Weighted Model Counting Without Parameter Variables

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1 Main Results

Notation. For any propositional formula ϕ and $p, q \in \mathbb{R}_{\geq 0}$,

$$[\phi]_q^p \coloneqq \begin{cases} p & \text{if } \phi \\ q & \text{otherwise.} \end{cases}$$

Formal definition of a previous WMC instance (CNF, literal weight function) and the new definition (set of $2^X \to \mathbb{R}_{\geq 0}$ pseudo-Boolean functions and a constant). Note that the constant idea is borrowed from `bklm16`.

References.

- 'Apply' is quadratic [1].
- Related work without publicly available implementations:
  - direct compilation to SDDs [2]
  - direct compilation to PSDDs, also eliminating parameter variables (a thesis)
  - maybe two more papers

Notes.

- Let X_P be the set of parameter variable and X_I be the set of indicator variables.
- Parameter variables are either taken from the LMAP file (for encodings produced by Ace) or assumed to be the variables that have both weights equal to 1.
- If a parameter variable in a clause is 'negated', we can ignore the clause. We assume that there are no clauses with more than one instance of parameter variables.
- The second **foreach** loop can be performed in constant time by representing ϕ' as a list and assuming that the two 'clauses' are adjacent in that list (and incorporating it into the first loop).
- The d map is constructed in $\mathcal{O}(|X_P| \log |X_P|)$ time (we want to use a data structure based on binary search trees rather than hashing).
- `rename` can be implemented in $\mathcal{O}(\log |X_P|)$ time.
- Apparently, the DPMC paper already shows that taking the first offered decomposition tree is best.
- This may look like preprocessing, but all the transformations are local and thus can be incorporated into an encoding algorithm with no slowdown. In fact, if anything, the resulting algorithm would be slightly faster, as it would have less data to output.
- It is already well-known that WMC is FPT.

Algorithm 1: WMC instance transformation

Data: an (old-format) WMC instance (ϕ, X_I, X_P, W)
Result: a (new-format) WMC instance (ϕ', ω)
$\phi' \leftarrow \emptyset$;
$\omega \leftarrow 1$;
let $d\colon X_P \to \mathbb{N}$ be defined as $v \mapsto |\{u \in X_P \mid u \leq v\}|$;
foreach *clause* $c \in \phi$ **do**
  if $c \cap X_P = \{v\}$ *for some* v **and** $W(v) \neq 1$ **then**
    if $|c| = 1$ **then**
      $\omega \leftarrow \omega \times W(v)$;
    else
      $\phi' \leftarrow \phi' \cup \left\{ \left[\bigwedge_{l \in c \setminus \{v\}} \neg l \right]_1^{W(v)} \right\}$;
  else if $\{v \mid \neg v \in c\} \cap X_P = \emptyset$ **then**
    $\phi' \leftarrow \phi' \cup \{[c]_0^1\}$;
foreach *indicator variable* $v \in X_I$ **do**
  if $\{[v]_1^p, [\neg v]_1^q\} \subseteq \phi'$ *for some* p *and* q **then**
    $\phi' \leftarrow \phi' \setminus \{[v]_1^p, [\neg v]_1^q\} \cup \{[v]_q^p\}$;
replace every variable v in ϕ' with `rename(`v`)`;
return (ϕ', ω);
Function `rename(`v`)`**:**
  $S \leftarrow \{u \in X_P \mid u \leq v\}$;
  if $S = \emptyset$ **then return** v;
  return $v - d(\max S)$;

2 Parameterised Complexity of DPMC

Summary:

- We establish DPMC inference as fixed-parameter tractable.
- We show that, in the case of cw, the parameter is the same as in the BN lower bound,
- and, in the case of d02, the DPMC parameter is exponentially higher.
- We experimentally show that cw+DPMC is best on low-to-moderate treewidth instances, and cd06+c2d overtakes cw+DPMC on higher treewidth instances.

Define:

- treewidth of a graph,
- Boolean formula in CNF (perhaps this is too trivial to define),
- primal graph of a CNF formula (a.k.a. Gaifman/(variable) interaction/connectivity/clique/representing graph),
- Bayesian network (I already have a definition of these last two),
- moralisation of a Bayesian network (or of any DAG),

Theorem 1 ([4], rephrased)**.** *BN Inference (for all algorithms that accept arbitrary instances) has a lower bound that's linear in the size of the BN and exponential in the treewidth of its moralisation (provide the exact formula).*

Definition 1 ([3], verbatim)**.** Let X be a set of Boolean variables and ϕ be a CNF formula over X. A *project-join tree* (PJT) of ϕ is a tuple (T, r, γ, π) where:

- T is a tree with root $r \in \mathcal{V}(T)$,
- $\gamma\colon \mathcal{L}(T) \to \phi$ is a bijection between the leaves of T and the clauses of ϕ, and
- $\pi\colon \mathcal{V}(T) \setminus \mathcal{L}(T) \to 2^X$ is a labelling function on internal nodes.

Moreover, (T, r, γ, π) must satisfy the following two properties:

1. $\{\pi(n) : n \in \mathcal{V}(T) \setminus \mathcal{L}(T)\}$ is a partition of X, and
2. for each internal node $n \in \mathcal{V}(T) \setminus \mathcal{L}(T)$, variable $x \in \pi(n)$, and clause $c \in \phi$ such that x appears in c, the leaf node $\gamma^{-1}(c)$ must be a descendant of n in T.

Definition 2 ([3])**.**

$$\texttt{Vars}(n) \coloneqq \begin{cases} \texttt{Vars}(\gamma(n)) & \text{if } n \in \mathcal{L}(T) \\ \left(\bigcup_{o \in \mathcal{C}(n)} \texttt{Vars}(o)\right) \setminus \pi(n) & \text{if } n \notin \mathcal{L}(T). \end{cases}$$

$$\texttt{size}(n) \coloneqq \begin{cases} |\texttt{Vars}(n)| & \text{if } n \in \mathcal{L}(T) \\ |\texttt{Vars}(n) \cup \pi(n)| & \text{if } n \notin \mathcal{L}(T). \end{cases}$$

The *width* of a PJT (T, r, γ, π) is $\texttt{width}(T) \coloneqq \max_{n \in \mathcal{V}(T)} \texttt{size}(n)$.

Theorem 2 ([3], paraphrased, 'ADD width is equal to the treewidth of the primal graph')**.** *Given a CNF formula ϕ with a tree decomposition of its primal graph of width w, Algorithm 2 (from [3]) returns a PJT of ϕ of width at most $w + 1$.*

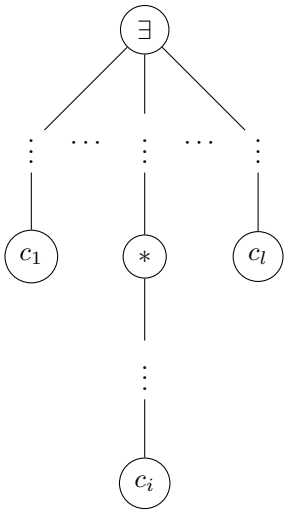

Figure 1: An example PJT with clauses $c_1, \ldots, c_i, \ldots, c_l$ that all contain the same variable, its projection node $\exists$, and a node under consideration $*$.

Lemma 1. *The number of variables that can appear in the ADDs within a PJT node is* $\leq k+1$.

Proof. All such variables are introduced in a 'clause' below and projected either in this node or in one of its ancestors. Each such variable would be in the bag of each clause that introduced it. By the tree decomposition properties, it will also be in the bag of both the projection node and the current node because they are on the path between the clause nodes (see the figure). □

Theorem 3. DPMC *is FPT w.r.t. the PJT width. More specifically,* DPMC *running time is* $\mathcal{O}(4^k m(n+k))$*, where* k *is the width,* n *is the number of clauses, and* m *is the number of variables.*

Proof.

- An ADD with k variables can have up to $2^0+2^1+\cdots+2^k = 2^{k+1}-1 = \mathcal{O}(2^k)$ nodes (including leaves).
- Multiplying m ADDs can then take up to $(m-1)(2^{k+1}-1)^2 = \mathcal{O}(m4^k)$ (since one multiplication takes up to $(2^{k+1}-1)^2$ time and the and the result will have up to $2^{k+1}-1$ nodes because the domain stays the same).
- Each projection consists of two linear operations and a quadratic operation, so projecting m variables will take $\mathcal{O}(4^k m)$ time.
- In total, operations on a PJT node with m ADDs, k variables, n of which are projected takes $\mathcal{O}(4^k(m+n))$ time.
- The total time taken by the DPMC execution stage is the sum of the time taken 'inside' each PJT node. The number of such nodes is upper bounded by m, so the total time complexity of DPMC is $\mathcal{O}(4^k m(n+k))$.

□

References

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[3] Dudek, J. M., Phan, V. H. N., and Vardi, M. Y. DPMC: weighted model counting by dynamic programming on project-join trees. In *Principles and Practice of Constraint Programming - 26th International Conference, CP 2020, Louvain-la-Neuve, Belgium, September 7-11, 2020, Proceedings* (2020), H. Simonis, Ed., vol. 12333 of *Lecture Notes in Computer Science*, Springer, pp. 211–230.

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