Weighted Model Counting Without Parameter Variables

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Abstract. The abstract should briefly summarize the contents of the paper in 150–250 words.

Keywords: First keyword · Second keyword · Another keyword.

1 Introduction

TODO: potentially related [21]

WMC has emerged as a powerful computational framework for problems in many domains, e.g., probabilistic graphical models such as Bayesian networks and Markov random fields [3, 7, 8, 14, 31], neuro-symbolic artificial intelligence [37], probabilistic programs [25], and probabilistic logic programs [20]. It has been extended to support continuous variables [6], infinite domains [4], first-order logic [23, 36], and arbitrary semirings [5, 26].

Related Work. Many WMC inference algorithms such as Ace[1], c2d [15], and miniC2D [28] work by compilation to tractable representations such as arithmetic circuits, deterministic, decomposable negation normal form [13], and sentential decision diagrams (SDDs) [16]. Some attempts have previously been made to skip this intermediate stage of representing a Bayesian network as a WMC instance and compile directly to a more convenient representation. Specifically, direct compilation from Bayesian networks to SDDs [12] and from structured Bayesian networks (i.e., a generalisation of Bayesian networks) to probabilistic SDDs [35] have been considered. To the best of the authors' knowledge, neither compilation approach has a publicly available implementation.

Our work is similar in that it introduces a new representation for computational problems that is based on pseudo-Boolean functions—pseudo-Boolean projection (PBP). We formally show that every WMC problem instance has a corresponding PBP instance. We argue that this new format is more appropriate for WMC solvers whose execution is build on pseudo-Boolean function manipulation, i.e., ADDMC [18] and DPMC [19]. Finally, we show how most methods of encoding probabilistic inference on Bayesian networks to WMC can be adapted to the PBP format. This transformation significantly improves inference speed, outperforming all other considered algorithm-encoding combinations.

[1] http://reasoning.cs.ucla.edu/ace/

Notes.

- we also show correctness
- and introduce a new notation
- TODO: briefly mention what a pseudo-Boolean function is

2 Weighted Model Counting

We begin with an overview of some notation and terminology. Throughout the paper, we use set theoretic notation for many concepts in logic. A *clause* is a set of literals that are part of an implicit disjunction. Similarly, a *formula* in CNF is a set of clauses that are part of an implicit conjunction. We identify a *model* with a set of variables that correspond to the positive literals in the model (and all other variables are the negative literals of the model). We can then define the *cardinality* of a model as the cardinality of this set. For example, let $\phi = (\neg a \vee b) \wedge a$ be a propositional formula over variables a and b. Then an equivalent set-theoretic representation of ϕ is $\{\{\neg a, b\}, \{a\}\}$. Any subset of $\{a, b\}$ is an interpretation of ϕ, e.g., $\{a, b\}$ is a model of ϕ (written $\{a, b\} \models \phi$) of cardinality two, while $\emptyset$ is an interpretation but not a model. We can now formally define WMC.

Definition 1 (WMC). *A* WMC instance *is a tuple (ϕ, X_I, X_P, w), where X_I is the set of* indicator *variables, X_P is the set of* parameter *variables (with $X_I \cap X_P = \emptyset$), ϕ is a propositional formula in CNF over $X_I \cup X_P$, and $w\colon X_I \cup X_P \cup \{\neg x \mid x \in X_I \cup X_P\} \to \mathbb{R}$ is the weight function. The* answer *of the instance is $\sum_{Y \models \phi} \prod_{Y \models l} w(l)$.*

That is, the answer to a WMC instance is the sum of the weights of all models of ϕ, where the weight of a model is defined as the product of the weights of all (positive and negative) literals in it. Our definition of WMC is largely based on the standard definition [10], but explicitly partitions variables into indicator and parameter variables. In practice, we identify this partition in one of two ways. If an encoding is generated by Ace, then variable types are explicitly identified in the LMAP file generated alongside the encoding. Otherwise, we define a variable x to be a parameter variable if $w(x) = w(\neg x) = 1$. Next, we formally define a variation of the WMC problem used by some of the Bayesian network encodings [7, 8].

Definition 2. *Let ϕ be a formula over a set of variables X. Then $Y \subseteq X$ is a* minimum-cardinality model *of ϕ if $Y \models \phi$ and $|Y| \leq |Z|$ for all $Z \models \phi$.*

Definition 3 (Minimum-Cardinality WMC). *A* minimum-cardinality WMC *instance consists of the same tuple as a WMC instance, but its answer is defined to be $\sum_{Y \models \phi,\ |Y|=k} \prod_{Y \models l} w(l)$ (where $k = \min_{Y \models \phi} |Y|$) if ϕ is satisfiable, and zero otherwise.*

2.1 Bayesian Network Encodings

A *Bayesian network* is a directed acyclic graph with random variables as vertices and edges as conditional dependencies. As is common in related literature [14, 31], we assume that each variable has a finite number of values. We call a Bayesian network *binary* if every variable has two values. If all variables have finite numbers of values, the probability function associated with each variable v can be represented as a *conditional probability table* (CPT), i.e., a table with a row for each combination of values that v and its parent vertices can take. Each row then also has a *probability*, i.e., a number in $[0, 1]$.

WMC is a well-established technique for Bayesian network inference, particularly effective on networks where most variables have only a few possible values [14]. Many ways of encoding a Bayesian network into a WMC instance have been proposed. We will refer to them based on initials of the authors and the year of publication. Darwiche was the first to suggest the `d02` [14] encoding that, in many ways, remains the foundation behind most other encodings. He also introduced the distinction between *indicator* and *parameter variables*; the former represent variable-value pairs in the Bayesian network, while the latter are associated with probabilities in the CPTs. The encoding `sbk05` [31] is the only encoding that deviates from this arrangement: for each variable in the Bayesian network, one indicator variable acts simultaneously as a parameter variable. Chavira and Darwiche propose `cd05` [7] where they shift from WMC to minimum-cardinality WMC because that allows the encoding to have fewer variables and clauses. In particular, they propose a way to use the same parameter variable to represent all probabilities in a CPT that are equal and keep only clauses that 'imply' parameter variables (i.e., omit clauses where a parameter variable implies indicator variables). In their next encoding, `cd06` [8], the same authors optimise the aforementioned implication clauses, choosing the smallest sufficient selection of indicator variables. A decade later, Bart et al. present `bklm16` [3] that improves upon `cd06` in two ways. First, they optimise the number of indicator variables used per Bayesian network variable from a linear to a logarithmic amount. Second, they introduce a scaling factor that can 'absorb' one probability per Bayesian network variable. However, for this work, we choose to disable the latter improvement since this scaling factor is often small enough to be indistinguishable from zero without the use of arbitrary precision arithmetic. Indeed, even a small Bayesian network with seven mutually independent binary variables, 0.1 and 0.9 probabilities each, is already big enough for the scaling factor to be exactly equal to zero (as produced by the `bklm16` encoder[2]).

3 Pseudo-Boolean Functions

In this work we propose a more expressive representation for WMC based on pseudo-Boolean functions. Pseudo-Boolean functions, most commonly represented as algebraic decision diagrams (ADDs) [2] (although a tensor-based ap-

[2] http://www.cril.univ-artois.fr/kc/bn2cnf.html

proach has also been suggested [17, 19]), have seen extensive use in value iteration for Markov decision processes [24], both exact and approximate Bayesian network inference [9, 22], and sum-product network [29] to Bayesian network conversion [38]. ADDs have been extended to compactly represent additive and multiplicative structure [34], sentences in first-order logic [32], and continuous variables [33], the last of which was also applied to weighted model integration, i.e., the WMC extension for continuous variables [6, 27].

Since two-valued pseudo-Boolean functions will be used extensively henceforth, we introduce some notation. For any propositional formula ϕ over a set of variables X and $p, q \in \mathbb{R}$, let $[\phi]_q^p \colon 2^X \to \mathbb{R}$ be a pseudo-Boolean function defined as

$$[\phi]_q^p(Y) \coloneqq \begin{cases} p & \text{if } Y \models \phi \\ q & \text{otherwise} \end{cases}$$

for any $Y \subseteq X$. Next, we define some useful operations on pseudo-Boolean functions.

Definition 4 (Operations). *Let $f, g \colon 2^X \to \mathbb{R}$ be pseudo-Boolean functions, $x, y \in X$, $Y = \{y_i\}_{i=1}^n \subseteq X$, and $r \in \mathbb{R}$. Operations such as addition and multiplication are defined pointwise as*

$$(f + g)(Y) \coloneqq f(Y) + g(Y), \quad \textit{and} \quad (f \cdot g)(Y) \coloneqq f(Y) \cdot g(Y).$$

Note that this means that binary operations on pseudo-Boolean functions inherit properties such as associativity and commutativity. By regarding a real number as a constant pseudo-Boolean function, we can reuse the same definitions to define scalar *operations as*

$$(r + f)(Y) = r + f(Y), \quad \textit{and} \quad (r \cdot f)(Y) = r \cdot f(Y).$$

Restrictions $f|_{x=0}, f|_{x=1} \colon 2^X \to \mathbb{R}$ *of f are defined as*

$$f|_{x=0}(Y) \coloneqq f(Y \setminus \{x\}), \quad \textit{and} \quad f|_{x=1}(Y) \coloneqq f(Y \cup \{x\})$$

for all $Y \subseteq X$.

Projection $\exists_x$ *is an endomorphism $\exists_x \colon \mathbb{R}^{2^X} \to \mathbb{R}^{2^X}$ defined as*

$$\exists_x f \coloneqq f|_{x=1} + f|_{x=0}.$$

Since projection is commutative (i.e., $\exists_x \exists_y f = \exists_y \exists_x f$) [18, 19], we can define $\exists_Y \colon \mathbb{R}^{2^X} \to \mathbb{R}^{2^X}$ as $\exists_Y \coloneqq \exists_{y_1} \exists_{y_2} \dots \exists_{y_n}$. Throughout the paper, projection is assumed to have the lowest precedence (e.g., $\exists_x fg = \exists_x (fg)$).

Finally, below we list some properties of the operations on pseudo-Boolean functions discussed in this section that can be conveniently represented using our syntax. The proofs of all these properties follow directly from the definitions.

Proposition 1 (Basic Properties). *For any propositional formulas ϕ and ψ, and $a, b, c, d \in \mathbb{R}$,*

- $[\phi]_b^a = [\neg\phi]_a^b$;
- $c + [\phi]_b^a = [\phi]_{b+c}^{a+c}$;
- $c \cdot [\phi]_b^a = [\phi]_{bc}^{ac}$;
- $[\phi]_b^a \cdot [\phi]_d^c = [\phi]_{bd}^{ac}$;
- $[\phi]_0^1 \cdot [\psi]_0^1 = [\phi \wedge \psi]_0^1$.

And for any pair of pseudo-Boolean functions $f, g\colon 2^X \to \mathbb{R}$ *and* $x \in X$, $(fg)|_{x=i} = f|_{x=i} \cdot g|_{x=i}$ *for* $i = 0, 1$.

4 Pseudo-Boolean Projection

We introduce a new type of computational problem called *pseudo-Boolean projection* based on two-valued pseudo-Boolean functions. While the same computational framework can handle any pseudo-Boolean functions, two-valued functions are particularly convenient because DPMC can be easily adapted to use them as input, and they are easily representable in both text files and using the syntax proposed in this paper.

Definition 5 (PBP Instance). *A PBP instance is a tuple* (F, X, ω)*, where* X *is the set of variables,* F *is a set of two-valued pseudo-Boolean functions* $2^X \to \mathbb{R}$*, and* $\omega \in \mathbb{R}$ *is the scaling factor. The* answer *of the instance is* $\omega \cdot \left(\exists_X \prod_{f \in F} f\right)(\emptyset)$.

Adding scaling factor ω to the definition allows us to remove clauses that consist entirely of a single parameter variable. The idea of extracting some of the structure of the WMC instance into an external multiplicative factor was inspired by the `bklm16` encoding.

4.1 From WMC to PBP

In this section we describe an algorithm for transforming WMC instances to the PBP format while removing all parameter variables. The algorithm works on four out of the five Bayesian network encodings. There is no obvious way to adjust it to work with `sbk05` because the roles of indicator and parameter (i.e., 'chance') variables overlap [31]. The algorithm is based on several observations that will be made more precise in Section 4.2. First, all weights except for $\{w(p) \mid p \in X_P\}$ are redundant as they either duplicate an already-defined weight or are equal to one. Second, each clause has at most one parameter variable. Third, if the parameter variable is negated, we can ignore the clause (this idea first appears in the `cd05` paper [7]). Note that while we formulate our algorithm as a sequel to the WMC encoding procedure primarily because the implementations of Bayesian network WMC encodings are all closed-source, as all transformations in the algorithm are local, it can be efficiently incorporated into a WMC encoding algorithm with no slowdown.

The algorithm is listed as Algorithm 1. The main part of the algorithms is the first loop that iterates over clauses. If a clause consists of a single parameter

Algorithm 1: WMC to PBP transformation

Data: WMC (or minimum-cardinality WMC) instance (ϕ, X_I, X_P, w)
Result: PBP instance (F, X_I, ω)

$F \leftarrow \emptyset$;
$\omega \leftarrow 1$;
foreach *clause* $c \in \phi$ **do**
  if $c \cap X_P = \{p\}$ *for some* p **and** $w(p) \neq 1$ **then**
    if $|c| = 1$ **then**
      $\omega \leftarrow \omega \times w(p)$;
    else
      $F \leftarrow F \cup \left\{ \left[\bigwedge_{l \in c \setminus \{p\}} \neg l \right]_1^{w(p)} \right\}$;
  else if $\{p \mid \neg p \in c\} \cap X_P = \emptyset$ **then**
    $F \leftarrow F \cup \{[c]_0^1\}$;
foreach $v \in X_I$ *such that* $\{[v]_1^p, [\neg v]_1^q\} \subseteq F$ *for some* p *and* q **do**
  $F \leftarrow F \setminus \{[v]_1^p, [\neg v]_1^q\} \cup \{[v]_q^p\}$;

variable, we incorporate it into ω. If a clause is of the form $\alpha \Rightarrow p$, where $p \in X_P$ and α is a conjunction of literals over X_I, we transform it into a pseudo-Boolean function $[\alpha]_1^{w(p)}$. If a clause (say, $c \in \phi$) has no parameter variables, we reformulate it into a pseudo-Boolean function $[c]_0^1$. Finally, if a clause has negative parameter literals, we skip it.

As all 'weighted' pseudo-Boolean functions produced by the first loop are of the form $[\alpha]_1^p$ (for some $p \in \mathbb{R}$ and formula α), the second loop merges two functions into one whenever α is a literal. Note that taking into account the order in which clauses are typically generated by encoding algorithms allows us to do this in linear time (i.e., the two mergeable functions will be generated one after the other).

4.2 Correctness Proofs

In this section we outline key properties that a (WMC or minimum-cardinality WMC) encoding has to satisfy for Algorithm 1 to output an equivalent PBP instance. We divide the correctness proof into two theorems: Theorem 2 for WMC encodings (i.e., `bklm16` and `d02`) and Theorem 3 for minimum-cardinality WMC encodings (i.e., `cd05` and `cd06`). We begin by listing some properties of pseudo-Boolean functions and establishing a canonical transformation from WMC to PBP.

Theorem 1 (Early Projection [18, 19]). *Let X and Y be sets of variables. For all pseudo-Boolean functions $f\colon 2^X \to \mathbb{R}$ and $g\colon 2^Y \to \mathbb{R}$, if $x \in X \setminus Y$, then $\exists_x(f \cdot g) = (\exists_x f) \cdot g$.*

Lemma 1. *For any pseudo-Boolean function $f\colon 2^X \to \mathbb{R}$, we have that $(\exists_X f)(\emptyset) = \sum_{Y \subseteq X} f(Y)$.*

Proof. If $X = \{x\}$, then

$$(\exists_x f)(\emptyset) = (f|_{x=1} + f|_{x=0})(\emptyset) = f|_{x=1}(\emptyset) + f|_{x=0}(\emptyset) = \sum_{Y \subseteq \{x\}} f(Y).$$

This easily extends to $|X| > 1$ by the definition of projection on sets of variables.

Proposition 2. *Let (ϕ, X_I, X_P, w) be a WMC instance. Then*

$$\left(\left\{[c]_0^1 \mid c \in \phi\right\} \cup \left\{[x]_{w(\neg x)}^{w(x)} \;\middle|\; x \in X_I \cup X_P\right\}, X_I \cup X_P, 1\right) \tag{1}$$

is a PBP instance with the same answer.

Proof. Let $f = \prod_{c \in \phi}[c]_0^1$, and $g = \prod_{x \in X_I \cup X_P}[x]_{w(\neg x)}^{w(x)}$. Then the answer to the WMC instance (1) is

$$(\exists_{X_I \cup X_P} fg)(\emptyset) = \sum_{Y \subseteq X_I \cup X_P} (fg)(Y) = \sum_{Y \subseteq X_I \cup X_P} f(Y)g(Y)$$

by Lemma 1. Note that

$$f(Y) = \begin{cases} 1 & \text{if } Y \models \phi, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(Y) = \prod_{Y \models l} w(l),$$

which means that $\sum_{Y \subseteq X_I \cup X_P} f(Y)g(Y) = \sum_{Y \models \phi} \prod_{Y \models l} w(l)$ as required.

Theorem 2 (Correctness for WMC). *Algorithm 1, when given a WMC instance (ϕ, X_I, X_P, w), returns PBP instance with the same answer, provided the following conditions are satisfied:*

1. *for all indicator variables $i \in X_I$, $w(i) = w(\neg i) = 1$,*
2. *and either*
   (a) *for all parameter variables $p \in X_P$, there is a non-empty family of literals $(l_i)_{i=1}^n$ such that*
      i. *$w(\neg p) = 1$,*
      ii. *$l_i \in X_I$ or $\neg l_i \in X_I$ for all $i = 1, \ldots, n$,*
      iii. *and $\{c \in \phi \mid p \in c \text{ or } \neg p \in c\} = \{p \vee \bigvee_{i=1}^n \neg l_i\} \cup \{l_i \vee \neg p \mid i = 1, \ldots, n\}$;*
   (b) *or for all parameter variables $p \in X_P$,*
      i. *$w(p) + w(\neg p) = 1$,*
      ii. *for any clause $c \in \phi$, $|c \cap X_P| \leq 1$,*
      iii. *there is no clause $c \in \phi$ such that $\neg p \in c$,*
      iv. *if $\{p\} \in \phi$, then there is no clause $c \in \phi$ such that $c \neq \{p\}$ and $p \in c$,*
      v. *and for any $c, d \in \phi$ such that $c \neq d$, $p \in c$ and $p \in d$, $\bigwedge_{l \in c \setminus \{p\}} \neg l \wedge \bigwedge_{l \in d \setminus \{p\}} \neg l$ is false.*

Condition 1 applies to all four WMC encodings under consideration and ensures that only parameter variables carry nontrivial weights. Condition 2a (for `d02`), while spelled out precisely in the theorem, simply states that each parameter variable is equivalent to a conjunction of indicator literals. Condition 2b is for encodings that have implications rather than equivalences associated with parameter variables (which, in this case, is `bklm16`). It ensures that each clause has at most one positive parameter literal and no negative ones, and that at most one implication clause per any parameter variable $p \in X_P$ can 'force p to be positive'.

Proof. By Proposition 2,

$$\left(\left\{[c]_0^1 \mid c \in \phi\right\} \cup \left\{[x]_{w(\neg x)}^{w(x)} \;\middle|\; x \in X_I \cup X_P\right\}, X_I \cup X_P, 1\right) \tag{2}$$

is a PBP instance with the same answer as the given WMC instance. By Definition 5, its answer is $\left(\exists_{X_I \cup X_P} \left(\prod_{c\in\phi}[c]_0^1\right) \prod_{x \in X_I \cup X_P} [x]_{w(\neg x)}^{w(x)}\right)(\emptyset)$. Since both Conditions 2a and 2b ensure that each clause in ϕ has at most one parameter variable, we can partition ϕ into $\phi_* \coloneqq \{c \in \phi \mid \texttt{Vars}(c) \cap X_P = \emptyset\}$ and $\phi_p \coloneqq \{c \in \phi \mid \texttt{Vars}(c) \cap X_P = \{p\}\}$ for all $p \in X_P$. We can then use Theorem 1 to reorder the answer into $\left(\exists_{X_I} \left(\prod_{x\in X_I}[x]_{w(\neg x)}^{w(x)}\right) \left(\prod_{c\in\phi_*}[c]_0^1\right) \prod_{p\in X_P} \exists_p [p]_{w(\neg p)}^{w(p)} \prod_{c\in\phi_p}[c]_0^1\right)(\emptyset)$.

Let us first consider how the unfinished WMC instance (F, X_I, ω) after the loop on Lines 3 to 10 differs from (2). Note that Algorithm 1 leaves each $c \in \phi_*$ unchanged, i.e., adds $[c]_0^1$ to F. We can then fix an arbitrary $p \in X_P$ and let F_p be the set of functions added to F as a replacement of ϕ_p. It is sufficient to show that

$$\omega \prod_{f\in F_p} f = \exists_p [p]_{w(\neg p)}^{w(p)} \prod_{c\in\phi_p} [c]_0^1. \tag{3}$$

Note that under Condition 2a,

$$\bigwedge_{c\in\phi_p} c \equiv p \Leftrightarrow \bigwedge_{i=1}^{n} l_i$$

for some family of indicator variable literals $(l_i)_{i=1}^n$. Thus,

$$\exists_p [p]_{w(\neg p)}^{w(p)} \prod_{c\in\phi_p} [c]_0^1 = \exists_p [p]_1^{w(p)} \left[p \Leftrightarrow \bigwedge_{i=1}^{n} l_i\right]_0^1.$$

If $w(p) = 1$, then

$$\exists_p [p]_1^{w(p)} \left[p \Leftrightarrow \bigwedge_{i=1}^{n} l_i\right]_0^1 = \left[p \Leftrightarrow \bigwedge_{i=1}^{n} l_i\right]_0^1\Bigg|_{p=1} + \left[p \Leftrightarrow \bigwedge_{i=1}^{n} l_i\right]_0^1\Bigg|_{p=0}. \tag{4}$$

Since for any input, $\bigwedge_{i=1}^n l_i$ is either true or false, exactly one of the two summands in Eq. (4) will be equal to one, and the other will be equal to zero, and

so

$$\left[p \Leftrightarrow \bigwedge_{i=1}^{n} l_i\right]_0^1\Bigg|_{p=1} + \left[p \Leftrightarrow \bigwedge_{i=1}^{n} l_i\right]_0^1\Bigg|_{p=0} = 1,$$

where 1 is a pseudo-Boolean function that always returns one. On the other side of Eq. (3), since $F_p = \emptyset$, and ω is unchanged, we get $\omega \prod_{f \in F_p} f = 1$, and so Eq. (3) is satisfied under Condition 2a when $w(p) = 1$.

If $w(p) \neq 1$, then

$$F_p = \left\{ \left[\bigwedge_{i=1}^{n} l_i\right]_1^{w(p)} \right\},$$

and $\omega = 1$, and so we want to show that $[\bigwedge_{i=1}^{n} l_i]_1^{w(p)} = \exists_p [p]_1^{w(p)} [p \Leftrightarrow \bigwedge_{i=1}^{n} l_i]_0^1$, and indeed

$$\exists_p [p]_1^{w(p)} \left[p \Leftrightarrow \bigwedge_{i=1}^{n} l_i\right]_0^1 = w(p) \cdot \left[\bigwedge_{i=1}^{n} l_i\right]_0^1 + \left[\bigwedge_{i=1}^{n} l_i\right]_1^0 = \left[\bigwedge_{i=1}^{n} l_i\right]_1^{w(p)}.$$

This finishes the proof of the correctness of the first 'foreach' loop under Condition 2a.

Now let us assume Condition 2b. We still want to prove Eq. (3). If $w(p) = 1$, then $F_p = \emptyset$, and $\omega = 1$, and so the left-hand side of Eq. (3) is equal to one. Then the right-hand side is

$$\exists_p [p]_0^1 \prod_{c \in \phi_p} [c]_0^1 = \exists_p \left[p \wedge \bigwedge_{c \in \phi_p} c\right]_0^1 = \exists_p [p]_0^1 = 0 + 1 = 1$$

since $p \in c$ for every clause $c \in \phi_p$.

If $w(p) \neq 1$, and $\{p\} \in \phi_p$, then, by Condition 2(b)iv, $\phi_p = \{\{p\}\}$, and Algorithm 1 produces $F_p = \emptyset$ and $\omega = w(p)$, and so

$$\exists_p [p]_{w(\neg p)}^{w(p)} [p]_0^1 = \exists_p [p]_0^{w(p)} = w(p) = \omega \prod_{f \in F_p} f.$$

The only remaining case is when $w(p) \neq 1$ and $\{p\} \notin \phi_p$. Then $\omega = 1$, and $F_p = \left\{ \left[\bigwedge_{l \in c \setminus \{p\}} \neg l\right]_1^{w(p)} \;\middle|\; c \in \phi_p \right\}$, so we need to show that

$$\prod_{c \in \phi_p} \left[\bigwedge_{l \in c \setminus \{p\}} \neg l\right]_1^{w(p)} = \exists_p [p]_{1-w(p)}^{w(p)} \prod_{c \in \phi_p} [c]_0^1.$$

We can rearrange the right-hand side as

$$\exists_p [p]_{1-w(p)}^{w(p)} \prod_{c \in \phi_p} [c]_0^1 = \exists_p [p]_{1-w(p)}^{w(p)} \left[\bigwedge_{c \in \phi_p} c \right]_0^1 = \exists_p [p]_{1-w(p)}^{w(p)} \left[p \vee \bigwedge_{c \in \phi_p} c \setminus \{p\} \right]_0^1$$
$$= w(p) + (1 - w(p)) \left[\bigwedge_{c \in \phi_p} c \setminus \{p\} \right]_0^1 = w(p) + \left[\bigwedge_{c \in \phi_p} c \setminus \{p\} \right]_0^{1-w(p)}$$
$$= \left[\bigwedge_{c \in \phi_p} c \setminus \{p\} \right]_{w(p)}^1 = \left[\neg \bigwedge_{c \in \phi_p} c \setminus \{p\} \right]_1^{w(p)} = \left[\bigvee_{c \in \phi_p} \neg (c \setminus \{p\}) \right]_1^{w(p)}$$
$$= \left[\bigvee_{c \in \phi_p} \neg \bigvee_{l \in c \setminus \{p\}} l \right]_1^{w(p)} = \left[\bigvee_{c \in \phi_p} \bigwedge_{l \in c \setminus \{p\}} \neg l \right]_1^{w(p)} .$$

By Condition 2(b)v, $\bigwedge_{l \in c \setminus \{p\}} \neg l$ can be true for at most one $c \in \phi_p$, and so

$$\left[\bigvee_{c \in \phi_p} \bigwedge_{l \in c \setminus \{p\}} \neg l \right]_1^{w(p)} = \prod_{c \in \phi_p} \left[\bigwedge_{l \in c \setminus \{p\}} \neg l \right]_1^{w(p)}$$

which is exactly what we needed to show. This ends the proof that the first loop of Algorithm 1 preserves the answer under both Condition 2a and Condition 2b. Finally, the loop on Lines 11 to 12 of Algorithm 1 replaces $[v]_1^p [\neg v]_1^q$ with $[v]_q^p$ (for some $v \in X_I$ and $p, q \in \mathbb{R}$), but, of course, $[v]_1^p [\neg v]_1^q = [v]_1^p [v]_q^1 = [v]_q^p$, i.e., the answer is unchanged.

Theorem 3 (Minimum-Cardinality Correctness). *Let (ϕ, X_I, X_P, w) be a minimum-cardinality WMC instance that satisfies Condition 1 and Conditions 2(b)ii to 2(b)v of Theorem 2 as well as the following:*

1. *for all parameter variables $p \in X_P$, $w(\neg p) = 1$.*
2. *all models of $\{c \in \phi \mid c \cap X_P = \emptyset\}$ (as subsets of X_I) have the same cardinality;*
3. *$\min_{Z \subseteq X_P} |Z|$ such that $Y \cup Z \models \phi$ is the same for all $Y \models \{c \in \phi \mid c \cap X_P = \emptyset\}$.*

Then Algorithm 1, when applied to (ϕ, X_I, X_P, w), outputs a PBP instance with the same answer.

In this case, we have to add some assumptions about the cardinality of models. Condition 2 states that all models of the indicator-only part of the formula have the same cardinality. Bayesian network encodings such as `cd05` and `cd06` satisfy this condition by assigning an indicator variable to each possible variable-value pair and requiring each random variable to be paired with exactly

one value. Condition 3 then says that the smallest number of parameter variables needed to turn an indicator-only model into a full model is the same for all indicator-only models. As some ideas duplicate between the proofs of Theorems 2 and 3, the following proof is slightly less explicit and assumes that $\omega = 1$.

Proof. Let (F, X_I, ω) be the tuple returned by Algorithm 1 and note that

$$F = \left\{ [c]_0^1 \mid c \in \phi,\, c \cap X_P = \emptyset \right\} \cup \left\{ \left[\bigwedge_{l \in c \setminus \{p\}} \neg l \right]_1^{w(p)} \;\middle|\; p \in X_P,\, p \in c \in \phi,\, c \neq \{p\} \right\}.$$

We split the proof into two parts. In the first part, we show that there is a bijection between minimum-cardinality models of ϕ and $Y \subseteq X_I$ such that $\left(\prod_{f \in F} f\right)(Y) \neq 0$ (with the assumption that $w(p) \neq 0$ for all $p \in X_P$).[3] Let $Y \subseteq X_I$ and $Z \subseteq X_I \cup X_P$ be related via this bijection. Then in the second part we will show that

$$\prod_{Z \models l} w(l) = \left(\prod_{f \in F} f \right)(Y). \tag{5}$$

On the one hand, if $Z \subseteq X_I \cup X_P$ is a minimum-cardinality model of ϕ, then $\left(\prod_{f \in F}\right)(Z \cap X_I) \neq 0$ under the given assumptions. On the other hand, if $Y \subseteq X_I$ is such that $\left(\prod_{f \in F}\right)(Y) \neq 0$, then $Y \models \{c \in \phi \mid c \cap X_P = \emptyset\}$. Let $Y \subseteq Z \subseteq X_I \cup X_P$ be the smallest superset of Y such that $Z \models \phi$ (it exists by Condition 2(b)iii of Theorem 2). We need to show that Z has minimum cardinality. Let Y' and Z' be defined equivalently to Y and Z. We will show that $|Z| = |Z'|$. Note that $|Y| = |Y'|$ by Condition 2, and $|Z \setminus Y| = |Z' \setminus Y'|$ by Condition 3. Combining that with the general property that $|Z| = |Y| + |Z \setminus Y|$ finishes the first part of the proof.

For the second part, let us consider the multiplicative influence of a single parameter variable $p \in X_P$ on Eq. (5). If the left-hand side is multiplied by $w(p)$ (i.e., $p \in Z$), then there must be some clause $c \in \phi$ such that $Z \setminus \{p\} \not\models c$. But then $Y \models \bigwedge_{l \in c \setminus \{p\}} \neg l$, and so the right-hand side is multiplied by $w(p)$ as well (exactly once because of Condition 2(b)v of Theorem 2). This argument works in the other direction as well.

5 Experimental Evaluation

We run a set of experiments, comparing all five original Bayesian network encodings (`bklm16`, `cd05`, `cd06`, `d02` `sbk05`) as well as the first four with Algorithm 1

[3] The assumption is only there to make the proof more convenient and can be omitted by reformulating the proof in a slightly more cumbersome way.

applied afterwards.[4] For each encoding `e`, we will write `e++` to denote the combination of encoding a Bayesian network as a WMC instance using `e` and transforming it into a PBP instance using Algorithm 1. Along with DPMC[5], we also include WMC algorithms used in the papers that introduce each encoding: Ace for `cd05`, `cd06`, and `d02`; Cachet[6] [30] for `sbk05`; and c2d[7] [15] with query-dnnf[8] for `bklm16`. We focus on the following questions:

- Can parameter variable elimination improve inference speed?
- How does DPMC combined with encodings without (and with) parameter variables compare with other WMC algorithms and other encodings?
- Which instances is our approach particularly successful on (compared to other algorithms and encodings and to the same encoding before our transformation)?
- What proportion of variables is typically eliminated?
- Do some encodings benefit from this transformation more than others?

5.1 Setup

DPMC is run with tree decomposition based planning and ADD based execution—the best-performing combination in the original set of experiments [19]. We use a single iteration of htd [1] to generate approximately optimal tree decompositions—we found that this configuration is efficient enough to handle huge instances, and yet the width of the returned decomposition is unlikely to differ from optimal by more than one or two. We also enabled DPMC's greedy mode. This mode (which was not part of the original paper [19]) optimises the order in which pseudo-Boolean functions are multiplied by prioritising functions with small representations.

For experimental data, we use Bayesian networks available with Ace and Cachet. We split them into the following groups:

- DQMR (390 instances) and
- Grid networks (450 instances) as described by Sang et al. [31];
- Mastermind (144 instances) and
- Random Blocks (256 instances) by Chavira et al. [11];
- other binary Bayesian networks (50 instances) including Plan Recognition [31], Friends and Smokers, Students and Professors [11], and `tcc4f`;
- non-binary classic networks (176 instances): `alarm`, `diabetes`, `hailfinder`, `mildew`, `munin1`–`4`, `pathfinder`, `pigs`, and `water`.

To perform Bayesian network inference with DPMC (or with any other WMC algorithm not based on compilation such as Cachet), one needs to select a probability to compute [19, 30]. If a network comes with an evidence file, we compute

[4] Recall that `cd05` and `cd06` are incompatible with DPMC.
[5] https://github.com/vardigroup/DPMC
[6] https://cs.rochester.edu/u/kautz/Cachet/
[7] http://reasoning.cs.ucla.edu/c2d/
[8] http://www.cril.univ-artois.fr/kc/d-DNNF-reasoner.html

the probability of this evidence. Otherwise, let X be the variable last mentioned in the Bayesian network file. If true is one of the values of X, then we compute $\Pr(X = \mathsf{true})$, otherwise we choose the first-mentioned value of X.

The experiments were run on a computing cluster with Intel Xeon E5-2630, Intel Xeon E7-4820, and Intel Xeon Gold 6138 processors with a 1000 s timeout separately on both encoding and inference, and a 32 GiB memory limit.[9]

5.2 Results

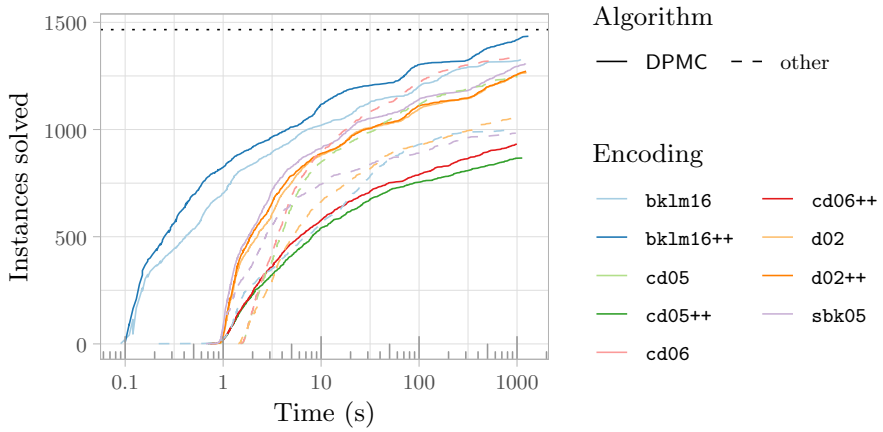

Fig. 1. Cumulative numbers of instances solved by each algorithm-encoding pair over time. The dotted line denotes the total number of instances used.

Figure 1 shows DPMC + bklm16++ to be the best-performing combination across all time limits up to 1000 s with Ace + cd06 and DPMC + bklm16 not far behind. Overall, DPMC+bklm16++ is 3.35 times faster than DPMC+bklm16 and 2.96 times faster than Ace+cd06. Table 1 further shows that DPMC+bklm16++ solves almost a hundred more instances than any other combination, and is the fastest in 69.1 % of them.

The scatter plots in Fig. 2 show that how DPMC + bklm16++ (and perhaps DPMC more generally) compares to Ace + cd06 depends significantly on the data set: the former is a clear winner on DQMR and Grid instances, while the latter performs well on Mastermind and Random Blocks. Perhaps because the underlying WMC algorithm remains the same, the difference between DPMC + bklm16 with and without applying Algorithm 1 is quite noisy, i.e, with most instances scattered around the line of equality. However, our transformation does enable DPMC to solve many instances that were previously beyond its reach.

[9] Each instance was run on the same processor across all algorithms and encodings.

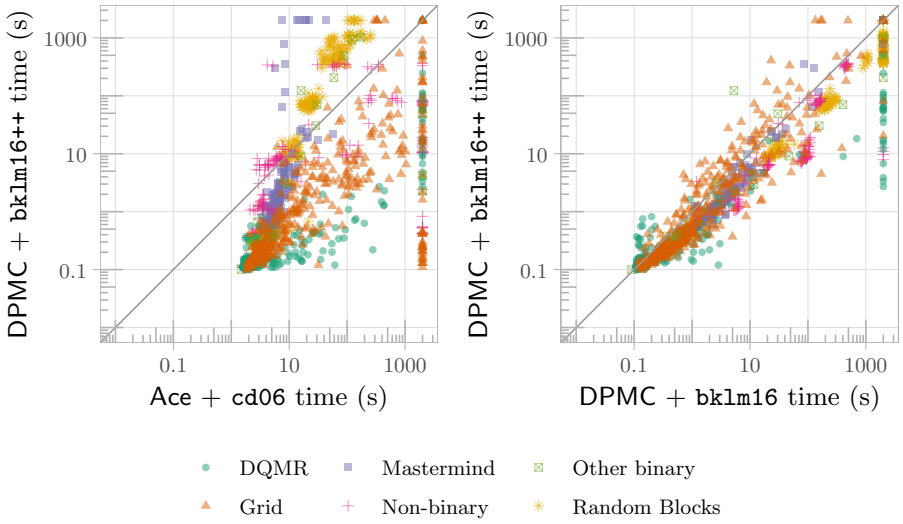

Fig. 2. An instance-by-instance comparison between DPMC + `bklm16++` (the best combination according to Fig. 1) and the second and third best performing combinations: Ace + `cd06` and DPMC + `bklm16`.

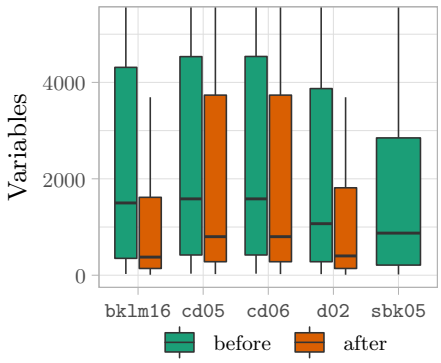

Fig. 3. Box plots of the numbers of variables in each encoding across all benchmark instances before and after applying Algorithm 1. Outliers and the top parts of some whiskers are omitted.

Table 1. The numbers of instances (out of 1466) that each algorithm and encoding combination solved faster than any other combination and in total.

Combination	Fastest	Solved
Ace + `cd05`	27	1247
Ace + `cd06`	135	1340
Ace + `d02`	56	1060
DPMC + `bklm16`	241	1327
DPMC + `bklm16++`	**992**	**1435**
DPMC + `cd05++`	0	867
DPMC + `cd06++`	0	932
DPMC + `d02`	1	1267
DPMC + `d02++`	7	1272
DPMC + `sbk05`	31	1308
c2d + `bklm16`	0	997
Cachet + `sbk05`	49	983

We also record numbers of variables in each encoding before and after applying Algorithm 1. Figure 3 shows a significant reduction in the number of variables. For instance, the median number of variables in instances encoded with `bklm16` was reduced four times: from 1499 to 376. While `bklm16++` results in the overall lowest number of variables, the difference between `bklm16++` and `d02++` seems small. Indeed, the numbers of variables in these two encodings are equal for binary Bayesian networks (i.e., most of our data). Nonetheless, `bklm16++` is still much faster than `d02++` when run with DPMC.

Overall, transforming WMC instances to the PBP format allows us to significantly simplify each instance. This transformation is particularly effective on `bklm16`, allowing it to surpass `cd06` and become the new state of the art. While there is a similarly significant reduction in the number of variables for `d02`, the performance of DPMC + `d02` is virtually unaffected. Finally, while our transformation makes it possible to use `cd05` and `cd06` with DPMC, the two combinations remain inefficient.

6 Conclusions

Notes.

- Benefits of my approach:
  - smaller primal graph, so easier to perform tree decomposition
  - Variable order is less likely to be obstructed by all the unnecessary 'parameter' variables.
  - There are others, but they're not that important.
- Future work: do other WMC encodings also have this variable partition property?
- Do I need to formally consider extending a pseudo-Boolean function to a bigger domain?
- Benefits of having this proof in the paper:
  - It puts all encodings on a common ground.
  - It illustrates the convenience of our notation for reasoning about (certain types of) pseudo-Boolean functions.
  - It's too big and too important to be left for the appendix.

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