What Boolean Algebras Can Teach Us About Weighted Model Counting

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1 Introduction

Contributions.

- WMC defines a measure over a BA.
- WMC with weights on literals imposes an independence assumption. (Measures are 'slightly' more expressive than WMC with weights on models because they apply to non-atomic BAs.)
- A BA can be augmented with new literals in order to support any measure.
- (Maybe) a lower bound on the number of new literals needed in order to support any measure.
- Alternatively, one can use coproducts and pushouts to define a BA with precisely the right independence and conditional independence conditions. (This requires a relaxed version of WMC.)
- This results in a smaller problem for WMC algorithms (w.r.t. both the number of literals and the length of the theory) and is optimal for, e.g., Bayesian networks.
- (Maybe) this results in faster inference (?)

Notable previous/related work.

- Hailperin's approach to probability logic [9]
- Nilsson's (somewhat successful) probabilistic logic [14]
- Logical induction: a big paper with a good overview of previous attempts to assign probabilities to logical sentences in a sensible way [7]
- Measures on Boolean algebras
  - On possibility and probability measures in finite Boolean algebras [2]
  - Representation of conditional probability measures [12]

2 Preliminaries

Definition 1. A *Boolean algebra* (BA) is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ consisting of a set $\mathbf{B}$ with binary operations *meet* $\wedge$ and *join* $\vee$, unary operation $\neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;

- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;
- $\vee$ distributes over $\wedge$ and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

For clarity and succinctness, we will occasionally use three other operations that can be defined using the original three[1]:

$$\begin{aligned} a \to b &= \neg a \vee b, \\ a \leftrightarrow b &= (a \wedge b) \vee (\neg a \wedge \neg b), \\ a + b &= (a \wedge \neg b) \vee (\neg a \wedge b). \end{aligned}$$

We can also define a partial order $\leq$ on $\mathbf{B}$ as $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$) for $a, b \in \mathbf{B}$. Furthermore, let $a < b$ denote $a \leq b$ and $a \neq b$. For the rest of this paper, let $\mathbf{B}$ refer to the BA $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$. For any $S \subseteq \mathbf{B}$, we write $\bigvee S$ for $\bigvee_{x \in S} x$ and call it the *supremum* of S. Similarly, $\bigwedge S = \bigwedge_{x \in S} x$ is the *infimum*. By convention, $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$. For any $a, b \in \mathbf{B}$, we say that a and b are *disjoint* if $a \wedge b = 0$.

Definition 2 ([10, 13])**.** An element $a \neq 0$ of $\mathbf{B}$ is an *atom* if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. Equivalently, $a \neq 0$ is an atom if there is no $x \in \mathbf{B}$ such that $0 < x < a$. We say that $\mathbf{B}$ is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([6])**.** *For any two distinct atoms a, $b \in \mathbf{B}$, $a \wedge b = 0$.*

Lemma 2 ([8])**.** *The following are equivalent:*

- $\mathbf{B}$ *is atomic.*
- *For any $x \in \mathbf{B}$, $x = \bigvee_{atoms\ a \leq x} a$.*
- 1 *is the supremum of all atoms.*

Lemma 3 ([8])**.** *All finite BAs are atomic.*

Definition 3 ([5, 10])**.** A *measure* on $\mathbf{B}$ is a function $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ such that:

- $m(0) = 0$;
- $m(a \vee b) = m(a) + m(b)$ for all $a, b \in \mathbf{B}$ whenever $a \wedge b = 0$.

If $m(1) = 1$, we call m a *probability measure.* Also, if $m(x) > 0$ for all $x \neq 0$, then m is *strictly positive.*

Definition 4 ([8])**.** An *ideal* is a non-empty subset $I \subseteq \mathbf{B}$ such that

- $i \vee j \in I$ for all $i, j \in I$;
- $i \wedge a \in I$ for all $i \in I$ and $a \in \mathbf{B}$.

For any $p \in \mathbf{B}$, the *principal ideal of* p—denoted by (p)—is the smallest ideal that contains p. It can also be expressed as $(p) = \{a \in \mathbf{B} \mid a \leq p\}$.

[1]We use + to denote symmetric difference because it is the additive operation of a Boolean ring.

Definition 5 ([8])**.** Let I be an ideal in $\mathbf{B}$. The *quotient algebra of* $\mathbf{B}$ *modulo the ideal* I $\mathbf{B}/I$ is a BA of equivalence classes of elements of $\mathbf{B}$ with respect to the equivalence relation

$$a \sim b \quad \iff \quad a + b \in I$$

for all $a, b \in \mathbf{B}$. Elements of $\mathbf{B}/I$ are usually denoted by a/I (for some $a \in \mathbf{B}$) with the understanding that if $b \sim a$ (for some $b \in \mathbf{B}$), then $b/I = a/I$. The three algebraic operations on $\mathbf{B}/I$ are defined as

$$\begin{aligned} a/I \wedge b/I &= a \wedge b/I, \\ a/I \vee b/I &= a \vee b/I, \\ \neg(a/I) &= (\neg a)/I. \end{aligned}$$

Definition 6 ([8])**.** Let $\mathbf{A}$ and $\mathbf{B}$ be BAs. A *(Boolean) homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $f\colon \mathbf{A} \to \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,
- $f(x \vee y) = f(x) \vee f(y)$,
- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Lemma 4 ([8])**.** *Let $I \subseteq \mathbf{B}$ be an ideal. The map $f\colon \mathbf{B} \to \mathbf{B}/I$ defined by $f(x) = x/I$ is a homomorphism.*

Lemma 5 (Homomorphisms preserve order [8])**.** *Let $f\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism between two BAs $\mathbf{A}$ and $\mathbf{B}$. Then, for any $x, y \in \mathbf{A}$, if $x \leq y$, then $f(x) \leq f(y)$.*

Lemma 6 ([16])**.** *For any $a, b \in \mathbf{B}$, $a \leq b$ if and only if $a \wedge \neg b = 0$.*

Lemma 7 ([8])**.** *Let $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure. Then for all $a, b \in \mathbf{B}$, if $a \leq b$, then $m(a) \leq m(b)$.*

3 WMC as a Measure

Definition 7. Let $\mathcal{L}$ be a propositional (or first-order) logic, and let Δ be a theory in $\mathcal{L}$. We can define an equivalence relation on formulas in $\mathcal{L}$ as

$$\alpha \sim \beta \quad \text{if and only if} \quad \Delta \vdash \alpha \leftrightarrow \beta$$

for all $\alpha, \beta \in \mathcal{L}$. Let $[\alpha]$ denote the equivalence class of $\alpha \in \mathcal{L}$ with respect to $\sim$. We can then let $B(\Delta) = \{[\alpha] \mid \alpha \in \mathcal{L}\}$ and define the structure of a BA on $B(\Delta)$ as

$$\begin{aligned} [\alpha] \vee [\beta] &= [\alpha \vee \beta], \\ [\alpha] \wedge [\beta] &= [\alpha \wedge \beta], \\ \neg[\alpha] &= [\neg \alpha], \\ 1 &= [\alpha \to \alpha], \\ 0 &= [\alpha \wedge \neg \alpha] \end{aligned}$$

for all $\alpha, \beta \in \mathcal{L}$. Then $B(\Delta)$ is the *Lindenbaum-Tarski algebra* of Δ [11, 17].

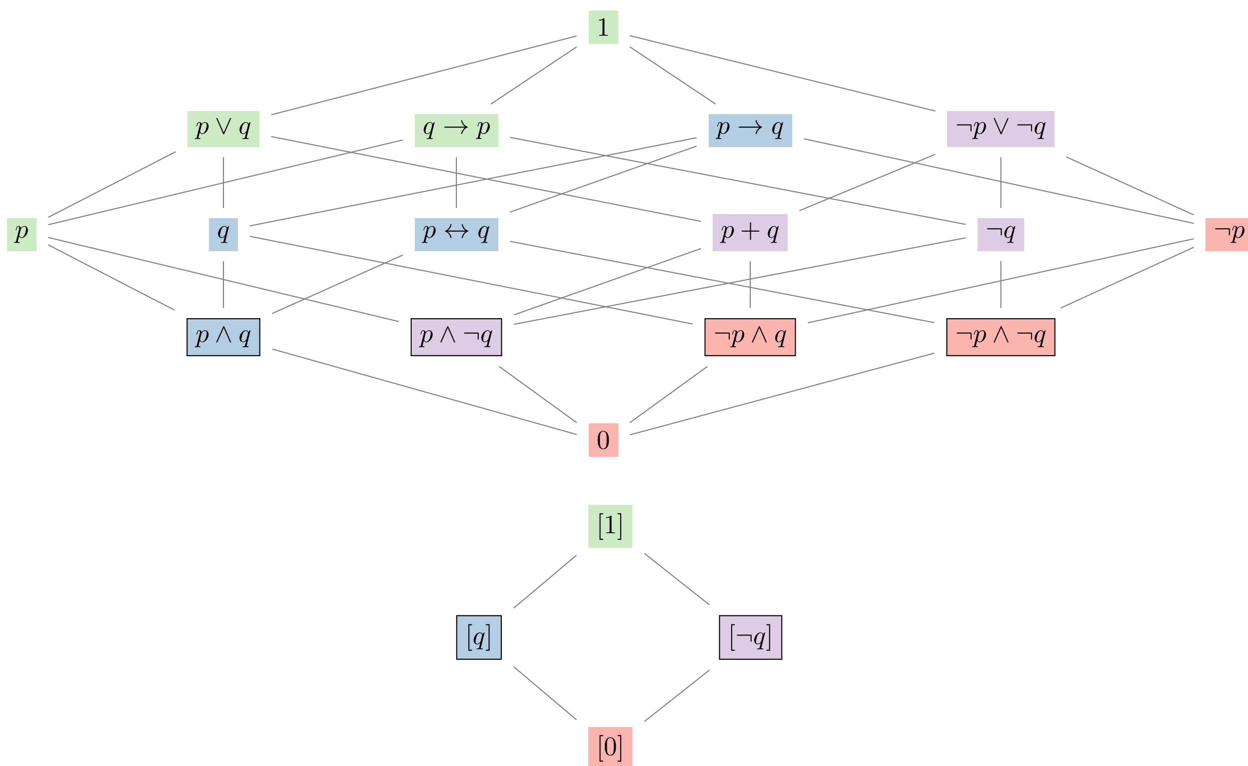

Figure 1: Two BAs from Example 1: $B(\mathcal{L})$ at the top and $B(\Delta)$ at the bottom. An edge between elements a and b (with a positioned lower than b) means that $a < b$. Each element of $B(\Delta)$ is an equivalence class of elements of $B(\mathcal{L})$, and the colours show which elements of $B(\mathcal{L})$ belong to which class. In both algebras, atoms have borders around them.

Example 1. Let $\mathcal{L}$ be a propositional logic with p and q as its only atoms. Then $L = \{p, q, \neg p, \neg q\}$ is its set of literals. Let $w : L \to \mathbb{R}_{\geq 0}$ be the *weight function* defined by

$$\begin{aligned} w(p) &= 0.3, \\ w(\neg p) &= 0.7, \\ w(q) &= 0.2, \\ w(\neg q) &= 0.8. \end{aligned}$$

Let Δ be a theory in $\mathcal{L}$ with a sole axiom p. Then Δ has two models, i.e., $\{p, q\}$ and $\{p, \neg q\}$. The *weighted model count* (WMC) [3] of Δ is then

$$\sum_{\omega \models \Delta} \prod_{\omega \models l} w(l) = w(p)w(q) + w(p)w(\neg q) = 0.3.$$

The corresponding BA $B(\Delta)$ can then be constructed using Definition 7. Alternatively, one can first construct the free BA generated by the set $\{p, q\}$—this corresponds to $B(\mathcal{L})$ in Fig. 1—and then take a quotient with respect to either the filter generated by p or the ideal[2] generated by $\neg p$. In any case, the resulting BA is pictured at the bottom of Fig. 1.

Each element of $B(\mathcal{L})$ can also be seen as a subset of the set of all models of $\mathcal{L}$, with 0 representing $\emptyset$, 1 representing the set of all (four) models, each atom representing a single model, and each edge going upward representing a subset relation. Thus, the Boolean-algebraic way of calculating the WMC of Δ consists of:

1. Identifying an element $a \in B(\mathcal{L})$ that corresponds to Δ.
2. Finding all atoms of $B(\mathcal{L})$ that are 'dominated' by a according to the partial order.
3. Using w to calculate the weight of each such atom.
4. Adding the weights of these atoms.

This motivates the following definition of WMC generalised to BAs.

Definition 8. Let $\mathbf{B}$ be an atomic BA, and let $M \subset \mathbf{B}$ be its set of atoms. Let $L \subset \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge L'$ for some $L' \subseteq L$, and let $w \colon L \to \mathbb{R}_{\geq 0}$ be arbitrary. The *weighted model count* $\mathrm{WMC}_w \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}_w(x) = \begin{cases} 0 & \text{if } x = 0 \\ \prod_{l \in L'} w(l) & \text{if } M \ni x = \bigwedge L' \\ \sum_{\text{atoms } a \leq x} \mathrm{WMC}_w(a) & \text{otherwise} \end{cases}$$

for any $x \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{NWMC}_w \colon \mathbf{B} \to [0, 1]$ as $\mathrm{NWMC}_w(x) = \frac{\mathrm{WMC}_w(x)}{\mathrm{WMC}_w(1)}$ for all $x \in \mathbf{B}$. For both WMC_w and NWMC_w, we will drop the subscript when doing so results in no potential confusion.

Theorem 1. WMC *is a measure, and* NWMC *is a probability measure.*

Proof. First, note that WMC is non-negative and $\mathrm{WMC}(0) = 0$ by definition. Next, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

[2] More details on these concepts can be found in many books on BAs [8, 11].

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Lemma 2 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned}\mathrm{WMC}(x \vee y) &= \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i\right) \vee \left(\bigvee_{j \in J} y_j\right)\right) = \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j) \\ &= \mathrm{WMC}(x) + \mathrm{WMC}(y),\end{aligned}$$

finishing the proof that WMC is a measure. This immediately shows that NWMC is a probability measure since, by definition, $\mathrm{NWMC}(1) = 1$. □

Given a theory Δ in a logic $\mathcal{L}$, the usual way of using WMC to compute the probability of a query q is [1, 15]

$$\Pr_{\Delta, w}(q) = \frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}.$$

In our algebraic formulation, this can be computed in two different ways:

- as $\frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}$ in $B(\mathcal{L})$,
- and as $\mathrm{NWMC}_w([q])$ in $B(\Delta)$.

But how does the measure defined on $B(\mathcal{L})$ transfer to $B(\Delta)$?

Lemma 8. *For any measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ and elements $a, b \in \mathbf{B}$,*

$$m(a \vee b) = m(a) + m(b) - m(a \wedge b).$$

Proof. By Definition 3,

$$\begin{aligned} m(a) &= m(a \wedge b) + m(a \wedge \neg b), \\ m(b) &= m(a \wedge b) + m(\neg a \wedge b), \\ m(a \vee b) &= m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b), \end{aligned}$$

so

$$m(a) + m(b) - m(a \wedge b) = m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b) = m(a \vee b)$$

as required. □

Lemma 9. *For any $a, b \in \mathbf{B}$ and any principal ideal (p), if $a/(p) = b/(p)$, then $a \vee p = b \vee p$.*

Proof. Note that

$$a/(p) = b/(p) \iff a + b \in (p) \iff a + b \leq p \iff (a + b) \vee p = p$$

by Definitions 4 and 5, and the definition of $\leq$. So $p = (a \wedge \neg b) \vee (\neg a \wedge b) \vee p$, and thus

$$0 = p \wedge \neg p = (a \wedge \neg b \wedge \neg p) \vee (\neg a \wedge b \wedge \neg p) \vee (p \wedge \neg p) = (a \wedge \neg(b \vee p)) \vee (b \wedge \neg(a \vee p)).$$

It follows that

$$a \wedge \neg(b \vee p) = 0 \quad \text{and} \quad b \wedge \neg(a \vee p) = 0.$$

Focusing on the first equation,

$$\neg a = (\neg a \vee a) \wedge [\neg a \vee \neg(b \vee p)] = \neg[a \wedge (b \vee p)],$$

and so $a = a \wedge (b \vee p)$, and

$$a \vee p = (a \vee p) \wedge (b \vee p) = (a \wedge b) \vee p.$$

By similar arguments, $b \vee p = (a \wedge b) \vee p$ as well which shows that $a \vee p = b \vee p$ as required. □

Outdated. $m(a \wedge \neg p)$ is better than $m(a \vee p)$.

Proposition 1 (Measures on quotients). *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure, and let (p) be a principal ideal. Let $m^* \colon \mathbf{B}/(p) \to \mathbb{R}_{\geq 0}$ be defined as*

$$m^*(a/(p)) = m(a \vee p)$$

for any $a \in \mathbf{B}$. The function m^* is well-defined. Furthermore, it is a measure on $\mathbf{B}/(p)$ if and only if $m(p) = 0$. Moreover, if it is a measure, then the following properties transfer from m to m^*:

- *if m is a* probability measure*, then so is m^*;*
- *if m is* strictly positive*, then so is m^*.*

Proof. Lemma 9 proves that the function is well-defined. Next, note that

$$m^*(0/(p)) = m(0 \vee p) = m(p),$$

so $m^*(0/(p)) = 0$ if and only if $m(p) = 0$. For the second part of Definition 3, let $a/(p), b/(p) \in \mathbf{B}/(p)$ be such that

$$a/(p) \wedge b/(p) = a \wedge b/(p) = 0/(p).$$

This condition is equivalent to $a \wedge b \in (p)$ and $(a \wedge b) \vee p = p$ by well-known properties of quotients and ideals [8], Definition 4, and the definition of $\leq$, respectively. Now

$$\begin{aligned} m^*(a/(p) \vee b/(p)) = m^*(a \vee b/(p)) = m(a \vee b \vee p) &= m((a \vee p) \vee (b \vee p)) \\ &= m(a \vee p) + m(b \vee p) - m((a \vee p) \wedge (b \vee p)) \\ &= m^*(a/(p)) + m^*(b/(p)) - m((a \vee p) \wedge (b \vee p)) \end{aligned}$$

by Lemma 8. However

$$(a \vee p) \wedge (b \vee p) = (a \wedge b) \vee p = p,$$

so $m^*(a/(p) \vee b/(p)) = m^*(a/(p)) + m^*(b/(p))$ if and only if $m(p) = 0$.

The two remaining properties are easy to prove:

- If $m(1) = 1$, then $m^*(1/(p)) = m(1 \vee p) = m(1) = 1$.
- Suppose that m is strictly positive, and let $a/(p) \in \mathbf{B}/(p)$ be such that $a/(p) \neq 0/(p)$. Then

  $$m^*(a/(p)) = m(a \vee p) \geq m(a) > 0,$$

  where the first inequality comes from an elementary property of $\leq$ that $x \leq x \vee y$ for any $x, y \in \mathbf{B}$ [16] and Lemma 7; and the second inequality follows because $a/(p) \neq 0/(p)$ implies that $a \neq 0$, and m is assumed to be strictly positive.

□

3.1 Lemma Galore

This section made me realise that I was using the wrong definition

Lemma 10. *Let (p) be a principal ideal. Then for any $a \in \mathbf{B}$, $(a \wedge \neg p)/(p) = a/(p)$.*

Proof. Note that

$$(a \wedge \neg p)/(p) = a/(p) \iff (a \wedge \neg p) + a \in (p) \iff (a \wedge \neg p) + a \leq p.$$

We also have that

$$(a \wedge \neg p) + a = (a \wedge \neg p \wedge \neg a) \vee (\neg(a \wedge \neg p) \wedge a) = (\neg a \vee p) \wedge a = (\neg a \wedge a) \vee (p \wedge a) = p \wedge a.$$

And, since $p \wedge a \leq p$, we have that $(a \wedge \neg p) + a \leq p$ as required. ☐

Lemma 11. *Let (p) be a principal ideal. For any $a, b \in \mathbf{B}$, $a/(p) \leq b/(p)$ if and only if $a \wedge \neg p \leq b \wedge \neg p$.*

Proof. Let us begin with the 'only of' direction. Lemma 10 tells us that $(a \wedge \neg p)/(p) = a/(p)$. Combining this with Lemmas 4 and 5 shows that

$$a \wedge \neg p \leq b \wedge \neg p \implies (a \wedge \neg p)/(p) \leq (b \wedge \neg p)/(p) \iff a/(p) \leq b/(p)$$

as required.

For the other direction, let $a, b \in \mathbf{B}$ be such that $a/(p) \leq b/(p)$. Then, by Lemma 6,

$$[a/(p)] \wedge \neg[b/(p)] = (a \wedge \neg b)/(p) = 0/(p),$$

i.e.,

$$a \wedge \neg b \in (p) \iff a \wedge \neg b \leq p \iff a \wedge \neg b \wedge \neg p = 0$$

by Definition 4 and Lemma 6. We need to show that $a \wedge \neg p \leq b \wedge \neg p$. By Lemma 6, this is equivalent to $a \wedge \neg p \wedge \neg(b \wedge \neg p) = 0$. But

$$a \wedge \neg p \wedge \neg(b \wedge \neg p) = a \wedge \neg p \wedge (\neg b \vee p) = (a \wedge \neg p \wedge \neg b) \vee (a \wedge \neg p \wedge p) = a \wedge \neg p \wedge \neg b,$$

and we already have that $a \wedge \neg p \wedge \neg b = 0$ by assumption. ☐

Lemma 12. *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure, let $p \in \mathbf{B}$ be such that $m(p) = 0$, and let $m^* \colon \mathbf{B}/(p) \to \mathbb{R}_{\geq 0}$ be a measure defined by $m^*(a/(p)) = m(a \vee p)$. Then for any $a \in \mathbf{B}$, if $a/(p)$ is an atom in $\mathbf{B}/(p)$, then $a \wedge \neg p$ is an atom in $\mathbf{B}$ such that $m^*(a/(p)) = m(a \wedge \neg p)$.*

Proof. First, we want to show that if $a/(p)$ is an atom, then $a \wedge \neg p$ is an atom. We can instead prove the contrapositive statement, i.e., if there exists a $b \in \mathbf{B}$ such that $0 < b < a \wedge \neg p$, then there exists a $b' \in \mathbf{B}$ such that $0/(p) < b'/(p) < a/(p)$. We will show that, in fact, we set $b' = b$. Lemmas 4 and 5 already tell us that $b/(p) \leq a/(p)$, so we only need to show that $0/(p) < b/(p) \neq a/(p)$. For the first part, note that

$$0/(p) < b/(p) \iff b/(p) \neq 0/(p) \iff b \notin (p) \iff b \nleq p \iff b \wedge \neg p \neq 0$$

by Lemma 6. But if $b \wedge \neg p = 0$, then $b \wedge a \wedge \neg p = 0$. This contradicts either that $b \leq a \wedge \neg p$ (i.e., $b \wedge a \wedge \neg p = b$) or that $b \neq 0$. For the second part, i.e., $b/(p) \neq a/(p)$, we will show that if $b/(p) = a/(p)$, and $b \leq a \wedge \neg p$, then $b = a \wedge \neg p$. Indeed,

$$b/(p) = a/(p) \iff a + b \in (p) \iff a + b \leq p \iff (a + b) \wedge \neg p = 0,$$

and

$$(a + b) \wedge \neg p = [(a \wedge \neg b) \vee (\neg a \wedge b)] \wedge \neg p = (a \wedge \neg b \wedge \neg p) \vee (\neg a \wedge b \wedge \neg p),$$

so $(a+b)\wedge\neg p=0$ implies that $a\wedge\neg b\wedge\neg p=0$ which is equivalent to $a\wedge\neg p\leq b$. Therefore we have that $a\wedge\neg p\leq b\leq a\wedge\neg p$, so $b=a\wedge\neg p$ which, by contradiction, shows that $b/(p)\neq a/(p)$ and finishes the proof that $0/(p)<b/(p)<a/(p)$.

In order to show that $m^*(a/(p))=m(a\wedge\neg p)$, note that $a\wedge p$, $a\wedge\neg p$, and $\neg a\wedge p$ are pairwise disjoint and their supremum is $a\vee p$, so we have that

$$m^*(a/(p))=m(a\vee p)=m(a\wedge p)+m(a\wedge\neg p)+m(\neg a\wedge p).$$

Furthermore, since $a\wedge p\leq p$, $m(a\wedge p)\leq m(p)=0$. Similarly, $m(\neg a\wedge p)=0$, so $m^*(a/(p))=m(a\wedge\neg p)$ as required. □

Lemma 13. *Let* $\mathbf{B}$ *be a complete BA. For any* $a,b\in\mathbf{B}$*, if* $a/(p)=b/(p)$*, then* $a\wedge\neg p=b\wedge\neg p$*. As a consequence,* $a\wedge\neg p\leq b$*.*

Proof. As in the proof of Lemma 12, $a/(p)=b/(p)$ implies that $a\wedge\neg p\leq b$. Since $\mathbf{B}$ is complete, let $b=\bigwedge\{c\in\mathbf{B}\mid c/(p)=a/(p)\}$; then we still have that $b/(p)=a/(p)$. But then $b\leq a\wedge\neg p\leq b$, so $a\wedge\neg p=b$. This defines $a\wedge\neg p$ independently of a as the least element in $\{c\in\mathbf{B}\mid c/(p)=a/(p)\}$. □

Corollary 1. *For any complete BA* $\mathbf{B}$*, if* $a\in\mathbf{B}$ *is an atom, then* $a/(p)$ *is either an atom or* $0/(p)$*. In the former case,* $a=a\wedge\neg p$*.*

Proof. Since Lemma 13 tells us that for all $b\in\mathbf{B}$, if $b/(p)=a/(p)$, then $b\geq a\wedge\neg p$, if there is an atom $b\in\mathbf{B}$ such that $b/(p)=a/(p)$, then it must be $a\wedge\neg p$. If a is an atom, then $a\wedge\neg p\leq a$ implies that either $a=a\wedge\neg p$ or $a\wedge\neg p=0$. The latter is equivalent to $a/(p)=0/(p)$ by Lemma 6. The former, combined with the assumption that a is an atom and Lemma 11, implies that $a/(p)$ is an atom. □

4 What Measures Are WMC-Computable?

4.1 WMC Requires Independent Literals

Theorem 2. *Let* $\mathbf{B}$ *be a finite measure algebra with measure* $m\colon\mathbf{B}\to\mathbb{R}_{\geq 0}$*. Let* $L\subset\mathbf{B}$ *be defined as*

$$L=\{l_i\mid i\in[n]\}\cup\{\neg l_i\mid i\in[n]\}$$

for some $n\in\mathbb{N}$*. Finally, assume that* $\mathbf{B}$ *has* 2^n *atoms, where each atom* $a\in\mathbf{B}$ *is an infimum*

$$a=\bigwedge_{i=1}^{n}a_i$$

such that $a_i\in\{l_i,\neg l_i\}$ *for* $i\in[n]$*. Then there exists a weight function* $w\colon L\to\mathbb{R}_{\geq 0}$ *that makes* m *a WMC measure if and only if*

$$m(l\wedge l')=m(l)m(l') \tag{2}$$

for all distinct $l,l'\in L$ *such that* $l\neq\neg l'$*.*

Remark. Note that if $n=1$, then Eq. (2) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. Let us begin with the 'if' part of the statement. Let $w\colon L\to\mathbb{R}_{\geq 0}$ be defined by

$$w(l)=m(l) \tag{3}$$

for all $l\in L$. We are going to show that NWMC $=m$. First, note that NWMC$(0)=0=m(0)$ by the definitions of both NWMC and m. Second, let

$$a=\bigwedge_{i=1}^{n}a_i \tag{4}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\mathrm{NWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)} = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} w(a_i) = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} m(a_i) = \frac{1}{\mathrm{WMC}(1)} m\left(\bigwedge_{i=1}^{n} a_i\right) = \frac{m(a)}{\mathrm{WMC}(1)}$$

by Definition 8 and Eqs. (2) to (4). Now we just need to show that $\mathrm{WMC}(1) = 1$. Indeed,

$$\begin{aligned}\mathrm{WMC}(1) &= \sum_{\text{atoms } a \in \mathbf{B}} \mathrm{WMC}(a) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} w(a_i) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} m(a_i) \\ &= \sum_{\text{atoms } a \in \mathbf{B}} m\left(\bigwedge_{i=1}^{n} a_i\right) = \sum_{\text{atoms } a \in \mathbf{B}} m(a) = m\left(\bigvee_{\text{atoms } a \in \mathbf{B}}\right) = m(1) = 1.\end{aligned}$$

Finally, note that if NWMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

For the other direction, we are given a weight function $w \colon L \to \mathbb{R}_{\geq 0}$ that induces a measure $m = \mathrm{NWMC} \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (2) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$. We will first prove an auxiliary result that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \tag{5}$$

is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j). \tag{6}$$

First, note that k_i can be expressed as

$$k_i = (k_i \wedge k_j) \vee (k_i \wedge \neg k_j)$$

with the condition that

$$(k_i \wedge k_j) \wedge (k_i \wedge \neg k_j) = 0,$$

so, by properties of a measure,

$$m(k_i) = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j). \tag{7}$$

Applying Eq. (7) and the equivalent expression for $m(k_j)$ allows us to rewrite Eq. (5) as

$$\begin{aligned}m(k_i \wedge k_j) &= [m(k_i \wedge k_j) + m(k_i \wedge \neg k_j)] \cdot [m(k_i \wedge k_j) + m(\neg k_i \wedge k_j)] \\ &= m(k_i \wedge k_j)^2 + m(k_i \wedge k_j)[m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j)] + m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)\end{aligned}$$

Dividing both sides by $m(k_i \wedge k_j)$ gives

$$1 = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j) + \frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)}. \tag{8}$$

Since $k_i \wedge k_j \wedge k_i \wedge \neg k_j = 0$, and

$$(k_i \wedge k_j) \vee (k_i \wedge \neg k_j) = k_i \wedge (k_j \vee \neg k_j) = k_i \wedge 1 = k_i,$$

we have that

$$m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) = m(k_i).$$

Similarly, $k_i \wedge \neg k_i \wedge k_j = 0$, and

$$k_i \vee (\neg k_i \wedge k_j) = (k_i \vee \neg k_i) \wedge (k_i \vee k_j) = k_i \vee k_j,$$

so

$$m(k_i) + m(\neg k_i \wedge k_j) = m(k_i \vee k_j).$$

Finally, note that

$$(k_i \vee k_j) \wedge \neg(k_i \vee k_j) = 0,$$

and

$$(k_i \vee k_j) \vee \neg(k_i \vee k_j) = 1,$$

so

$$m(k_i \vee k_j) + m(\neg(k_i \vee k_j)) = m(1) = 1.$$

This allows us to rewrite Eq. (8) as

$$\frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)} = 1 - m(k_i \vee k_j) = m(\neg(k_i \vee k_j)) = m(\neg k_i \wedge \neg k_j)$$

which immediately gives us Eq. (6).

Now recall that $m = \mathrm{NWMC}$ and note that Eq. (6) can be multiplied by $\mathrm{WMC}(1)^2$ to turn the equation into one for WMC instead of NWMC. Then

$$\begin{aligned}
\mathrm{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \leq k_i \wedge k_j} \mathrm{WMC}(a) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\
&= \sum_{\text{atoms } a \leq k_i \wedge k_j} w(a_i)w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \leq k_i \wedge k_j} w(k_i)w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) \\
&= w(k_i)w(k_j) \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i)w(k_j)C,
\end{aligned}$$

where C denotes the part of $\mathrm{WMC}(k_i \wedge k_j)$ that will be the same for $\mathrm{WMC}(\neg k_i \wedge k_j)$, $\mathrm{WMC}(k_i \wedge \neg k_j)$, and $\mathrm{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (6) becomes

$$w(k_i)w(k_j)w(\neg k_i)w(\neg k_j)C^2 = w(k_i)w(\neg k_j)w(\neg k_i)w(k_j)C^2$$

which is trivially true. By showing that WMC satisfies Eq. (6), we also showed that it satisfies Eq. (5), finishing the second part of the proof. □

4.2 Extending the Algebra

A well-known way to overcome this limitation of independence is by adding more literals [3], i.e., extending the set L covered by the WMC weight function $w \colon L \to \mathbb{R}_{\geq 0}$. Let us translate this idea to the language of Boolean algebras.

Theorem 3. *Let $\mathbf{B}$ be a finite Boolean algebra freely generated by some set of 'literals' L, and let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be an arbitrary measure. We know that $\mathbf{B}$ has $n = 2^{|L|}$ atoms. Let $(a_i)_{i=1}^n$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i \mid i \in [n]\} \cup \{\neg\phi_i \mid i \in [n]\}$ be the set L extended with $2n$ new literals. Let $\mathbf{B}'$ be the unique Boolean algebra with*

$$\{\phi_i \wedge a_i \mid i \in [n]\} \cup \{\neg\phi_i \wedge a_i \mid i \in [n]\}$$

as its set of atoms. Let $\iota \colon \mathbf{B} \to \mathbf{B}'$ be the inclusion homomorphism (i.e., $\iota(a) = a$ for all $a \in \mathbf{B}$). Let $w \colon L' \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = \begin{cases} \frac{m(a_i)}{2} & \text{if } l = \phi_i \text{ or } l = \neg\phi_i \text{ for some } i \in [n] \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m' \colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In simpler terms, any measure can be computed using WMC by extending the Boolean algebra with more literals. More precisely, we are given the red part in

$$\begin{tikzcd}
\mathbb{R}_{\geq 0} & & \\
\mathbf{B} \arrow[u, "m"] \arrow[r, hook, "\iota"] & \mathbf{B}' \arrow[ul, "m'"'] & \\
L \arrow[u, hook] \arrow[r, hook] & L' \arrow[u, hook] \arrow[r, "w"] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

and construct the black part in such a way that the triangle commutes.

Proof. Since $\mathbf{B}$ is freely generated by L, each atom $a_i \in \mathbf{B}$ is an infimum of elements in L, i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either

$$b = \phi_i \wedge a_i \quad \text{or} \quad b = \neg\phi_i \wedge a_i$$

for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L'. Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies

$$\phi_i \wedge a_i \leq \iota(b)$$

if and only if it satisfies

$$\neg\phi_i \wedge a_i \leq \iota(b).$$

Then, according to the definition of w,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \leq \iota(b) \quad \text{if and only if} \quad a_i \leq b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. ☐

Now we can show that the construction in Theorem 3 is smallest possible.

Conjecture 1. *Let $\mathbf{B}$ and $\mathbf{B}'$ be Boolean algebras, and $\iota \colon \mathbf{B} \to \mathbf{B}'$ be the inclusion map such that $\mathbf{B}$ is freely generated by L, all atoms of $\mathbf{B}'$ can be expressed as meets of elements of L', and the following subset relations are satisfied:*

$$\begin{array}{ccc}
\mathbf{B} & \xrightarrow{\iota} & \mathbf{B}' \\
\cup & & \cup \\
L & \subset & L'
\end{array}$$

If, for any measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$*, one can construct a weight function* $w\colon L' \to \mathbb{R}_{\geq 0}$ *such that the WMC measure* $\mathrm{WMC}\colon \mathbf{B'} \to \mathbb{R}_{\geq 0}$ *with respect to* w *satisfies*

$$m = \mathrm{WMC} \circ \iota,$$

then $|L' \setminus L| \geq 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [4] and Sang et al. [15]. Suppose we have a discrete probability distribution with n variables, and the ith variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^n v_i$ models. My expansion would then use

$$\sum_{i=1}^n v_i + 2 \prod_{i=1}^n v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [4] would use

$$\sum_{i=1}^n v_i + \sum_{i=1}^n \prod_{j=1}^i v_j$$

variables, while for the encoding by Sang et al. [15],

$$\sum_{i=1}^n v_i + \sum_{i=1}^n (v_i - 1) \prod_{j=1}^{i-1} v_j$$

variables would suffice.

5 Representing Independence and Conditional Independence

5.1 Independence (all known results)

Definition 9. Given a BA $\mathbf{A}$, a *subalgebra* is a subset $\mathbf{B} \subseteq \mathbf{A}$ that, together with the operations, zero, and one of $\mathbf{A}$, is a BA.

Definition 10 ([8])**.** Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be BAs such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Let $f\colon \mathbf{A} \to \mathbf{C}$ and $g\colon \mathbf{B} \to \mathbf{C}$ be homomorphisms. Then f is an *extension* of g if $f(x) = g(x)$ for all $x \in \mathbf{B}$. If f is an extension of each member of a family $\{g_i\}_{i \in I}$ of homomorphisms, then f is called a *common extension* of $\{g_i\}_{i \in I}$.

Definition 11 ([8])**.** Let $\{\mathbf{A}_i\}_{i \in I}$ be a family of subalgebras of a BA $\mathbf{A}$. If for any BA $\mathbf{B}$ with a family of homomorphisms $\{f_i\colon \mathbf{A}_i \to \mathbf{B}\}_{i \in I}$ there exists a unique common extension of $\{f_i\colon \mathbf{A}_i \to \mathbf{B}\}_{i \in I}$ ($f\colon \mathbf{A} \to \mathbf{B}$ in the diagram),

$$\begin{tikzcd} \mathbf{A}_i \arrow[r, hook] \arrow[dr, "f_i"'] & \mathbf{A} \arrow[d, dashed, "f"] \\ & \mathbf{B} \end{tikzcd}$$

then $\mathbf{A}$ is the *internal sum*[3] of $\{\mathbf{A}_i\}_{i \in I}$. We will denote it as $\bigoplus_{i \in I} \mathbf{A}_i$.

Proposition 2 ([16])**.** *Let* $\mathbf{A}$ *be the internal sum of a family of BAs* $\{\mathbf{A}_i\}_{i \in I}$*, and let* $\{m_i\colon \mathbf{A}_i \to \mathbb{R}_{\geq 0}\}_{i \in I}$ *be a family of measures. Then there exists a unique measure* $m\colon \mathbf{A} \to \mathbb{R}_{\geq 0}$ *such that, for any finite subset* $J \subseteq I$ *and family of elements* $\{x_j \in \mathbf{A}_j\}_{j \in J}$*,*

$$m\left(\bigwedge_{j \in J} x_j\right) = \prod_{j \in J} m_j(x_j).$$

[3] A slightly more general version of this definition is also known as the free product, the Boolean product, and the coproduct in the category of BAs [8, 11, 16].

5.2 Conditional Independence

Definition 12 ([11])**.** Let $\mathbf{A}$ be a BA. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, and let $(\mathbf{A}_i)_{i\in I}$ be a family of subalgebras of $\mathbf{A}$ such that $\mathbf{A}_i \cap \mathbf{A}_j = \mathbf{B}$ for all $i \neq j$ in I. Let $\{\iota_i \colon \mathbf{B} \to \mathbf{A}_i\}$ be a family of inclusion homomorphisms. Then $\mathbf{A}$ is the *amalgamated free product*[4] *of* $\{\mathbf{A}_i\}_{i\in I}$ *over* $\mathbf{B}$ if, for any Boolean algebra $\mathbf{C}$ with a family of homomorphisms $\{f_i \colon \mathbf{A}_i \to \mathbf{C}\}_{i\in I}$ such that $f_i \circ \iota_i = f_j \circ \iota_j$ for all $i, j \in I$, there is a unique homomorphism $f \colon \mathbf{A} \to \mathbf{C}$ such that the triangle in

$$\begin{tikzcd}
\mathbf{B} \arrow[r, hook, "\iota_i"] & \mathbf{A}_i \arrow[r, hook] \arrow[d, "f_i"'] & \mathbf{A} \arrow[dl, dashed, "f"] \\
& \mathbf{C} &
\end{tikzcd}$$

commutes for all $i \in I$. We will denote this product as

$$\mathbf{A} = \bigoplus_{\substack{\mathbf{B} \\ i\in I}} \mathbf{A}_i.$$

5.2.1 New Results

Theorem 4. *Let $\{\mathbf{A}_i\}_{i\in I}$ be a family of BAs with measures $\{m_i \colon \mathbf{A}_i \to \mathbb{R}_{\geq 0}\}_{i\in I}$, let $\mathbf{B}$ be a BA, and let $\mathbf{A}$ be the amalgamated free product of $\{\mathbf{A}_i\}_{i\in I}$ over $\mathbf{B}$. Then there is a unique measure $m \colon \mathbf{A} \to \mathbb{R}_{\geq 0}$ such that, for any element $b \in \mathbf{B}$, finite subset $J \subseteq I$, and elements $\{a_j \in \mathbf{A}_j\}_{j\in J}$,*

$$m\left(b \wedge \bigwedge_{j\in J} a_j\right) = \prod_{j\in J} m_j(b \wedge a_j).$$

Theorem 5. *The number of weights needed to encode a Bayesian network using coproducts and pushouts is equal to the number of entries in the tables of the network (and the resulting theory is shorter).*

Theorem 6 (Pushouts of free BAs are free)**.** *Let*

$$\mathbf{A} = \bigoplus_{\substack{\mathbf{B} \\ i\in I}} \mathbf{A}_i$$

be an amalgamated free product such that $\{\mathbf{A}_i\}_{i\in I}$ are free BAs with $\{S_i\}_{i\in I}$ as their respective sets of generators. Let $S = \bigcup_{i\in I} S_i$. Then $\mathbf{A}$ is a free BA with generating set S.

Proof. Suppose we have a map from S to an arbitrary BA $\mathbf{C}$, as in

$$\begin{tikzcd}
S_i \arrow[r, hook] \arrow[d, hook] & S \arrow[d, hook] \arrow[ddr, bend left] & \\
\mathbf{A}_i \arrow[r, hook] \arrow[drr, dashed, bend right] & \mathbf{A} \arrow[dr, dashed] & \\
& & \mathbf{C}.
\end{tikzcd}$$

We want to show that there exists a unique homomorphism $\mathbf{A} \to \mathbf{C}$. For $i \in I$, from $S_i \hookrightarrow S$ and $S \to \mathbf{C}$ we get a map $S_i \to \mathbf{C}$, so—by the definition of a free BA—there is a unique homomorphism $\mathbf{A}_i \to \mathbf{C}$. Furthermore, a family of homomorphisms $\{\mathbf{A}_i \to \mathbf{C}\}_{i\in I}$ uniquely determine a homomorphism $\mathbf{A} \to \mathbf{C}$ by the universal mapping property of a (wide) pushout. Thus $\mathbf{A}$ is a free BA with generating set S. $\square$

Corollary 2. *Similarly, coproducts of free BAs are free.*

[4] Also known as a (wide) pushout in the category of BAs.

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