Weighted Model Counting with Conditional Measures

Paulius Dilkas

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1 Introduction

- Thesis: many important computational problems are solved by encoding them as WMC. But if the weight function is extended to 'conditional probabilities', the problem becomes easier because current approaches need workarounds in order to be independent.
- Shorter thesis: many important problems are encoded as WMC, but they can be encoded in a better way if we allow for conditional probabilities.
- The claim behind this paper is that allowing for conditional probabilities in the context of weighted model counting seems to be a good idea. Bayesian networks and ADDMC are only particular examples. This should also work with Cachet.
- Potential criticism may be that this is a step backwards and doesn't allow us to use SAT-based techniques for probabilistic inference. However, they can still be used for the 'theory+query' part.
  - Zero-probability weights and one-probability weights can be interpreted as logical clauses. This doesn't affect ADDMC but could be useful for other solvers.

F What are the main claims, what are the main takeaways, intuitive [???] of theorems to follow. To do this, we appeal to algebraic constructions to define the main concepts for introducing measures on Boolean algebras.

- Algorithms[1]
  - ADDMC [16] (rediscovered the multiplicativity of BAs in different words) (with optimal settings)
  - Cachet [41]
  - c2d [14]
  - d4 [33] (closed source, boo!)
  - miniC2D [38]
- Contributions
  - WMC defines a measure over a BA.
  - WMC with weights on literals imposes an independence assumption. (Measures are 'slightly' more expressive than WMC with weights on models because they apply to non-atomic BAs.)
  - A BA can be augmented with new literals in order to support any measure.
  - (Maybe) a lower bound on the number of new literals needed in order to support any measure.
  - This results in a smaller problem for WMC algorithms (w.r.t. both the number of literals and the length of the theory) and thus faster inference.

[1] `http://beyondnp.org/pages/solvers/model-counters-exact/`

- Notable previous/related work
  - Hailperin's approach to probability logic [23]
  - Nilsson's (somewhat successful) probabilistic logic [36, 37]
  - Logical induction: a big paper with a good overview of previous attempts to assign probabilities to logical sentences in a sensible way [20]
  - Measures on Boolean algebras
    * On possibility and probability measures in finite Boolean algebras [6]
    * Representation of conditional probability measures [31]

F2 The paper at this stage is very technical—the danger is that WMC/SRL people may not be able to follow it and so would be hard to get accepted. Without clear target audience, [???] get work accepted. My main high level suggestion is that let us tease apart what you have and see if a story emerges. That is, let us attempt to write a paper **with examples** and see whether with significant motivation, we have a story emerging. Below, sample text needed to adequately motivate [???] for WMC/SRL community.

2 Preliminaries

F2 Explain with examples: models = elements [atoms] of algebra.

- Perhaps reorder the section of preliminaries into paragraphs, i.e., a paragraph for order, for homomorphisms, etc. This would take up less space.
- Make up my mind about a, b vs. x, y and stick to it (maybe x, y?).
- Terminology: 'with generating set S' $\rightarrow$ 'over S'.
- Notation: if L denotes literals, then it doesn't denote a generating set. Literals should be U.

Definition 1. A *Boolean algebra* (BA) is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ consisting of a set $\mathbf{B}$ with binary operations *meet* $\wedge$ and *join* $\vee$, unary operation $\neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;
- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;
- $\vee$ distributes over $\wedge$ and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

For clarity and succinctness, we will occasionally use three other operations that can be defined using the original three[2]:

$$
\begin{aligned}
a \rightarrow b &= \neg a \vee b, \\
a \leftrightarrow b &= (a \wedge b) \vee (\neg a \wedge \neg b), \\
a + b &= (a \wedge \neg b) \vee (\neg a \wedge b).
\end{aligned}
$$

We can also define a partial order $\leq$ on $\mathbf{B}$ as $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$) for all $a, b \in \mathbf{B}$. Furthermore, let $a < b$ denote $a \leq b$ and $a \neq b$. For the rest of this paper, let $\mathbf{B}$ refer to the

[2] We use $+$ to denote symmetric difference because it is the additive operation of a Boolean ring.

BA $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$. For any $S \subseteq \mathbf{B}$, we write $\bigvee S$ for $\bigvee_{x \in S} x$ and call it the *supremum* of S. Similarly, $\bigwedge S = \bigwedge_{x \in S} x$ is the *infimum*. By convention, $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$. For any $a, b \in \mathbf{B}$, we say that a and b are *disjoint* if $a \wedge b = 0$.

Definition 2 ([27, 34])**.** An element $a \neq 0$ of $\mathbf{B}$ is an *atom* if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. Equivalently, $a \neq 0$ is an atom if there is no $x \in \mathbf{B}$ such that $0 < x < a$. We say that $\mathbf{B}$ is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([19])**.** *For any two distinct atoms a, $b \in \mathbf{B}$, $a \wedge b = 0$.*

Lemma 2 ([21])**.** *The following are equivalent:*

- $\mathbf{B}$ *is atomic.*
- *For any $x \in \mathbf{B}$, $x = \bigvee_{atoms\ a \leq x} a$.*
- 1 *is the supremum of all atoms.*

Lemma 3 ([21])**.** *All finite BAs are atomic.*

Definition 3 ([18, 27])**.** A *measure* on $\mathbf{B}$ is a function $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ such that:

- $m(0) = 0$;
- $m(a \vee b) = m(a) + m(b)$ for all $a, b \in \mathbf{B}$ whenever $a \wedge b = 0$.

If $m(1) = 1$, we call m a *probability measure.* Also, if $m(x) > 0$ for all $x \neq 0$, then m is *strictly positive.*

Definition 4 ([21])**.** Let $\mathbf{A}$ and $\mathbf{B}$ be BAs. A *(Boolean) homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $f\colon \mathbf{A} \to \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,
- $f(x \vee y) = f(x) \vee f(y)$,
- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Lemma 4 (Homomorphisms preserve order [21])**.** *Let $f\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism between two BAs $\mathbf{A}$ and $\mathbf{B}$. Then, for any $x, y \in \mathbf{A}$, if $x \leq y$, then $f(x) \leq f(y)$.*

Lemma 5 ([43])**.** *For any $a, b \in \mathbf{B}$, $a \leq b$ if and only if $a \wedge \neg b = 0$.*

Lemma 6 ([21])**.** *Let $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure. Then for all $a, b \in \mathbf{B}$, if $a \leq b$, then $m(a) \leq m(b)$.*

Definition 5 ([30])**.** Let S be a set, and let $\mathbf{B}$ be a BA. Then $\mathbf{B}$ is a *free BA over S* if there is a map $S \to \mathbf{B}$ such that for any BA $\mathbf{C}$ and map $S \to \mathbf{C}$, there is a unique homomorphism $\mathbf{B} \to \mathbf{C}$ that makes

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathbf{B} \arrow[d, dashed] \\ & \mathbf{C}. \end{tikzcd}$$

commute. A BA $\mathbf{B}$ is *free* if S exists.

Lemma 7 ([43])**.** *A finite BA is free if and only if it has 2^{2^n} elements for some $n \in \mathbb{N}$. It then has 2^n atoms and n generators.*

3 Weighted Model Counting as a Measure

- Any function $2^U \to \mathbb{R}_{\geq 0}$ induces a measure on 2^{2^U}—the full Boolean algebra. This measure is not necessarily probabilistic.

F2 Need to explain how WMC and NWMC connects to standard definitions of WMC and $\Pr(\phi \mid e) = \mathrm{WMC}(\phi \wedge e) / \mathrm{WMC}(e)$.

F2 You need to explain what precisely these mean in logic and models and weight functions are usually defined and understood.

- Be careful about mentioning ideals, filters, and quotients.
- Note that we're going to use logical notation for BAs. Give an example of the two notations.

Definition 6. Let $\mathcal{L}$ be a propositional (or first-order) logic, and let Δ be a theory in $\mathcal{L}$. We can define an equivalence relation on formulas in $\mathcal{L}$ as

$$\alpha \sim \beta \quad \text{if and only if} \quad \Delta \vdash \alpha \leftrightarrow \beta$$

for all $\alpha, \beta \in \mathcal{L}$. Let $[\alpha]$ denote the equivalence class of $\alpha \in \mathcal{L}$ with respect to $\sim$. We can then let $B(\Delta) = \{[\alpha] \mid \alpha \in \mathcal{L}\}$ and define the structure of a BA on $B(\Delta)$ as

$$\begin{aligned} [\alpha] \vee [\beta] &= [\alpha \vee \beta], \\ [\alpha] \wedge [\beta] &= [\alpha \wedge \beta], \\ \neg[\alpha] &= [\neg\alpha], \\ 1 &= [\alpha \to \alpha], \\ 0 &= [\alpha \wedge \neg\alpha] \end{aligned}$$

for all $\alpha, \beta \in \mathcal{L}$. Then $B(\Delta)$ is the *Lindenbaum-Tarski algebra* of Δ [30, 45].

Example 1. Let $\mathcal{L}$ be a propositional logic with p and q as its only atoms. Then $L = \{p, q, \neg p, \neg q\}$ is its set of literals. Let $w : L \to \mathbb{R}_{\geq 0}$ be the *weight function* defined by

$$\begin{aligned} w(p) &= 0.3, \\ w(\neg p) &= 0.7, \\ w(q) &= 0.2, \\ w(\neg q) &= 0.8. \end{aligned}$$

Let Δ be a theory in $\mathcal{L}$ with a sole axiom p. Then Δ has two models, i.e., $\{p, q\}$ and $\{p, \neg q\}$. The *weighted model count* (WMC) [10] of Δ is then

$$\sum_{\omega \models \Delta} \prod_{\omega \models l} w(l) = w(p)w(q) + w(p)w(\neg q) = 0.3.$$

The corresponding BA $B(\Delta)$ can then be constructed using Definition 6. Alternatively, one can first construct the free BA generated by the set $\{p, q\}$ and then take a quotient with respect to either the filter generated by p or the ideal[3] generated by $\neg p$.

Each element of $B(\mathcal{L})$ can also be seen as a subset of the set of all models of $\mathcal{L}$, with 0 representing $\emptyset$, 1 representing the set of all (four) models, each atom representing a single model, and each edge going upward representing a subset relation. Thus, the Boolean-algebraic way of calculating the WMC of Δ consists of:

[3]More details on these concepts can be found in many books on BAs [21, 30].

1. Identifying an element $a \in B(\mathcal{L})$ that corresponds to Δ.
2. Finding all atoms of $B(\mathcal{L})$ that are 'dominated' by a according to the partial order.
3. Using w to calculate the weight of each such atom.
4. Adding the weights of these atoms.

This motivates the following definition of WMC generalised to BAs.

- Why is Step 1 always possible?
- Clarify what $B(L)$ means and whether $B(\Delta)$ is even necessary.
- Find a reference for the set/subset thing.
- This should be replaced with inner sums (a.k.a. free products).
- Mention that the subsequent definition can be reduced to a single formula (i.e., without cases).
- Any measure is a WMC measure if all atoms are in L.

Definition 7. Let $\mathbf{B}$ be an atomic BA, and let $M \subset \mathbf{B}$ be its set of atoms. Let $L \subset \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge L'$ for some $L' \subseteq L$, and let $w \colon L \to \mathbb{R}_{\geq 0}$ be arbitrary. The *weighted model count* $\mathrm{WMC}_w \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}_w(x) = \begin{cases} 0 & \text{if } x = 0 \\ \prod_{l \in L'} w(l) & \text{if } M \ni x = \bigwedge L' \\ \sum_{\text{atoms } a \leq x} \mathrm{WMC}_w(a) & \text{otherwise} \end{cases}$$

for any $x \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{NWMC}_w \colon \mathbf{B} \to [0, 1]$ as $\mathrm{NWMC}_w(x) = \frac{\mathrm{WMC}_w(x)}{\mathrm{WMC}_w(1)}$ for all $x \in \mathbf{B}$. For both WMC_w and NWMC_w, we will drop the subscript when doing so results in no potential confusion. Finally, we say that a measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is a *WMC measure* (or is *WMC-definable*) if there exists a subset $L \subset \mathbf{B}$ and a weight function $w \colon L \to \mathbb{R}_{\geq 0}$ such that $m = \mathrm{WMC}_w$.

Theorem 1. WMC *is a measure, and* NWMC *is a probability measure.*

Proof. First, note that WMC is non-negative and $\mathrm{WMC}(0) = 0$ by definition. Next, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Lemma 2 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned}\mathrm{WMC}(x \vee y) = \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i\right) \vee \left(\bigvee_{j \in J} y_j\right)\right) &= \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j) \\ &= \mathrm{WMC}(x) + \mathrm{WMC}(y),\end{aligned}$$

finishing the proof that WMC is a measure. This immediately shows that NWMC is a probability measure since, by definition, $\mathrm{NWMC}(1) = 1$. $\square$

Given a theory Δ in a logic $\mathcal{L}$, the usual way of using WMC to compute the probability of a query q is [4, 42]

$$\Pr_{\Delta,w}(q) = \frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}.$$

In our algebraic formulation, this can be computed in two different ways:

- as $\frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}$ in $B(\mathcal{L})$,
- and as $\mathrm{NWMC}_w([q])$ in $B(\Delta)$.

But how does the measure defined on $B(\mathcal{L})$ transfer to $B(\Delta)$?

4 Limitations of Literal-Based Weighted Model Counting

F Give a concrete example of something impossible to represent using WMC.

F Can you say something here about factorized vs non-factorized weight function definitions? That is, factorized is when w maps literals to $R_{\geq 0}$, non-factorized is when w maps models to $R_{\geq 0}$ and

  - come up with nice example when non-factorized weights are intuitive;
  - clarify that the factorized definition have is w.r.t. models, in case some one gets confused. [It doesn't have to be, if the BA is not free—P.]

- What Measures are WMC-Definable: proofs need to be updated and propositions could be phrased in a better way, but the gist should be the same.

4.1 WMC Requires Independent Literals

Lemma 8. *For any measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ and elements $a, b \in \mathbf{B}$,*

$$m(a \wedge b) = m(a)m(b) \tag{2}$$

if and only if

$$m(a \wedge b) \cdot m(\neg a \wedge \neg b) = m(a \wedge \neg b) \cdot m(\neg a \wedge b). \tag{3}$$

Proof. First, note that $a = (a \wedge b) \vee (a \wedge \neg b)$ and $(a \wedge b) \wedge (a \wedge \neg b) = 0$, so, by properties of a measure,

$$m(a) = m(a \wedge b) + m(a \wedge \neg b). \tag{4}$$

Applying Eq. (4) and the equivalent expression for $m(b)$ allows us to rewrite Eq. (2) as

$$m(a \wedge b) = [m(a \wedge b) + m(a \wedge \neg b)][m(a \wedge b) + m(\neg a \wedge b)]$$

which is equivalent to

$$m(a \wedge b)[1 - m(a \wedge b) - m(a \wedge \neg b) - m(\neg a \wedge b)] = m(a \wedge \neg b)m(\neg a \wedge b). \tag{5}$$

Since $a \wedge b$, $a \wedge \neg b$, $\neg a \wedge b$, $\neg a \wedge \neg b$ are pairwise disjoint and their supremum is 1,

$$m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b) + m(\neg a \wedge \neg b) = 1,$$

and this allows us to rewrite Eq. (5) into Eq. (3). As all transformations are invertible, the two expressions are equivalent. $\square$

This theorem needs a special case for zero weights.

Theorem 2. *Let* $\mathbf{B}$ *be a free BA over* $\{l_i\}_{i=1}^n$ *(for some* $n \in \mathbb{N}$*) with measure* $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$*, and let* $L = \{l_i\}_{i=1}^n \cup \{\neg l_i\}_{i=1}^n$*. Then there exists a weight function* $w\colon L \to \mathbb{R}_{\geq 0}$ *such that* $m = \mathrm{WMC}_w$ *if and only if*

$$m(l \wedge l') = m(l)m(l') \tag{6}$$

for all distinct $l, l' \in L$ *such that* $l \neq \neg l'$*.*

Remark. Note that if $n = 1$, then Eq. (6) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. ($\Leftarrow$) Let $w\colon L \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = m(l) \tag{7}$$

for all $l \in L$. We are going to show that $\mathrm{WMC}_w = m$. First, note that $\mathrm{WMC}_w(0) = 0 = m(0)$ by the definitions of both WMC_w and m. Second, let

$$a = \bigwedge_{i=1}^n a_i \tag{8}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\mathrm{WMC}(a) = \prod_{i=1}^n w(a_i) = \prod_{i=1}^n m(a_i) = m\left(\bigwedge_{i=1}^n a_i\right) = m(a)$$

by Definition 7 and Eqs. (6) to (8). Finally, note that if WMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

($\Rightarrow$) For the other direction, we are given a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ that induces a measure $m = \mathrm{WMC}_w\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (6) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$ for some $i, j \in [n]$. We then want to show that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \tag{9}$$

which is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j) \tag{10}$$

by Lemma 8. Then

$$\begin{aligned}
\mathrm{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \leq k_i \wedge k_j} \mathrm{WMC}(a) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\
&= \sum_{\text{atoms } a \leq k_i \wedge k_j} w(a_i)w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \leq k_i \wedge k_j} w(k_i)w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) \\
&= w(k_i)w(k_j) \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i)w(k_j)C,
\end{aligned}$$

where C denotes the part of $\mathrm{WMC}(k_i \wedge k_j)$ that will be the same for $\mathrm{WMC}(\neg k_i \wedge k_j)$, $\mathrm{WMC}(k_i \wedge \neg k_j)$, and $\mathrm{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (10) becomes

$$w(k_i)w(k_j)w(\neg k_i)w(\neg k_j)C^2 = w(k_i)w(\neg k_j)w(\neg k_i)w(k_j)C^2$$

which is trivially true. $\square$

4.2 Extending the Algebra

F2 If you prove (b) above, you could motivate why weights on literals is attractive and whether there is a way to augment the expressivity of WMC while still maintaining literal level weights. Hence this action.

F2 You need to explain the significance of this result.

Given this requirement for independence, a well-known way to represent probability distributions that do not consist entirely of independent variables is by adding more literals [10], i.e., extending the set L covered by the WMC weight function $w\colon L \to \mathbb{R}_{\geq 0}$. Let us translate this idea to the language of BAs.

Theorem 3. *Let $\mathbf{B}$ be a free BA over a finite set S, and let $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be an arbitrary measure. Let $L = \{s \mid s \in S\} \cup \{\neg s \mid s \in S\}$. By Lemma 7, we know that $\mathbf{B}$ has $n = 2^{|S|}$ atoms. Let $\{a_i\}_{i=1}^n$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i\}_{i=1}^n \cup \{\neg\phi_i\}_{i=1}^n$ be the set L extended with $2n$ new elements. Let $\mathbf{B}'$ be the unique Boolean algebra with $\{\phi_i \wedge a_i\}_{i=1}^n \cup \{\neg\phi_i \wedge a_i\}_{i=1}^n$ as its set of atoms. Let $\iota\colon \mathbf{B} \hookrightarrow \mathbf{B}'$ be the inclusion homomorphism. Let $w\colon L' \to \mathbb{R}_{\geq 0}$ be defined by*

$$w(l) = \begin{cases} m(a_i)/2 & \text{if } l = \phi_i \text{ or } l = \neg\phi_i \text{ for some } i \in [n] \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m' = \mathrm{WMC}_w\colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In other words, any measure can be computed using WMC by extending the BA with more literals. More precisely, we are given the left-hand column in

$$\begin{tikzcd}
\mathbb{R}_{\geq 0} & & \\
\mathbf{B} \arrow[u, "m"] \arrow[r, hook, "\iota"] & \mathbf{B}' \arrow[ul, "m'"'] & \\
L \arrow[u, hook] \arrow[r, hook] & L' \arrow[u, hook] \arrow[r, "w"] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

and construct the remaining part in such a way that the triangle commutes.

- Make J depend on i.
- Find a reference for this first claim in the following proof.

Proof. Since $\mathbf{B}$ is free over S, each atom $a_i \in \mathbf{B}$ is an infimum of elements in L, i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either $b = \phi_i \wedge a_i$ or $b = \neg\phi_i \wedge a_i$ for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L'. Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies $\phi_i \wedge a_i \le \iota(b)$ if and only if it satisfies $\neg\phi_i \wedge a_i \le \iota(b)$. Then, according to the definition of w,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \le \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \le \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \le \iota(b) \quad \text{if and only if} \quad a_i \le b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. $\square$

Now we can show that the construction in Theorem 3 is smallest possible.

Conjecture 1. *Let $\mathbf{B}$ and $\mathbf{B}'$ be Boolean algebras, and $\iota\colon \mathbf{B} \hookrightarrow \mathbf{B}'$ be the inclusion map such that $\mathbf{B}$ is free over L, all atoms of $\mathbf{B}'$ can be expressed as meets of elements of L', and the following subset relations are satisfied:*

$$\begin{array}{ccc} \mathbf{B} & \overset{\iota}{\hookrightarrow} & \mathbf{B}' \\ \cup & & \cup \\ L & \subset & L' \end{array}$$

If, for any measure $m\colon \mathbf{B} \to \mathbb{R}_{\ge 0}$, one can construct a weight function $w\colon L' \to \mathbb{R}_{\ge 0}$ such that the WMC measure $\mathrm{WMC}\colon \mathbf{B}' \to \mathbb{R}_{\ge 0}$ with respect to w satisfies

$$m = \mathrm{WMC} \circ \iota,$$

then $|L' \setminus L| \ge 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [13] and Sang et al. [42]. Suppose we have a discrete probability distribution with n variables, and the ith variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^n v_i$ models. My expansion would then use

$$\sum_{i=1}^n v_i + 2\prod_{i=1}^n v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [13] would use

$$\sum_{i=1}^n v_i + \sum_{i=1}^n \prod_{j=1}^i v_j$$

variables, while for the encoding by Sang et al. [42],

$$\sum_{i=1}^n v_i + \sum_{i=1}^n (v_i - 1) \prod_{j=1}^{i-1} v_j$$

variables would suffice.

5 Previous Work

5.1 Bayesian Network Encodings

- All encodings (including my non-binary encoding) live on a measure Boolean algebra where the measure is not probabilistic (so $\Pr(\neg A) \neq 1 - \Pr(A)$).
- `cd05` relaxes the encoding so much that extra models become possible. They are supposed to be filtered out by the algorithm, but mine can't do that because it doesn't deal with models. Same for `cd06` because it's based on `cd05`.
- `sbk05` uses my trick with dividing probabilities. That could explain small inaccuracies in its answers.
- Encodings:
  - `d02` [13]
  - `sbk05` [42]
  - `cd05` [8]
  - `cd06` [9] (supposed to be the best)
  - `db20` (mine)

5.2 Algebraic Decision Diagrams and ADDMC

6 The Algebra of Functions on Boolean Algebras

- Could clarify why $[\lambda_{X=0}] = \overline{[\lambda_{X=1}]}$. This follows from the definition. Or should I avoid the overline notation?
- Prove that we have an algebra of functions with two (?) additional operations. Or maybe just cite the original paper for this.

Let V denote the set of random variables in a Bayesian network. For any random variable $X \in V$, let $\mathrm{pa}(X)$ denote the set of parents of X and $\mathrm{im}\, X$ denote the set of possible values.

Definition 8 (Indicator variables)**.** Let $X \in V$ be a random variable. If X is binary (i.e., $|\mathrm{im}\, X| = 2$), we can arbitrary identify one of the values as 1 and the other one as 0 (i.e, $\mathrm{im}\, X \cong \{0, 1\}$). Then X can be represented by a single *indicator variable* $\lambda_{X=1}$. If we interpret $2^{\{\lambda_{X=1}\}}$ as a Boolean algebra, we can let $\lambda_{X=0} = \neg\lambda_{X=1}$ in the algebraic notation, or, equivalently, $\lambda_{X=0} = \emptyset \in 2^{\{\lambda_{X=1}\}}$ in the set-theoretic notation. On the other hand, if X is not binary, we represent X with $|\mathrm{im}\, X|$ indicator variables, one for each value. We let

$$E(X) = \begin{cases} \{\lambda_{X=1}\} & \text{if } |\mathrm{im}\, X| = 2 \\ \{\lambda_{X=x} \mid x \in \mathrm{im}\, X\} & \text{otherwise.} \end{cases}$$

denote the set of indicator variables for X and

$$E^*(X) = E(X) \cup \bigcup_{Y \in \mathrm{pa}(X)} E(Y).$$

denote the set of indicator variables for X and its parents in the Bayesian network. Finally, let

$$U = \bigcup_{X \in V} E(X)$$

denote the set of all indicator variables for all random variables in the Bayesian network.

Definition 9 (Operations on functions)**.** Let $A\colon 2^X \to \mathbb{R}_{\geq 0}$ and $B\colon 2^Y \to \mathbb{R}_{\geq 0}$ be Boolean functions, $\alpha \in \mathbb{R}_{\geq 0}$, and $x \in X$. We define the following operations:

Addition: $A + B$ is a function $A + B\colon 2^{X\cup Y} \to \mathbb{R}_{\geq 0}$ such that

$$(A + B)(\tau) = A(\tau \cap X) + B(\tau \cap Y)$$

for all $\tau \in 2^{X\cup Y}$.

Inverse: $\overline{A}$ is a function $\overline{A}\colon 2^X \to \mathbb{R}_{\geq 0}$ such that

$$\overline{A}(\tau) = 1 - A(\tau)$$

for all $\tau \in 2^X$.

Multiplication: $A \cdot B$ is a function $A \cdot B\colon 2^{X\cup Y} \to \mathbb{R}_{\geq 0}$ such that

$$(A \cdot B)(\tau) = A(\tau \cap X) \cdot B(\tau \cap Y)$$

for all $\tau \in 2^{X\cup Y}$.

Scalar multiplication: αA is a function $\alpha A\colon 2^X \to \mathbb{R}_{\geq 0}$ such that

$$(\alpha A)(\tau) = \alpha \cdot A(\tau)$$

for all $\tau \in 2^X$.

Projection: $\exists_x A$ is a function $\exists_x A\colon 2^{X\setminus\{x\}} \to \mathbb{R}_{\geq 0}$ such that

$$(\exists_x A)(\tau) = A(\tau) + A(\tau \cup \{x\})$$

for all $\tau \in 2^{X\setminus\{x\}}$.

Note that both addition and multiplication commute.

Definition 10 (Special functions)**.**

- unit $1\colon 2^{\emptyset} \to \mathbb{R}_{\geq 0}$, $1(\tau) = 1$.
- zero $0\colon 2^{\emptyset} \to \mathbb{R}_{\geq 0}$, $0(\tau) = 0$.
- constant $[a]\colon 2^{\{a\}} \to \mathbb{R}_{\geq 0}$,

$$[a](\tau) = \begin{cases} 1 & \text{if } a \in \tau \\ 0 & \text{if } a \notin \tau. \end{cases}$$

Remark. For any function $A\colon 2^X \to \mathbb{R}_{\geq 0}$, $A + \overline{A} = 1$.

Henceforth, for any function $A\colon 2^X \to \mathbb{R}_{\geq 0}$ and any set τ, we will write $A(\tau)$ to mean $A(\tau \cap X)$.

7 Encoding Bayesian Networks Using Conditional Measures

- We assume that the first literal after 'w' is positive.
- The last two numbers are the positive and the negative probabilities, respectively—sometimes they add to one, and sometimes the negative probability is one, regardless of the value of the first probability.
- We assume that all variables in the Bayesian network have at least two values.

$\phi \leftarrow 1$;
for $X \in V$ **do**
  let $\mathrm{pa}(X) = \{Y_1, \dots, Y_n\}$;
  $\mathrm{CPT}_X \leftarrow 0$;
  if $|\operatorname{im} X| = 2$ **then**
    for $(y_1, \dots, y_n) \in \prod_{i=1}^n \operatorname{im} Y_i$ **do**
      $p_1 \leftarrow \Pr(X = 1 \mid Y_1 = y_1, \dots, Y_n = y_n)$;
      $p_0 \leftarrow \Pr(X \neq 1 \mid Y_1 = y_1, \dots, Y_n = y_n)$;
      $\mathrm{CPT}_X \leftarrow \mathrm{CPT}_X + p_1[\lambda_{X=1}] \cdot \prod_{i=1}^n [\lambda_{Y_i=y_i}] + p_0\overline{[\lambda_{X=1}]} \cdot \prod_{i=1}^n [\lambda_{Y_i=y_i}]$;
  else
    let $\operatorname{im} X = \{x_1, \dots, x_m\}$;
    for $x \in \operatorname{im} X$ **and** $(y_1, \dots, y_n) \in \prod_{i=1}^n \operatorname{im} Y_i$ **do**
      $p_x \leftarrow \Pr(X = x \mid Y_1 = y_1, \dots, Y_n = y_n)$;
      $\mathrm{CPT}_X \leftarrow \mathrm{CPT}_X + p_x[\lambda_{X=x}] \cdot \prod_{i=1}^n [\lambda_{Y_i=y_i}] + \overline{[\lambda_{X=1}]} \cdot \prod_{i=1}^n [\lambda_{Y_i=y_i}]$;
    $\mathrm{CPT}_X \leftarrow \mathrm{CPT}_X \cdot (\sum_{i=1}^m [\lambda_{X=x_i}]) \cdot \prod_{i=1}^m \prod_{j=i+1}^m (\overline{[\lambda_{X=x_i}]} + \overline{[\lambda_{X=x_j}]})$;
  $\phi \leftarrow \phi \cdot \mathrm{CPT}_X$;
return ϕ;

- Have an example of how the ADDs function in this situation. If not for the paper, then at least for slides. Use the framework to check its correctness.
- The function ϕ created by the algorithm can be seen as a measure on the BA 2^U.
- We extend the Gaifman graph to add edges when two variables occur in the same CPT (e.g., including the edge from A to B when the CPT is $\Pr(A \mid B)$).

Lemma 9. *Let $X \in V$ be a random variable with parents $\mathrm{pa}(X) = \{Y_1, \dots, Y_n\}$. Then $\mathrm{CPT}_X \colon 2^{E^*(X)} \to \mathbb{R}_{\geq 0}$ is such that for any $x \in \operatorname{im} X$ and $(y_1, \dots, y_n) \in \prod_{i=1}^n \operatorname{im} Y_i$,*

$$\mathrm{CPT}_X\left(\lambda_{X=x} \wedge \bigwedge_{i=1}^n \lambda_{Y_i=y_i}\right) = \Pr(X = x \mid Y_1 = y_1, \dots, Y_n = y_n).$$

Proof. Let

$$\tau = \lambda_{X=x} \wedge \bigwedge_{i=1}^n \lambda_{Y_i=y_i}.$$

If X is binary, then CPT_X is a sum of $2\prod_{i=1}^n |\operatorname{im} Y_i|$ terms, one for each possible assignment of values to variables $X, Y_1, \dots, Y_n$. Exactly one of these terms is nonzero when applied to τ, and it is equal to $\Pr(X = x \mid Y_1 = y_1, \dots, Y_n = y_n)$ by definition.

If X is not binary, then

$$\left(\sum_{i=1}^m [\lambda_{X=x_i}]\right)(\tau) = 1,$$

and

$$\left(\prod_{i=1}^m \prod_{j=i+1}^m (\overline{[\lambda_{X=x_i}]} + \overline{[\lambda_{X=x_j}]})\right)(\tau) = 1,$$

so, by a similar argument as before,

$$\mathrm{CPT}_X(\tau) = \Pr(X = x \mid Y_1 = y_1, \dots, Y_n = y_n).$$

□

Proposition 1. *$\phi\colon 2^U \to \mathbb{R}_{\geq 0}$ represents the full probability distribution of the Bayesian network, i.e., if $V = \{X_1, \dots, X_n\}$, then*

$$\phi(\tau) = \begin{cases} \Pr(X_1 = x_1, \dots, X_n = x_n) & \text{if } \tau = \bigwedge_{i=1}^n \lambda_{X_i = x_i} \text{ for some } (x_1, \dots, x_n) \in \prod_{i=1}^n \operatorname{im} X_i \\ 0 & \text{otherwise,} \end{cases}$$

for all $\tau \in 2^U$.

Proof. If

$$\tau = \bigwedge_{X \in V} \lambda_{X = v_X}$$

for some $(v_X)_{X \in V} \in \prod_{X \in V} \operatorname{im} X$, then

$$\phi(\tau) = \prod_{X \in V} \Pr\left(X = v_X \,\middle|\, \bigwedge_{Y \in \mathrm{pa}(X)} Y = v_Y \right) = \Pr\left(\bigwedge_{X \in V} X = v_X \right)$$

by Lemma 9 and the definition of a Bayesian network. Otherwise there must be some non-binary random variable $X \in V$ such that $|E(X) \cap \tau| \neq 1$. If $E(X) \cap \tau = \emptyset$, then

$$\left(\sum_{i=1}^m [\lambda_{X = x_i}] \right)(\tau) = 0,$$

and so $\mathrm{CPT}_X(\tau) = 0$, and $\phi(\tau) = 0$. If $|E(X) \cap \tau| > 1$, then we must have two different values $x_1, x_2 \in \operatorname{im} X$ such that $\{\lambda_{X=x_1}, \lambda_{X=x_2}\} \subseteq \tau$ which means that

$$(\overline{[\lambda_{X=x_1}]} + \overline{[\lambda_{X=x_2}]})(\tau) = 0,$$

and so, again, $\mathrm{CPT}_X(\tau) = 0$, and $\phi(\tau) = 0$. □

Theorem 4. *Let $\phi\colon 2^U \to \mathbb{R}_{\geq 0}$ be a function generated by the algorithm. Then*

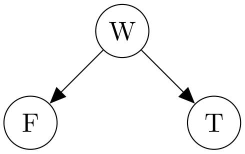
$$(\exists_U (\phi \cdot [\lambda_{X=x}]))(\emptyset) = \Pr(X = x).$$

Proof. Let $V = \{X, Y_1, \dots, Y_n\}$. Then

$$\begin{aligned} (\exists_U (\phi \cdot [\lambda_{X=x}]))(\emptyset) &= \sum_{\tau \in 2^U} (\phi \cdot [\lambda_{X=x}])(\tau) = \sum_{\lambda_{X=x} \in \tau \in 2^U} \phi(\tau) = \sum_{\lambda_{X=x} \in \tau \in 2^U} \left(\prod_{Y \in V} \mathrm{CPT}_Y \right)(\tau) \\ &= \sum_{(y_1, \dots, y_n) \in \prod_{i=1}^n \operatorname{im} Y_i} \Pr(X = x, Y_1 = y_1, \dots, Y_n = y_n) = \Pr(X = x) \end{aligned}$$

by the following arguments:

- the proof of Theorem 1 in the ADDMC paper [16];
- if $\lambda_{X=x} \notin \tau \in 2^U$, then $(\phi \cdot [\lambda_{X=x}])(\tau) = \phi(\tau) \cdot [\lambda_{X=x}](\tau \cap \{\lambda_{X=x}\}) = \phi(\tau) \cdot 0 = 0$;
- Proposition 1;



w	$\Pr(W = w)$
1	0.5
0	0.5

w	f	$\Pr(F = f \mid W = w)$
1	1	0.6
1	0	0.4
0	1	0.1
0	0	0.9

w	t	$\Pr(T = t \mid W = w)$
1	l	0.2
1	m	0.4
1	h	0.4
0	l	0.6
0	m	0.3
0	h	0.1

Figure 1: An example Bayesian network with its CPTs

- marginalisation of a probability distribution.

□

Example 2. The Bayesian network in Fig. 1 has

$$\begin{aligned}
V &= \{W, F, T\},\\
\mathrm{pa}(W) &= \emptyset,\\
\mathrm{pa}(F) &= \mathrm{pa}(T) = \{W\},\\
\operatorname{im} W &= \operatorname{im} F = \{0, 1\},\\
\operatorname{im} T &= \{l, m, h\},\\
E(W) &= \{\lambda_{W=1}\},\\
E(F) &= \{\lambda_{F=1}\},\\
E(T) &= \{\lambda_{T=l}, \lambda_{T=m}, \lambda_{T=h}\},\\
E^*(W) &= \{\lambda_{W=1}\},\\
E^*(F) &= \{\lambda_{F=1}, \lambda_{W=1}\},\\
E^*(T) &= \{\lambda_{T=l}, \lambda_{T=m}, \lambda_{T=h}, \lambda_{W=1}\},\\
\mathrm{CPT_W} &= 0.5[\lambda_{W=1}] + 0.5\overline{[\lambda_{W=1}]} = 0.5 \cdot 1,\\
\mathrm{CPT_F} &= 0.6[\lambda_{F=1}] \cdot [\lambda_{W=1}] + 0.4[\lambda_{F=0}] \cdot [\lambda_{W=1}] + 0.1[\lambda_{F=1}] \cdot [\lambda_{W=0}] + 0.9[\lambda_{F=0}] \cdot [\lambda_{W=0}]\\
&= 0.6[\lambda_{F=1}] \cdot [\lambda_{W=1}] + 0.4\overline{[\lambda_{F=1}]} \cdot [\lambda_{W=1}] + 0.1[\lambda_{F=1}] \cdot \overline{[\lambda_{W=1}]} + 0.9\overline{[\lambda_{F=1}]} \cdot \overline{[\lambda_{W=1}]},\\
\mathrm{CPT_T} &= ([\lambda_{T=l}] + [\lambda_{T=m}] + [\lambda_{T=h}]) \cdot (\overline{[\lambda_{T=l}]} + \overline{[\lambda_{T=m}]}) \cdot (\overline{[\lambda_{T=l}]} + \overline{[\lambda_{T=h}]}) \cdot (\overline{[\lambda_{T=m}]} + \overline{[\lambda_{T=h}]}) \cdot (\dots),
\end{aligned}$$

and can be encoded in a DIMACS-like CNF format as

$$\begin{array}{lrrll}
\lambda_{T=l} & \lambda_{T=m} & \lambda_{T=h} & & 0\\
 & -\lambda_{T=l} & -\lambda_{T=m} & & 0\\
 & -\lambda_{T=l} & -\lambda_{T=h} & & 0\\
 & -\lambda_{T=m} & -\lambda_{T=h} & & 0\\
w & \lambda_{W=1} & & 0.5 & 0.5\\
w & \lambda_{F=1} & \lambda_{W=1} & 0.6 & 0.4\\
w & \lambda_{F=1} & -\lambda_{W=1} & 0.1 & 0.9\\
w & \lambda_{T=l} & \lambda_{W=1} & 0.2 & 1\\
w & \lambda_{T=m} & \lambda_{W=1} & 0.4 & 1\\
w & \lambda_{T=h} & \lambda_{W=1} & 0.4 & 1\\
w & \lambda_{T=l} & \lambda_{W=0} & 0.6 & 1\\
w & \lambda_{T=m} & \lambda_{W=0} & 0.3 & 1\\
w & \lambda_{T=h} & \lambda_{W=0} & 0.1 & 1
\end{array}$$

with each λ replaced with a unique positive integer.

8 Experimental Comparison

- We don't compare 'compile times' because our encoding time is linear, so we would easily beat everyone else.
- When the Bayesian network has an evidence file, we compute the probability of evidence. Otherwise, let X denote the last-mentioned node in the Bayesian network. If `true` is a valid value of X, we compute the marginal probability of $X = \texttt{true}$. Otherwise, we pick the first value of X and calculate its marginal probability. This applies to the Grid data set (as intended) and also to two instances of Plan Reconstruction and roughly half of the instances from 2004-PGM that have empty evidence files.
- After the experiments are finished, note the processor, memory per thread, and add the following acknowledgment.
- All other encodings are implemented in Ace 3.0[4] and should be compiled with `-encodeOnly` (i.e., don't compile the CNF into an AC) and `-noEclause` (i.e., only use standard syntax) flags.
- Datasets
  - binary Bayesian networks from Sang et al.[5] [42]
    - Grid (networks) (ratio 75 means that 75% of the nodes are deterministic),
    - Plan recognition (problems),
    - Deterministic quick medical reference (what do the numbers mean? the README doesn't say).
  - Bayesian networks available with Ace
    - `2004-pgm` [11] (binary)
    - `2005-ijcai` [8]. The Genie/Smile files have their own citation data that I should probably extract. This is the only dataset that has some non-binary networks.
    - `2006-ijar` [11] (binary)

9 Explaining The Performance Benefits

- `d02` has

  $$\sum_{X \in V} |\operatorname{im} X| + |\operatorname{im} X| \prod_{Y \in \operatorname{pa}(X)} |\operatorname{im} Y|$$

  variables and

  $$\sum_{X \in V} 1 + \binom{|\operatorname{im} X|}{2} + |\operatorname{im} X|(2 + |\operatorname{pa}(X)|) \prod_{Y \in \operatorname{pa}(X)} |\operatorname{im} Y|$$

  clauses (along with one ADD per variable to encode the weights).
- `sbk05` is a bit harder to evaluate due to a handful of small optimisations in the encoding. Could find an upper bound anyway.
- `db20` (my encoding) has

  $$\sum_{X \in V} |\operatorname{im} X|$$

  variables (less for binary) and

  $$\sum_{X \in V} |\operatorname{im} X| + 1 + \binom{|\operatorname{im} X|}{2}$$

  ADDs.

[4] `http://reasoning.cs.ucla.edu/ace/`
[5] `https://www.cs.rochester.edu/u/kautz/Cachet/`

- Let:
  - $N = |V|$ (i.e., the number of nodes in the Bayesian network),
  - $D = \max_{X \in V} |\mathrm{pa}(X)|$ (i.e., the maximum in-degree or the number of parents),
  - $V = \max_{X \in V} |\operatorname{im} X|$ (i.e., the maximum number of values per variables).
- Then my encoding has $\mathcal{O}(NV)$ variables and $\mathcal{O}(NV^2)$ ADDs while `d02` has $\mathcal{O}(NV^{D+1})$ variables and $\mathcal{O}(NDV^{D+1})$ ADDs.

Calculate numVariables/numClauses (or the other way around) for each instance and plot this ratio vs runtime (for each encoding, or at least mine and D02)

10 Conclusion and Future Work

- Extra benefit: one does not need to come up with a way to turn some probability distribution to into a fully independent one.
- Important future work: replacing ADDs with AADDs[6] is likely to bring performance benefits. Other extensions:
  - FOADDs can represent first order statements;
  - XADDs can replace WMI for continuous variables;
  - ADDs with intervals can do approximations.
- Filtering out ADDs that have nothing to do with the answer helps tremendously, but I'm purposefully not doing that. Perhaps a heuristic could do the same thing?
- Encodings for everything else
  - probabilistic programs [24]
  - ProbLog [17]
    * For the ProbLog to WMC conversion, check out this guy: `https://users.ics.aalto.fi/ttj/`.
    * proof-based [35]
    * rule-based [26]
    * For ground ProbLog, we can encode a program

      ```
      p :: a :- b
      q :: a :- c
      ```

      into $P(a \mid b) = p$, $P(a \mid c) = q$ instead of having clauses $b \Rightarrow a$, $c \Rightarrow a$. Some logical structure is likely to remain.
- Bayesian networks are often solved in a compile once, query many times fashion. This can be achieved using ADDMC by selecting a subset S of variable we may want to query over and running ADDMC while excluding S from variable elimination/projection/$\exists$.
- More references
  - Measures on/in Boolean algebras: Horn and Tarski [25], Jech [28]
  - On Boolean algebras and their role in analysis [47]

[6] `https://github.com/ssanner/dd-inference`

– Infinite domains

  * Markov Logic in Infinite Domains (Singla and Domingos) [44]
  * Objective Bayesian probabilistic logic (Williamson) [46]
  * Unifying Logic and Probability (Russell) [40]

– Logical induction [20]

– Quantum probabilistic logic programming [2]

– WMC

  * algebraic model counting [29]
  * Explanation-Based Approximate Weighted Model Counting for Probabilistic Logics [39]
  * OUWMC [3]
  * Formula-Based Probabilistic Inference [22]
  * Parallel Probabilistic Inference by WMC [12]
  * Semiring Programming [5]
  * theoretical extension: WMC beyond two-variable logic [32]
  * from weighted to unweighted model counting [7]
  * theory behind WMC algorithms: solving #SAT and Bayesian inference with backtracking search [1]

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