Statistical Relational Models as Polyadic Measure Algebras

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1 Propositional Logic and Boolean Algebras

1.1 Preliminaries

Definition 1. A *Boolean algebra* is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ of a set $\mathbf{B}$ with operations $\wedge, \vee, \neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;
- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;
- $\vee$ distributes over $\wedge$ and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

Let $a, b \in \mathbf{B}$ be arbitrary. Let $\leq$ be a partial order on $\mathbf{B}$ defined by $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$), and let $a < b$ denote $a \leq b$ and $a \neq b$.

Definition 2 ([5, 6])**.** An element $a \neq 0$ of a Boolean algebra $\mathbf{B}$ is an *atom* if there is no $x \in \mathbf{B}$ such that $0 < x < a$. Equivalently, $a \neq 0$ is an atom if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. A Boolean algebra is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([2])**.** *For any two distinct atoms a, b in a Boolean algebra, $a \wedge b = 0$.*

Lemma 2 ([3])**.** *All finite Boolean algebras are atomic.*

Theorem 1 ([2])**.** *Let $\mathbf{B}$ be a finite Boolean algebra. Then every $a \in \mathbf{B} \setminus \{0\}$ can be uniquely expressed as $a = \bigvee_{i \in I} m_i$ for some set of atoms $\{m_i\}_{i \in I}$.*

Definition 3 ([1])**.** A *(strictly positive) measure* on a Boolean algebra $\mathbf{B}$ is a function $m : \mathbf{B} \to [0, 1]$ such that:

1. $m(1) = 1$, and $m(x) > 0$ for $x \neq 0$;
2. $m(x \vee y) = m(x) + m(y)$ for all $x, y \in \mathbf{B}$ whenever $x \wedge y = 0$.

1.2 New Results

Definition 4. Let $\mathbf{B}$ be a finite Boolean algebra, let L be a subset of $\mathbf{B}$ such that every atom m can be uniquely expressed as $m = \bigwedge_{i \in I} l_i$ for some $\{l_i\}_{i \in I} \subseteq L$, and let $w : L \to \mathbb{R}_{>0}$ be arbitrary. The *weighted model count* $\mathrm{WMC} : \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}(a) = \begin{cases} 0 & \text{if } a = 0 \\ \prod_{i \in I} w(l_i) & \text{if } M \ni a = \bigwedge_{i \in I} l_i \\ \sum_{i \in I} \mathrm{WMC}(m_i) & \text{if } \mathbf{B} \setminus (M \cup \{0\}) \ni a = \bigvee_{i \in I} m_i \end{cases}$$

for any $a \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{nWMC} : \mathbf{B} \to [0,1]$ as $\mathrm{nWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)}$ for all $a \in \mathbf{B}$.

Proposition 1. nWMC *is a measure for any finite Boolean algebra* $\mathbf{B}$.

Proof. First, note that Property 1 of Definition 3 is satisfied by the definition of nWMC. Next, in order to prove Property 2, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{nWMC}(x \vee y) = \mathrm{nWMC}(x) + \mathrm{nWMC}(y)$$

which is equivalent to

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Theorem 1 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned} \mathrm{WMC}(x \vee y) = \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i \right) \vee \left(\bigvee_{j \in J} y_j \right) \right) = \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j) \\ = \mathrm{WMC}(x) + \mathrm{WMC}(y), \end{aligned}$$

finishing the proof. □

2 First-Order Logic and Polyadic Algebras

2.1 Preliminaries

What follows is a summary of [4].

Let $\mathbf{B}$ be a Boolean algebra (of propositions). Let X be the (non-empty) domain of discourse. Let I be an index set, elements of which can be interpreted as variables. The elements of X^I are functions from I to X. For any $x \in X^I$ and $i \in I$, we write x_i to represent $x(i) \in X$. Let $\mathbf{A}^*$ be the set of all functions $X^I \to \mathbf{B}$, and note that it forms a Boolean algebra with operations defined pointwise.

Let T be the semigroup of all $I \to I$ transformations. For any $\tau \in T$, let $\tau_* : X^I \to X^I$ be defined by

$$(\tau_* x)_i = x_{\tau i}$$

for all $x \in X^I$ and $i \in I$. For any (Boolean/polyadic) algebra $\mathbf{C}$, let $\mathrm{End}(\mathbf{C})$ denote the set of all its endomorphisms. We can then define $\mathbf{S}$ to be a map $\mathbf{S} : T \to \mathrm{End}(\mathbf{A}^*)$ defined by

$$\mathbf{S}(\tau)p(x) = p(\tau_* x)$$

for all $x \in X^I$ and $p \in \mathbf{A}^*$.

For any $J \subseteq I$, let J_* be the relation on X^I defined by

$$xJ_*y \quad \iff \quad x_i = y_i \quad \text{for all } i \in I \setminus J$$

for all $x, y \in X^I$. For any $J \subseteq I$, we then define $\exists(J)$ to be a transformation $\mathbf{A}^* \to \mathbf{A}^*$ defined by

$$\exists(J)p(x) = \bigvee_{\substack{y \in X^I, \\ xJ_*y}} p(y)$$

for all $p \in \mathbf{A}^*$, provided this supremum exists for all $x \in X^I$[1].

Finally, a *functional polyadic (Boolean) algebra*[2] is a subalgebra $\mathbf{A}$ of $\mathbf{A}^*$ such that:

- $\mathbf{S}(\tau)p \in \mathbf{A}$ for all $p \in \mathbf{A}$ and $\tau \in T$;
- $\exists(J)p \in \mathbf{A}$ for all $p \in \mathbf{A}$ and $J \subseteq I$.

Definition 5. Similarly to $\exists$, a *constant* c is a map $c : \mathcal{P}(I) \to \mathrm{End}(\mathbf{A})$ (Boolean endomorphisms?) such that:

- $c(\emptyset) = \mathrm{id}_{\mathbf{A}}$;
- $c(J \cup K) = c(J)c(K)$;
- $c(J)\exists(K) = \exists(K)c(J \setminus K)$;
- $\exists(J)c(K) = c(K)\exists(J \setminus K)$;
- $c(J)\mathbf{S}(\tau) = \mathbf{S}(\tau)c(\tau^{-1}J)$

for all $J, K \in \mathcal{P}(I)$ and $\tau \in T$. If J is a singleton set $\{i\}$, we will simply write $c(i)$ instead of $c(J)$.

2.2 New Results

Proposition 2. *Let $\mathbf{B}$ be a finite Boolean algebra with a measure $m : \mathbf{B} \to [0, 1]$. Let $\mathbf{A}$ be a $\mathbf{B}$-valued functional polyadic algebra with domain X and variables I. Let $m^* : \mathbf{A} \to \mathbb{R}_{\geq 0}$ be defined by*

$$m^*(p) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I :\, y \leq p(x)}} m(y)$$

for all $p \in \mathbf{A}$. Then m^* is a measure on $\mathbf{A}$.

Remark. While definitions of m^* such as

$$m^*(p) = m\left(\bigvee_{x \in X^I} p(x)\right)$$

might look tempting, they are not additive.

Proof. First, we can show that $m^*(1) = 1$ by observing that

$$m^*(1) = \sum_{\text{atoms } y \in \mathbf{B}} m(y) = m\left(\bigvee_{\text{atoms } y \in \mathbf{B}} y\right) = m(1) = 1,$$

[1] The universal quantifier $\forall(J)$ is then defined as $\forall(J)p = \neg(\exists(J)\neg p)$ for all $p \in \mathbf{A}^*$.

[2] To be more explicit, a $\mathbf{B}$-valued functional polyadic algebra with domain X and variables I.

where we use Theorem 1 and express $1 \in \mathbf{B}$ as the supremum of all atoms in $\mathbf{B}$ [2]. Clearly $m^*(p) \geq 0$ for all $p \in \mathbf{A}$, so we can restrict the codomain of m^* to $[0,1]$.

Next, we want to show that $m^*(p) > 0$ for all $p \in \mathbf{A} \setminus \{0\}$. If $p \neq 0$, then there must be some $x' \in X^I$ such that $p(x') \neq 0$. But then, since finite Boolean algebras are atomic, there must also be an atom $y \in \mathbf{B}$ such that $y \leq p(x')$. Therefore, $m^*(p) \geq m(y) > 0$, finishing this part of the proof.

Let $p, q \in \mathbf{A}$ be such that $p \wedge q = 0$. We want to show that $m^*(p \vee q) = m^*(p) \vee m^*(q)$. First, note that

$$y \leq (p \vee q)(x) = p(x) \vee q(x)$$

if and only if

$$y = (p(x) \vee q(x)) \wedge y = (p(x) \wedge y) \vee (q(x) \wedge y)$$

by Definition 1. Also note that

$$(p(x) \wedge y) \wedge (q(x) \wedge y) = p(x) \wedge q(x) \wedge y = (p \wedge q)(x) \wedge y = 0 \wedge y = 0,$$

so

$$m(y) = m((p(x) \wedge y) \vee (q(x) \wedge y)) = m(p(x) \wedge y) + m(q(x) \wedge y)$$

by Definition 3 which then leads to

$$\begin{aligned} m^*(p \vee q) &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(y) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(p(x) \wedge y) + m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(q(x) \wedge y). \end{aligned}$$

Since y is an atom,

$$p(x) \wedge y = \begin{cases} y & \text{if } y \leq p(x) \\ 0 & \text{otherwise,} \end{cases}$$

so

$$\begin{aligned} m^*(p \vee q) &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x) \text{ and } y \leq p(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B}\text{s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x) \text{ and } y \leq q(x)}} m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq p(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B}\text{s.t.} \\ \exists x \in X^I \colon y \leq q(x)}} m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq p(x)}} m(y) + \sum_{\substack{\text{atoms } y \in \mathbf{B}\text{s.t.} \\ \exists x \in X^I \colon y \leq q(x)}} m(y) = m^*(p) + m^*(q), \end{aligned}$$

finishing the proof that m^* is a measure. □

3 From First-Order Logic to Polyadic Algebras

3.1 Preliminaries

Definition 6 ([3]). An *ideal* in a Boolean algebra $\mathbf{B}$ is a subset $M \subseteq \mathbf{B}$ such that:

- $0 \in M$;
- $x \vee y \in M$ for all $x, y \in M$;

- $x \wedge y \in M$ for all $x \in M$ and $y \in \mathbf{B}$.

For any subset $S \subseteq \mathbf{B}$, the *ideal generated by* S is the smallest ideal M such that $S \subseteq M$.

Note that Definition 6 gives us a simple characterisation of an ideal generated by a set of atoms.

Lemma 3. *Let $\mathbf{B}$ be a Boolean algebra, and let $S \subseteq \mathbf{B}$ be a set of atoms. The ideal I generated by S is defined by the following:*

- $0 \in I$,
- $S \subseteq I$,
- $x \vee y \in I$ *for all* $x, y \in I$.

Definition 7 ([3])**.** Let $\mathbf{B}$ be a Boolean algebra, and let I be an ideal in $\mathbf{B}$. The *quotient algebra* $\mathbf{B}/I$ is a Boolean algebra on equivalence classes of elements of $\mathbf{B}$ (with operations defined pointwise) based on the equivalence relation

$$x \sim y \quad \iff \quad x + y \in I$$

where $x + y = (x \wedge \neg y) \vee (y \wedge \neg x)$ is the symmetric difference operation (written as a sum because it can interpreted as the 'additive' part of the corresponding Boolean ring).

3.2 New Results

In order to make the example algebras easily describable, our example programs will have to be tiny. Consider the following ProbLog [7] program:

$$\begin{aligned}
&1.0 :: \mathsf{p}(a, b).\\
&0.5 :: \mathsf{p}(X, X) :- \mathsf{p}(X, Y);\ \mathsf{p}(Y, X).
\end{aligned}$$

Let $L = \{\mathsf{p}(a, a), \mathsf{p}(a, b), \mathsf{p}(b, a), \mathsf{p}(b, b)\}$ be the set of all possible ground atoms. Let $\mathbf{B}$ be the Boolean algebra freely generated by L (see, e.g., [3] for more on free Boolean algebras). Then $\mathbf{B}$ will have sixteen atoms ranging from $\mathsf{p}(a, a) \wedge \mathsf{p}(a, b) \wedge \mathsf{p}(b, a) \wedge \mathsf{p}(b, b)$ to $\neg\mathsf{p}(a, a) \wedge \neg\mathsf{p}(a, b) \wedge \neg\mathsf{p}(b, a) \wedge \neg\mathsf{p}(b, b)$. The weight function $w : L \to \mathbb{R}_{\geq 0}$ defined by

$$w(l) = \begin{cases} 1 & \text{if } l = \mathsf{p}(a, b) \\ 0.5 & \text{if } l \in \{\mathsf{p}(a, a), \mathsf{p}(b, b)\} \\ 0 & \text{if } l = \mathsf{p}(b, a) \\ 1 - w(l') & \text{if } l = \neg l' \end{cases}$$

for all $l \in L$ defines a WMC measure over $\mathbf{B}$. Note that while we could define an ideal generated by $\{\mathsf{p}(b, a), \neg\mathsf{p}(a, b)\}$ and take the quotient of $\mathbf{B}$ by that ideal to get a Boolean algebra with a strictly positive measure, this would put zero-probability queries outside of our algebras, i.e., we would not be able to ask a question whose answer is zero.

Finally, let $\mathbf{A}$ be the functional polyadic algebra $X^I \to \mathbf{B}$ for $I = \{1, 2\}$ and $X = \{a, b\}$[3]. The elements of X^I can then be represented as tuples (x_1, x_2) for some $x_1, x_2 \in X$. See Table 1 for example elements of $\mathbf{A}$ which consists of a single predicate function p and operators $\exists, \mathbf{S}, a, b, \neg, \wedge, \vee$, the last three of which are defined pointwise.

[3] X cannot (or should not) have constants that do not occur in $\mathbf{B}$.

Table 1: Example elements of $\mathbf{A}$ as maps $X^I \to \mathbf{B}$, with $a : \mathcal{P}(I) \to \mathrm{End}(\mathbf{A})$ as one of two possible constants.

Element of $\mathbf{A}$	Action on X^I
$p = \mathbf{S}(\mathrm{id})p = \exists(\emptyset)p = a(\emptyset)p = b(\emptyset)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, x_2)$
$\exists(1)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, x_2) \vee \mathsf{p}(b, x_2)$
$\exists(2)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, a) \vee \mathsf{p}(x_1, b)$
$\exists(I)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, a) \vee \mathsf{p}(a, b) \vee \mathsf{p}(b, a) \vee \mathsf{p}(b, b)$
$\mathbf{S}(\{1 \mapsto 1, 2 \mapsto 1\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, x_1)$
$\mathbf{S}(\{1 \mapsto 2, 2 \mapsto 1\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_2, x_1)$
$\mathbf{S}(\{1 \mapsto 2, 2 \mapsto 2\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_2, x_2)$
$a(1)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, x_2)$
$a(2)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, a)$
$a(I)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, a)$

4 Homomorphisms

Definition 8 ([3]). Let $\mathbf{A}$ and $\mathbf{B}$ be Boolean algebras. A *Boolean homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $f : \mathbf{A} \to \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,
- $f(x \vee y) = f(x) \vee f(y)$,
- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Definition 9 ([4]). Given two polyadic algebras $\mathbf{A}$ and $\mathbf{B}$, a *polyadic homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a Boolean homomorphism $f : \mathbf{A} \to \mathbf{B}$ such that

- $f\mathbf{S}(\tau)p = \mathbf{S}(\tau)fp$,
- $f\exists(J)p = \exists(J)fp$

for all $\tau \in T$, $p \in \mathbf{A}$, and $J \subseteq I$.

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