Weighted Model Counting/Integration from the Perspective of Boolean Algebras

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1 Introduction

Previous/related work:

- Hailperin's approach to probability logic [10]
- Nilsson's (somewhat successful) probabilistic logic [17]
- Semiring programming [3]
- WMI [2]
- Measures on Boolean algebras: overview articles (from most cited to least cited)
  - Horn and Tarski [11]
  - Concerning measures on Boolean algebras [7]
  - Jech – Measures on Boolean algebras (arXiv) [13]
- Measures on Boolean algebras: more specific articles
  - On possibility and probability measures in finite Boolean algebras [4]
  - Representation of conditional probability measures [15]

2 Preliminaries

Definition 1. A *Boolean algebra* (BA) is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ consisting of a set $\mathbf{B}$ with binary operations *meet* $\wedge$ and *join* $\vee$, unary operation $\neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;
- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;
- $\vee$ distributes over $\wedge$ and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

For clarity and succinctness, we will occasionally use three other operations that can be defined using the original three[1]:

$$\begin{aligned} a \to b &= \neg a \vee b, \\ a \leftrightarrow b &= (a \wedge b) \vee (\neg a \wedge \neg b), \\ a + b &= (a \wedge \neg b) \vee (\neg a \wedge b). \end{aligned}$$

We can also define a partial order $\leq$ on $\mathbf{B}$ as $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$) for $a, b \in \mathbf{B}$. Furthermore, let $a < b$ denote $a \leq b$ and $a \neq b$. For the rest of this paper, let $\mathbf{B}$ refer to the BA $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$. For any $S \subseteq \mathbf{B}$, we write $\bigvee S$ for $\bigvee_{x \in S} x$ and call it the *supremum* of S. Similarly, $\bigwedge S = \bigwedge_{x \in S} x$ is the *infimum*. By convention, $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$.

Definition 2 ([12, 16])**.** An element $a \neq 0$ of $\mathbf{B}$ is an *atom* if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. Equivalently, $a \neq 0$ is an atom if there is no $x \in \mathbf{B}$ such that $0 < x < a$. The BA $\mathbf{B}$ is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([8])**.** *For any two distinct atoms a, $b \in \mathbf{B}$, $a \wedge b = 0$.*

Lemma 2 ([9])**.** *The following are equivalent:*

- $\mathbf{B}$ *is atomic.*
- *For any $x \in \mathbf{B}$,*
$$x = \bigvee_{\text{atoms } a \leq x} a.$$
- 1 *is the supremum of all atoms.*

Lemma 3 ([9])**.** *All finite BAs are atomic.*

Definition 3 ([7, 12])**.** A *measure* on $\mathbf{B}$ is a function $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ such that:

- $m(0) = 0$;
- $m(x \vee y) = m(x) + m(y)$ for all $x, y \in \mathbf{B}$ whenever $x \wedge y = 0$.

If $m(1) = 1$, we call m a *probability measure*. Also, if $m(x) > 0$ for all $x \neq 0$, then m is *strictly positive*.

Definition 4 ([9])**.** An *ideal* is a non-empty subset $I \subseteq \mathbf{B}$ such that

- $i \vee j \in I$ for all $i, j \in I$;
- $i \wedge a \in I$ for all $i \in I$ and $a \in \mathbf{B}$.

For any $p \in \mathbf{B}$, the *principal ideal of p*—denoted by (p)—is the smallest ideal that contains p. It can also be expressed as $(p) = \{a \in \mathbf{B} \mid a \leq p\}$.

Definition 5 ([9])**.** Let I be an ideal in $\mathbf{B}$. The *quotient algebra of* $\mathbf{B}$ *modulo the ideal* I $\mathbf{B}/I$ is a BA of equivalence classes of elements of $\mathbf{B}$ with respect to the equivalence relation

$$a \sim b \quad \iff \quad a + b \in I$$

for all $a, b \in \mathbf{B}$. Elements of $\mathbf{B}/I$ are usually denoted by a/I (for some $a \in \mathbf{B}$) with the understanding that if $b \sim a$ (for some $b \in \mathbf{B}$), then $b/I = a/I$. The three algebraic operations on $\mathbf{B}/I$ are defined as

$$\begin{aligned} a/I \wedge b/I &= a \wedge b/I, \\ a/I \vee b/I &= a \vee b/I, \\ \neg(a/I) &= (\neg a)/I. \end{aligned}$$

[1]We use $+$ to denote symmetric difference because it is the additive operation of a Boolean ring.

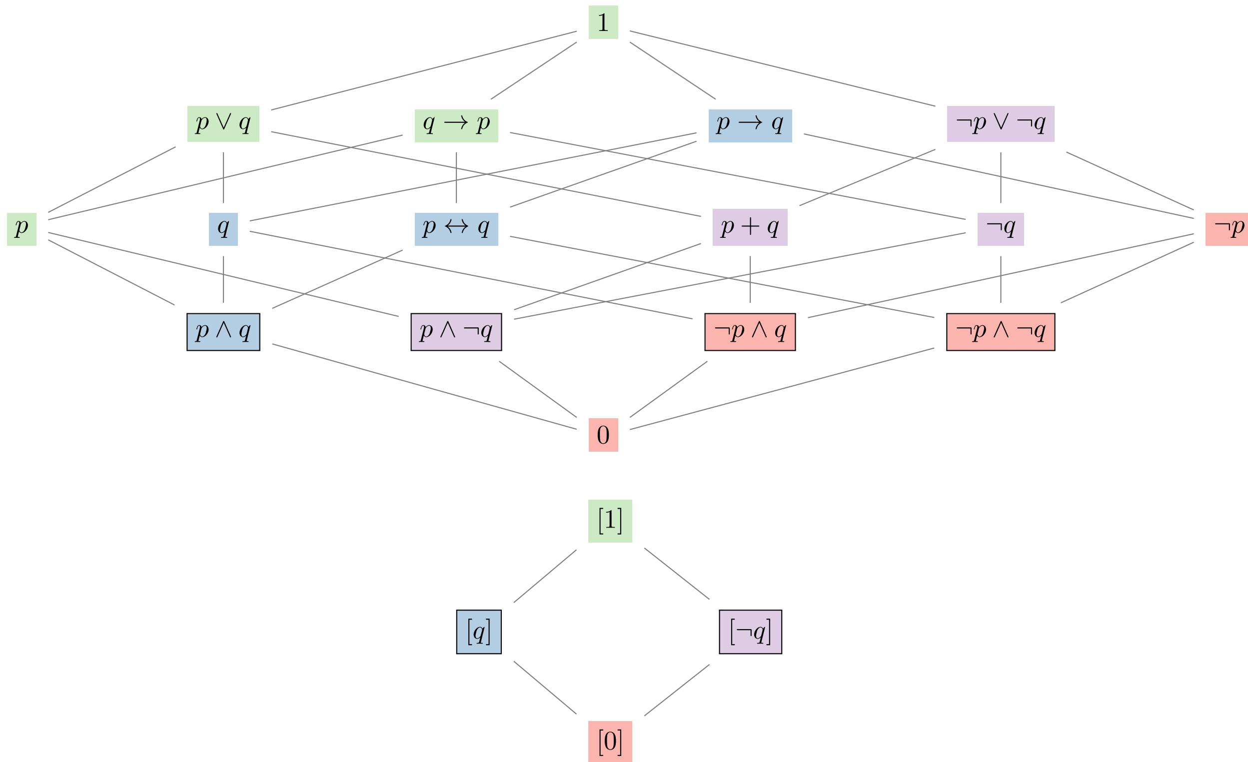

Figure 1: Two BAs from Example 1: $B(\mathcal{L})$ at the top and $B(\Delta)$ at the bottom. An edge between elements a and b (with a positioned lower than b) means that $a < b$. Each element of $B(\Delta)$ is an equivalence class of elements of $B(\mathcal{L})$, and the colours show which elements of $B(\mathcal{L})$ belong to which class. In both algebras, atoms have borders around them.

3 WMC as a Measure

Definition 6. Let $\mathcal{L}$ be a propositional (or first-order) logic, and let Δ be a theory in $\mathcal{L}$. We can define an equivalence relation on formulas in $\mathcal{L}$ as

$$\alpha \sim \beta \quad \text{if and only if} \quad \Delta \vdash \alpha \leftrightarrow \beta$$

for all $\alpha, \beta \in \mathcal{L}$. Let $[\alpha]$ denote the equivalence class of $\alpha \in \mathcal{L}$ with respect to $\sim$. We can then let $B(\Delta) = \{[\alpha] \mid \alpha \in \mathcal{L}\}$ and define the structure of a BA on $B(\Delta)$ as

$$\begin{aligned}
[\alpha] \vee [\beta] &= [\alpha \vee \beta], \\
[\alpha] \wedge [\beta] &= [\alpha \wedge \beta], \\
\neg[\alpha] &= [\neg\alpha], \\
1 &= [\alpha \to \alpha], \\
0 &= [\alpha \wedge \neg\alpha]
\end{aligned}$$

for all $\alpha, \beta \in \mathcal{L}$. Then $B(\Delta)$ is the *Lindenbaum-Tarski algebra* of Δ [14, 19].

Example 1. Let $\mathcal{L}$ be a propositional logic with p and q as its only atoms. Then $L = \{p, q, \neg p, \neg q\}$ is its set of literals. Let $w : L \to \mathbb{R}_{\geq 0}$ be the *weight function* defined by

$$\begin{aligned} w(p) &= 0.3, \\ w(\neg p) &= 0.7, \\ w(q) &= 0.2, \\ w(\neg q) &= 0.8. \end{aligned}$$

Let Δ be a theory in $\mathcal{L}$ with a sole axiom p. Then Δ has two models, i.e., $\{p, q\}$ and $\{p, \neg q\}$. The *weighted model count* (WMC) [5] of Δ is then

$$\sum_{\omega \models \Delta} \prod_{\omega \models l} w(l) = w(p)w(q) + w(p)w(\neg q) = 0.3.$$

The corresponding BA $B(\Delta)$ can then be constructed using Definition 6. Alternatively, one can first construct the free BA generated by the set $\{p, q\}$—this corresponds to $B(\mathcal{L})$ in Fig. 1—and then take a quotient with respect to either the filter generated by p or the ideal[2] generated by $\neg p$. In any case, the resulting BA is pictured at the bottom of Fig. 1.

Each element of $B(\mathcal{L})$ can also be seen as a subset of the set of all models of $\mathcal{L}$, with 0 representing $\emptyset$, 1 representing the set of all (four) models, each atom representing a single model, and each edge going upward representing a subset relation. Thus, the Boolean-algebraic way of calculating the WMC of Δ consists of:

1. Identifying an element $a \in B(\mathcal{L})$ that corresponds to Δ.
2. Finding all atoms of $B(\mathcal{L})$ that are 'dominated' by a according to the partial order.
3. Using w to calculate the weight of each such atom.
4. Adding the weights of these atoms.

This motivates the following definition of WMC generalised to BAs.

Definition 7. Let $\mathbf{B}$ be an atomic BA, and let $M \subset \mathbf{B}$ be its set of atoms. Let $L \subset \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge L'$ for some $L' \subseteq L$, and let $w \colon L \to \mathbb{R}_{\geq 0}$ be arbitrary. The *weighted model count* $\mathrm{WMC}_w \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}_w(x) = \begin{cases} 0 & \text{if } x = 0 \\ \prod_{l \in L'} w(l) & \text{if } M \ni x = \bigwedge L' \\ \sum_{\text{atoms } a \leq x} \mathrm{WMC}_w(a) & \text{otherwise} \end{cases}$$

for any $x \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{NWMC}_w \colon \mathbf{B} \to [0, 1]$ as $\mathrm{NWMC}_w(x) = \frac{\mathrm{WMC}_w(x)}{\mathrm{WMC}_w(1)}$ for all $x \in \mathbf{B}$. For both WMC_w and NWMC_w, we will drop the subscript when doing so results in no potential confusion.

Proposition 1. WMC *is a measure, and* NWMC *is a probability measure.*

Proof. First, note that WMC is non-negative and $\mathrm{WMC}(0) = 0$ by definition. Next, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

[2]More details on these concepts can be found in many books on BAs [9, 14].

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Lemma 2 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned} \mathrm{WMC}(x \vee y) = \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i\right) \vee \left(\bigvee_{j \in J} y_j\right)\right) &= \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j) \\ &= \mathrm{WMC}(x) + \mathrm{WMC}(y), \end{aligned}$$

finishing the proof that WMC is a measure. This immediately shows that NWMC is a probability measure since, by definition, $\mathrm{NWMC}(1) = 1$. $\square$

Given a theory Δ in a logic $\mathcal{L}$, the usual way of using WMC to compute the probability of a query q is [1, 18]

$$\Pr_{\Delta, w}(q) = \frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}.$$

In our algebraic formulation, this can be computed in two different ways:

- as $\frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}$ in $B(\mathcal{L})$,
- and as $\mathrm{NWMC}_w([q])$ in $B(\Delta)$.

But how does the measure defined on $B(\mathcal{L})$ transfer to $B(\Delta)$?

Lemma 4. *For any measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ and elements $a, b \in \mathbf{B}$,*

$$m(a \vee b) = m(a) + m(b) - m(a \wedge b).$$

Proof. By Definition 3,

$$\begin{aligned} m(a) &= m(a \wedge b) + m(a \wedge \neg b), \\ m(b) &= m(a \wedge b) + m(\neg a \wedge b), \\ m(a \vee b) &= m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b), \end{aligned}$$

so

$$m(a) + m(b) - m(a \wedge b) = m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b) = m(a \vee b)$$

as required. $\square$

Lemma 5. *For any $a, b \in \mathbf{B}$ and any principal ideal (p), if $a/(p) = b/(p)$, then $a \vee p = b \vee p$.*

Proof. If $a/F = b/F$, then $a \to b \in F$ by **??**. Then $p \leq a \to b$ by **??**, and $(a \to b) \wedge p = p$ by the definition of $\leq$. Then

$$1 = p \vee \neg p = ((a \to b) \vee \neg p) \wedge (p \vee \neg p) = \neg a \vee b \vee \neg p = \neg(a \wedge p) \vee b.$$

So

$$\neg b = (\neg(a \wedge p) \wedge \neg b) \vee (b \wedge \neg b) = \neg((a \wedge p) \vee b),$$

and thus $b = (a \wedge p) \vee b$, and

$$b \wedge p = (a \wedge p) \vee (b \wedge p) = (a \vee b) \wedge p.$$

Similarly, $a \wedge p = (a \vee b) \wedge p$ which proves that $a \wedge p = b \wedge p$. $\square$

Proposition 2 (Measures on quotients)**.** *Let $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure, and let (p) be a principal ideal. Let $m^*\colon \mathbf{B}/(p) \to \mathbb{R}_{\geq 0}$ be defined as*

$$m^*(a/(p)) = m(a \vee p)$$

for any $a \in \mathbf{B}$. The function m^* is well-defined. Furthermore, it is a measure on $\mathbf{B}/(p)$ if and only if $m(p) = 0$. Moreover, if it is a measure, then the following properties transfer from m to m^*:

- *if m is a probability measure, then so is m^*;*
- *if m is strictly positive, then so is m^*;*
- *if m is a WMC measure, then so is m^*.*

Proof. Lemma 5 proves that the function is well-defined. □

4 What Measures Are WMC-Computable?

4.1 WMC Requires Independent Literals

Proposition 3. *Let $\mathbf{B}$ be a finite measure algebra with measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$. Let $L \subset \mathbf{B}$ be defined as*

$$L = \{l_i \mid i \in [n]\} \cup \{\neg l_i \mid i \in [n]\}$$

for some $n \in \mathbb{N}$. Finally, assume that $\mathbf{B}$ has 2^n atoms, where each atom $a \in \mathbf{B}$ is an infimum

$$a = \bigwedge_{i=1}^{n} a_i$$

such that $a_i \in \{l_i, \neg l_i\}$ for $i \in [n]$. Then there exists a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ that makes m a WMC measure if and only if

$$m(l \wedge l') = m(l)m(l') \tag{2}$$

for all distinct $l, l' \in L$ such that $l \neq \neg l'$.

Remark. Note that if $n = 1$, then Eq. (2) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. Let us begin with the 'if' part of the statement. Let $w\colon L \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = m(l) \tag{3}$$

for all $l \in L$. We are going to show that NWMC $= m$. First, note that $\mathrm{NWMC}(0) = 0 = m(0)$ by the definitions of both NWMC and m. Second, let

$$a = \bigwedge_{i=1}^{n} a_i \tag{4}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\mathrm{NWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)} = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} w(a_i) = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} m(a_i) = \frac{1}{\mathrm{WMC}(1)} m\left(\bigwedge_{i=1}^{n} a_i\right) = \frac{m(a)}{\mathrm{WMC}(1)}$$

by Definition 7 and Eqs. (2) to (4). Now we just need to show that $\mathrm{WMC}(1) = 1$. Indeed,

$$\begin{aligned}\mathrm{WMC}(1) &= \sum_{\text{atoms } a \in \mathbf{B}} \mathrm{WMC}(a) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} w(a_i) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} m(a_i) \\ &= \sum_{\text{atoms } a \in \mathbf{B}} m\left(\bigwedge_{i=1}^{n} a_i\right) = \sum_{\text{atoms } a \in \mathbf{B}} m(a) = m\left(\bigvee_{\text{atoms } a \in \mathbf{B}}\right) = m(1) = 1.\end{aligned}$$

Finally, note that if NWMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

For the other direction, we are given a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ that induces a measure $m = \mathrm{NWMC}\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (2) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$. We will first prove an auxiliary result that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \tag{5}$$

is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j). \tag{6}$$

First, note that k_i can be expressed as

$$k_i = (k_i \wedge k_j) \vee (k_i \wedge \neg k_j)$$

with the condition that

$$(k_i \wedge k_j) \wedge (k_i \wedge \neg k_j) = 0,$$

so, by properties of a measure,

$$m(k_i) = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j). \tag{7}$$

Applying Eq. (7) and the equivalent expression for $m(k_j)$ allows us to rewrite Eq. (5) as

$$\begin{aligned} m(k_i \wedge k_j) &= [m(k_i \wedge k_j) + m(k_i \wedge \neg k_j)] \cdot [m(k_i \wedge k_j) + m(\neg k_i \wedge k_j)] \\ &= m(k_i \wedge k_j)^2 + m(k_i \wedge k_j)[m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j)] + m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j) \end{aligned}$$

Dividing both sides by $m(k_i \wedge k_j)$ gives

$$1 = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j) + \frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)}. \tag{8}$$

Since $k_i \wedge k_j \wedge k_i \wedge \neg k_j = 0$, and

$$(k_i \wedge k_j) \vee (k_i \wedge \neg k_j) = k_i \wedge (k_j \vee \neg k_j) = k_i \wedge 1 = k_i,$$

we have that

$$m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) = m(k_i).$$

Similarly, $k_i \wedge \neg k_i \wedge k_j = 0$, and

$$k_i \vee (\neg k_i \wedge k_j) = (k_i \vee \neg k_i) \wedge (k_i \vee k_j) = k_i \vee k_j,$$

so

$$m(k_i) + m(\neg k_i \wedge k_j) = m(k_i \vee k_j).$$

Finally, note that

$$(k_i \vee k_j) \wedge \neg(k_i \vee k_j) = 0,$$

and

$$(k_i \vee k_j) \vee \neg(k_i \vee k_j) = 1,$$

so

$$m(k_i \vee k_j) + m(\neg(k_i \vee k_j)) = m(1) = 1.$$

This allows us to rewrite Eq. (8) as

$$\frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)} = 1 - m(k_i \vee k_j) = m(\neg(k_i \vee k_j)) = m(\neg k_i \wedge \neg k_j)$$

which immediately gives us Eq. (6).

Now recall that $m = \text{NWMC}$ and note that Eq. (6) can be multiplied by $\text{WMC}(1)^2$ to turn the equation into one for WMC instead of NWMC. Then

$$\begin{aligned}
\text{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \leq k_i \wedge k_j} \text{WMC}(a) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\
&= \sum_{\text{atoms } a \leq k_i \wedge k_j} w(a_i) w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \leq k_i \wedge k_j} w(k_i) w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) \\
&= w(k_i) w(k_j) \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i) w(k_j) C,
\end{aligned}$$

where C denotes the part of $\text{WMC}(k_i \wedge k_j)$ that will be the same for $\text{WMC}(\neg k_i \wedge k_j)$, $\text{WMC}(k_i \wedge \neg k_j)$, and $\text{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (6) becomes

$$w(k_i) w(k_j) w(\neg k_i) w(\neg k_j) C^2 = w(k_i) w(\neg k_j) w(\neg k_i) w(k_j) C^2$$

which is trivially true. By showing that WMC satisfies Eq. (6), we also showed that it satisfies Eq. (5), finishing the second part of the proof. □

4.2 Extending the Algebra

A well-known way to overcome this limitation of independence is by adding more literals [5], i.e., extending the set L covered by the WMC weight function $w \colon L \to \mathbb{R}_{\geq 0}$. Let us translate this idea to the language of Boolean algebras.

Theorem 1. *Let $\mathbf{B}$ be a finite Boolean algebra freely generated by some set of 'literals' L, and let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be an arbitrary measure. We know that $\mathbf{B}$ has $n = 2^{|L|}$ atoms. Let $(a_i)_{i=1}^n$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i \mid i \in [n]\} \cup \{\neg\phi_i \mid i \in [n]\}$ be the set L extended with $2n$ new literals. Let $\mathbf{B}'$ be the unique Boolean algebra with*

$$\{\phi_i \wedge a_i \mid i \in [n]\} \cup \{\neg\phi_i \wedge a_i \mid i \in [n]\}$$

as its set of atoms. Let $\iota \colon \mathbf{B} \to \mathbf{B}'$ be the inclusion homomorphism (i.e., $\iota(a) = a$ for all $a \in \mathbf{B}$). Let $w \colon L' \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = \begin{cases} \frac{m(a_i)}{2} & \text{if } l = \phi_i \text{ or } l = \neg\phi_i \text{ for some } i \in [n] \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m' \colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In simpler terms, any measure can be computed using WMC by extending the Boolean algebra with more literals. More precisely, we are given the red part in

$$\begin{tikzcd}
\mathbb{R}_{\geq 0} & & \\
\mathbf{B} \arrow[u, "m"] \arrow[r, "\iota"] & \mathbf{B}' \arrow[ul, "m'"'] & \\
\cup & \cup & \\
L \arrow[r, phantom, "\subset"] & L' \arrow[r, "w"] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

and construct the black part in such a way that the triangle commutes.

Proof. Since $\mathbf{B}$ is freely generated by L, each atom $a_i \in \mathbf{B}$ is an infimum of elements in L, i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either

$$b = \phi_i \wedge a_i \quad \text{or} \quad b = \neg\phi_i \wedge a_i$$

for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L'. Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies

$$\phi_i \wedge a_i \leq \iota(b)$$

if and only if it satisfies

$$\neg\phi_i \wedge a_i \leq \iota(b).$$

Then, according to the definition of w,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \leq \iota(b) \quad \text{if and only if} \quad a_i \leq b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. $\square$

Now we can show that the construction in Theorem 1 is smallest possible.

Conjecture 1. *Let* $\mathbf{B}$ *and* $\mathbf{B}'$ *be Boolean algebras, and* $\iota\colon \mathbf{B} \to \mathbf{B}'$ *be the inclusion map such that* $\mathbf{B}$ *is freely generated by* L*, all atoms of* $\mathbf{B}'$ *can be expressed as meets of elements of* L'*, and the following subset relations are satisfied:*

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\iota} & \mathbf{B}' \\ \cup & & \cup \\ L & \subset & L' \end{array}$$

If, for any measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$*, one can construct a weight function* $w\colon L' \to \mathbb{R}_{\geq 0}$ *such that the WMC measure* $\mathrm{WMC}\colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$ *with respect to* w *satisfies*

$$m = \mathrm{WMC} \circ \iota,$$

then $|L' \setminus L| \geq 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [6] and Sang et al. [18]. Suppose we have a discrete probability distribution with n variables, and the ith variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^{n} v_i$ models. My expansion would then use

$$\sum_{i=1}^{n} v_i + 2\prod_{i=1}^{n} v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [6] would use

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n}\prod_{j=1}^{i} v_j$$

variables, while for the encoding by Sang et al. [18],

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n}(v_i - 1)\prod_{j=1}^{i-1} v_j$$

variables would suffice.

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