What Boolean Algebras Can Teach Us About Weighted Model Counting

Paulius Dilkas

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1 Introduction

Contributions.

- WMC defines a measure over a BA.

- WMC with weights on literals imposes an independence assumption. (Measures are 'slightly' more expressive than WMC with weights on models because they apply to non-atomic BAs.)

- A BA can be augmented with new literals in order to support any measure.

- (Maybe) a lower bound on the number of new literals needed in order to support any measure.

- Alternatively, one can use coproducts and pushouts to define a BA with precisely the right independence and conditional independence conditions. (This requires a relaxed version of WMC.)

- This results in a smaller problem for WMC algorithms (w.r.t. both the number of literals and the length of the theory) and is optimal for, e.g., Bayesian networks.

- (Maybe) this results in faster inference (?)

Notable previous/related work.

- Hailperin's approach to probability logic [9]

- Nilsson's (somewhat successful) probabilistic logic [14]

- Logical induction: a big paper with a good overview of previous attempts to assign probabilities to logical sentences in a sensible way [7]

- Measures on Boolean algebras

  - On possibility and probability measures in finite Boolean algebras [2]
  - Representation of conditional probability measures [12]

2 Preliminaries

Definition 1. A *Boolean algebra* (BA) is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ consisting of a set $\mathbf{B}$ with binary operations *meet* $\wedge$ and *join* $\vee$, unary operation $\neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;

- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;

- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;

- $\vee$ distributes over $\wedge$ and vice versa;

- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

For clarity and succinctness, we will occasionally use three other operations that can be defined using the original three[1]:

$$a \to b = \neg a \vee b,$$
$$a \leftrightarrow b = (a \wedge b) \vee (\neg a \wedge \neg b),$$
$$a + b = (a \wedge \neg b) \vee (\neg a \wedge b).$$

We can also define a partial order $\leq$ on $\mathbf{B}$ as $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$) for all $a, b \in \mathbf{B}$. Furthermore, let $a < b$ denote $a \leq b$ and $a \neq b$. For the rest of this paper, let $\mathbf{B}$ refer to the BA $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$. For any $S \subseteq \mathbf{B}$, we write $\bigvee S$ for $\bigvee_{x \in S} x$ and call it the *supremum* of S. Similarly, $\bigwedge S = \bigwedge_{x \in S} x$ is the *infimum*. By convention, $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$. For any $a, b \in \mathbf{B}$, we say that a and b are *disjoint* if $a \wedge b = 0$.

Definition 2 ([10, 13]). An element $a \neq 0$ of $\mathbf{B}$ is an *atom* if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. Equivalently, $a \neq 0$ is an atom if there is no $x \in \mathbf{B}$ such that $0 < x < a$. We say that $\mathbf{B}$ is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 3 ([6]). *For any two distinct atoms a, $b \in \mathbf{B}$, $a \wedge b = 0$.*

Lemma 4 ([8]). *The following are equivalent:*

- $\mathbf{B}$ *is atomic.*

- *For any $x \in \mathbf{B}$, $x = \bigvee_{atoms\ a \leq x} a$.*

- 1 *is the supremum of all atoms.*

Lemma 5 ([8]). *All finite BAs are atomic.*

Definition 6 ([5, 10]). A *measure* on $\mathbf{B}$ is a function $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ such that:

- $m(0) = 0$;

- $m(a \vee b) = m(a) + m(b)$ for all $a, b \in \mathbf{B}$ whenever $a \wedge b = 0$.

If $m(1) = 1$, we call m a *probability measure*. Also, if $m(x) > 0$ for all $x \neq 0$, then m is *strictly positive*.

Definition 7 ([8]). Let $\mathbf{A}$ and $\mathbf{B}$ be BAs. A *(Boolean) homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $f \colon \mathbf{A} \to \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,

- $f(x \vee y) = f(x) \vee f(y)$,

- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Lemma 8 (Homomorphisms preserve order [8]). *Let $f \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism between two BAs $\mathbf{A}$ and $\mathbf{B}$. Then, for any $x, y \in \mathbf{A}$, if $x \leq y$, then $f(x) \leq f(y)$.*

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[1]We use $+$ to denote symmetric difference because it is the additive operation of a Boolean ring.

Lemma 9 ([16]). *For any $a, b \in \mathbf{B}$, $a \leq b$ if and only if $a \wedge \neg b = 0$.*

Lemma 10 ([8]). *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure. Then for all $a, b \in \mathbf{B}$, if $a \leq b$, then $m(a) \leq m(b)$.*

Definition 11 ([11]). Let S be a set, and let $\mathbf{B}$ be a BA. Then $\mathbf{B}$ is a *free BA over S* if there is a map $S \to \mathbf{B}$ such that for any BA $\mathbf{C}$ and map $S \to \mathbf{C}$, there is a unique homomorphism $\mathbf{B} \to \mathbf{C}$ that makes

$$
\begin{array}{ccc}
S & \longrightarrow & \mathbf{B} \\
 & \searrow & \downarrow \\
 & & \mathbf{C}.
\end{array}
$$

commute. A BA $\mathbf{B}$ is *free* if S exists.

Lemma 12 ([16]). *A finite BA is free if and only if it has 2^{2^n} elements for some $n \in \mathbb{N}$. It then has 2^n atoms and n generators.*

3 WMC as a Measure

Definition 13. Let $\mathcal{L}$ be a propositional (or first-order) logic, and let Δ be a theory in $\mathcal{L}$. We can define an equivalence relation on formulas in $\mathcal{L}$ as

$$\alpha \sim \beta \quad \text{if and only if} \quad \Delta \vdash \alpha \leftrightarrow \beta$$

for all $\alpha, \beta \in \mathcal{L}$. Let $[\alpha]$ denote the equivalence class of $\alpha \in \mathcal{L}$ with respect to $\sim$. We can then let $B(\Delta) = \{[\alpha] \mid \alpha \in \mathcal{L}\}$ and define the structure of a BA on $B(\Delta)$ as

$$
\begin{aligned}
[\alpha] \vee [\beta] &= [\alpha \vee \beta], \\
[\alpha] \wedge [\beta] &= [\alpha \wedge \beta], \\
\neg[\alpha] &= [\neg\alpha], \\
1 &= [\alpha \to \alpha], \\
0 &= [\alpha \wedge \neg\alpha]
\end{aligned}
$$

for all $\alpha, \beta \in \mathcal{L}$. Then $B(\Delta)$ is the *Lindenbaum-Tarski algebra* of Δ [11, 17].

Example 14. Let $\mathcal{L}$ be a propositional logic with p and q as its only atoms. Then $L = \{p, q, \neg p, \neg q\}$ is its set of literals. Let $w : L \to \mathbb{R}_{\geq 0}$ be the *weight function* defined by

$$
\begin{aligned}
w(p) &= 0.3, \\
w(\neg p) &= 0.7, \\
w(q) &= 0.2, \\
w(\neg q) &= 0.8.
\end{aligned}
$$

Let Δ be a theory in $\mathcal{L}$ with a sole axiom p. Then Δ has two models, i.e., $\{p, q\}$ and $\{p, \neg q\}$. The *weighted model count* (WMC) [3] of Δ is then

$$\sum_{\omega \models \Delta} \prod_{\omega \models l} w(l) = w(p)w(q) + w(p)w(\neg q) = 0.3.$$

The corresponding BA $B(\Delta)$ can then be constructed using Definition 13. Alternatively, one can first construct the free BA generated by the set $\{p, q\}$ and then take a quotient with respect to either the filter generated by p or the ideal[2] generated by $\neg p$.

Each element of $B(\mathcal{L})$ can also be seen as a subset of the set of all models of $\mathcal{L}$, with 0 representing $\emptyset$, 1 representing the set of all (four) models, each atom representing a single model, and each edge going upward representing a subset relation. Thus, the Boolean-algebraic way of calculating the WMC of Δ consists of:

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[2]More details on these concepts can be found in many books on BAs [8, 11].

1. Identifying an element $a \in B(\mathcal{L})$ that corresponds to Δ.

2. Finding all atoms of $B(\mathcal{L})$ that are 'dominated' by a according to the partial order.

3. Using w to calculate the weight of each such atom.

4. Adding the weights of these atoms.

This motivates the following definition of WMC generalised to BAs.

Definition 15. Let $\mathbf{B}$ be an atomic BA, and let $M \subset \mathbf{B}$ be its set of atoms. Let $L \subset \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge L'$ for some $L' \subseteq L$, and let $w \colon L \to \mathbb{R}_{\geq 0}$ be arbitrary. The *weighted model count* $\mathrm{WMC}_w \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}_w(x) = \begin{cases} 0 & \text{if } x = 0 \\ \prod_{l \in L'} w(l) & \text{if } M \ni x = \bigwedge L' \\ \sum_{\text{atoms } a \leq x} \mathrm{WMC}_w(a) & \text{otherwise} \end{cases}$$

for any $x \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{NWMC}_w \colon \mathbf{B} \to [0,1]$ as $\mathrm{NWMC}_w(x) = \frac{\mathrm{WMC}_w(x)}{\mathrm{WMC}_w(1)}$ for all $x \in \mathbf{B}$. For both WMC_w and NWMC_w, we will drop the subscript when doing so results in no potential confusion. Finally, we say that a measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is a *WMC measure* (or is *WMC-definable*) if there exists a subset $L \subset \mathbf{B}$ and a weight function $w \colon L \to \mathbb{R}_{\geq 0}$ such that $m = \mathrm{WMC}_w$.

Theorem 16. WMC *is a measure, and* NWMC *is a probability measure.*

Proof. First, note that WMC is non-negative and $\mathrm{WMC}(0) = 0$ by definition. Next, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Lemma 4 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i \right) \vee \left(\bigvee_{j \in J} y_j \right) \right) = \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j)$$
$$= \mathrm{WMC}(x) + \mathrm{WMC}(y),$$

finishing the proof that WMC is a measure. This immediately shows that NWMC is a probability measure since, by definition, $\mathrm{NWMC}(1) = 1$. $\qquad \square$

4

Given a theory Δ in a logic $\mathcal{L}$, the usual way of using WMC to compute the probability of a query q is [1, 15]

$$\Pr_{\Delta,w}(q) = \frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}.$$

In our algebraic formulation, this can be computed in two different ways:

- as $\frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}$ in $B(\mathcal{L})$,

- and as $\mathrm{NWMC}_w([q])$ in $B(\Delta)$.

But how does the measure defined on $B(\mathcal{L})$ transfer to $B(\Delta)$?

4 What Measures Are WMC-Definable?

4.1 WMC Requires Independent Literals

Lemma 17. *For any measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ and elements $a, b \in \mathbf{B}$,*

$$m(a \wedge b) = m(a)m(b) \tag{2}$$

if and only if

$$m(a \wedge b) \cdot m(\neg a \wedge \neg b) = m(a \wedge \neg b) \cdot m(\neg a \wedge b). \tag{3}$$

Proof. First, note that $a = (a \wedge b) \vee (a \wedge \neg b)$ and $(a \wedge b) \wedge (a \wedge \neg b) = 0$, so, by properties of a measure,

$$m(a) = m(a \wedge b) + m(a \wedge \neg b). \tag{4}$$

Applying Eq. (4) and the equivalent expression for $m(b)$ allows us to rewrite Eq. (2) as

$$m(a \wedge b) = [m(a \wedge b) + m(a \wedge \neg b)][m(a \wedge b) + m(\neg a \wedge b)]$$

which is equivalent to

$$m(a \wedge b)[1 - m(a \wedge b) - m(a \wedge \neg b) - m(\neg a \wedge b)] = m(a \wedge \neg b)m(\neg a \wedge b). \tag{5}$$

Since $a \wedge b$, $a \wedge \neg b$, $\neg a \wedge b$, $\neg a \wedge \neg b$ are pairwise disjoint and their supremum is 1,

$$m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b) + m(\neg a \wedge \neg b) = 1,$$

and this allows us to rewrite Eq. (5) into Eq. (3). As all transformations are invertible, the two expressions are equivalent. $\qquad\square$

Theorem 18. *Let $\mathbf{B}$ be a free BA over $\{l_i\}_{i=1}^n$ (for some $n \in \mathbb{N}$) with measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and let $L = \{l_i\}_{i=1}^n \cup \{\neg l_i\}_{i=1}^n$. Then there exists a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ such that $m = \mathrm{WMC}_w$ if and only if*

$$m(l \wedge l') = m(l)m(l') \tag{6}$$

for all distinct $l, l' \in L$ such that $l \neq \neg l'$.

Remark. Note that if $n = 1$, then Eq. (6) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. ($\Leftarrow$) Let $w\colon L \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = m(l) \tag{7}$$

for all $l \in L$. We are going to show that $\mathrm{WMC}_w = m$. First, note that $\mathrm{WMC}_w(0) = 0 = m(0)$ by the definitions of both WMC_w and m. Second, let

$$a = \bigwedge_{i=1}^{n} a_i \tag{8}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\mathrm{WMC}(a) = \prod_{i=1}^{n} w(a_i) = \prod_{i=1}^{n} m(a_i) = m\left(\bigwedge_{i=1}^{n} a_i\right) = m(a)$$

by Definition 15 and Eqs. (6) to (8). Finally, note that if WMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

($\Rightarrow$) For the other direction, we are given a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ that induces a measure $m = \mathrm{WMC}_w\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (6) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$ for some $i, j \in [n]$. We then want to show that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \tag{9}$$

which is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j) \tag{10}$$

by Lemma 17. Then

$$\mathrm{WMC}(k_i \wedge k_j) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \mathrm{WMC}(a) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n]} w(a_m)$$

$$= \sum_{\text{atoms } a \leq k_i \wedge k_j} w(a_i)w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \leq k_i \wedge k_j} w(k_i)w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m)$$

$$= w(k_i)w(k_j) \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i)w(k_j)C,$$

where C denotes the part of $\mathrm{WMC}(k_i \wedge k_j)$ that will be the same for $\mathrm{WMC}(\neg k_i \wedge k_j)$, $\mathrm{WMC}(k_i \wedge \neg k_j)$, and $\mathrm{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (10) becomes

$$w(k_i)w(k_j)w(\neg k_i)w(\neg k_j)C^2 = w(k_i)w(\neg k_j)w(\neg k_i)w(k_j)C^2$$

which is trivially true. $\qquad\qquad\square$

4.2 Extending the Algebra

Given this requirement for independence, a well-known way to represent probability distributions that do not consist entirely of independent variables is by adding more literals [3], i.e., extending the set L covered by the WMC weight function $w\colon L \to \mathbb{R}_{\geq 0}$. Let us translate this idea to the language of BAs.

Theorem 19. *Let $\mathbf{B}$ be a free BA over a finite set S, and let $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be an arbitrary measure. Let $L = \{s \mid s \in S\} \cup \{\neg s \mid s \in S\}$. By Lemma 12, we know that $\mathbf{B}$ has $n = 2^{|S|}$ atoms. Let $\{a_i\}_{i=1}^{n}$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i\}_{i=1}^{n} \cup \{\neg \phi_i\}_{i=1}^{n}$ be the set L extended with $2n$ new elements. Let $\mathbf{B}'$ be the unique Boolean algebra with $\{\phi_i \wedge a_i\}_{i=1}^{n} \cup \{\neg \phi_i \wedge a_i\}_{i=1}^{n}$ as its set of atoms. Let $\iota\colon \mathbf{B} \hookrightarrow \mathbf{B}'$ be the inclusion homomorphism. Let $w\colon L' \to \mathbb{R}_{\geq 0}$ be defined by*

$$w(l) = \begin{cases} m(a_i)/2 & \text{if } l = \phi_i \text{ or } l = \neg\phi_i \text{ for some } i \in [n] \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m' = \mathrm{WMC}_w \colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In other words, any measure can be computed using WMC by extending the BA with more literals. More precisely, we are given the left-hand column in

and construct the remaining part in such a way that the triangle commutes.

Proof. Since $\mathbf{B}$ is free over S, each atom $a_i \in \mathbf{B}$ is an infimum of elements in L, i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either $b = \phi_i \wedge a_i$ or $b = \neg\phi_i \wedge a_i$ for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L'. Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies $\phi_i \wedge a_i \leq \iota(b)$ if and only if it satisfies $\neg\phi_i \wedge a_i \leq \iota(b)$. Then, according to the definition of w,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \leq \iota(b) \quad \text{if and only if} \quad a_i \leq b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. $\qquad\square$

Now we can show that the construction in Theorem 19 is smallest possible.

Conjecture 20. *Let $\mathbf{B}$ and $\mathbf{B}'$ be Boolean algebras, and $\iota \colon \mathbf{B} \hookrightarrow \mathbf{B}'$ be the inclusion map such that $\mathbf{B}$ is free over L, all atoms of $\mathbf{B}'$ can be expressed as meets of elements of L', and the following subset relations are satisfied:*

$$\mathbf{B} \stackrel{\iota}{\hookrightarrow} \mathbf{B}'$$
$$\cup \qquad \cup$$
$$L \quad \subset \quad L'$$

If, for any measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, one can construct a weight function $w \colon L' \to \mathbb{R}_{\geq 0}$ such that the WMC measure $\mathrm{WMC} \colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$ with respect to w satisfies

$$m = \mathrm{WMC} \circ \iota,$$

then $|L' \setminus L| \geq 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [4] and Sang et al. [15]. Suppose we have a discrete probability distribution with n variables, and the ith variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^{n} v_i$ models. My expansion would then use

$$\sum_{i=1}^{n} v_i + 2 \prod_{i=1}^{n} v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [4] would use

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \prod_{j=1}^{i} v_j$$

variables, while for the encoding by Sang et al. [15],

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n} (v_i - 1) \prod_{j=1}^{i-1} v_j$$

variables would suffice.

5 Representing Independence and Conditional Independence

5.1 Preliminaries

Definition 21. Given a BA $\mathbf{A}$, a *subalgebra* is a subset $\mathbf{B} \subseteq \mathbf{A}$ that, together with the operations, zero, and one of $\mathbf{A}$, is a BA.

Definition 22 ([8]). Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be BAs such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Let $f\colon \mathbf{A} \to \mathbf{C}$ and $g\colon \mathbf{B} \to \mathbf{C}$ be homomorphisms. Then f is an *extension* of g if $f(x) = g(x)$ for all $x \in \mathbf{B}$. If f is an extension of each member of a family $\{g_i\}_{i \in I}$ of homomorphisms, then f is called a *common extension* of $\{g_i\}_{i \in I}$.

Definition 23 ([8]). Let $\{\mathbf{A}_i\}_{i \in I}$ be a family of subalgebras of a BA $\mathbf{A}$. If for any BA $\mathbf{B}$ with a family of homomorphisms $\{f_i\colon \mathbf{A}_i \to \mathbf{B}\}_{i \in I}$ there exists a unique common extension of $\{f_i\colon \mathbf{A}_i \to \mathbf{B}\}_{i \in I}$ ($f\colon \mathbf{A} \to \mathbf{B}$ in the diagram),

$$\begin{array}{ccc} \mathbf{A}_i & \lhook\joinrel\longrightarrow & \mathbf{A} \\ & f_i \searrow & \downarrow f \\ & & \mathbf{B} \end{array}$$

then $\mathbf{A}$ is the *internal sum*[3] of $\{\mathbf{A}_i\}_{i \in I}$. We will denote it as $\bigoplus_{i \in I} \mathbf{A}_i$.

Proposition 24 ([16]). *Let $\mathbf{A}$ be the internal sum of a family of BAs $\{\mathbf{A}_i\}_{i \in I}$, and let $\{m_i\colon \mathbf{A}_i \to \mathbb{R}_{\geq 0}\}_{i \in I}$ be a family of measures. Then there exists a unique measure $m\colon \mathbf{A} \to \mathbb{R}_{\geq 0}$ such that, for any finite subset $J \subseteq I$ and family of elements $\{x_j \in \mathbf{A}_j\}_{j \in J}$,*

$$m\left(\bigwedge_{j \in J} x_j\right) = \prod_{j \in J} m_j(x_j).$$

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[3]A slightly more general version of this definition is also known as the free product, the Boolean product, and the coproduct in the category of BAs [8, 11, 16].

Definition 25 ([11])**.** Let $\mathbf{A}$ be a BA. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, and let $\{\mathbf{A}_i\}_{i \in I}$ be a family of subalgebras of $\mathbf{A}$ such that $\mathbf{A}_i \cap \mathbf{A}_j = \mathbf{B}$ for all $i \neq j$ in I. Let $\{\iota_i \colon \mathbf{B} \to \mathbf{A}_i\}$ be a family of inclusion homomorphisms. Then $\mathbf{A}$ is the *amalgamated free product*[4] *of* $\{\mathbf{A}_i\}_{i \in I}$ *over* $\mathbf{B}$ if, for any Boolean algebra $\mathbf{C}$ with a family of homomorphisms $\{f_i \colon \mathbf{A}_i \to \mathbf{C}\}_{i \in I}$ such that $f_i \circ \iota_i = f_j \circ \iota_j$ for all $i, j \in I$, there is a unique homomorphism $f \colon \mathbf{A} \to \mathbf{C}$ such that the triangle in

$$\mathbf{B} \overset{\iota_i}{\hookrightarrow} \mathbf{A}_i \longrightarrow \mathbf{A}$$

$$f_i \downarrow \qquad \swarrow f$$

$$\mathbf{C}$$

commutes for all $i \in I$. We will denote this product as

$$\mathbf{A} = \bigoplus_{\substack{\mathbf{B} \\ i \in I}} \mathbf{A}_i.$$

5.2 New Results

Theorem 26. *Let* $\{S_i\}_{i=0}^{n}$ *be a finite set of finite sets for some* $n > 1$ *such that for all distinct positive integers* i *and* j, $S_i \cap S_j = S_0$, *and let*

$$\mathbf{A} = \bigoplus_{\substack{\mathcal{F}(S_0) \\ 1 \le i \le n}} \mathcal{F}(S_i).$$

Let $(m_i \colon \mathcal{F}(S_i) \to \mathbb{R}_{\ge 0})_{i=1}^{n}$ *be arbitrary measures. Then there is a unique measure* $m \colon \mathbf{A} \to \mathbb{R}_{\ge 0}$ *such that, for any element* $b \in \mathcal{F}(S_0)$, *subset* $J \subseteq \{1, 2, \ldots, n\}$, *and elements* $\{a_j \in \mathcal{F}(S_j \setminus S_0)\}_{j \in J}$,

$$m \left(b \wedge \bigwedge_{j \in J} a_j \right) = \prod_{j \in J} m_j (b \wedge a_j).$$

Proof. □

Theorem 27. *The number of weights needed to encode a Bayesian network using coproducts and pushouts is equal to the number of entries in the tables of the network (and the resulting theory is shorter).*

Theorem 28 (Pushouts of free BAs are free)**.** *Let*

$$\mathbf{A} = \bigoplus_{\substack{\mathbf{B} \\ i \in I}} \mathbf{A}_i$$

be an amalgamated free product such that $\{\mathbf{A}_i\}_{i \in I}$ *are free BAs with* $\{S_i\}_{i \in I}$ *as their respective sets of generators. Let* $S = \bigcup_{i \in I} S_i$. *Then* $\mathbf{A}$ *is a free BA with generating set* S.

Proof. Suppose we have a map from S to an arbitrary BA $\mathbf{C}$, as in

$$S_i \longrightarrow S$$
$$\downarrow \qquad \downarrow$$
$$\mathbf{A}_i \longrightarrow \mathbf{A}$$
$$\mathbf{C}.$$

We want to show that there exists a unique homomorphism $\mathbf{A} \to \mathbf{C}$. For all $i \in I$, from $S_i \hookrightarrow S$ and $S \to \mathbf{C}$ we get a map $S_i \to \mathbf{C}$, so—by the definition of a free BA—there is a unique homomorphism $\mathbf{A}_i \to \mathbf{C}$. Furthermore, a family of homomorphisms $\{\mathbf{A}_i \to \mathbf{C}\}_{i \in I}$ uniquely determine a homomorphism $\mathbf{A} \to \mathbf{C}$ by the universal mapping property of a (wide) pushout. Thus $\mathbf{A}$ is a free BA with generating set S. □

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[4]Also known as a (wide) pushout in the category of BAs.

Corollary 29. *Similarly, coproducts of free BAs are free.*

6 How to represent a Bayesian network

Let $G = (V, E)$ be a DAG, and let $(X_v)_{v \in V}$ be the discrete random variables (RVs). Let $(d_v)_{v \in V}$ denote the numbers of values that each RV can take and assume that d_v is a power of two for all $v \in V$.
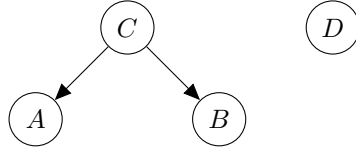
1. For each $v \in V$, let A_v be a set of $\log_2 d_v$ Boolean variables. (If d_v is not a power of two, we would need to manually forbid each illegal value, e.g., by adding formulas such as $\neg(x_1 \wedge x_2 \wedge x_3 \wedge \neg x_4)$ to the theory. Alternatively, we could set their probability to zero, but the former should be better.)

2. For $v \in V$:

   (a) If v has no parents, we can directly provide a probability measure on $\mathcal{F}(A_v)$.

   (b) Otherwise, we need a probability measure on

   $$\mathcal{F}\left(A_v \cup \bigcup_{(u,v) \in E} A_u\right).$$

   For any element $a_u \wedge a_v$ in this free Boolean algebra such that $a_v \in A_v \setminus \{0, 1\}$, and $a_u \notin A_v$, we can set
   $$m(a_u \wedge a_v) = \Pr(a_v \mid a_u) \Pr(a_u).$$

3. When a #SAT algorithm asks for the weight of a model $\bigwedge_{v \in V} l_v$, split it into segments that correspond to each leaf, take the probabilities from each such node, and multiply them. NB: The segments will intersect, e.g., with

$a \wedge b \wedge c \wedge d$ will be split into $(a \wedge c) \wedge (b \wedge c) \wedge d$.
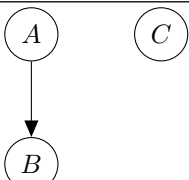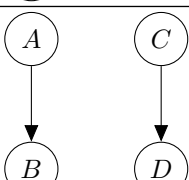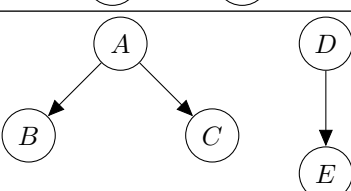
6.1 Bayesian networks as Boolean algebras

To turn a Bayesian network into a Boolean algebra, run $f(\{v \in V \mid \text{outdeg}(v) = 0\})$, where f works as follows on an arbitrary $A \subseteq V$:

1. $S \leftarrow \{u \mid (u, v) \in E, v \in A\}$ (i.e., S is the set of parents of the elements in A).

2. If $S = \emptyset$, return $\sum A$ (a coproduct).

3. Otherwise, return $\sum_{f(S)} A$ (a pushout).

Notes on the implementation of the algorithm

- We extend the Gaifman graph to add edges when two variables occur in the same CPT (e.g., including the edge from A to B when the CPT is $\Pr(A \mid B)$).

- Extra benefit: one does not need to come up with a way to turn some probability distribution to into a fully independent one.

- Trivial CPTs such as $\Pr(A \mid B) = \Pr(\neg A \mid B) = 0.5$, the ADDs of which simplify to a single number, are put in the first cluster.

Table 1: Examples of how Bayesian networks encode as Boolean algebras. Here, A denotes $\mathcal{F}\{a\}$ and AB denotes $\mathcal{F}\{a, b\}$.

Bayesian network	Boolean algebra
A	A
A B	$A + B$
$A \rightarrow B$	AB
$A \rightarrow B$, C	$B +_A C$
$A \rightarrow B$, $C \rightarrow D$	$B +_{A+C} D$
$B \leftarrow A \rightarrow C$, $D \rightarrow E$	$B +_{A+D} C +_{A+D} E$

- TODO: Look into ways to theoretically justify the performance benefits (I have some suggestions in my notes).

- TODO: when performing experiments, measure memory usage as well.

- TODO: Along with comparing the (3?) encodings, I could also compare this to using ADDs directly on Bayesian networks.

- TODO: Feedback: Give a concrete example of something impossible to represent using WMC.

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