Weighted Model Counting with Conditional Weights for Bayesian Networks

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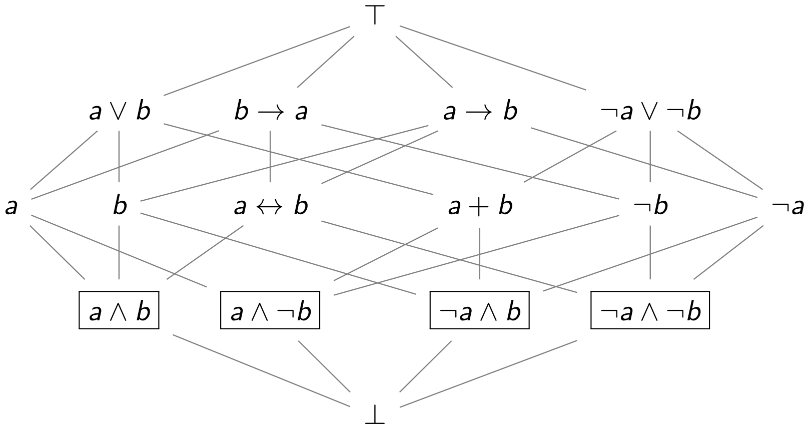
Boolean Algebras and Propositional Logic

Let $U = \{a, b\}$. Then 2^{2^U} is a Boolean algebra with the following Hasse diagram ($x \leq y$ if $x \subseteq y$ or, equivalently, $x = x \wedge y$).



Some Definitions

- A measure is a function $\mu\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ such that:
  - $\mu(\perp) = 0$;
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- A measure μ is factorable if there exists a factored weight function ν that induces μ.

WMC as a Measure on a Boolean Algebra

- Weighted model count (WMC) of a theory Δ, i.e.,

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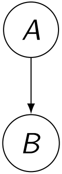
- Traditional workaround: expanding the Boolean algebra.
  - But we don't need to do that!
  - Instead, we can use conditional weight functions in the spirit of conditional probabilities.
  - Intuition: $\Pr(a, b) = \Pr(a) \Pr(b \mid a)$ instead of $\Pr(a, b) = \Pr(a) \Pr(b)$ (when appropriate).

Example: Encoding Bayesian Networks



a	$\Pr(A = a)$
1	0.5
0	0.5

a	b	$\Pr(B = b \mid A = a)$
1	1	0.6
1	0	0.4
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Figure: A Bayesian network with its conditional probability tables

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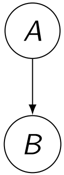

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Figure: A Bayesian network with its conditional probability tables

Let $U = \{\lambda_{A=1}, \lambda_{B=1}\}$. The weight function $\nu \colon 2^U \to \mathbb{R}_{\geq 0}$ for this network can be defined as $\nu := \nu_A \cdot \nu_B$, where $\nu_A = 0.5$, and

$$\begin{aligned}\nu_B = {} & 0.6[\lambda_{B=1}] \cdot [\lambda_{A=1}] + 0.4\overline{[\lambda_{B=1}]} \cdot [\lambda_{A=1}] \\ & + 0.1[\lambda_{B=1}] \cdot \overline{[\lambda_{A=1}]} + 0.9\overline{[\lambda_{B=1}]} \cdot \overline{[\lambda_{A=1}]}.\end{aligned}$$

Experimental Results