Statistical Relational Models as Polyadic Measure Algebras

Paulius Dilkas

2nd March 2020

1 Propositional Logic and Boolean Algebras

1.1 Preliminaries

Definition 1. A *Boolean algebra* is a tuple $(\mathbf{B}, \land, \lor, \neg, 0, 1)$ of a set $\mathbf{B}$ with operations $\land, \lor, \neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\land$ and $\lor$ are associative and commutative;
- $a \lor (a \land b) = a$, and $a \land (a \lor b) = a$;
- 0 is the identity of $\lor$, and 1 is the identity of $\land$;
- $\lor$ distributes over $\land$ and vice versa;
- $a \lor \neg a = 1$, and $a \land \neg a = 0$.

Let $a, b \in \mathbf{B}$ be arbitrary. Let $\leq$ be a partial order on $\mathbf{B}$ defined by $a \leq b$ if $a = b \land a$ (or, equivalently, $a \lor b = b$), and let $a < b$ denote $a \leq b$ and $a \neq b$.

Definition 2 ([5, 6])**.** An element $a \neq 0$ of a Boolean algebra $\mathbf{B}$ is an *atom* if there is no $x \in \mathbf{B}$ such that $0 < x < a$. Equivalently, $a \neq 0$ is an atom if, for all $x \in \mathbf{B}$, either $x \land a = a$ or $x \land a = 0$. A Boolean algebra is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([2])**.** *For any two distinct atoms a, b in a Boolean algebra, $a \land b = 0$.*

Lemma 2 ([3])**.** *All finite Boolean algebras are atomic.*

Theorem 1 ([2])**.** *Let $\mathbf{B}$ be a finite Boolean algebra. Then every $a \in \mathbf{B} \setminus \{0\}$ can be uniquely expressed as $a = \bigvee_{i \in I} m_i$ for some set of atoms $\{m_i\}_{i \in I}$.*

Definition 3 ([1])**.** A *(strictly positive) measure* on a Boolean algebra $\mathbf{B}$ is a function $m : \mathbf{B} \to [0, 1]$ such that:

1. $m(1) = 1$, and $m(x) > 0$ for $x \neq 0$;
2. $m(x \lor y) = m(x) + m(y)$ for all $x, y \in \mathbf{B}$ whenever $x \land y = 0$.

1.2 New Results

Definition 4. Let $\mathbf{B}$ be a finite Boolean algebra, let L be a subset of $\mathbf{B}$ such that every atom m can be uniquely expressed as $m = \bigwedge_{i \in I} l_i$ for some $\{l_i\}_{i \in I} \subseteq L$, and let $w : L \to \mathbb{R}_{>0}$ be arbitrary. The *weighted model count* $\mathrm{WMC} : \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}(a) = \begin{cases} 0 & \text{if } a = 0 \\ \prod_{i \in I} w(l_i) & \text{if } M \ni a = \bigwedge_{i \in I} l_i \\ \sum_{i \in I} \mathrm{WMC}(m_i) & \text{if } \mathbf{B} \setminus (M \cup \{0\}) \ni a = \bigvee_{i \in I} m_i \end{cases}$$

for any $a \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{nWMC} : \mathbf{B} \to [0,1]$ as $\mathrm{nWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)}$ for all $a \in \mathbf{B}$.

Proposition 1. nWMC *is a measure for any finite Boolean algebra* $\mathbf{B}$.

Proof. First, note that Property 1 of Definition 3 is satisfied by the definition of nWMC. Next, in order to prove Property 2, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{nWMC}(x \vee y) = \mathrm{nWMC}(x) + \mathrm{nWMC}(y)$$

which is equivalent to

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Theorem 1 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned} \mathrm{WMC}(x \vee y) &= \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i\right) \vee \left(\bigvee_{j \in J} y_j\right)\right) = \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j) \\ &= \mathrm{WMC}(x) + \mathrm{WMC}(y), \end{aligned}$$

finishing the proof. $\square$

2 First-Order Logic and Polyadic Algebras

2.1 Preliminaries

What follows is a summary of [4].

Let $\mathbf{B}$ be a Boolean algebra (of propositions). Let X be the (non-empty) domain of discourse. Let I be an index set, elements of which can be interpreted as variables. The elements of X^I are functions from I to X. For any $x \in X^I$ and $i \in I$, we write x_i to represent $x(i) \in X$. Let $\mathbf{A}^*$ be the set of all functions $X^I \to \mathbf{B}$, and note that it forms a Boolean algebra with operations defined pointwise.

Let T be the semigroup of all $I \to I$ transformations. For any $\tau \in T$, let $\tau_* : X^I \to X^I$ be defined by

$$(\tau_* x)_i = x_{\tau i}$$

for all $x \in X^I$ and $i \in I$. We can then define $\mathbf{S}$ to be a map from T to Boolean endomorphisms of $\mathbf{A}^*$ defined by

$$\mathbf{S}(\tau)p(x) = p(\tau_* x)$$

for all $x \in X^I$ and $p \in \mathbf{A}^*$.

For any $J \subseteq I$, let J_* be the relation on X^I defined by

$$xJ_*y \iff x_i = y_i \quad \text{for all } i \in I \setminus J$$

for all $x, y \in X^I$. For any $J \subseteq I$, we then define $\exists(J)$ to be a transformation $\mathbf{A}^* \to \mathbf{A}^*$ defined by

$$\exists(J)p(x) = \bigvee_{\substack{y \in X^I, \\ xJ_*y}} p(y)$$

for all $p \in \mathbf{A}^*$, provided this supremum exists for all $x \in X^I$[1].

Finally, a *functional polyadic (Boolean) algebra*[2] is a subalgebra $\mathbf{A}$ of $\mathbf{A}^*$ such that:

- $\mathbf{S}(\tau)p \in \mathbf{A}$ for all $p \in \mathbf{A}$ and $\tau \in T$;
- $\exists(J)p \in \mathbf{A}$ for all $p \in \mathbf{A}$ and $J \subseteq I$.

Definition 5. Similarly to $\exists$, a *constant* c is a mapping from $\mathcal{P}(I)$ to Boolean endomorphisms of $\mathbf{A}$ such that:

- $c(\emptyset) = \mathrm{id}_{\mathbf{A}}$;
- $c(J \cup K) = c(J)c(K)$;
- $c(J)\exists(K) = \exists(K)c(J \setminus K)$;
- $\exists(J)c(K) = c(K)\exists(J \setminus K)$;
- $c(J)\mathbf{S}(\tau) = \mathbf{S}(\tau)c(\tau^{-1}J)$

for all $J, K \in \mathcal{P}(I)$ and $\tau \in T$. If J is a singleton set $\{i\}$, we will simply write $c(i)$ instead of $c(J)$.

2.2 New Results

Proposition 2. *Let $\mathbf{B}$ be a finite Boolean algebra with a measure $m : \mathbf{B} \to [0, 1]$. Let $\mathbf{A}$ be a $\mathbf{B}$-valued functional polyadic algebra with domain X and variables I. Let $m^* : \mathbf{A} \to \mathbb{R}_{\geq 0}$ be defined by*

$$m^*(p) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I :\, y \leq p(x)}} m(y)$$

for all $p \in \mathbf{A}$. Then m^* is a measure on $\mathbf{A}$.

Remark. While definitions of m^* such as

$$m^*(p) = m\left(\bigvee_{x \in X^I} p(x)\right)$$

might look tempting, they are not additive.

Proof. First, we can show that $m^*(1) = 1$ by observing that

$$m^*(1) = \sum_{\text{atoms } y \in \mathbf{B}} m(y) = m\left(\bigvee_{\text{atoms } y \in \mathbf{B}} y\right) = m(1) = 1,$$

[1]The universal quantifier $\forall(J)$ is then defined as $\forall(J)p = \neg(\exists(J)\neg p)$ for all $p \in \mathbf{A}^*$.

[2]To be more explicit, a $\mathbf{B}$-valued functional polyadic algebra with domain X and variables I.

where we use Theorem 1 and express $1 \in \mathbf{B}$ as the supremum of all atoms in $\mathbf{B}$ [2]. Clearly $m^*(p) \geq 0$ for all $p \in \mathbf{A}$, so we can restrict the codomain of m^* to $[0,1]$.

Next, we want to show that $m^*(p) > 0$ for all $p \in \mathbf{A} \setminus \{0\}$. If $p \neq 0$, then there must be some $x' \in X^I$ such that $p(x') \neq 0$. But then, since finite Boolean algebras are atomic, there must also be an atom $y \in \mathbf{B}$ such that $y \leq p(x')$. Therefore, $m^*(p) \geq m(y) > 0$, finishing this part of the proof.

Let $p, q \in \mathbf{A}$ be such that $p \wedge q = 0$. We want to show that $m^*(p \vee q) = m^*(p) \vee m^*(q)$. First, note that

$$y \leq (p \vee q)(x) = p(x) \vee q(x)$$

if and only if

$$y = (p(x) \vee q(x)) \wedge y = (p(x) \wedge y) \vee (q(x) \wedge y)$$

by Definition 1. Also note that

$$(p(x) \wedge y) \wedge (q(x) \wedge y) = p(x) \wedge q(x) \wedge y = (p \wedge q)(x) \wedge y = 0 \wedge y = 0,$$

so

$$m(y) = m((p(x) \wedge y) \vee (q(x) \wedge y)) = m(p(x) \wedge y) + m(q(x) \wedge y)$$

by Definition 3 which then leads to

$$\begin{aligned} m^*(p \vee q) &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(y) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(p(x) \wedge y) + m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(q(x) \wedge y). \end{aligned}$$

Since y is an atom,

$$p(x) \wedge y = \begin{cases} y & \text{if } y \leq p(x) \\ 0 & \text{otherwise,} \end{cases}$$

so

$$\begin{aligned} m^*(p \vee q) = &\sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x) \text{ and } y \leq p(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B}\text{s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x) \text{ and } y \leq q(x)}} m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq p(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B}\text{s.t.} \\ \exists x \in X^I \colon y \leq q(x)}} m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq p(x)}} m(y) + \sum_{\substack{\text{atoms } y \in \mathbf{B}\text{s.t.} \\ \exists x \in X^I \colon y \leq q(x)}} m(y) = m^*(p) + m^*(q), \end{aligned}$$

finishing the proof that m^* is a measure. □

References

[1] Haim Gaifman. Concerning measures on Boolean algebras. *Pacific Journal of Mathematics*, 14(1):61–73, 1964.

[2] M. Ganesh. *Introduction to fuzzy sets and fuzzy logic*. PHI Learning Pvt. Ltd., 2006.

[3] Steven Givant and Paul R. Halmos. *Introduction to Boolean algebras*. Springer Science & Business Media, 2008.

[4] Paul R. Halmos. *Algebraic logic.* Courier Dover Publications, 2016.

[5] Thomas Jech. *Set theory, Second Edition.* Perspectives in Mathematical Logic. Springer, 1997.

[6] Ken Levasseur and Al Doerr. *Applied Discrete Structures.* Lulu.com, 2012.