Weighted Model Counting as a Special Case of Polyadic Measure Algebras

Paulius Dilkas

10th March 2020

1 Propositional Logic and Boolean Algebras

1.1 Preliminaries

Definition 1. A *Boolean algebra* is a tuple $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ of a set $\mathbf{B}$ with operations $\wedge, \vee, \neg$ and elements $0, 1 \in \mathbf{B}$ such that the following axioms hold for all $a, b, \in \mathbf{B}$:

- both $\wedge$ and $\vee$ are associative and commutative;
- $a \vee (a \wedge b) = a$, and $a \wedge (a \vee b) = a$;
- 0 is the identity of $\vee$, and 1 is the identity of $\wedge$;
- $\vee$ distributes over $\wedge$ and vice versa;
- $a \vee \neg a = 1$, and $a \wedge \neg a = 0$.

Let $a, b \in \mathbf{B}$ be arbitrary. Let $\leq$ be a partial order on $\mathbf{B}$ defined by $a \leq b$ if $a = b \wedge a$ (or, equivalently, $a \vee b = b$), and let $a < b$ denote $a \leq b$ and $a \neq b$.

Which definition do I actually need?

Definition 2 ([5, 6])**.** An element $a \neq 0$ of a Boolean algebra $\mathbf{B}$ is an *atom* if there is no $x \in \mathbf{B}$ such that $0 < x < a$. Equivalently, $a \neq 0$ is an atom if, for all $x \in \mathbf{B}$, either $x \wedge a = a$ or $x \wedge a = 0$. A Boolean algebra is *atomic* if for every $a \in \mathbf{B} \setminus \{0\}$, there is an atom x such that $x \leq a$.

Lemma 1 ([2])**.** *For any two distinct atoms a, b in a Boolean algebra, $a \wedge b = 0$.*

Lemma 2 ([3])**.** *All finite Boolean algebras are atomic.*

Theorem 1. *Let $\mathbf{B}$ be a finite Boolean algebra. Then every $x \in \mathbf{B} \setminus \{0\}$ can be uniquely expressed as*

$$x = \bigvee_{\text{atoms } a \leq x} a.$$

Proof. A simple consequence of the theorem that every finite Boolean algebra is isomorphic to a field of subsets of a set, where the cardinality of the set is equal to the number of atoms in the Boolean algebra. $\square$

Remove the requirement for being strictly positive

Definition 3 ([1])**.** A *(strictly positive) measure* on a Boolean algebra $\mathbf{B}$ is a function $m : \mathbf{B} \to [0, 1]$ such that:

1. $m(1) = 1$, and $m(x) > 0$ for $x \neq 0$;
2. $m(x \vee y) = m(x) + m(y)$ for all $x, y \in \mathbf{B}$ whenever $x \wedge y = 0$.

1.2 New Results

Allow weight to be zero

Definition 4. Let $\mathbf{B}$ be a finite Boolean algebra, and let $M \subseteq \mathbf{B}$ be its set of atoms. Let $L \subseteq \mathbf{B}$ be such that every atom $m \in M$ can be uniquely expressed as $m = \bigwedge_{i \in I} l_i$ for some $\{l_i\}_{i \in I} \subseteq L$, and let $w : L \to \mathbb{R}_{>0}$ be arbitrary. The *weighted model count* $\mathrm{WMC} : \mathbf{B} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{WMC}(a) = \begin{cases} 0 & \text{if } a = 0 \\ \prod_{i \in I} w(l_i) & \text{if } M \ni a = \bigwedge_{i \in I} l_i \\ \sum_{i \in I} \mathrm{WMC}(m_i) & \text{if } \mathbf{B} \setminus (M \cup \{0\}) \ni a = \bigvee_{i \in I} m_i \end{cases}$$

for any $a \in \mathbf{B}$. Furthermore, we define the *normalised weighted model count* $\mathrm{nWMC} : \mathbf{B} \to [0,1]$ as $\mathrm{nWMC}(a) = \frac{\mathrm{WMC}(a)}{\mathrm{WMC}(1)}$ for all $a \in \mathbf{B}$.

Proposition 1. nWMC *is a measure for any finite Boolean algebra* $\mathbf{B}$*.*

Proof. First, note that Property 1 of Definition 3 is satisfied by the definition of nWMC. Next, in order to prove Property 2, let $x, y \in \mathbf{B}$ be such that $x \wedge y = 0$. We want to show that

$$\mathrm{nWMC}(x \vee y) = \mathrm{nWMC}(x) + \mathrm{nWMC}(y)$$

which is equivalent to

$$\mathrm{WMC}(x \vee y) = \mathrm{WMC}(x) + \mathrm{WMC}(y). \tag{1}$$

If, say, $x = 0$, then Eq. (1) becomes

$$\mathrm{WMC}(y) = \mathrm{WMC}(0) + \mathrm{WMC}(y) = \mathrm{WMC}(y)$$

(and likewise for $y = 0$). Thus we can assume that $x \neq 0 \neq y$ and use Theorem 1 to write

$$x = \bigvee_{i \in I} x_i \quad \text{and} \quad y = \bigvee_{j \in J} y_j$$

for some sequences of atoms $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$. If $x_{i'} = y_{j'}$ for some $i' \in I$ and $j' \in J$, then

$$x \wedge y = \bigvee_{i \in I} \bigvee_{j \in J} x_i \wedge y_j = x_{i'} \wedge y_{j'} \neq 0,$$

contradicting the assumption. This is enough to show that

$$\begin{aligned} \mathrm{WMC}(x \vee y) = \mathrm{WMC}\left(\left(\bigvee_{i \in I} x_i\right) \vee \left(\bigvee_{j \in J} y_j\right)\right) &= \sum_{i \in I} \mathrm{WMC}(x_i) + \sum_{j \in J} \mathrm{WMC}(y_j) \\ = \mathrm{WMC}(x) + \mathrm{WMC}(y), \end{aligned}$$

finishing the proof. ☐

2 First-Order Logic and Polyadic Algebras

2.1 Preliminaries

What follows is a summary of [4].

Let $\mathbf{B}$ be a Boolean algebra (of propositions). Let X be the (non-empty) domain of discourse. Let I be an index set, elements of which can be interpreted as variables. The elements of X^I are functions from I to

X. For any $x \in X^I$ and $i \in I$, we write x_i to represent $x(i) \in X$. Let $\mathbf{A}^*$ be the set of all functions $X^I \to \mathbf{B}$, and note that it forms a Boolean algebra with operations defined pointwise.

Let T be the semigroup of all $I \to I$ transformations. For any $\tau \in T$, let $\tau_* : X^I \to X^I$ be defined by

$$(\tau_* x)_i = x_{\tau i}$$

for all $x \in X^I$ and $i \in I$. For any (Boolean/polyadic) algebra $\mathbf{C}$, let $\mathrm{End}(\mathbf{C})$ denote the set of all its endomorphisms. We can then define $\mathbf{S}$ to be a map $\mathbf{S} : T \to \mathrm{End}(\mathbf{A}^*)$ defined by

$$\mathbf{S}(\tau)p(x) = p(\tau_* x)$$

for all $x \in X^I$ and $p \in \mathbf{A}^*$.

For any $J \subseteq I$, let J_* be the relation on X^I defined by

$$xJ_*y \iff x_i = y_i \quad \text{for all } i \in I \setminus J$$

for all $x, y \in X^I$. For any $J \subseteq I$, we then define $\exists(J)$ to be a transformation $\mathbf{A}^* \to \mathbf{A}^*$ defined by

$$\exists(J)p(x) = \bigvee_{\substack{y \in X^I, \\ xJ_*y}} p(y)$$

for all $p \in \mathbf{A}^*$, provided this supremum exists for all $x \in X^I$[1].

Finally, a *functional polyadic (Boolean) algebra*[2] is a subalgebra $\mathbf{A}$ of $\mathbf{A}^*$ such that:

- $\mathbf{S}(\tau)p \in \mathbf{A}$ for all $p \in \mathbf{A}$ and $\tau \in T$;
- $\exists(J)p \in \mathbf{A}$ for all $p \in \mathbf{A}$ and $J \subseteq I$.

Definition 5. Similarly to $\exists$, a *constant* c is a map $c : \mathcal{P}(I) \to \mathrm{End}(\mathbf{A})$ (Boolean endomorphisms?) such that:

- $c(\emptyset) = \mathrm{id}_{\mathbf{A}}$;
- $c(J \cup K) = c(J)c(K)$;
- $c(J)\exists(K) = \exists(K)c(J \setminus K)$;
- $\exists(J)c(K) = c(K)\exists(J \setminus K)$;
- $c(J)\mathbf{S}(\tau) = \mathbf{S}(\tau)c(\tau^{-1}J)$

for all $J, K \in \mathcal{P}(I)$ and $\tau \in T$. If J is a singleton set $\{i\}$, we will simply write $c(i)$ instead of $c(J)$.

2.2 New Results

Proposition 2. *Let $\mathbf{B}$ be a finite Boolean algebra with a measure $m : \mathbf{B} \to [0, 1]$. Let $\mathbf{A}$ be a $\mathbf{B}$-valued functional polyadic algebra with domain X and variables I. Let $m^* : \mathbf{A} \to \mathbb{R}_{\geq 0}$ be defined by*

$$m^*(p) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I : y \leq p(x)}} m(y)$$

for all $p \in \mathbf{A}$. Then m^* is a measure on $\mathbf{A}$.

[1] The universal quantifier $\forall(J)$ is then defined as $\forall(J)p = \neg(\exists(J)\neg p)$ for all $p \in \mathbf{A}^*$.
[2] To be more explicit, a $\mathbf{B}$-valued functional polyadic algebra with domain X and variables I.

Remark. While definitions of m^* such as

$$m^*(p) = m\left(\bigvee_{x \in X^I} p(x)\right)$$

might look tempting, they are not additive.

Proof.

Update the proof w.r.t. definitions

First, we can show that $m^*(1) = 1$ by observing that

$$m^*(1) = \sum_{\text{atoms } y \in \mathbf{B}} m(y) = m\left(\bigvee_{\text{atoms } y \in \mathbf{B}} y\right) = m(1) = 1,$$

where we use Theorem 1 and express $1 \in \mathbf{B}$ as the supremum of all atoms in $\mathbf{B}$ [2]. Clearly $m^*(p) \geq 0$ for all $p \in \mathbf{A}$, so we can restrict the codomain of m^* to $[0, 1]$.

Next, we want to show that $m^*(p) > 0$ for all $p \in \mathbf{A} \setminus \{0\}$. If $p \neq 0$, then there must be some $x' \in X^I$ such that $p(x') \neq 0$. But then, since finite Boolean algebras are atomic, there must also be an atom $y \in \mathbf{B}$ such that $y \leq p(x')$. Therefore, $m^*(p) \geq m(y) > 0$, finishing this part of the proof.

Let $p, q \in \mathbf{A}$ be such that $p \wedge q = 0$. We want to show that $m^*(p \vee q) = m^*(p) \vee m^*(q)$. First, note that

$$y \leq (p \vee q)(x) = p(x) \vee q(x)$$

if and only if

$$y = (p(x) \vee q(x)) \wedge y = (p(x) \wedge y) \vee (q(x) \wedge y)$$

by Definition 1. Also note that

$$(p(x) \wedge y) \wedge (q(x) \wedge y) = p(x) \wedge q(x) \wedge y = (p \wedge q)(x) \wedge y = 0 \wedge y = 0,$$

so

$$m(y) = m((p(x) \wedge y) \vee (q(x) \wedge y)) = m(p(x) \wedge y) + m(q(x) \wedge y)$$

by Definition 3 which then leads to

$$\begin{aligned} m^*(p \vee q) &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(y) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(p(x) \wedge y) + m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x)}} m(q(x) \wedge y). \end{aligned}$$

Since y is an atom,

$$p(x) \wedge y = \begin{cases} y & \text{if } y \leq p(x) \\ 0 & \text{otherwise,} \end{cases}$$

so

$$\begin{aligned} m^*(p \vee q) = &\sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x) \text{ and } y \leq p(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{s.t.} \\ \exists x \in X^I \colon y \leq (p \vee q)(x) \text{ and } y \leq q(x)}} m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq p(x)}} m(p(x) \wedge y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{s.t.} \\ \exists x \in X^I \colon y \leq q(x)}} m(q(x) \wedge y) \\ &= \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ \exists x \in X^I \colon y \leq p(x)}} m(y) + \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{s.t.} \\ \exists x \in X^I \colon y \leq q(x)}} m(y) = m^*(p) + m^*(q), \end{aligned}$$

finishing the proof that m^* is a measure. $\square$

Lemma 3. *Given the setup of Proposition 2 and $p \in \mathbf{A}$, if $p(x) = p(y)$ for all $x, y \in X^I$ (i.e., p has no free variables), then*

$$m^*(p) = m(p(x))$$

(for some $x \in X^I$) is an alternative (i.e., equivalent and simpler) definition of m^*.

Proof. Fix some $x \in X^I$. Then

$$m(p(x)) = m\left(\bigvee_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.}\\ y \leq p(x)}} y\right) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.}\\ y \leq p(x)}} m(y) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.}\\ \exists x' \in X^I : y \leq p(x')}} m(y) = m^*(p),$$

where we use Theorem 1 for the first step, Definition 3 and Lemma 2 for the second step, the assumptions of Lemma 3 for the third step, and the definition of m^* for the fourth one. □

3 From First-Order Logic to Polyadic Algebras

3.1 Preliminaries

Definition 6 ([3])**.** An *ideal* in a Boolean algebra $\mathbf{B}$ is a subset $M \subseteq \mathbf{B}$ such that:

- $0 \in M$;
- $x \vee y \in M$ for all $x, y \in M$;
- $x \wedge y \in M$ for all $x \in M$ and $y \in \mathbf{B}$.

For any subset $S \subseteq \mathbf{B}$, the *ideal generated by* S is the smallest ideal M such that $S \subseteq M$.

Note that Definition 6 gives us a simple characterisation of an ideal generated by a set of atoms.

Lemma 4. *Let $\mathbf{B}$ be a Boolean algebra, and let $S \subseteq \mathbf{B}$ be a set of atoms. The ideal I generated by S is defined by the following:*

- *$0 \in I$,*
- *$S \subseteq I$,*
- *$x \vee y \in I$ for all $x, y \in I$.*

Definition 7 ([3])**.** Let $\mathbf{B}$ be a Boolean algebra, and let I be an ideal in $\mathbf{B}$. The *quotient algebra* $\mathbf{B}/I$ is a Boolean algebra on equivalence classes of elements of $\mathbf{B}$ (with operations defined pointwise) based on the equivalence relation

$$x \sim y \quad \iff \quad x + y \in I$$

where $x + y = (x \wedge \neg y) \vee (y \wedge \neg x)$ is the symmetric difference operation (written as a sum because it can interpreted as the 'additive' part of the corresponding Boolean ring).

3.2 New Results (an Example)

In order to make the example algebras easily describable, our example programs will have to be tiny. Consider the following ProbLog [7] program:

$$1.0 :: \mathsf{p}(a, b).$$
$$0.5 :: \mathsf{p}(X, X) :- \mathsf{p}(X, Y);\ \mathsf{p}(Y, X).$$

Table 1: Example elements of $\mathbf{A}$ as maps $X^I \to \mathbf{B}$, with $a : \mathcal{P}(I) \to \mathrm{End}(\mathbf{A})$ as one of two possible constants.

Element of $\mathbf{A}$	Action on X^I
$p = \mathbf{S}(\mathrm{id})p = \exists(\emptyset)p = a(\emptyset)p = b(\emptyset)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, x_2)$
$\exists(1)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, x_2) \vee \mathsf{p}(b, x_2)$
$\exists(2)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, a) \vee \mathsf{p}(x_1, b)$
$\exists(I)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, a) \vee \mathsf{p}(a, b) \vee \mathsf{p}(b, a) \vee \mathsf{p}(b, b)$
$\mathbf{S}(\{1 \mapsto 1, 2 \mapsto 1\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, x_1)$
$\mathbf{S}(\{1 \mapsto 2, 2 \mapsto 1\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_2, x_1)$
$\mathbf{S}(\{1 \mapsto 2, 2 \mapsto 2\})p$	$(x_1, x_2) \mapsto \mathsf{p}(x_2, x_2)$
$a(1)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, x_2)$
$a(2)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, a)$
$a(I)p$	$(x_1, x_2) \mapsto \mathsf{p}(a, a)$

Table 2: Step-by-step derivation of how a more complex element of $\mathbf{A}$ acts on elements of X^I

Element of $\mathbf{A}$	Action on X^I
p	$(x_1, x_2) \mapsto \mathsf{p}(x_1, x_2)$
$b(2)p$	$(x_1, x_2) \mapsto \mathsf{p}(x_1, b)$
$\neg b(2)p$	$(x_1, x_2) \mapsto \neg\mathsf{p}(x_1, b)$
$\exists(1)\neg b(2)p$	$(x_1, x_2) \mapsto \neg\mathsf{p}(a, b) \vee \neg\mathsf{p}(b, b) = \neg(\mathsf{p}(a, b) \wedge \mathsf{p}(b, b))$
$\forall(1)b(2)p = \neg\exists(1)\neg b(2)p$	$(x_1, x_2) \mapsto \neg\neg(\mathsf{p}(a, b) \wedge \mathsf{p}(b, b)) = \mathsf{p}(a, b) \wedge \mathsf{p}(b, b)$

Let $L = \{\mathsf{p}(a, a), \mathsf{p}(a, b), \mathsf{p}(b, a), \mathsf{p}(b, b)\}$ be the set of all possible ground atoms. Let $\mathbf{B}$ be the Boolean algebra freely generated by L (see, e.g., [3] for more on free Boolean algebras). Then $\mathbf{B}$ will have sixteen atoms ranging from $\mathsf{p}(a, a) \wedge \mathsf{p}(a, b) \wedge \mathsf{p}(b, a) \wedge \mathsf{p}(b, b)$ to $\neg\mathsf{p}(a, a) \wedge \neg\mathsf{p}(a, b) \wedge \neg\mathsf{p}(b, a) \wedge \neg\mathsf{p}(b, b)$. The weight function $w : L \to \mathbb{R}_{\geq 0}$ defined by

$$w(l) = \begin{cases} 1 & \text{if } l = \mathsf{p}(a, b) \\ 0.5 & \text{if } l \in \{\mathsf{p}(a, a), \mathsf{p}(b, b)\} \\ 0 & \text{if } l = \mathsf{p}(b, a) \\ 1 - w(l') & \text{if } l = \neg l' \end{cases}$$

for all $l \in L$ defines a WMC measure over $\mathbf{B}$. Note that while we could define an ideal generated by $\{\mathsf{p}(b, a), \neg\mathsf{p}(a, b)\}$ and take the quotient of $\mathbf{B}$ by that ideal to get a Boolean algebra with a strictly positive measure, this would put zero-probability queries outside of our algebras, i.e., we would not be able to ask a question whose answer is zero.

Finally, let $\mathbf{A}$ be the functional polyadic algebra $X^I \to \mathbf{B}$ for $I = \{1, 2\}$ and $X = \{a, b\}$[3]. The elements of X^I can then be represented as tuples (x_1, x_2) for some $x_1, x_2 \in X$. See Table 1 for example elements of $\mathbf{A}$ which consists of a single predicate function p and operators $\exists, \mathbf{S}, a, b, \neg, \wedge, \vee$, the last three of which are defined pointwise.

Let us calculate the probability $\Pr(\forall x_1 \in X, \mathsf{p}(x_1, b))$. The same expression can be translated into the notation for our polyadic algebra $\mathbf{A}$ as $m^*(\forall(1)b(2)p)$. Recall that $\forall(1)b(2)p = \neg\exists(1)\neg b(2)p$. The effect of this function on an arbitrary element of X^I is derived step-by-step in Table 2. Since the resulting function

[3] X cannot (or should not) have constants that do not occur in $\mathbf{B}$.

Table 3: Atoms $y \in \mathbf{B}$ (and their measures) such that $y \leq \mathsf{p}(a,b) \wedge \mathsf{p}(b,b)$

Atom $y \in \mathbf{B}$	$m(y)$
$\mathsf{p}(a,b) \wedge \mathsf{p}(b,b) \wedge \mathsf{p}(a,a) \wedge \mathsf{p}(b,a)$	$1 \times 0.5 \times 0.5 \times 0 = 0$
$\mathsf{p}(a,b) \wedge \mathsf{p}(b,b) \wedge \neg\mathsf{p}(a,a) \wedge \mathsf{p}(b,a)$	$1 \times 0.5 \times 0.5 \times 0 = 0$
$\mathsf{p}(a,b) \wedge \mathsf{p}(b,b) \wedge \mathsf{p}(a,a) \wedge \neg\mathsf{p}(b,a)$	$1 \times 0.5 \times 0.5 \times 1 = 0.25$
$\mathsf{p}(a,b) \wedge \mathsf{p}(b,b) \wedge \neg\mathsf{p}(a,a) \wedge \neg\mathsf{p}(b,a)$	$1 \times 0.5 \times 0.5 \times 1 = 0.25$

is constant (i.e., the logical formula has no free variables), Lemma 3 tells us that

$$m^*(\forall(1)b(2)p) = m(\mathsf{p}(a,b) \wedge \mathsf{p}(b,b)) = m\left(\bigvee_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ y \leq \mathsf{p}(a,b) \wedge \mathsf{p}(b,b)}} y \right) = \sum_{\substack{\text{atoms } y \in \mathbf{B} \text{ s.t.} \\ y \leq \mathsf{p}(a,b) \wedge \mathsf{p}(b,b)}} m(y).$$

The resulting sum is over four atoms; these atoms and their probabilities are listed in Table 3. Thus, we get that

$$m^*(\forall(1)b(2)p) = 0 + 0 + 0.25 + 0.25 = 0.5.$$

Proposition 3. *Let $\mathbf{B}$ be a finite measure algebra with measure $m : \mathbf{B} \to \mathbb{R}_{\geq 0}$. Let $L \subset \mathbf{B}$ be defined as*

$$L = \{l_i \mid i \in [n]\} \cup \{\neg l_i \mid i \in [n]\}$$

for some $n \in \mathbb{N}$. Finally, assume that $\mathbf{B}$ has 2^n atoms, where each atom $a \in \mathbf{B}$ is an infimum

$$a = \bigwedge_{i=1}^{n} a_i$$

such that $a_i \in \{l_i, \neg l_i\}$ for $i \in [n]$. Then there exists a weight function $w : L \to \mathbb{R}_{>0}$ that makes m a WMC measure if and only if

$$m(l \wedge l') = m(l)m(l') \tag{2}$$

for all distinct $l, l' \in L$ such that $l \neq \neg l'$.

Remark. Note that if $n = 1$, then Eq. (2) is vacuously satisfied and so any valid measure can be expressed as WMC.

Proof. Let us begin with the 'if' part of the statement. Let $w : L \to \mathbb{R}_{>0}$ be defined by

$$w(l) = m(l) \tag{3}$$

for all $l \in L$. We are going to show that nWMC $= m$. First, note that nWMC$(0) = 0 = m(0)$ by the definitions of both nWMC and m. Second, let

$$a = \bigwedge_{i=1}^{n} a_i \tag{4}$$

be an atom in $\mathbf{B}$ such that $a_i \in \{l_i, \neg l_i\}$ for all $i \in [n]$. Then

$$\text{nWMC}(a) = \frac{\text{WMC}(a)}{\text{WMC}(1)} = \frac{1}{\text{WMC}(1)} \prod_{i=1}^{n} w(a_i) = \frac{1}{\text{WMC}(1)} \prod_{i=1}^{n} m(a_i) = \frac{1}{\text{WMC}(1)} m\left(\bigwedge_{i=1}^{n} a_i \right) = \frac{m(a)}{\text{WMC}(1)}$$

by Definition 4 and Eqs. (2) to (4). Now we just need to show that $\mathrm{WMC}(1) = 1$. Indeed,

$$\begin{aligned}\mathrm{WMC}(1) &= \sum_{\text{atoms } a \in \mathbf{B}} \mathrm{WMC}(a) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} w(a_i) = \sum_{\text{atoms } a \in \mathbf{B}} \prod_{i=1}^{n} m(a_i) \\ &= \sum_{\text{atoms } a \in \mathbf{B}} m\left(\bigwedge_{i=1}^{n} a_i\right) = \sum_{\text{atoms } a \in \mathbf{B}} m(a) = m\left(\bigvee_{\text{atoms } a \in \mathbf{B}}\right) = m(1) = 1.\end{aligned}$$

Finally, note that if nWMC and m agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

For the other direction, we are given a weight function $w : L \to \mathbb{R}_{>0}$ that induces a measure $m = \mathrm{nWMC} : \mathbf{B} \to \mathbb{R}_{\geq 0}$, and we want to show that Eq. (2) is satisfied. Let $k_i, k_j \in L$ be such that $k_i \in \{l_i, \neg l_i\}$, $k_j \in \{l_j, \neg l_j\}$, and $i \neq j$. We will first prove an auxiliary result that

$$m(k_i \wedge k_j) = m(k_i)m(k_j) \tag{5}$$

is equivalent to

$$m(k_i \wedge k_j) \cdot m(\neg k_i \wedge \neg k_j) = m(k_i \wedge \neg k_j) \cdot m(\neg k_i \wedge k_j). \tag{6}$$

First, note that k_i can be expressed as

$$k_i = (k_i \wedge k_j) \vee (k_i \wedge \neg k_j)$$

with the condition that

$$(k_i \wedge k_j) \wedge (k_i \wedge \neg k_j) = 0,$$

so, by properties of a measure,

$$m(k_i) = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j). \tag{7}$$

Applying Eq. (7) and the equivalent expression for $m(k_j)$ allows us to rewrite Eq. (5) as

$$\begin{aligned}m(k_i \wedge k_j) &= [m(k_i \wedge k_j) + m(k_i \wedge \neg k_j)] \cdot [m(k_i \wedge k_j) + m(\neg k_i \wedge k_j)] \\ &= m(k_i \wedge k_j)^2 + m(k_i \wedge k_j)[m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j)] + m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)\end{aligned}$$

Dividing both sides by $m(k_i \wedge k_j)$ gives

$$1 = m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) + m(\neg k_i \wedge k_j) + \frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)}. \tag{8}$$

Since $k_i \wedge k_j \wedge k_i \wedge \neg k_j = 0$, and

$$(k_i \wedge k_j) \vee (k_i \wedge \neg k_j) = k_i \wedge (k_j \vee \neg k_j) = k_i \wedge 1 = k_i,$$

we have that

$$m(k_i \wedge k_j) + m(k_i \wedge \neg k_j) = m(k_i).$$

Similarly, $k_i \wedge \neg k_i \wedge k_j = 0$, and

$$k_i \vee (\neg k_i \wedge k_j) = (k_i \vee \neg k_i) \wedge (k_i \vee k_j) = k_i \vee k_j,$$

so

$$m(k_i) + m(\neg k_i \wedge k_j) = m(k_i \vee k_j).$$

Finally, note that

$$(k_i \vee k_j) \wedge \neg(k_i \vee k_j) = 0,$$

and

$$(k_i \vee k_j) \vee \neg(k_i \vee k_j) = 1,$$

so

$$m(k_i \vee k_j) + m(\neg(k_i \vee k_j)) = m(1) = 1.$$

This allows us to rewrite Eq. (8) as

$$\frac{m(k_i \wedge \neg k_j)m(\neg k_i \wedge k_j)}{m(k_i \wedge k_j)} = 1 - m(k_i \vee k_j) = m(\neg(k_i \vee k_j)) = m(\neg k_i \wedge \neg k_j)$$

which immediately gives us Eq. (6).

Now recall that $m = \mathrm{nWMC}$ and note that Eq. (6) can be multiplied by $\mathrm{WMC}(1)^2$ to turn the equation into one for WMC instead of nWMC. Then

$$\begin{aligned}
\mathrm{WMC}(k_i \wedge k_j) &= \sum_{\text{atoms } a \le k_i \wedge k_j} \mathrm{WMC}(a) = \sum_{\text{atoms } a \le k_i \wedge k_j} \prod_{m \in [n]} w(a_m) \\
&= \sum_{\text{atoms } a \le k_i \wedge k_j} w(a_i)w(a_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = \sum_{\text{atoms } a \le k_i \wedge k_j} w(k_i)w(k_j) \prod_{m \in [n] \setminus \{i,j\}} w(a_m) \\
&= w(k_i)w(k_j) \sum_{\text{atoms } a \le k_i \wedge k_j} \prod_{m \in [n] \setminus \{i,j\}} w(a_m) = w(k_i)w(k_j)C,
\end{aligned}$$

where C denotes the part of $\mathrm{WMC}(k_i \wedge k_j)$ that will be the same for $\mathrm{WMC}(\neg k_i \wedge k_j)$, $\mathrm{WMC}(k_i \wedge \neg k_j)$, and $\mathrm{WMC}(\neg k_i \wedge \neg k_j)$ as well. But then Eq. (6) becomes

$$w(k_i)w(k_j)w(\neg k_i)w(\neg k_j)C^2 = w(k_i)w(\neg k_j)w(\neg k_i)w(k_j)C^2$$

which is trivially true. By showing that WMC satisfies Eq. (6), we also showed that it satisfies Eq. (5), finishing the second part of the proof.

Adapt the proof to zero measure events

□

The big TODO list

- Extension to infinite (atomic?) Boolean algebras.
- How many extra variable do you need to add to make any probability distribution representable using WMC?
- Abstraction refinements as homomorphisms.
- Definition of a measure-preserving homomorphism from Jech's set theory book.
- A Boolean algebra is approximable if its Stone space is approximable.

4 Homomorphisms

Definition 8 ([3])**.** Let $\mathbf{A}$ and $\mathbf{B}$ be Boolean algebras. A *Boolean homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $f : \mathbf{A} \to \mathbf{B}$ such that:

- $f(x \wedge y) = f(x) \wedge f(y)$,
- $f(x \vee y) = f(x) \vee f(y)$,
- $f(\neg x) = \neg f(x)$

for all $x, y \in \mathbf{A}$.

Definition 9 ([4])**.** Given two polyadic algebras $\mathbf{A}$ and $\mathbf{B}$, a *polyadic homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a Boolean homomorphism $f : \mathbf{A} \to \mathbf{B}$ such that

- $f\mathbf{S}(\tau)p = \mathbf{S}(\tau)fp$,
- $f\exists(J)p = \exists(J)fp$

for all $\tau \in T$, $p \in \mathbf{A}$, and $J \subseteq I$.

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