_i) \\ &= \sum_{\text{atoms } a \in \mathbf{B}} m\left(\bigwedge_{i=1}^{n} a_i\right) = \sum_{\text{atoms } a \in \mathbf{B}} m(a) = m\left(\bigvee_{\text{atoms } a \in \mathbf{B}}\right) = m(1) = 1. \end{aligned}$$

Finally, note that if nWMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

For the other direction, we are given a weight function $w \colon L \to \mathbb{R}_{>0}$ that induces a measure $m = \mathrm{nWMC} \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (2) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$. We will first prove an auxiliary result that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \tag{5}$$

is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j). \tag{6}$$

First, note that k_i can be expressed as

$$k_i = (k_i \wedge k_j) \vee (k_i \wedge \neg k_j)$$

with the condition that

$$(k_i \wedge k_j) \wedge (k_i \wedge \neg k_j) = 0,$$

so, by properties of a measure,

$$m(k_i) = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j). \tag{7}$$

Applying Eq. (7) and the equivalent expression for $m(k_j)$ allows us to rewrite Eq. (5) as

$$\begin{aligned} m(k_i \wedge k_j) &= [m(k_i \wedge k_j) + m(k_i \wedge \neg k_j)] \cdot [m(k_i \wedge k_j) + m(\neg k_i \wedge k_j)] \\ &= m(k_i \wedge k_j)^2 + m(k_i \wedge k_j)[m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j)] + m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j) \end{aligned}$$

Dividing both sides by $m(k_i \wedge k_j)$ gives

$$1 = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j) + \frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)}. \tag{8}$$

Since $k_i \wedge k_j \wedge k_i \wedge \neg k_j = 0$, and

$$(k_i \wedge k_j) \vee (k_i \wedge \neg k_j) = k_i \wedge (k_j \vee \neg k_j) = k_i \wedge 1 = k_i,$$

we have that

$$m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) = m(k_i).$$

Similarly, $k_i \wedge \neg k_i \wedge k_j = 0$, and

$$k_i \vee (\neg k_i \wedge k_j) = (k_i \vee \neg k_i) \wedge (k_i \vee k_j) = k_i \vee k_j,$$

so

$$m(k_i) + m(\neg k_i \wedge k_j) = m(k_i \vee k_j).$$

Finally, note that

$$(k_i \vee k_j) \wedge \neg(k_i \vee k_j) = 0,$$

and

$$(k_i \vee k_j) \vee \neg(k_i \vee k_j) = 1,$$

so

$$m(k_i \vee k_j) + m(\neg(k_i \vee k_j)) = m(1) = 1.$$

This allows us to rewrite Eq. (8) as

$$\frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)} = 1 - m(k_i \vee k_j) = m(\neg(k_i \vee k_j)) = m(\neg k_i \wedge \neg k_j)$$

which immediately gives us Eq. (6).

Now recall that $m = \mathrm{nWMC}$ and note that Eq. (6) can be multiplied by $\mathrm{WMC}(1)^2$ to turn the equation into one for WMC instead of nWMC. Then

$$\begin{aligned}
\mathrm{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \leq k_i \wedge k_j} \mathrm{WMC}(a) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\
&= \sum_{\text{atoms } a \leq k_i \wedge k_j} w(a_i)w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \leq k_i \wedge k_j} w(k_i)w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) \\
&= w(k_i)w(k_j) \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i)w(k_j)C,
\end{aligned}$$

where C denotes the part of $\mathrm{WMC}(k_i \wedge k_j)$ that will be the same for $\mathrm{WMC}(\neg k_i \wedge k_j)$, $\mathrm{WMC}(k_i \wedge \neg k_j)$, and $\mathrm{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (6) becomes

$$w(k_i)w(k_j)w(\neg k_i)w(\neg k_j)C^2 = w(k_i)w(\neg k_j)w(\neg k_i)w(k_j)C^2$$

which is trivially true. By showing that WMC satisfies Eq. (6), we also showed that it satisfies Eq. (5), finishing the second part of the proof. □

3.2 Extending the Algebra

A well-known way to overcome this limitation of independence is by adding more literals [5], i.e., extending the set L covered by the WMC weight function $w \colon L \to \mathbb{R}_{>0}$. Let us translate this idea to the language of Boolean algebras.

Theorem 2. *Let $\mathbf{B}$ be a finite Boolean algebra freely generated by some set of 'literals' L, and let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be an arbitrary measure. We know that $\mathbf{B}$ has $n = 2^{|L|}$ atoms. Let $(a_i)_{i=1}^n$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i \mid i \in [n]\} \cup \{\neg\phi_i \mid i \in [n]\}$ be the set L extended with $2n$ new literals. Let $\mathbf{B}'$ be the unique Boolean algebra with*

$$\{\phi_i \wedge a_i \mid i \in [n]\} \cup \{\neg\phi_i \wedge a_i \mid i \in [n]\}$$

as its set of atoms. Let $\iota \colon \mathbf{B} \to \mathbf{B}'$ be the inclusion homomorphism (i.e., $\iota(a) = a$ for all $a \in \mathbf{B}$). Let $w \colon L' \to \mathbb{R}_{>0}$ be defined by

$$w(l) = \begin{cases} \frac{m(a_i)}{2} & \textit{if } l = \phi_i \textit{ or } l = \neg\phi_i \textit{ for some } i \in [n] \\ 1 & \textit{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m' \colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In simpler terms, any measure can be computed using WMC by extending the Boolean algebra with more literals. More precisely, we are given the red part in

$$\begin{tikzcd}
\mathbb{R}_{\geq 0} & \\
\mathbf{B} \arrow[u, "m"] \arrow[r, "\iota"] & \mathbf{B}' \arrow[ul, "m'"'] \\
\cup & \cup \\
L \arrow[r, phantom, "\subset"] & L' \arrow[r, "w"] & \mathbb{R}_{>0}
\end{tikzcd}$$

and construct the black part in such a way that the triangle commutes.

Proof. Since $\mathbf{B}$ is freely generated by L, each atom $a_i \in \mathbf{B}$ is an infimum of elements in L, i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either

$$b = \phi_i \wedge a_i \quad \text{or} \quad b = \neg\phi_i \wedge a_i$$

for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L'. Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies

$$\phi_i \wedge a_i \leq \iota(b)$$

if and only if it satisfies

$$\neg\phi_i \wedge a_i \leq \iota(b).$$

Then, according to the definition of w,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \leq \iota(b) \quad \text{if and only if} \quad a_i \leq b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. ☐

Now we can show that the construction in Theorem 2 is smallest possible.

Conjecture 1. *Let $\mathbf{B}$ and $\mathbf{B}'$ be Boolean algebras, and $\iota\colon \mathbf{B} \to \mathbf{B}'$ be the inclusion map such that $\mathbf{B}$ is freely generated by L, all atoms of $\mathbf{B}'$ can be expressed as meets of elements of L', and the following subset relations are satisfied:*

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "\iota"] & \mathbf{B}' \\
\cup & \cup \\
L \arrow[r, phantom, "\subset"] & L'
\end{tikzcd}$$

If, for any measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$*, one can construct a weight function* $w\colon L' \to \mathbb{R}_{>0}$ *such that the WMC measure* $\mathrm{WMC}\colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$ *with respect to* w *satisfies*

$$m = \mathrm{WMC} \circ \iota,$$

then $|L' \setminus L| \geq 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [6] and Sang et al. [17]. Suppose we have a discrete probability distribution with n variables, and the i-th variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^{n} v_i$ models. My expansion would then use

$$\sum_{i=1}^{n} v_i + 2\prod_{i=1}^{n} v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [6] would use

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n}\prod_{j=1}^{i} v_j$$

variables, while for the encoding by Sang et al. [17],

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n}(v_i - 1)\prod_{j=1}^{i-1} v_j$$

variables would suffice.

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