Generalising Weighted Model Counting to Boolean Algebras

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1 Introduction

Potential directions to explore (no good ones, really):

- Infinite BAs.
  - For example, the BA of finite and cofinite sets could be interesting.
  - OUWMC requires infinite logics to be compact (whatever that means). My algebraic angle suggests that completeness should be enough. Topologically, compactness implies completeness, but I have no idea how this translates to logics and algebras.
- WMI
- Measures with something other than $\mathbb{R}_{\geq 0}$ as the codomain (doesn't look promising).

Previous/related work:

- Hailperin's approach to probability logic [11]
- Nilsson's (somewhat successful) probabilistic logic [16]
- Logical induction: a big paper with a good overview of previous attempts to assign probabilities to logical sentences in a sensible way [9]
- Semiring programming [3]
- WMI [2]
- Measures on Boolean algebras
  - On possibility and probability measures in finite Boolean algebras [4]
  - Representation of conditional probability measures [14]

2 Preliminaries

Definition 1. A *Boolean algebra* (BA) is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ consisting of a set $\mathbf{B}$ with binary operations *meet* $\wedge$ and *join* $\vee$, unary operation $\neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;
- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;

- $\vee$ distributes over $\wedge$ and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

For clarity and succinctness, we will occasionally use three other operations that can be defined using the original three[1]:

$$\begin{aligned} a \to b &= \neg a \vee b, \\ a \leftrightarrow b &= (a \wedge b) \vee (\neg a \wedge \neg b), \\ a + b &= (a \wedge \neg b) \vee (\neg a \wedge b). \end{aligned}$$

We can also define a partial order $\leq$ on $\mathbf{B}$ as $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$) for $a, b \in \mathbf{B}$. Furthermore, let $a < b$ denote $a \leq b$ and $a \neq b$. For the rest of this paper, let $\mathbf{B}$ refer to the BA $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$. For any $S \subseteq \mathbf{B}$, we write $\bigvee S$ for $\bigvee_{x \in S} x$ and call it the *supremum* of S. Similarly, $\bigwedge S = \bigwedge_{x \in S} x$ is the *infimum*. By convention, $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$. For any $a, b \in \mathbf{B}$, we say that a and b are *disjoint* if $a \wedge b = 0$.

Definition 2 ([12, 15])**.** An element $a \neq 0$ of $\mathbf{B}$ is an *atom* if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. Equivalently, $a \neq 0$ is an atom if there is no $x \in \mathbf{B}$ such that $0 < x < a$. We say that $\mathbf{B}$ is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([8])**.** *For any two distinct atoms a, $b \in \mathbf{B}$, $a \wedge b = 0$.*

Lemma 2 ([10])**.** *The following are equivalent:*

- $\mathbf{B}$ *is atomic.*
- *For any $x \in \mathbf{B}$,*
$$x = \bigvee_{\text{atoms } a \leq x} a.$$
- 1 *is the supremum of all atoms.*

Lemma 3 ([10])**.** *All finite BAs are atomic.*

Definition 3 ([7, 12])**.** A *measure* on $\mathbf{B}$ is a function $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ such that:

- $m(0) = 0$;
- $m(a \vee b) = m(a) + m(b)$ for all $a, b \in \mathbf{B}$ whenever $a \wedge b = 0$.

If $m(1) = 1$, we call m a *probability measure*. Also, if $m(x) > 0$ for all $x \neq 0$, then m is *strictly positive*.

Definition 4 ([10])**.** An *ideal* is a non-empty subset $I \subseteq \mathbf{B}$ such that

- $i \vee j \in I$ for all $i, j \in I$;
- $i \wedge a \in I$ for all $i \in I$ and $a \in \mathbf{B}$.

For any $p \in \mathbf{B}$, the *principal ideal of p*—denoted by (p)—is the smallest ideal that contains p. It can also be expressed as $(p) = \{a \in \mathbf{B} \mid a \leq p\}$.

Definition 5 ([10])**.** Let I be an ideal in $\mathbf{B}$. The *quotient algebra of* $\mathbf{B}$ *modulo the ideal* I $\mathbf{B}/I$ is a BA of equivalence classes of elements of $\mathbf{B}$ with respect to the equivalence relation

$$a \sim b \quad \iff \quad a + b \in I$$

[1] We use $+$ to denote symmetric difference because it is the additive operation of a Boolean ring.

for all $a, b \in \mathbf{B}$. Elements of $\mathbf{B}/I$ are usually denoted by a/I (for some $a \in \mathbf{B}$) with the understanding that if $b \sim a$ (for some $b \in \mathbf{B}$), then $b/I = a/I$. The three algebraic operations on $\mathbf{B}/I$ are defined as

$$\begin{aligned} a/I \wedge b/I &= a \wedge b/I, \\ a/I \vee b/I &= a \vee b/I, \\ \neg(a/I) &= (\neg a)/I. \end{aligned}$$

Definition 6 ([10])**.** Let $\mathbf{A}$ and $\mathbf{B}$ be BAs. A *(Boolean) homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $f \colon \mathbf{A} \to \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,
- $f(x \vee y) = f(x) \vee f(y)$,
- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Lemma 4 ([10])**.** *Let $I \subseteq \mathbf{B}$ be an ideal. The map $f \colon \mathbf{B} \to \mathbf{B}/I$ defined by $f(x) = x/I$ is a homomorphism.*

Lemma 5 (Homomorphisms preserve order [10])**.** *Let $f \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism between two BAs $\mathbf{A}$ and $\mathbf{B}$. Then, for any $x, y \in \mathbf{A}$, if $x \leq y$, then $f(x) \leq f(y)$.*

Lemma 6 ([18])**.** *For any $a, b \in \mathbf{B}$, $a \leq b$ if and only if $a \wedge \neg b = 0$.*

Lemma 7 ([10])**.** *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure. Then for all $a, b \in \mathbf{B}$, if $a \leq b$, then $m(a) \leq m(b)$.*

3 WMC as a Measure

Definition 7. Let $\mathcal{L}$ be a propositional (or first-order) logic, and let Δ be a theory in $\mathcal{L}$. We can define an equivalence relation on formulas in $\mathcal{L}$ as

$$\alpha \sim \beta \quad \text{if and only if} \quad \Delta \vdash \alpha \leftrightarrow \beta$$

for all $\alpha, \beta \in \mathcal{L}$. Let $[\alpha]$ denote the equivalence class of $\alpha \in \mathcal{L}$ with respect to $\sim$. We can then let $B(\Delta) = \{[\alpha] \mid \alpha \in \mathcal{L}\}$ and define the structure of a BA on $B(\Delta)$ as

$$\begin{aligned} [\alpha] \vee [\beta] &= [\alpha \vee \beta], \\ [\alpha] \wedge [\beta] &= [\alpha \wedge \beta], \\ \neg[\alpha] &= [\neg\alpha], \\ 1 &= [\alpha \to \alpha], \\ 0 &= [\alpha \wedge \neg\alpha] \end{aligned}$$

for all $\alpha, \beta \in \mathcal{L}$. Then $B(\Delta)$ is the *Lindenbaum-Tarski algebra* of Δ [13, 19].

Example 1. Let $\mathcal{L}$ be a propositional logic with p and q as its only atoms. Then $L = \{p, q, \neg p, \neg q\}$ is its set of literals. Let $w : L \to \mathbb{R}_{\geq 0}$ be the *weight function* defined by

$$\begin{aligned} w(p) &= 0.3, \\ w(\neg p) &= 0.7, \\ w(q) &= 0.2, \\ w(\neg q) &= 0.8. \end{aligned}$$

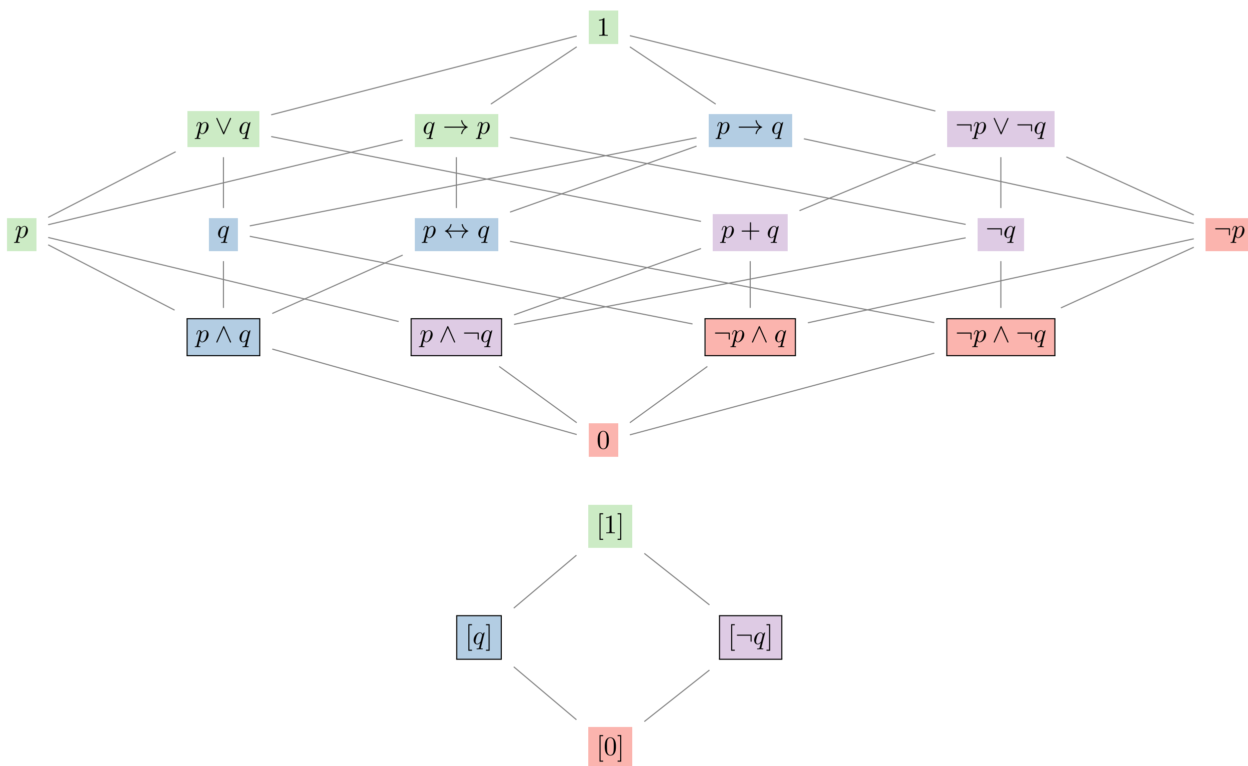

Figure 1: Two BAs from Example 1: $B(\mathcal{L})$ at the top and $B(\Delta)$ at the bottom. An edge between elements a and b (with a positioned lower than b) means that $a < b$. Each element of $B(\Delta)$ is an equivalence class of elements of $B(\mathcal{L})$, and the colours show which elements of $B(\mathcal{L})$ belong to which class. In both algebras, atoms have borders around them.

Let Δ be a theory in $\mathcal{L}$ with a sole axiom p. Then Δ has two models, i.e., $\{p, q\}$ and $\{p, \neg q\}$. The *weighted model count* (WMC) [5] of Δ is then

$$\sum_{\omega \models \Delta} \prod_{\omega \models l} w(l) = w(p)w(q) + w(p)w(\neg q) = 0.3.$$

The corresponding BA $B(\Delta)$ can then be constructed using Definition 7. Alternatively, one can first construct the free BA generated by the set $\{p, q\}$—this corresponds to $B(\mathcal{L})$ in Fig. 1—and then take a quotient with respect to either the filter generated by p or the ideal[2] generated by $\neg p$. In any case, the resulting BA is pictured at the bottom of Fig. 1.

Each element of $B(\mathcal{L})$ can also be seen as a subset of the set of all models of $\mathcal{L}$, with 0 representing $\emptyset$, 1 representing the set of all (four) models, each atom representing a single model, and each edge going upward representing a subset relation. Thus, the Boolean-algebraic way of calculating the WMC of Δ consists of:

1. Identifying an element $a \in B(\mathcal{L})$ that corresponds to Δ.
2. Finding all atoms of $B(\mathcal{L})$ that are 'dominated' by a according to the partial order.
3. Using w to calculate the weight of each such atom.
4. Adding the weights of these atoms.

This motivates the following definition of WMC generalised to BAs.

This should be replaced with inner sums (a.k.a. free products)

Definition 8. Let $\mathbf{B}$ be an atomic BA, and let $M \subset \mathbf{B}$ be its set of atoms. Let $L \subset \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge L'$ for some $L' \subseteq L$, and let $w \colon L \to \mathbb{R}_{\geq 0}$ be arbitrary. The *weighted model count* $\mathrm{WMC}_w \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}_w(x) = \begin{cases} 0 & \text{if } x = 0 \\ \prod_{l \in L'} w(l) & \text{if } M \ni x = \bigwedge L' \\ \sum_{\text{atoms } a \leq x} \mathrm{WMC}_w(a) & \text{otherwise} \end{cases}$$

for any $x \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{NWMC}_w \colon \mathbf{B} \to [0, 1]$ as $\mathrm{NWMC}_w(x) = \frac{\mathrm{WMC}_w(x)}{\mathrm{WMC}_w(1)}$ for all $x \in \mathbf{B}$. For both WMC_w and NWMC_w, we will drop the subscript when doing so results in no potential confusion.

Theorem 1. WMC *is a measure, and* NWMC *is a probability measure.*

Proof. First, note that WMC is non-negative and $\mathrm{WMC}(0) = 0$ by definition. Next, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Lemma 2 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

[2] More details on these concepts can be found in many books on BAs [10, 13].

for some sequences of atoms $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i\in I} \bigvee_{j\in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned}\mathrm{WMC}(x \vee y) = \mathrm{WMC}\left(\left(\bigvee_{i\in I} x_i\right) \vee \left(\bigvee_{j\in J} y_j\right)\right) &= \sum_{i\in I} \mathrm{WMC}(x_i) + \sum_{j\in J} \mathrm{WMC}(y_j) \\ &= \mathrm{WMC}(x) + \mathrm{WMC}(y),\end{aligned}$$

finishing the proof that WMC is a measure. This immediately shows that NWMC is a probability measure since, by definition, $\mathrm{NWMC}(1) = 1$. □

Given a theory Δ in a logic $\mathcal{L}$, the usual way of using WMC to compute the probability of a query q is [1, 17]

$$\Pr_{\Delta,w}(q) = \frac{\mathrm{WMC}_w(\Delta \wedge q)}{\mathrm{WMC}_w(\Delta)}.$$

In our algebraic formulation, this can be computed in two different ways:

- as $\frac{\mathrm{WMC}_w(\Delta\wedge q)}{\mathrm{WMC}_w(\Delta)}$ in $B(\mathcal{L})$,
- and as $\mathrm{NWMC}_w([q])$ in $B(\Delta)$.

But how does the measure defined on $B(\mathcal{L})$ transfer to $B(\Delta)$?

Lemma 8. *For any measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ and elements $a, b \in \mathbf{B}$,*

$$m(a \vee b) = m(a) + m(b) - m(a \wedge b).$$

Proof. By Definition 3,

$$\begin{aligned} m(a) &= m(a \wedge b) + m(a \wedge \neg b), \\ m(b) &= m(a \wedge b) + m(\neg a \wedge b), \\ m(a \vee b) &= m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b), \end{aligned}$$

so

$$m(a) + m(b) - m(a \wedge b) = m(a \wedge b) + m(a \wedge \neg b) + m(\neg a \wedge b) = m(a \vee b)$$

as required. □

Lemma 9. *For any $a, b \in \mathbf{B}$ and any principal ideal (p), if $a/(p) = b/(p)$, then $a \vee p = b \vee p$.*

Proof. Note that

$$a/(p) = b/(p) \iff a + b \in (p) \iff a + b \leq p \iff (a + b) \vee p = p$$

by Definitions 4 and 5, and the definition of $\leq$. So $p = (a \wedge \neg b) \vee (\neg a \wedge b) \vee p$, and thus

$$0 = p \wedge \neg p = (a \wedge \neg b \wedge \neg p) \vee (\neg a \wedge b \wedge \neg p) \vee (p \wedge \neg p) = (a \wedge \neg(b \vee p)) \vee (b \wedge \neg(a \vee p)).$$

It follows that

$$a \wedge \neg(b \vee p) = 0 \quad \text{and} \quad b \wedge \neg(a \vee p) = 0.$$

Focusing on the first equation,

$$\neg a = (\neg a \vee a) \wedge [\neg a \vee \neg(b \vee p)] = \neg[a \wedge (b \vee p)],$$

and so $a = a \wedge (b \vee p)$, and

$$a \vee p = (a \vee p) \wedge (b \vee p) = (a \wedge b) \vee p.$$

By similar arguments, $b \vee p = (a \wedge b) \vee p$ as well which shows that $a \vee p = b \vee p$ as required. □

Outdated. $m(a \wedge \neg p)$ is better than $m(a \vee p)$.

Proposition 1 (Measures on quotients)**.** *Let $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure, and let (p) be a principal ideal. Let $m^*\colon \mathbf{B}/(p) \to \mathbb{R}_{\geq 0}$ be defined as*

$$m^*(a/(p)) = m(a \vee p)$$

for any $a \in \mathbf{B}$. The function m^* is well-defined. Furthermore, it is a measure on $\mathbf{B}/(p)$ if and only if $m(p) = 0$. Moreover, if it is a measure, then the following properties transfer from m to m^*:

- *if m is a* probability measure*, then so is m^*;*
- *if m is* strictly positive*, then so is m^*.*

Proof. Lemma 9 proves that the function is well-defined. Next, note that

$$m^*(0/(p)) = m(0 \vee p) = m(p),$$

so $m^*(0/(p)) = 0$ if and only if $m(p) = 0$. For the second part of Definition 3, let $a/(p), b/(p) \in \mathbf{B}/(p)$ be such that

$$a/(p) \wedge b/(p) = a \wedge b/(p) = 0/(p).$$

This condition is equivalent to $a \wedge b \in (p)$ and $(a \wedge b) \vee p = p$ by well-known properties of quotients and ideals [10], Definition 4, and the definition of $\leq$, respectively. Now

$$\begin{aligned} m^*(a/(p) \vee b/(p)) = m^*(a \vee b/(p)) &= m(a \vee b \vee p) = m((a \vee p) \vee (b \vee p)) \\ &= m(a \vee p) + m(b \vee p) - m((a \vee p) \wedge (b \vee p)) \\ &= m^*(a/(p)) + m^*(b/(p)) - m((a \vee p) \wedge (b \vee p)) \end{aligned}$$

by Lemma 8. However

$$(a \vee p) \wedge (b \vee p) = (a \wedge b) \vee p = p,$$

so $m^*(a/(p) \vee b/(p)) = m^*(a/(p)) + m^*(b/(p))$ if and only if $m(p) = 0$.

The two remaining properties are easy to prove:

- If $m(1) = 1$, then $m^*(1/(p)) = m(1 \vee p) = m(1) = 1$.
- Suppose that m is strictly positive, and let $a/(p) \in \mathbf{B}/(p)$ be such that $a/(p) \neq 0/(p)$. Then

  $$m^*(a/(p)) = m(a \vee p) \geq m(a) > 0,$$

  where the first inequality comes from an elementary property of $\leq$ that $x \leq x \vee y$ for any $x, y \in \mathbf{B}$ [18] and Lemma 7; and the second inequality follows because $a/(p) \neq 0/(p)$ implies that $a \neq 0$, and m is assumed to be strictly positive.

□

3.1 Lemma Galore

This section made me realise that I was using the wrong definition

Lemma 10. *Let (p) be a principal ideal. Then for any $a \in \mathbf{B}$, $(a \wedge \neg p)/(p) = a/(p)$.*

Proof. Note that

$$(a \wedge \neg p)/(p) = a/(p) \quad \iff \quad (a \wedge \neg p) + a \in (p) \quad \iff \quad (a \wedge \neg p) + a \leq p.$$

We also have that

$$(a \wedge \neg p) + a = (a \wedge \neg p \wedge \neg a) \vee (\neg(a \wedge \neg p) \wedge a) = (\neg a \vee p) \wedge a = (\neg a \wedge a) \vee (p \wedge a) = p \wedge a.$$

And, since $p \wedge a \leq p$, we have that $(a \wedge \neg p) + a \leq p$ as required. □

Lemma 11. *Let (p) be a principal ideal. For any $a, b \in \mathbf{B}$, $a/(p) \leq b/(p)$ if and only if $a \wedge \neg p \leq b \wedge \neg p$.*

Proof. Let us begin with the 'only of' direction. Lemma 10 tells us that $(a \wedge \neg p)/(p) = a/(p)$. Combining this with Lemmas 4 and 5 shows that

$$a \wedge \neg p \leq b \wedge \neg p \quad \implies \quad (a \wedge \neg p)/(p) \leq (b \wedge \neg p)/(p) \quad \iff \quad a/(p) \leq b/(p)$$

as required.

For the other direction, let $a, b \in \mathbf{B}$ be such that $a/(p) \leq b/(p)$. Then, by Lemma 6,

$$[a/(p)] \wedge \neg[b/(p)] = (a \wedge \neg b)/(p) = 0/(p),$$

i.e.,

$$a \wedge \neg b \in (p) \quad \iff \quad a \wedge \neg b \leq p \quad \iff \quad a \wedge \neg b \wedge \neg p = 0$$

by Definition 4 and Lemma 6. We need to show that $a \wedge \neg p \leq b \wedge \neg p$. By Lemma 6, this is equivalent to $a \wedge \neg p \wedge \neg(b \wedge \neg p) = 0$. But

$$a \wedge \neg p \wedge \neg(b \wedge \neg p) = a \wedge \neg p \wedge (\neg b \vee p) = (a \wedge \neg p \wedge \neg b) \vee (a \wedge \neg p \wedge p) = a \wedge \neg p \wedge \neg b,$$

and we already have that $a \wedge \neg p \wedge \neg b = 0$ by assumption. $\square$

Lemma 12. *Let $m \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$ be a measure, let $p \in \mathbf{B}$ be such that $m(p) = 0$, and let $m^* \colon \mathbf{B}/(p) \to \mathbb{R}_{\geq 0}$ be a measure defined by $m^*(a/(p)) = m(a \vee p)$. Then for any $a \in \mathbf{B}$, if $a/(p)$ is an atom in $\mathbf{B}/(p)$, then $a \wedge \neg p$ is an atom in $\mathbf{B}$ such that $m^*(a/(p)) = m(a \wedge \neg p)$.*

Proof. First, we want to show that if $a/(p)$ is an atom, then $a \wedge \neg p$ is an atom. We can instead prove the contrapositive statement, i.e., if there exists a $b \in \mathbf{B}$ such that $0 < b < a \wedge \neg p$, then there exists a $b' \in \mathbf{B}$ such that $0/(p) < b'/(p) < a/(p)$. We will show that, in fact, we set $b' = b$. Lemmas 4 and 5 already tell us that $b/(p) \leq a/(p)$, so we only need to show that $0/(p) < b/(p) \neq a/(p)$. For the first part, note that

$$0/(p) < b/(p) \quad \iff \quad b/(p) \neq 0/(p) \quad \iff \quad b \notin (p) \quad \iff \quad b \not\leq p \quad \iff \quad b \wedge \neg p \neq 0$$

by Lemma 6. But if $b \wedge \neg p = 0$, then $b \wedge a \wedge \neg p = 0$. This contradicts either that $b \leq a \wedge \neg p$ (i.e., $b \wedge a \wedge \neg p = b$) or that $b \neq 0$. For the second part, i.e., $b/(p) \neq a/(p)$, we will show that if $b/(p) = a/(p)$, and $b \leq a \wedge \neg p$, then $b = a \wedge \neg p$. Indeed,

$$b/(p) = a/(p) \quad \iff \quad a + b \in (p) \quad \iff \quad a + b \leq p \quad \iff \quad (a + b) \wedge \neg p = 0,$$

and

$$(a + b) \wedge \neg p = [(a \wedge \neg b) \vee (\neg a \wedge b)] \wedge \neg p = (a \wedge \neg b \wedge \neg p) \vee (\neg a \wedge b \wedge \neg p),$$

so $(a + b) \wedge \neg p = 0$ implies that $a \wedge \neg b \wedge \neg p = 0$ which is equivalent to $a \wedge \neg p \leq b$. Therefore we have that $a \wedge \neg p \leq b \leq a \wedge \neg p$, so $b = a \wedge \neg p$ which, by contradiction, shows that $b/(p) \neq a/(p)$ and finishes the proof that $0/(p) < b/(p) < a/(p)$.

In order to show that $m^*(a/(p)) = m(a \wedge \neg p)$, note that $a \wedge p$, $a \wedge \neg p$, and $\neg a \wedge p$ are pairwise disjoint and their supremum is $a \vee p$, so we have that

$$m^*(a/(p)) = m(a \vee p) = m(a \wedge p) + m(a \wedge \neg p) + m(\neg a \wedge p).$$

Furthermore, since $a \wedge p \leq p$, $m(a \wedge p) \leq m(p) = 0$. Similarly, $m(\neg a \wedge p) = 0$, so $m^*(a/(p)) = m(a \wedge \neg p)$ as required. $\square$

Lemma 13. *Let $\mathbf{B}$ be a complete BA. For any $a, b \in \mathbf{B}$, if $a/(p) = b/(p)$, then $a \wedge \neg p = b \wedge \neg p$. As a consequence, $a \wedge \neg p \leq b$.*

Proof. As in the proof of Lemma 12, $a/(p) = b/(p)$ implies that $a \wedge \neg p \leq b$. Since $\mathbf{B}$ is complete, let $b = \bigwedge\{c \in \mathbf{B} \mid c/(p) = a/(p)\}$; then we still have that $b/(p) = a/(p)$. But then $b \leq a \wedge \neg p \leq b$, so $a \wedge \neg p = b$. This defines $a \wedge \neg p$ independently of a as the least element in $\{c \in \mathbf{B} \mid c/(p) = a/(p)\}$. □

Corollary 1. *For any complete BA $\mathbf{B}$, if $a \in \mathbf{B}$ is an atom, then $a/(p)$ is either an atom or $0/(p)$. In the former case, $a = a \wedge \neg p$.*

Proof. Since Lemma 13 tells us that for all $b \in \mathbf{B}$, if $b/(p) = a/(p)$, then $b \geq a \wedge \neg p$, if there is an atom $b \in \mathbf{B}$ such that $b/(p) = a/(p)$, then it must be $a \wedge \neg p$. If a is an atom, then $a \wedge \neg p \leq a$ implies that either $a = a \wedge \neg p$ or $a \wedge \neg p = 0$. The latter is equivalent to $a/(p) = 0/(p)$ by Lemma 6. The former, combined with the assumption that a is an atom and Lemma 11, implies that $a/(p)$ is an atom. □

4 What Measures Are WMC-Computable?

Proofs need to be updated and propositions could be phrased in a better way, but the gist should be the same.

4.1 WMC Requires Independent Literals

Theorem 2. *Let $\mathbf{B}$ be a finite measure algebra with measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$. Let $L \subset \mathbf{B}$ be defined as*

$$L = \{l_i \mid i \in [n]\} \cup \{\neg l_i \mid i \in [n]\}$$

for some $n \in \mathbb{N}$. Finally, assume that $\mathbf{B}$ has 2^n atoms, where each atom $a \in \mathbf{B}$ is an infimum

$$a = \bigwedge_{i=1}^{n} a_i$$

such that $a_i \in \{l_i, \neg l_i\}$ for $i \in [n]$. Then there exists a weight function $w\colon L \to \mathbb{R}_{\geq 0}$ that makes m a WMC measure if and only if

$$m(l \wedge l') = m(l)m(l') \tag{2}$$

for all distinct $l, l' \in L$ such that $l \neq \neg l'$.

Remark. Note that if $n = 1$, then Eq. (2) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. Let us begin with the 'if' part of the statement. Let $w\colon L \to \mathbb{R}_{\geq 0}$ be defined by

$$w(l) = m(l) \tag{3}$$

for all $l \in L$. We are going to show that $\mathrm{NWMC} = m$. First, note that $\mathrm{NWMC}(0) = 0 = m(0)$ by the definitions of both NWMC and m. Second, let

$$a = \bigwedge_{i=1}^{n} a_i \tag{4}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\mathrm{NWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)} = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} w(a_i) = \frac{1}{\mathrm{WMC}(1)} \prod_{i=1}^{n} m(a_i) = \frac{1}{\mathrm{WMC}(1)} m\left(\bigwedge_{i=1}^{n} a_i\right) = \frac{m(a)}{\mathrm{WMC}(1)}$$

by Definition 8 and Eqs. (2) to (4). Now we just need to show that $\mathrm{WMC}(1) = 1$. Indeed,

$$\begin{aligned}\mathrm{WMC}(1) &= \sum_{\text{atoms } a \in \mathbf{B}} \mathrm{WMC}(a) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} w(a_i) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} m(a_i) \\ &= \sum_{\text{atoms } a \in \mathbf{B}} m\left(\bigwedge_{i=1}^{n} a_i\right) = \sum_{\text{atoms } a \in \mathbf{B}} m(a) = m\left(\bigvee_{\text{atoms } a \in \mathbf{B}}\right) = m(1) = 1.\end{aligned}$$

Finally, note that if NWMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

For the other direction, we are given a weight function $w \colon L \to \mathbb{R}_{\geq 0}$ that induces a measure $m = \mathrm{NWMC} \colon \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (2) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$. We will first prove an auxiliary result that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \tag{5}$$

is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j). \tag{6}$$

First, note that k_i can be expressed as

$$k_i = (k_i \wedge k_j) \vee (k_i \wedge \neg k_j)$$

with the condition that

$$(k_i \wedge k_j) \wedge (k_i \wedge \neg k_j) = 0,$$

so, by properties of a measure,

$$m(k_i) = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j). \tag{7}$$

Applying Eq. (7) and the equivalent expression for $m(k_j)$ allows us to rewrite Eq. (5) as

$$\begin{aligned}m(k_i \wedge k_j) &= [m(k_i \wedge k_j) + m(k_i \wedge \neg k_j)] \cdot [m(k_i \wedge k_j) + m(\neg k_i \wedge k_j)] \\ &= m(k_i \wedge k_j)^2 + m(k_i \wedge k_j)[m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j)] + m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)\end{aligned}$$

Dividing both sides by $m(k_i \wedge k_j)$ gives

$$1 = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j) + \frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)}. \tag{8}$$

Since $k_i \wedge k_j \wedge k_i \wedge \neg k_j = 0$, and

$$(k_i \wedge k_j) \vee (k_i \wedge \neg k_j) = k_i \wedge (k_j \vee \neg k_j) = k_i \wedge 1 = k_i,$$

we have that

$$m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) = m(k_i).$$

Similarly, $k_i \wedge \neg k_i \wedge k_j = 0$, and

$$k_i \vee (\neg k_i \wedge k_j) = (k_i \vee \neg k_i) \wedge (k_i \vee k_j) = k_i \vee k_j,$$

so

$$m(k_i) + m(\neg k_i \wedge k_j) = m(k_i \vee k_j).$$

Finally, note that

$$(k_i \vee k_j) \wedge \neg(k_i \vee k_j) = 0,$$

and

$$(k_i \vee k_j) \vee \neg(k_i \vee k_j) = 1,$$

so

$$m(k_i \vee k_j) + m(\neg(k_i \vee k_j)) = m(1) = 1.$$

This allows us to rewrite Eq. (8) as

$$\frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)} = 1 - m(k_i \vee k_j) = m(\neg(k_i \vee k_j)) = m(\neg k_i \wedge \neg k_j)$$

which immediately gives us Eq. (6).

Now recall that $m = \mathrm{NWMC}$ and note that Eq. (6) can be multiplied by $\mathrm{WMC}(1)^2$ to turn the equation into one for WMC instead of NWMC. Then

$$\begin{aligned}
\mathrm{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \le k_i \wedge k_j} \mathrm{WMC}(a) = \sum_{\text{atoms } a \le k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\
&= \sum_{\text{atoms } a \le k_i \wedge k_j} w(a_i)w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \le k_i \wedge k_j} w(k_i)w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) \\
&= w(k_i)w(k_j) \sum_{\text{atoms } a \le k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i)w(k_j)C,
\end{aligned}$$

where C denotes the part of $\mathrm{WMC}(k_i \wedge k_j)$ that will be the same for $\mathrm{WMC}(\neg k_i \wedge k_j)$, $\mathrm{WMC}(k_i \wedge \neg k_j)$, and $\mathrm{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (6) becomes

$$w(k_i)w(k_j)w(\neg k_i)w(\neg k_j)C^2 = w(k_i)w(\neg k_j)w(\neg k_i)w(k_j)C^2$$

which is trivially true. By showing that WMC satisfies Eq. (6), we also showed that it satisfies Eq. (5), finishing the second part of the proof. □

4.2 Extending the Algebra

A well-known way to overcome this limitation of independence is by adding more literals [5], i.e., extending the set L covered by the WMC weight function $w\colon L \to \mathbb{R}_{\ge 0}$. Let us translate this idea to the language of Boolean algebras.

Theorem 3. *Let $\mathbf{B}$ be a finite Boolean algebra freely generated by some set of 'literals' L, and let $m\colon \mathbf{B} \to \mathbb{R}_{\ge 0}$ be an arbitrary measure. We know that $\mathbf{B}$ has $n = 2^{|L|}$ atoms. Let $(a_i)_{i=1}^n$ denote those atoms in some arbitrary order. Let $L' = L \cup \{\phi_i \mid i \in [n]\} \cup \{\neg\phi_i \mid i \in [n]\}$ be the set L extended with $2n$ new literals. Let $\mathbf{B}'$ be the unique Boolean algebra with*

$$\{\phi_i \wedge a_i \mid i \in [n]\} \cup \{\neg\phi_i \wedge a_i \mid i \in [n]\}$$

as its set of atoms. Let $\iota\colon \mathbf{B} \to \mathbf{B}'$ be the inclusion homomorphism (i.e., $\iota(a) = a$ for all $a \in \mathbf{B}$). Let $w\colon L' \to \mathbb{R}_{\ge 0}$ be defined by

$$w(l) = \begin{cases} \frac{m(a_i)}{2} & \text{if } l = \phi_i \text{ or } l = \neg\phi_i \text{ for some } i \in [n] \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in L'$, and note that this defines a WMC measure $m'\colon \mathbf{B}' \to \mathbb{R}_{\ge 0}$. Then

$$m(a) = (m' \circ \iota)(a)$$

for all $a \in \mathbf{B}$.

In simpler terms, any measure can be computed using WMC by extending the Boolean algebra with more literals. More precisely, we are given the red part in

$$\begin{tikzcd}
\mathbb{R}_{\geq 0} & & \\
\mathbf{B} \arrow[u, "m"] \arrow[r, "\iota"] & \mathbf{B}' \arrow[ul, "m'"'] & \\
\cup & \cup & \\
L \arrow[r, phantom, "\subset"] & L' \arrow[r, "w"] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

and construct the black part in such a way that the triangle commutes.

Proof. Since $\mathbf{B}$ is freely generated by L, each atom $a_i \in \mathbf{B}$ is an infimum of elements in L, i.e.,

$$a_i = \bigwedge_{j \in J} a_{i,j}$$

for some $\{a_{i,j}\}_{j \in J} \subset L$. Moreover, each atom $b \in \mathbf{B}'$ can be represented as either

$$b = \phi_i \wedge a_i \quad \text{or} \quad b = \neg\phi_i \wedge a_i$$

for some atom $a_i \in \mathbf{B}$, also making it an infimum over a subset of L'. Then, for any $b \in \mathbf{B}$,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) \prod_{j \in J} w(a_{i,j}),$$

recognising that, for any $\iota(b)$, any atom $a_i \in \mathbf{B}$ satisfies

$$\phi_i \wedge a_i \leq \iota(b)$$

if and only if it satisfies

$$\neg\phi_i \wedge a_i \leq \iota(b).$$

Then, according to the definition of w,

$$(m' \circ \iota)(b) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} (w(\phi_i) + w(\neg\phi_i)) = \sum_{\substack{\text{atoms } a_i \in \mathbf{B}: \\ \phi_i \wedge a_i \leq \iota(b)}} m(a_i) = m(b),$$

provided that

$$\phi_i \wedge a_i \leq \iota(b) \quad \text{if and only if} \quad a_i \leq b,$$

but this is equivalent to

$$\phi_i \wedge a_i = \phi_i \wedge a_i \wedge b \quad \text{if and only if} \quad a_i = a_i \wedge b$$

which is true because $\phi_i \notin L$. ☐

Now we can show that the construction in Theorem 3 is smallest possible.

Conjecture 1. *Let $\mathbf{B}$ and $\mathbf{B}'$ be Boolean algebras, and $\iota\colon \mathbf{B} \to \mathbf{B}'$ be the inclusion map such that $\mathbf{B}$ is freely generated by L, all atoms of $\mathbf{B}'$ can be expressed as meets of elements of L', and the following subset relations are satisfied:*

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "\iota"] & \mathbf{B}' \\
\cup & \cup \\
L \arrow[r, phantom, "\subset"] & L'
\end{tikzcd}$$

If, for any measure $m\colon \mathbf{B} \to \mathbb{R}_{\geq 0}$*, one can construct a weight function* $w\colon L' \to \mathbb{R}_{\geq 0}$ *such that the WMC measure* $\mathrm{WMC}\colon \mathbf{B}' \to \mathbb{R}_{\geq 0}$ *with respect to* w *satisfies*

$$m = \mathrm{WMC} \circ \iota,$$

then $|L' \setminus L| \geq 2^{|L|+1}$.

Let us note how our lower bound on the number of added literals compares to two methods of translating a discrete probability distribution into a WMC problem over a propositional knowledge base proposed by Darwiche [6] and Sang et al. [17]. Suppose we have a discrete probability distribution with n variables, and the ith variable has v_i values, for each $i \in [n]$. Interpreted as a logical system, it has $\prod_{i=1}^{n} v_i$ models. My expansion would then use

$$\sum_{i=1}^{n} v_i + 2\prod_{i=1}^{n} v_i$$

variables, i.e., a variable for each possible variable-value assignment, and two additional variables for each model. Without making any independence assumptions, the encoding by Darwiche [6] would use

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \prod_{j=1}^{i} v_j$$

variables, while for the encoding by Sang et al. [17],

$$\sum_{i=1}^{n} v_i + \sum_{i=1}^{n} (v_i - 1) \prod_{j=1}^{i-1} v_j$$

variables would suffice.

5 Implications for Lifted Inference

5.1 Preliminaries

Definition 9. Given a BA $\mathbf{A}$, a *subalgebra* is a subset $\mathbf{B} \subseteq \mathbf{A}$ that, together with the operations, zero, and one of $\mathbf{A}$, is a BA.

Definition 10 ([10])**.** Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be BAs such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Let $f\colon \mathbf{A} \to \mathbf{C}$ and $g\colon \mathbf{B} \to \mathbf{C}$ be homomorphisms. Then f is an *extension* of g if $f(x) = g(x)$ for all $x \in \mathbf{B}$. If f is an extension of each member of a family $\{g_i\}_{i \in I}$ of homomorphisms, then f is called a *common extension* of $\{g_i\}_{i \in I}$.

Definition 11 ([10])**.** Let $\{\mathbf{A}_i\}_{i \in I}$ be a family of subalgebras of a BA $\mathbf{A}$ with a family of inclusion maps $\{\iota_i\colon \mathbf{A}_i \to \mathbf{A}\}_{i \in I}$. If for any BA $\mathbf{B}$ with a family of homomorphisms $\{f_i\colon \mathbf{A} \to \mathbf{B}\}_{i \in I}$ there exists a unique common extension of $\{f_i\colon \mathbf{A} \to \mathbf{B}\}_{i \in I}$ ($f\colon \mathbf{A} \to \mathbf{B}$ in the diagram),

$$\begin{tikzcd} \mathbf{A}_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & \mathbf{A} \arrow[d, dashed, "f"] \\ & \mathbf{B} \end{tikzcd}$$

then $\mathbf{A}$ is the *internal sum*[3] of $\{\mathbf{A}_i\}_{i \in I}$. We will denote it as $\bigoplus_{i \in I} \mathbf{A}_i$.

[3] A slightly more general version of this definition is also known as the free product, the Boolean product, and the coproduct in the category of BAs [10, 13, 18].

Proposition 2 ([18]). *Let $\mathbf{A}$ be the internal sum of a family of BAs $\{\mathbf{A}_i\}_{i \in I}$, and let $\{m_i \colon \mathbf{A}_i \to \mathbb{R}_{\geq 0}\}_{i \in I}$ be a family of measures. Then there exists a unique measure $m \colon \mathbf{A} \to \mathbb{R}_{\geq 0}$ such that, for any finite subset $J \subseteq I$ and family of elements $\{x_j \in \mathbf{A}_j\}_{j \in J}$,*

$$m\left(\bigwedge_{j \in J} x_j\right) = \prod_{j \in J} m_j(x_j).$$

5.2 New Results

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