

On the Limitations of Weighted Model Counting

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1 Introduction

2 WMC as a Measure

2.1 Preliminaries

Definition 1. A *Boolean algebra* is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ of a set \mathbf{B} with operations \wedge, \vee, \neg and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b \in \mathbf{B}$:

- both \wedge and \vee are associative and commutative;
- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of \vee , and 1 is the identity of \wedge ;
- \vee distributes over \wedge and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

Let $a, b \in \mathbf{B}$ be arbitrary. Let \leq be a partial order on \mathbf{B} defined by $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$), and let $a < b$ denote $a \leq b$ and $a \neq b$.

Which definition do I actually need?

Definition 2 ([7, 8]). An element $a \neq 0$ of a Boolean algebra \mathbf{B} is an *atom* if there is no $x \in \mathbf{B}$ such that $0 < x < a$. Equivalently, $a \neq 0$ is an atom if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. A Boolean algebra is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([4]). For any two distinct atoms a, b in a Boolean algebra, $a \wedge b = 0$.

Lemma 2 ([5]). All finite Boolean algebras are atomic.

Theorem 1. Let \mathbf{B} be a finite Boolean algebra. Then every $x \in \mathbf{B} \setminus \{0\}$ can be uniquely expressed as

$$x = \bigvee_{\text{atoms } a \leq x} a.$$

Proof. A simple consequence of the theorem that every finite Boolean algebra is isomorphic to a field of subsets of a set, where the cardinality of the set is equal to the number of atoms in the Boolean algebra. \square

Remove the requirement for being strictly positive

Definition 3 ([3]). A (*strictly positive*) *measure* on a Boolean algebra \mathbf{B} is a function $m : \mathbf{B} \rightarrow [0, 1]$ such that:

1. $m(1) = 1$, and $m(x) > 0$ for $x \neq 0$;
2. $m(x \vee y) = m(x) + m(y)$ for all $x, y \in \mathbf{B}$ whenever $x \wedge y = 0$.

2.2 New Results

Allow weight to be zero

Definition 4. Let \mathbf{B} be a finite Boolean algebra, and let $M \subseteq \mathbf{B}$ be its set of atoms. Let $L \subseteq \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge_{i \in I} l_i$ for some $\{l_i\}_{i \in I} \subseteq L$, and let $w : L \rightarrow \mathbb{R}_{>0}$ be arbitrary. The *weighted model count* $\text{WMC} : \mathbf{B} \rightarrow \mathbb{R}_{\geq 0}$ is defined as

$$\text{WMC}(a) = \begin{cases} 0 & \text{if } a = 0 \\ \prod_{i \in I} w(l_i) & \text{if } M \ni a = \bigwedge_{i \in I} l_i \\ \sum_{i \in I} \text{WMC}(m_i) & \text{if } \mathbf{B} \setminus (M \cup \{0\}) \ni a = \bigvee_{i \in I} m_i \end{cases}$$

for any $a \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\text{nWMC} : \mathbf{B} \rightarrow [0, 1]$ as $\text{nWMC}(a) = \frac{\text{WMC}(a)}{\text{WMC}(1)}$ for all $a \in \mathbf{B}$.

Proposition 1. nWMC is a measure for any finite Boolean algebra \mathbf{B} .

Proof. First, note that Property 1 of Definition 3 is satisfied by the definition of nWMC . Next, in order to prove Property 2, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\text{nWMC}(x \vee y) = \text{nWMC}(x) + \text{nWMC}(y)$$

which is equivalent to

$$\text{WMC}(x \vee y) = \text{WMC}(x) + \text{WMC}(y). \quad (1)$$

If, say, $x = 0$, then Eq. (1) becomes

$$\text{WMC}(y) = \text{WMC}(0) + \text{WMC}(y) = \text{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Theorem 1 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned} \text{WMC}(x \vee y) &= \text{WMC} \left(\left(\bigvee_{i \in I} x_i \right) \vee \left(\bigvee_{j \in J} y_j \right) \right) = \sum_{i \in I} \text{WMC}(x_i) + \sum_{j \in J} \text{WMC}(y_j) \\ &= \text{WMC}(x) + \text{WMC}(y), \end{aligned}$$

finishing the proof. □

3 What Measures Are WMC-Computable?

3.1 WMC Requires Independent Literals

Proposition 2. Let \mathbf{B} be a finite measure algebra with measure $m : \mathbf{B} \rightarrow \mathbb{R}_{\geq 0}$. Let $L \subset \mathbf{B}$ be defined as

$$L = \{l_i \mid i \in [n]\} \cup \{\neg l_i \mid i \in [n]\}$$

for some $n \in \mathbb{N}$. Finally, assume that \mathbf{B} has 2^n atoms, where each atom $a \in \mathbf{B}$ is an infimum

$$a = \bigwedge_{i=1}^n a_i$$

such that $a_i \in \{l_i, \neg l_i\}$ for $i \in [n]$. Then there exists a weight function $w : L \rightarrow \mathbb{R}_{>0}$ that makes m a WMC measure if and only if

$$m(l \wedge l') = m(l)m(l') \quad (2)$$

for all distinct $l, l' \in L$ such that $l \neq \neg l'$.

Remark. Note that if $n = 1$, then Eq. (2) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. Let us begin with the ‘if’ part of the statement. Let $w : L \rightarrow \mathbb{R}_{>0}$ be defined by

$$w(l) = m(l) \quad (3)$$

for all $l \in L$. We are going to show that $\text{nWMC} = m$. First, note that $\text{nWMC}(0) = 0 = m(0)$ by the definitions of both nWMC and m . Second, let

$$a = \bigwedge_{i=1}^n a_i \quad (4)$$

be an atom in \mathbf{B} such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\text{nWMC}(a) = \frac{\text{WMC}(a)}{\text{WMC}(1)} = \frac{1}{\text{WMC}(1)} \prod_{i=1}^n w(a_i) = \frac{1}{\text{WMC}(1)} \prod_{i=1}^n m(a_i) = \frac{1}{\text{WMC}(1)} m\left(\bigwedge_{i=1}^n a_i\right) = \frac{m(a)}{\text{WMC}(1)}$$

by Definition 4 and Eqs. (2) to (4). Now we just need to show that $\text{WMC}(1) = 1$. Indeed,

$$\begin{aligned} \text{WMC}(1) &= \sum_{\text{atoms } a \in \mathbf{B}} \text{WMC}(a) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^n w(a_i) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^n m(a_i) \\ &= \sum_{\text{atoms } a \in \mathbf{B}} m\left(\bigwedge_{i=1}^n a_i\right) = \sum_{\text{atoms } a \in \mathbf{B}} m(a) = m\left(\bigvee_{\text{atoms } a \in \mathbf{B}} a\right) = m(1) = 1. \end{aligned}$$

Finally, note that if nWMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

For the other direction, we are given a weight function $w : L \rightarrow \mathbb{R}_{>0}$ that induces a measure $m = \text{nWMC} : \mathbf{B} \rightarrow \mathbb{R}_{\geq 0}$, and we want to show that Eq. (2) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$. We will first prove an auxiliary result that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \quad (5)$$

is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j). \quad (6)$$

First, note that k_i can be expressed as

$$k_i = (k_i \wedge k_j) \vee (k_i \wedge \neg k_j)$$

with the condition that

$$(k_i \wedge k_j) \wedge (k_i \wedge \neg k_j) = 0,$$

so, by properties of a measure,

$$m(k_i) = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j). \quad (7)$$

Applying Eq. (7) and the equivalent expression for $m(k_j)$ allows us to rewrite Eq. (5) as

$$\begin{aligned} m(k_i \wedge k_j) &= [m(k_i \wedge k_j) + m(k_i \wedge \neg k_j)] \cdot [m(k_i \wedge k_j) + m(\neg k_i \wedge k_j)] \\ &= m(k_i \wedge k_j)^2 + m(k_i \wedge k_j)[m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j)] + m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j) \end{aligned}$$

Dividing both sides by $m(k_i \wedge k_j)$ gives

$$1 = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j) + \frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)}. \quad (8)$$

Since $k_i \wedge k_j \wedge k_i \wedge \neg k_j = 0$, and

$$(k_i \wedge k_j) \vee (k_i \wedge \neg k_j) = k_i \wedge (k_j \vee \neg k_j) = k_i \wedge 1 = k_i,$$

we have that

$$m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) = m(k_i).$$

Similarly, $k_i \wedge \neg k_i \wedge k_j = 0$, and

$$k_i \vee (\neg k_i \wedge k_j) = (k_i \vee \neg k_i) \wedge (k_i \vee k_j) = k_i \vee k_j,$$

so

$$m(k_i) + m(\neg k_i \wedge k_j) = m(k_i \vee k_j).$$

Finally, note that

$$(k_i \vee k_j) \wedge \neg(k_i \vee k_j) = 0,$$

and

$$(k_i \vee k_j) \vee \neg(k_i \vee k_j) = 1,$$

so

$$m(k_i \vee k_j) + m(\neg(k_i \vee k_j)) = m(1) = 1.$$

This allows us to rewrite Eq. (8) as

$$\frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)} = 1 - m(k_i \vee k_j) = m(\neg(k_i \vee k_j)) = m(\neg k_i \wedge \neg k_j)$$

which immediately gives us Eq. (6).

Now recall that $m = \text{nWMC}$ and note that Eq. (6) can be multiplied by $\text{WMC}(1)^2$ to turn the equation into one for WMC instead of nWMC . Then

$$\begin{aligned} \text{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \leq k_i \wedge k_j} \text{WMC}(a) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\ &= \sum_{\text{atoms } a \leq k_i \wedge k_j} w(a_i)w(a_j) \prod_{m \in [n] \setminus \{i, j\}} w(a_m) = \sum_{\text{atoms } a \leq k_i \wedge k_j} w(k_i)w(k_j) \prod_{m \in [n] \setminus \{i, j\}} w(a_m) \\ &= w(k_i)w(k_j) \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n] \setminus \{i, j\}} w(a_m) = w(k_i)w(k_j)C, \end{aligned}$$

where C denotes the part of $\text{WMC}(k_i \wedge k_j)$ that will be the same for $\text{WMC}(\neg k_i \wedge k_j)$, $\text{WMC}(k_i \wedge \neg k_j)$, and $\text{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (6) becomes

$$w(k_i)w(k_j)w(\neg k_i)w(\neg k_j)C^2 = w(k_i)w(\neg k_j)w(\neg k_i)w(k_j)C^2$$

which is trivially true. By showing that WMC satisfies Eq. (6), we also showed that it satisfies Eq. (5), finishing the second part of the proof. \square

3.2 Extending the Algebra

3.2.1 Preliminaries

Definition 5 ([5]). Let \mathbf{A} and \mathbf{B} be Boolean algebras. A *Boolean homomorphism* from \mathbf{A} to \mathbf{B} is a map $f : \mathbf{A} \rightarrow \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,
- $f(x \vee y) = f(x) \vee f(y)$,
- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Definition 6 ([6]). Given two polyadic algebras \mathbf{A} and \mathbf{B} , a *polyadic homomorphism* from \mathbf{A} to \mathbf{B} is a Boolean homomorphism $f : \mathbf{A} \rightarrow \mathbf{B}$ such that

- $f\mathbf{S}(\tau)p = \mathbf{S}(\tau)fp$,
- $f\exists(J)p = \exists(J)fp$

for all $\tau \in T$, $p \in \mathbf{A}$, and $J \subseteq I$.

3.2.2 New Results

A well-known way to overcome this limitation of independence is by adding more literals [1], i.e., extending the set L covered by the WMC weight function $w : L \rightarrow \mathbb{R}_{>0}$. Let us translate this idea to the language of Boolean algebras.

Theorem 2. Let \mathbf{B} be a finite Boolean algebra freely generated by some set of ‘literals’ L , and let $m : \mathbf{B} \rightarrow \mathbb{R}_{\geq 0}$ be an arbitrary measure. We know that \mathbf{B} has $n = 2^{|L|}$ atoms. Let $(a_i)_{i=1}^n$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i \mid i \in [n]\} \cup \{\neg\phi_i \mid i \in [n]\}$ be the set L extended with $2n$ new literals. Let \mathbf{B}' be the unique Boolean algebra with

$$\{\phi_i \wedge a_i \mid i \in [n]\} \cup \{\neg\phi_i \wedge a_i \mid i \in [n]\}$$

as its set of atoms. Let $\iota : \mathbf{B} \rightarrow \mathbf{B}'$ be the inclusion homomorphism (i.e., $\iota(a) = a$ for all $a \in \mathbf{B}$). Let $w : L' \rightarrow \mathbb{R}_{>0}$ be defined by

$$w(l) = \begin{cases} \frac{m(a_i)}{2} & \text{if } l = \phi_i \text{ or } l = \neg\phi_i \text{ for some } i \in [n] \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m' : \mathbf{B}' \rightarrow \mathbb{R}_{\geq 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In simpler terms, any measure can be computed using WMC by extending the Boolean algebra with more literals. More precisely, we are given the red part in

$$\begin{array}{ccc} \mathbb{R}_{\geq 0} & & \\ \uparrow m & \nwarrow m' & \\ \mathbf{B} & \xrightarrow{\iota} & \mathbf{B}' \\ \cup & & \cup \\ L & \subset & L' \xrightarrow{w} \mathbb{R}_{>0} \end{array}$$

and construct the black part in such a way that the triangle commutes.

Proof. Since \mathbf{B} is freely generated by L , each atom $a_i \in \mathbf{B}$ is an infimum of elements in L , i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either

$$b = \phi_i \wedge a_i \quad \text{or} \quad b = \neg\phi_i \wedge a_i$$

for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L' . Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies

$$\phi_i \wedge a_i \leq \iota(b)$$

if and only if it satisfies

$$\neg\phi_i \wedge a_i \leq \iota(b).$$

Then, according to the definition of w ,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \leq \iota(b) \quad \text{if and only if} \quad a_i \leq b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. □

Now we can show that the construction in Theorem 2 is smallest possible.

Conjecture 1. *Let \mathbf{B} and \mathbf{B}' be Boolean algebras, and $\iota : \mathbf{B} \rightarrow \mathbf{B}'$ be the inclusion map such that \mathbf{B} is freely generated by L , all atoms of \mathbf{B}' can be expressed as meets of elements of L' , and the following subset relations are satisfied:*

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\iota} & \mathbf{B}' \\ \cup & & \cup \\ L & \subset & L' \end{array}$$

If, for any measure $m : \mathbf{B} \rightarrow \mathbb{R}_{\geq 0}$, one can construct a weight function $w : L' \rightarrow \mathbb{R}_{> 0}$ such that the WMC measure $\text{WMC} : \mathbf{B}' \rightarrow \mathbb{R}_{\geq 0}$ with respect to w satisfies

$$m = \text{WMC} \circ \iota,$$

then $|L' \setminus L| \geq 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [2] and Sang et al. [9]. Suppose we have a discrete probability distribution with n variables, and

the i -th variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^n v_i$ models. My expansion would then use

$$\sum_{i=1}^n v_i + 2 \prod_{i=1}^n v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [2] would use

$$\sum_{i=1}^n v_i + \sum_{i=1}^n \prod_{j=1}^i v_j$$

variables, while for the encoding by Sang et al. [9],

$$\sum_{i=1}^n v_i + \sum_{i=1}^n (v_i - 1) \prod_{j=1}^{i-1} v_j$$

variables would suffice.

The big TODO list

- Extension to infinite (atomic?) Boolean algebras.
- Compare my polyadic measures with first-order WMC.
- Abstraction refinements as homomorphisms.
- Definition of a measure-preserving homomorphism from Jech's set theory book.
- A Boolean algebra is approximable if its Stone space is approximable.

References

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