Weighted Model Counting with Conditional Measures

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1 Introduction

- The Main Narrative
  1. When weights are defined on literals, the measure on the free BA is fully independent.
  2. This means that the BA itself must be larger (i.e., have additional 'meaningless' literals) to turn any probability distribution into an independent one.
  3. We show how we can define conditional weights on literals, allowing us to encode any probability distribution into a Boolean algebra that's not necessarily independent and thus can be smaller.
  4. We demonstrate a specific example of this by presenting a new way to encode Bayesian networks into instances of WMC and adapting a WMC algorithm (ADDMC) to run on the new format.
  5. We show that this results in significantly faster inference.
  6. We show that our encoding results in asymptotically fewer literals and fewer ADDs, and thus a simpler problem.
  7. (Maybe) we experimentally demonstrate a phase transition based on the number of variables per ADD.
- Potential criticism may be that this is a step backwards and doesn't allow us to use SAT-based techniques for probabilistic inference. However, they can still be used for the 'theory+query' part.
  - Zero-probability weights and one-probability weights can be interpreted as logical clauses. This doesn't affect ADDMC but could be useful for other solvers.

F What are the main claims, what are the main takeaways, intuitive [???] of theorems to follow. To do this, we appeal to algebraic constructions to define the main concepts for introducing measures on Boolean algebras.

- Algorithms[1]
  - ADDMC [18] (rediscovered the multiplicativity of BAs in different words) (with optimal settings)
  - Cachet [38]
  - c2d [16]
  - d4 [31] (closed source, boo!)
  - miniC2D [35]
- Notable previous/related work
  - Hailperin's approach to probability logic [22]

[1] `http://beyondnp.org/pages/solvers/model-counters-exact/`

Table 1: A comparison of Boolean-algebraic (BA) and set-theoretic (ST) concepts for 2^X for some set X

BA name	BA symbol	ST symbol	ST name
bottom	$\bot$	$\emptyset$	empty set
top	$\top$	X	
meet, and	$\wedge$	$\cap$	intersection
join, or	$\vee$	$\cup$	union
complement, not	$\neg$	^{c}	complement
	$\leq$	$\subseteq$	subset relation, set inclusion
atom			singleton, unit set

  - Nilsson's (somewhat successful) probabilistic logic [33, 34]
  - Logical induction: a big paper with a good overview of previous attempts to assign probabilities to logical sentences in a sensible way [20]
  - Measures on Boolean algebras
    * On possibility and probability measures in finite Boolean algebras [7]
    * Representation of conditional probability measures [29]

- Intuitively, a measure is just like a probability, except it's in $\mathbb{R}_{\geq 0}$ instead of $[0, 1]$.

2 Boolean Algebras and Power Sets

Notation: make up my mind about a, b vs. x, y and stick to it (maybe x, y?).

Let X be a set and let 2^X be its power set. We can equivalently interpret 2^X as a Boolean algebra. See Table 1 for a summary of the differences in terminology and notation. We will use both.

2.1 The Space of Functions on Boolean Algebras

Definition 1 (Operations on functions)**.** Let $A\colon 2^X \to \mathbb{R}_{\geq 0}$ and $B\colon 2^Y \to \mathbb{R}_{\geq 0}$ be arbitrary functions, $\alpha \in \mathbb{R}_{\geq 0}$, and $x \in X$. We define the following operations:

Addition: $A + B$ is a function $A + B\colon 2^{X \cup Y} \to \mathbb{R}_{\geq 0}$ such that

$$(A + B)(\tau) = A(\tau \cap X) + B(\tau \cap Y)$$

for all $\tau \in 2^{X \cup Y}$.

Inverse: $\overline{A}$ is a function $\overline{A}\colon 2^X \to \mathbb{R}_{\geq 0}$ such that

$$\overline{A}(\tau) = 1 - A(\tau)$$

for all $\tau \in 2^X$.

Multiplication: $A \cdot B$ is a function $A \cdot B\colon 2^{X \cup Y} \to \mathbb{R}_{\geq 0}$ such that

$$(A \cdot B)(\tau) = A(\tau \cap X) \cdot B(\tau \cap Y)$$

for all $\tau \in 2^{X \cup Y}$.

Scalar multiplication: αA is a function $\alpha A\colon 2^X \to \mathbb{R}_{\geq 0}$ such that

$$(\alpha A)(\tau) = \alpha \cdot A(\tau)$$

for all $\tau \in 2^X$.

Projection: $\exists_x A$ is a function $\exists_x A\colon 2^{X\setminus\{x\}} \to \mathbb{R}_{\geq 0}$ such that

$$(\exists_x A)(\tau) = A(\tau) + A(\tau \cup \{x\})$$

for all $\tau \in 2^{X\setminus\{x\}}$.

Observation 1. *Let U be a set, and $\mathcal{V} = \{A\colon 2^X \to \mathbb{R}_{\geq 0} \mid X \subseteq U\}$. Then $\mathcal{V}$ is a semi-vector space with three additional operations: inverse, (non-scalar) multiplication, and projection. Specifically, note that both addition and multiplication are both associative and commutative.*

Definition 2 (Special functions)**.**

- unit $1\colon 2^{\emptyset} \to \mathbb{R}_{\geq 0}$, $1(\emptyset) = 1$.
- zero $0\colon 2^{\emptyset} \to \mathbb{R}_{\geq 0}$, $0(\emptyset) = 0$.
- constant $[a]\colon 2^{\{a\}} \to \mathbb{R}_{\geq 0}$, $[a](\emptyset) = 0$, $[a](\{a\}) = 1$.

Henceforth, for any function $A\colon 2^X \to \mathbb{R}_{\geq 0}$ and any set τ, we will write $A(\tau)$ to mean $A(\tau \cap X)$.

Remark. For any function $A\colon 2^X \to \mathbb{R}_{\geq 0}$, $A + \overline{A} = 1$.

3 Weighted Model Counting as a Measure

Describe Table 2 as an example.

Feedback: explain with examples: models = atoms of algebra, formulas = all elements.

Definition 3. Let U be a set.

- A *measure* is a function $M\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ such that $M(\bot) = 0$ and

  $$M(a \vee b) = M(a) + M(b)$$

  for all $a, b \in 2^{2^U}$ whenever $a \wedge b = \bot$.
- A *weight function* is a function $W\colon 2^U \to \mathbb{R}_{\geq 0}$.
- A weight function is *factored* if $W = \prod_{x \in U} W_x$ for some functions $W_x\colon 2^{\{x\}} \to \mathbb{R}_{\geq 0}$.
- A measure $M\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ is *factorable* if there exists a factored weight function $W\colon 2^U \to \mathbb{R}_{\geq 0}$ that induces M.

Proposition 1. *Every weight function induces a measure, i.e, for any weight function $W\colon 2^U \to \mathbb{R}_{\geq 0}$, the function $M_W\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$, defined as*

$$M_W(x) = \begin{cases} 0 & \textit{if } x = \bot \\ W(u) & \textit{if } x = \{u\} \\ \sum_{\{u\} \leq x} W(u) & \textit{otherwise,} \end{cases}$$

is a measure.

Proof.

Prove that M_W satisfies the definition of a measure.

□

Table 2: (in the same order)

	Set-theoretic notation	Boolean-algebraic notation
Atoms (elements of U)	a, b	a, b
Models (elements of 2^U)	$\emptyset, \{a\}, \{b\}, \{a, b\}$	$\neg a \wedge \neg b, a \wedge \neg b, \neg a \wedge b, a \wedge b$
	$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$	$\top$
	$\{\emptyset, \{a\}, \{b\}\}, \{\emptyset, \{a\}, \{a, b\}\}$	$\neg a \vee \neg b, a \to b$
	$\{\emptyset, \{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}$	$b \to a, a \vee b$
Formulas (elements of 2^{2^U})	$\{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a, b\}\}$	$\neg b, \neg a, a \leftrightarrow b$
	$\{\{a\}, \{b\}\}, \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}$	$(a \wedge \neg b) \vee (b \wedge \neg a), a, b$
	$\{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}$	$\neg a \wedge \neg b, a \wedge \neg b, \neg a \wedge b, a \wedge b$
	$\emptyset$	$\perp$

The process of calculating the value of $M_W(x)$ for some $x \in 2^{2^U}$ with a given definition of W is known as *weighted model counting*.

Example 1 (Relation to the classical view of WMC)**.** Let $\mathcal{L}$ be a propositional logic with a set of atoms $U = \{p, q\}$. Let $w : \{p, q, \neg p, \neg q\} \to \mathbb{R}_{\geq 0}$ be the *weight function* defined by

$$w(p) = 0.3, \quad w(\neg p) = 0.7, \quad w(q) = 0.2, \quad w(\neg q) = 0.8.$$

Let Δ be a theory in $\mathcal{L}$ with a sole axiom p. Then Δ has two models, i.e., $\{p, q\}$ and $\{p, \neg q\}$. The *weighted model count* (WMC) [12] of Δ is then

$$\mathrm{WMC}(\Delta) = \sum_{\omega \models \Delta} \prod_{\omega \models l} w(l) = w(p)w(q) + w(p)w(\neg q) = 0.3. \tag{1}$$

Alternatively, we can define $W_p \colon 2^{\{p\}} \to \mathbb{R}_{\geq 0}$ as

$$W_p(\{p\}) = 0.3, \quad W_p(\emptyset) = 0.7$$

and $W_q \colon 2^{\{q\}} \to \mathbb{R}_{\geq 0}$ as

$$W_q(\{q\}) = 0.2, \quad W_q(\emptyset) = 0.8.$$

Let M be the measure on 2^{2^U} induced by $W = W_p \cdot W_q$. Then, equivalently to Eq. (1), we can write

$$M(p) = W(\{p, q\}) + W(\{p\}) = W_p(\{p\})W_q(\{q\}) + W_p(\{p\})W_q(\emptyset) = 0.3.$$

Given the theory Δ, we can compute the probability of a query q as [5, 39]

$$\Pr_{\Delta, w}(q) = \frac{\mathrm{WMC}(\Delta \wedge q)}{\mathrm{WMC}(\Delta)}.$$

The same thing can be accomplished using the algebraic formulation, with M replacing WMC.

For the rest of the paper, we let U be the set of (logical) atoms. We use set-theoretic notation for 2^U and Boolean-algebraic notation for 2^{2^U}, except for (Boolean) atoms in 2^{2^U} that are denoted as $\{x\}$ for some model $x \in 2^U$.

4 Limitations of Factorable Measures

F Give a concrete example of something impossible to represent using WMC.

- F Can you say something here about factorized vs non-factorized weight function definitions? That is, factorized is when w maps literals to $R_{\geq 0}$, non-factorized is when w maps models to $R_{\geq 0}$ and
  - come up with nice example when non-factorized weights are intuitive;
  - clarify that the factorized definition have is w.r.t. models, in case some one gets confused. [It doesn't have to be, if the BA is not free—P.]

Lemma 1. *For any measure $M\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ and elements $a, b \in 2^{2^U}$,*

$$M(a \wedge b) = M(a)M(b) \tag{2}$$

if and only if

$$M(a \wedge b) \cdot M(\neg a \wedge \neg b) = M(a \wedge \neg b) \cdot M(\neg a \wedge b). \tag{3}$$

Proof. First, note that $a = (a \wedge b) \vee (a \wedge \neg b)$ and $(a \wedge b) \wedge (a \wedge \neg b) = 0$, so, by properties of a measure,

$$M(a) = M(a \wedge b) + M(a \wedge \neg b). \tag{4}$$

Applying Eq. (4) and the equivalent expression for $M(b)$ allows us to rewrite Eq. (2) as

$$M(a \wedge b) = [M(a \wedge b) + M(a \wedge \neg b)][M(a \wedge b) + M(\neg a \wedge b)]$$

which is equivalent to

$$M(a \wedge b)[1 - M(a \wedge b) - M(a \wedge \neg b) - M(\neg a \wedge b)] = M(a \wedge \neg b)M(\neg a \wedge b). \tag{5}$$

Since $a \wedge b$, $a \wedge \neg b$, $\neg a \wedge b$, $\neg a \wedge \neg b$ are pairwise disjoint and their supremum is 1,

$$M(a \wedge b) + M(a \wedge \neg b) + M(\neg a \wedge b) + M(\neg a \wedge \neg b) = 1,$$

and this allows us to rewrite Eq. (5) into Eq. (3). As all transformations are invertible, the two expressions are equivalent. □

Theorem 1. *A measure $M\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ is factorable if and only if*

$$M(u \wedge v) = M(u)M(v) \tag{6}$$

for all distinct $u, v \in U \cup \{\neg w \mid w \in U\}$ such that $u \neq \neg v$.

Proof.

Check if the proof is valid for zero weights.

($\Leftarrow$) For each $x \in U$, let $W_x\colon 2^{\{x\}} \to \mathbb{R}_{\geq 0}$ be defined by $W_x(\{x\}) = M(x)$ and $W_x(\emptyset) = M(\neg x)$. Let M_W be the measure induced by

$$W = \prod_{x \in U} W_x.$$

We will show that $M = M_W$. First, note that $M_w(\bot) = 0 = M(\bot)$ by Definition 3 and Proposition 1. Second, let

$$a = \bigwedge_{u \in U} a_u$$

be an atom in 2^{2^U} such that $a_u \in \{u, \neg u\}$ for all $u \in U$. Then

$$M_W(a) = W(\{a_u \mid u \in U\}) = \prod_{u \in U} W_u(\{a_u\}) = \prod_{u \in U} M(a_u) = M\left(\bigwedge_{u \in U} a_u\right) = M(a)$$

Finally, note that if M_W and M agree on all atoms, then they must also agree on all other non-zero elements of the Boolean algebra.

($\Rightarrow$) For the other direction, we are given a factored weight function

$$W = \prod_{x \in U} W_x,$$

and we want to show that its induced measure M_W satisfies Eq. (6). Let $k_u, k_v \in U \cup \{\neg w \mid w \in U\}$ be such that $k_u \in \{u, \neg u\}$, $k_v \in \{v, \neg v\}$, and $u \neq v$. We then want to show that

$$M_W(k_u \wedge k_v) = M_W(k_u) M_W(k_v) \tag{7}$$

which is equivalent to

$$M_W(k_u \wedge k_v) \cdot M_W(\neg k_u \wedge \neg k_v) = M_W(k_u \wedge \neg k_v) \cdot M_w(\neg k_u \wedge k_v) \tag{8}$$

by Lemma 1. Then

$$\begin{aligned} M_W(k_u \wedge k_v) &= \sum_{\{a\} \leq k_u \wedge k_v} W(a) = \sum_{\{a\} \leq k_u \wedge k_v} \prod_{x \in U} W_x(a) \\ &= \sum_{\{a\} \leq k_u \wedge k_v} W_u(a_u) W_v(a_v) \prod_{x \in U \setminus \{u,v\}} W_x(a) = \sum_{\{a\} \leq k_u \wedge k_v} W_u(k_u) W_v(k_v) \prod_{x \in U \setminus \{u,v\}} W_x(a) \\ &= W_u(k_u) W_v(k_v) \sum_{\{a\} \leq k_u \wedge k_v} \prod_{x \in U \setminus \{u,v\}} W_x(a) = W_u(k_u) W_v(k_v) C, \end{aligned}$$

where C denotes the part of $M_W(k_u \wedge k_v)$ that will be the same for $M_W(\neg k_u \wedge k_v)$, $M_W(k_u \wedge \neg k_v)$, and $M_W(\neg k_u \wedge \neg k_v)$ as well. But then Eq. (8) becomes

$$W_u(k_u) W_v(k_v) W_u(\neg k_u) W_v(\neg k_v) C^2 = W_u(k_u) W_v(\neg k_v) W_u(\neg k_u) W_v(k_v) C^2$$

which is trivially true. $\square$

5 Previous Work

5.1 Bayesian Network Encodings

- Encodings:
  - `d02` [15]
  - `sbk05` [39]
  - `cd05` [9]
  - `cd06` [10] (supposed to be the best)
  - `db20` (mine)
- `cd05` relaxes the encoding so much that extra models become possible. They are supposed to be filtered out by the algorithm, but mine can't do that because it doesn't deal with models. Same for `cd06` because it's based on `cd05`.

5.2 Algebraic Decision Diagrams and Their Use in Probabilistic Inference

Cover:

- ADDs provide an efficient way to manipulate functions from BAs/power sets [2].
- background reading
  - Compiling Bayesian Networks Using Variable Elimination (Chavira and Darwiche) [11]
  - On the Relationship between Sum-Product Networks and Bayesian Networks (Zhao et al.) [45]

6 Encoding Bayesian Networks Using Conditional Weights

Let V denote the set of random variables in a Bayesian network. For any random variable $X \in V$, let $\mathrm{pa}(X)$ denote the set of parents of X and $\mathrm{im}\,X$ denote the set of possible values.

Definition 4 (Indicator variables)**.** Let $X \in V$ be a random variable. If X is binary (i.e., $|\,\mathrm{im}\,X| = 2$), we can arbitrary identify one of the values as 1 and the other one as 0 (i.e, $\mathrm{im}\,X \cong \{0,1\}$). Then X can be represented by a single *indicator variable* $\lambda_{X=1}$. For notational simplicity, for any set S, whenever we write $\lambda_{X=0} \in S$ or $S = \{\lambda_{X=0}, \dots\}$, we actually mean $\lambda_{X=1} \notin S$,

On the other hand, if X is not binary, we represent X with $|\,\mathrm{im}\,X|$ indicator variables, one for each value. We let

$$E(X) = \begin{cases} \{\lambda_{X=1}\} & \text{if } |\,\mathrm{im}\,X| = 2 \\ \{\lambda_{X=x} \mid x \in \mathrm{im}\,X\} & \text{otherwise.} \end{cases}$$

denote the set of indicator variables for X and

$$E^*(X) = E(X) \cup \bigcup_{Y \in \mathrm{pa}(X)} E(Y).$$

denote the set of indicator variables for X and its parents in the Bayesian network. Finally, let

$$U = \bigcup_{X \in V} E(X)$$

denote the set of all indicator variables for all random variables in the Bayesian network.

Algorithm 1: Encoding a Bayesian network as a function $2^U \to \mathbb{R}_{\geq 0}$

```
Data: a Bayesian network with vertices V and probability distribution Pr
Result: a function φ: 2^U → ℝ_{≥0}
φ ← 1;
for X ∈ V do
    let pa(X) = {Y_1, ..., Y_n};
    CPT_X ← 0;
    if |im X| = 2 then
        for (y_1, ..., y_n) ∈ ∏_{i=1}^n im Y_i do
            p_1 ← Pr(X = 1 | Y_1 = y_1, ..., Y_n = y_n);
            p_0 ← Pr(X ≠ 1 | Y_1 = y_1, ..., Y_n = y_n);
            CPT_X ← CPT_X + p_1[λ_{X=1}] · ∏_{i=1}^n [λ_{Y_i=y_i}] + p_0 \overline{[λ_{X=1}]} · ∏_{i=1}^n [λ_{Y_i=y_i}];
    else
        let im X = {x_1, ..., x_m};
        for x ∈ im X and (y_1, ..., y_n) ∈ ∏_{i=1}^n im Y_i do
            p_x ← Pr(X = x | Y_1 = y_1, ..., Y_n = y_n);
            CPT_X ← CPT_X + p_x[λ_{X=x}] · ∏_{i=1}^n [λ_{Y_i=y_i}] + \overline{[λ_{X=1}]} · ∏_{i=1}^n [λ_{Y_i=y_i}];
        CPT_X ← CPT_X · (∑_{i=1}^m [λ_{X=x_i}]) · ∏_{i=1}^m ∏_{j=i+1}^m (\overline{[λ_{X=x_i}]} + \overline{[λ_{X=x_j}]});
    φ ← φ · CPT_X;
return φ;
```

Describe Algorithm 1

6.1 Proof of Correctness

Lemma 2. *Let $X \in V$ be a random variable with parents* $\mathrm{pa}(X) = \{Y_1, \dots, Y_n\}$*. Then* $\mathrm{CPT}_X \colon 2^{E^*(X)} \to \mathbb{R}_{\geq 0}$ *is such that for any $x \in \mathrm{im}\, X$ and $(y_1, \dots, y_n) \in \prod_{i=1}^n \mathrm{im}\, Y_i$,*

$$\mathrm{CPT}_X(\{\lambda_{X=x}\} \cup \{\lambda_{Y_i=y_i} \mid i = 1, \dots, n\}) = \Pr(X = x \mid Y_1 = y_1, \dots, Y_n = y_n).$$

Proof. Let $\tau = \{\lambda_{X=x}\} \cup \{\lambda_{Y_i=y_i} \mid i = 1, \dots, n\}$. If X is binary, then CPT_X is a sum of $2 \prod_{i=1}^n |\mathrm{im}\, Y_i|$ terms, one for each possible assignment of values to variables $X, Y_1, \dots, Y_n$. Exactly one of these terms is nonzero when applied to τ, and it is equal to $\Pr(X = x \mid Y_1 = y_1, \dots, Y_n = y_n)$ by definition.

If X is not binary, then

$$\left(\sum_{i=1}^m [\lambda_{X=x_i}] \right)(\tau) = 1,$$

and

$$\left(\prod_{i=1}^m \prod_{j=i+1}^m (\overline{[\lambda_{X=x_i}]} + \overline{[\lambda_{X=x_j}]}) \right)(\tau) = 1,$$

so, by a similar argument as before,

$$\mathrm{CPT}_X(\tau) = \Pr(X = x \mid Y_1 = y_1, \dots, Y_n = y_n).$$

□

Proposition 2. *$\phi \colon 2^U \to \mathbb{R}_{\geq 0}$ represents the full probability distribution of the Bayesian network, i.e., if $V = \{X_1, \dots, X_n\}$, then*

$$\phi(\tau) = \begin{cases} \Pr(X_1 = x_1, \dots, X_n = x_n) & \text{if } \tau = \{\lambda_{X_i=x_i} \mid i = 1, \dots, n\} \text{ for some } (x_1, \dots, x_n) \in \prod_{i=1}^n \mathrm{im}\, X_i \\ 0 & \text{otherwise,} \end{cases}$$

for all $\tau \in 2^U$.

Proof. If $\tau = \{\lambda_{X=v_X} \mid X \in V\}$ for some $(v_X)_{X \in V} \in \prod_{X \in V} \mathrm{im}\, X$, then

$$\phi(\tau) = \prod_{X \in V} \Pr\left(X = v_X \;\middle|\; \bigwedge_{Y \in \mathrm{pa}(X)} Y = v_Y \right) = \Pr\left(\bigwedge_{X \in V} X = v_X \right)$$

by Lemma 2 and the definition of a Bayesian network. Otherwise there must be some non-binary random variable $X \in V$ such that $|E(X) \cap \tau| \neq 1$. If $E(X) \cap \tau = \emptyset$, then

$$\left(\sum_{i=1}^m [\lambda_{X=x_i}] \right)(\tau) = 0,$$

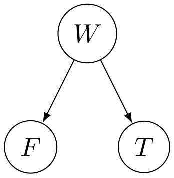
and so $\mathrm{CPT}_X(\tau) = 0$, and $\phi(\tau) = 0$. If $|E(X) \cap \tau| > 1$, then we must have two different values $x_1, x_2 \in \mathrm{im}\, X$ such that $\{\lambda_{X=x_1}, \lambda_{X=x_2}\} \subseteq \tau$ which means that

$$(\overline{[\lambda_{X=x_1}]} + \overline{[\lambda_{X=x_2}]})(\tau) = 0,$$

and so, again, $\mathrm{CPT}_X(\tau) = 0$, and $\phi(\tau) = 0$. □

Theorem 2. *Let $\phi \colon 2^U \to \mathbb{R}_{\geq 0}$ be a function generated by the algorithm. Then*

$$(\exists_U (\phi \cdot [\lambda_{X=x}]))(\emptyset) = \Pr(X = x).$$



w	$\Pr(W = w)$
1	0.5
0	0.5

w	f	$\Pr(F = f \mid W = w)$
1	1	0.6
1	0	0.4
0	1	0.1
0	0	0.9

w	t	$\Pr(T = t \mid W = w)$
1	l	0.2
1	m	0.4
1	h	0.4
0	l	0.6
0	m	0.3
0	h	0.1

Figure 1: An example Bayesian network with its CPTs

Proof. Let $V = \{X, Y_1, \dots, Y_n\}$. Then

$$\begin{aligned}(\exists_U(\phi \cdot [\lambda_{X=x}]))(\emptyset) &= \sum_{\tau \in 2^U} (\phi \cdot [\lambda_{X=x}])(\tau) = \sum_{\lambda_{X=x} \in \tau \in 2^U} \phi(\tau) = \sum_{\lambda_{X=x} \in \tau \in 2^U} \left(\prod_{Y \in V} \mathrm{CPT}_Y \right)(\tau) \\ &= \sum_{(y_1,\dots,y_n) \in \prod_{i=1}^n \operatorname{im} Y_i} \Pr(X = x, Y_1 = y_1, \dots, Y_n = y_n) = \Pr(X = x)\end{aligned}$$

by the following arguments:

- the proof of Theorem 1 in the ADDMC paper [18];
- if $\lambda_{X=x} \not\in \tau \in 2^U$, then $(\phi \cdot [\lambda_{X=x}])(\tau) = \phi(\tau) \cdot [\lambda_{X=x}](\tau \cap \{\lambda_{X=x}\}) = \phi(\tau) \cdot 0 = 0$;
- Proposition 2;
- marginalisation of a probability distribution.

□

6.2 An Example

Reformulate this as a running example throughout the section.

The Bayesian network in Fig. 1 has

$$\begin{aligned}
V &= \{W, F, T\}, \\
\mathrm{pa}(W) &= \emptyset, \\
\mathrm{pa}(F) &= \mathrm{pa}(T) = \{W\}, \\
\operatorname{im} W &= \operatorname{im} F = \{0, 1\}, \\
\operatorname{im} T &= \{l, m, h\}, \\
E(W) &= \{\lambda_{W=1}\}, \\
E(F) &= \{\lambda_{F=1}\}, \\
E(T) &= \{\lambda_{T=l}, \lambda_{T=m}, \lambda_{T=h}\}, \\
E^*(W) &= \{\lambda_{W=1}\}, \\
E^*(F) &= \{\lambda_{F=1}, \lambda_{W=1}\}, \\
E^*(T) &= \{\lambda_{T=l}, \lambda_{T=m}, \lambda_{T=h}, \lambda_{W=1}\}, \\
\mathrm{CPT_W} &= 0.5[\lambda_{W=1}] + 0.5\overline{[\lambda_{W=1}]} = 0.5 \cdot 1, \\
\mathrm{CPT_F} &= 0.6[\lambda_{F=1}] \cdot [\lambda_{W=1}] + 0.4[\lambda_{F=0}] \cdot [\lambda_{W=1}] + 0.1[\lambda_{F=1}] \cdot [\lambda_{W=0}] + 0.9[\lambda_{F=0}] \cdot [\lambda_{W=0}] \\
&= 0.6[\lambda_{F=1}] \cdot [\lambda_{W=1}] + 0.4\overline{[\lambda_{F=1}]} \cdot [\lambda_{W=1}] + 0.1[\lambda_{F=1}] \cdot \overline{[\lambda_{W=1}]} + 0.9\overline{[\lambda_{F=1}]} \cdot \overline{[\lambda_{W=1}]}, \\
\mathrm{CPT_T} &= ([\lambda_{T=l}] + [\lambda_{T=m}] + [\lambda_{T=h}]) \cdot (\overline{[\lambda_{T=l}]} + \overline{[\lambda_{T=m}]}) \cdot (\overline{[\lambda_{T=l}]} + \overline{[\lambda_{T=h}]}) \cdot (\overline{[\lambda_{T=m}]} + \overline{[\lambda_{T=h}]}) \cdot (\dots),
\end{aligned}$$

and can be represented in a textual format as

$$
\begin{array}{lrrll}
\lambda_{T=l} & \lambda_{T=m} & \lambda_{T=h} & & 0 \\
 & -\lambda_{T=l} & -\lambda_{T=m} & & 0 \\
 & -\lambda_{T=l} & -\lambda_{T=h} & & 0 \\
 & -\lambda_{T=m} & -\lambda_{T=h} & & 0 \\
w & \lambda_{W=1} & & 0.5 & 0.5 \\
w & \lambda_{F=1} & \lambda_{W=1} & 0.6 & 0.4 \\
w & \lambda_{F=1} & -\lambda_{W=1} & 0.1 & 0.9 \\
w & \lambda_{T=l} & \lambda_{W=1} & 0.2 & 1 \\
w & \lambda_{T=m} & \lambda_{W=1} & 0.4 & 1 \\
w & \lambda_{T=h} & \lambda_{W=1} & 0.4 & 1 \\
w & \lambda_{T=l} & \lambda_{W=0} & 0.6 & 1 \\
w & \lambda_{T=m} & \lambda_{W=0} & 0.3 & 1 \\
w & \lambda_{T=h} & \lambda_{W=0} & 0.1 & 1
\end{array}
$$

with each λ replaced with a unique positive integer. This format is based on the format used by the Cachet solver [38] to encode WMC problems, which extends the DIMACS format for CNF formulas with weight clauses. Subsequently, we extend it in two ways:

- a single weight clause now supports an arbitrary number of literals,
- and each weight clause has two probabilities instead of one (i.e., we no longer assume that $\Pr(v) + \Pr(\neg v) = 1$ for all variables $v \in U$).

The way we use this encoding, it is always the case that either both probabilities sum to one, or the second probability (i.e., the probability for the complement of the variable) is equal to one.

6.3 Adjusting ADDMC to Work with the New Format

ADDMC constructs the *Gaifman graph* of the input CNF formula as an aid for the algorithm's heuristics. This graph has as vertices the variables of the formula, and there is an edge between two variables u and v if there is a clause in the formula that contains both u and v. We extend this definition to functions on Boolean algebras, i.e., the factors of ϕ. For any pair of distinct variables $u, v \in U$, we draw an edge between them in the Gaifman graph if there is a function $A\colon 2^X \to \mathbb{R}_{\geq 0}$ that is a factor of ϕ such that $u \in X$ and $v \in X$. For instance, a factor such as CPT_X will enable edges between all distinct pairs of variables in $E^*(X)$.

7 Experimental Comparison

- We don't compare 'compile times' because our encoding time is linear, so we would easily beat everyone else.
- When the Bayesian network has an evidence file, we compute the probability of evidence. Otherwise, let X denote the last-mentioned node in the Bayesian network. If `true` is a valid value of X, we compute the marginal probability of $X =$ `true`. Otherwise, we pick the first value of X and calculate its marginal probability. This applies to the Grid data set (as intended) and also to two instances of Plan Reconstruction and roughly half of the instances from 2004-PGM that have empty evidence files.
- After the experiments are finished, note the processor, memory per thread, and add the following acknowledgment.
- All other encodings are implemented in Ace 3.0[2] and should be compiled with `-encodeOnly` (i.e., don't compile the CNF into an AC) and `-noEclause` (i.e., only use standard syntax) flags.

[2] `http://reasoning.cs.ucla.edu/ace/`

- Datasets
  - binary Bayesian networks from Sang et al.[3] [39]
    - Grid (networks) (ratio 75 means that 75% of the nodes are deterministic),
    - Plan recognition (problems),
    - Deterministic quick medical reference (what do the numbers mean? the README doesn't say).
  - Bayesian networks available with Ace
    - `2004-pgm` [13] (binary)
    - `2005-ijcai` [9]. The Genie/Smile files have their own citation data that I should probably extract. This is the only dataset that has some non-binary networks.
    - `2006-ijar` [13] (binary)

8 Explaining The Performance Benefits

- `d02` has

  $$\sum_{X \in V} |\operatorname{im} X| + |\operatorname{im} X| \prod_{Y \in \operatorname{pa}(X)} |\operatorname{im} Y|$$

  variables and

  $$\sum_{X \in V} 1 + \binom{|\operatorname{im} X|}{2} + |\operatorname{im} X|(2 + |\operatorname{pa}(X)|) \prod_{Y \in \operatorname{pa}(X)} |\operatorname{im} Y|$$

  clauses (along with one ADD per variable to encode the weights).

- `sbk05` is a bit harder to evaluate due to a handful of small optimisations in the encoding. Could find an upper bound anyway.

- `db20` (my encoding) has

  $$\sum_{X \in V} |\operatorname{im} X|$$

  variables (less for binary) and

  $$\sum_{X \in V} |\operatorname{im} X| + 1 + \binom{|\operatorname{im} X|}{2}$$

  ADDs.

- Let:
  - $N = |V|$ (i.e., the number of nodes in the Bayesian network),
  - $D = \max_{X \in V} |\operatorname{pa}(X)|$ (i.e., the maximum in-degree or the number of parents),
  - $V = \max_{X \in V} |\operatorname{im} X|$ (i.e., the maximum number of values per variables).

- Then my encoding has $\mathcal{O}(NV)$ variables and $\mathcal{O}(NV^2)$ ADDs while `d02` has $\mathcal{O}(NV^{D+1})$ variables and $\mathcal{O}(NDV^{D+1})$ ADDs.

Calculate numVariables/numClauses (or the other way around) for each instance and plot this ratio vs runtime (for each encoding, or at least mine and D02). The new CP paper kind of beat me to it...

[3] `https://www.cs.rochester.edu/u/kautz/Cachet/`

9 Conclusion and Future Work

- Bayesian networks and ADDMC are only particular examples. This should also work with Cachet.
- Extra benefit: one does not need to come up with a way to turn some probability distribution to into a fully independent one.
- Important future work: replacing ADDs with AADDs[4] [40] is likely to bring performance benefits. Other extensions:
  - FOADDs can represent first order statements;
  - XADDs can replace WMI for continuous variables;
  - ADDs with intervals can do approximations.
- Filtering out ADDs that have nothing to do with the answer helps tremendously, but I'm purposefully not doing that. Perhaps a heuristic could do the same thing?
- Encodings for everything else
  - probabilistic programs [23]
  - ProbLog [19]
    - For the ProbLog to WMC conversion, check out this guy: `https://users.ics.aalto.fi/ttj/`.
    - proof-based [32]
    - rule-based [25]
    - For ground ProbLog, we can encode a program
      ```
      p :: a :- b
      q :: a :- c
      ```
      into $P(a \mid b) = p$, $P(a \mid c) = q$ instead of having clauses $b \Rightarrow a$, $c \Rightarrow a$. Some logical structure is likely to remain.
- Bayesian networks are often solved in a compile once, query many times fashion. This can be achieved using ADDMC by selecting a subset S of variable we may want to query over and running ADDMC while excluding S from variable elimination/projection/$\exists$.
- More references
  - Measures on/in Boolean algebras: Horn and Tarski [24], Jech [26]
  - On Boolean algebras and their role in analysis [43]
  - Infinite domains
    - Markov Logic in Infinite Domains (Singla and Domingos) [41]
    - Objective Bayesian probabilistic logic (Williamson) [42]
    - Unifying Logic and Probability (Russell) [37]
  - Logical induction [20]
  - Quantum probabilistic logic programming [3]
  - WMC
    - algebraic model counting [28]
    - Explanation-Based Approximate Weighted Model Counting for Probabilistic Logics [36]
    - OUWMC [4]

[4] `https://github.com/ssanner/dd-inference`

* Formula-Based Probabilistic Inference [21]
* Parallel Probabilistic Inference by WMC [14]
* Semiring Programming [6]
* theoretical extension: WMC beyond two-variable logic [30]
* from weighted to unweighted model counting [8]
* theory behind WMC algorithms: solving #SAT and Bayesian inference with backtracking search [1]

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