On the Limitations of Weighted Model Counting

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1st April 2020

1 Introduction

Potential directions to explore (no good ones, really):

- Infinite BAs.

  - For example, the BA of finite and cofinite sets could be interesting.
  - OUWMC requires infinite logics to be compact (whatever that means). My algebraic angle suggests that completeness should be enough. Topologically, compactness implies completeness, but I have no idea how this translates to logics and algebras.

- WMI

- Measures with something other than $\mathbb{R}_{\geq 0}$ as the codomain (doesn't look promising).

Previous/related work:

- Hailperin's approach to probability logic [11]

- Nilsson's (somewhat successful) probabilistic logic [18]

- Logical induction: a big paper with a good overview of previous attempts to assign probabilities to logical sentences in a sensible way [9]

- Semiring programming [3]

- WMI [2]

- Measures on Boolean algebras

  - On possibility and probability measures in finite Boolean algebras [4]
  - Representation of conditional probability measures [16]

2 Preliminaries

Definition 1. A *Boolean algebra* (BA) is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ consisting of a set $\mathbf{B}$ with binary operations *meet* $\wedge$ and *join* $\vee$, unary operation $\neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;

- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;

- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;

- $\lor$ distributes over $\land$ and vice versa;

- $a \lor \neg a = 1$, and $a \land \neg a = 0$.

For clarity and succinctness, we will occasionally use three other operations that can be defined using the original three[1]:

$$a \to b = \neg a \lor b,$$
$$a \leftrightarrow b = (a \land b) \lor (\neg a \land \neg b),$$
$$a + b = (a \land \neg b) \lor (\neg a \land b).$$

We can also define a partial order $\leq$ on $\mathbf{B}$ as $a \leq b$ if $a = b \land a$ (or, equivalently, $a \lor b = b$) for $a, b \in \mathbf{B}$. Furthermore, let $a < b$ denote $a \leq b$ and $a \neq b$. For the rest of this paper, let $\mathbf{B}$ refer to the BA $(\mathbf{B}, \land, \lor, \neg, 0, 1)$. For any $S \subseteq \mathbf{B}$, we write $\bigvee S$ for $\bigvee_{x \in S} x$ and call it the *supremum* of S. Similarly, $\bigwedge S = \bigwedge_{x \in S} x$ is the *infimum*. By convention, $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$. For any $a, b \in \mathbf{B}$, we say that a and b are *disjoint* if $a \land b = 0$.

Definition 2 ([14, 17]). An element $a \neq 0$ of $\mathbf{B}$ is an *atom* if, for all $x \in \mathbf{B}$, either $x \land a = a$ or $x \land a = 0$. Equivalently, $a \neq 0$ is an atom if there is no $x \in \mathbf{B}$ such that $0 < x < a$. We say that $\mathbf{B}$ is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([8]). *For any two distinct atoms $a, b \in \mathbf{B}$, $a \land b = 0$.*

Lemma 2 ([10]). *The following are equivalent:*

- $\mathbf{B}$ *is atomic.*

- *For any $x \in \mathbf{B}$,*

$$x = \bigvee_{atoms\ a \leq x} a.$$

- 1 *is the supremum of all atoms.*

Lemma 3 ([10]). *All finite BAs are atomic.*

Definition 3 ([7, 14]). A *measure* on $\mathbf{B}$ is a function $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ such that:

- $m(0) = 0$;

- $m(a \lor b) = m(a) + m(b)$ for all $a, b \in \mathbf{B}$ whenever $a \land b = 0$.

If $m(1) = 1$, we call m a *probability measure*. Also, if $m(x) > 0$ for all $x \neq 0$, then m is *strictly positive*.

Lemma 4 ([13]). *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure. For any $a, b \in \mathbf{B}$, if $a \leq b$, then $m(a) \leq m(b)$.*

Definition 4 ([10]). An *ideal* is a non-empty subset $I \subseteq \mathbf{B}$ such that

- $i \lor j \in I$ for all $i, j \in I$;

- $i \land a \in I$ for all $i \in I$ and $a \in \mathbf{B}$.

For any $p \in \mathbf{B}$, the *principal ideal of p*—denoted by (p)—is the smallest ideal that contains p. It can also be expressed as $(p) = \{a \in \mathbf{B} \mid a \leq p\}$.

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[1]We use $+$ to denote symmetric difference because it is the additive operation of a Boolean ring.

Definition 5 ([10])**.** Let I be an ideal in $\mathbf{B}$. The *quotient algebra of $\mathbf{B}$ modulo the ideal I* $\mathbf{B}/I$ is a BA of equivalence classes of elements of $\mathbf{B}$ with respect to the equivalence relation

$$a \sim b \quad \Longleftrightarrow \quad a + b \in I$$

for all $a, b \in \mathbf{B}$. Elements of $\mathbf{B}/I$ are usually denoted by a/I (for some $a \in \mathbf{B}$) with the understanding that if $b \sim a$ (for some $b \in \mathbf{B}$), then $b/I = a/I$. The three algebraic operations on $\mathbf{B}/I$ are defined as

$$a/I \wedge b/I = a \wedge b/I,$$
$$a/I \vee b/I = a \vee b/I,$$
$$\neg(a/I) = (\neg a)/I.$$

Definition 6 ([10])**.** Let $\mathbf{A}$ and $\mathbf{B}$ be BAs. A *(Boolean) homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $f \colon \mathbf{A} \to \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,

- $f(x \vee y) = f(x) \vee f(y)$,

- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Lemma 5 ([10])**.** *Let $I \subseteq \mathbf{B}$ be an ideal. The map $f \colon \mathbf{B} \to \mathbf{B}/I$ defined by $f(x) = x/I$ is a homomorphism.*

Lemma 6 (Homomorphisms preserve order [10])**.** *Let $f \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism between two BAs $\mathbf{A}$ and $\mathbf{B}$. Then, for any $x, y \in \mathbf{A}$, if $x \leq y$, then $f(x) \leq f(y)$.*

Lemma 7 ([21])**.** *For any $a, b \in \mathbf{B}$, $a \leq b$ if and only if $a \wedge \neg b = 0$.*

Lemma 8 ([10])**.** *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure. Then for all $a, b \in \mathbf{B}$, if $a \leq b$, then $m(a) \leq m(b)$.*

3 WMC as a Measure

Definition 7. Let $\mathcal{L}$ be a propositional (or first-order) logic, and let Δ be a theory in $\mathcal{L}$. We can define an equivalence relation on formulas in $\mathcal{L}$ as

$$\alpha \sim \beta \quad \text{if and only if} \quad \Delta \vdash \alpha \leftrightarrow \beta$$

for all $\alpha, \beta \in \mathcal{L}$. Let $[\alpha]$ denote the equivalence class of $\alpha \in \mathcal{L}$ with respect to $\sim$. We can then let $B(\Delta) = \{[\alpha] \mid \alpha \in \mathcal{L}\}$ and define the structure of a BA on $B(\Delta)$ as

$$[\alpha] \vee [\beta] = [\alpha \vee \beta],$$
$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta],$$
$$\neg[\alpha] = [\neg \alpha],$$
$$1 = [\alpha \to \alpha],$$
$$0 = [\alpha \wedge \neg \alpha]$$

for all $\alpha, \beta \in \mathcal{L}$. Then $B(\Delta)$ is the *Lindenbaum-Tarski algebra* of Δ [15, 22].
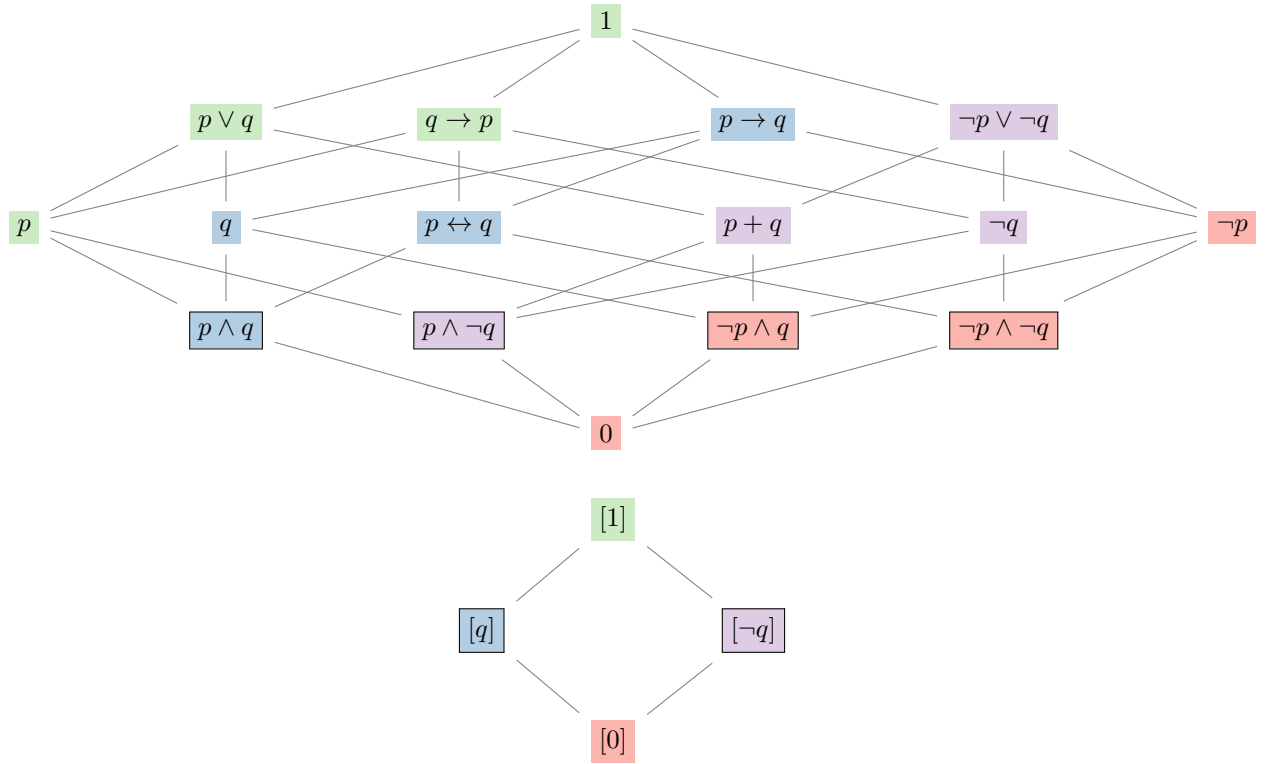
Figure 1: Two BAs from Example 1: $B(\mathcal{L})$ at the top and $B(\Delta)$ at the bottom. An edge between elements a and b (with a positioned lower than b) means that $a < b$. Each element of $B(\Delta)$ is an equivalence class of elements of $B(\mathcal{L})$, and the colours show which elements of $B(\mathcal{L})$ belong to which class. In both algebras, atoms have borders around them.

Example 1. Let $\mathcal{L}$ be a propositional logic with p and q as its only atoms. Then $L = \{p, q, \neg p, \neg q\}$ is its set of literals. Let $w : L \to \mathbb{R}_{\geq 0}$ be the *weight function* defined by

$$
\begin{aligned}
w(p) &= 0.3, \\
w(\neg p) &= 0.7, \\
w(q) &= 0.2, \\
w(\neg q) &= 0.8.
\end{aligned}
$$

Let Δ be a theory in $\mathcal{L}$ with a sole axiom p. Then Δ has two models, i.e., $\{p, q\}$ and $\{p, \neg q\}$. The *weighted model count* (WMC) [5] of Δ is then

$$
\sum_{\omega \models \Delta} \prod_{\omega \models l} w(l) = w(p)w(q) + w(p)w(\neg q) = 0.3.
$$

The corresponding BA $B(\Delta)$ can then be constructed using Definition 7. Alternatively, one can first construct the free BA generated by the set $\{p, q\}$—this corresponds to $B(\mathcal{L})$ in Fig. 1—and then take a quotient with respect to either the filter generated by p or the ideal[2] generated by $\neg p$. In any case, the resulting BA is pictured at the bottom of Fig. 1.

Each element of $B(\mathcal{L})$ can also be seen as a subset of the set of all models of $\mathcal{L}$, with 0 representing $\emptyset$, 1 representing the set of all (four) models, each atom representing a single model, and each edge going upward representing a subset relation. Thus, the Boolean-algebraic way of calculating the WMC of Δ consists of:

1. Identifying an element $a \in B(\mathcal{L})$ that corresponds to Δ.

2. Finding all atoms of $B(\mathcal{L})$ that are 'dominated' by a according to the partial order.

3. Using w to calculate the weight of each such atom.

4. Adding the weights of these atoms.

This motivates the following definition of WMC generalised to BAs.

> This should be replaced with inner sums (a.k.a. free products)

Definition 8. Let $\mathbf{B}$ be an atomic BA, and let $M \subset \mathbf{B}$ be its set of atoms. Let $L \subset \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge L'$ for some $L' \subseteq L$, and let $w : L \to \mathbb{R}_{\geq 0}$ be arbitrary. The *weighted model count* $\mathrm{WMC}_w : \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$
\mathrm{WMC}_w(x) = \begin{cases} 0 & \text{if } x = 0 \\ \prod_{l \in L'} w(l) & \text{if } M \ni x = \bigwedge L' \\ \sum_{\text{atoms } a \leq x} \mathrm{WMC}_w(a) & \text{otherwise} \end{cases}
$$

for any $x \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{NWMC}_w : \mathbf{B} \to [0, 1]$ as $\mathrm{NWMC}_w(x) = \frac{\mathrm{WMC}_w(x)}{\mathrm{WMC}_w(1)}$ for all $x \in \mathbf{B}$. For both WMC_w and NWMC_w, we will drop the subscript when doing so results in no potential confusion.

Proposition 1. WMC *is a measure, and* NWMC *is a probability measure.*

Proof. First, note that WMC is non-negative and $\mathrm{WMC}(0) = 0$ by definition. Next, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$
\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}
$$

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[2]More details on these concepts can be found in many books on BAs [10, 15].

If, say, $x = 0$, then Eq. (1) becomes

$$\text{WMC}(y) = \text{WMC}(0) + \text{WMC}(y) = \text{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Lemma 2 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\text{WMC}(x \vee y) = \text{WMC}\left(\left(\bigvee_{i \in I} x_i \right) \vee \left(\bigvee_{j \in J} y_j \right) \right) = \sum_{i \in I} \text{WMC}(x_i) + \sum_{j \in J} \text{WMC}(y_j)$$
$$= \text{WMC}(x) + \text{WMC}(y),$$

finishing the proof that WMC is a measure. This immediately shows that NWMC is a probability measure since, by definition, $\text{NWMC}(1) = 1$. $\qquad\square$

Given a theory Δ in a logic $\mathcal{L}$, the usual way of using WMC to compute the probability of a query q is [1, 20]

$$\Pr_{\Delta, w}(q) = \frac{\text{WMC}_w(\Delta \wedge q)}{\text{WMC}_w(\Delta)}.$$

In our algebraic formulation, this can be computed in two different ways:

- as $\frac{\text{WMC}_w(\Delta \wedge q)}{\text{WMC}_w(\Delta)}$ in $B(\mathcal{L})$,

- and as $\text{NWMC}_w([q])$ in $B(\Delta)$.

But how does the measure defined on $B(\mathcal{L})$ transfer to $B(\Delta)$?

Lemma 9. *For any measure $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ and elements $a, b \in \mathbf{B}$,*

$$m(a \vee b) = m(a) + m(b) - m(a \wedge b).$$

Proof. By Definition 3,

$$m(a) = m(a \wedge b) + m(a \wedge \neg b),$$
$$m(b) = m(a \wedge b) + m(\neg a \wedge b),$$
$$m(a \vee b) = m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b),$$

so

$$m(a) + m(b) - m(a \wedge b) = m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b) = m(a \vee b)$$

as required. $\qquad\square$

Lemma 10. *For any $a, b \in \mathbf{B}$ and any principal ideal (p), if $a/(p) = b/(p)$, then $a \vee p = b \vee p$.*

Proof. Note that

$$a/(p) = b/(p) \quad \Longleftrightarrow \quad a + b \in (p) \quad \Longleftrightarrow \quad a + b \leq p \quad \Longleftrightarrow \quad (a+b) \vee p = p$$

by Definitions 4 and 5, and the definition of $\leq$. So $p = (a \wedge \neg b) \vee (\neg a \wedge b) \vee p$, and thus

$$0 = p \wedge \neg p = (a \wedge \neg b \wedge \neg p) \vee (\neg a \wedge b \wedge \neg p) \vee (p \wedge \neg p) = (a \wedge \neg(b \vee p)) \vee (b \wedge \neg(a \vee p)).$$

It follows that

$$a \wedge \neg(b \vee p) = 0 \quad \text{and} \quad b \wedge \neg(a \vee p) = 0.$$

Focusing on the first equation,

$$\neg a = (\neg a \vee a) \wedge [\neg a \vee \neg(b \vee p)] = \neg[a \wedge (b \vee p)],$$

and so $a = a \wedge (b \vee p)$, and

$$a \vee p = (a \vee p) \wedge (b \vee p) = (a \wedge b) \vee p.$$

By similar arguments, $b \vee p = (a \wedge b) \vee p$ as well which shows that $a \vee p = b \vee p$ as required. □

> Outdated. $m(a \wedge \neg p)$ is better than $m(a \vee p)$.

Proposition 2 (Measures on quotients)**.** *Let* $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ *be a measure, and let* (p) *be a principal ideal. Let* $m^* \colon \mathbf{B}/(p) \to \mathbb{R}_{\geq 0}$ *be defined as*

$$m^*(a/(p)) = m(a \vee p)$$

for any $a \in \mathbf{B}$*. The function* m^* *is well-defined. Furthermore, it is a measure on* $\mathbf{B}/(p)$ *if and only if* $m(p) = 0$*. Moreover, if it is a measure, then the following properties transfer from* m *to* m^**:*

- *if* m *is a* probability measure, *then so is* m^*;

- *if* m *is* strictly positive, *then so is* m^*.

Proof. Lemma 10 proves that the function is well-defined. Next, note that

$$m^*(0/(p)) = m(0 \vee p) = m(p),$$

so $m^*(0/(p)) = 0$ if and only if $m(p) = 0$. For the second part of Definition 3, let $a/(p), b/(p) \in \mathbf{B}/(p)$ be such that

$$a/(p) \wedge b/(p) = a \wedge b/(p) = 0/(p).$$

This condition is equivalent to $a \wedge b \in (p)$ and $(a \wedge b) \vee p = p$ by well-known properties of quotients and ideals [10], Definition 4, and the definition of $\leq$, respectively. Now

$$m^*(a/(p) \vee b/(p)) = m^*(a \vee b/(p)) = m(a \vee b \vee p) = m((a \vee p) \vee (b \vee p))$$
$$= m(a \vee p) + m(b \vee p) - m((a \vee p) \wedge (b \vee p))$$
$$= m^*(a/(p)) + m^*(b/(p)) - m((a \vee p) \wedge (b \vee p))$$

by Lemma 9. However

$$(a \vee p) \wedge (b \vee p) = (a \wedge b) \vee p = p,$$

so $m^*(a/(p) \vee b/(p)) = m^*(a/(p)) + m^*(b/(p))$ if and only if $m(p) = 0$.

The two remaining properties are easy to prove:

- If $m(1) = 1$, then $m^*(1/(p)) = m(1 \vee p) = m(1) = 1$.

- Suppose that m is strictly positive, and let $a/(p) \in \mathbf{B}/(p)$ be such that $a/(p) \neq 0/(p)$. Then

$$m^*(a/(p)) = m(a \vee p) \geq m(a) > 0,$$

  where the first inequality comes from an elementary property of $\leq$ that $x \leq x \vee y$ for any $x, y \in \mathbf{B}$ [21] and Lemma 4; and the second inequality follows because $a/(p) \neq 0/(p)$ implies that $a \neq 0$, and m is assumed to be strictly positive.

□

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3.1   Lemma Galore

This section made me realise that I was using the wrong definition

Lemma 11. *Let (p) be a principal ideal. Then for any $a \in \mathbf{B}$, $(a \wedge \neg p)/(p) = a/(p)$.*

Proof. Note that

$$(a \wedge \neg p)/(p) = a/(p) \quad \Longleftrightarrow \quad (a \wedge \neg p) + a \in (p) \quad \Longleftrightarrow \quad (a \wedge \neg p) + a \leq p.$$

We also have that

$$(a \wedge \neg p) + a = (a \wedge \neg p \wedge \neg a) \vee (\neg(a \wedge \neg p) \wedge a) = (\neg a \vee p) \wedge a = (\neg a \wedge a) \vee (p \wedge a) = p \wedge a.$$

And, since $p \wedge a \leq p$, we have that $(a \wedge \neg p) + a \leq p$ as required. □

Lemma 12. *Let (p) be a principal ideal. For any $a, b \in \mathbf{B}$, $a/(p) \leq b/(p)$ if and only if $a \wedge \neg p \leq b \wedge \neg p$.*

Proof. Let us begin with the 'only of' direction. Lemma 11 tells us that $(a \wedge \neg p)/(p) = a/(p)$. Combining this with Lemmas 5 and 6 shows that

$$a \wedge \neg p \leq b \wedge \neg p \quad \Longrightarrow \quad (a \wedge \neg p)/(p) \leq (b \wedge \neg p)/(p) \quad \Longleftrightarrow \quad a/(p) \leq b/(p)$$

as required.

For the other direction, let $a, b \in \mathbf{B}$ be such that $a/(p) \leq b/(p)$. Then, by Lemma 7,

$$[a/(p)] \wedge \neg[b/(p)] = (a \wedge \neg b)/(p) = 0/(p),$$

i.e.,

$$a \wedge \neg b \in (p) \quad \Longleftrightarrow \quad a \wedge \neg b \leq p \quad \Longleftrightarrow \quad a \wedge \neg b \wedge \neg p = 0$$

by Definition 4 and Lemma 7. We need to show that $a \wedge \neg p \leq b \wedge \neg p$. By Lemma 7, this is equivalent to $a \wedge \neg p \wedge \neg(b \wedge \neg p) = 0$. But

$$a \wedge \neg p \wedge \neg(b \wedge \neg p) = a \wedge \neg p \wedge (\neg b \vee p) = (a \wedge \neg p \wedge \neg b) \vee (a \wedge \neg p \wedge p) = a \wedge \neg p \wedge \neg b,$$

and we already have that $a \wedge \neg p \wedge \neg b = 0$ by assumption. □

Lemma 13. *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure, let $p \in \mathbf{B}$ be such that $m(p) = 0$, and let $m^* \colon \mathbf{B}/(p) \to \mathbb{R}_{\geq 0}$ be a measure defined by $m^*(a/(p)) = m(a \vee p)$. Then for any $a \in \mathbf{B}$, if $a/(p)$ is an atom in $\mathbf{B}/(p)$, then $a \wedge \neg p$ is an atom in $\mathbf{B}$ such that $m^*(a/(p)) = m(a \wedge \neg p)$.*

Proof. First, we want to show that if $a/(p)$ is an atom, then $a \wedge \neg p$ is an atom. We can instead prove the contrapositive statement, i.e., if there exists a $b \in \mathbf{B}$ such that $0 < b < a \wedge \neg p$, then there exists a $b' \in \mathbf{B}$ such that $0/(p) < b'/(p) < a/(p)$. We will show that, in fact, we set $b' = b$. Lemmas 5 and 6 already tell us that $b/(p) \leq a/(p)$, so we only need to show that $0/(p) < b/(p) \neq a/(p)$. For the first part, note that

$$0/(p) < b/(p) \quad \Longleftrightarrow \quad b/(p) \neq 0/(p) \quad \Longleftrightarrow \quad b \notin (p) \quad \Longleftrightarrow \quad b \nleq p \quad \Longleftrightarrow \quad b \wedge \neg p \neq 0$$

by Lemma 7. But if $b \wedge \neg p = 0$, then $b \wedge a \wedge \neg p = 0$. This contradicts either that $b \leq a \wedge \neg p$ (i.e., $b \wedge a \wedge \neg p = b$) or that $b \neq 0$. For the second part, i.e., $b/(p) \neq a/(p)$, we will show that if $b/(p) = a/(p)$, and $b \leq a \wedge \neg p$, then $b = a \wedge \neg p$. Indeed,

$$b/(p) = a/(p) \quad \Longleftrightarrow \quad a + b \in (p) \quad \Longleftrightarrow \quad a + b \leq p \quad \Longleftrightarrow \quad (a + b) \wedge \neg p = 0,$$

and

$$(a + b) \wedge \neg p = [(a \wedge \neg b) \vee (\neg a \wedge b)] \wedge \neg p = (a \wedge \neg b \wedge \neg p) \vee (\neg a \wedge b \wedge \neg p),$$

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so $(a + b) \wedge \neg p = 0$ implies that $a \wedge \neg b \wedge \neg p = 0$ which is equivalent to $a \wedge \neg p \leq b$. Therefore we have that $a \wedge \neg p \leq b \leq a \wedge \neg p$, so $b = a \wedge \neg p$ which, by contradiction, shows that $b/(p) \neq a/(p)$ and finishes the proof that $0/(p) < b/(p) < a/(p)$.

In order to show that $m^*(a/(p)) = m(a \wedge \neg p)$, note that $a \wedge p$, $a \wedge \neg p$, and $\neg a \wedge p$ are pairwise disjoint and their supremum is $a \vee p$, so we have that

$$m^*(a/(p)) = m(a \vee p) = m(a \wedge p) + m(a \wedge \neg p) + m(\neg a \wedge p).$$

Furthermore, since $a \wedge p \leq p$, $m(a \wedge p) \leq m(p) = 0$. Similarly, $m(\neg a \wedge p) = 0$, so $m^*(a/(p)) = m(a \wedge \neg p)$ as required. $\qquad\square$

Lemma 14. *Let* $\mathbf{B}$ *be a complete BA. For any* $a, b \in \mathbf{B}$*, if* $a/(p) = b/(p)$*, then* $a \wedge \neg p = b \wedge \neg p$*. As a consequence,* $a \wedge \neg p \leq b$*.*

Proof. As in the proof of Lemma 13, $a/(p) = b/(p)$ implies that $a \wedge \neg p \leq b$. Since $\mathbf{B}$ is complete, let $b = \bigwedge \{c \in \mathbf{B} \mid c/(p) = a/(p)\}$; then we still have that $b/(p) = a/(p)$. But then $b \leq a \wedge \neg p \leq b$, so $a \wedge \neg p = b$. This defines $a \wedge \neg p$ independently of a as the least element in $\{c \in \mathbf{B} \mid c/(p) = a/(p)\}$. $\qquad\square$

Corollary 1. *For any complete BA* $\mathbf{B}$*, if* $a \in \mathbf{B}$ *is an atom, then* $a/(p)$ *is either an atom or* $0/(p)$*. In the former case,* $a = a \wedge \neg p$*.*

Proof. Since Lemma 14 tells us that for all $b \in \mathbf{B}$, if $b/(p) = a/(p)$, then $b \geq a \wedge \neg p$, if there is an atom $b \in \mathbf{B}$ such that $b/(p) = a/(p)$, then it must be $a \wedge \neg p$. If a is an atom, then $a \wedge \neg p \leq a$ implies that either $a = a \wedge \neg p$ or $a \wedge \neg p = 0$. The latter is equivalent to $a/(p) = 0/(p)$ by Lemma 7. The former, combined with the assumption that a is an atom and Lemma 12, implies that $a/(p)$ is an atom. $\qquad\square$

4 What Measures Are WMC-Computable?

Proofs need to be updated and propositions could be phrased in a better way, but the gist should be the same.

4.1 WMC Requires Independent Literals

Proposition 3. *Let* $\mathbf{B}$ *be a finite measure algebra with measure* $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$*. Let* $L \subset \mathbf{B}$ *be defined as*

$$L = \{l_i \mid i \in [n]\} \cup \{\neg l_i \mid i \in [n]\}$$

for some $n \in \mathbb{N}$*. Finally, assume that* $\mathbf{B}$ *has* 2^n *atoms, where each atom* $a \in \mathbf{B}$ *is an infimum*

$$a = \bigwedge_{i=1}^{n} a_i$$

such that $a_i \in \{l_i, \neg l_i\}$ *for* $i \in [n]$*. Then there exists a weight function* $w \colon L \to \mathbb{R}_{\geq 0}$ *that makes* m *a WMC measure if and only if*

$$m(l \wedge l') = m(l)m(l') \tag{2}$$

for all distinct $l, l' \in L$ *such that* $l \neq \neg l'$*.*

Remark. Note that if $n = 1$, then Eq. (2) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. Let us begin with the 'if' part of the statement. Let $w\colon L \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = m(l) \tag{3}$$

for all $l \in L$. We are going to show that NWMC $= m$. First, note that NWMC$(0) = 0 = m(0)$ by the definitions of both NWMC and m. Second, let

$$a = \bigwedge_{i=1}^{n} a_i \tag{4}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\mathrm{NWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)} = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} w(a_i) = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} m(a_i) = \frac{1}{\mathrm{WMC}(1)} m\left(\bigwedge_{i=1}^{n} a_i\right) = \frac{m(a)}{\mathrm{WMC}(1)}$$

by Definition 8 and Eqs. (2) to (4). Now we just need to show that WMC$(1) = 1$. Indeed,

$$\mathrm{WMC}(1) = \sum_{\text{atoms } a \in \mathbf{B}} \mathrm{WMC}(a) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} w(a_i) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} m(a_i)$$

$$= \sum_{\text{atoms } a \in \mathbf{B}} m\left(\bigwedge_{i=1}^{n} a_i\right) = \sum_{\text{atoms } a \in \mathbf{B}} m(a) = m\left(\bigvee_{\text{atoms } a \in \mathbf{B}}\right) = m(1) = 1.$$

Finally, note that if NWMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

For the other direction, we are given a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ that induces a measure $m = \mathrm{NWMC}\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (2) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$. We will first prove an auxiliary result that

$$m(k_i \wedge k_j) = m(k_i) m(k_j) \tag{5}$$

is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j). \tag{6}$$

First, note that k_i can be expressed as

$$k_i = (k_i \wedge k_j) \vee (k_i \wedge \neg k_j)$$

with the condition that

$$(k_i \wedge k_j) \wedge (k_i \wedge \neg k_j) = 0,$$

so, by properties of a measure,

$$m(k_i) = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j). \tag{7}$$

Applying Eq. (7) and the equivalent expression for $m(k_j)$ allows us to rewrite Eq. (5) as

$$m(k_i \wedge k_j) = [m(k_i \wedge k_j) + m(k_i \wedge \neg k_j)] \cdot [m(k_i \wedge k_j) + m(\neg k_i \wedge k_j)]$$

$$= m(k_i \wedge k_j)^2 + m(k_i \wedge k_j)[m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j)] + m(k_i \wedge \neg k_j) m(\neg k_i \wedge k_j)$$

Dividing both sides by $m(k_i \wedge k_j)$ gives

$$1 = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j) + \frac{m(k_i \wedge \neg k_j) m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)}. \tag{8}$$

Since $k_i \wedge k_j \wedge k_i \wedge \neg k_j = 0$, and

$$(k_i \wedge k_j) \vee (k_i \wedge \neg k_j) = k_i \wedge (k_j \vee \neg k_j) = k_i \wedge 1 = k_i,$$

we have that

$$m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) = m(k_i).$$

Similarly, $k_i \wedge \neg k_i \wedge k_j = 0$, and

$$k_i \vee (\neg k_i \wedge k_j) = (k_i \vee \neg k_i) \wedge (k_i \vee k_j) = k_i \vee k_j,$$

so

$$m(k_i) + m(\neg k_i \wedge k_j) = m(k_i \vee k_j).$$

Finally, note that

$$(k_i \vee k_j) \wedge \neg(k_i \vee k_j) = 0,$$

and

$$(k_i \vee k_j) \vee \neg(k_i \vee k_j) = 1,$$

so

$$m(k_i \vee k_j) + m(\neg(k_i \vee k_j)) = m(1) = 1.$$

This allows us to rewrite Eq. (8) as

$$\frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)} = 1 - m(k_i \vee k_j) = m(\neg(k_i \vee k_j)) = m(\neg k_i \wedge \neg k_j)$$

which immediately gives us Eq. (6).

Now recall that $m = \text{NWMC}$ and note that Eq. (6) can be multiplied by $\text{WMC}(1)^2$ to turn the equation into one for WMC instead of NWMC. Then

$$
\begin{aligned}
\text{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \leq k_i \wedge k_j} \text{WMC}(a) = \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\
&= \sum_{\text{atoms } a \leq k_i \wedge k_j} w(a_i)w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \leq k_i \wedge k_j} w(k_i)w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) \\
&= w(k_i)w(k_j) \sum_{\text{atoms } a \leq k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i)w(k_j)C,
\end{aligned}
$$

where C denotes the part of $\text{WMC}(k_i \wedge k_j)$ that will be the same for $\text{WMC}(\neg k_i \wedge k_j)$, $\text{WMC}(k_i \wedge \neg k_j)$, and $\text{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (6) becomes

$$w(k_i)w(k_j)w(\neg k_i)w(\neg k_j)C^2 = w(k_i)w(\neg k_j)w(\neg k_i)w(k_j)C^2$$

which is trivially true. By showing that WMC satisfies Eq. (6), we also showed that it satisfies Eq. (5), finishing the second part of the proof. $\qquad \square$

4.2 Extending the Algebra

A well-known way to overcome this limitation of independence is by adding more literals [5], i.e., extending the set L covered by the WMC weight function $w \colon L \to \mathbb{R}_{\geq 0}$. Let us translate this idea to the language of Boolean algebras.

Theorem 1. *Let* $\mathbf{B}$ *be a finite Boolean algebra freely generated by some set of 'literals' L, and let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be an arbitrary measure. We know that $\mathbf{B}$ has $n = 2^{|L|}$ atoms. Let $(a_i)_{i=1}^n$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i \mid i \in [n]\} \cup \{\neg \phi_i \mid i \in [n]\}$ be the set L extended with $2n$ new literals. Let $\mathbf{B}'$ be the unique Boolean algebra with*

$$\{\phi_i \wedge a_i \mid i \in [n]\} \cup \{\neg \phi_i \wedge a_i \mid i \in [n]\}$$

as its set of atoms. Let $\iota\colon \mathbf{B} \to \mathbf{B}'$ be the inclusion homomorphism (i.e., $\iota(a) = a$ for all $a \in \mathbf{B}$). Let $w\colon L' \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = \begin{cases} \frac{m(a_i)}{2} & \text{if } l = \phi_i \text{ or } l = \neg\phi_i \text{ for some } i \in [n] \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m'\colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In simpler terms, any measure can be computed using WMC by extending the Boolean algebra with more literals. More precisely, we are given the red part in

$$
\begin{array}{ccc}
& \mathbb{R}_{\geq 0} & \\
{\scriptstyle m}\nwarrow & & \nwarrow{\scriptstyle m'} \\
\mathbf{B} & \xrightarrow{\;\iota\;} & \mathbf{B}' \\
\cup & & \cup \\
L & \subset \quad L' & \xrightarrow{\;w\;} \mathbb{R}_{\geq 0}
\end{array}
$$

and construct the black part in such a way that the triangle commutes.

Proof. Since $\mathbf{B}$ is freely generated by L, each atom $a_i \in \mathbf{B}$ is an infimum of elements in L, i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either

$$b = \phi_i \wedge a_i \quad \text{or} \quad b = \neg\phi_i \wedge a_i$$

for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L'. Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies

$$\phi_i \wedge a_i \leq \iota(b)$$

if and only if it satisfies

$$\neg\phi_i \wedge a_i \leq \iota(b).$$

Then, according to the definition of w,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \leq \iota(b) \quad \text{if and only if} \quad a_i \leq b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. $\qquad\square$

Now we can show that the construction in Theorem 1 is smallest possible.

Conjecture 1. *Let* $\mathbf{B}$ *and* $\mathbf{B}'$ *be Boolean algebras, and* $\iota\colon \mathbf{B} \to \mathbf{B}'$ *be the inclusion map such that* $\mathbf{B}$ *is freely generated by* L, *all atoms of* $\mathbf{B}'$ *can be expressed as meets of elements of* L', *and the following subset relations are satisfied:*

$$\mathbf{B} \xrightarrow{\;\iota\;} \mathbf{B}'$$
$$\cup \qquad\quad \cup$$
$$L \quad \subset \quad L'$$

If, for any measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, *one can construct a weight function* $w\colon L' \to \mathbb{R}_{\geq 0}$ *such that the WMC measure* $\mathrm{WMC}\colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$ *with respect to* w *satisfies*

$$m = \mathrm{WMC} \circ \iota,$$

then $|L' \setminus L| \geq 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [6] and Sang et al. [20]. Suppose we have a discrete probability distribution with n variables, and the ith variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^n v_i$ models. My expansion would then use

$$\sum_{i=1}^n v_i + 2\prod_{i=1}^n v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [6] would use

$$\sum_{i=1}^n v_i + \sum_{i=1}^n \prod_{j=1}^i v_j$$

variables, while for the encoding by Sang et al. [20],

$$\sum_{i=1}^n v_i + \sum_{i=1}^n (v_i - 1)\prod_{j=1}^{i-1} v_j$$

variables would suffice.

5 Polyadic Measure Algebras

Potential directions to explore:

- Representing independence and exchangeability. This seems important.

- (More detail below.) Inequalities as bounds for probabilities. This seems to be somewhat explored with other setups.

- Implementation in SageMath. I would need to define prenex normal forms, equality, and lots of other things.

- Alternative compact ways to define a probability distribution over N models without assuming that everything is independent. Declaring a measure on a FO formula defines a linear equation over the probabilities of models, so using this method by itself would require $N - 1$ equations, but maybe combining this with information about independence and exchangeability can help.

We show how the measure on the models of a FO theory can be extended to a measure over FO formulas. Two outcomes:

- For any $p, q \in \mathbf{A}$, if $p \wedge q = 0$, then $m^*(p \vee q) = m^*(p) + m^*(q)$.

- If $p \leq q$, then $m^*(p) \leq m^*(q)$. This can be useful in two situations:

  - If $m^*(q) = 0$ or $m^*(p) = 1$, then this immediately tells us the measure on the other sentence.
  - If we want to find $m^*(p)$, finding $q, r \in \mathbf{A}$ such that $r \leq p \leq q$ bounds the answer.

Definition 9 ([12]). Given two polyadic algebras $\mathbf{A}$ and $\mathbf{B}$, a *polyadic homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a Boolean homomorphism $f \colon \mathbf{A} \to \mathbf{B}$ such that

- $f\mathbf{S}(\tau)p = \mathbf{S}(\tau)fp$,

- $f\exists(J)p = \exists(J)fp$

for all $\tau \in T$, $p \in \mathbf{A}$, and $J \subseteq I$.

5.1 The Set-Up

5.1.1 Preliminaries

What follows is a summary of [12].

Let $\mathbf{B}$ be a Boolean algebra (of propositions). Let X be the (non-empty) domain of discourse. Let I be an index set, elements of which can be interpreted as variables. The elements of X^I are functions from I to X. For any $x \in X^I$ and $i \in I$, we write x_i to represent $x(i) \in X$. Let $\mathbf{A}^*$ be the set of all functions $X^I \to \mathbf{B}$, and note that it forms a Boolean algebra with operations defined pointwise.

Let T be the semigroup of all $I \to I$ transformations. For any $\tau \in T$, let $\tau_* : X^I \to X^I$ be defined by

$$(\tau_* x)_i = x_{\tau i}$$

for all $x \in X^I$ and $i \in I$. For any (Boolean/polyadic) algebra $\mathbf{C}$, let $\mathrm{End}(\mathbf{C})$ denote the set of all its endomorphisms. We can then define $\mathbf{S}$ to be a map $\mathbf{S} : T \to \mathrm{End}(\mathbf{A}^*)$ defined by

$$\mathbf{S}(\tau)p(x) = p(\tau_* x)$$

for all $x \in X^I$ and $p \in \mathbf{A}^*$.

For any $J \subseteq I$, let J_* be the relation on X^I defined by

$$x J_* y \quad \Longleftrightarrow \quad x_i = y_i \quad \text{for all } i \in I \setminus J$$

for all $x, y \in X^I$. For any $J \subseteq I$, we then define $\exists(J)$ to be a transformation $\mathbf{A}^* \to \mathbf{A}^*$ defined by

$$\exists(J)p(x) = \bigvee_{\substack{y \in X^I, \\ x J_* y}} p(y)$$

for all $p \in \mathbf{A}^*$, provided this supremum exists for all $x \in X^{I}$[3].

Finally, a *functional polyadic (Boolean) algebra*[4] is a subalgebra $\mathbf{A}$ of $\mathbf{A}^*$ such that:

- $\mathbf{S}(\tau)p \in \mathbf{A}$ for all $p \in \mathbf{A}$ and $\tau \in T$;

---

[3] The universal quantifier $\forall(J)$ is then defined as $\forall(J)p = \neg(\exists(J)\neg p)$ for all $p \in \mathbf{A}^*$.

[4] To be more explicit, a $\mathbf{B}$-valued functional polyadic algebra with domain X and variables I.

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- $\exists(J)p \in \mathbf{A}$ for all $p \in \mathbf{A}$ and $J \subseteq I$.

Proposition 4. *For all finite $J, K \subseteq I$, finite $\sigma, \tau \in T$, and all $p \in \mathbf{A}$,*

- $\exists(\emptyset)p = p$;

- $\exists(J)\exists(K) = \exists(J \cup K)$;

- $\mathbf{S}(\mathrm{id})p = p$;

- $(\sigma\tau)p = \sigma(\tau p)$;

- *if $\sigma_{|I \setminus J} = \tau_{|I \setminus J}$, then $\sigma\exists(J) = \tau\exists(J)$;*

- *if τ is injective on $\tau^{-1}J$, then $\exists(J)\tau = \tau\exists(\tau^{-1}J)$;*

- *for every $p \in \mathbf{A}$, there exists a finite $J \subseteq I$ such that $\exists(i)p = p$ whenever $i \notin J$.*

Definition 10. Similarly to $\exists$, a *constant c* is a map $c : \mathcal{P}(I) \to \mathrm{End}(\mathbf{A})$ such that:

- $c(\emptyset) = \mathrm{id}_{\mathbf{A}}$;

- $c(J \cup K) = c(J)c(K)$;

- $c(J)\exists(K) = \exists(K)c(J \setminus K)$;

- $\exists(J)c(K) = c(K)\exists(J \setminus K)$;

- $c(J)\mathbf{S}(\tau) = \mathbf{S}(\tau)c(\tau^{-1}J)$

for all $J, K \in \mathcal{P}(I)$ and $\tau \in T$. If J is a singleton set $\{i\}$, we will simply write $c(i)$ instead of $c(J)$.

5.1.2 New Results

Proposition 5. *Let $\mathbf{B}$ be a finite Boolean algebra with a measure $m : \mathbf{B} \to [0,1]$. Let $\mathbf{A}$ be a $\mathbf{B}$-valued functional polyadic algebra with domain X and variables I. Let $m^* : \mathbf{A} \to \mathbb{R}_{\geq 0}$ be defined by*

$$m^*(p) = \sum_{\substack{atoms\ y \in \mathbf{B}\ s.t. \\ \exists x \in X^I : y \leq p(x)}} m(y)$$

for all $p \in \mathbf{A}$. Then m^* is a measure on $\mathbf{A}$.

Remark. While definitions of m^* such as

$$m^*(p) = m\left(\bigvee_{x \in X^I} p(x)\right)$$

might look tempting, they are not additive.

Proof. First, we can show that $m^*(1) = 1$ by observing that

$$m^*(1) = \sum_{atoms\ y \in \mathbf{B}} m(y) = m\left(\bigvee_{atoms\ y \in \mathbf{B}} y\right) = m(1) = 1,$$

where we use Lemma 2 and express $1 \in \mathbf{B}$ as the supremum of all atoms in $\mathbf{B}$ [8]. Clearly $m^*(p) \geq 0$ for all $p \in \mathbf{A}$, so we can restrict the codomain of m^* to $[0,1]$.

Next, we want to show that $m^*(p) > 0$ for all $p \in \mathbf{A} \setminus \{0\}$. If $p \neq 0$, then there must be some $x' \in X^I$ such that $p(x') \neq 0$. But then, since finite Boolean algebras are atomic, there must also be an atom $y \in \mathbf{B}$ such that $y \leq p(x')$. Therefore, $m^*(p) \geq m(y) > 0$, finishing this part of the proof.

Let $p, q \in \mathbf{A}$ be such that $p \wedge q = 0$. We want to show that $m^*(p \vee q) = m^*(p) \vee m^*(q)$. First, note that

$$y \leq (p \vee q)(x) = p(x) \vee q(x)$$

if and only if

$$y = (p(x) \vee q(x)) \wedge y = (p(x) \wedge y) \vee (q(x) \wedge y)$$

by Definition 1. Also note that

$$(p(x) \wedge y) \wedge (q(x) \wedge y) = p(x) \wedge q(x) \wedge y = (p \wedge q)(x) \wedge y = 0 \wedge y = 0,$$

so

$$m(y) = m((p(x) \wedge y) \vee (q(x) \wedge y)) = m(p(x) \wedge y) + m(q(x) \wedge y)$$

by Definition 3 which then leads to

$$m^*(p \vee q) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I: y \leq (p \vee q)(x)}} m(y) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I: y \leq (p \vee q)(x)}} m(p(x) \wedge y) + m(q(x) \wedge y)$$

$$= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I: y \leq (p \vee q)(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I: y \leq (p \vee q)(x)}} m(q(x) \wedge y).$$

Since y is an atom,

$$p(x) \wedge y = \begin{cases} y & \text{if } y \leq p(x) \\ 0 & \text{otherwise}, \end{cases}$$

so

$$m^*(p \vee q) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I: y \leq (p \vee q)(x) \text{ and } y \leq p(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{s.t.} \\ \exists x \in X^I: y \leq (p \vee q)(x) \text{ and } y \leq q(x)}} m(q(x) \wedge y)$$

$$= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I: y \leq p(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{s.t.} \\ \exists x \in X^I: y \leq q(x)}} m(q(x) \wedge y)$$

$$= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I: y \leq p(x)}} m(y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{s.t.} \\ \exists x \in X^I: y \leq q(x)}} m(y) = m^*(p) + m^*(q),$$

finishing the proof that m^* is a measure. $\qquad\square$

Lemma 15. *Given the setup of Proposition 5 and $p \in \mathbf{A}$, if $p(x) = p(y)$ for all $x, y \in X^I$ (i.e., p has no free variables), then*

$$m^*(p) = m(p(x))$$

(for some $x \in X^I$) is an alternative (i.e., equivalent and simpler) definition of m^*.

Proof. Fix some $x \in X^I$. Then

$$m(p(x)) = m\left(\bigvee_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ y \leq p(x)}} y \right) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ y \leq p(x)}} m(y) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x' \in X^I: y \leq p(x')}} m(y) = m^*(p),$$

where we use Lemma 2 for the first step, Definition 3 and Lemma 3 for the second step, the assumptions of Lemma 15 for the third step, and the definition of m^* for the fourth one. $\qquad\square$

Table 1: Example elements of $\mathbf{A}$ as maps $X^I \to \mathbf{B}$, with $a : \mathcal{P}(I) \to \text{End}(\mathbf{A})$ as one of two possible constants.

Element of $\mathbf{A}$	Action on X^I
$p = \mathbf{S}(\text{id})p = \exists(\emptyset)p = a(\emptyset)p = b(\emptyset)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, x_2)$
$\exists(1)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, x_2) \vee \mathsf{p}(b, x_2)$
$\exists(2)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, a) \vee \mathsf{p}(x_1, b)$
$\exists(I)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, a) \vee \mathsf{p}(a, b) \vee \mathsf{p}(b, a) \vee \mathsf{p}(b, b)$
$\mathbf{S}(\{1 \mapsto 1, 2 \mapsto 1\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, x_1)$
$\mathbf{S}(\{1 \mapsto 2, 2 \mapsto 1\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_2, x_1)$
$\mathbf{S}(\{1 \mapsto 2, 2 \mapsto 2\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_2, x_2)$
$a(1)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, x_2)$
$a(2)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, a)$
$a(I)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, a)$

Proposition 6. *The following identities are true for any $p \in \mathbf{A}$, $J \subseteq I$, $\tau \in T$, and constant $c : \mathcal{P}(I) \to \text{End}(A)$:*

- $c\neg = \neg c$ *(i.e., constants commute with negation);*

- $\mathbf{S}(\tau)\neg = \neg\mathbf{S}(\tau)$;

- $\forall(J)p \leq c(J)p \leq \exists(J)p$;

Proof.   • Obvious.

- Ditto.

- The first part can be derived from $p \leq q \implies \neg q \leq \neg p$. The second part comes from the definition of $\exists$. $\qquad\square$

5.2 How Probabilities Are Computed

In order to make the example algebras easily describable, our example programs will have to be tiny. Consider the following ProbLog [19] program:

$$1.0 :: \mathsf{p}(a, b).$$
$$0.5 :: \mathsf{p}(X, X) :- \mathsf{p}(X, Y); \mathsf{p}(Y, X).$$

Let $L = \{\mathsf{p}(a, a), \mathsf{p}(a, b), \mathsf{p}(b, a), \mathsf{p}(b, b)\}$ be the set of all possible ground atoms. Let $\mathbf{B}$ be the Boolean algebra freely generated by L (see, e.g., [10] for more on free Boolean algebras). Then $\mathbf{B}$ will have sixteen atoms ranging from $\mathsf{p}(a, a) \wedge \mathsf{p}(a, b) \wedge \mathsf{p}(b, a) \wedge \mathsf{p}(b, b)$ to $\neg\mathsf{p}(a, a) \wedge \neg\mathsf{p}(a, b) \wedge \neg\mathsf{p}(b, a) \wedge \neg\mathsf{p}(b, b)$. The weight function $w : L \to \mathbb{R}_{\geq 0}$ defined by

$$w(l) = \begin{cases} 1 & \text{if } l = \mathsf{p}(a, b) \\ 0.5 & \text{if } l \in \{\mathsf{p}(a, a), \mathsf{p}(b, b)\} \\ 0 & \text{if } l = \mathsf{p}(b, a) \\ 1 - w(l') & \text{if } l = \neg l' \end{cases}$$

for all $l \in L$ defines a WMC measure over $\mathbf{B}$. Note that while we could define an ideal generated by $\{\mathsf{p}(b, a), \neg\mathsf{p}(a, b)\}$ and take the quotient of $\mathbf{B}$ by that ideal to get a Boolean algebra with a strictly positive measure, this would put zero-probability queries outside of our algebras, i.e., we would not be able to ask a question whose answer is zero.

Table 2: Step-by-step derivation of how a more complex element of **A** acts on elements of X^I

Element of **A**	Action on X^I
p	$(x_1, x_2) \mapsto \mathsf{p}(x_1, x_2)$
$b(2)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, b)$
$\neg b(2)p$	$(x_1, x_2) \mapsto \neg\mathsf{p}(x_1, b)$
$\exists(1)\neg b(2)p$	$(x_1, x_2) \mapsto \neg\mathsf{p}(a, b) \vee \neg\mathsf{p}(b, b) = \neg(\mathsf{p}(a, b) \wedge \mathsf{p}(b, b))$
$\forall(1)b(2)p = \neg\exists(1)\neg b(2)p$	$(x_1, x_2) \mapsto \neg\neg(\mathsf{p}(a, b) \wedge \mathsf{p}(b, b)) = \mathsf{p}(a, b) \wedge \mathsf{p}(b, b)$

Table 3: Atoms $y \in \mathbf{B}$ (and their measures) such that $y \leq \mathsf{p}(a, b) \wedge \mathsf{p}(b, b)$

Atom $y \in \mathbf{B}$	$m(y)$
$\mathsf{p}(a, b) \wedge \mathsf{p}(b, b) \wedge \mathsf{p}(a, a) \wedge \mathsf{p}(b, a)$	$1 \times 0.5 \times 0.5 \times 0 = 0$
$\mathsf{p}(a, b) \wedge \mathsf{p}(b, b) \wedge \neg\mathsf{p}(a, a) \wedge \mathsf{p}(b, a)$	$1 \times 0.5 \times 0.5 \times 0 = 0$
$\mathsf{p}(a, b) \wedge \mathsf{p}(b, b) \wedge \mathsf{p}(a, a) \wedge \neg\mathsf{p}(b, a)$	$1 \times 0.5 \times 0.5 \times 1 = 0.25$
$\mathsf{p}(a, b) \wedge \mathsf{p}(b, b) \wedge \neg\mathsf{p}(a, a) \wedge \neg\mathsf{p}(b, a)$	$1 \times 0.5 \times 0.5 \times 1 = 0.25$

Finally, let **A** be the functional polyadic algebra $X^I \to \mathbf{B}$ for $I = \{1, 2\}$ and $X = \{a, b\}^5$. The elements of X^I can then be represented as tuples (x_1, x_2) for some $x_1, x_2 \in X$. See Table 1 for example elements of **A** which consists of a single predicate function p and operators $\exists, \mathbf{S}, a, b, \neg, \wedge, \vee$, the last three of which are defined pointwise.

Let us calculate the probability $\Pr(\forall x_1 \in X, \mathsf{p}(x_1, b))$. The same expression can be translated into the notation for our polyadic algebra **A** as $m^*(\forall(1)b(2)p)$. Recall that $\forall(1)b(2)p = \neg\exists(1)\neg b(2)p$. The effect of this function on an arbitrary element of X^I is derived step-by-step in Table 2. Since the resulting function is constant (i.e., the logical formula has no free variables), Lemma 15 tells us that

$$m^*(\forall(1)b(2)p) = m(\mathsf{p}(a, b) \wedge \mathsf{p}(b, b)) = m\left(\bigvee_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ y \leq \mathsf{p}(a,b) \wedge \mathsf{p}(b,b)}} y\right) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ y \leq \mathsf{p}(a,b) \wedge \mathsf{p}(b,b)}} m(y).$$

The resulting sum is over four atoms; these atoms and their probabilities are listed in Table 3. Thus, we get that

$$m^*(\forall(1)b(2)p) = 0 + 0 + 0.25 + 0.25 = 0.5.$$

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$^5 X$ cannot (or should not) have constants that do not occur in **B**.

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