Engineering Mathematics-III

Solution 2071 to 2076

AC

Diploma in Engineering All
II Yr. I Part

EXAM 2076 (REGULAR/BACK)

Program: Diploma in civil / Architecture / Computer/ Electronic/IT Engineering

Full Marks: 80

Year/Part: II/I (New + Old Course)

Pass Marks: 32

Subject : Engineering Mathematics - III

Time: 3hrs

Candidates are required to give their answer in their own words as far as practicable. The figures in the margin indicates the full marks.

Attempt All questions.

Group 'A' **[3×(5+5) = 30]**

1. a) Define trigonometric and fourier series. Determine the Fourier coefficient a_0 by Euler's formula.

Solution:

Trigonometric Series

A series of the form

$a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \ldots$ where, $a_0, a_1, a_2, a_3, + \ldots, b_1, b_2, b_3, \ldots$ are constants, is called trigonometric series.

The series can also be written as

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Fourier Series

A series of sines and cosines of an angle and its multiple of the form

$$\frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots +$$
$$b_1\sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called the Fourier series, where $a_0, a_1, a_2, \ldots a_n, \ldots b_1, b_2, b_3, \ldots, b_n \ldots$ are constants are Fourier coefficients.

Determination of Fourier coefficients a_0 by Euler's Formula

The Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + a_n \cos nx + \ldots + b_1\sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots \quad \text{(i)}$$

To find a_0

Integrating both sides of (i) from x = 0 to x = 2π, we have

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1\int_0^{2\pi}\cos x\,dx + a_2\int_0^{2\pi}\cos 2x\,dx + \ldots +$$

$$a_n\int_0^{2\pi}\cos nx\,dx + \ldots + b_1\int_0^{2\pi}\sin x\,dx + b_2\int_0^{2\pi}\sin 2x\,dx + \ldots$$

$$+ b_n\int_0^{2\pi}\sin nx\,dx$$

$$= \frac{a_0}{2}\int_0^{2\pi} dx$$

$$= \frac{a_0}{2}[x]_0^{2\pi}$$

$$= \frac{a_0}{2}\times 2\pi$$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

b) Find the Fourier series for the function

$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ 1 & \pi < x < 2\pi \end{cases}$$

Sol^n: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ...(i)

where, $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx$

$$= 0 + \frac{1}{\pi}[x]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}(2\pi - \pi)$$

$\therefore a_0 = 1$

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 0\times\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 1.\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$$

$$= 0 + \frac{1}{\pi}\left[\frac{\sin 2n\pi - \sin\pi}{n}\right]$$

$$\therefore a_n = 0$$

and $$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 0\times\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 1\sin nx\,dx$$

$$= 0 + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2n\pi}{n} + \frac{\cos n\pi}{n}\right]$$

$$= \frac{1}{n\pi}[-1 + (-1)^n]$$

$$= \begin{cases} \dfrac{-2}{\pi n} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Substituting these values in (i), we get

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sin x - \frac{2}{3\pi}\sin 3x - \frac{2}{5\pi}\sin 5x - \ldots$$

$$= \frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$$

2. a) Define a group. In a group, prove that $(a * b)^{-1} = b^{-1} * a^{-1}$
Also prove that inverse of each of the element of a group is unique.

Solution:

An algebraic structure (G, ∗) with a set G under the binary operation '∗', and denoted by (G, ∗) is known as a group if it is closure, associative, has an identity element and an inverse element.

To prove $(a * b)^{-1} = b^{-1} * a^{-1}$

For a, b ∈ G, we have a * b ∈ G.

Let us consider

$$(a * b) * (b^{-1} * a^{-1}) = a * [b * (b^{-1} * a^{-1})]$$
$$= a * [(b * b^{-1}) * a^{-1}]$$
$$= a * (e * a^{-1})$$
$$= a * a^{-1} = e.$$

Similarly, we can show that $(b^{-1} * a^{-1}) * (a * b) = e$.

This shows that $b^{-1} * a^{-1}$ is the inverse of a * b. Since (a * b) has unique inverse, so $(a * b)^{-1} = b^{-1} * a^{-1}$. Proved.

To prove that inverse of each of the element of a group is unique.

Let x be an inverse of a ∈ G.

Then, $a * x = e = x * a$, ...(i)

where e is the identity element.

If possible, let a ∈ G has another inverse, say y.

Then, $a * y = e = y * a$ ....(ii)

From equation (i) and (ii), we have

$a * x = a * y$ [∵ each equal to e]

Now, $a * x = a * y \Rightarrow x * (a * x) = x * (a * y)$

$\Rightarrow (x * a) * x = (x * a) * y$

$\Rightarrow e * x = e * y$, from equation (i).

$\Rightarrow x = y$

This shows that inverse of every a ∈ G is unique.

b) **Given a set G = {0, 1, 2, 3, 4} and binary operation addition modulo 5 ($+_5$) is defined on G. Prepare a Caley table for it. Find the identity and inverse element of 3 and 4.**

Solution:

Here, the given set G = {0, 1, 2, 3, 4} and the given operation is addition modulo 5. The Cayley's table for the set G under the operation multiplication modulo 5 is given below:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Clearly, from the above table, we see that the identity element for the given set under addition modulo 5 is 0.

From the above table, we also see that $3 +_5 2 = 2 +_5 3 = 0$ (identity element) and $4 +_5 1 = 1 +_5 4 = 0$ (identity element). Hence, the inverse element of 3 is 2 and that of 4 is 1 under addition modulo 5.

3. a) By using the definition of partial derivatives find f_x and f_y for

$$f(x, y) = x^2y - xy^3$$

Solution:

Here, $f(x, y) = x^2y - xy^3$

$$\therefore \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2y - y^3(x + \Delta x) - (x^2y - xy^3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2y + 2xy\,\Delta x + (\Delta x)^2y - xy^3 - \Delta xy^3 - x^2y + xy^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2xy\,\Delta x + (\Delta x)^2y - \Delta xy^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\,(2\,xy + \Delta x\,y - y^3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2\,xy + \Delta x\,y - y^3)$$

$$= 2xy - y^3$$

$$\therefore f_x = \frac{\partial f}{\partial x} = 2xy - y^3$$

and $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2(y + \Delta y) - x(y + \Delta y)^3 - (x^2y - xy^3)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2y + x^2\Delta y - xy^3 - 3xy^2\Delta y - 3xy(\Delta y)^2 - x(\Delta y)^3 - x^2y + xy^3}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2\Delta y - 3xy^2\Delta y - 3xy(\Delta y)^2 - x(\Delta y)^3}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y\{x^2 - 3xy^2 - 3xy(\Delta y) - x(\Delta y)^2\}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \{x^2 - 3xy^2 - 3xy(\Delta y) - x(\Delta y)^2\}$$

$$= x^2 - 3xy^2$$

$$\therefore f_y = \frac{\partial f}{\partial y} = x^2 - 3xy^2$$

b) If $u = \sqrt{x^2 + y^2 + z^2}$, then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

Solution:

Here, $u = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial u}{\partial x} = \frac{2x}{2(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{x}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2 + z^2)^{1/2} . 1 - \frac{x}{2}(x^2 + y^2 + z^2)^{-1/2} . 2x}{(x^2 + y^2 + z^2)}$$

$$= \frac{(x^2 + y^2 + z^2)\,(x^2 + y^2 + z^2)^{1/2} - x^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{x^2 + y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{2(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{y}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2 + z^2)^{1/2} . 1 - \frac{y}{2}(x^2 + y^2 + z^2)^{-1/2} . 2y}{(x^2 + y^2 + z^2)}$$

$$= \frac{(x^2 + y^2 + z^2)^{1/2}\,(x^2 + y^2 + z^2)^{1/2} - y^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{x^2 + y^2 + z^2 - y^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial u}{\partial z} = \frac{2z}{2(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{(x^2 + y^2 + z^2)^{1/2} . 1 - \frac{z}{2}(x^2 + y^2 + z^2)^{-1/2} . 2z}{x^2 + y^2 + z^2}$$

$$= \frac{(x^2 + y^2 + z^2)^{1/2}\,(x^2 + y^2 + z^2)^{1/2} - z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

Now,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}+\frac{z^2+x^2}{(x^2+y^2+z^2)^{3/2}}+\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{y^2+z^2+z^2+x^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{(x^2+y^2+z^2)^{1/2}} = \frac{2}{u}$$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$ Proved.

Group 'B' **[5×10=50]**

4. Solve (1 + cos x) dy = (1 – cos x) dx

Solution:

Here, the given differential equation is

$$(1+\cos x)\,dy = (1-\cos x)\,dx$$

or, $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

or, $\frac{dy}{dx} = \frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{1+\left(2\cos^2\frac{x}{2}-1\right)}$

or, $\frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$

or, $\frac{dy}{dx} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}$

or, $\frac{dy}{dx} = \tan^2\frac{x}{2}$

or, $\frac{dy}{dx} = \sec^2\frac{x}{2} - 1$

or, $dy = \sec^2\frac{x}{2}\,dx - dx$

Integrating, we get

$y = 2\tan\frac{x}{2} - x + c$ which is the required solution.

5. Solve: $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

Solution:

Here, the given differential equation is

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

or, $\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$

Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

From equation (i), we have

or, $v + x\frac{dv}{dx} = v - \tan v$

or, $x\frac{dv}{dx} = -\tan v$

or, $\frac{dx}{x} + \cot v\, dv = 0$

Integrating, we get

$\log x + \log \sin v = \log c$

or, $\log (x \sin v) = \log c$

or, $x \sin v = c$

or, $x \sin\left(\frac{y}{x}\right) = c$

which is the required solution.

6. Form a P.D.E., $z = \phi(x + iy) + \phi(x - iy)$

Solution:

Here, $z = \phi(x + iy) + \phi(x - iy)$

Differentiating partially with respect to x, we have

$$\frac{\partial z}{\partial x} = \phi'(x + iy) + \phi'(x - iy)$$

$$\frac{\partial^2 z}{\partial x^2} = \phi''(x + iy) + \phi''(x - iy) \quad \ldots(i)$$

Differentiating partially with respect to x, we have

$$\frac{\partial z}{\partial y} = \phi'(x + iy).i + \phi'(x - iy).(-i)$$

or, $\frac{\partial^2 z}{\partial^2 y} = \phi'(x + iy).i^2 + \phi'(x - iy).(-i)^2$

or, $\frac{\partial^2 z}{\partial^2 y} = \phi'(x + iy).(-1) + \phi'(x - iy).(-1)$

or, $\frac{\partial^2 z}{\partial^2 y} = (-1)[\phi''(x+iy) + \phi''(x-iy)]$

or, $(-1)\frac{\partial^2 z}{\partial^2 y} = [\phi''(x+iy) + \phi''(x-iy)]$ ...(ii)

Subtracting (ii) from (i), we have

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial^2 y} = 0$, which is the required differential equation.

7. **By using the ratio test, test the convergence or divergence of the series**

$$\sum_{n=1}^{\infty} \frac{n!}{(2n-1)!}$$

Solution:

Here, $u_n = \frac{n!}{(2n-1)!}$ and $u_{n+1} = \frac{(n+1)!}{[2(n+1)-1]!} = \frac{(n+1)!}{(2n+2-1)!} = \frac{(n+1)!}{(2n+1)!}$

Now, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+1)!} \times \frac{(2n-1)!}{n!}$

$$= \lim_{n\to\infty} \frac{(n+1)\times n!}{(2n+1)\times 2n\,(2n-1)!} \times \frac{(2n-1)!}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{(2n+1)\times 2n}$$

$= 0 < 1$, so the given series is convergence.

8. **Test the following series for convergence by Cauchy's root test:**

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \ldots$$

Solution:

Given that the series is

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots + \frac{n^2-1}{n^2+1}x^n + \ldots, x > 0.$$

Here, $u_n = \frac{n^2-1}{n^2+1}x^n$

$$\therefore \lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2-1}{n^2+1}x^n}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}}x^n}$$

$$= x$$

Hence the series is convergent for $x < 1$ and divergent for $x > 1$.

When $x = 1$, the series will be

$$1 + \frac{3}{5} + \frac{8}{10} + \frac{15}{17} + \ldots + \frac{n^2 - 1}{n^2 + 1} + \ldots, \text{ and its } n^{th} \text{ term is } v_n = \frac{n^2 - 1}{n^2 + 1}$$

Now, $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{n^2 - 1}{n^2 + 1}$

$$= \lim_{n\to\infty} \frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n^2}}$$

$= 1 \neq 0$, so the series is divergent for $x = 1$.

Hence, the series convergent for $x < 1$ and divergent for $x \geq 1$.

9. Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{10^n}$

Solution:

We have

$$u_n = \frac{(x-3)^n}{10^n} \text{ and } u_{n+1} = \frac{(x-3)^{n+1}}{10^{n+1}}$$

Now, $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{10^{n+1}} \times \frac{10^n}{(x-3)^n}\right|$

$$= \lim_{n\to\infty} \left|\frac{x-3}{10}\right|$$

$$= \left|\frac{x-3}{10}\right|$$

So the series is convergent for

$$\left|\frac{x-3}{10}\right| < 1$$

or, $-1 < \frac{x-3}{10} < 1$

or, $-10 < x - 3 < 10$

or, $-10 + 3 < x < 10 + 3$

or, $-7 < x < 13$

and divergent for $\left|\frac{x-3}{10}\right| > 1$.

For $x = -7$,

$$\sum_{n=1}^{\infty}\frac{(x-3)^n}{10^n}=\sum_{n=1}^{\infty}\frac{(-7-3)^n}{10^n}=\sum_{n=1}^{\infty}\frac{(-10)^n}{10^n}=\sum_{n=1}^{\infty}(-1)^n \text{ whose value is } -1$$

or 0 according as n is even or odd and which is not convergent.

For $x = -7$,

$$\sum_{n=1}^{\infty}\frac{(x-3)^n}{10^n}=\sum_{n=1}^{\infty}\frac{(13-3)^n}{10^n}=\sum_{n=1}^{\infty}\frac{(10)^n}{10^n}=\sum_{n=1}^{\infty}1=n \text{ which diverges.}$$

$\therefore$ Interval of convergence $-7 < x < 13$, *i.e.*, $(-7, 13)$ and radius of convergence $= \dfrac{13-(-7)}{2} = 10$ Ans

10. Define periodic function and its period. Find the least period (fundational period) of $\tan\frac{3x}{5}$.

Solution:

Periodic function

A function f which satisfies $f(x + k) = f(x)$ for all x belonging to the domain and $k > 0$ is said to be periodic function. The smallest value of k is known as the period of the function.

For second part

Let $f(x) = \tan\frac{3x}{5}$

If p is the smallest period of f(x), then

$$f(x + p) = f(x)$$

or, $\tan\left(\dfrac{3(x+p)}{5}\right) = \tan\dfrac{3x}{5}$

or, $\tan\left(\dfrac{3x+3p}{5}\right) = \tan\left(\dfrac{3x}{5}+\pi\right)$ [$\because$ fundamental period of tangent function is π]

or, $\dfrac{3x+3p}{5} = \dfrac{3x}{5}+\pi$

or, $\dfrac{3x}{5}+\dfrac{3p}{5} = \dfrac{3x}{5}+\pi$

or, $\dfrac{3p}{5} = \pi$

or, **Error! Bookmark not defined.** $p = \dfrac{5\pi}{3}$

Hence, the smallest period of $\tan\frac{3x}{5}$ is $\frac{5\pi}{3}$.

11. Find the Taylor's series expansion of $f(x) = \frac{1}{1-x}$ at $x = 0$.

Solution:

Let $f(x) = \frac{1}{1-x}$ $\qquad \therefore f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2}$ $\qquad \therefore f'(0) = 1$

$f''(x) = \frac{2}{(1-x)^3}$ $\qquad \therefore f''(0) = 2$

$f'''(x) = \frac{6}{(1-x)^4}$ $\qquad \therefore f'''(0) = 6$

$f^{iv}(x) = \frac{24}{(1-x)^5}$ $\qquad \therefore f^{iv}(0) = 24$

$f^{v}(x) = \frac{120}{(1-x)^6}$ $\qquad \therefore f^{iv}(0) = 120$

... ...

Now, using Maclaurin's theorem, we have:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \ldots$$

$$\text{or, } \frac{1}{1-x} = 1 + x + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(6) + \frac{x^4}{4!}(24) + \frac{x^5}{5!}(120) + \ldots$$

$$= 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \text{ Ans}$$

12. Find $\frac{du}{dt}$ of $u = e^{xyz}, x = t^3, y = \frac{1}{t}, z = e^t$

Solution:

Here, $u = e^{xyz}, x = t^3, y = \frac{1}{t}, z = e^t$

$$\frac{\partial u}{\partial x} = e^{xyz} yz \qquad \frac{\partial u}{\partial y} = e^{xyz} zx \qquad \frac{\partial u}{\partial z} = e^{xyz} xy$$

$$\frac{dx}{dt} = 3t^2, \qquad \frac{dy}{dt} = -\frac{1}{t^2} \qquad \frac{dz}{dt} = e^t$$

Hence, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$= e^{xyz} yz \times 3t^2 + e^{xyz} zx \times -\frac{1}{t^2} + e^{xyz} xy \times e^t$$

$$= e^{xyz}(yz \times 3t^2 + zx \times -\frac{1}{t^2} + xy \times e^t)$$

$$= e^{t^2 \times 1/t\ e^t}(3t^2 \times \frac{1}{t}e^t - e^t t^3 \times \frac{1}{t^2} + t^3 \frac{1}{t} \times e^t)$$

$$= e^{te^t}(3te^t - te^t + t^2 e^t)$$

$$= e^{te^t}(2te^t + t^2 e^t)$$

$$= e^{te^t}\ te^t(2 + t)$$

$$= t\, e^{te^t + e^t}(2 + t)$$

$$= t\, e^{e^t(t + 1)}(2 + t)$$

12. A binary operation is defined on the set of real number as below:

$m * n = 2mn + m + 3n$ for all $m, n \in R$.

Find the identity element and inverse element of 2 and – 4.

Solution:

Let e be the identity element, then for all $x \in R$,

$x * e = e * x = x$

or, $2xe + x + 3e = x$

or, $2xe + 3e = 0$

or, $e(2x + 3) = 0$

or, $2e(2x + 3) = 0$

or, $e = 0$ $[\because 2x + 3 \neq 0]$

$\therefore$ 0 is the identity element.

Again, let a be the inverse of 2, then

$2 * a = 0$

or, $2.2a + 2 + 3a = 0$

or, $7a + 2 = 0$

or, $a = -\frac{2}{7}$

$\therefore$ Inverse of 2 is $-\frac{2}{7}$.

Let b be the inverse of – 4, then

$-4 * b = 0$

or, $2(-4)b + (-4) + 3b = 0$

or, $-8b - 4 + 3b = 0$

or, $-5b - 4 = 0$

or, $b = -\frac{4}{5}$

$\therefore$ Inverse of – 4 is $-\frac{4}{5}$.

EXAM 2075 (REGULAR/BACK)

Group 'A' [3×(5+5) = 30]

1. a) **Define partial derivative of a function. Using definition, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x, y) = xy + y^2$.**

Solution:

Partial derivative of a function

A partial derivative of a function of two or three variables is the ordinary derivative with respect to one of the variables when all the rest variables are kept constant.

For Numerical Part

We have, $f(x, y) = xy + y^2$

By the definition

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)y + y^2 - (xy + y^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{xy + y\Delta x + y^2 - xy - y^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{y\Delta x}{\Delta x}$$

$$= y$$

$$\therefore \frac{\partial f}{\partial x} = y$$

and $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x(y + \Delta y) + (y + \Delta y)^2 - (xy + y^2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{xy + x\Delta y + y^2 + 2y\,\Delta y + (\Delta y)^2 - xy - y^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y(x + 2y + \Delta y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} (x + 2y + \Delta y)$$

$$= x + 2y$$

$$\therefore \frac{\partial f}{\partial y} = x + 2y$$

b) Define homogeneous function. State and prove Euler's theorem for a two variable cases.

Solution:

Homogeneous function

A function f(x, y) of two independent variables x, and y is said to be homogeneous of degree n if it can be expressed in either of the form

$$x^n \phi(y/x) \text{ or } y^n \phi(x/y).$$

Alternatively, f(x, y) is said to be homogeneous function of degree n if $f(xt, yt) = t^n f(x, y)$ for all values of t. Similarly, a function f(x, y, z) of three independent variables x, y and z is said to be homogeneous of degree n if it can be expressed in either of the form

$$x^n\phi\left(\frac{y}{x}, \frac{z}{x}\right) \text{ or, } y^n\phi\left(\frac{x}{y}, \frac{z}{y}\right) \text{ or } z^n\phi\left(\frac{x}{z}, \frac{y}{z}\right)$$

Euler's theorem on homogeneous function of two independent variables:

If f(x, y) be a homogeneous function of two independent variables x and y of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\, f(x, y)$.

Proof:

Since f(x, y) is a homogeneous function of x and y of degree n, so

let $f(x, y) = x^n f(y/x) = x^n \phi(v)$, where $v = \frac{y}{x}$

$$\therefore \frac{\partial f}{\partial x} = n\, x^{n-1}\phi(v) + x^n \phi'(v)\frac{\partial v}{\partial x}$$

$$= n\, x^{n-1}\phi(v) + x^n \phi'(v)\frac{-y}{x^2} \qquad \left[\because v = \frac{y}{x} \Rightarrow \frac{\partial v}{\partial x} = \frac{-y}{x^2}\right]$$

$$= n\, x^{n-1}\phi(v) + x^{n-2} \phi'(v)$$

and $\frac{\partial f}{\partial x} = x^n \phi'(v)\frac{\partial v}{\partial y} = x^n\phi'(v)\frac{1}{x} = x^{n-1}\phi'(v)$

$$\text{Now, } x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x[n\, x^{n-1}\phi(v) + x^{n-2}\phi'(v)] + y[x^{n-1}\phi'(v)]$$

$$= n\, x^n\phi(v) - x^{n-1}y\,\phi'(v) + x^{n-1}y\,\phi'(v)$$

$$= n\, x^n \phi(v)$$

$$= n\, f(y/x)$$

$$= n\, f(x, y)$$

$$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\, f(x, y)$$

Note: *If f(x, y, z) be a homogeneous function of three independent variables x, y and z of degree n, then* $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = n\, f(x, y, z)$.

2. a) **Define ordinary differential equation. Solve by separation of variables. (any one)**

i) $\frac{dy}{dx} = \frac{x+y}{x+y+1}$ ii) $\frac{dy}{dx} + 1 = e^{x+y}$

Solution:

(i) The given differential equation is

$$\frac{dy}{dx} = \frac{x+y}{x+y+1} \quad \text{.......(i)}$$

Put $x + y = v$

$$\therefore \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Hence, equation (i) will reduce to

$$\frac{dv}{dx} - 1 = \frac{v}{v+1}$$

or, $\frac{dv}{dx} = \frac{v}{v+1} + 1$

or, $\frac{dv}{dx} = \frac{v+v+1}{v+1}$

or, $\frac{dv}{dx} = \frac{2v+1}{v+1}$

or, $\frac{v+1}{2v+1}\,dv = dx$

or, $\frac{1}{2}\left(1 + \frac{1}{2v+1}\right) dv = dx$

or, $\left(1 + \frac{1}{2v+1}\right) dv = 2dx$

Integrating, we get

$$v + \frac{1}{2}\log(2v+1) = 2x + k$$

or, $2v + \log(2v+1) = 2x + 2k$ .....(iii)

Restoring the value of $v = x + y$ in (ii), we get,

$$2(x+y) + \log(2x + 2y + 1) = 2x + 2k,$$

or, $2y + \log(2x + 2y + 1) = c$, (where $c = 2k$) which is required solution.

ii) The given differential equation is

$$\frac{dy}{dx} + 1 = e^{x+y} \quad .....(i)$$

put $x + y = v$

$$\therefore \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

or, $\frac{dv}{dx} - 1 = \frac{dy}{dx}$

After substitution equation (i) gives

$$\frac{dv}{dx} - 1 + 1 = e^v$$

or, $\frac{dv}{dx} = e^v$

or, $e^{-v}\, dv = dx$

Integrating, we get

$$-e^{-v} = x + c$$

or, $-e^{-x+y} = x + c$

or, $-\frac{1}{e^{x+y}} = x + c$

or, $(x + c)\, e^{x+y} + 1 = 0$ which is the required solution.

b) Show the given equation is homogeneous and solve:

$$x \sin\frac{y}{x}\, dy = \left(y \sin\frac{y}{x} - x\right) dx$$

Solution:

The given differential equation is

$$x \sin\left(\frac{y}{x}\right) dy = \left(y \sin\frac{y}{x} - x\right) dx$$

or, $\frac{dy}{dx} = \frac{y \sin\frac{y}{x} - x}{x \sin\frac{y}{x}} \quad ....(i)$

Put $y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

After substitution, equation (i) gives

$$v + x\frac{dv}{dx} = \frac{vx \sin v - x}{x \sin v}$$

or, $v + x\frac{dv}{dx} = \frac{v \sin v - 1}{\sin v}$

or, $x\frac{dv}{dx} = \frac{v \sin v - 1}{\sin v} - v$

or, $x\frac{dv}{dx} = \frac{v \sin v - 1 - v \sin v}{\sin v}$

or, $x\frac{dv}{dx} = \frac{-1}{\sin v}$

or, $\sin v \, dv + \frac{dx}{x} = 0$

Integrating, we get

$-\cos v + \log x = c$

or, $-\cos y/x + \log x = c$

or, $\log x = \cos y/x + c$ which is the required solution.

OR

Show that the given equation is exact and solve:

$(2ax + by)\, y\, dx + (ax + 2by)\, x\, dy = 0$

Solution:

The given differential equation is

$(2ax + by)y\, dx + (ax + 2by)x\, dy = 0$

or, $(2axy + by^2)dx + (ax^2 + 2bxy)dy = 0$

Comparing this equation with $Mdx + Ndy = 0$, we get

$M = 2axy + by^2$ and $N = ax^2 + 2bxy$

$\frac{\partial M}{\partial y} = 2ax + 2by$ and $\frac{\partial N}{\partial x} = 2ax + 2by$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, thus the given differential equation is exact.

Now, $\int M\, dx = \int (2axy + by^2)\, dx$

$= ax^2y + bxy^2$

There is no term free from x in N so, we need not find $\int N\, dy$.

Hence, the required solution is $ax^2y + bxy^2 = c$, which is the required solution.

3. a) Discuss the convergence of the given geometric series for $r = 1, -1$ and $|r| < 1, |r| > 1$:

$r + r^2 + r^3 + ... + r^{n-1} + ...$

Solution:

Here, the given series is

$r + r^2 + r^3 + ... + r^{n-1} + ...$

The sequence $\{s_n\}$ of partial sum is given by

$s_n = r + r^2 + r^3 + ... + r^n$

$$= \frac{r(1 - r^n)}{1 - r}$$

$$= \frac{r}{1 - r} - \frac{r^n + 1}{1 - r}$$

Since $r^n \to 0$ as $n \to \infty$ if $-1 < r < 1$, therefore $\{s_n\} \to \frac{r}{1 - r}$, so the given series converges to $\frac{r}{1 - r}$ if $|r| < 1$.

If $r = 1$, then $r^n = 1$ for all $n \in N$ and then $s_n = n$, which shows that $s_n \to +\infty$. In this case $\sum r^n$ diverges to $+\infty$.

If $r > 1$, then $r^n > 1$ for all $n \in N$ and then $s_n > n$, which shows that $s_n \to +\infty$. In this case the given series diverges to $+\infty$.

If $r = -1$, $s_n = -1$ or 0 according as n is odd or even, so $\{s_n\}$ oscillates finitely and hence the given series oscillates infinitely.

If $r < -1$, $\{s_n\}$ oscillates infinitely and hence the given series oscillates infinitely.

b) Test the convergence of series by comparison test or ratio test. (any one)

i) $\sum \frac{\sqrt{n}}{n^2 + 1}$ **ii)** $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \dots \dots \dots$

olution:

i) $\sum \frac{\sqrt{n}}{n^2 + 1}$

The given series is $\sum u_n = \sum \frac{\sqrt{n}}{n^2 + 1}$

Let $\sum v_n = \sum \frac{1}{n^{3/2}}$

So, $$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \left(\frac{\frac{\sqrt{n}}{n^2 + 1}}{\frac{1}{n^{3/2}}} \right)$$

$$= \lim_{n \to \infty} \frac{\sqrt{n} \, . \, n^{3/2}}{n^2 + 1}$$

$$= \lim_{n \to \infty} \frac{n^2}{n^2 + 1}$$

$$= \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}}$$

$= 1$ which is finite and non-zero.

Hence, $\sum u_n$ and $\sum v_n$ converge or diverge together. As the series $\sum \frac{1}{n^p}$ convergent for $p > 1$, $\sum v_n$ is convergent. Hence, the given series is convergent.

ii) $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \ldots\ldots\ldots$

The given series is $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \ldots\ldots\ldots$

Here, $u_n = \frac{n}{n+1}x^{n+1}$ and $u_{n+1} = \frac{n+1}{n+2}x^{n+2}$

Now, $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{n+1}{n+2}x^{n+2} \times \frac{(n+1)}{n\,x^{n+1}}$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)}x$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)^2}{n^2\left(1+\frac{2}{n}\right)}x$$

$$= x$$

Hence, the series convergence for $x < 1$ and divergence for $x > 1$.

If $x = 1$, then the series is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots\ldots\ldots$ and its n^{th} term is $v_n = \frac{n}{n+1}$.

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)}$$

$= 1 \neq 0$, so thr series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots\ldots\ldots$ is divergent.

Hence, the given series is convergent for $x < 1$ and divergent for $x \geq 1$. Ans

Group 'B'

[5 × 10 = 50]

4. Find $\frac{du}{dt}$ (Any One):

i) $u = z + \sin(xy)$, $x = t$, $y = \log t$, $z = e^{t-1}$

ii) $u = x^3 - y^3$, $x = \cos t$, $y = \sin t$

Solution

i) $u = z + \sin(xy)$, $x = t$, $y = \log t$, $z = e^{t-1}$

$$\frac{\partial u}{\partial x} = \cos(xy) \cdot y, \qquad \frac{\partial u}{\partial y} = \cos(xy) \cdot x, \qquad \frac{\partial u}{\partial z} = 1$$

$$\text{And } \frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = \frac{1}{t}, \qquad \frac{dz}{dt} = e^{t-1}$$

$$\therefore \frac{du}{dt} = [\cos(xy) \cdot y] \cdot 1 + [\cos(xy) \cdot x] \cdot \frac{1}{t} + 1 \cdot e^{t-1}$$

$$= \cos(t \log t) \times \log t + \cos(t \log t) \times t \cdot \frac{1}{t} + 1 \cdot e^{t-1}$$

$$= \log t \cos(t \log t) + \cos(t \log t) + e^{t-1}$$

$$= \cos(t \log t)(\log t + 1) + e^{t-1} \text{ Ans.}$$

ii) $u = x^3 - y^3$, $x = \cos t$, $y = \sin t$

Differentiating $u = x^3 - y^3$ partially with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = 3x^2 \text{ and } \frac{\partial u}{\partial y} = -3y^2$$

Again differentiating $x = \cos t$, $y = \sin t$ with respect to t, we get

$$\frac{dx}{dt} = -\sin t \qquad \text{and } \frac{dy}{dt} = \cos t$$

We have,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 3x^2(-\sin t) + (-3y^2)\cos t$$

$$= 3\cos^2 t(-\sin t) - 3\sin^2 t \cos t$$

$$= -3\cos^2 t \sin t - 3\sin^2 t \cos t$$

$$= -3\sin t \cos t(\cos t + \sin t) \text{ Ans}$$

5. Form a partial differential equation:

$(lx + my + nz) = f(x^2 + y^2 + z^2)$.

Solution:

The given differential equation is

$$lx + my + nz = f(x^2 + y^2 + z^2) \qquad \ldots(i)$$

Differentiating (i) partially with respect to x and y, we get

$$l + n\frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2)\,(2x + 2z\frac{\partial z}{\partial x})$$

$$l + np = f'(x^2 + y^2 + z^2)\,(2x + 2zp) \qquad \text{...(ii)}$$

and $m + n\frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2)\,(2y + 2z\frac{\partial z}{\partial y})$

or, $m + nq = f'(x^2 + y^2 + z^2)\,(2y + 2z\,q)$ ....(iii)

Dividing equation (ii) by (iii), we get

$$\frac{l + np}{m + nq} = \frac{2x + 2zp}{2y + 2zq}$$

or, $\frac{l + np}{m + nq} = \frac{x + zp}{y + zq}$

or, $(l + np)\,(y + zq) = (m + nq)\,(x + zp)$

or, $l\,y + lzq + nyp + nzpq = mx + mzp + nxq + nz\,pq$

or, $(ny - mz)p + (l\,z - nx)q = mx - ly$, which is the required differential equation.

6. Solve the partial differentiate equation: $y^3p - xyq = x(z - 2y)$.

Solution:

The given differential equation is

$$y^2p - xyq = x(z - 2y)$$

Comparing equation (i) with Pp + Qq = R, we have

$P = y^2$, $\quad Q = -xy$, $\quad R = x(z - 2y)$

Here, the auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)}$ ....(i)

Considering first two members of the equations

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

or, $\frac{dx}{y} = \frac{dy}{-x}$

or, $x\,dx = -y\,dy$

Integrating, we get

$$\frac{x^2}{2} = -\frac{y^2}{2} + k$$

$x^2 + y^2 = c_1$, where $c_1 = 2k$ ...(ii)

From last two equations of (i), we get

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

or, $\frac{dy}{-y} = \frac{dz}{(z-2y)}$

or, $-z\,dy + 2y\,dy = y\,dz$

or, $y\,dy = y\,dz + z\,dy$

On integrating, we get

$$\frac{y^2}{2} = yz + k$$

or, $y^2 - 2yz = 2k$

or, $y^2 - 2yz = c_2$ ...(iii)

From (ii) and (iii), we have

$x^2 + y^2 = f(y^2 - 2yz)$, which is the required solution.

7. Find the interval and radius of convergence of the power series:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Solution:

The given series is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

Here, $u_n = \frac{x^{n-1}}{(n-1)!}$ and $u_{n+1} = \frac{x^n}{n!}$

Now, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^n}{n!} \times \frac{(n-1)!}{x^{n-1}}\right|$

$$= \lim_{n\to\infty}\left|\frac{x}{n}\right|$$

$$= 0 < 1$$

Hence, the series is convergent for all values of x. The interval of convergent is $(-\infty, \infty)$ and radius of convergence is $R = \infty$.

8. Find the Taylor's series expansion of $f(x) = \sqrt{x}$ about $x = 1$.

Solution:

Here, $f(x) = \sqrt{x} = x^{1/2}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(1) = \frac{1}{2}$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{4}x^{-3/2} \qquad f''(1) = -\frac{1}{4}$$

$$f'''(x) = -\frac{1}{4}\left(-\frac{3}{2}\right)x^{-5/2} = \frac{3}{8}x^{-5/2} \qquad f'''(1) = \frac{3}{8}$$

$$f^{iv}(x) = \frac{3}{8}\left(-\frac{5}{2}\right)x^{-7/2} = -\frac{15}{16}x^{-7/2} \qquad f^{iv}(1) = -\frac{15}{16}$$

$$f^{v}(x) = -\frac{15}{16}\left(-\frac{7}{2}\right)x^{-9/2} = \frac{105}{32}x^{-9/2} \qquad f^{v}(1) = \frac{105}{32}$$

... ...

The Taylor's series expansion of $f(x) = \sqrt{x}$ about $x = 1$ is given by

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{iv}(1) + \ldots$$

$$\text{or, } \sqrt{x} = 1 + (x-1)\frac{1}{2} + \frac{(x-1)^2}{2!}\left(-\frac{1}{4}\right) + \frac{(x-1)^3}{3!}\frac{3}{8} + \frac{(x-1)^4}{4!}\left(-\frac{15}{16}\right) + \ldots$$

$$= 1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16} - \frac{5(x-1)^4}{128} + \ldots$$

9. Obtain the Fourier series: $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 2 & 0 < x < 2 \end{cases}$

Solution:

The Fourier series of the function f(x) in the interval (– 2, 2) is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right]$$

$$\text{where, } a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0\,dx + \frac{1}{2}\int_{0}^{1} 2\,dx$$

$$= 0 + \frac{1}{2}[2x]_0^2$$

$$= \frac{1}{2} \times 2 \times 2$$

$$= 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0.\cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi x}{2}dx$$

$$= 0 + \int_{0}^{2} \cos\frac{n\pi x}{2}dx$$

$$= \left[\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{2}{n\pi}\left[\frac{\sin 2n\pi}{2}\right] = 0$$

and $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx$

$$= \frac{1}{2}\int_{-2}^{0} 0.\cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2} 2\sin\frac{n\pi x}{2}dx$$

$$= \left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{2}{n\pi}\left[\frac{-\cos n\pi + \cos 0}{n}\right]$$

$$= \frac{2}{n\pi}\left[\frac{1-(-1)^n}{n}\right]$$

Substituting these values in(i)

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\frac{1-(-1)^n}{n}\right)\sin\frac{n\pi x}{2}$$

$$= 1 + \frac{4}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \dots\right)$$

10. Find the Fourier series for the function defined as

$$f(x) = \begin{cases} \pi & -1 < x < 0 \\ -\pi & 0 \le x < 1 \end{cases}$$

Solution:

The Fourier series of the function f(x) in the interval (– 1, 1) is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{1} + b_n \sin\frac{n\pi x}{1}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\pi x + b_n \sin n\pi x]$$

where, $a_0 = \int_{-1}^{1} f(x)\,dx$

$$= \int_{-1}^{0} \pi\,dx + \int_{0}^{1} (-\pi)\,dx$$

$$= [\pi x]_{-1}^{0} + [-\pi x]_{0}^{1}$$

$$= 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx$$

$$= \int_{-1}^{1} f(x)\cos n\pi x\,dx$$

$$= \int_{-1}^{0} \pi.\cos n\pi x\,dx + \int_{0}^{1} (-\pi)\cos n\pi x\,dx$$

$$= \left[\frac{\pi \sin n\pi x}{n\pi}\right]_{-1}^{0} - \left[\frac{\pi \sin n\pi x}{n\pi}\right]_{0}^{1}$$

$$= \left[\frac{\sin n\pi x}{n}\right]_{-1}^{0} - \left[\frac{\sin n\pi x}{n}\right]_{0}^{1}$$

$$= \frac{1}{n}(\sin n\pi x - \sin n\pi x)$$

$$= 0$$

and $b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx$

$$= \int_{-1}^{1} f(x)\sin n\pi x\,dx$$

$$= \int_{-1}^{0} \pi \sin n\pi x \, dx + \int_{1}^{1} (-\pi) \sin n\pi x \, dx$$

$$= \left[\frac{\pi \cos n\pi x}{-n\pi}\right]_{-1}^{0} - \left[\frac{\pi \cos n\pi x}{-n\pi}\right]_{0}^{1}$$

$$= \left[\frac{\cos n\pi x}{-n}\right]_{-1}^{0} - \left[\frac{\pi \cos nx}{-n}\right]_{0}^{1}$$

$$= \frac{1}{-n}\{1 - \cos(-n\pi)\} - \{\cos n\pi - 1\}$$

$$= \frac{1}{-n}(1 - \cos n\pi - \cos n\pi + 1)$$

$$= \frac{2}{n}(\cos n\pi - 1)$$

Substituting these values in(i), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(\cos n\pi - 1) \sin n\pi x$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} \sin n\pi x$$ which is the required series.

11. Define binary operation. Show that multiplication (×) is binary on the set $S = \{1, \omega, \omega^2\}$, where ω is the cube root of unity.

Solution:

Here, the given set is $S = \{1, \omega, \omega^2\}$ and the operation is ordinary multiplication '×'.

The Cayley's table is given below:

×	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

This shows that all the entries in the Cayley's table are the elements of the set. Thus the given operation ordinary multiplication '×' is a binary operation on the given set S.

12. Find the identity element for the binary operation is defined as $x * y = x + y + 1$ for every $x, y \in R$. Also find the inverse of 2 and -3.

Sol^n: Let e be the identity element, so that for all $x \in R$,

$x * e = e * x = x$

or, $x + e + 1 = x$

or, $e + 1 = 0$

or, $e = -1$

Thus the identity element $e = -1$.

Now, 'a' be the inverse element of 2 under the operation $*$, then

$2 * a = e$

or, $2 + a + 1 = -1$ [$\because e = -1$]

or, $a = -4$

So, -4 is the inverse element of 2.

Again, let b be the inverse element of -3 under $*$, then

$-3 * b = e$

or, $-3 + b + 1 = -1$ [$\because e = -1$]

or, $b = 1$

So, 1 is the inverse element of -1.

13. Let $G = R - \{-1\}$, the set of real numbers without -1. An operation $*$ is defined on G by $x * y = x + y + xy$ for all $x, y \in G$. Show that $(G, *)$ is a group.

Solution:

The given set $G = \mathbf{R} - \{-1\}$.

i) For closure property

For $x, y \in G$, $x * y = x + y + xy \in G$, so G is closed under the given operation $*$.

ii) For associative property

For $x, y, z \in G$, $x * (y * z) = x * (y + z + yz)$

$= x + y + z + yz + x(y + z + yz)$

$= x + y + z + yz + xy + xz + xyz$

and $(x * y) * z = (x + y + xy) * z$

$= x + y + xy + z + (x + y + z)z$

$= x + y + z + yz + xy + xz + xyz$

Thus, $x * (y * z) = (x * y) * z$, so associative law is satisfied by G under the given operation $*$.

iii) **For the identity element**

For all $x \in G = R - \{1\}$, $x + 0 = 0 + x = x$ and $0 \in G$, so 0 is the identity element of G.

iv) **For inverse element**

Again, let x' be the inverse of x, then

$$x * x' = 0$$

or, $x + x' + xx' = 0$

or, $x'(1 + x) = -x$

or, $x' = \frac{-x}{1+x}$, which is the inverse element of x for $x \neq -1$.

Hence, each elements of G has its inverse element in G.

Hence, the set G forms the group under the given operation.

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EXAM 2074 (REGULAR/BACK)

Attempt any two questions from group A and three questions from group B.
Attempt All questions.

Group 'A' $[3 \times (5+5) = 30]$

1. **Define Fourier series in the interval $(-\pi, \pi)$. Find the Fourier series of the function** $f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$

Solution:

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(i)$$

By Euler's formula, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, fx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} (-1)\, dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\, dx$$

$$= \frac{1}{\pi}[-x]_{-\pi}^{-\pi/2} + \frac{1}{\pi}[x]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi/2} -\cos nx\, dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} \cos nx\, dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{-\sin\frac{\pi}{2}}{n}\right] + \frac{1}{\pi}\left[\frac{-\sin\frac{n\pi}{2}}{n}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} -\sin nx\, dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0.dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi}^{\pi/2}$$

$$f(x) = \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi - \cos n\pi + \cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$\therefore f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\cos\frac{n\pi}{2} - \cos nx\right]\sin n\pi$$

$$= \frac{2}{p}\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right] \text{ Ans.}$$

a) Using Maclaurin's series expand the function of f(x) = tan x

Solution:

Let $f(x) = \tan x$ $\qquad \therefore f(0) = 0$

$f'(x) = \sec^2 x$ $\qquad \therefore f'(0) = 1$

$= 1 + \tan^2 x$

$= 1 + [f(x)]^2$

$f''(x) = 2f(x).f'(x)$ $\qquad \therefore f''(0) = 0$

$f'''(x) = 2f'(x).f'(x) + 2f(x)f''(x)$ $\qquad \therefore f'''(0) = 2$

$= 2[f'(x)]^2 + 2f(x)f''(x)$

$f^{iv}(x) = 4f'(x)f''(x) + 2f'(x)f''(x) + 2f(x)f'''(x)$ $\qquad \therefore f^{iv}(0) = 0$

$f^{v}(x) = 4f''(x).f''(x) + 4f'(x)f'''(x) + 2f''(x)f''(x) + 2f'(x)f'''(x)$

$+ 2f(x)f^{iv}(x) + 2f(x)f'''(x)$ $\qquad \therefore f^{v}(0) = 16$

Now, using Maclaurin's series, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \ldots$$

$$\text{or, } \tan x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

b) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Solution:

The given series is

$$\sum u_n = \sum \frac{\sqrt{n}}{n^2+1}$$

Let $\sum v_n = \sum \frac{1}{n^{3/2}}$

So, $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} \right)$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\,.\,n^{3/2}}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}}$$

$= 1$ which is finite and non-zero.

Hence, $\sum u_n$ and $\sum v_n$ converge or diverge together. As the series $\sum \frac{1}{n^p}$ is convergent for $p > 1$, $\sum v_n$ is convergent. Hence, the given series is convergent.

3. **a) If $f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3$; find f_{xx}, f_{xy}, f_{yx} f_{yy}.**

b) Find total differential $\frac{du}{dt}$ if $u = (x + y)e^{xy}$, $x = t$, $y = \frac{1}{t^2}$.

Solution:

a) Here, $f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3$

$$f_x = 3x^2 + 6xy + 3y^2$$

$$f_y = 3x^2 + 6xy + 3y^2$$
$$f_{xx} = 6x + 6y$$
$$f_{xy} = 6x + 6y$$
$$f_{yx} = 6x + 6y$$
$$f_{yy} = 6x + 6y$$

b) Here, $u = (y + x)e^{xy}$ ...(i)

and $x = t, y = \frac{1}{t^2}$ ...(ii)

Differentiating (i) partially with respect to x and y respectively, we get

$$\frac{du}{dx} = 1.e^{xy} + (y + x).e^{xy}.x = e^{xy}(1 + y^2 + xy)$$

$$\frac{du}{dy} = 1.\ e^{xy} + (y + x)\ e^{xy}\ .x = e^{xy}(1 + xy + x^2)$$

Again Differentiating (ii) with respect to, x and y, we get

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = \frac{-2}{t^3}$$

Now, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$

$$= e^{xy}(1 + y^2 + xy).1 + e^{xy}(1 + xy + x^2)\left(-\frac{2}{t^3}\right)$$

$$= e^{xy}\left[(1 + y^2 + xy) + (1 + xy + x^2)\left(-\frac{2}{t^3}\right)\right]$$

$$= e^{t\times\frac{1}{t^2}}\left[1 + y^2 + xy - \frac{2}{x^3} - \frac{2y}{x^2} - \frac{2}{x}\right]$$

$$= e^{\frac{1}{t}}\left[1 + \frac{1}{t^4} + t\frac{1}{t^2} - \frac{2}{t^3} - \frac{2}{t^2\times t^2} - \frac{2}{t}\right]$$

$$= e^{\frac{1}{t}}\left[1 + \frac{1}{t^4} + \frac{1}{t} - \frac{2}{t^3} - \frac{2}{t^4} - \frac{2}{t}\right]$$

$$= e^{\frac{1}{t}}\left(\frac{t^4 + 1 + t^3 - 2t - 2 - 2t^3}{t^4}\right)$$

$$= e^{\frac{1}{t}}\left(\frac{t^4 - t^3 - 2t - 2}{t^4}\right)$$

Group 'B' [5 × 10 = 50]

4. **Solve : $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}$**

Solution:

The given equation is

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x} \qquad \ldots(i)$$

The equation (i) is of the form $\frac{dy}{dx} + py = Q$, where, $P = \frac{1}{x}$ and $Q = \frac{1}{x}$

$\therefore$ I. F. $= e^{\int(1/x)\,dx}$

$= e^{\log x}$

$= x$

Now, multiplying both sides of (i) by x, we get

$$x\frac{dy}{dx} + y = 1$$

or, $d(xy) = 1\,dx$

Integrating , we get

$xy = x + c$, which is the required solution.

5. Solve the homogeneous differential equation: $\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$.

Solution:

The given differential equation is

or, $$\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3} \quad \text{...(i)}$$

The equation (i) being homogeneous differential equation, so put $y = vx$

$\therefore$ $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

After substituting the equation (i) becomes

$$v + x\frac{dv}{dx} = \frac{x^2.\,vx}{x^3 + v^3x^3}$$

or, $$v + x\frac{dv}{dx} = \frac{v}{1 + v^3}$$

or, $$x\frac{dv}{dx} = \frac{v}{1 + v^3} - v$$

or, $$x\frac{dv}{dx} = \frac{v - v - v^4}{1 + v^3}$$

or, $$x\frac{dv}{dx} = -\frac{v^4}{1 + v^3}$$

or, $$\frac{1 + v^3}{v^4}dv = -\frac{dx}{x}$$

or, $$\left(\frac{1}{v^4} + \frac{1}{v}\right) + \frac{dx}{x} = 0$$

Integrating, we get

$$-\frac{1}{3v^3} + \log v + \log x = k$$

or, $$-\frac{1}{3v^3} + \log vx = k$$

or, $\log \frac{y}{x} \cdot x = \frac{1}{3\left(\frac{y}{x}\right)^3} + k$

or, $3 \log y = \frac{x^3}{y^3} + 3k$

or, $3 \log y = \frac{x^3}{y^3} + c$ (c = 3k) which is the required equation.

6. Show that the equation $(x^3 + 3x^2y)dx + (3x^2y + y^3)dx = 0$ is exact and solve it.

Solution:

The given differential equation is

$$(x^3 + 3x\,y^2)dx + (3x^2y + y^3)dy = 0$$

Comparing this equation with Mdx + Ndy = 0, we get

$$M = x^3 + 3xy^2 \text{ and } N = 3x^2y + y^3$$

Now, $\frac{\partial M}{\partial y} = 6xy$ and $\frac{\partial N}{\partial x} = 6xy$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, thus the given differential equation is exact.

Now, $\int M\,dx = \int (x^3 + 3xy^2)\,dx$

$$= \frac{x^4}{4} + 3x^2y^2$$

Taking the term free from x in N and integrating it with respect to y,

$$\int y^3\,dy = \frac{y^4}{4}$$

Hence, the required solution is

$$\frac{x^4}{4} + 3x^2y^2 + \frac{y^4}{4} = k$$

or, $x^4 + 12x^2y^2 + y^4 = 4k$

or, $x^4 + 12x^2y^2 + y^4 = c$, where c = 4k.

7. Discuss the convergence of the series:

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots + \frac{n^2-1}{n^2+1}x^n + \ldots$$

Solution:

Here, ignoring the first term $u_n = \frac{n^2-1}{(n^2+1)}x^n$ and

$$u_{n+1} = \frac{(n+1)^2 - 1}{(n+1)^2 + 1}x^{n+1}$$

Now, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2 - 1}{(n+1)^2 + 1} \times \frac{x^{n+1}}{x^n}$

$$= \lim_{n\to\infty} \frac{(n+1)^2 - 1}{(n+1)^2 + 1} \times \frac{n^2+1}{n^2-1} . x$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 - \frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}} \times \frac{1+\frac{1}{n^2}}{1-\frac{1}{n^2}} . x$$

$= x,$

So, the series is convergent if $x < 1$ and divergent if $x > 1$.

When $x = 1$, the series is $1 + \frac{3}{5} + \frac{8}{10} + \frac{15}{17} + \ldots \ldots + \frac{n^2-1}{n^2+1}$

n^{th} term $U_n = \frac{n^2-1}{n^2+1}$

Now, $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} = 1 \neq 0$, hence the series is divergent for $x = 1$.

Hence, the series convergent for $x < 1$ and divergent for $x \geq 1$.

8. Show that the series $1 - \frac{2}{3} + \frac{3}{3\cdot 2} - \frac{4}{3\cdot 3} + \ldots$ is absolutely convergent.

Solution:

The given series is $1 - \frac{2}{3} + \frac{3}{3.2} - \frac{4}{3.3} + \ldots$

Here, $|u_n| = \frac{n}{3(n-1)}$ and $u_{n+1} = \frac{n+1}{3n}$ [∵ After ignoring the first term]

Now, $\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{(n+1)}{3n} \times \frac{3(n-1)}{n}$

$$= \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right). 3n\left(1-\frac{1}{n}\right)}{3n \times n}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right). 3\left(1-\frac{1}{n}\right)}{3}$$

$= 1$

Hence, by the ratio test the series $\sum |u_n|$ is convergent. Thus the given series is absolutely convergent.

9. **Form the partial differential equation by eliminating f(x) from the equation $x + y + z = f(x^2 + y^2 + z^2)$.**

Solution:

The given differential equation is

$$x + y + z = f(x^2 + y^2 + z^2) \quad \text{...(i)}$$

Differentiating (i) partially with respect to x and y, we get

$$1 + \frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2)\,(2x + 2z\frac{\partial z}{\partial x})$$

$$1 + p = f'(x^2 + y^2 + z^2)\,(2x + 2zp) \quad \text{...(ii)}$$

and $$1 + \frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2)\,(2y + 2z\frac{\partial z}{\partial y})$$

or, $$1 + q = f'(x^2 + y^2 + z^2)\,(2y + 2z\,q) \quad \text{....(iii)}$$

Dividing equation (ii) by (iii), we get

$$\frac{1+p}{1+q} = \frac{2x + 2zp}{2y + 2zq}$$

or, $$\frac{1+p}{1+q} = \frac{x + zp}{y + zq}$$

or, $(1 + p)(y + zq) = (1 + q)(x + zp)$

or, $y + zq + yp + zpq = x + zp + xq + z\,pq$

or, $(y - z)p + (z - x)q = x - y$, which is the required differential equation.

10. **Solve the partial differential equation: $\frac{y^2 z}{x} p + zxq = y^2$**

Solution:

The given differential equation is

$$\frac{y^2 z}{x} p + zxq = y^2 \quad \text{.....(i)}$$

Comparing equation (i) with Pp + Qq = R, we have

$$P = \frac{y^2 z}{x}, \qquad Q = zx, \qquad R = y^2$$

The auxiliary equations are

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{zx} = \frac{dz}{y^2} \qquad [\because \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}]$$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{zx} = \frac{dz}{y^2} \quad \text{....(ii)}$$

Considering first members of the equations, we have

$$\frac{x\,dx}{y^2 z} = \frac{dy}{zx}$$

or, $\frac{x\,dx}{y^2} = \frac{dy}{x}$

or, $x^2\,dx = y^2\,dy$

Integrating, we get

$$\frac{x^3}{3} = \frac{y^3}{3} + k$$

or, $x^3 - y^3 = 3k$

or, $x^3 - y^3 = c_1$ ....(i)

Considering first and last members of the equations, we have

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

or, $x\,dx = y\,dy$

Integrating, we get

$$\frac{x^2}{2} = \frac{y^2}{2} + k$$

or, $x^2 - y^2 = 2k$

or, $x^2 - y^2 = c_2$

From equation (i) and (ii), the required solution is

$$f(x^3 - y^3, x^2 - y^2) = 0$$

11. Find the smallest period of sin 5x.

Solution:

Let $f(x) = \sin 5x$

If p is the smallest period of f(x), then

$$f(x + p) = f(x)$$

or, $\sin 5(x + p) = \sin 5x$

or, $\sin 5(x + p) = \sin (5x + 2\pi)$ [$\because$ fundamental period of sine functi is 2π]

$\therefore$ $5x + 5p = 5x + 2\pi$

or, $5p = 2\pi$

or, $p = \frac{2\pi}{5}$

Thus, the smallest period of sin 5x is $\frac{2\pi}{5}$.

12. **Prove that**

i) the identity element of group (G, ∗) is unique.

ii) In a group, the inverse of an element is unique.

Solution:

i) The identity element of group (G, ∗) is unique.

Let e be an identity element in a group (G, ∗). If possible, let e' be another identity element. Then we have

$e * e' = e' * e = e'$, assuming e as the identity element

Also, $e * e' = e' * e = e$, assuming e' as the identity element.

Hence, $e = e'$, *i.e.*, there is one and only one identity element.

ii) **Every element in a group (G, ∗) has unique inverse.**

Let x be an inverse of $a \in G$.

Then, $a * x = e = x * a$, ...(i)

where e is the identity element.

If possible, let $a \in G$ has another inverse, say y.

Then, $a * y = e = y * a$ ....(ii)

From equation (i) and (ii), we have

$a * x = a * y$ [∵ each equal to e]

Now, $a * x = a * y \Rightarrow x * (a * x) = x * (a * y)$

$\Rightarrow (x * a) * x = (x * a) * y$

$\Rightarrow e * x = e * y$, from equation (i).

$\Rightarrow x = y$

This shows that inverse of every $a \in G$ is unique.

13. **Given algebraic structure (G, ∗) with G = R − {1} the set of numbers without the unit number and '∗' stands for the binary operation defined by $x * y = x + y - xy$ for all $x, y \in G$. Find the identity element and inverse elements of 3 and −2.**

Solution:

Let e be the identity element.

So, for all $x \in R - \{1\}$

$x * e = e * x = x$

or, $x + e - xe = x$

or, $e - xe = 0$

or, $e(1 - x) = 0$

or, $e = 0$ [∵ $x - 1 \neq 0$]

∴ 0 is the identity element of G.

Again, let a be the inverse of 3, then

$$3 * a = 0$$

or, $3 + a - 3a = 0$

or, $3 - 2a = 0$

or, $a = \frac{3}{2}$

$\therefore$ Inverse of 3 is $\frac{3}{2}$.

Let b be the inverse of – 2, then

$$-2 * b = 0$$

or, $-2 + b + 3b = 0$

or, $-2 + 3b = 0$

or, $b = \frac{2}{3}$

$\therefore$ Inverse of – 2 is $\frac{2}{3}$.

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EXAM 2073 (REGULAR/BACK)

Attempt All questions.

Group 'A' $[3 \times (5+5) = 30]$

1. a) **If the binary operation '*' set of rational number Q is defined by $x * y = x + y - xy$, for every $x, y \in Q$. Show that ' *' is commutative and associative.**

Solution:

Let $x, y \in Q$,

$$x * y = x + y - xy = y + x - yx = y * x$$

Hence, $x * y = y * x$, so $*$ is commutative.

Also, for $x, y, z \in Q$,

$$x * (y * z) = x * (y + z - yz)$$
$$= x + y + z - yz - x(y + z - yz)$$
$$= x + y + z - yz - xy - xz + xyz$$

and $(x * y) * z = (x + y - xy) * z$

$$= x + y - yz + z - (x + y - xy)z$$
$$= x + y + z - yz - xy - xz + xyz$$

Hence, $x * (y * z) = (x * y) * z$, so the $*$ is associative.

b) **Let $(G, *)$ be a group and $g \in G$. Then inverse of g is unique.**

Soluton:

Let x be an inverse of $g \in G$.

Then, $g * x = e = x * g$, ...(i)

where e is the identity element.

If possible, let $g \in G$ has another inverse, say y.

Then, $g * y = e = y * g$ ....(ii)

From equation (i) and (ii), we have

$g * x = g * y$ [$\because$ each equal to e]

Now, $g * x = g * y \Rightarrow x * (g * x) = x * (g * y)$

$$\Rightarrow (x * g) * x = (x * g) * y$$
$$\Rightarrow e * x = e * y, \text{ from equation (i).}$$
$$\Rightarrow x = y$$

This shows that inverse of every $g \in G$ is unique.

2. a) Solve: $\frac{dy}{dx} = e^{x+y} - 1$

Solution:

The given differential equation is

$$\frac{dy}{dx} + 1 = e^{x+y} \quad .....(i)$$

put $x + y = v$

$\therefore \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dv}{dx} - 1 = \frac{dy}{dx}$

After substitution equation (i) gives

$$\frac{dv}{dx} - 1 + 1 = e^v$$

or, $\frac{dv}{dx} = e^v$

or, $e^{-v}\, dv = dx$

Integrating, we get

$$-e^{-v} = x + c$$

or, $-e^{-x+y} = x + c$

or, $-\frac{1}{e^{x+y}} = x + c$

or, $(x + c)\, e^{x+y} + 1 = 0$ which is the required solution.

b) Show that the given equation is exact and solve:
$(2xy + e^y)dx + (x^2 + xe^y)\, dy = 0$

Solution:

The given differential equation is

$$(2xy + e^y)dx + (x^2 + xe^y)\, dy = 0$$

Comparing this equation with $Mdx + Ndy = 0$, we get

$$M = 2xy + e^y \text{ and } N = x^2 + xe^y$$

Now, $\frac{\partial M}{\partial y} = 2x + e^y$ and $\frac{\partial N}{\partial x} = 2x + e^y$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, thus the given differential equation is exact.

Now, $\int M\, dx = \int (2xy + e^y)\, dx = \frac{2x^2y}{2} + xe^y = x^2y + xe^y$

Taking the term free from x in N and integrating it with respect to y,(but there is no term free from x, so we need not to integrate with respect to y)

Hence, the required solution is $x^2y + xe^y = c$ Ans

3. a) **Define convergent and divergent of the series. Test whether the series is convergent or divergent: $\frac{1}{2}+\frac{3}{5}+\frac{5}{8}+\ldots$**

Solution:

Convergent and divergent series

An infinite series $\sum u_n$ is said to be convergent if the sequence $\{S_n\}$ of partial sums is convergent. If $\{S_n\} \to S$, we say that $\sum u_n \to S$. The series is said to be divergent or oscillatory according as $\{S_n\}$ diverges or oscillates.

For second Part

The given series is $\frac{1}{2}+\frac{3}{5}+\frac{5}{8}+\ldots$

Here, $u_n = \frac{2n-1}{3n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2n-1}{3n-1}$$

$$= \lim_{n\to\infty} \frac{n\left(2-\frac{1}{n}\right)}{n\left(3-\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)}{\left(3-\frac{1}{n}\right)}$$

$= \frac{2}{3} \neq 0$, so the given series is divergent.

b) **Show that the series is conditionally convergent:**

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$$

Solution:

Here the given series is $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$

Here, $\sum |u_n| = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots$

$= \sum \frac{1}{n^{1/2}}$, which is a p-series and $n = \frac{1}{2} < 1$.

Thus the series is divergent.

Now, $\sum u_n$ is an alternating series and $u_n = \frac{1}{\sqrt{n}}$ and $u_{n+1} = \frac{1}{\sqrt{n+1}}$

Clearly $u_n > u_{n+1}$ for all n and $\{u_n\}$ is monotonically decreasing.

Also, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

Thus by Leibnitz's test, the series is convergent.

Hence, the given series is conditionally convergent.

Group 'B' **[5 × 10 = 50**

4. Form a partial differential equation (Any one):

i) $z = ax + by + a^2 + b^2$ ii) $lx + my + nz = f(x^2 + y^2 + z^2)$

Solution:

i) $z = ax + by + a^2 + b^2$

The given equation is

$$z = ax + by + a^2 + b^2 \quad \text{....(i)}$$

Equation (i) contains two arbitrary constants a and b.

Differentiating (i) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = a$$

or, $p = a$ ....(ii)

Differentiating (i) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = b$$

or, $q = b$ ....(iii)

Substituting values of a and b, in given equation, we get

$z = px + qy + p^2 + q^2$, which is the required differential equation.

ii) $lx + my + nz = f(x^2 + y^2 + z^2)$

The given differential equation is

$$lx + my + nz = f(x^2 + y^2 + z^2) \quad \text{...(i)}$$

Differentiating (i) partially with respect to x and y, we get

$$l + n\frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2)\,(2x + 2z\frac{\partial z}{\partial x})$$

$$l + np = f'(x^2 + y^2 + z^2)\,(2x + 2zp) \quad \text{...(ii)}$$

and $m + n\frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2)\,(2y + 2z\frac{\partial z}{\partial y})$

or, $m + nq = f'(x^2 + y^2 + z^2)\,(2y + 2z\,q)$ ....(iii)

Dividing equation (ii) by (iii), we get

$$\frac{l + np}{m + nq} = \frac{2x + 2zp}{2y + 2zq}$$

or, $\dfrac{l + np}{m + nq} = \dfrac{x + zp}{y + zq}$

or, $(l + np)(y + zq) = (m + nq)(x + zp)$

or, $ly + lzq + nyp + nzpq = mx + mzp + nxq + nz\,pq$

or, $(ny - mz)p + (lz - nx)q = mx - ly$, which is the required differential equation.

5. Solve the partial differential equation:

$$\mathbf{x(y - z)\,p + y(z - x)\,q = z(x - y)}$$

Solution:

The given differential equation is

$$x(y - z)\,p + y(z - x)\,q = z(x - y)$$

which is of the form $P\,p + Q\,q = R$, where $P = x(y - z)$, $Q = y(z - x)$ and $R = z(x - y)$.

Here, the auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)} = \frac{dx + dy + dz}{x(y - z) + y(z - x) + z(x - y)}$$

$$= \frac{dx + dy + dz}{0} \qquad \text{...(i)}$$

$\therefore$ $dx + dy + dz = 0$,

which on integration gives

$$x + y + z = c_1 \qquad \text{...(ii)}$$

Again, (i) can be written as

$$\frac{\frac{dx}{x}}{y - z} = \frac{\frac{dy}{y}}{z - x} = \frac{\frac{dz}{z}}{x - y} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$\therefore$ $\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z} = 0$

On integration, we get

$$\log x + \log y + \log z = \log c_2$$

or, $\log(xyz) = \log c_2$

or, $xyz = c_2 \qquad \text{...(iii)}$

From (ii) and (iii), the general solution is

$$xyz = f(x + y + z)$$

6. **Find the interval and radius of convergence of power series:**

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Solution:

We have, $u_n = \frac{(-1)^{n+1}x^n}{n}$ and $u_{n+1} = \frac{(-1)^{n+2}x^{n+1}}{n+1}$

Now, $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{n+1} \times \frac{n}{(-1)^{n+1}x^n}\right|$

$$= \lim_{n\to\infty} \left| -\frac{n}{n+1} x \right|$$

$$= \lim_{n\to\infty} \left| -\frac{1}{1+\frac{1}{n}} x \right|$$

$$= |x|$$

So the series is convergent for $|x| < 1$, *i.e.*, $-1 < x < 1$ and divergent for $|x| > 1$. when $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots\ldots$ .i.e., an alternating series with $u_n = \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. So by Leibnitz's test the series is convergent. For $x = -1$, the given series is

$$-1 - \frac{(-1)^2}{2} + \frac{(-1)^3}{3} - \frac{(-1)^4}{4} + \frac{(-1)^5}{5} + \ldots\ldots\ldots$$

$$= -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \ldots\ldots\ldots$$

$$= -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots\ldots\right)$$

Which is divergent (by p – test $\frac{1}{n^p}$ is divergent for $p < 1$.)

Hence the interval of convergent is $(-1, 1]$, radius of convergence 1 and centre of convergence is 0.

7. **Assuming the convergence of Taylor's series, find the Maclaurin's expansion sin x and log (1 + x).**

Solution:

Let $f(x) = \log(1 + x)$ $\quad\therefore f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\quad\therefore f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\quad\therefore f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\quad\therefore f'''(0) = 2$

$f^{iv}(x) = -\frac{6}{(1+x)^4}$ $\quad\therefore f^{iv}(0) = -6$

$f^{v}(x) = \frac{24}{(1+x)^5}$ $\quad\therefore f^{v}(0) = 24$

... ...

Now, using Maclaurin's theorem, we have

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \ldots$$

$$\text{or,}\quad \log(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \ldots$$

8. Using the definition, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from $f(x, y) = xy + y^2$.

Solution:

Here, $f(x, y) = xy + y^2$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)y + y^2 - (xy + y^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{xy + y\Delta x + y^2 - xy - y^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{y\Delta x}{Dx}$$

$$= y$$

$$\therefore \frac{\partial f}{\partial x} = y$$

$$\text{and}\quad \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x(y+\Delta y) + (y+\Delta y)^2 - (xy + y^2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{xy + x\Delta y + y^2 + 2y\,\Delta y + (\Delta y)^2 - xy - y^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y(x + 2y + \Delta y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} (x + 2y + \Delta y)$$

$$= x + 2y$$

$$\therefore \frac{\partial f}{\partial y} = x + 2y$$

9. Find f_x, f_y and f_z where f(x, y, z) is given by $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

Solution:

Here, $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$f_x = \frac{2x}{2(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{x}{(x^2 + y^2 + z^2)^{1/2}}$$

$$f_y = \frac{2y}{2(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{y}{(x^2 + y^2 + z^2)^{1/2}}$$

$$f_z = \frac{2z}{2(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{z}{(x^2 + y^2 + z^2)^{1/2}}$$

10. Let $u = \frac{x^4 + y^4}{x + y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$.

Solution:

Here, $u = \frac{x^4 + y^4}{x + y} = \frac{x^4\left(1 + \frac{y^4}{x^4}\right)}{\left(1 + \frac{y}{x}\right)} = x^3\,\phi\left(\frac{y}{x}\right)$

Thus u is a homogeneous function of degree 3. So, by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u \qquad \text{...(i)}$$

$$\frac{\partial u}{\partial x} = \frac{(x + y).\,4x^3 - (x^4 + y^4)\,.\,1}{(x + y)^2}$$

$$= \frac{4x^4 + 4x^3y - x^4 - y^4}{(x + y)^2}$$

$$= \frac{3x^4 + 4x^3y - y^4}{(x + y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x + y).\,4y^3 - (x^4 + y^4)\,.\,1}{(x + y)^2}$$

$$= \frac{4xy^3 + 4y^4 - x^4 - y^4}{(x + y)^2}$$

$$= \frac{4xy^3 + 3x^4 - x^4}{(x + y)^2}$$

$$\text{Now, } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x(3x^4 + 4x^3y - y^4)}{(x+y)^2} + \frac{y(4xy^3 + 3y^4 - x^4)}{(x+y)^2}$$

$$= \frac{3x^5 + 4x^4y - xy^4 + 4xy^4 + 3y^5 - x^4y}{(x+y)^2}$$

$$= \frac{3x^5 + 3x^4y + 3xy^4 + 3y^5}{(x+y)^2}$$

$$= \frac{3(x^5 + x^4y + xy^4 + y^5)}{(x+y)^2}$$

$$= \frac{3(x+y)(x^4 + y^4)}{(x+y)^2}$$

$$= \frac{3(x^4 + y^4)}{x+y}$$

$$= 3u$$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$, proved.

11. Define Fourier series. Determine the Fourier coefficients a_0 and a_n by Euler's Formula.

Solution:

A series of sines and cosines of an angle and its multiple of the form

$$\frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots +$$
$$b_1\sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called the Fourier series, where $a_0, a_1, a_2, \ldots a_n, \ldots b_1, b_2, b_3, \ldots, b_n \ldots$ are constants are Fourier coefficients.

Determination of Fourier coefficients (Euler's Formula)

The Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + a_n \cos nx + \ldots + b_1\sin x +$$
$$b_2 \sin 2x + \ldots + b_n \sin nx + \ldots \qquad \text{(i)}$$

To find a_0

Integrating both sides of (i) from $x = 0$ to $x = 2\pi$, we have

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1\int_0^{2\pi}\cos x\,dx + a_2\int_0^{2\pi}\cos 2x\,dx + \dots +$$

$$a_n\int_0^{2\pi}\cos nx\,dx + \dots + b_1\int_0^{2\pi}\sin x\,dx + b_2\int_0^{2\pi}\sin 2x\,dx + \dots +$$

$$b_n\int_0^{2\pi}\sin nx\,dx$$

$$= \frac{a_0}{2}\int_0^{2\pi} dx$$

$$= \frac{a_0}{2}[x]_0^{2\pi}$$

$$= \frac{a_0}{2}\times 2\pi$$

$$\therefore\ a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

To find a_n:

Multiplying both sides of (i) by cos nx and integrating from x = 0, x = 2π, we have

$$\int_0^{2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos nx\,dx + a_1\int_0^{2\pi}\cos x\cos nx\,dx + \dots +$$

$$a_n\int_0^{2\pi}\cos^2 nx\,dx + \dots + b_1\int_0^{2\pi}\sin x\cos nx\,dx +$$

$$b_2\int_0^{2\pi}\sin 2x\cos nx\,dx + \dots$$

$$= a_n\int_0^{2\pi}\cos^2 nx$$

$$= a_n\pi$$

$$\therefore\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

By taking n = 1, 2, 3, …. we can find the values of a_1, a_2, a_3…

12. **Find the Fourier series of given function in the given interval:**

$$f(x) = \begin{cases} 1 & \text{for } -\pi \le x \le 0 \\ -1 & \text{for } 0 \le x \le \pi \end{cases}$$

Solution:

Here, for $x > 0$, $f(x) = -1$, then $f(-x) = 1 = -(-1) = -f(x)$

and for $x < 0$, $f(x) = 1$, then $-x > 0$, $f(-x) = -1 = -f(x)$

So, in either case $f(-x) = -f(x)$, hence $f(x)$ is odd function.

The Fourier series corresponding to the odd function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad \ldots(i)$$

$$\text{where } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$\text{Now, } b_n = \frac{1}{\pi}\int_{-\pi}^{0} (1) \sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} (-1) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-1}{n} + \frac{\cos n\pi}{n}\right] + \frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

Substituting these values of b_n in (i), we get

$$f(x) = -\frac{4}{\pi}\sin x - \frac{4}{3\pi}\sin 3x - \frac{4}{5\pi}\sin 5x - \ldots$$

$$= -\frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right]$$

OR, $f(x) = x^2$, for $-\pi < x < \pi$.

Solution:

Here, $f(x) = x^2$

Now, $f(-x) = (-x)^2$, $f(x)$ is even function. The Fourier series corresponding to the even function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ldots(i)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

Now, $a_0 = \frac{1}{\pi}\sum_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^3}{3} - 0\right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} - \frac{2x(-\cos nx)}{n^2} + \frac{2(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2p\frac{\cos nx}{n^2}\right]$$

$$= \frac{4\pi \cos nx}{\pi\ \ n^2}$$

$$= \frac{4\cos n\pi}{n^2}$$

$$= \begin{cases} \frac{4}{n^2} & \text{if n is even} \\ \frac{-4}{n^2} & \text{if n is odd} \end{cases}$$

$$\therefore a_n = \frac{4(-1)^n}{n^2}$$

Substituting these values of a_0, and a_n in (i), we get

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

or, $$x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - \ldots\right)$$

13. **Find the Fourier series of f(x) defined in the interval (– 2, 2) as**

$$f(x) = \begin{cases} 0 & \text{for } -2 < x < 0 \\ 2 & \text{for } 0 < x < 2 \end{cases}$$

Solution:

The Fourier series of the function f(x) in the interval (– 2, 2) is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right]$$

$$\text{where, } a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0\,dx + \frac{1}{2}\int_{0}^{1} 2\,dx$$

$$= 0 + \frac{1}{2}[2x]_0^2$$

$$= \frac{1}{2}\times 2\times 2$$

$$= 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\ \cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0.\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi x}{2}\,dx$$

$$= 0 + \int_{0}^{2}\cos\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{2}{n\pi}\left[\frac{\sin 2n\pi}{2}\right] = 0$$

and $b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin \frac{n\pi x}{2} dx$

$$= \frac{1}{2}\int_{-2}^{0} 0.\cos \frac{n\pi x}{2} dx + \frac{1}{2}\int_{0}^{2} 2 \sin \frac{n\pi x}{2} dx$$

$$= \left[\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{2}{n\pi}\left[\frac{-\cos n\pi + \cos 0}{n}\right]$$

$$= \frac{2}{n\pi}\left[\frac{1-(-1)^n}{n}\right]$$

Substituting these values in(i), we have

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\frac{1-(-1)^n}{n}\right) \sin \frac{n\pi x}{2}$$

$$= 1 + \frac{4}{\pi}\left(\sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} + \frac{1}{5} \sin \frac{5\pi x}{2} + \ldots\right)$$

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EXAM 2072 (BACK)

Attempt All questions.

Group 'A' $[(5+5)\times 3 = 30]$

1. a) **Prepare Cayley table for the set {0, 1, 2, 3, 4, 5} under the operation multiplication modulo 6. Identify the identity element and the inverse of each element if possible.**

Solution:

Here, the given set S = {0, 1, 2, 3, 4, 5} and the given operation is multiplication modulo 6. The Cayley's table for the set S under the operation multiplication modulo 6 is given below:

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Clearly, from the above table, the identity element for the given set under multiplication modulo 6 is 1.

From the table we also see that the inverse of 1 is 1 and the inverse 5 is 5. The inverse of the other elements 0, 2, 3 and 4 do not exist.

b) **Define group. Prove that the identity element of group is unique. Also show that inverse element of group is unique.**

Solution:

An algebraic structure (G, ∗) with a set G under the binary operation '∗', and denoted by (G, ∗) is known as a group if it is closure, associative, has an identity element and an inverse element.

To prove identity element of group (G, ∗) is unique.

Let e be an identity element in a group (G, ∗). If possible, let e' be another identity element. Then we have

e∗e' = e'∗ e = e', assuming e as the identity element

Also, e∗e' = e'∗e = e, assuming e' as the identity element.

Hence, e = e', *i.e.*, there is one and only one identity element.

To prove every element in a group (G, $*$) has unique inverse.

Let x be an inverse of $a \in G$. ...(i)

Then, $a * x = e = x * a$,

where e is the identity element.

If possible, let $a \in G$ has another inverse, say y. ....(ii)

Then, $a * y = e = y * a$

From equation (i) and (ii), we have

[$\because$ each equal to e]

$a * x = a * y$

Now, $a * x = a * y \Rightarrow x * (a * x) = x * (a * y)$

$\Rightarrow (x * a) * x = (x * a) * y$

$\Rightarrow e * x = e * y$, from equation (i).

$\Rightarrow x = y$

This shows that inverse of every $a \in G$ is unique.

2. a) If $u = \sqrt{x^2 + y^2 + z^2}$, then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

Solution:

Here, $u = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial u}{\partial x} = \frac{2x}{2(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2 + z^2)^{1/2} . 1 - \frac{x}{2}(x^2 + y^2 + z^2)^{-1/2} . 2x}{(x^2 + y^2 + z^2)}$$

$$= \frac{(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{1/2} - x^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{x^2 + y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{2(x^2 + y^2 + z^2)^{1/2}} = \frac{y}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2 + z^2)^{1/2} . 1 - \frac{y}{2}(x^2 + y^2 + z^2)^{-1/2} . 2y}{(x^2 + y^2 + z^2)}$$

$$= \frac{(x^2 + y^2 + z^2)^{1/2}(x^2 + y^2 + z^2)^{1/2} - y^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{x^2 + y^2 + z^2 - y^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial u}{\partial z} = \frac{2z}{2(x^2 + y^2 + z^2)^{1/2}} = \frac{z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{(x^2+y^2+z^2)^{1/2}.1-\frac{z}{2}(x^2+y^2+z^2)^{-1/2}.2z}{x^2+y^2+z^2}$$

$$=\frac{(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)^{1/2}-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}}=\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

Now,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}+\frac{z^2+x^2}{(x^2+y^2+z^2)^{3/2}}+\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{y^2+z^2+z^2+x^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{2}{(x^2+y^2+z^2)^{1/2}}$$

$$=\frac{2}{u}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{u}\text{ proved.}$$

b) Let $u=\frac{x^4+y^4}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u$

Solution:

Here, $u=\frac{x^4+y^4}{x+y}=\frac{x^4\left(1+\frac{y^4}{x^4}\right)}{\left(1+\frac{y}{x}\right)}=x^3\,\phi\left(\frac{y}{x}\right)$

Thus u is a homogeneous function of degree 3. So, by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u \quad \text{...(i)}$$

$$\frac{\partial u}{\partial x}=\frac{(x+y).4x^3-(x^4+y^4).1}{(x+y)^2}$$

$$=\frac{4x^4+4x^3y-x^4-y^4}{(x+y)^2}$$

$$=\frac{3x^4+4x^3y-y^4}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x+y).\,4y^3 - (x^4+y^4).\,1}{(x+y)^2}$$

$$= \frac{4xy^3 + 4y^4 - x^4 - y^4}{(x+y)^2}$$

$$= \frac{4xy^3 + 3x^4 - x^4}{(x+y)^2}$$

Now, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x(3x^4 + 4x^3y - y^4)}{(x+y)^2} + \frac{y(4xy^3 + 3y^4 - x^4)}{(x+y)^2}$

$$= \frac{3x^5 + 4x^4y - xy^4 + 4xy^4 + 3y^5 - x^4y}{(x+y)^2}$$

$$= \frac{3x^5 + 3x^4y + 3xy^4 + 3y^5}{(x+y)^2}$$

$$= \frac{3(x^5 + x^4y + xy^4 + y^5)}{(x+y)^2}$$

$$= \frac{3(x+y)(x^4+y^4)}{(x+y)^2}$$

$$= \frac{3(x^4+y^4)}{x+y}$$

$$= 3u$$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$, proved.

3. a) Define p-series. Test the series for convergence by applying ratio test of $\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \ldots$

Solution:

The given series is $\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \ldots$

Here, $u_n = \frac{n^2}{3^n}$ and $u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$

Now, $\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{3^{n+1}} \times \frac{3^n}{n^2}$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^2$$

$$= \frac{1}{3}\left(1 + \frac{1}{n}\right)^2$$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^2 = \frac{1}{3} < 1.$

Hence, by ratio test the series is convergent.

b) **Find the interval and radius of convergence of the power series:**
$1 + 2x + 3x^2 + 4x^3 + \ldots$

Solution:

Here, $u_n = nx^{n-1}$ and $u_{n+1} = (n+1)x^n$

$$\text{Now, } \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,x^n}{nx^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}x\right|$$

$$= |x|$$

So, the series is convergence for $|x| < 1$, *i.e.*, $-1 < x < 1$ and divergent for $|x| > 1$.

For $x = 1$, the series will be

$1 + 2 + 3 + 4 + \ldots$which is a divergent series.

For $x = -1$, the series will be

$1 - 2 + 3 - 4 + \ldots$which is not a convergent series.

Hence, the interval of convergence is $(-1, 1)$ and the radius of convergence is $\dfrac{1-(-1)}{2} = 1$.

Group 'B' **[5 × 10 = 50]**

4. **Solve: $(xy^2 + x)dx + (yx^2 + y)\,dy = 0$**

Solution:

The given differential equation is

$$(xy^2 + x)dx + (yx^2 + y)\,dy = 0$$

or, $x(y^2 + 1)dx + y(x^2 + 1)dy = 0$

or, $\dfrac{x\,dx}{x^2+1} + \dfrac{y\,dy}{y^2+1} = 0$

or, $\dfrac{2x\,dx}{x^2+1} + \dfrac{2y\,dy}{y^2+1} = 0$

Integrating, we get

$$\log(x^2 + 1) + \log(y^2 + 1) = \log c$$

or, $\log[(x^2 + 1)(y^2 + 1)] = \log c$

or, $(x^2 + 1)(y^2 + 1) = c$, which is the required solution.

5. **Solve: $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$**

Solution:

The given differential equation is

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \qquad \ldots\text{(i)}$$

Put $y = vx$

$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

After substitution, (i) gives

$$v + x\frac{dv}{dx} = v + \tan v$$

or, $x\frac{dv}{dx} = \tan v$

or, $\cot v\, dv = \frac{dx}{x}$

Integrating, we get

$$\log \sin v = \log x + \log c$$

or, $\log \sin v = \log cx$

or, $\sin v = cx$

or, $\sin\frac{y}{x} = cx$ which is the required solution

6. Form the partial differential equations $z = ax + by + a^2 + b^2$

Solution:

The given equation is

$$z = ax + by + a^2 + b^2 \quad \text{....(i)}$$

Equation (i) contains two arbitrary constants a and b.

Differentiating (i) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = a$$

or, $p = a$ ....(ii)

Differentiating (i) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = b$$

or, $q = b$ ....(iii)

Substituting values of a and b, in given equation, we get

$z = px + qy + p^2 + q^2$, which is the required differential equation.

7. Show that the given equation is exact and solve:
$(2ax + by)\, y\, dy + (ax + 2by)x\, dy = 0$

Solution

The given differential equation is

$$(2ax + by)y\, dy + (ax + 2by)x\, dy = 0$$

or, $(2axy + by^2)dy + (ax^2 + 2bxy)dy = 0$

Comparing this equation with $Mdx + Ndy = 0$, we get

$$M = 2axy + by^2 \text{ and } N = ax^2 + 2bxy$$

Now, $\frac{\partial M}{\partial y} = 2axy + 2by$ and $\frac{\partial N}{\partial x} = 2ax + 2by$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, thus the given differential equation is exact.

Now, $\int M\,dx = \int (2axy + by^2)\,dx$
$= ax^2y + bxy^2$

Taking the term free from x in N and integrating it with respect to y,

$\int 2bxy\,dy = bxy^2$

Hence, the required solution is

$ax^2y + bxy^2 + bxy^2 = c$

or, $ax^2y + bxy^2 + bxy^2 = c$

or, $ax^2y + bxy^2 + bxy^2 = c$.

8. Using the definition, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from $f(x, y) = x^2y$.

Solution:

Here, $f(x, y) = x^2y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 y - x^2y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[x^2 + 2x\Delta x + (\Delta x)^2]y - x^2y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2y + 2xy\Delta x + y(\Delta x)^2 - x^2y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2xy\Delta x + y(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2xy + y\Delta x)\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2xy + y\Delta x)$$

$$= 2xy$$

$$\therefore \quad \frac{\partial f}{\partial x} = 2xy$$

and $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2(y + \Delta y) - x^2y}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2y + x^2\Delta y - x^2y}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2\Delta y}{\Delta y}$$

$$= \lim_{\Delta y \to 0} (x^2)$$

$$= x^2$$

$$\therefore \quad \frac{\partial f}{\partial y} = x^2$$

9. Find the Maclaurin's series expansion of cos x.

Solution:

Here, $f(x) = \cos x$ $\qquad f(0) = \cos 0 = 1$

$f'(x) = -\sin x$ $\qquad f'(0) = \sin 0 = 0$

$f''(x) = -\cos x$ $\qquad f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x$ $\qquad f'''(0) = -\sin 0 = 0$

$f^{iv}(x) = \cos x$ $\qquad f^{iv}(0) = -\cos 0 = 1$

... ...

The Maclaurin's series is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$$

or, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

10. Find the smallest positive period p of sin nx.

Solution:

Let $f(x) = \sin nx$

If is the smallest period of $f(x) = \sin nx$ then,

$f(x + p) = f(x)$

or, $\sin n(x + p) = \sin nx$

or, $\sin (nx + np) = \sin(nx + 2\pi)$

or, $nx + np = nx + 2\pi$

or, $np = 2\pi$

or, $p = \frac{2\pi}{n}$

Thus the smallest positive period of $f(x) = \sin nx$ is $\frac{2\pi}{n}$.

11. **Find the Fourier series of given function in the given interval:**

$$f(x) = \begin{cases} 0 & \text{for } -2 < x < 0 \\ 2 & \text{for } 0 < x < 2 \end{cases}$$

Solution:

The Fourier series of the function f(x) in the interval (– 2, 2) is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2} \right]$$

$$\text{where, } a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0\,dx + \frac{1}{2}\int_{0}^{1} 2\,dx$$

$$= 0 + \frac{1}{2}[2x]_0^2$$

$$= \frac{1}{2} \times 2 \times 2$$

$$= 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0.\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi x}{2}\,dx$$

$$= 0 + \int_{0}^{2} \cos\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} \right]_0^2$$

$$= \frac{2}{n\pi}\left[\frac{\sin 2n\pi}{2}\right] = 0$$

$$\text{and } b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0.\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} 2\sin\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{2}{n\pi}\left[\frac{-\cos n\pi + \cos 0}{n}\right]$$

$$= \frac{2}{n\pi}\left[\frac{1-(-1)^n}{n}\right]$$

Substituting these values in(i)

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\frac{1-(-1)^n}{n}\right)\sin\frac{n\pi x}{2}$$

$$= 1 + \frac{4}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \ldots\right)$$

12. Test whether the function is odd or even. Also find the corresponding Fourier series: $\begin{cases} -2x & \text{for } -\pi < x < 0 \\ 2x & \text{for } 0 < x < \pi \end{cases}$

Solution:

Here, for $x > 0$, $f(x) = 2x$, then $f(-x) = -2x = -f(x)$

and for $x < 0$, $f(x) = -2x$, then $f(-x) = 2x = -(-2x) = -f(x)$

So, in either cases $f(-x) = -f(x)$, hence f(x) is odd function.

The Fourier series corresponding to the odd function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\,dx \qquad \ldots(i)$$

where, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Now, $b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-2x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\sin nx\,dx$

$$= \frac{2}{\pi}\int_{-\pi}^{0} -x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$\frac{2}{\pi}\left[\frac{x\cos nx}{n}-\frac{\sin nx}{n^2}\right]_{-\pi}^{0}+\frac{2}{\pi}\left[\frac{-x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$=\frac{2}{\pi}\left(\frac{-\pi\cos n\pi}{n}-0\right)+\frac{2}{\pi}\left(\frac{-\pi\cos n\pi}{n}-0\right)$$

$$=\frac{-2\pi\cos n\pi}{n}-\frac{2\cos n\pi}{n}$$

$$=\frac{-4\cos n\pi}{n}$$

Substituting these values in (i), we get

$$f(x)=\sum_{n=1}^{\infty}\left(\frac{-4\cos n\pi}{n}\right)\sin nx$$

$$=4\left[\frac{\sin nx}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\dots\right]$$

13. Find $\frac{du}{dt}$: $u = x^2 + y^2$, $x = 2t + 1$, $y = t^2 + 2$

Solution:

Here, $u = x^2 + y^2$, $x = 2t + 1$ and $y = t^2 + 2$

Now, $\frac{\partial u}{\partial x} = 2x$, $\qquad \frac{\partial u}{\partial y} = 2y$,

Also, $\frac{dx}{dt} = 2$, $\qquad \frac{dy}{dt} = 2t$

Hence, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$= 2x \times 2 + 2y \times 2t$$

$$= 4(2t + 1) + 4(t^2 + 2)\,t$$

$$= 8t + 4 + 4t^3 + 8t$$

$$= 4t^3 + 16t + 4$$

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EXAM 2071 (REGULAR/BACK)

Attempt All questions.

Group 'A' **$[(5+5)\times 3=30]$**

1. a) **Prepare Cayley table for the set {0, 1, 2, 3} under the operation multiplication modulo 4. Identify the identity element and inverse of each element if possible.**

Solution:

Here, the given set S = {0, 1, 2, 3} and the given operation is multiplication modulo 4. The Cayley's table for the set S under the operation multiplication modulo 4 is given below:

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Clearly, from the above table, we see that $1 \times_4 a = a \times_4 1 = a$ for each element a of the set S. Hence, 1 is the identity element of the set under the multiplication modulo 4.

The element 0 has no inverse element.

We have, $1 \times_4 1 = 1$, so inverse of 1 is 1.

The element 2 has no inverse element

We have, $3 \times_4 3 = 1$, so inverse of 3 is 3.

b) **Define group. Prove that the identity element of group is unique. also show that inverse of each element of group is unique.**

Solution:

An algebraic structure (G, *) with a set G under the binary operation '*', and denoted by (G, *) is known as a group if it is closure, associative, has an identity element and an inverse element.

To prove identity element of group (G, *) is unique.

Let e be an identity element in a group (G, *). If possible, let e' be another identity element. Then we have

$e*e' = e'*e = e'$, assuming e as the identity element

Also, $e*e' = e'*e = e$, assuming e' as the identity element.

Hence, $e = e'$, *i.e.*, there is one and only one identity element.

To prove every element in a group (G, *) has unique inverse.

Let x be an inverse of $a \in G$.

Then, $a * x = e = x * a$, ...(i)

where e is the identity element.

If possible, let $a \in G$ has another inverse, say y.

Then, $a * y = e = y * a$ ....(ii)

From equation (i) and (ii), we have

$$a * x = a * y \quad [\because \text{ each equal to } e]$$

Now, $a * x = a * y \Rightarrow x * (a * x) = x * (a * y)$

$$\Rightarrow (x * a) * x = (x * a) * y$$

$$\Rightarrow e * x = e * y, \text{ from equation (i).}$$

$$\Rightarrow x = y$$

This shows that inverse of every $a \in G$ is unique.

2. a) Solve: $(x + y + 1)\dfrac{dy}{dx} = 1$

Solution:

Here, the given differential equation is

or, $(x + y + 1)\dfrac{dy}{dx} = 1$ ......(i)

Put $x + y + 1 = v$

$\therefore \quad 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$

After substitution, equation (i) gives

$$v\left(\frac{dv}{dx} - 1\right) = 1$$

or, $\dfrac{dv}{dx} - 1 = \dfrac{1}{v}$

or, $\dfrac{dv}{dx} = 1 + \dfrac{1}{v}$

or, $\dfrac{dv}{dx} = \dfrac{1 + v}{v}$

or, $\dfrac{v}{1 + v}\,dv = dx$

or, $\left(1 - \dfrac{1}{1 + v}\right)dv = dx$

Integrating, we get

$$v - \log v = x + k$$

or, $x + y + 1 - \log(x + y + 1) = x + k$

or, $y = \log(x + y + 1) + c$

where $c = k - 1$ which is the required solution.

b) Show that the given equation is exact and solve:

$(x + y - 1)\, dx + (x - y - 2)\, dy = 0$

Soluton:

The given differential equation is

$$(x + y - 1)dx + (x - y - 2)dy = 0$$

Comparing this equation with Mdx + Ndy = 0, we get

$$M = x + y - 1 \text{ and } N = x - y - 2$$

Now, $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, thus the given differential equation is exact.

Now, $\int M\, dx = \int(x + y - 1)\, dx = \frac{x^2}{2} + xy - x$

Taking the term free from x in N and integrating it with respect to y,

$\int(-y - 2)\, dx = -\frac{y^2}{2} - 2y$

Hence, the required solution is

$$\frac{x^2}{2} + xy - x - \frac{y^2}{2} - 2y = k$$

or, $\quad x^2 + 2xy - 2x - 4y = 4k$

or, $\quad x^2 - y^2 + 2xy - 2x - 4y = c$. Ans

3. a) **Define convergent and the divergent series. Test whether the series is convergent or divergent:**

$$1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \ldots.$$

Solution:

Convergent and the divergent series

An infinite series $\sum u_n$ is said to be convergent if the sequence $\{S_n\}$ of partial sums is convergent. If $\{S_n\} \to S$, we say that $\sum u_n \to S$. The series is said to be divergent or oscillatory according as $\{S_n\}$ diverges or oscillates

For second part

The given series is

$$1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \ldots. = \sum_{n=0}^{\infty} \frac{1}{n^2 + 1}$$

Here, $u_n = \frac{1}{n^2 + 1}$

Let $v_n = \frac{1}{n^2}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^2+1} \times n^2$

$$= \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= 1$$

which is finite and non-zero. Thus the series $\sum u_n$ and $\sum v_n$ converge or diverge together. But the series $\sum v_n$ is convergent and hence the given series is convergent.

b) Show the series is conditionally convergent:

$$\mathbf{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.}$$

Solution:

The given series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

The term of the series are alternatively positive and negative.

$$|u_n| = \frac{1}{n} \text{ and } |u_{n+1}| = \frac{1}{n+1}$$

$\therefore \quad |u_{n+1}| < |u_n|$ as $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n\to\infty} u_n = 0$

Thus the given series is convergent.

But $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is divergent (p-series with n = 1).

Hence, the given series is conditionally convergent.

Group 'B' **[5 × 10 = 50]**

4. Form the partial differential equation:(Any one)

i) $z = ke^{ax} \sin ay$ **ii) $lx + my + nz = f(x^2 + y^2 + z^2)$**

Solution:

i) $z = ke^{ax} \sin ay$

The given equation is

$$z = ke^{ax} \sin ay \qquad \ldots(i)$$

Equation (i) contains two arbitrary constants a and k.

Differentiating (i) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = kae^{ax} \sin ay \quad \text{....(ii)}$$

Differentiating (i) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = kae^{ax} \cos ay \quad \text{....(iii)}$$

Again differentiating (ii) with respect to x, we get

$$\frac{\partial^2 z}{\partial x^2} = ka^2e^{ax} \sin ay \quad \text{....(iv)}$$

and differentiating (iii) with respect to y, we get

$$\frac{\partial^2 z}{\partial y^2} = -ka^2e^{ax} \sin ay \quad \text{....(v)}$$

From (iv) and (v) on adding, we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = ka^2e^{ax} \sin ay - ka^2e^{ax} \sin ay$$

or, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, which is the required differential equation.

ii) $lx + my + nz = f(x^2 + y^2 + z^2)$

The given differential equation is

$$lx + my + nz = f(x^2 + y^2 + z^2) \quad \text{...(i)}$$

Differentiating (i) partially with respect to x and y, we get

$$l + n\frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2)\,(2x + 2z\frac{\partial z}{\partial x})$$

$$l + np = f'(x^2 + y^2 + z^2)\,(2x + 2zp) \quad \text{...(ii)}$$

and $m + n\frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2)\,(2y + 2z\frac{\partial z}{\partial y})$

or, $m + nq = f'(x^2 + y^2 + z^2)\,(2y + 2z\,q)$ ....(iii)

Dividing equation (ii) by (iii), we get

$$\frac{l + np}{m + nq} = \frac{2x + 2zp}{2y + 2zq}$$

or, $\frac{l + np}{m + nq} = \frac{x + zp}{y + zq}$

or, $(l + np)(y + zq) = (m + nq)(x + zp)$

or, $ly + lzq + nyp + nzpq = mx + mzp + nxq + nz\,pq$

or, $(ny - mz)p + (lz - nx)q = mx - ly$, which is the required differential equation.

5. Solve the partial differential equations: (Any one)

i) $\frac{\partial z}{\partial x} xz + \frac{\partial z}{\partial y} yz = xy$ ii) $xp - yq + x^2 - y^2 = 0$

Solution:

i) $\frac{\partial z}{\partial x} xz + \frac{\partial z}{\partial y} yz = xy$

The given differential equation is

$$\frac{\partial z}{\partial x} xz + \frac{\partial z}{\partial y} yz = xy$$

or, $xzp + yzq = xy$

Comparing this equation with $Pp + Qq = R$, we have

$$P = xz,\ Q = yz \text{ and } R = xy$$

The auxiliary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

From the first two relation,

$$\frac{dx}{x} = \frac{dy}{y}$$

or, $\log x = \log y + \log c_1$

or, $\log x - \log y = \log c_1$

or, $\log\left(\frac{x}{y}\right) = \log c_1$

or, $\frac{x}{y} = c_1$ ...(i)

From the last two relation,

$$\frac{dy}{z} = \frac{dz}{x}$$

or, $c_1 y\, dy = zdz$ [using (i)]

or, $2c_1 y^2 = 2zdz$

or, $c_1 y^2 = z^2 + c_2$

or, $xy = z^2 + c_2$ [using (i)]

or, $xy - z^2 = c_2$ ...(ii)

From (i) and (ii), the solution is

$$f\left(\frac{x}{y}, xy - z^2\right) = 0 \text{ Ans.}$$

ii) $xp - yq + x^2 - y^2 = 0$

The given differential equation is

$$xp - yq + x^2 - y^2 = 0$$

or, $xp - yq = y^2 - x^2$

Comparing this equation with Pp + Qq = R, we have

$$P = x,\ Q = -y \text{ and } R = y^2 - x^2$$

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$$

From the first two relation,

$$\frac{dx}{x} = \frac{dy}{-y}$$

or, $\log x = -\log y + \log c_1$

or, $\log x + \log y = \log c_1$

or, $\log(xy) = \log c_1$

or, $xy = c_1$ ...(i)

From the last two relation,

$$\frac{dy}{-y} = \frac{dz}{y^2 - x^2}$$

or, $(y^2 - x^2)dy = -y\,dz$

or, $(y^2 - \frac{c_1^2}{y^2})dy = -y\,dz$ [using (i)]

or, $(y - \frac{c_1^2}{y^3})dy = -\,dz$

or, $\frac{y^2}{2} + \frac{x^2 y^2}{2y^2}) = -z + c_2$ [using (i)]

or, $\frac{y^2}{2} + \frac{x^2}{2} + z = c_2$ ...(ii)

Hence, from (i) and (ii), we have

$f(xy) = \frac{y^2}{2} + \frac{x^2}{2} + z$, which is the required solution.

6. Find the interval and radius of convergent of the power series (Any one)

i) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(n+1)}{2} x^{n-1}$ **ii)** $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

Solution:

i) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(n+1)}{2} x^{n-1}$

Here, $u_n = = \frac{(-1)^{n-1} n(n+1)\, x^{n-1}}{2}$ and $u_{n+1} = \frac{(-1)^n (n+1)(n+2)\, x^n}{2}$

Now,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(n+1)(n+2)x^n}{2}\times\frac{2}{(-1)^{n-1}n(n+1)x^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(n+2)x}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n\left(1+\frac{2}{n}\right)x}{n}\right|$$

$$= |x|$$

So, the series is convergent for $|x| < 1$, *i.e.*, $-1 < x < 1$ and divergent for $|x| > 1$.

For $x = 1$, the given series is $1 - 3 + 6 - 10 + 15 - \ldots$ which is an alternating series with $u_n = \frac{n(n+1)}{2}$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n(n+1)}{2} = \infty$.

Hence, by Leibnitz's test the series is not convergent. Similarly, for $x = -1$, the series is not convergent.

Hence the series is converges for $-1 < x < 1$ and the interval of convergence is $(-1, 1)$, radius of convergence = 1 and centre of convergent = 0.

ii) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$

We have

$$u_n = \frac{(x-2)^n}{10^n} \text{ and } u_{n+1} = \frac{(x-2)^{n+1}}{10^{n+1}}$$

$$\text{Now, } \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{10^{n+1}}\times\frac{10^n}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x-2}{10}\right|$$

$$= \left|\frac{x-2}{10}\right|$$

So the series is convergent for

$$\left|\frac{x-2}{10}\right| < 1$$

or, $-1 < \frac{x-2}{10} < 1$

or, $-10 < x - 2 < 10$

or, $-10 + 2 < x < 10 + 2$

or, $-8 < x < 12$

and divergent for $\left|\frac{x-2}{10}\right| > 1$.

For $x = -8$,

$\sum_{n=1}^{\infty}\frac{(x-2)^n}{10^n} = \sum_{n=1}^{\infty}\frac{(-8-2)^n}{10^n} = \sum_{n=1}^{\infty}\frac{(-10)^n}{10^n} = \sum_{n=1}^{\infty}(-1)^n$ whose value is -1 or 0 according as n is even or odd and which is not convergent.

For $x = 12$,

$\sum_{n=1}^{\infty}\frac{(x-2)^n}{10^n} = \sum_{n=1}^{\infty}\frac{(12-2)^n}{10^n} = \sum_{n=1}^{\infty}\frac{(10)^n}{10^n} = \sum_{n=1}^{\infty} 1 = n$ which diverges.

$\therefore$ Interval of convergence $-8 < x < 12$, i.e., $(-8, 12)$ and radius of convergence $= \frac{12-(-8)}{2} = 10$ Ans.

7. **Assuming the convergence of Taylor series, find the Maclaurin's expansion of sin x.**

Solution:

Let $f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{iv}(x) = \sin x$	$f^{iv}(0) = 0$
$f^{v}(x) = \cos x$	$f^{v}(0) = 1$
... ...	... ...

Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots$$

or, $\sin x = 0 + x + 0 + \frac{x_3}{3!}(-1) + 0 + \frac{x^5}{5!} - \ldots$

or, $\sin x = x - \frac{x_3}{3!} + \frac{x^5}{5!} - \ldots$

8. **Using the definition, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from $f(x, y) = x^2 - xy$.**

Solution:

Here, $f(x, y) = x^2 - xy$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - y(x+\Delta x) - (x^2 - xy)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - xy - y\Delta x - x^2 + xy}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 - y\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x - y)$$

$$= 2x - y$$

$$\therefore \quad \frac{\partial f}{\partial x} = 2x - y$$

and $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2 - x(y+\Delta y) - (x^2 - xy)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2 - xy - x\Delta y - x^2 + xy}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{-x\Delta y}{\Delta y}$$

$$= -x$$

$$\therefore \quad \frac{\partial f}{\partial y} = -x$$

9. Find $\frac{du}{dt}$ (any one) ;

i) $u = e^{xyz}, x = t^3, \frac{1}{t}, z = e^t$

ii) $u = x^2 + y^2 + z^2, x = 2t + 1, y = t + 5, z = 7t$

iii) $u = x^2 + y^2 + z^2, x = 2t + 1, y = t + 5, z = 7t$

Solution:

i) Here, $u = e^{xyz}, x = t^3, y = \frac{1}{t}, z = e^t$

$$\frac{\partial u}{\partial x} = e^{xyz}\, yz \qquad \frac{\partial u}{\partial y} = e^{xyz}\, zx \qquad \frac{\partial u}{\partial z} = e^{xyz}\, xy$$

$$\frac{dx}{dt} = 3t^2, \qquad \frac{dy}{dt} = -\frac{1}{t^2} \qquad \frac{dz}{dt} = e^t$$

Hence, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$= e^{xyz}\, yz \times 3t^2 + e^{xyz}\, zx \times -\frac{1}{t^2} + e^{xyz}\, xy \times e^t$$

$$= e^{xyz}(yz \times 3t^2 + zx \times -\frac{1}{t^2} + xy \times e^t)$$

$$= e^{t^2 \times 1/t\ e^t}(3t^2 \times \frac{1}{t} e^t - e^t t^3 \times \frac{1}{t^2} + t^3 \frac{1}{t} \times e^t)$$

$$= e^{te^t}(3te^t - te^t + t^2 e^t)$$

$$= e^{te^t}(2te^t + t^2 e^t)$$

$$= e^{te^t}\, te^t(2 + t)$$

$$= t\, e^{te^t + e^t}(2 + t)$$

$$= t\, e^{e^t(t+1)}(2 + t)$$

ii) Here, $u = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 2y$

$\frac{dx}{dt} = -\sin t + \cos t$, $\frac{dy}{dt} = -\sin t - \cos t$

Hence, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$= 2x(-\sin t + \cos t) + 2y(-\sin t - \cos t)$$

$$= 2[(\cos t + \sin t)(-\sin t + \cos t) + (\cos t - \sin t)(-\sin t - \cos t)$$

$$= 2[(\cos t + \sin t)(\cos t - \sin t) + (-\sin t + \cos t)(-\sin t - \cos t)$$

$$= 2[(\cos^2 t - \sin^2 t) + (-\sin t)^2 - \cos^2 t)]$$

$$= 2[\cos^2 t - \sin^2 t + \sin^2 t - \cos^2 t]$$

$$= 2 \times 0$$

$$= 0$$

$\therefore$ At $t = 0$, $\frac{du}{dt} = 0$

iii) Here, $u = x^2 + y^2 + z^2$, $x = 2t + 1$, $y = t + 5$, $z = 7t$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 2y \qquad \frac{\partial u}{\partial z} = 2z$

$\frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = 1 \qquad \frac{dz}{dt} = 7$

Hence,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 2x \,.\, 2 + 2y \,.\, 1 + 2z \,.\, 7$$

$$= 4x, + 2y + 14z$$

$$= 4(2t + 1) + 2(t + 5) + 14.\ 7t$$

$$= 8t + 4 + 2t + 10 + 98t$$

$$= 108t + 14$$

10. **Let $u = \sin^{-1}\dfrac{x^2 + y^2}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$**

Solution:

Here, $u = \sin^{-1}\dfrac{x^2 + y^2}{x + y}$

or, $\sin u = \dfrac{x^2 + y^2}{x + y}$

or, $z = \dfrac{x^2 + y^2}{x + y}$, where $z = \sin u$

or, $z(x, y) = \dfrac{x^2 + y^2}{x + y}$

or, $z(xt, yt) = \dfrac{(xt)^2 + (yt)^2}{xt + yt}$

$$= \frac{x^2t^2 + y^2t^2}{xt + yt}$$

$$= \frac{t^2(x^2 + y^2)}{t(x + y)}$$

$$= t^1\frac{x^2 + y^2}{x + y}$$

$$= tz$$

Thus z is a homogeneous function of degree 1. Hence, by the Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.\ z$$

or, $x\dfrac{\partial}{\partial x}(\sin u) + y\dfrac{\partial}{\partial y}(\sin u) = \sin u$

or, $x\cos u\dfrac{\partial u}{\partial x} + y\cos u\dfrac{\partial u}{\partial y} = \sin u$

or, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{\sin u}{\cos u}$

$= \tan u$, hence proved.

11. Find the Fourier series of given function in the given interval:

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le \pi \\ 1 & \text{for } \pi \le x \le 2\pi \end{cases}$$

Solution:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ...(i)

where, $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx$

$= 0 + \frac{1}{\pi}[x]_{\pi}^{2\pi}$

$= \frac{1}{\pi}(2\pi - \pi)$

$\therefore a_0 = 1$

$a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_0^{\pi} 0 \times \cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 1.\cos nx\,dx$

$= 0 + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$

$= \frac{1}{\pi}\left[\frac{\sin 2n\pi - \sin \pi}{n}\right]$

$\therefore a_n = 0$

and $b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_0^{\pi} 0 \times \sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 1 \sin nx\,dx$

$= 0 + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{\pi}^{2\pi}$

$$= \frac{1}{\pi}\left[-\frac{\cos 2n\pi}{n} + \frac{\cos n\pi}{n}\right]$$

$$= \frac{1}{n\pi}[-1 + (-1)^n]$$

$$= \begin{cases} \frac{-2}{\pi n} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Substituting these values in (i), we get

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sin x - \frac{2}{3\pi}\sin 3x - \frac{2}{5\pi}\sin 5x - \dots$$

$$= \frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

12. **Test whether the function is odd or even. Also find the corresponding Fourier series:** $\begin{cases} -2x & \text{for } -\pi < x < 0 \\ 2x & \text{for } 0 < x < \pi \end{cases}$

Solution:

Here, for $x > 0$, $f(x) = 2x$, then $f(-x) = -2x = -f(x)$

and for $x < 0$, $f(x) = -2x$, then $f(-x) = 2x = -(-2x) = -f(x)$

So, in either cases $f(-x) = -f(x)$, hence $f(x)$ is odd function.

The Fourier series corresponding to the odd function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\, dx \qquad \dots(i)$$

where, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Now, $b_n = \frac{1}{\pi}\int_{-\pi}^{0} (-2x) \sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} 2x \sin nx\, dx$

$$= \frac{2}{\pi}\int_{-\pi}^{0} -x \sin nx\, dx + \frac{2}{\pi}\int_{0}^{\pi} x \sin nx\, dx$$

$$\frac{2}{\pi}\left[\frac{x\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{-\pi \cos n\pi}{n} - 0\right) + \frac{2}{\pi}\left(\frac{-\pi \cos n\pi}{n} - 0\right)$$

$$= \frac{-2\pi \cos n\pi}{n} - \frac{2 \cos n\pi}{n}$$

$$= \frac{-4 \cos n\pi}{n}$$

Substituting these values in (i), we get

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{-4 \cos n\pi}{n}\right) \sin nx$$

$$= 4\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right]$$

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