Solution Manual to

Based on New Syllabus of CTEVT 2078

CTEVT

Engineering MATHEMATICS III

Diploma (Third Semester)

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UNIT 1

Applications of Derivatives

Exercise 1.1

1. Find from definition, the derivative of

a. $\cos^{-1}x$ b. $\tan^{-1}x$ c. $\ln\cos^{-1}x$

Solution

a. Let $f(x) = \cos^{-1}x$

We have,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\therefore \frac{d}{dx}(\cos^{-1}x) = \lim_{h\to 0}\frac{\cos^{-1}(x+h)-\cos^{-1}x}{h} \quad \ldots(i)$$

Put $\cos^{-1}x = y$ and $\cos^{-1}(x+h) = y+k$

then $x = \cos y$ and $x + h = \cos(y+k)$

$\therefore$ $h = \cos(y+k) - \cos y$

Also, as $h \to 0$, $k \to 0$

From (i)

$$\frac{d}{dx}(\cos^{-1}x) = \lim_{h\to 0}\frac{(y+k)-y}{h}$$

$$= \lim_{h\to 0}\frac{k}{h}$$

$$= \lim_{k\to 0}\left\{\frac{k}{\cos(y+k)-\cos y}\right\}$$

$$= \lim_{k\to 0}\left\{\frac{k}{2\sin\left(\frac{y+k+y}{2}\right)\sin\left(\frac{y-y-k}{2}\right)}\right\}$$

$$= \lim_{k\to 0}\left\{\frac{-1}{\sin\left(y+\frac{k}{2}\right).\ \sin\frac{k}{2}\Big/\frac{k}{2}}\right\}$$

$$= \frac{-1}{\sin y\cdot 1}$$

$$= \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$= \frac{-1}{\sqrt{1-x^2}}$$

b. Let $f(x) = \tan^{-1}x$

$\therefore$ $f(x+h) = \tan^{-1}(x+h)$

We have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore \quad \frac{d}{dx}(\tan^{-1}x) = \lim_{h \to 0} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{h} \qquad \ldots(i)$$

Put $\tan^{-1}x = y$ & $\tan^{-1}(x+h) = y + k$

Then, $x = \tan y$ & $(x+h) = \tan(y+k)$

$\therefore$ $h = \tan(y+k) - \tan y$

Also, as $h \to 0, k \to 0$

Now, from (i)

$$\frac{d}{dx}(\tan^{-1}x) = \lim_{h \to 0} \frac{y+k-y}{h}$$

$$= \lim_{h \to 0} \frac{k}{h} = \lim_{k \to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= \lim_{k \to 0} \left\{ \frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}} \right\}$$

$$= \lim_{k \to 0} \left\{ \frac{k}{\frac{\sin(y+k)\cos y - \cos(y+k)\sin y}{\cos(y+k)\cos y}} \right\}$$

$$= \lim_{k \to 0} \left\{ \frac{k \cos(y+k)\cos y}{\sin(y+k-y)} \right\}$$

$$= \lim_{k \to 0} \left\{ \frac{\cos(y+k)\cos y}{\frac{\sin y}{k}} \right\}$$

$$= \cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

c. Let $f(x) = \ln \cos^{-1}x$

We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\therefore \quad \frac{d}{dx}(\ln \cos^{-1}x) = \lim_{h \to 0} \frac{\ln\cos^{-1}(x+h) - \ln\cos^{-1}x}{h} \qquad \ldots(i)$$

Put $\cos^{-1}x = y$ and $\cos^{-1}(x+h) = y + k$

then $x = \cos y$ and $x + h = \cos(y+k)$

$\therefore$ $h = \cos(y+k) - \cos y$

Also, as $h \to 0, k \to 0$

From (i)

$$\frac{d}{dx}(\ln \cos^{-1}x) = \lim_{h \to 0} \frac{\ln(y+k) - \ln y}{h}$$

$$= \lim_{h \to 0} \frac{\ln\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{k \to 0} \left\{ \frac{\ln\left(1 + \frac{k}{y}\right)}{\frac{k}{y}} \cdot \frac{\frac{k}{y}}{h} \right\}$$

$$= 1 . \frac{1}{y} \lim_{k \to 0} \left\{ \frac{k}{\cos(y+k) - \cos y} \right\}$$

$$= \frac{1}{y} \lim_{k \to 0} \left\{ \frac{k}{2\sin\left(\frac{y+k+y}{2}\right) \sin\left(\frac{y-y-k}{2}\right)} \right\}$$

$$= \frac{1}{y} \lim_{k \to 0} \left\{ \frac{-1}{\sin\left(y+\frac{k}{2}\right) . \sin\frac{\frac{k}{2}}{\frac{k}{2}}} \right\}$$

$$= \frac{1}{y} \cdot \frac{-1}{\sin y \cdot 1} = \frac{-1}{\cos^{-1}x\sqrt{1-\cos^2 y}}$$

$$= \frac{-1}{\cos^{-1} x\sqrt{1-x^2}}$$

2. Find the derivatives of the following functions:

a. $\sin^{-1} x^3$ b. $\cos^{-1}\sqrt{x}$ c. $\sin^{-1}(3x-4x^3)$

d. $\cos^{-1}(4x^3-3x)$ e. $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ f. $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

g. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$. h. $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ i. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

Solution

a. $\sin^{-1} x^3$

Let $y = \sin^{-1} x^3$,

Differentiating both sides w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x^3)$$

$$= \frac{d(\sin^{-1} x^3)}{dx^3} \cdot \frac{dx^3}{dx}$$

$$= \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2$$

$$= \frac{3x^2}{\sqrt{1-x^6}}$$

b. $\cos^{-1}\sqrt{x}$

Let $y = \cos^{-1}\sqrt{x}$

Differentiating both sides w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1}\sqrt{x})$$

$$= \frac{d(\cos^{-1}\sqrt{x})}{d(\sqrt{x})} \cdot \frac{d(\sqrt{x})}{dx}$$

$$= -\frac{1}{\sqrt{1-x}} \cdot \frac{1}{2} x^{\frac{-1}{2}}$$

$$= -\frac{1}{2\sqrt{x}\sqrt{1-x}}$$

c. $\sin^{-1}(3x-4x^3)$

Let $y = \sin^{-1}(3x-4x^3)$

Put: $x = \sin\theta$ then

$y = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$

$= 3\theta$

$= 3\sin^{-1} x$

Differentiating both sides w.r.t. 'x' we get

$$\frac{dy}{dx} = 3.\frac{d}{dx}(\sin^{-1} x)$$

$$= \frac{3}{\sqrt{1-x^2}}$$

d. $\cos^{-1}(4x^3 - 3x)$

Let $y = \cos^{-1}(4x^3 - 3x)$

Put: $x = \cos\theta$, then

$$y = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$$
$$= \cos^{-1}(\cos 3\theta)$$
$$= 3\theta$$
$$= 3\cos^{-1} x$$

Differentiating both sides w.r.t 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx}(3\cos^{-1} x)$$

$$= 3.\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-3}{\sqrt{1-x^2}}$$

e. $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Put: $x = \tan\theta$, the $y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

or, $y = \tan^{-1}(\tan 2\theta)$

or, $y = 2\theta$

or, $y = 2.\tan^{-1} x$

Differentiating both sides w.r.t 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} x) = \frac{2}{1+x^2}$$

f. $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

Let $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

$$= \tan^{-1}\left(\frac{2\sin\frac{x}{2}.\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Differentiating both sides w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

$$\therefore \frac{d}{dx}\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) = \frac{1}{2}$$

g. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$

Then

$$\frac{1-x^2}{1+x^2} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

or, $y = \cos^{-1}(\cos 2\theta)$

or, $y = 2\theta$

$\therefore$ $y = 2\tan^{-1}x$ $\qquad [\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\frac{d(\tan^{-1}x)}{dx} = \frac{2}{1+x^2}.$$

h. $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put: $x = \tan\theta$, then

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1}x$$

Differentiating both sides w.r.t., 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x\right)$$

$$= \frac{1}{2}\cdot\frac{1}{(1+x^2)}$$

$$= \frac{1}{2(1+x^2)}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

i. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

Let $y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$$= \tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}$$

$$= \tan^{-1}\sqrt{\tan^2\frac{x}{2}}$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Differentiating both sides w.r.t., x,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

3. Find the derivative of

a. $\ln(\sin^{-1} x)$ b. $\sin^{-1}(\cos x)$ c. $\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution

a. $\ln(\sin^{-1} x)$

Let $y = \ln(\sin^{-1} x)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\ln(\sin^{-1} x)\}$$

$$= \frac{d\{\ln(\sin^{-1} x)\}}{d(\sin^{-1} x)} \cdot \frac{d(\sin^{-1} x)}{dx}$$

$$= \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$$

b. $\sin^{-1}(\cos x)$

Let $y = \sin^{-1}(\cos x)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\sin^{-1}(\cos x)\}$$

$$= \frac{d\{\sin^{-1}(\cos x)\}}{d(\cos x)} \cdot \frac{d(\cos x)}{dx}$$

$$= \frac{1}{\sqrt{1-\cos^2 x}} \cdot (-\sin x)$$

$$= -1$$

c. $\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $y = \tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$= \tan^{-1} x + 2\tan^{-1} x$$

$$= 3\tan^{-1} x$$

$$\frac{dy}{dx} = 3\frac{d}{dx}(\tan^{-1} x)$$

$$= \frac{3}{1+x^2}$$

4. a. Find the derivative of $\tan^{-1} x$ with respect to $\cot^{-1} x$.

b. Find the derivative of $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$.

Solution

a. $$\frac{d(\tan^{-1}x)}{d(\cot^{-1}x)} = \frac{\frac{d}{dx}(\tan^{-1}x)}{\frac{d}{dx}(\cot^{-1}x)} = \frac{\frac{1}{1+x^2}}{-\frac{1}{1+x^2}} = -1$$

b. Let $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

and $z = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

We have to find $\dfrac{dy}{dz}$.

Put $x = \tan\theta$. Then

$$\begin{aligned} y &= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \\ &= \cot^{-1}(\cos 2\theta) \\ &= 2\theta \\ &= 2\tan^{-1}x \end{aligned}$$

and,

$$\begin{aligned} z &= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) \\ &= \tan^{-1}(\tan 3\theta) \\ &= 3\theta \\ &= 3\tan^{-1}x \end{aligned}$$

Now,

$$\begin{aligned} \frac{dy}{dz} &= \frac{d(2\tan^{-1}x)}{d(3\tan^{-1}x)} \\ &= \frac{2}{3}\cdot\frac{d(\tan^{-1}x)}{d(\tan^{-1}x)} \\ &= \frac{2}{3} \end{aligned}$$

Exercise 1.2

Find the derivatives:

1. ln (sinh x)

Solution

Let $y = \ln(\sinh x)$

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\sinh x)$$

$$= \frac{d\{\ln(\sinh x)\}}{d(\sinh x)} \cdot \frac{d(\sinh x)}{dx}$$

$$= \frac{1}{\sinh x} \cdot \cosh x$$

$$= \coth x$$

2. ln (cosh $2x$)

Solution

Let $y = \ln(\cosh 2x)$

then

$$\frac{dy}{dx} = \frac{d}{dx}\{\ln(\cosh 2x)\}$$

$$= \frac{1}{\cosh 2x}\sinh 2x.\ 2$$

$$= 2\tanh 2x.$$

3. $e^{\cosh x}$

Solution

Let $y = e^{\cosh x}$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cosh x})$$

$$= \frac{d(e^{\cosh x})}{d(\cosh x)} \cdot \frac{d(\cosh x)}{dx}$$

$$= e^{\cosh x} \cdot \sinh x$$

4. $e^{\cosh^{-1} x}$

Solution

Let $y = e^{\cosh^{-1} x}$

$$= e^{\cosh^{-1} x}\ \frac{dy}{dx} = \frac{d}{dx}(e^{\cosh^{-1} x})$$

$$= e^{\cosh^{-1} x}\frac{1}{\sqrt{x^2 - 1}}$$

5. $\sinh^{-1} x - \operatorname{cosech}^{-1} x$

Solution

Let $y = \sinh^{-1} x - \operatorname{cosech}^{-1} x$

Then

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1} x) - \frac{d}{dx}(\operatorname{cosech}^{-1} x)$$

$$= \frac{1}{\sqrt{1 + x^2}} + \frac{1}{x\sqrt{1 + x^2}}$$

$$= \frac{x + 1}{x\sqrt{x^2 + 1}}$$

6. Arc cosh (sec x)

Solution

Let $y = \cosh^{-1}(\sec x)$

Then

$$\frac{dy}{dx} = \frac{d}{dx}\{\cosh^{-1}(\sec x)\}$$

$$= \frac{d\{\cosh^{-1}(\sec x)\}}{d(\sec x)} \cdot \frac{d(\sec x)}{dx}$$

$$= \frac{1}{\sqrt{\sec^2 x - 1}} \cdot \sec x \cdot \tan x$$

$$= \frac{\sec x \cdot \tan x}{\tan x}$$

$$= \sec x$$

7. $\tan^{-1}(\sinh x)$

Solution

Then

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sinh x)$$

$$= \frac{d(\tan^{-1}(\sinh x))}{d(\sinh x)} \cdot \frac{d\sinh x}{dx}$$

$$= \frac{1}{1+\sinh^2 x} \cdot \cosh x$$

$$= \frac{1}{\cosh^2 x} \cdot \cosh x = \operatorname{sech} x.$$

8. $\cosh^{-1}(\sinh x)$

Solution

Then

$$\frac{dy}{dx} = \frac{d}{dx}\{\cosh^{-1}(\sinh x)\}$$

$$= \frac{1}{\sqrt{\sinh^2 x - 1}} \cdot \frac{d}{dx}(\sinh x)$$

$$= \frac{\cosh x}{\sqrt{\sinh^2 x - 1}}$$

9. $x\tanh^{-1}\sqrt{x}$

Solution

Let $y = 2\tanh^{-1}\sqrt{x}$

Then

$$\frac{dy}{dx} = \frac{d}{dx}(x\tanh^{-1}\sqrt{x})$$

$$= x\frac{d}{dx}(\tanh^{-1}\sqrt{x}) + \tanh^{-1}\sqrt{x}\frac{d}{dx}x$$

$$= x.\frac{1}{1-x} \cdot \frac{d}{dx}\sqrt{x} + \tanh^{-1}\sqrt{x}$$

$$= \frac{x}{1-x} \cdot \frac{1}{2\sqrt{x}} + \tanh^{-1}\sqrt{x}$$

$$= \frac{x}{1-x} \cdot \frac{1}{2\sqrt{x}} + \tanh^{-1}\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1-x)} + \tanh^{-1}\sqrt{x}$$

10. $x^{\cosh\frac{x}{a}}$

Solution

Let $y = x^{\cosh\frac{x}{a}}$

Then $\ln y = \ln x^{\cosh\frac{x}{a}} = \cosh\frac{x}{a}.\ln x$

Also,

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\cosh\frac{x}{a}.\ln x\right)$$

or, $$\frac{1}{y}\frac{dy}{dx} = \cosh\frac{x}{a}.\frac{1}{x} + \ln x.\sinh\frac{x}{a}.\frac{1}{a}$$

or, $$\frac{dy}{dx} = x^{\cosh\frac{x}{a}}\left(\frac{1}{x}\cosh\frac{x}{a} + \frac{1}{a}\ln x.\sinh\frac{x}{a}\right)$$

11. $\left(\cosh\frac{x}{a}\right)^{\ln x}$

Solution

Let $y = \left(\cosh\frac{x}{a}\right)^{\ln x}$

Then $\ln y = \ln\left(\cosh\frac{x}{a}\right)^{\ln x} = \ln x.\ln\left(\cosh\frac{x}{a}\right)$

and $\frac{d}{dx}(\ln y) = \frac{d}{dx}\left\{\ln x.\ln\left(\cosh\frac{x}{a}\right)\right\}$

or, $$\frac{1}{y}\frac{dy}{dx} = \ln x.\frac{d}{dx}\left\{\ln\left(\cosh\frac{x}{a}\right)\right\} + \ln\left(\cosh\frac{x}{a}\right)\frac{d}{dx}(\ln x)$$

or, $$\frac{dy}{dx} = y\left[\ln x.\frac{1}{\cosh\frac{x}{a}}.\sinh\frac{x}{a}.\frac{1}{a} + \ln\left(\cosh\frac{x}{a}\right).\frac{1}{x}\right]$$

$$= \left(\cosh\frac{x}{a}\right)^{\ln x}\left[\frac{1}{a}\ln x.\tanh\frac{x}{a} + \frac{1}{x}\ln\left(\cosh\frac{x}{a}\right)\right]$$

12. $x^{\tanh\frac{x}{a}}$

Solution

Let $y = x^{\tanh\frac{x}{a}}$

Taking logarithm on both sides, we get

$\ln y = \tanh\frac{x}{a}\cdot\ln x$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\tanh\frac{x}{a}\cdot\ln x\right)$$

or, $$\frac{d(\ln y)}{dy}\cdot\frac{dy}{dx} = \tanh\left(\frac{x}{a}\right)\cdot\frac{d}{dx}(\ln x) + \ln x\frac{d\left(\tanh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}\cdot\frac{d\left(\frac{x}{a}\right)}{dx}$$

or, $\frac{1}{y}\cdot\frac{dy}{dx} = \tanh\frac{x}{a}\cdot\frac{1}{x} + \ln x\cdot\operatorname{sech}^2\frac{x}{a}\cdot\frac{1}{a}$

or, $\frac{dy}{dx} = y\left(\frac{1}{x}\tanh\frac{x}{a} + \frac{1}{a}\ln x\cdot\operatorname{sech}^2\frac{x}{a}\right)$

$\therefore \frac{dy}{dx} = x^{\tanh\frac{x}{a}}\left(\frac{1}{x}\tanh\frac{x}{a} + \frac{1}{a}\ln x\operatorname{sech}^2\frac{x}{a}\right)$

13. $(\cosh x)^{\sinh^{-1}x}$

Solution

Let $y = (\cos h\, x)^{\sinh^{-1}x}$

Taking log on both sides, we get,

$\log y = \sin h^{-1} x\cdot\log\cosh x$

Differentiating both sides with respect to x, we get

$\frac{d}{dy}(\log y) = \frac{d}{dx}(\sinh^{-1}x\log\cosh x)$

or, $\frac{d}{dy}(\log y).\frac{dy}{dx} = \sinh^{-1}x\frac{d}{dx}(\log\cosh x) + \log\cosh x\frac{d}{dx}(\sinh^{-1}x)$

or, $\frac{1}{y}\frac{dy}{dx} = \sinh^{-1}x\,.\,\frac{d(\log\cosh x)}{d(\cosh x)}\cdot\frac{d(\cosh x)}{dx} + \log\cosh x\frac{1}{\sqrt{1+x^2}}$

or, $\frac{dy}{dx} = y\left\{\sinh^{-1}x\,.\,\frac{1}{\cosh x}\cdot\sinh x + \log\cosh x\frac{1}{\sqrt{1+x^2}}\right\}$

$= (\cosh x)^{\sinh^{-1}x}\left\{\sinh^{-1}x\cdot\tanh x + \frac{1}{\sqrt{1+x^2}}\log\cosh x\right\}$

14. $(\ln x)^{\sinh x}$

Solution

Let $y = (\ln x)^{\sinh x}$

Taking 'ln' on both sides, we get,

$\ln y = \sinh x\ln(\ln x)$

Differentiating both sides with respect to 'x'

$\frac{d}{dx}(\ln y) = \frac{d}{dx}\{\sinh x\ln(\ln x)\}$

$\frac{d}{dy}(\ln y).\frac{dy}{dx} = \sinh x.\frac{d}{dx}\{\ln(\ln x)\} + \ln(\ln x)\frac{d}{dx}(\sinh x)$

or, $\frac{1}{y}\frac{dy}{dx} = \sinh x\,.\,\frac{d\{\ln(\ln x)\}}{d(\ln x)}\cdot\frac{d(\ln x)}{dx} + \ln(\ln x)\,.\cosh x$

or, $\frac{1}{y}\frac{dy}{dx} = \sinh x\frac{1}{\ln x}\cdot\frac{1}{x} + \cosh x\,.\ln(\ln x)$

or, $\frac{dy}{dx} = y\left\{\frac{\sinh x}{x\ln x} + \cosh x.\ln(\ln x)\right\}$

$\therefore \frac{dy}{dx} = (\ln x)^{\sinh x}\left\{\frac{\sinh x}{x\ln x} + \cosh x\cdot\ln(\ln x)\right\}$

Exercise 1.3

1. Find Δy, dy and $\Delta y - dy$ when $y = x^2 + 5x$ when $x = 2$ and $dx = 0.1$.

Solution

Here, $y = x^2 + 5x$, $x = 2$ and $dx = 0.1$

Again,

$y + \Delta y = (x + \Delta x)^2 + 5(x + \Delta x)$

or, $\Delta y = x^2 + 2 \cdot x \cdot \Delta x + (\Delta x)^2 + 5x + 5\Delta x - x^2 - 5x$

or, $\Delta y = 2 \cdot x \cdot \Delta x + (\Delta x)^2 + 5\Delta x = 0.1\ (2 \times 2 + 0.1 + 5) = 0.91$

$dy = (2x + 5)\, dx = (2 \times 2 + 5) \times 0.1 = 0.9$

Now,

$\Delta y - dy = 0.91 - 0.9 = 0.01$

2. What is the exact change in the value of $y = x^2$ when x changes from 10 to 10.1? What is the approximate change in y?

Solution

Here, $y = x^2$

$y + \Delta y = (x + \Delta x)^2$

or, $\Delta y = (x + \Delta x)^2 - x^2$ ... (i)

Here, $\Delta x = dx = 10.1 - 10 = 0.1$ and $x = 10$.

From (i), exact change $\Delta y = (10 + 0.1)^2 - 10^2 = (10.1)^2 - 10^2 = 2.01$

Again, $dy = 2x\, dx$

$\therefore$ Approximate change in y $(dy) = 2 \times 10 \times 0.1 = 2$

3. If the radius of sphere changes from 2 cm to 2.01 cm, find the approximate increase in its volume.

Solution

We have, volume of sphere $(V) = \frac{4}{3}\pi r^3$

Here, $r = 2$ cm, $r + \Delta r = 2.01$ cm

$\therefore$ $\Delta r = dr = 2.01 - 2 = 0.01$ cm

Approximate change in volume (dV) $= \frac{4}{3}\pi \cdot 3r^2 \cdot dr$

$= \frac{4}{3}\pi \times 3 \times 2^2 \times 0.01$

$= 0.16\pi \text{ cm}^3$

4. If the radius of a circle is increased from 5 to 5.1 cm, find the approximate increase in area.

Solution

Here, $r = 5$ cm, $r + \Delta r = 5.1$ cm

$\Delta r = dr = 5.1 - 5 = 0.1$ cm

We have,

$A = \pi r^2$

$dA = \pi \cdot 2r \cdot dr$

Approximate increase in area (dA) $= \pi \times 2 \times 5 \times 0.1 = \pi \text{ cm}^2$

5. The radius of a circle increases from 10 m to 10.1 m. Estimate the increase in the circle's area. Also find true change ΔA.

Solution

Here, $r = 10$ m

$r + \Delta r = 10.1$ m

$\therefore$ $\Delta r = dr = 10.1 - 10 = 0.1$ m

We have, $A = \pi r^2$

$dA = \pi \cdot 2r\, dr$

Approximate change in area $(dA) = \pi \times 2 \times 10 \times 0.1 = 2\pi$ m^2

Again,

$A + \Delta A = \pi(r + \Delta r)^2$

or, $\Delta A = \pi(r + \Delta r)^2 - \pi r^2$

$= \pi r^2 + 2 \cdot r \cdot \Delta r \cdot \pi + \pi \cdot (\Delta r)^2 - \pi r^2$

$= \Delta r\,(2\pi r + \pi \cdot \Delta r)$

True change $= 0.1\,(2\pi \times 10 + \pi \times 0.1) = 2.01\pi$ m^2

6. Find the approximate increase in the area of a cube if the edge increases from 10 cm to 10.2 cm. Calculate the percentage error in the use of differential approximation.

Solution

Here, $x = 10$ cm, $x + \Delta x = 10.2$ cm

$\therefore$ $dx = \Delta x = 10.2$ cm $- 10$ cm $= 0.2$ cm

We have, area of cube $(A) = 6x^2$

$dA = 6 \cdot 2x \cdot dx$

$= 6 \times 2 \times 10 \times 0.2$

$= 24$ cm^2

Again,

$A + \Delta A = 6(x + \Delta x)^2$

or, $\Delta A = 6(x + \Delta x)^2 - 6x^2$

$= 6(10.2)^2 - 6 \times 10^2$

$= 24.24$

Error $=$ Actual increase $-$ Approximate increase

$= 24.24 - 24$

$= 0.24$

Percentage error $= \dfrac{\text{error}}{A} \times 100 = \dfrac{0.24}{6 \times 10^2} \times 100 = 0.04\%$

Exercise 1.4

1. Find the slope and inclination with x–axis of the tangent of the following curves
   a. $3y = x^3 + 1$ at $x = 1$
   b. $y = -x^4 - 3x$ at $(-1, 2)$
   c. $x^2 + y^2 = 25$ at $(4, -3)$

Solution

a. Here,

$3y = x^3 + 1$

or, $3\frac{dy}{dx} = 3x^2$

or, $\frac{dy}{dx} = x^2$

At $x = 1, \frac{dy}{dx} = 1^2 = 1$

Slope at $x = 1$ is 1.

If θ is the inclination of tangent with x-axis then

$\tan\theta = 1$

or; $\tan\theta = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

b. Here, $y = -x^4 - 3x$ at $(-1, 2)$

$\frac{dy}{dx} = -4x^3 - 3$

At $x = -1, \frac{dy}{dx} = -4(-1)^3 - 3 = 4 - 3 = 1.$

Slope = 1

If θ is the inclination of tangent with x-axis then

$\tan\theta = 1 = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

c. Here,

$x^2 + y^2 = 25$ at $(4, -3)$

or, $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$

or, $2x + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x}{y}$

At $(x, y) = (4, -3), \frac{dy}{dx} = \frac{-(4)}{-3} = \frac{4}{3}$

$\therefore$ Slope $= \frac{4}{3}$

If θ be the inclination of tangent with x-axis then

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right).$

2. At what angle does the curve $y(1+x) = x$ cut x–axis?

Solution

Given, $y(1+x) = x$ ...(i)

The curve meets the x-axis where $y = 0$. So, putting $y = 0$ in (i), we get

From (i)

$$y = \frac{x}{1+x}$$

$$\frac{dy}{dx} = \frac{(1+x)\frac{dx}{dx} - x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x).1 - x}{(1+x)^2}$$

$$= \frac{1}{(1+x)^2}$$

At $x = 0$;

$$\frac{dy}{dx} = \frac{1}{(1+0)^2} = 1$$

If θ be the angle made by tangent with x-axis then,

$$\tan\theta = 1 = \tan\frac{\pi}{4}$$

$$\therefore \quad \theta = \frac{\pi}{4}$$

3. Find the equations of the tangents and normals to the following curves.

a. $y = 2x^3 - 5x^2 + 8$ at (2, 4)

b. $x^2 + y^2 = 25$ at (3, 4)

Solution

a. Given curve is $y = 2x^3 - 5x^2 + 8$

Differentiating both sides with respect to 'x'

$$\frac{dy}{dx} = 6x^2 - 10x$$

At (2, 4),

$$\frac{dy}{dx} = 6 \times 2^2 - 10 \times 2 = 4$$

i.e. slope (m) = 4

The equation of tangent at (2, 4) and having slope 4 is

$y - y_1 = m(x - x_1)$

or, $y - 4 = 4(x - 2)$

or, $y - 4 = 4x - 8$

or, $4x - y - 4 = 0$

$\therefore$ $4x - y = 4$

Slope of normal $= -\dfrac{1}{\frac{dy}{dx}} = -\dfrac{1}{4}$

The equation of normal at $(x_1, y_1) = (2, 4)$ is

$y - y_1 = m(x - x_1)$

or, $y - 4 = -\frac{1}{4}(x - 2)$

or, $4y - 16 = -x + 2$

$\therefore$ $x + 4y = 18$

b. Here, $x^2 + y^2 = 25$

Differentiating both sides w.r.t. x,

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

or, $2x + 2y\frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = -\frac{x}{y}$

At $(x, y) = (3, 4)$; $\frac{dy}{dx} = -\frac{3}{4}$

∴ Slope of tangent (m) $= -\frac{3}{4}$

The equation of tangent at $(x_1, y_1) = (3, 4)$ is

$y - y_1 = m(x - x_1)$

or, $y - 4 = -\frac{3}{4}(x - 3)$

or, $4y - 16 = -3x + 9$

∴ $3x + 4y = 25$

Again, slope of normal $= -\frac{1}{\text{Slope of tangent}} = -\frac{1}{\frac{-3}{4}} = \frac{4}{3}$

The equation of normal at $(x_1, y_1) = (3, 4)$ is

$y - y_1 = \frac{4}{3}(x - x_1)$

or, $y - 4 = \frac{4}{3}(x - 3)$

or, $3y - 12 = 4x - 12$

∴ $4x - 3y = 0$

4. Find the points on the following curves where the tangents are parallel to the x-axis

a. $y = x^2 + 4x + 1$

b. $y = x^3 - 2x^2 + 1$

Solution

a. Here, $y = x^2 + 4x + 1$

$\frac{dy}{dx} = 2x + 4$

For tangent parallel to x-axis, we have,

$\frac{dy}{dx} = 0$

or, $2x + 4 = 0$

or, $x = -2$

When, $x = -2$, from (i)

$y = (-2)^2 + 4(-2) + 1$
$= 4 - 8 + 1 = -3$

∴ Required point $= (-2, -3)$.

b. Here, $y = x^3 - 2x^2 + 1$ ... (i)

$\frac{dy}{dx} = 3x^2 - 4x$

For tangent parallel to x-axis,

$\frac{dy}{dx} = 0$

or, $3x^2 - 4x = 0$

or, $x = 0, \frac{4}{3}$

When $x = 0$, from (i)

$y = 0^3 - 2 \times 0^2 + 1 = 1$

When $x = \frac{4}{3}$, from (i)

$y = \left(\frac{4}{3}\right)^3 - 2\left(\frac{4}{3}\right)^2 + 1 = \frac{-5}{27}$

$\therefore$ Required points are $(0, 1)$ and $\left(\frac{4}{3}, \frac{-5}{27}\right)$.

5. Find the points on the curve $x^2 + y^2 = 25$ at which the tangents are parallel to the (a) x–axis (b) y–axis.

Solution

Given curve is $x^2 + y^2 = 25$ ... (i)

Differentiating both sides w.r. to x,

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$

$2x + 2y\frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = -\frac{x}{y}$

a. For tangents parallel to x-axis, we have

$\frac{dy}{dx} = 0$

or, $-\frac{x}{y} = 0$

$\therefore$ $x = 0$

Putting the value of x in (i), we get

$0 + y^2 = 25$

or, $y = \pm 5$

$\therefore$ Required points are $(0, \pm 5)$

b. For the tangent parallel to y-axis, we have

$\frac{dx}{dy} = 0$

or, $\frac{-y}{x} = 0$

or, $y = 0$

Then, from (i)

$x^2 = 25$

or, $x = \pm 5$

$\therefore$ Required points are $(\pm 5, 0)$.

6. a. Find the equation of the tangent to the curve $y = 2x^2 - 3x + 1$ which is parallel to the line $x - y + 5 = 0$.

b. Find the equation of the tangent to the curve $y = 3x^2 - 3x + 5$ which is perpendicular to the line $x + 3y + 5 = 0$.

Solution

a. Given curve is $y = 2x^2 - 3x + 1$ ... (i)

$\frac{dy}{dx} = 4x - 3$

Slope of the line $x - y + 5 = 0$ is $-\dfrac{\text{coeff. of } x}{\text{coeff. of } y} = -\dfrac{1}{-1} = 1.$

Then,

$4x - 3 = 1$

or, $4x = 4$

or, $x = 1$

Then, from (i),

$y = 2 \times 1^2 - 3 \times 1 + 1 = 0$

The equation of tangent through the point $(x_1, y_1) = (1, 0)$ and having slope $(m) = 1$ is

$y - y_1 = m(x - x_1)$

or, $y - 0 = 1\,(x - 1)$

or, $y = x - 1$

$\therefore$ $x - y - 1 = 0$

b. Given curve is $y = 3x^2 - 3x + 5$ ... (ii)

$\dfrac{dy}{dx} = 6x - 3 = m_1$ (say)

Slope of the line $x + 3y + 5 = 0$ is $-\dfrac{1}{3}$.

But the tangent at the curve (i) is perpendicular to $x + 3y + 5 = 0$, so

$m_1 \cdot m_2 = -1$

or, $(6x - 3) \times \left(\dfrac{-1}{3}\right) = -1$

or, $2x - 1 = 1$

or, $2x = 2$

$\therefore$ $x = 1$

So,

$m_1 = 6 \times 1 - 3 = 3$

Then, from (i),

$y = 3 \times 1^2 - 3 \times 1 + 5 = 5$

Now, the equation of tangent at the point $(x_1\, y_1) = (1, 5)$ and having the slope $(m_1) = 3$ is

$y - y_1 = m_1(x - x_1)$

or, $y - 5 = 3(x - 1)$

or, $y - 5 = 3x - 3$

or, $0 = 3x - y + 5 - 3$

$\therefore$ $3x - y + 2 = 0$

7. Find the angle of intersection of the following curves.

a. $y^2 = x^3$ and $y = 2x$ at $(0, 0)$

b. $xy = 6$ and $x^2y = 12$

Solution

a. Given curves are

$y = 2x$ ...(i)

$y^2 = x^3$

$y = x^{3/2}$ ...(ii)

Now, differentiating both sides of (i) with respect to 'x'

$\dfrac{dy}{dx} = 2$

At $(0, 0)$; $\dfrac{dy}{dx} = 2$

i.e. $m_1 = 2$

Again, differentiating both sides of (ii) with respect to 'x'

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

At (0, 0): $\frac{dy}{dx} = 0$

i.e. $m_2 = 0$

If θ be the angle of intersection then,

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$= \frac{2-0}{1+2\times 0} = 2$

$\therefore \theta = \tan^{-1}(2)$

b. Given,

$xy = 6$ ... (i)

And $x^2y = 12$

or, $y = \frac{12}{x^2}$ ... (ii)

From (i) and (ii),

$x^2y = 12$

or, $x^2\cdot\left(\frac{6}{x}\right) = 12$ [Using (i)]

or, $x = 2$

Putting $x = 2$ in (i), $y = 3$

The point of intersection = (2, 3).

Diff. (i) w.r.t. x,

$y + x\frac{dy}{dx} = 0$

$\Rightarrow m_1 = \frac{dy}{dx} = -\frac{y}{x} = -\frac{3}{2}$ at the point (2, 3).

Again, diff. (ii) w.r.t. to x,

$2xy + x^2\frac{dy}{dx} = 0$

$\Rightarrow m_2 = \frac{dy}{dx} = -\frac{2y}{x} = -2\cdot\frac{3}{2} = -3$ at (2, 3)

Let, θ be the angle between the two curves. Then,

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$= \frac{-\frac{3}{2}+3}{1+\frac{9}{2}}$

$= \frac{\frac{6-3}{2}}{\frac{2+9}{2}}$

$= \frac{3}{11}$

$\therefore \theta = \tan^{-1}\left(\frac{3}{11}\right)$

Exercise 1.5

1. a. Show that the function $f(x) = 3x^3 - 24x + 1$ is increasing at $x = 4$ and decreasing at $x = \frac{1}{2}$.

b. Examine whether the function $f(x) = 2x^3 - x^2 + 4$ at $x = 1$ and at $x = \frac{1}{4}$.

c. Show that $f(x) = x - \frac{1}{x}$ is increasing for all $x \in \mathbb{R}$ except at $x = 0$.

d. Show that the function $f(x) = x^2 - 6x + 3$ is decreasing on the interval (0, 2).

Solution

a. Given function is

$f(x) = 3x^3 - 24x + 1$

$f'(x) = 9x^2 - 24$

When $x = 4$,

$f'(4) = 9 \times 4^2 - 24 = 120 > 4$

So, $f(x)$ is increasing at $x = 4$.

When $x = \frac{1}{2}$,

$$f'\left(\frac{1}{2}\right) = 9 \times \left(\frac{1}{2}\right)^2 - 24 = -21.72 < 0.$$

So, $f(x)$ is decreasing at $x = \frac{1}{2}$.

b. $f(x) = 2x^3 - x^2 + 4$

$f'(x) = 6x^2 - 2x$

When $x = 1$,

$f'(1) = 6 \times 1^2 - 2 \times 1 = 4 > 0$

So, $f(x)$ is increasing at $x = 1$.

When $x = \frac{1}{4}$,

$$f'\left(\frac{1}{4}\right) = 6 \times \left(\frac{1}{4}\right)^2 - 2 \times \frac{1}{4} = -0.125 < 0.$$

So, $f(x)$ is decreasing at $x = \frac{1}{4}$.

c. Here,

$$f(x) = x - \frac{1}{x} = x - x^{-1}$$

$$f'(x) = \frac{d}{dx}(x - x^{-1})$$

$$= 1 - (-1)\, x^{-1-1}$$

$= 1 + \frac{1}{x^2}$, which is always positive for any real x except $x = 0$.

$\therefore$ $f(x)$ is increasing for all $x \in$ R except at $x = 0$

d. $f(x) = x^2 - 6x + 3$

$f'(x) = 2x - 6$

Here, $f'(x) = 2(x - 3) < 0$ for $x < 3$

i.e. $f(x)$ is decreasing on $(-\infty, 3)$. Since $(0, 2) \subset (-\infty, 3)$, so $f(x)$ is decreasing on $(0, 2)$.

2. Find the intervals in which the following functions are increasing or decreasing.

a. $f(x) = x^2 - 2x$ b. $f(x) = -x^2 - 3x + 3$

c. $f(x) = x^3 - 3x^2 - 9x$ d. $f(x) = -x^3 + 12x + 5, -3 \le x \le 3$

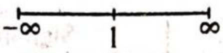
Solution

a. $f(x) = x^2 - 2x$

$f'(x) = 2x - 2$

For critical points,

$f'(x) = 0$



or, $2x - 2 = 0$

$\therefore$ $x = 1$

The point $x = 1$ divides the whole real line in two sub-intervals.

Now,

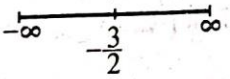
Intervals	Sign of f '(x)	Nature
$(-\infty, 1)$	–ve	decreasing
$(1, \infty)$	+ve	increasing

$\therefore$ $f(x)$ is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$.

b. $f(x) = -x^2 - 3x + 3$

$f'(x) = -2x - 3$

For critical points,



$f'(x) = 0$

or, $-2x - 3 = 0$

or, $x = -\frac{3}{2}$

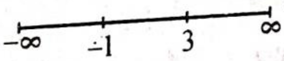
Now,

Intervals	Sign of f '(x)	Nature
$\left(-\infty, -\frac{3}{2}\right)$	+ve	increasing
$\left(-\frac{3}{2}, \infty\right)$	–ve	decreasing

$\therefore$ $f(x)$ is increasing on $\left(-\infty, -\frac{3}{2}\right)$ and decreasing on $\left(-\frac{3}{2}, \infty\right)$.

c. $f(x) = x^3 - 3x^2 - 9x$

$f'(x) = 3x^2 - 6x - 9$



For critical points,

$f'(x) = 0$

or, $3x^2 - 6x - 9 = 0$

or, $x^2 - 2x - 3 = 0$

or, $x^2 - 3x + x - 3 = 0$

or, $x(x - 3) + 1(x - 3) = 0$

or, $(x - 3)(x + 1) = 0$

$\therefore$ $x = 3, -1$

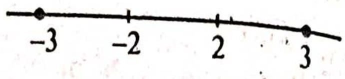
Now,

Intervals	Sign of f '(x)	Nature of $f(x)$
$(-\infty, -1)$	+ve	increasing
$(-1, 3)$	–ve	decreasing
$(3, \infty)$	+ve	increasing

$\therefore$ $f(x)$ is increasing on $(-\infty, -1) \cup (3, \infty)$ and decreasing on $(-1, 3)$.

d. $f(x) = -x^3 + 12x + 5, -3 \le x \le 3$

$f'(x) = -3x^2 + 12$



For critical points,

$f'(x) = 0$

or, $-3x^2 + 12 = 0$

or, $x^2 - 4 = 0$

$\therefore$ $x = \pm 2.$

Now,

Intervals	Sign of f '(x)	Nature of $f(x)$
$[-3, -2)$	–ve	decreasing
$(-2, 2)$	+ve	increasing
$(2, 3]$	–ve	decreasing

$\therefore$ $f(x)$ is increasing on $(-2, 2)$ and decreasing on $[-3, -2) \cup (2, 3]$

3. Find the absolute maximum and minimum values of each function on the given interval.

a. $f(x) = \frac{2}{3}x - 5, -2 \le x \le 3$

b. $f(x) = x^2 - 1, -1 \le x \le 2$

c. $f(x) = x^3 - 3x^2 + 5$ on $[-2, 2]$

d. $h(x) = x^{\frac{2}{3}}$ on $[-2, 3]$

Solution

a. $f(x) = \frac{2}{3}x - 5$ in $[-2, 3]$

$f'(x) = \frac{2}{3}$

Since $f'(x) \neq 0$ for any values of x. So, there is no critical points. So we have to calculate $f(x)$ at the end points only.

When $x = -2$,

$$f(-2) = \frac{2}{3}(-2) - 5 = -\frac{19}{3}$$

When $x = 3$,

$$f(3) = \frac{2}{3} \times 3 - 5 = -3$$

$\therefore$ Absolute max. value $= -3$ at $x = 3$.

Absolute min. value $= -\frac{19}{3}$ at $x = -2$.

b. $f(x) = x^2 - 1$ in $[-2, 2]$

$f'(x) = 2x$

For critical points,

$f'(x) = 0$

or, $2x = 0$

$x = 0$

When $x = -2, f(-2) = (-2)^2 - 1 = 3$

When $x = 0, f(0) = 0^2 - 1 = -1$

When $x = 2, f(2) = 2^2 - 1 = 3$

$\therefore$ Absolute max. value $= 3$ at $x = 2$

Absolute min. value $= -1$ at $x = 0$.

c. $f(x) = x^3 - 3x^2 + 5$ on $[-2, 2]$

$f'(x) = 3x^2 - 6x$

For critical points, we have,

$f'(x) = 0$

$3x^2 - 6x = 0$

or, $x^2 - 2x = 0$

or, $x(x - 2) = 0$

$\therefore$ $x = 0, 2$

When $x = -2$,

$f(-2) = (-2)^3 - 3 \times (-2)^2 + 5 = -15$

When $x = 0$,

$f(0) = 0^3 - 3 \times 0^2 + 5 = 5$

When $x = 2$,

$f(2) = 2^3 - 3 \times 2^2 + 5 = 1$

$\therefore$ Absolute max. value $= 5$ at $x = 0$

Absolute min. value $= -15$ at $x = -2$.

d. $h(x) = = x^{\frac{2}{3}}$

$h'(x) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$

Since $h'(x)$ does not exist at $x = 0$.

So 0 is a critical point.

When $x = -2$,

$h(-2) = (-2)^{\frac{2}{3}} = 4^{\frac{1}{3}} = \sqrt[3]{4}$

When $x = 0$,

$h(0) = 0^{\frac{2}{3}} = 0$

When $x = 3$,

$h(3) = 3^{\frac{2}{3}} = \sqrt[3]{9}$

$\therefore$ Absolute max. value $= \sqrt[3]{9}$ at $x = 3$

Absolute min. value $= 0$ at $x = 0$.

4. Find the local maximum and minimum values and point of inflection of following functions.

a. $f(x) = 3x^2 - 6x + 4$

b. $f(x) = x^3 - 3x + 1$

c. $f(x) = x^3 - 3x^2 - 9x + 27$

d. $f(x) = 2x^3 - 15x^2 + 36x + 5$

e. $y = 4x^3 - 6x^2 - 9x + 1$ on the interval $(-1, 2)$.

f. $y = x + \frac{25}{x}$

Solution

a. Here,

$f(x) = 3x^2 - 6x + 4$

$f'(x) = 6x - 6$

$f''(x) = 6$

For minimum and maximum values, we have,

$f'(x) = 0$

or, $6x - 6 = 0$

or, $x = 1$

When $x = 1$, $f''(1) = 6 > 0$

So, $f(x)$ has minimum value at $x = 1$.

Minimum value $= f(1) = 3 \cdot 1^2 - 6 \cdot 1 + 4 = 1$

Since $f''(x) = 6 \neq 0$; so

$f(x)$ has no point of inflection.

b. Let $y = x^3 - 3x + 1$

$$\frac{dy}{dx} = 3x^2 - 3$$

$$\frac{d^2y}{dx^2} = 6x$$

For max. or min. $\frac{dy}{dx} = 0$

$\Rightarrow$ $3x^2 - 3 = 0$

$\therefore$ $x = \pm 1$

when $x = 1$

$\frac{d^2y}{dx^2} = 6 \times 1 = 6 > 0$ (minima) &

The minimum value at $x = 1$ is

$y = 1^3 - 3 \times 1 + 1 = -1$

when $x = -1$

$\frac{d^2y}{dx^2} = 6 \times -1 = -6 < 0$ (maxima) &

The maximum value at $x = -1$ is

$y = (-1)^3 - 3 \times -1 + 1 = 3$

For point of inflection,

$f''(x) = 0$

or, $6x = 0$

$\therefore$ $x = 0$

c. Let $f(x) = x^3 - 3x^2 - 9x + 27$ ... (i)

Differentiating both sides w.r. to 'x', we get

$$\frac{d}{dx}f(x) = \frac{d}{dx}(x^3 - 3x^2 - 9x + 27)$$

i.e. $f'(x) = 3x^2 - 6x - 9$ ... (ii)

Again, differentiating both sides w.r. to 'x' we get

$f''(x) = 6x - 6$ ... (iii)

For the maximum or minimum values of $f(x)$,

$f'(x) = 0$

$\therefore$ $3x^2 - 6x - 9 = 0$

or, $3x^2 - 9x + 3x - 9 = 0$

or, $3x(x - 3) + 3(x - 3) = 0$

or, $(x - 3)(3x + 3) = 0$

$\therefore$ $x = 3, -1$

When $x = 3$, $f''(x) = 6 \times 3 - 6 = 12 > 0$.

So $f(x)$ has a minimum value at $x = 3$ and the minimum value of $f(x)$ is

$f(3) = 3^3 - 3(3)^2 - 9 \times 3 + 27$

$= 27 - 27 - 27 + 27 = 0$

And when $x = -1$, $f''(x) = 6 \times (-1) - 6 = -12 < 0$.

So $f(x)$ has a maximum value at $x = -1$ and the maximum value of $f(x)$ is

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 27$
$= -1 - 3 + 9 + 27 = 32$

Thus, the maximum value of $f(x) = 32$ at $x = -1$,
And, the minimum value of $f(x) = 0$ at $x = 3$
For point of inflection
$f''(x) = 0$
or, $6x - 6 = 0$
$\therefore$ $x = 1$

d. Here, $f(x) = 2x^3 - 15x^2 + 36x + 5$
$f'(x) = 6x^2 - 30x + 36$
$f''(x) = 12x - 30$
For stationary points, we have $f'(x) = 0$
or, $6x^2 - 30x + 36 = 0$
or, $x^2 - 5x + 6 = 0$
or, $(x-2)(x-3) = 0$
$\therefore$ $x = 2, 3$
At $x = 2$
$f''(2) = 12 \times 2 - 30 = -6 < 0$
So, $f(x)$ has maximum value at $x = 2$
Maximum value $= f(2)$
$= 2 \times 2^3 - 15 \times 2^2 + 36 \times 2 + 5$
$= 33$

At $x = 3$
$f''(3) = 12 \times 3 - 30 = 6 > 0$
So, $f(x)$ has minimum value at $x = 3$
Minimum value $= f(3)$
$= 2 \times 3^3 - 15 \times 3^2 + 36 \times 3 + 5$
$= 32$

For point of inflection,
$f''(x) = 0$
or, $12x - 30 = 0$
$\therefore$ $x = \frac{5}{2}$.

e. Given curve is
$y = 4x^3 - 6x^2 - 9x + 1$ ... (i)
Differentiating both sides with respect to 'x', we get
$$\frac{dy}{dx} = \frac{d}{dx}(4x^3 - 6x^2 - 9x + 1)$$
$$\frac{dy}{dx} = 12x^2 - 12x - 9 \quad \text{... (ii)}$$
Again, differentiating both sides with respect to 'x', we get
$$\frac{d^2y}{dx^2} = 24x - 12 \quad \text{... (iii)}$$
For maxima or minima $\frac{dy}{dx} = 0$
i.e. $12x^2 - 12x - 9 = 0$
or, $4x^2 - 4x - 3 = 0$
or, $4x^2 - 6x + 2x - 3 = 0$
or, $2x(2x - 3) + 1(2x - 3) = 0$
or, $(2x - 3)(2x + 1) = 0$
$\therefore$ $x = \frac{3}{2}$ or $-\frac{1}{2}$

At $x = -\frac{1}{2}$

$$\frac{d^2y}{dx^2} = 24\left(-\frac{1}{2}\right) - 12 = -24 < 0$$

$\therefore$ The given function is maximum at $x = -\frac{1}{2}$.

Maximum value is

$$y_{max} = 4\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 9\left(-\frac{1}{2}\right) + 1$$

$$= 4\left(-\frac{1}{8}\right) - 6\left(\frac{1}{4}\right) + \frac{9}{2} + 1$$

$$= -\frac{1}{2} - \frac{3}{2} + \frac{9}{2} + 1$$

$$= \frac{-1 - 3 + 9 + 2}{2}$$

$$= \frac{7}{2}$$

At $x = \frac{3}{2}$

$$\frac{d^2y}{dx^2} = 24 \times \frac{3}{2} - 12 = 24 > 0$$

So, the given function is minimum at $x = \frac{3}{2}$ and its minimum value is,

$$y_{min} = 4\left(\frac{3}{2}\right)^3 - 6\left(\frac{3}{2}\right)^2 - 9\left(\frac{3}{2}\right) + 1$$

$$= 4 \times \frac{27}{8} - 6 \times \frac{9}{4} - \frac{27}{2} + 1$$

$$= \frac{27}{2} - \frac{27}{2} - \frac{27}{2} + 1$$

$$= -\frac{27}{2} + 1$$

$$= \frac{-27 + 2}{2} = \frac{-25}{2}$$

For the point of inflection,

$$\frac{d^2y}{dx^2} = 0$$

i.e. $24x - 12 = 0$

$\therefore$ $x = \frac{1}{2}$

f. Let, $f(x) = x + \frac{25}{x}$

$f'(x) = 1 - 25\,x^{-2}$,

$f''(x) = 50\,x^{-3} = \frac{50}{x^3}$

For max. or min. $f'(x) = 0 \Rightarrow x = \pm 5$,

When $x = 5$

$f'(5) = \frac{50}{125} > 0$ (minima)

and the minimum value at $x = 5$ is $f(5) = 5 + \frac{25}{5} = 5 + 5 = 10$

Again, when $x = -5$, $f''(-5) = \frac{-50}{125} < 0$ (max.)

and the maximum value at $x = -5$ is $f(x) = -5 + \frac{25}{-5} = -5 + -5 = -10$

At last

It has no point of inflection since $\frac{d^2y}{dx^2} = f''(x) \neq 0$ for any real values of x.

5. Show that the function $x^3 + 6x + 30$ has neither a maximum value nor a minimum value.

Solution

Let, $y = x^3 + 6x + 30$

$$\frac{dy}{dx} = 3x^2 + 6$$

$$\frac{d^2y}{dx^2} = 6x$$

and $\frac{d^3y}{dx^3} = 6 \neq 0$. This shows that it has neither maxima nor minima.

6. Find the interval in which the given functions are concave upwards and downwards.

a. $f(x) = x^2 - 3x + 1$

b. $f(x) = x^3 - 3x^2 + 5$

c. $f(x) = x^4 - 8x^3 + 18x^2 - 24$

Solution

a. $f(x) = x^2 - 3x + 1$

$f'(x) = 2x - 3$

$f''(x) = 2$ which is always positive for all x in $\mathbb{R}$.

So, $f(x)$ is concave upwards on $(-\infty, \infty)$.

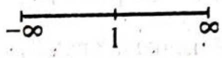
b. $f(x) = x^3 - 3x^2 + 5$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

For point of inflection,

$f''(x) = 0$



or, $6x - 6 = 0$

∴ $x = 1$.

Intervals	Sign of $f''(x)$	Nature
$(-\infty, 1)$	−ve	concave downwards
$(1, \infty)$	+ve	concave upwards

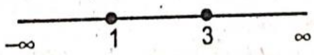
∴ $f(x)$ is concave downwards on $(-\infty, 1)$ and concave upwards on $(1, \infty)$.

c. Given function is $f(x) = x^4 - 8x^3 + 18x^2 - 24$

∴ $f'(x) = 4x^3 - 24x^2 + 36x$

And

$$\begin{aligned} f''(x) &= 12x^2 - 48x + 36 \\ &= 12(x^2 - 4x + 3) \\ &= 12(x^2 - 3x - x + 3) \\ &= 12(x - 3)(x - 1) \end{aligned}$$



For the points of inflection, $f''(x) = 0$

$12(x - 3)(x - 1) = 0 \Rightarrow x = 1, 3$

∴ Required points of inflection are $x = 1$ and $x = 3$.

Intervals	Sign of $f''(x)$	Remarks
$(-\infty, 1)$	+ ve	Concave upwards
$(1, 3)$	– ve	Concave downwards
$(3, \infty)$	+ ve	Concave upwards

$\therefore$ $f(x)$ is concave upwards on $(-\infty, 1) \cup (3, \infty)$ and downwards on $(1, 3)$.

7. A man who has 144 m of fencing material wishes to enclose a rectangular garden. Find the maximum area he can enclose.

Solution

Let, the sides of the rectangular garden be x and y.

Then, perimeter of the garden $= 2(x + y)$

Given, perimeter = 144 m.

$\therefore$ $2(x + y) = 144$ m

$y = 72 - x$ ...(i)

Let A be the area of the rectangular garden. Then,

$A = xy = x(72 - x)$ [Using (i)]

or, $A = 72x - x^2$

Differentiating w.r.t. 'x', we get,

$$\frac{dA}{dx} = \frac{d(72x - x^2)}{dx}$$

or, $\dfrac{dA}{dx} = 72 - 2x$

Again, differentiating w.r.t. 'x', we get

$$\frac{d^2A}{dx^2} = 0 - 2 = -2$$

For maximum or minimum area of the rectangular garden, $\dfrac{dA}{dx} = 0$

i.e. $72 - 2x = 0$

or, $2x = 72$

or, $x = 36$.

When, $x = 36, \dfrac{d^2A}{dx^2} = -2 < 0$.

$\therefore$ Area A is maximum when $x = 36$ and when $x = 36, y = 72 - 36 = 36$.

Hence, the maximum area $= xy = 36 \times 36 = 1296$ m^2

8. A gardener having 120 m of fencing wishes to enclose a rectangular plot of land and also to erect a fence across the land parallel to two of the sides. Find the maximum area he can enclose.

Solution

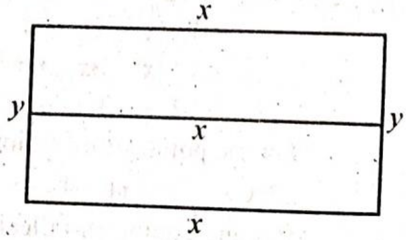
Let x be the length and y be the breadth of rectangular plot of land.

By question,

$3x + 2y = 120$

or, $2y = 120 - 3x$

$y = \dfrac{120 - 3x}{2} = 60 - \dfrac{3}{2}x$ ...(i)



We have,

Area (A) = length × breadth

or, $A = x \times \left(60 - \dfrac{3}{2}x\right) = 60x - \dfrac{3}{2}x^2$

$\dfrac{dA}{dx} = 60 - \dfrac{3}{2} \cdot 2x = 60 - 3x$

$\dfrac{d^2A}{dx^2} = -3$

For max. or min, we have,

$\frac{dA}{dx} = 0$

or, $60 - 3x = 0$

or, $3x = 60$

$\therefore$ $x = 20$

When $x = 20$,

$\frac{d^2A}{dx^2} = -3 < 0$

So, area is maximum when $x = 20$.

Max. Area $= A_{max}$

$= 20\left(60 - \frac{3}{2} \times 20\right)$

$= 20 \times 30$

$= 600 \text{ m}^2$

9. Using derivatives, find two numbers whose sum is 20 and sum of whose squares is minimum.

Solution

Let two numbers be x and y.

By question,

$x + y = 20$

$y = 20 - x$ ...(i)

Let

$S = x^2 + y^2$

or, $S = x^2 + (20 - x)^2$ [Using (i)]

or, $S = x^2 + 400 - 40x + x^2$

or, $S = 2x^2 - 40x + 400$

$\frac{dS}{dx} = 4x - 40$

$\frac{d^2S}{dx^2} = 4$

For max. or min. values, we have,

$\frac{dS}{dx} = 0$

or, $4x - 40 = 0$

or, $x = 10$

When $x = 10$, $\frac{d^2S}{dx^2} = 4 < 0$

So, S is minimum when $x = 10$.

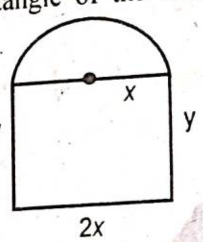
When $x = 10$, from (i) by $y = 20 - 10 = 10$

Required two numbers are 10 and 10.

10. A window is in the form of a rectangle surmounted by a semi circle. If the total perimeter is 9 m, find the radius of the semicircle for the greatest window area.

Solution

Let $2x$ be the length and y be the breadth of the rectangle of the window which is surmounted by a semi–circle as shown in the figure.



Total perimeter = 9m

or, $y + 2x + y + \pi x = 9$

or, $2y = 9 - 2x - \pi x$

or, $y = \frac{9 - 2x - \pi x}{2}$ ...(i)

Area of the window (A) = Area of rectangle + Area of semi-circle

$$= 2x \cdot y + \frac{1}{2} \cdot \pi \cdot x^2$$

$$= \frac{\pi x^2}{2} + 2x\left(\frac{9 - 2x - \pi x}{2}\right) \qquad \text{[using (i)]}$$

$$= \frac{\pi x^2}{2} + 9x - 2x^2 - \pi x^2$$

$$= 9x - 2x^2 - \frac{\pi x^2}{2}$$

$$\frac{dA}{dx} = 9 - 4x - \pi x$$

$$\frac{d^2A}{dx^2} = -4 - \pi$$

For maximum area, $\frac{dA}{dx} = 0$

or, $9 - 4x - \pi x = 0$

or, $(4 + \pi)x = 9$

$\therefore$ $x = \frac{9}{4 + \pi}$

When $x = \frac{9}{4 + \pi}$, $\frac{d^2A}{dx^2} = -4 - \pi < 0$

So, area is maximum when $x = \frac{9}{4 + \pi}$

Hence, for the greatest window area, the radius of semi-circle $= \frac{9}{4 + \pi}$ m.

11. A closed cylindrical can is to be made so that its volume is 26 cm^3. Find its radius and height if the surface is to be a minimum.

Solution

Let r be the radius and h be the height of cylindrical can.

We have,

$v = \pi r^2 h$

or, $26 = \pi r^2 h$

or, $h = \frac{26}{\pi r^2}$ ... (i)

Again, we have,

$S = 2\pi rh + 2\pi r^2$

or, $S = 2\pi r\left(\frac{26}{\pi r^2}\right) + 2\pi r^2$

or, $S = \frac{52}{r} + 2\pi r^2$

$$\frac{dS}{dr} = -\frac{52}{r^2} + 4\pi r$$

$$\frac{d^2S}{dr^2} = \frac{104}{r^3} + 4\pi$$

For max. or min. values, we have,

$$\frac{dS}{dr} = 0$$

or, $-\frac{52}{r^2} + 4\pi r = 0$

or, $-52 + 4\pi r^3 = 0$

or, $r^3 = \frac{52}{4\pi}$

or, $r^3 = \frac{13}{\pi}$

or, $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}$

When $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}$,

$\frac{d^2S}{dr^2} = \frac{104}{\frac{13}{\pi}} + 4\pi$ which is positive.

So, S is minimum when $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}$

When $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}$, from (i)

$$h = \frac{26}{\pi\left[\left(\frac{13}{\pi}\right)^{\frac{1}{3}}\right]^2}$$

$$= 2\left(\frac{13}{\pi}\right)\left(\frac{13}{\pi}\right)^{-\frac{2}{3}}$$

$$= 2\left(\frac{13}{\pi}\right)^{\frac{1}{3}}.$$

Exercise 1.6

1. The distance s, in meters, travelled in t seconds by a particle moving in a straight line is given by $s = t^3 - 2t^2$. Find the velocity and acceleration of the particle when $t = 2$ seconds.

Solution

Given,

$s = t^3 - 2t^2$

$\frac{ds}{dt} = 3t^2 - 4t$

When t = 2 sec,

$\frac{ds}{dt} = 3 \times 2^2 - 4 \times 2 = 12 - 8 = 4$ m/sec.

Also, $\frac{d^2s}{dt^2} = 6t - 4$

When $t = 2$ sec,

$\frac{d^2s}{dt^2} = 6 \times 2 - 4 = 8$ m/sec^2.

$\therefore$ Velocity = 4 m/sec, acceleration = 8 m/sec^2.

2. Find the rate of change of the volume of a cylinder of radius r and height h will respect to a change in the radius.

Solution

Let V be the volume of cylinder.

We have, $V = \pi r^2 h$.

Now,

$$\frac{dV}{dr} = \frac{d}{dr}(\pi r^2 h)$$

$$= \pi h \frac{d}{dr}(r^2) \qquad [\because h \text{ is constant}]$$

$$= 2\pi r h$$

Therefore, the rate of change of V with respect to a change in radius is $2\pi rh$.

3. The radius of a circular plate is increasing at the rate of 0.20 cm/sec. At what rate is the area increasing when the radius of the plate is 25 cm?

Solution

Let A be the area and 'r' be the radius of circular plate at time 't'.

Given that; $\frac{dr}{dt} = 0.20$ cm/sec

Since, $A = \pi r^2$

$\therefore \frac{dA}{dt} = \frac{d(\pi r^2)}{dt} = \pi \frac{dr^2}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$

when, $r = 25$ cm and $\frac{dr}{dt} = 0.20$ cm/sec

$\therefore \frac{dA}{dt} = \pi \times 2 \times 25 \times 0.20 = 10\pi$

Hence, the area is increasing at the rate of $10\,\pi$ cm^2/sec.

4. a. A spherical balloon is inflated at the rate of 10 cubic cm/sec. At what rate is the radius increasing when the radius is 6 cm.

b. The radius of a sphere is increasing at a rate of 1 cm/sec. Find the rate of change of volume at this time when the radius is 3 cm.

Solution

a. Let r be the radius and V be the volume of spherical balloon at time t. By question,

$$\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec}$$

When r = 6 cm, $\frac{dr}{dt} = ?$

We have, $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

When r = 6 cm

$$10 = \frac{4}{3}\pi \times 3 \times 6^2 \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{5}{72\pi} \text{ cm/sec}$$

The radius is increasing at the rate of $\frac{5}{72\pi}$ cm/sec

b Let the volume be V and radius be r at time t, then

$$V = \frac{4}{3}\pi r^3$$

and $\frac{dV}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

Here,

$\frac{dr}{dt} = 1$, when $r = 3$

Hence

$$\frac{dV}{dt} = 4\pi(3)^2 \cdot 1 = 36\pi \text{ cu. cm/ sec}$$

It means that the volume is changing at the rate of 36π cu cm/sec.

5. A spherical ball of salt is dissolving in water in such a way that the rate of decrease in volume at any instant is proportional to the surface. Prove that the radius is decreasing at the constant rate.

Solution

Let r be the radius, V be the volume and be S be the surface area of the spherical ball of salt at time t. By question,

$$\therefore \frac{dV}{dt} \propto S$$

or, $\frac{dV}{dt} = -ks$ ...(i)

where –k is a constant.

We have, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

Then (i) can be written as

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k \cdot 4\pi r^2$$

or, $\frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = -k \cdot 4\pi r^2$

or, $\frac{dr}{dt} = -k$ which is a constant.

$\therefore$ The radius is decreasing at a constant rate.

6. The side of a square is increasing at the rate of 0.2 cm/sec. Find the rate of increase of the (a) perimeter of the square and (b) area of square when the side of the square is 12 cm.

Solution

Let x be the length of side, A be the area and P be the perimeter of the square in time t. By given

$\frac{dx}{dt} = 0.2$ cm/sec

a. We know

$P = 4x$

$\frac{dP}{dt} = 4 \cdot \frac{dx}{dt}$

When $x = 12$ cm, $\frac{dP}{dt} = 4 \times 0.2 = 0.8$ cm/sec

$\therefore$ The perimeter is increasing at the rate of 0.8 cm/sec.

b. We have

$A = x^2$

$\frac{dA}{dt} = 2x\frac{dx}{dt}$

When $x = 12$ cm, $\frac{dA}{dt} = 2 \times 12 \times 0.2 = 4.8$ cm²/sec.

$\therefore$ The area is increasing at the rate of 4.8 cm^2/sec.

7. If the volume of an expanding cube is increasing at the rate of 4ft^3/min, how fast is its surface area increasing when the surface area is 24 sq.ft.?

Solution

Let x be the length of side, V be the volume and S be the surface area at time t.

By question, $\frac{dV}{dt} = 4ft^3/min$

When S = 24 sq.ft, $\frac{dS}{dt} = ?$

We have, $V = x^3$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

or, $4 = 3x^2\frac{dx}{dt}$

$\therefore \frac{dx}{dt} = \frac{4}{3x^2}$ ...(i)

Again, we have, $S = 6x^2$

$\frac{dS}{dt} = 6 \cdot 2x \cdot \frac{dx}{dt}$

$= 12x \times \frac{4}{3x^2}$ [using (i)]

$= \frac{16}{x}$

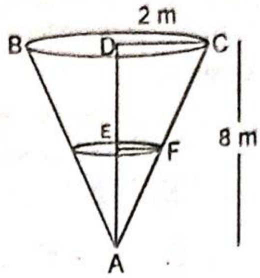
When S = 24 i.e. $6x^2 = 24 \Rightarrow x = 2$, $\frac{dS}{dt} = \frac{16}{2} = 8$ ft²/min

Hence, the surface area is increasing at the rate of 8 ft²/min.

8. Water flows into an inverted conical vessel at the rate of 24 m^3/m. When the depth of water depth of water is 4 m, how fast is the level rising, assuming that the height of the vessel is 8 m and the radius at the top is 2 m?

Solution

Let ABC be the conical tank into which water is flowing. Let h be the height AE of the water and r be the radius EF of the water surface at time t.



Now,

ΔACD and ΔAFE are similar.

So, $\frac{AE}{AD} = \frac{EF}{DC}$

or, $\frac{h}{8} = \frac{r}{2}$

$\Rightarrow$ $r = \frac{h}{4}$

Let V be the volume of water in the tank. By given, $\frac{dV}{dt} = 24m^3/m$. Then,

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 \cdot h$$

$$= \frac{\pi h^3}{48}$$

$$\frac{dV}{dt} = \frac{\pi}{48} \cdot 3h^2 \frac{dh}{dt}$$

$$= \frac{\pi h^2}{16} \cdot \frac{dh}{dt}$$

When h = 4m,

$$24 = \frac{\pi \times 4^2}{16} \times \frac{dh}{dt}$$

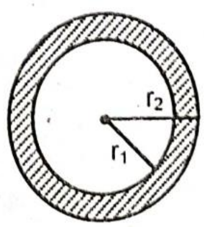
$$\therefore \quad \frac{dh}{dt} = \frac{24}{\pi}$$

Hence, the level is rising at the rate of $\frac{24}{\pi}$ m/min.

9. Two concentric circles are expanding in such a way that the radius of the inner circle is increasing at the rate of 10 cm/sec and that of the outer circle at the rate of 7 cm/sec. At a certain time, the radii of the inner and the outer circles are respectively 24 cm and 30 cm. At that time, is the area between the circles increasing or decreasing?

Solution

Let r_1 and r_2 be the radius of inner and outer circle respectively in time t.



By given,

$$\frac{dr_1}{dt} = 10 \text{ cm/sec}$$

and $$\frac{dr_2}{dt} = 7 \text{ cm/sec}$$

If A be the area between the outer and inner circles at time t, then

$$A = \text{area of outer circle} - \text{area of inner circle}$$

$$= \pi r_2^2 - \pi r_1^2$$

$$\frac{dA}{dt} = 2\pi r_2 \frac{dr_2}{dt} - 2\pi r_1 \frac{dr_1}{dt}$$

When, $r_1 = 24$ cm and $r_2 = 30$ cm

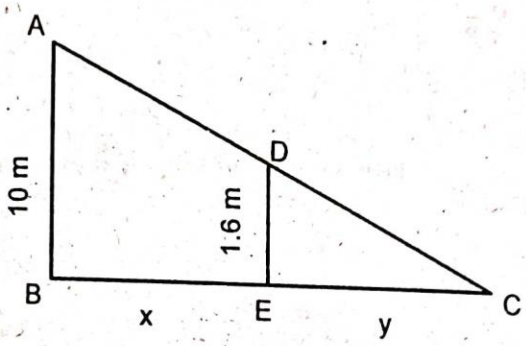
$$\frac{dA}{dt} = 2\pi \times 30 \times 7 - 2\pi \times 24 \times 10$$
$$= -60\pi \text{ cm}^2/\text{sec}$$

This shows that the area between the circles is decreasing at the rate of 60π cm^2/sec.

10. A 1.6 m man walks away from a 10 m lamp post at the rate of 3 m/sec. How fast is his shadow lengthening when he is 18 m from the post?

Solution

Let, AB be the lamp post and DE be the position of the man after time 't'.
If BE = x and shadow EC = y, then from similar triangles ABC and DEC;



$$\frac{AB}{DE} = \frac{BC}{EC}$$

$$\Rightarrow \frac{10}{1.6} = \frac{x+y}{y}$$

$$\Rightarrow 6.25 = \frac{x+y}{y}$$

$$\Rightarrow 5.25y = x$$

$$\therefore 5.25\frac{dy}{dx} = \frac{dx}{dt}$$

when, $\frac{dx}{dt} = 3$m/sec and $x = 18$ m, we have

$$5.25\frac{dy}{dx} = 3$$

$$\therefore \frac{dy}{dx} = \frac{3}{5.25} = \frac{4}{7} \text{ m/sec}$$

Hence, the shadow of man is lengthening at the rate of $\frac{4}{7}$ m/sec.

11. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one–sixth of the radius of the base. How fast is the height of the sand–cone increasing when the height is 4 cm?

Solution

Let V be the volume, 'r' be the radius and 'h' be the height of sand-cone at time 't'.
Then we have

$$\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}.$$

Also,

$$h = \frac{1}{6}r$$

We have,

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi (6h)^2 \cdot h$$
$$= \frac{1}{3}\pi \cdot 36h^3$$
$$= 12\pi h^3$$

Now, $$\frac{dV}{dt} = 12\pi \cdot 3h^2 \cdot \frac{dh}{dt}$$
$$= 36\pi h^2 \frac{dh}{dt}$$

when, $h = 4$ cm and $\frac{dV}{dt} = 12$ cm³/s, we have

$$12 = 36\pi \times 16 \times \frac{dh}{dt}$$

or, $$\frac{dh}{dt} = \frac{12}{36\pi \times 16} = \frac{1}{48\pi} \text{ cm/sec}$$

Hence, the height of the sand-cone is increasing at the rate $\frac{1}{48\pi}$ cm/sec.

12. A kite is 24 m high and there are 25 m of chord out. If the kite moves horizontally at the rate of 36 km/h. directly away from the person who is flying it, how fast is the chord paid out?

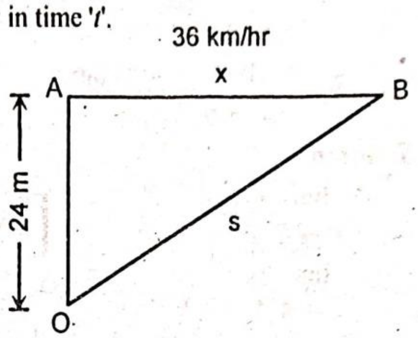
Solution

Let us draw a figure as shown in alongside.

Here, OA = 24 m

Let B be the position of the kite moving horizontally in time 't'.

Also, suppose OB = s and AB = x, $\frac{dx}{dt} = 36$ km/hr



From figure, we have

$$OB^2 = OA^2 + AB^2$$

$\Rightarrow$ $s^2 = 24^2 + x^2$ ... (i)

when, $s = 25$

$$25^2 = 24^2 + x^2$$

$\therefore$ $x^2 = 25^2 - 24^2 = 49$

$\Rightarrow$ $x = 7$

From (i), $2s\frac{ds}{dt} = 0 + 2x\frac{dx}{dt}$

$\Rightarrow$ $s\frac{ds}{dt} = x\frac{dx}{dt}$

when $x = 7\text{m} = 0.007$ km, $s = 25$ m $= 0.025$ km

$$0.025 \times \frac{ds}{dt} = 0.007 \times 36$$

$\Rightarrow$ $\frac{ds}{dt} = \frac{7 \times 36}{25} = 10.08$ km. p.h.

Exercise 1.7

1. Evaluate the following using L Hospital's rule.

a. $\lim_{x \to 4} \frac{x^2-16}{x-4}$

b. $\lim_{x \to 3} \frac{x^3-27}{x^2-9}$

c. $\lim_{x \to -5} \frac{x^2-25}{x+5}$

d. $\lim_{x \to 2} \frac{x^2-5x+6}{x^2-x-2}$

e. $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$

f. $\lim_{x \to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$

g. $\lim_{x \to 0} \frac{1-\cos x}{x+x^2}$

h. $\lim_{x \to 0} \frac{e^x-x-1}{x^2}$

i. $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

j. $\lim_{x \to 0} \frac{x - \sin x \cos x}{x^3}$

k. $\lim_{x \to 0} \frac{(e^x-1)\tan x}{x^2}$

l. $\lim_{t \to 0} \frac{t \sin t}{1-\cos t}$

Solution

a. $\lim_{x \to 4} \frac{x^2-16}{x-4}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 4} \frac{2x}{1}$

$= 2 \times 4 = 8$

b. $\lim_{x \to 3} \frac{x^3-27}{x^2-9}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 3} \frac{3x^2}{2x}$

$= \frac{3 \times 3^2}{2 \times 3} = \frac{9}{2}$

c. $\lim_{x \to -5} \frac{x^2-25}{x+5}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to -5} \frac{2x}{1}$

$= 2 \times (-5)$

$= -10.$

d. $\lim_{x \to 2} \frac{x^2-5x+6}{x^2-x-2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 2} \frac{2x-5}{2x-1}$

$= \frac{2 \times 2 - 5}{2 \times 2 - 1} = -\frac{1}{3}.$

e. $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1}$

$= \frac{1}{2\sqrt{1+0}} = \frac{1}{2}.$

f. $\lim_{x \to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{-\frac{1}{4(1+x)^{\frac{3}{2}}}}{2}$

$= -\frac{1}{8(1+0)^{\frac{3}{2}}}$

$= -\frac{1}{8}$

g. $\lim_{x \to 0} \frac{1-\cos x}{x+x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{\sin x}{1+2x}$

$= \frac{\sin 0}{1+2\times 0}$

$= \frac{0}{1}$

$= 0.$

h. $\lim_{x \to 0} \frac{e^x - x - 1}{x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{e^x - 1}{2x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{e^x}{2}$

$= \frac{e^0}{2}$

$= \frac{1}{2}$

i. $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0} \frac{2\sec x \,.\, \sec x \tan x}{\sin x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{\sin x}$

$= \lim_{x\to 0} \frac{2(\sec^2 x \,.\, \sec^2 x + \tan x \,.\, 2\sec x \,.\, \sec x \,.\, \tan x)}{\cos x}$

$= \lim_{x\to 0} \frac{2(\sec^4 x + 2\sec^2 x \tan^2 x)}{\cos x} = \frac{2(1+0)}{1} = 2$

j. $\lim_{x \to 0} \frac{x - \sin x \cos x}{x^3}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{x - \frac{1}{2}\sin 2x}{x^3}$

$= \lim_{x \to 0} \frac{1 - \cos 2x}{3x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{2\sin 2x}{6x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{\sin 2x}{3x}$

$= \lim_{x \to 0} \frac{2\cos 2x}{3}$

$= \frac{2\times 1}{3}$

$= \frac{2}{3}$

k. $\lim_{x \to 0} \frac{(e^x - 1)\tan x}{x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{(e^x - 1)\cdot \sec^2 x + e^x \tan x}{2x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x \to 0} \frac{(e^x - 1)\cdot 2\sec^2 x \tan x + e^x \sec^2 x + e^x \sec^2 x + e^x \tan x}{2}$

$= \frac{0+1+1+0}{2}$

$= \frac{2}{2}$

$= 1.$

l. $\lim_{t \to 0} \frac{t \sin t}{1 - \cos t}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{t \to 0} \frac{t \cos t + \sin t}{\sin t}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{t \to 0} \frac{t(-\sin t) + \cos t + \cos t}{\cos t}$

$= \frac{0 + 1 + 1}{1}$

$= 2$

2. Evaluate the following limits using L Hospital's rule.

a. $\lim_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1}$

b. $\lim_{x \to \infty} \frac{2x^2 + 3x}{1 + 5x^2}$

c. $\lim_{x \to \infty} \frac{x^5}{e^x}$

d. $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$

Solution

a. $\lim_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{10x - 3}{14x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{10}{14}$

$= \frac{5}{7}$

b. $\lim_{x \to \infty} \frac{2x^2 + 3x}{1 + 5x^2}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{4x + 3}{10x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{4}{10}$

$= \frac{2}{5}$

c. $\lim_{x \to \infty} \frac{x^5}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{5x^4}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{20x^3}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{60x^2}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{120x}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{120}{e^x}$

$= 0$

d. $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$

$= \lim_{x \to \frac{\pi}{2}} \frac{\sec^2 x}{3 \sec^2 3x}$

$= \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 3x}{3 \cos^2 x}$

$= \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 6x}{3(1 + \cos 2x)}$

$= \lim_{x \to \frac{\pi}{2}} \frac{-6 \sin 6x}{-6 \sin 2x}$

$= \lim_{x \to \frac{\pi}{2}} \frac{\sin 6x}{\sin 2x}$

$= \lim_{x \to \frac{\pi}{2}} \frac{6 \cos 6x}{2 \cos 2x}$

$= \frac{6 \cos 3\pi}{2 \cos \pi}$

$= \frac{-6}{-2} = 3$

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UNIT 2

Partial Derivatives

Exercise 2.1

1. Use definition (first principles) of partial derivatives to find $f_x(x, y)$ and $f_y(x, y)$ for the following functions.

a. $f(x, y) = xy$

b. $f(x, y) = 3x^2y^2$

c. $f(x, y) = xy + y^2$

d. $f(x, y) = x^2 - xy$

Solution

a. Given

$f(x, y) = xy$

We have,

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)y - xy}{h}$$

$$= \lim_{h \to 0} \frac{xy + hy - xy}{h}$$

$$= \lim_{h \to 0} \frac{hy}{h}$$

$$= \lim_{h \to 0} (y)$$

$$= y$$

Again, we have,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{x(y+k) - xy}{k}$$

$$= \lim_{k \to 0} \frac{xy + xk - xy}{k}$$

$$= \lim_{k \to 0} \frac{xk}{k}$$

$$= \lim_{k \to 0} (x)$$

$$= x$$

b. Given

$f(x, y) = 3x^2y^2$

We have,

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2y^2 - 3x^2y^2}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2)y^2 - 3x^2y^2}{h}$$

$$= \lim_{h \to 0} \frac{3x^2y^2 + 6xy^2h + 3h^2y^2 - 3x^2y^2}{h}$$

$$= \lim_{h \to 0} \frac{h(6xy^2 + 3hy^2)}{h}$$

$$= \lim_{h \to 0} (6xy^2 + 3hy^2)$$

$$= 6xy^2$$

Again, we have,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{3x^2(y+k)^2 - 3x^2y^2}{k}$$

$$= \lim_{k \to 0} \frac{3x^2(y^2 + 2yk + k^2) - 3x^2y^2}{k}$$

$$= \lim_{k \to 0} \frac{3x^2y^2 + 6x^2yk + 3x^2k^2 - 3x^2y^2}{k}$$

$$= \lim_{k \to 0} \frac{k(6x^2y + 3x^2k)}{k}$$

$$= \lim_{k \to 0} (6x^2y + 3x^2k)$$

$$= 6x^2y + 0$$

$$= 6x^2y$$

c. Given

$f(x, y) = xy + y^2$

We have,

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)y + y^2 - (xy + y^2)}{h}$$

$$= \lim_{h \to 0} \frac{xy + hy + y^2 - xy - y^2}{h}$$

$$= \lim_{h \to 0} \frac{hy}{h}$$

$$= \lim_{h \to 0} (y)$$

$$= y$$

Again, we have,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{x(y+k) + (y+k)^2 - (xy + y^2)}{k}$$

$$= \lim_{k \to 0} \frac{xy + xk + y^2 + 2yk + k^2 - xy - y^2}{k}$$

$$= \lim_{k \to 0} \frac{k(x + 2y + k)}{k}$$

$$= x + 2y + 0$$

$$= x + 2y$$

d Given

$f(x, y) = x^2 - xy$

We have,

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - (x+h)y - (x^2 - xy)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - xy + hy - x^2 + xy}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - hy}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h - y)}{h}$$

$$= \lim_{h \to 0} (2x + h - y)$$

$$= 2x + 0 - y$$

$$= 2x - y$$

Again, we have,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{x^2 - x(y+k) - (x^2 - xy)}{k}$$

$$= \lim_{k \to 0} \frac{x^2 - xy - xk - x^2 + xy}{k}$$

$$= \lim_{k \to 0} \frac{-xk}{k} = \lim_{k \to 0} (-x)$$

$$= -x$$

2. Using definition, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ from $f(x, y, z) = xyz$.

Solution

Given,

$f(x, y, z) = xyz$

We have,

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)yz - xyz}{h}$$

$$= \lim_{h \to 0} \frac{xyz + hyz - xyz}{h}$$

$$= \lim_{h \to 0} \frac{hyz}{h}$$

$$= yz$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k}$$

$$= \lim_{k \to 0} \frac{x(y+k)z - xyz}{k}$$

$$= \lim_{k \to 0} \frac{xyz + kxz - xyz}{k}$$

$$= \lim_{k \to 0} \frac{kxz}{k}$$

$$= xz$$

$$\frac{\partial f}{\partial z} = \lim_{l \to 0} \frac{f(x, y, z+l) - f(x, y, z)}{l}$$

$$= \lim_{l \to 0} \frac{xy(z+l) - xyz}{l}$$

$$= \lim_{l \to 0} \frac{xyz + xyl - xyz}{l}$$

$$= \lim_{l \to 0} \frac{xyl}{l} = xy$$

3. Find the first order partial derivatives of the following functions.

a. $f(x, y) = 5x^4y^5$

b. $f(x, y) = ax^2 + 2hxy + by^2$

c. $f(x, y) = x^3 + x^2y^2 - 2y^3$

d. $f(x, y) = \sqrt{x^2 + y^2}$

e. $f(x, y) = x \sin xy$

f. $f(x, y) = \dfrac{3xy}{x + \cos y}$

g. $f(x, y) = \sin^{-1}(3x + 4y)$

h. $f(x, y) = \ln(2x + 5y)$

i. $f(x, y, z) = x^3y^5z^2$

j. $f(x, y, z) = \ln(3x + y + 2z)$

Solution

a. Given,

$f(x, y) = 5x^4y^5$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(5x^4y^5) = 20x^3y^5$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(5x^4y^5) = 25x^4y^4$$

b. Given,

$f(x, y) = ax^2 + 2hxy + by^2$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(ax^2 + 2hxy + by^2)$$

$$= 2ax + 2hy$$

$$= 2(2x + hy)$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(ax^2 + 2hxy + by^2)$$

$$= 2hx + 2by$$

$$= 2(hx + by)$$

c. Given,

$f(x, y) = x^3 + x^2y^2 - 2y^3$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 + x^2y^2 - 2y^3)$$

$$= 3x^2 + 2xy^2$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 + x^2y^2 - 2y^3)$$

$$= 2x^2y - 6y^2$$

d. Given,

$f(x, y) = \sqrt{x^2 + y^2}$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\sqrt{x^2 + y^2})$$

$$= \frac{\partial (x^2 + y^2)^{1/2}}{\partial (x^2 + y^2)} \cdot \frac{\partial (x^2 + y^2)}{\partial x}$$

$$= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x$$

$$\frac{x}{\sqrt{x^2 + y^2}}$$

Similarly,

$$f_y = \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

e. Given,

$f(x, y) = x \sin xy$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \sin xy)$$

$$= x\frac{\partial}{\partial x}(\sin xy) + \sin xy\frac{\partial}{\partial x}(x) \quad \text{[Product rule]}$$

$$= x \cdot \cos xy \cdot y + \sin xy$$

$$= xy \cos xy + \sin xy$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \sin xy)$$

$$= x\frac{\partial}{\partial y}(\sin xy)$$

$$= x \cdot \cos xy \cdot x$$

$$= x^2 \cos xy$$

f. Given,

$$f(x, y) = \frac{3xy}{x + \cos y}$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{3xy}{x+\cos y}\right)$$

$$= \frac{(x+\cos y)\frac{\partial}{\partial x}(3xy) - 3xy\frac{\partial}{\partial x}(x+\cos y)}{(x+\cos y)^2}$$

$$= \frac{(x+\cos y)\cdot 3y - 3xy \cdot 1}{(x+\cos y)^2}$$

$$= \frac{3xy + 3y\cos y - 3xy}{(x+\cos y)^2}$$

$$= \frac{3y\cos y}{(x+\cos y)^2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{3xy}{x+\cos y}\right)$$

$$= \frac{(x+\cos y)\frac{\partial}{\partial y}(3xy) - 3xy\frac{\partial}{\partial y}(x+\cos y)}{(x+\cos y)^2}$$

$$= \frac{(x+\cos y)\cdot 3x - 3xy\cdot(-\sin y)}{(x+\cos y)^2}$$

$$= \frac{3x^2 + 3x\cos y + 3xy\sin y}{(x+\cos y)^2}$$

g. Given,

$f(x, y) = \sin^{-1}(3x + 4y)$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{\sin^{-1}(3x+4y)\}$$

$$= \frac{\partial\{\sin^{-1}(3x+4y)\}}{\partial(3x+4y)} \cdot \frac{\partial(3x+4y)}{\partial x}$$

$$= \frac{3}{\sqrt{1-(3x+4y)^2}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\{\sin^{-1}(3x+4y)\}$$
$$= \frac{\partial\{\sin^{-1}(3x+4y)\}}{\partial(3x+4y)} \cdot \frac{\partial(3x+4y)}{\partial y}$$
$$= \frac{4}{\sqrt{1-(3x+4y)^2}}$$

h. Given,

$$f(x, y) = \ln(2x+5y)$$
$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{\ln(2x+5y)\}$$
$$= \frac{\partial\{\ln(2x+5y)\}}{\partial(2x+5y)} \cdot \frac{\partial(2x+5y)}{\partial x}$$
$$= \frac{1}{2x+5y} \cdot 2$$
$$= \frac{2}{2x+5y}$$

Similarly,

$$f_y = \frac{\partial f}{\partial y} = \frac{5}{2x+5y}$$

i. Given,

$$f(x, y, z) = x^3y^5z^2$$
$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3y^5z^2) = 3x^2y^5z^2$$
$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y^5z^2) = 5x^3y^4z^2$$
$$f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^3y^5z^2) = 2x^3y^5z$$

j. Given,

$$f(x, y, z) = \ln(3x+y+2z)$$
$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{\ln(3x+y+2z)\}$$
$$= \frac{\partial\{\ln(3x+y+2z)\}}{\partial(3x+y+2z)} \cdot \frac{\partial(3x+y+2z)}{\partial x}$$
$$= \frac{1}{3x+y+2z} \cdot 3$$
$$= \frac{3}{3x+y+2z}$$

In the same way,

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{3x+y+2z}$$
$$f_z = \frac{\partial f}{\partial z} = \frac{2}{2x+y+2z}$$

4. Find the second order partial derivatives of the following functions.

a. $f(x, y) = x^3y^4 + 3x^2y^2 - 1$

b. $f(x, y) = \sin(2x+3y)$

c. $f(x, y) = x^5 e^{3y}$

Solution

a. $f(x, y) = x^3y^4 + 3x^2y^2 - 1$

$f_x = 3x^3y^4 + 6xy^2$

$f_y = 4x^3y^3 + 6x^2y$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(3x^2y^4 + 6xy^2) = 6xy^4 + 6y^2$

$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(3x^3y^4 + 6xy^2) = 12x^2y^3 + 12xy$

$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(4x^3y^3 + 6x^2y) = 12x^2y^3 + 12xy$

$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(4x^3y^3 + 6x^2y) = 12x^3y^2 + 6x^2$

b. $f(x, y) = \sin(2x + 3y)$

$f_x = \frac{\partial}{\partial x}\{\sin(2x + 3y)\}$

$= \frac{\partial\{\sin(2x + 3y)\}}{\partial(2x + 3y)} \cdot \frac{\partial(2x + 3y)}{\partial x}$

$= \cos(2x + 3y) \cdot 2$

$= 2\cos(2x + 3y)$

$f_y = \frac{\partial}{\partial y}\{\sin(2x + 3y)\}$

$= \frac{\partial\{\sin(2x + 3y)\}}{\partial(2x + 3y)} \cdot \frac{\partial(2x + 3y)}{\partial y}$

$= \cos(2x + 3y) \cdot 3$

$= 3\cos(2x + 3y)$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\{2\cos(2x + 3y)\} = -4\sin(2x + 3y)$

$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\{2\cos(2x + 3y)\} = -2\sin(2x + 3y)$

$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\{3\cos(2x + 3y)\} = -6\sin(2x + 3y)$

$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\{3\cos(2x + 3y)\} = -9\sin(2x + 3y)$

c. $f(x, y) = x^5 e^{3y}$

$f_x = 5x^4 e^{3y}$

$f_y = 3x^5 e^{3y}$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(5x^4 e^{3y}) = 20x^3e^{3y}$

$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(5x^4 e^{3y}) = 15x^4e^{3y}$

$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(3x^5 e^{3y}) = 15x^4e^{3y}$

$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(3x^5 e^{3y}) = 9x^5e^{3y}$

5. Calculate all the first order and second order partial derivatives of $f(x, y) = e^{-x} \sin(x+y)$.

Solution

Given,

$f(x, y) = e^{-x} \sin(x+y)$

$f_x = \frac{\partial}{\partial x}\{e^{-x}\sin(x+y)\}$

$= e^{-x}\frac{\partial}{\partial x}\{\sin(x+y)\} + \sin(x+y)\frac{\partial}{\partial x}(e^{-x})$

$= e^{-x}\cos(x+y)\cdot 1 + \sin(x+y)(-1)\cdot e^{-x}$

$= e^{-x}\{\cos(x+y) - \sin(x+y)\}$

$f_y = \frac{\partial}{\partial y}\{e^{-x}\sin(x+y)\}$

$= e^{-x}\frac{\partial\{\sin(x+y)\}}{\partial(x+y)}\cdot\frac{\partial(x+y)}{\partial y}$

$= e^{-x}\cos(x+y)\cdot 1$

$= e^{-x}\cos(x+y)$

$f_{xx} = \frac{\partial}{\partial x}(f_x)$

$= e^{-x}\frac{\partial}{\partial x}\{\cos(x+y) - \sin(x+y)\} + \{\cos(x+y) - \sin(x+y)\}\frac{\partial}{\partial x}(e^{-x})$

$= e^{-x}\{-\sin(x+y) - \cos(x+y) - \cos(x+y) + \sin(x+y)\}$

$= -2e^{-x}\cos(x+y)$

$f_{yy} = \frac{\partial}{\partial y}(f_y)$

$= \frac{\partial}{\partial y}\{e^{-x}\cos(x+y)\}$

$= e^{-x}\frac{\partial\{\cos(x+y)\}}{\partial(x+y)}\cdot\frac{\partial(x+y)}{\partial y}$

$= e^{-x}\{-\sin(x+y)\}\cdot 1$

$= -e^{-x}\sin(x+y)$

$f_{xy} = (f_x)_y$

$= \frac{\partial}{\partial y}(f_x)$

$= \frac{\partial}{\partial y}[e^{-x}\{\cos(x+y) - \sin(x+y)\}]$

$= e^{-x}\{-\sin(x+y) - \cos(x+y)\}\cdot 1$

$= -e^{-x}\{\sin(x+y) + \cos(x+y)\}$

$f_{yx} = (f_y)_x$

$= \frac{\partial}{\partial x}(f_y)$

$= \frac{\partial}{\partial x}\{e^{-x}\cos(x+y)\}$

$= e^{-x}\frac{\partial}{\partial x}\{\cos(x+y)\} + \cos(x+y)\frac{\partial}{\partial x}(e^{-x})$

$= e^{-x}\{-\sin(x+y)\}\cdot 1 + \cos(x+y)\cdot e^{-x}\cdot(-1)$

$= -e^{-x}\{\sin(x+y) + \cos(x+y)\}$

6. a. If $u = \ln(x^3 + y^3 + z^3 - 3xyz)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

b. If $z = \frac{x^2 + y^2}{x + y}$, then show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Solution

a. Given,

$$u = \ln(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\{\ln(x^3 + y^3 + z^3 - 3xyz)\}$$

$$= \frac{\partial\{\ln(x^3 + y^3 + z^3 - 3xyz)\}}{\partial(x^3 + y^3 + z^3 - 3xyz)} \cdot \frac{\partial(x^3 + y^3 + z^3 - 3xyz)}{\partial x}$$

$$= \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

Now,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz} + \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$= \frac{3}{x + y + z}$$

b. Given,

$$z = \frac{x^2 + y^2}{x + y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2 + y^2}{x + y}\right)$$

$$= \frac{(x + y)\frac{\partial}{\partial x}(x^2 + y^2) - (x^2 + y^2)\frac{\partial}{\partial x}(x + y)}{(x + y)^2}$$

$$= \frac{(x + y) \cdot 2x - (x^2 + y^2)}{(x + y)^2}$$

$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x + y)^2}$$

$$= \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2 + y^2}{x + y}\right)$$

$$= \frac{(x + y)\frac{\partial}{\partial y}(x^2 + y^2) - (x^2 + y^2)\frac{\partial}{\partial y}(x + y)}{(x + y)^2}$$

$$= \frac{(x + y) \cdot 2y - (x^2 + y^2)}{(x + y)^2}$$

$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x + y)^2}$$

$$= \frac{y^2 + 2xy - x^2}{(x + y)^2}$$

$$\text{LHS} = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$$

$$= \left(\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right)^2$$

$$= \left\{\frac{2x^2-2y^2}{(x+y)^2}\right\}^2$$

$$= 4\left\{\frac{x^2-y^2}{(x+y)^2}\right\}^2$$

$$= 4\left\{\frac{(x+y)(x-y)}{(x+y)^2}\right\}^2$$

$$= 4\left(\frac{x-y}{x+y}\right)^2$$

$$\text{RHS} = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= 4\left(1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right)$$

$$= 4\left\{\frac{(x+y)^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right\}$$

$$= 4\left\{\frac{x^2+2xy+y^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right\}$$

$$= 4\left\{\frac{x^2-2xy+y^2}{(x+y)^2}\right\}$$

$$= 4\left(\frac{x-y}{x+y}\right)^2$$

$\therefore$ LHS = RHS

7. a. If $z = \ln(x^2+y^2)$, then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

b. If $z = \cos(3x+4y)$, then show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

Solution

a. Given,

$$z = \ln(x^2+y^2)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\{\ln(x^2+y^2)\}$$

$$= \frac{\partial\{\ln(x^2+y^2)\}}{\partial(x^2+y^2)} \cdot \frac{\partial(x^2+y^2)}{\partial x}$$

$$= \frac{1}{x^2+y^2} \cdot 2x$$

$$= \frac{2x}{x^2+y^2}$$

Similarly,

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$$

Again,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial x}(2x) - 2x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\cdot 2 - 2x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$= \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

Similarly,

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

Now,

$$\text{LHS} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{-2x^2+2y^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2}$$

$$= 0$$

$$= \text{RHS}$$

b. Given,

$$z = \cos(3x+4y)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\{\cos(3x+4y)\}$$

$$= \frac{\partial\{\cos(3x+4y)\}}{\partial(3x+4y)}\cdot\frac{\partial(3x+4y)}{\partial y}$$

$$= -\sin(3x+4y)\cdot 4$$

$$= -4\sin(3x+4y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\{\cos(3x+4y)\}$$

$$= -\sin(3x+4y)\cdot 3$$

$$= -3\sin(3x+4y)$$

$$\text{LHS} = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\{-4\sin(3x+4y)\}$$

$$= -\frac{\partial\{\sin(3x+4y)\}}{\partial(3x+4y)}\cdot\frac{\partial(3x+4y)}{\partial x}$$

$$= -4\cos(3x+4y)\cdot 3$$

$$= -12\cos(3x+4y)$$

$$\text{RHS} = \frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\{-3\sin(3x+4y)\}$$

$$= -3\cos(3x+4y)\cdot 4$$

$$= -12\cos(3x+4y)$$

$\therefore$ LHS = RHS

8. a. If $f(x, y) = x^3y^2 + x^5 + y^3$, then find f_x and f_y at the point (1, 2).
   b. If $f(x, y) = x^4y^2 - x^3y^3$, then find f_{xx} and f_{xxx} at the point (2, 1).
   b. If $f(x, y, z) = \dfrac{x}{x+y+z}$, then find $f_x(-1, 1, 2)$ and $f_y(-1,1, 2)$.

Solution

a. Given,

$f(x, y) = x^3y^2 + x^5 + y^3$

$f_x \quad = 3x^2y^2 + 5x^4$

At (1, 1),

$f_x(1, 1) = 3 \times 1^2 \times 1^2 + 5 \times 1^4 = 8$

$f_y \quad = 2x^3y + 3y^2$

At (1, 1),

$f_y(1, 1) = 2 \times 1^3 \times 1 + 3 \times 1^2 = 5$

b. Given,

$f(x, y) = x^4y^2 - x^3y^3$

$f_x \quad = 4x^3y^2 - 3x^2y^3$

$f_{xx} \quad = 12x^2y^2 - 6xy^3$

$f_{xxx} \quad = 24xy^2 - 6y^3$

At (2, 1),

$f_{xx} \quad = 12 \times 2^2 \times 1^2 - 6 \times 2 \times 1^3 = 36$

$f_{xxx} \quad = 24 \times 2 \times 1^2 - 6 \times 1^3 = 42$

c. Given,

$$f(x, y, z) = \frac{x}{x+y+z}$$

$$f_x \quad = \frac{\partial}{\partial x}\left(\frac{x}{x+y+z}\right)$$

$$= \frac{(x+y+z)\frac{\partial}{\partial x}(x) - x \cdot \frac{\partial}{\partial x}(x+y+z)}{(x+y+z)^2}$$

$$= \frac{(x+y+z) \cdot 1 - x \cdot 1}{(x+y+z)^2}$$

$$= \frac{y+z}{(x+y+z)^2}$$

At (–1, 1, 2),

$$f_x \quad = \frac{1+2}{(-1+1+2)^2} = \frac{3}{4}$$

Again,

$$f_y \quad = \frac{\partial}{\partial y}\left(\frac{x}{x+y+z}\right)$$

$$= \frac{\partial}{\partial y}\{x(x+y+z)^{-1}\}$$

$$= x\frac{\partial(x+y+z)^{-1}}{\partial(x+y+z)} \cdot \frac{\partial(x+y+z)}{\partial y}$$

$$= -x(x+y+z)^{-2} \cdot 1$$

$$= \frac{-x}{(x+y+z)^2}$$

At (– 1,1, 2)

$$f_y \quad = \frac{-(-1)}{(-1+1+2)^2}$$

$$= \frac{1}{4}$$

Exercise 2.2

1. Verify Euler's theorem for the following functions.

a. $u = x^2 + y^2$

b. $u = ax^2 + 2hxy + by^2$

c. $u = \dfrac{xy}{x+y}$

d. $u = x^2 e^{\frac{y}{x}}$

e. $u = x^3 + y^3 + z^3 - 3xyz$

f. $u = \dfrac{x^2 + z^2}{xy + yz}$

g. $u = \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$

Solution

a. Given,

$u = f(x, y) = x^2 + y^2$

Now,

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2$$
$$= \lambda^2 (x^2 + y^2)$$
$$= \lambda^2 f(x, y)$$

This shows that $u = f(x, y)$ is a homogeneous function of degree 2.

Thus $n = 2$.

Next, to verify Euler's theorem we first calculate $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ then verify

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Now,

$u = x^2 + y^2$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(2x) + y(2y)$$
$$= 2x^2 + 2y^2$$
$$= 2(x^2 + y^2)$$
$$= 2u \qquad [\because u = x^2 + y^2]$$

Hence, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u.$

Thus, Euler's theorem is verified.

b. Given,

$u = ax^2 + 2hxy + by^2$

Let $f(x, y) = ax^2 + 2hxy + by^2$

Now,

$$f(\lambda x, \lambda y) = a(\lambda x)^2 + 2h(\lambda x)(\lambda y) + b(\lambda y)^2$$
$$= \lambda^2 (ax^2 + 2hxy + by^2)$$
$$= \lambda^2 f(x, y)$$

Hence, u is a homogeneous function of degree 2. To verify Euler's theorem, we have to show

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u \text{ as } n=2.$$

Now,

$$\text{LHS} = x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$
$$= x\frac{\partial}{\partial x}(ax^2+2hxy+by^2)+y\frac{\partial}{\partial y}(ax^2+2hxy+by^2)$$
$$= x(2ax+2hy)+y(2hx+2by)$$
$$= 2ax^2+2hxy+2hxy+2by^2$$
$$= 2ax^2+4hxy+2by^2$$
$$= 2(ax^2+2hxy+by^2)$$
$$= 2u$$
$$= \text{RHS}$$

c. Given

$$u=f(x+y)=\frac{xy}{x+y}$$

Now,

$$f(\lambda x,\lambda y) = \frac{(\lambda x)(\lambda y)}{\lambda x+\lambda y}$$
$$= \frac{\lambda^2 xy}{\lambda(x+y)}$$
$$= \lambda\left(\frac{xy}{x+y}\right)$$
$$= \lambda f(x,y)$$

This shows that u is a homogeneous function of degree 1. To verify Euler's theorem, we have to show

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

i.e. $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u \qquad [\because n=1]$$

Now,

$$\text{LHS} = x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$
$$= x\frac{\partial}{\partial x}\left(\frac{xy}{x+y}\right)+y\frac{\partial}{\partial y}\left(\frac{xy}{x+y}\right)$$
$$= x\cdot\left\{\frac{(x+y)\cdot y-xy\cdot 1}{(x+y)^2}\right\}+y\cdot\left\{\frac{(x+y)\cdot x-xy\cdot 1}{(x+y)^2}\right\}$$
$$= \frac{x(xy+y^2-xy)}{(x+y)^2}+\frac{y(x^2+xy-xy)}{(x+y)^2}$$
$$= \frac{xy^2}{(x+y)^2}+\frac{x^2y}{(x+y)^2}$$
$$= \frac{xy^2+x^2y}{(x+y)^2}$$
$$= \frac{xy(x+y)}{(x+y)^2}$$
$$= \frac{xy}{x+y}$$
$$= u$$
$$= \text{RHS}$$

d. Given

$$u = x^2 e^{\frac{y}{x}} = x^2 \phi\left(\frac{y}{x}\right)$$

So, u is a homogeneous function of degree 2.

Thus, $n = 2$.

To verify Euler's theorem, we have to show

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

i.e. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

Now,

$$\begin{aligned}
\text{LHS} &= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \\
&= x\frac{\partial}{\partial x}\left(x^2 e^{\frac{y}{x}}\right) + y\frac{\partial}{\partial y}\left(x^2 e^{\frac{y}{x}}\right) \\
&= x\cdot\left\{x^2\cdot\frac{\partial}{\partial x}\left(e^{\frac{y}{x}}\right) + e^{\frac{y}{x}}\cdot\frac{\partial}{\partial x}(x^2)\right\} + y\cdot x^2 e^{\frac{y}{x}}\cdot\frac{1}{x} \\
&= x\cdot\left\{x^2\cdot e^{\frac{y}{x}}\cdot y\cdot\left(-\frac{1}{x^2}\right) + e^{\frac{y}{x}}\cdot 2x^2\right\} + xy\,e^{\frac{y}{x}} \\
&= -xy\,e^{\frac{y}{x}} + 2x^2 e^{\frac{y}{x}} + xy\,e^{\frac{y}{x}} \\
&= 2\left(x^2 e^{\frac{y}{x}}\right) \\
&= 2u \\
&= \text{RHS}
\end{aligned}$$

e. Given,

$$u = f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$$

Now,

$$\begin{aligned}
f(\lambda x, \lambda y, \lambda z) &= (\lambda x)^3 + (\lambda y)^3 + (\lambda z)^3 - 3(\lambda x)(\lambda y)(\lambda z) \\
&= \lambda^3(x^3 + y^3 + z^3 - 3xyz) \\
&= \lambda^3 f(x, y, z)
\end{aligned}$$

This shows that u is a homogeneous function of degree 3. To verify Euler's theorem, we have to show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

i.e. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$$

Now,

$$\begin{aligned}
\text{LHS} &= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} \\
&= x\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + y\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz) + z\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz) \\
&= x(3x^2 - 3yz) + y(3y^2 - 3zx) + z(3z^2 - 3xy) \\
&= 3(x^3 - xyz + y^3 - xyz + z^3 - xyz) \\
&= 3(x^3 + y^3 + z^3 - 3xyz) \\
&= 3u \\
&= \text{RHS}
\end{aligned}$$

f. $u = \dfrac{x^2 + z^2}{xy + yz}$

Let $u = f(x, y, z) = \dfrac{x^2 + z^2}{xy + yz}$

Now,

$$f(\lambda x, \lambda y, \lambda z) = \frac{(\lambda x)^2 + (\lambda z)^2}{(\lambda x)(\lambda y) + (\lambda y)(\lambda z)}$$
$$= \frac{\lambda^2 (x^2 + z^2)}{\lambda^2 (xy + yz)}$$
$$= \frac{x^2 + z^2}{xy + yz}$$
$$= \lambda^{\circ} \left(\frac{x^2 + z^2}{xy + yz}\right)$$
$$= \lambda^{\circ} f(x, y, z)$$

This shows that u is a homogeneous function of degree 0. To verify the Euler's theorem, we have to show

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

i.e. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$ $\qquad [\because n = 0]$

$$\text{LHS} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$
$$= x\frac{\partial}{\partial x}\left(\frac{x^2 + z^2}{xy + yz}\right) + y\frac{\partial}{\partial y}\left(\frac{x^2 + z^2}{xy + yz}\right) + z\frac{\partial}{\partial z}\left(\frac{x^2 + z^2}{xy + yz}\right)$$
$$= 0$$

g. Given

$$u = \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$$

i.e., $u(x, y) = \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$

Now,

$$f(\lambda x, \lambda y) = \frac{\sqrt{\lambda x} + \sqrt{\lambda y}}{\sqrt{\lambda x} - \sqrt{\lambda y}}$$
$$= \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$$
$$= \lambda^{\circ}\left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}\right)$$
$$= \lambda^{\circ} f(x + y)$$

So, u is a homogeneous function of degree 0.

To verify Euler's theorem, we have to show

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

i.e. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ $\qquad [\because n = 0]$

Now,

$$\text{LHS} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$
$$= x\frac{\partial}{\partial x}\left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}\right) + y\frac{\partial}{\partial y}\left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}\right)$$

$$= x\cdot\left\{\frac{(\sqrt{x}-\sqrt{y})\cdot\frac{1}{2\sqrt{x}}-(\sqrt{x}+\sqrt{y})\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}-\sqrt{y})^2}\right\}+y\cdot\left\{\frac{(\sqrt{x}-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}-(\sqrt{x}+\sqrt{y})\cdot\left(-\frac{1}{2\sqrt{y}}\right)}{(\sqrt{x}-\sqrt{y})^2}\right\}$$

$$= \frac{x\left(\frac{\sqrt{x}-\sqrt{y}-\sqrt{x}-\sqrt{y}}{2\sqrt{x}}\right)+y\left(\frac{\sqrt{x}-\sqrt{y}+\sqrt{x}+\sqrt{y}}{2\sqrt{y}}\right)}{(\sqrt{x}-\sqrt{y})^2}$$

$$= \frac{\frac{x(-2y)+y(2x)}{2\sqrt{x}\sqrt{y}}}{(\sqrt{x}-\sqrt{y})^2}$$

$$= \frac{-2xy+2xy}{2\sqrt{x}\sqrt{y}\,(\sqrt{x}-\sqrt{y})^2}$$

$$= 0$$

$$= \text{RHS}$$

2. a. Check $f(x, y) = x^n \cdot \tan^{-1}\left(\frac{y}{x}\right)$ for homogeneity and verify Euler's theorem if homogenous.

b. If $u = \frac{x^4+y^4}{x+y}$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3u$.

c. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \tan u$.

d. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u$.

e. If $u = x^n \ln\left(\frac{y}{x}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$.

Solution

a. Given,

$$f(x, y) = x^n \cdot \tan^{-1}\left(\frac{y}{x}\right)$$

$$= x^n\,\phi\left(\frac{y}{x}\right)$$

So, $f(x, y)$ is a homogeneous function of degree n. To verify Euler's theorem, we have to show;

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = nf$$

$$\text{LHS} = x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$$

$$= x\frac{\partial}{\partial x}\left\{x^n\cdot\tan^{-1}\left(\frac{y}{x}\right)\right\}+y\frac{\partial}{\partial y}\left\{x^n\cdot\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$= x\left\{x^n\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)+\tan^{-1}\left(\frac{y}{x}\right)\cdot nx^{n-1}\right\}+x^n y\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

$$= -x^{n-1}y\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}+n\,x^n\cdot\tan^{-1}\left(\frac{y}{x}\right)+x^{n-1}y\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$= nx^n\tan^{-1}\left(\frac{y}{x}\right)$$

$$= nf$$

b. Let $u = u(x, y) = \dfrac{x^4 + y^4}{x + y}$

Now,

$$u(\lambda x, \lambda y) = \frac{(\lambda x)^4 + (\lambda y)^4}{\lambda x + \lambda y}$$
$$= \frac{\lambda^4 (x^4 + y^4)}{\lambda(x + y)}$$
$$= \lambda^3 \left(\frac{x^4 + y^4}{x + y}\right)$$
$$= \lambda^3 u(x, y)$$

So, u is a homogeneous function of dgree 3.

By Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$\therefore$ $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u.$$

c. Given,

$$u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$$

or, $$\sin u = \frac{x^2 + y^2}{x + y}$$
$$= \frac{x^2\left(1 + \frac{y^2}{x^2}\right)}{x\left(1 + \frac{y}{x}\right)}$$
$$= \frac{x\left\{1 + \left(\frac{y}{x}\right)^2\right\}}{\left(1 + \frac{y}{x}\right)}$$
$$= x\,\phi\left(\frac{y}{x}\right)$$

Thus, $\sin u$ is a homogeneous function of degree 1.

Let $\sin u = v$.

By Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \cdot v$$

or, $$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \sin u$$

or, $$x\frac{\partial}{\partial u}(\sin u)\frac{\partial u}{\partial x} + y\frac{\partial}{\partial u}(\sin u)\frac{\partial u}{\partial y} = \sin u$$

or, $$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

or, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}$$

$\therefore$ $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

d. Given,

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

or, $$\tan u = \frac{x^3+y^3}{x-y}$$

$$= \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)}$$

$$= \frac{x^2\left\{1+\left(\frac{y}{x}\right)^3\right\}}{\left(1-\frac{y}{x}\right)}$$

$$= x^2\,\phi\left(\frac{y}{x}\right)$$

Thus, $\tan u$ is a homogeneous function of degree 2.

Let $\tan u = v$.

By Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2\cdot v$$

or, $$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

or, $$x\frac{\partial}{\partial u}(\tan u)\frac{\partial u}{\partial x} + y\frac{\partial}{\partial u}(\tan u)\frac{\partial u}{\partial y} = 2\tan u$$

or, $$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

or, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

or, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cdot\frac{\sin u}{\cos u}\cdot\cos^2 u$$

$\therefore$ $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

e. Given,

$$u = x^n \ln\left(\frac{y}{x}\right)$$

$$= x^n\,\phi\left(\frac{y}{x}\right)$$

This shows that u is a homogenous function of degree n. Now, by Euler's theorem, we have.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

3. Using total derivative, find $\frac{du}{dt}$ if $u = x^2 + y^2$, $x = at^2$ and $y = 2at$.

Solution

Given,

$u = x^2 + y^2$

$x = at^2$

$y = 2at$

We have,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(x^2+y^2)\cdot\frac{d}{dt}(at^2) + \frac{\partial}{\partial y}(x^2+y^2)\frac{d}{dt}(2at)$$

$$= 2x\cdot 2at + 2y\cdot 2a$$ [Using values of x and y]

$$= 2\cdot at^2\cdot 2at + 2\cdot 2at\cdot 2a$$

$$= 4a^2t^3 + 8a^2t$$

$$= 4a^2t(t^2+2)$$

4. Find the total derivative $\frac{du}{dt}$ if $u = (x+y)e^{xy}, x = t, y = \frac{1}{t^2}$.

Solution

Given,

$u = (x+y)e^{xy},$

$x = t,$

$y = \frac{1}{t^2}$

We have,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}\{(x+y)e^{xy}\}\cdot\frac{d}{dt}(t) + \frac{\partial}{\partial y}\{(x+y)e^{xy}\}\frac{d}{dt}\left(\frac{1}{t^2}\right)$$

$$= \{(x+y)\cdot ye^{xy} + e^{xy}\}\cdot 1 + \{(x+y)\cdot xe^{xy} + e^{xy}\}\left(-\frac{2}{t^3}\right)$$

$$= e^{xy}\left[\{y(x+y)+1\} + \{x(x+y)+1\}\left(-\frac{2}{t^3}\right)\right]$$

$$= e^{t\cdot\frac{1}{t^2}}\left[\frac{1}{t^2}\left(t+\frac{1}{t^2}\right) + 1 + \left\{t\left(t+\frac{1}{t^2}\right)+1\right\}\left(-\frac{2}{t^3}\right)\right]$$

$$= e^{\frac{1}{t}}\left(\frac{1}{t}+\frac{1}{t^4}+1-\frac{2}{t}-\frac{2}{t^4}-\frac{2}{t^3}\right)$$

$$= e^{\frac{1}{t}}\left(1-\frac{1}{t}-\frac{1}{t^4}-\frac{2}{t^3}\right) = e^{\frac{1}{t}}\left(\frac{t^4-t^3-1-2t}{t^4}\right)$$

$$= e^{\frac{1}{t}}\left(\frac{t^4-t^3-2t-1}{t^4}\right).$$

5. Find $\frac{du}{dt}$ if

a. $u = x^2+y^2, x = \cos t + \sin t, y = \cos t - \sin t$ at $t = 0$.

b. $u = e^{xyz}, x = t^3, y = \frac{1}{t}, z = e^t$.

Solution

a. Given,

$u = x^2+y^2$

$x = \cos t + \sin t$

$y = \cos t - \sin t$

At $t = 0$,

$x = \cos 0 + \sin 0 = 1$

$y = \cos 0 - \sin 0 = 1$

We have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(x^2+y^2)\cdot\frac{d}{dt}(\cos t + \sin t) + \frac{\partial}{\partial y}(x^2+y^2)\frac{d}{dt}(\cos t - \sin t)$$

$$= 2x\cdot(-\sin t + \cos t) + 2y\cdot(-\sin t - \cos t)$$

$$\left.\frac{du}{dt}\right|_{t=0} = 2\cdot 1\cdot(-\sin 0 + \cos 0) + 2\cdot 1\cdot(-\sin 0 - \cos 0)$$

$$= 2(0+1) + 2(-0-1)$$

$$= 2 - 2$$

$$= 0$$

b. Given,

$u = e^{xyz}$

$x = t^3$

$y = \frac{1}{t}$

$z = e^t$

We have,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{\partial}{\partial x}(e^{xyz})\cdot\frac{d}{dt}(t^3) + \frac{\partial}{\partial y}(e^{xyz})\cdot\frac{d}{dt}\left(\frac{1}{t}\right) + \frac{\partial}{\partial z}(e^{xyz})\cdot\frac{d}{dt}(e^t)$$

$$= yz\cdot e^{xyz}\cdot 3t^2 + xz\cdot e^{xyz}\cdot\left(-\frac{1}{t^2}\right) + xy\cdot e^{xyz}\cdot e^t$$

$$= e^{xyz}\left[3t^2yz - \frac{xz}{t^2} + xye^t\right]$$

$$= e^{\left(t^3\cdot\frac{1}{t}\cdot e^t\right)}\left[3t^2\cdot\frac{1}{t}\cdot e^t - \frac{t^3\cdot e^t}{t^2} + t^3\cdot\frac{1}{t}\cdot e^t\right]$$

$$= e^{t^2e^t}(3te^t - te^t + t^2e^t)$$

$$= e^{t^2e^t}\cdot e^t(3t - t + t^2)$$

$$= e^{t^2e^t+t}(2t+t^2)$$

$$= te^{t(te^t+1)}(t+2)$$

6. a. Find $\frac{dz}{dt}$ if $z = f(x, y) = xy$, $x = \cos t$, $y = \sin t$, at $t = 0$.

b. Find $\frac{dw}{dt}$ if $w = f(x, y, z) = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$, at $t = \frac{\pi}{2}$.

c. Find $\frac{dw}{dt}$ if $w = f(x, y, z) = \ln(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$, at $t = 1$.

Solution

a Given,

$z = xy$

$x = \cos t$

$y = \sin t$,

At $t = 0$

$x = \cos 0 = 1$

$y = \sin 0 = 0$

We have,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(xy)\cdot\frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(xy)\frac{d}{dt}(\sin t)$$

$$= y(-\sin t) + x\cdot\cos t.$$

$$\left.\frac{du}{dt}\right|_{t=0} = 0(-\sin 0) + 1\cdot\cos 0 = 1$$

b. Given,

$w = xy + z,$

$x = \cos t,$

$y = \sin t,$

$z = t,$

At $t = \frac{\pi}{2}$

$x = \cos\frac{\pi}{2} = 0$

$y = \sin\frac{\pi}{2} = 1$

$z = \frac{\pi}{2},$

We have,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{\partial}{\partial x}(xy+z)\cdot\frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(xy+z)\cdot\frac{d}{dt}(\sin t) + \frac{\partial}{\partial z}(xy+z)\cdot\frac{d}{dt}(t)$$

$$= y\cdot(-\sin t) + x\cdot\cos t + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = 1\cdot\left(-\sin\frac{\pi}{2}\right) + 0\cdot\cos 0 + 1$$

$$= -1 + 1$$

$$= 0$$

c. Given,

$w = \ln(x^2 + y^2 + z^2)$

$x = \cos t$

$y = \sin t$

$z = 4\sqrt{t}$

We have,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{2x}{x^2+y^2+z^2}(-\sin t) + \frac{2y}{x^2+y^2+z^2}\cdot\cos t + \frac{2z}{x^2+y^2+z^2}\cdot 4\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{t}}$$

$$= \frac{-2x\sin t + 2y\cot t + 4z\cdot\frac{1}{4\sqrt{t}}\cdot 4}{x^2+y^2+z^2}$$

$$= \frac{-2x\cdot y + 2y\cdot x + 4z\cdot\frac{1}{z}\cdot 4}{x^2+y^2+z^2}$$

$$= \frac{16}{x^2+y^2+z^2} = \frac{16}{\cos^2 t + \sin^2 t + (4\sqrt{t})^2}$$

$$= \frac{16}{1+16t}$$

When $t = 1$,

$$\frac{dw}{dt} = \frac{16}{1+16} = \frac{16}{17}$$

7. Using chain rule, find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s.

a. $w = x^2 + y^2, x = 3r + 2s, y = 2r + 3s$

b. $w = xy + \ln z, x = \frac{s^2}{r}, y = r + s, z = \cos r$.

Solution

a. Given,

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}\\ &= 2x\cdot 3 + 2y\cdot 2\\ &= 6x + 4y\\ &= 6(3r+2s) + 4(2r+3s)\\ &= 18r + 12s + 8r + 12s\\ &= 26r + 24s\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}\\ &= 2x\cdot 2 + 2y\cdot 3\\ &= 4x + 6y\\ &= 4(3r+2s) + 6(2r+3s)\\ &= 12r + 8s + 12r + 18s\\ &= 24r + 26s\end{aligned}$$

b. Given,

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}\\ &= y\cdot\left(-\frac{s^2}{r^2}\right) + x\cdot 1 + \frac{1}{z}\cdot(-\sin r)\\ &= (r+s)\left(-\frac{s^2}{r^2}\right) + \frac{s^2}{r} + \left(\frac{-\sin r}{\cos r}\right)\\ &= \frac{-rs^2 - s^3 + rs^2}{r^2} - \tan r\\ &= -\frac{s^3}{r^2} - \tan r\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}\\ &= y\cdot\frac{2s}{r} + x\cdot 1 + \frac{1}{z}\cdot(0)\\ &= x + y\cdot\frac{2s}{r} = \frac{s^2}{r} + (r+s)\cdot\frac{2s}{r}\\ &= \frac{s^2 + 2rs + 2s^2}{r}\\ &= \frac{3s^2 + 2rs}{r} = \frac{s(3s+2r)}{r}\end{aligned}$$

8. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ when $w = x^3 + y^3, x = r + s$ and $y = r - s$ at $r = 1$ and $s = 2$.

Solution

Given,

$w = x^3 + y^3$

$x = r + s$

$y = r - s$

When $r = 1$ and $s = 2$

$x = 1 + 2 = 3$

$y = 1 - 2 = -1$

We have,

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}\\ &= 3x^2\cdot 1 + 3y^2\cdot 1\\ &= 3x^2 + 3y^2\\ &= 3\times 3^2 + 3\times(-1)^2\\ &= 27 + 3\\ &= 30\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}\\ &= 3x^2\cdot 1 + 3y^2\cdot(-1)\\ &= 3x^2 - 3y^2\\ &= 3\times 3^2 - 3\times(-1)^2\\ &= 27 - 3\\ &= 24\end{aligned}$$

9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $y^3 + z^3 + yz - xy = 2$ at $(x, y, z) = (1, 1, 1)$

Solution

Given,

$y^3 + z^3 + yz - xy = 2$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$$3z^2 \frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} - y = 0$$

or, $$\frac{\partial z}{\partial x} = \frac{y}{y + 3z^2}$$

And,

$$3y^2 + 3z^2 \frac{\partial z}{\partial y} + z + y\frac{\partial z}{\partial y} - x = 0$$

or, $$\frac{\partial z}{\partial y} = \frac{x - 3y^2 - z}{3z^2 + y}$$

At $(x, y, z) = (1, 1, 1)$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + 3 \times 1^2} = \frac{1}{4}$$

and

$$\frac{\partial z}{\partial y} = \frac{1 - 3 - 1}{3 + 1} = -\frac{3}{4}$$

10. Using partial derivatives, find $\frac{dy}{dx}$

a. $x^2 + y^2 = 25$

b. $ax^2 + 2hyx + by^2 = 0$

c. $x + y = \sin(x + y)$

Solution

a. Let $f(x, y) = x^2 + y^2 - 25$

$f_x = 2x$

$f_y = 2y$

We have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x}{2y} = -\frac{x}{y}$$

b. Let $f(x, y) = ax^2 + 2hyx + by^2$

$f_x = 2ax + 2hy$

$f_y = 2by + 2hx$

We have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(ax + hy)}{hx + by}$$

c. Let $f(x, y) = x + y - \sin(x + y)$

$f_x = 1 - \cos(x + y)$

$f_y = 1 - \cos(x + y)$

We have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\{1 - \cos(x + y)\}}{\{1 - \cos(x + y)\}} = -1$$

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UNIT 3

Applications of Anti-derivatives

Exercise 3.1

1. Evaluate the following integrals.

a. $\int \frac{1}{x^2+9}\,dx$

b. $\int \frac{1}{x^2-4}\,dx$

c. $\int \frac{1}{e^x+e^{-x}}\,dx$

d. $\int \frac{x}{x^4+3}\,dx$

Solution

a. $\int \frac{1}{x^2+9}\,dx = \int \frac{1}{x^2+3^2}\,dx$

$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

b. $\int \frac{1}{x^2-4}\,dx = \int \frac{1}{x^2-2^2}\,dx$

$= \frac{1}{2\cdot 2}\ln\left|\frac{x-2}{x+2}\right| + c$

$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + c$

c. $\int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{e^x}{(e^x)^2+1}\,dx$

Put $e^x = t$

Then $e^x dx = dt$

$\therefore \int \frac{e^x}{(e^x)^2+1}\,dx = \int \frac{dt}{t^2+1}$

$= \frac{1}{1}\tan^{-1}(t) + c$

$= \tan^{-1}(e^x) + c$

d. $\int \frac{x}{x^4+3}\,dx = \int \frac{x}{(x^2)^2+3}\,dx$

Put $x^2 = t$

then $2x\,dx = dt$

or, $x\,dx = \frac{1}{2}dt$

$$\therefore \int \frac{x}{(x^2)^2+3}dx = \int \frac{\frac{1}{2}dt}{t^2+(\sqrt{3})^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c$$

$$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2}{\sqrt{3}}\right)+c$$

2. Evaluate the following integrals.

a. $\int \frac{1}{x^2+10x-11}dx$ b. $\int \frac{1}{1+x+x^2}dx$

c. $\int \frac{1}{4x^2+24x+45}dx$ d. $\int \frac{1}{1+x-x^2}dx$

e. $\int \frac{\cos x}{\sin^2 x+4\sin x+5}dx$ f. $\int \frac{2x}{x^2+4x-12}dx$

g. $\int \frac{2x+3}{x^2+4x+5}dx$ h. $\int \frac{x^2}{x^2-4}dx$

Solution

a. $$\int \frac{1}{x^2+10x-11}dx = \int \frac{1}{(x+5)^2-5^2-11}dx$$

$$= \int \frac{1}{(x+5)^2-36}dx$$

$$= \int \frac{1}{(x+5)^2-6^2}dx$$

$$= \frac{1}{2\times 6}\ln\left|\frac{x+5-6}{x+5+6}\right|+c$$

$$= \frac{1}{12}\ln\left|\frac{x-1}{x+11}\right|+c$$

b. $$\int \frac{1}{1+x+x^2}dx = \int \frac{1}{x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}dx$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1}dx$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

c. $\int \frac{1}{4x^2+24x+45} dx = \int \frac{1}{(2x)^2 + 2\cdot 2x\cdot 6 + 6^2 - 6^2 + 45} dx$

$$= \int \frac{1}{(2x+6)^2+9} dx$$

Put $2x + 6 = t$

then $2dx = dt$

or, $dx = \frac{dt}{2}$

$\int \frac{1}{(2x+6)^2+9} dx = \int \frac{\frac{1}{2}dt}{t^2+3^2}$

$$= \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x+6}{3}\right) + c$$

d. $\int \frac{1}{1+x-x^2} dx = \int \frac{1}{1-(x^2-x)} dx$

$$= \int \frac{1}{1-\left\{(x)^2 - 2\cdot x\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}} dx$$

$$= \int \frac{1}{1-\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}} dx$$

$$= \int \frac{1}{\frac{5}{4} - \left(x-\frac{1}{2}\right)^2} dx$$

$$= \int \frac{1}{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2} dx$$

$$= \frac{1}{2\cdot\frac{\sqrt{5}}{2}} \ln\left|\frac{\frac{\sqrt{5}}{2} + x - \frac{1}{2}}{\frac{\sqrt{5}}{2} - x + \frac{1}{2}}\right| + c$$

$$= \frac{1}{\sqrt{5}} \ln\left|\frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}\right| + c$$

e. Let

$$I = \int \frac{\cos x}{\sin^2 x + 4\sin x + 5}\,dx$$

$$= \int \frac{\cos x}{(\sin x + 2)^2 + 1}\,dx$$

Put $y = \sin x + 2$, then
$dy = \cos x\,dx$
Now,

$$I = \int \frac{\cos x}{\sin^2 x + 4\sin x + 5}\,dx$$

$$= \int \frac{1}{y^2 + 1}\,dz$$

$= \tan^{-1} y + c$
$= \tan^{-1}(\sin x + 2) + c.$

f. $\int \frac{x}{x^2 + 4x - 12}\,dx$

$$= \int \frac{(2x + 4 - 4)}{x^2 + 4x - 12}\,dx$$

$$= \int \frac{(2x + 4)}{x^2 + 4x - 12}\,dx - 4\int \frac{1}{x^2 + 4x - 12}\,dx$$

$$= \ln|x^2 + 4x - 12| - 4\int \frac{1}{(x+2)^2 - 16}\,dx$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c\right]$$

$$= \ln|x^2 + 4x - 12| - 4\int \frac{1}{(x+2)^2 - 4^2}\,dx$$

$$= \ln|x^2 + 4x - 12| - 4\cdot\frac{1}{2\cdot 4}\ln\left|\frac{x+2-4}{x+2+4}\right| + c$$

$$= \ln|x^2 + 4x - 12| - \frac{1}{2}\ln\left|\frac{x-2}{x+6}\right| + c$$

b. $\int \frac{2x + 3}{x^2 + 4x + 5}\,dx$

$$= \int \frac{2x + 4 - 1}{x^2 + 4x + 5}\,dx$$

$$= \int \frac{2x + 4}{x^2 + 4x + 5}\,dx - \int \frac{1}{x^2 + 4x + 5}\,dx$$

$$= \ln|x^2 + 4x + 5| - \int \frac{1}{(x+2)^2 + 1^2}\,dx$$

$$= \ln|x^2 + 4x + 5| - \tan^{-1}(x + 2) + c$$

h. $\int \frac{x^2}{x^2 - 4}\,dx$

$$= \int \frac{x^2 - 4 + 4}{x^2 - 4}\,dx$$

$$= \int \frac{x^2 - 4}{x^2 - 4}\,dx + 4\int \frac{1}{x^2 - 4}\,dx$$

$$= \int dx + 4\int \frac{1}{x^2 - 2^2}\,dx$$

$$= x + \frac{4}{2\cdot 2}\ln\left|\frac{x-2}{x+2}\right| + c$$

$$= x + \ln\left|\frac{x-2}{x+2}\right| + c$$

3. Evaluate the following integrals.

a. $\int \frac{1}{\sqrt{x^2+x-2}}\,dx$ b. $\int \frac{1}{\sqrt{2ax+x^2}}\,dx$

c. $\int \frac{1}{\sqrt{3+3x+x^2}}\,dx$ d. $\int \frac{1}{\sqrt{2ax-x^2}}\,dx$

e. $\int \frac{1}{\sqrt{1-x-x^2}}\,dx$ f. $\int \frac{x+2}{\sqrt{x^2+2x-1}}\,dx$

g. $\int \frac{2x+3}{\sqrt{x^2+x+1}}\,dx$ h. $\int \frac{5-2x}{\sqrt{6+x-x^2}}\,dx$

Solution

a. $\int \frac{1}{\sqrt{x^2+x-2}}\,dx$

$$= \int \frac{1}{\sqrt{x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-2}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{9}{4}}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}}\,dx$$

$$\left[\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)+c\right]$$

$$= \ln\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}\right| + c$$

$$= \ln\left|x+\frac{1}{2}+\sqrt{x^2+x+\frac{1}{4}-\frac{9}{4}}\right| + c$$

$$= \ln\left|x+\frac{1}{2}+\sqrt{x^2+x-2}\right| + c$$

b. $$\int \frac{1}{\sqrt{2ax+x^2}}\,dx = \int \frac{1}{\sqrt{x^2+2ax+a^2-a^2}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+a)^2-a^2}}\,dx \quad \left[\because \int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= \ln\left|x+a+\sqrt{(x+a)^2-a^2}\right| + c$$

$$= \ln\left|x+a+\sqrt{2ax+x^2}\right| + c$$

c. $$\int \frac{1}{\sqrt{3+3x+x^2}}\,dx = \int \frac{1}{\sqrt{x^2+2\cdot x\cdot\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+3}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{9}{4}+3}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{3}{2}\right)^2+\frac{3}{4}}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$

$$= \ln\left|x+\frac{3}{2}+\sqrt{\left(x+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right| + c$$

$$= \ln\left|x+\frac{3}{2}+\sqrt{x^2+3x+3}\right| + c$$

$$= \ln\left|2x+3+2\sqrt{3+3x+x^2}\right| + c$$

d. $$\int \frac{1}{\sqrt{2ax-x^2}}\,dx = \int \frac{1}{\sqrt{-(x^2-2ax)}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2-2ax+a^2-a^2)}}\,dx$$

$$= \int \frac{1}{\sqrt{a^2-(x-a)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x-a}{a}\right) + c \quad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c\right]$$

e. $$\int \frac{1}{\sqrt{1-x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x^2+x)}}\,dx$$

$$= \int \frac{1}{\sqrt{1-\left(x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{1-\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right) + c = \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c$$

f. $\int \frac{x+2}{\sqrt{x^2+2x-1}}\,dx \quad = \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x-1}}\,dx$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x-1}}\,dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x-1}}\,dx$$

$$\left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c\right]$$

$$= \frac{1}{2} \times 2\sqrt{x^2+2x-1} + \int \frac{1}{\sqrt{(x+1)^2-2}}\,dx$$

$$= \sqrt{x^2+2x-1} + \int \frac{1}{\sqrt{(x+1)^2-(\sqrt{2})^2}}\,dx$$

$$= \sqrt{x^2+2x-1} + \left|(x+1) - \sqrt{(x+1)^2-2}\right| + c$$

$$= \sqrt{x^2+2x-1} + \ln\left|x+1+\sqrt{x^2+2x-1}\right| + c$$

g. $\int \frac{2x+3}{\sqrt{x^2+x+1}}\,dx \quad = \int \frac{2x+1+2}{\sqrt{x^2+x+1}}\,dx$

$$= \int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx + 2\int \frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{1}{\sqrt{x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}}\,dx$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1}}\,dx$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$

$$= 2\sqrt{x^2+x+1} + 2\ln\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right| + c$$

$$= 2\sqrt{x^2+x+1} + 2\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + c$$

h. Let

$$\int \frac{5-2x}{\sqrt{6+x-x^2}}\,dx = p\int \frac{1-2x}{\sqrt{6+x-x^2}}\,dx + q\int \frac{1}{\sqrt{6+x-x^2}}\,dx \quad \cdots(i)$$

Equating like term

$-2p = -2$ and $p+q=5$

$\therefore$ $p=1$ or, $1+q=5$

$\therefore q=4$

Putting the value of p and q in (i) we get

$$\int \frac{5-2x}{\sqrt{6+x-x^2}}\,dx = \int \frac{1-2x}{\sqrt{6+x-x^2}}\,dx + 4\int \frac{1}{\sqrt{6+x-x^2}}\,dx$$

$$= 2\sqrt{6+x-x^2} + 4\int \frac{1}{\sqrt{6+(x^2-x)}}\,dx$$

$$= 2\sqrt{6+x-x^2} + 4\int \frac{1}{\sqrt{6+\left\{x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right\}}}\,dx$$

$$= 2\sqrt{6+x-x^2} + 4\int \frac{1}{\sqrt{6-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}}\,dx$$

$$= 2\sqrt{6+x-x^2} + 4\int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}\,dx$$

$$= 2\sqrt{6+x-x^2} + 4\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{5}{2}}\right) + c$$

$$= 2\sqrt{6+x-x^2} + 4\sin^{-1}\left(\frac{2x-1}{5}\right) + c$$

4. Evaluate the following integrals.

a. $\displaystyle\int \frac{1}{(2x+1)\sqrt{4x+3}}\,dx$

b. $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$

c. $\displaystyle\int \frac{1}{\sqrt{1+e^{-2x}}}\,dx$

Solution

a. Put $\sqrt{4x+3} = z$. Then

$4x+3 = z^2$

and $4dx = 2z\,dz$

i.e., $dx = \frac{1}{2}z\,dz$

$$\therefore \int \frac{1}{(2x+1)\sqrt{4x+3}}\,dx = \int \frac{\frac{1}{2}z}{\left[2\left(\frac{z^2-3}{4}\right)+1\right]\sqrt{z}}\,dz$$

$$= \frac{2}{1}\times\frac{1}{2}\int \frac{1}{z^2-1}\,dz$$

$$= \int \frac{1}{z^2-1}\,dz$$

$$= \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right| + c.$$

b. $$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{1+x}{1-x}\times\frac{1+x}{1+x}}\,dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}} + c$$

$$= \sin^{-1}x - \frac{1}{2}\cdot 2\sqrt{1-x^2} + c$$

$$= \sin^{-1}x - \sqrt{1-x^2} + c$$

c. Let

$$I = \int \frac{dx}{\sqrt{1+e^{-2x}}} = \int \frac{dx}{\sqrt{1+\frac{1}{e^{2x}}}}$$

$$= \int \frac{dx}{\sqrt{\frac{e^{2x}+1}{e^{2x}}}} = \int \frac{dx}{\frac{\sqrt{e^{2x}+1}}{e^x}}$$

$$= \int \frac{e^x dx}{\sqrt{(e^x)^2+1}}$$

Let $y = e^x$

$dy = e^x\,dx.$

Then,

$$I = \int \frac{dy}{\sqrt{y^2+1}}$$

$$= \log\left(y+\sqrt{y^2+1}\right) + C$$

$$= \log\left(e^x+\sqrt{(e^x)^2+1}\right) + C$$

$$= \log\left(e^x+\sqrt{e^{2x}+1}\right) + C$$

Exercise 3.2

1. Evaluate the following integrals.

a. $\int\sqrt{x^2-9}\,dx$

b. $\int\sqrt{x^2-2x+5}\,dx$

c. $\int\sqrt{4+8x-5x^2}\,dx$

d. $\int\sqrt{2ax-x^2}\,dx$

e. $\int\sqrt{4x^2-4x+5}\,dx$

f. $\int\sqrt{(x-\alpha)(\beta-x)}\,dx$

Solution

a. $\int\sqrt{x^2-9}\,dx = \int\sqrt{x^2-3^2}\,dx$

$$= \frac{x}{2}\sqrt{x^2-3^2} - \frac{3^2}{2}\ln|x+\sqrt{x^2-9}| + c$$

$$\left[\because \int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= \frac{x}{2}\sqrt{x^2-9} - \frac{9}{2}\ln|x+\sqrt{x^2-9}| + c$$

b. $\int\sqrt{x^2-2x+5}\,dx = \int\sqrt{x^2-2x+4+1}\,dx$

$$= \int\sqrt{(x-1)^2+4}\,dx$$

$$= \sqrt{(x-1)^2+2^2}\,dx$$

$$= \frac{x-1}{2}\sqrt{(x-1)^2+2^2} + \frac{2^2}{2}\ln|x-1+\sqrt{5-2x+x^2}| + c$$

$$\left[\because \int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}| + c\right]$$

$$= \frac{x-1}{2}\sqrt{(x-1)^2+2^2} + 2\ln|x-1+\sqrt{5-2x+x^2}| + c$$

$$= \frac{1}{2}(x-1)\sqrt{5-2x+x^2} + 2\ln|x-1+\sqrt{5-2x+x^2}| + c$$

c. $\int\sqrt{4+8x-5x^2}\,dx = \int\sqrt{5\left(\frac{4}{5}+\frac{8}{5}x-x^2\right)}\,dx$

$$= \sqrt{5}\int\sqrt{\left(\frac{6}{5}\right)^2-\left(x-\frac{4}{5}\right)^2}\,dx$$

$$= \sqrt{5}\left(\frac{x-\frac{4}{5}}{2}\right)\sqrt{\left(\frac{6}{5}\right)^2-\left(x-\frac{4}{5}\right)^2} + \frac{\left(\frac{6}{5}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{4}{5}}{\frac{6}{5}}\right) + c$$

$$= \frac{1}{10}(5x-4)\sqrt{4+8x-5x^2} + \frac{18}{5\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{6}\right) + c.$$

d. Let

$$I = \int\sqrt{2ax-x^2}\,dx$$

$$= \int\sqrt{a^2-a^2+2ax-x^2}\,dx$$

$$= \int\sqrt{a^2-(x-a)^2}\,dx$$

Put $x-a=t$

Then, $dx = dt$

$$I = \int \sqrt{a^2 - t^2}\, dt$$

$$= \frac{t}{2}\sqrt{a^2 - t^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right) + c$$

$$= \left(\frac{x-a}{2}\right)\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$$

$$= \frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x-a}{a}\right) + c$$

e. Let $I = \int \sqrt{4x^2 - 4x + 5}\, dx$

$$= \int \sqrt{(2x)^2 - 2\cdot 2x + 1 + 4}\, dx$$

$$= \int \sqrt{(2x-1)^2 + 2^2}\, dx$$

Put $2x - 1 = t$

Then, $2dx = dt$

or, $dx = \frac{1}{2}dt$

$$\therefore\ I = \int \sqrt{t^2 + 2^2}\,\frac{1}{2}\, dt$$

$$= \frac{1}{2}\left\{\frac{t}{2}\sqrt{t^2 + 2^2} + \frac{2^2}{2}\ln\left|t + \sqrt{t^2 + 2^2}\right|\right\} + c$$

$$= \frac{t}{4}\sqrt{t^2 + 4} + \ln\left|t + \sqrt{t^2 + 2^2}\right| + c$$

$$= \left(\frac{2x-1}{4}\right)\sqrt{(2x-1)^2 + 4} + \ln\left|2x - 1 + \sqrt{(2x-1)^2 + 4}\right| + c$$

$$= \frac{1}{4}(2x-1)\sqrt{4x^2 - 4x + 5} + \ln\left(2x - 1 + \sqrt{4x^2 - 4x + 5}\right) + c$$

f. Let $I = \int \sqrt{(x-\alpha)(\beta - x)}$

Put $x - \alpha = t \Rightarrow x = t + \alpha$

$\therefore$ $dx = dt$

Now

$$I = \int \sqrt{t(\beta - (t + \alpha))}\, dt = \int \sqrt{(\beta - \alpha)t - t^2\alpha)}\, dt$$

$$= \int \sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - \left[\left(\frac{\beta-\alpha}{2}\right)^2 - 2\cdot\frac{\beta-\alpha}{2}\cdot t + t^2\right]}\, dt$$

$$= \int \sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - \left(t - \frac{\beta-\alpha}{2}\right)^2}\, dt$$

Again, put $t - \frac{\beta-\alpha}{2} = y$

Then $dt = dy$

$$\therefore\ I = \int \sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - y^2}\, dy$$

$$= \frac{y}{2}\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - y^2} + \frac{\left(\frac{\beta-\alpha}{2}\right)^2}{2}\sin^{-1}\left(\frac{y}{\left(\frac{\beta-\alpha}{2}\right)}\right) + c$$

$$= \frac{t-\left(\frac{\beta-\alpha}{2}\right)}{2}\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2-\left(t-\frac{\beta-\alpha}{2}\right)^2}+\frac{\left(\frac{\beta-\alpha}{2}\right)^2}{2}\sin^{-1}\left(\frac{\left(t-\frac{\beta-\alpha}{2}\right)}{\left(\frac{\beta-\alpha}{2}\right)}\right)+c$$

$$= \frac{2x-2\alpha-\beta+\alpha}{4}\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2-\left(x-\alpha-\frac{\beta-\alpha}{2}\right)^2}+\frac{\left(\frac{\beta-\alpha}{2}\right)^2}{2}\sin^{-1}\left(\frac{2x-2\alpha-\beta+\alpha}{\beta-\alpha}\right)+c$$

$$= \frac{2x-\alpha-\beta}{4}\sqrt{(x-\alpha)(x-\beta)}+\frac{(\beta-\alpha)^2}{8}\sin^{-1}\left(\frac{2x-\alpha-\beta}{\beta-\alpha}\right)+c$$

2. Evaluate the following integrals.

a. $\int (x-3)\sqrt{x^2-1}\,dx$ b. $\int (x+2)\sqrt{x^2-10x-11}\,dx$

c. $\int (x-1)\sqrt{x^2-x+1}\,dx$ d. $\int (3x-2)\sqrt{x^2-x+1}\,dx$

e. $\int (2x+1)\sqrt{4x^2+20x+21}\,dx$ f. $\frac{1}{x-\sqrt{x^2-1}}\,dx$

Solution

a. $\int (x-3)\sqrt{x^2-1}\,dx$ $= \int x\sqrt{x^2-1}\,dx - 3\int\sqrt{x^2-1}\,dx$

$$= \frac{1}{2}\int 2x\sqrt{x^2-1}\,dx - 3\int\sqrt{x^2-1^2}\,dx$$

$$\left[\because \int f'(x)\sqrt{f(x)}\,dx = \frac{2}{3}[f(x)]^{3/2}+c\right]$$

$$= \frac{1}{2}\times\frac{2}{3}(x^2-1)^{3/2} - 3\left\{\frac{x}{2}\sqrt{x^2-1}-\frac{1^2}{2}\ln\left|x+\sqrt{x^2-1}\right|\right\}+c$$

$$= \frac{1}{3}(x^2-1)^{\frac{3}{2}} - \frac{3x}{2}\sqrt{x^2-1}+\frac{3}{2}\ln|x+\sqrt{x^2-1}|$$

b. $\int (x+2)\sqrt{x^2-10x-11}\,dx$

Let $I = \int (x+2)\sqrt{x^2-10x-11}\,dx$

$= p\int (2x-10)\sqrt{x^2-10x-11}\,dx + q\int\sqrt{x^2-10x-11}\,dx$

Equating like term, we get

$2p = 1$ and $2 = -10p+q$

$\therefore$ $p = \frac{1}{2}$ or, $2 = -10\times\frac{1}{2}+q$

or, $2 = -5+q$

$\therefore q = 7$

Thus,

$$I = \frac{1}{2}\int (2x-10)\sqrt{x^2-10x-11}\,dx + 7\int\sqrt{x^2-10x-11}\,dx$$

$$= \frac{1}{2}\times\frac{2}{3}(x^2-10x-11)^{3/2}+7\int\sqrt{x^2-2\cdot x\cdot 5+5^2-5^2-11}\,dx$$

$$= \frac{1}{3}(x^2-10x-11)^{3/2}+7\int\sqrt{(x-5)^2-6^2}\,dx$$

$$= \frac{1}{3}(x^2-10x-11)^{3/2}+7\int\sqrt{t^2-6^2}\,dt \quad \text{[Putting } x-5=t \text{ then } dx=dt]$$

$$= \frac{1}{3}(x^2-10x-11)^{3/2}+7\left\{\frac{t}{2}\sqrt{t^2-6^2}-\frac{6^2}{2}\ln\left|t+\sqrt{t^2-6^2}\right|\right\}+c$$

$=\frac{1}{3}(x^2-10x-11)^{3/2}+\frac{7}{2}(x-5)\sqrt{(x-5)^2-6^2}-\frac{7\times6\times6}{2}\ln|x-5+\sqrt{(x-5)^2-6^2}|+c$

$=\frac{1}{3}(x^2-10x-11)^{3/2}+\frac{7}{5}(x-5)\sqrt{x^2-10x-11}-126\ln\left|x-5+\sqrt{x^2-10x-11}\right|+c$

Let $I=\int(x-1)\sqrt{x^2-x+1}\,dx$

and $\int(x-1)\sqrt{x^2-x+1}\,dx=P\int(2x-1)\sqrt{x^2-x+1}\,dx+q\int\sqrt{x^2-x+1}\,dx$

Equating like term, we get

$2p=1$ and $-p+q=-1$

$p=\frac{1}{2}$ or, $q=-1+\frac{1}{2}$

$\therefore q=-\frac{1}{2}$

$I=\frac{1}{2}\int(2x-1)\sqrt{x^2-x+1}\,dx-\frac{1}{2}\int\sqrt{x^2-x+1}\,dx$

$=\frac{1}{2}\times\frac{2}{3}(x^2-x+1)^{3/2}-\frac{1}{2}\int\sqrt{x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\,dx$

$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{2}\int\sqrt{\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+1}\,dx$

$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{2}\int\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$

$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{4}\int\sqrt{(2x-1)^2+(\sqrt{3})^2}\,dx$

$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{4}\int\sqrt{(t)^2+(\sqrt{3})^2}\,dt$ [Putting $2x-1=t$ then $dx=\frac{1}{2}dt$]

$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{8}\left\{\frac{t}{2}\sqrt{t^2+(\sqrt{3})^2}+\left(\frac{\sqrt{3}}{2}\right)^2\ln\left|t+\sqrt{t^2+(\sqrt{3})^2}\right|\right\}+c$

$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{16}(2x-1)\sqrt{(2x-1)^2+3}-\frac{3}{16}\ln\left|2x-1+\sqrt{(2x-1)^2+3}\right|+c$

$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{16}(2x-1)\sqrt{4x^2-4x+4}-\frac{3}{16}\ln\left|2x-1+\sqrt{4x^2-4x+4}\right|+c$

$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{8}(2x-1)\sqrt{x^2-x+1}-\frac{3}{16}\ln\left|2x-1+2\sqrt{x^2-x+1}\right|+c$

d. Let

$I=\int(3x-2)\sqrt{x^2-x+1}\,dx$

$=m\int(2x-1)\sqrt{x^2-x+1}\,dx+n\int\sqrt{x^2-x+1}\,dx$

Equating like terms, we get

$2m=3$ and $-m+n=-2$

or, $m=\frac{3}{2}$ and $n=-\frac{1}{2}$

$\therefore I=\frac{3}{2}\int(2x-1)\sqrt{x^2-x+1}\,dx-\frac{1}{2}\int\sqrt{x^2-x+1}\,dx$

$=\frac{3}{2}\int(2x-1)\sqrt{x^2-x+1}\,dx-\frac{1}{2}\int\sqrt{x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\,dx$

$$=\frac{3}{2}\times\frac{2}{3}(x^2-x+1)^{\frac{3}{2}}-\frac{1}{2}\int\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$=(x^2-x+1)^{\frac{3}{2}}-\frac{1}{2}\int\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$=(x^2-x+1)^{\frac{3}{2}}-\frac{1}{2}$$

$$\left\{\frac{\left(x-\frac{1}{2}\right)}{2}\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\ln\left|x-\frac{1}{2}+\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right|\right\}+c$$

$$=(x^2-x+1)^{\frac{3}{2}}-\frac{1}{8}(2x-1)\sqrt{x^2-x+1}-\frac{3}{16}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x+1}\right|+c.$$

e. Let $I=\int(2x+1)\sqrt{4x^2+20x+21}\,dx$

and $\int(2x+1)\sqrt{4x^2+20x+21}\,dx$

$$=p\int(8x+20)\sqrt{4x^2+20x+21}\,dx+q\int\sqrt{4x^2+20x+21}\,dx$$

Equating like term, we get

$8p=2$ and $20p+q=1$

$\therefore\ p=\frac{1}{4}$ or, $20\frac{1}{4}+q=1$

or, $q=1-5=-4$

$$\therefore\ I=\frac{1}{4}\int(8x+20)\sqrt{4x^2+20x+21}\,dx-4\int\sqrt{4x^2+20x+21}\,dx$$

$$=\frac{1}{4}\times\frac{2}{3}(4x^2+20x+21)^{3/2}-4\int\sqrt{(2x+5)^2-4}\,dx$$

Put $2x+5=t$

then $dx=\frac{1}{2}dt$

$$\therefore\ \int\sqrt{(2x+5)^2-4}\,dx=\int\sqrt{t^2-4}\times\frac{1}{2}\,dt$$

$$=\frac{1}{2}\left\{\frac{t}{2}\sqrt{t^2-4}-\frac{4}{2}\ln|t+\sqrt{t^2-4}|\right\}$$

$$=\frac{1}{4}(2x+5)\sqrt{(2x+5)^2-4}-\ln|2x+5+\sqrt{(2x+5)^2-4}|+c$$

$$=\frac{1}{4}(2x+5)\sqrt{4x^2+20x+21}-\ln|2x+5+\sqrt{4x^2+20x+21}|+c$$

$$\therefore\ I=\frac{1}{6}(4x^2+20x+21)^{3/2}-4\left\{\frac{1}{4}(2x+5)\sqrt{4x^2+20x+21}-\ln|2x+5+\sqrt{4x^2+20x+21}|\right\}+c$$

$$=\frac{1}{6}(4x^2+20x+21)^{3/2}-(2x+5)\sqrt{4x^2+20x+21}+4\ln|(2x+5)+\sqrt{4x^2+20x+21}|+c$$

$$\text{Let } I=\int\frac{1}{x-\sqrt{x^2-1}}\,dx=\int\frac{x+\sqrt{x^2-1}}{x^2-(x^2-1)}\,dx$$

$$=\int[x+\sqrt{x^2-1}]\,dx$$

$$=\int x\,dx+\int\sqrt{x^2-1}\,dx$$

$$=\frac{x^2}{2}+\left[\frac{x\sqrt{x^2-1}}{2}+\frac{1}{2}\ln|x+\sqrt{x^2-1}|\right]+c$$

$$=\frac{x^2}{2}+\frac{1}{2}\left(x\sqrt{x^2-1}+\ln|x+\sqrt{x^2-1}|\right)+c.$$

Exercise 3.3

Evaluate the following integrals.

1. $\int \frac{1}{2+\sin x}\, dx$

Solution

Let $I = \int \frac{1}{2+\sin x}\, dx$

$$= \int \frac{1}{2\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+2\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\left(2\sin^2\frac{x}{2}+2\cos^2\frac{x}{2}+2\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2}\right)\sec^2\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{2\tan^2\frac{x}{2}+2+2\tan\frac{x}{2}}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+\tan\frac{x}{2}+1}\, dx$$

Put $\tan\frac{x}{2} = t$

Then, $\sec^2\frac{x}{2}\cdot\frac{1}{2}\, dx = dt$

or, $\sec^2\frac{x}{2}\, dx = 2\, dt$

$\therefore$ $I = \frac{1}{2}\int \frac{2dt}{t^2+t+1} = \int \frac{2dt}{t^2+2\cdot t\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right)+c$$

2. $\int \frac{1}{1+2\sin x}\,dx$

Solution

$$\text{Let } I = \int \frac{1}{1+2\sin x}\,dx$$

$$= \int \frac{1}{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+2\cdot 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+1+4\tan\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t$

Then, $\sec^2\frac{x}{2}\cdot\frac{1}{2}\,dx = dt$

or, $\sec^2\frac{x}{2}\,dx = 2\,dt$

$$\therefore \quad I = \int \frac{2}{t^2+1+4t}\,dt$$

$$= 2\int \frac{1}{t^2+4t+1}\,dt$$

$$= 2\int \frac{1}{t^2+2\cdot t\cdot 2+2^2-2^2+1}\,dt$$

$$= 2\int \frac{1}{(t+2)^2-3}\,dt$$

$$= 2\int \frac{1}{(t+2)^2-(\sqrt{3})^2}\,dt$$

$$= 2\cdot\frac{1}{2\cdot\sqrt{3}}\ln\left|\frac{t+2-\sqrt{3}}{t+2+\sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}\right| + c$$

3. $\int \frac{1}{5-4\sin x}\,dx$

Solution

$$\text{Let } I = \int \frac{1}{5-4\sin x}\,dx$$

$$= \int \frac{1}{5\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)-4\cdot 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2 \frac{x}{2}}{5 \tan^2 \frac{x}{2} - 8 \tan \frac{x}{2} + 5} dx$$

$$= \frac{1}{5} \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} - \frac{8}{5} \tan \frac{x}{2} + 1} dx$$

Put $\tan \frac{x}{2} = t$

Then, $\sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dt$

or, $\sec^2 \frac{x}{2} dx = 2\, dt$

$$\therefore \quad I = \frac{1}{5} \int \frac{2dt}{t^2 - \frac{8}{5}t + 1}$$

$$= \frac{2}{5} \int \frac{1}{t^2 + 2t \cdot \frac{4}{5} + \left(\frac{4}{5}\right)^2 - \left(\frac{4}{5}\right)^2 + 1} dt$$

$$= \frac{2}{5} \int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt$$

$$= \frac{2}{5} \times \frac{1}{\frac{3}{5}} \tan^{-1} \left(\frac{t - \frac{4}{5}}{\frac{3}{5}} \right) + c$$

$$= \frac{2}{3} \tan^{-1} \left[\frac{5 \tan \frac{x}{2} - 4}{3} \right] + c$$

4. $\int \frac{1}{2 + \cos x} dx$

Solution

Let $I = \int \frac{1}{2 + \cos x} dx$

$$= \int \frac{1}{2\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right) + \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} dx$$

$$= \int \frac{1 \times \sec^2 \frac{x}{2}}{\left(3 \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right) \sec^2 \frac{x}{2}} dx$$

$$= \int \frac{\sec^2 \frac{x}{2}}{3 + \tan^2 \frac{x}{2}} dx$$

Put $\tan\frac{x}{2} = t$

Then, $\sec^2\frac{x}{2} \times \frac{1}{2}\,dx = dt$

or, $\sec^2\frac{x}{2} = 2dt$

$$\therefore \quad I = \int \frac{2dt}{3+t^2} = 2\int \frac{2dt}{(\sqrt{3})^2 + t^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + c$$

5. $\int \frac{1}{4+5\cos x}\,dx$

Solution

Let $I = \int \frac{1}{4+5\cos x}\,dx$

$$= \int \frac{1}{4\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 5\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}\,dx$$

$$= \int \frac{1 \times \sec^2\frac{x}{2}}{\left(9\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)\cdot \sec^2\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{9 - \tan^2\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t$

Then, $\sec^2\frac{x}{2}\,dx = 2dt$

$$\therefore \quad I = \int \frac{2dt}{3^2 - t^2}$$

$$= 2\int \frac{dt}{3^2 - t^2}$$

$$= 2\cdot\frac{1}{2\times 3} \ln\left|\frac{3+t}{3-t}\right| + c$$

$$= \frac{1}{3} \ln\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right| + c$$

6. $\int \frac{1}{1-2\cos x}\,dx$

Solution

Let $I = \int \frac{1}{1-2\cos x}\,dx$

$$= \int \frac{1}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\left(3\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)\cdot\sec^2\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{3\tan^2\frac{x}{2}-1}\,dx$$

Put $\sqrt{3}\tan\frac{x}{2} = t$

Then, $\sqrt{3}\sec^2\frac{x}{2}\frac{1}{2}\,dx = dt$

or, $\sec^2\frac{x}{2}\,dx = \frac{2}{\sqrt{3}}\,dt$

$\therefore$ I $= \frac{2}{\sqrt{3}}\int \frac{1}{t^2-1}\,dt$

$$= \frac{2}{\sqrt{3}}\cdot\frac{1}{2\cdot 1}\ln\left|\frac{t-1}{t+1}\right| + c$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}\tan\frac{x}{2}-1}{\sqrt{3}\tan\frac{x}{2}+1}\right| + c$$

7. $\int \frac{1}{1+\sin x+\cos x}\,dx$

Solution

Let I $= \int \frac{1}{1+\sin x+\cos x}\,dx$

$$= \int \frac{1}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{1+\tan\frac{x}{2}}\,dx$$

Put $1+\tan\frac{x}{2} = t$

Then, $\sec^2 \frac{x}{2} \cdot \frac{1}{2}\, dx = dt$

or, $\sec^2 \frac{x}{2}\, dx = 2\, dt$

$\therefore \quad I = \frac{1}{2} \int \frac{2dt}{t}$

$= \int \frac{1}{t}\, dt$

$= \ln |t| + c$

$= \ln \left| \tan \frac{x}{2} + 1 \right| + c$

8. $\int \frac{1}{4 - 5\sin^2 x}\, dx$

Solution

Let $I = \int \frac{1}{4 - 5\sin^2 x}\, dx$

$= \int \frac{1}{4(\sin^2 x + \cos^2 x) - 5\sin^2 x}\, dx$

$= \int \frac{\sec^2 x}{(4\cos^2 x - \sin^2 x)\sec^2 x}\, dx$

$= \int \frac{\sec^2 x}{4 - \tan^2 x}\, dx$

Put $\tan x = t$

Then, $\sec^2 x \cdot dx = dt$

$\therefore \quad I = \int \frac{1}{4 - t^2}\, dt$

$= \int \frac{1}{2^2 - t^2}\, dt$

$= \frac{1}{2 \cdot 2} \ln \left| \frac{2 + t}{2 - t} \right| + c$

$= \frac{1}{4} \ln \left| \frac{2 + \tan x}{2 - \tan x} \right| + c$

9. $\int \frac{\sin 2x}{(\sin x + \cos x)^2}\, dx$

Solution

Let $I = \int \frac{\sin 2x}{(\sin x + \cos x)^2}\, dx$

$= \int \frac{\sin 2x + 1 - 1}{(\sin x + \cos x)^2}\, dx$

$$= \int \frac{(\sin x + \cos x)^2 - 1}{(\sin x + \cos x)^2}\, dx$$

$$= \int dx - \int \frac{1}{(\sin x + \cos x)^2}\, dx$$

$$= x - \int \frac{\sec^2 x}{(1 + \tan x)^2}\, dx$$

Put $1 + \tan x = t$

Then $\sec^2 x\, dx = dt$

$$\int \frac{\sec^2 x}{(1 + \tan x)^2}\, dx = \int \frac{dt}{t^2}$$

$$= -\frac{1}{t}$$

$$= -\frac{1}{1 + \tan x}$$

$$\therefore I = x + \frac{1}{1 + \tan x} + c$$

10. $\int \frac{1}{3\sin x + 4\cos x}\, dx$

Solution

Let $I = \int \frac{1}{3\sin x + 4\cos x}\, dx$

Put $3 = r \cdot \cos\theta,\ 4 = r \cdot \sin\theta$

Then $r = \sqrt{3^2 + 4^2} = 5$ and $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$$\therefore I = \int \frac{1}{r\cos\theta \cdot \sin x + r\sin\theta \cdot \cos x}\, dx$$

$$= \frac{1}{r} \int \frac{1}{\sin(x + \theta)}\, dx$$

$$= \frac{1}{5} \int \operatorname{cosec}(x + \theta)\, dx$$

$$= \frac{1}{5} \ln\left|\tan\frac{x + \theta}{2}\right| + c, \text{ where } \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

11. $\int \frac{dx}{3\sin x - 5\cos x}$

Solution

Let

$$I = \int \frac{dx}{3\sin x - 5\cos x}$$

$$= \int \frac{dx}{3 \cdot 2\sin\frac{x}{2}\cos\frac{x}{2} - 5\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{6 \sin \frac{x}{2} \cos \frac{x}{2} - 5 \cos^2 \frac{x}{2} + 5 \sin^2 \frac{x}{2}} \times \frac{\sec^2 \frac{x}{2}}{\sec^2 \frac{x}{2}}$$

$$= \int \frac{\sec^2 \frac{x}{2} dx}{5 \tan^2 \frac{x}{2} + 6 \tan \frac{x}{2} - 5}$$

$$= \frac{1}{5} \int \frac{\sec^2 \frac{x}{2} dx}{\tan^2 \frac{x}{2} + \frac{6}{5} \tan \frac{x}{2} - 1}$$

Put $\tan \frac{x}{2} = y$

$$\frac{1}{2} \sec^2 \frac{x}{2}\ dx = dy$$

$$\sec^2 \frac{x}{2} dx = 2\ dy$$

$$\therefore \quad I = \frac{1}{5} \int \frac{2dy}{y^2 + \frac{6}{5} y - 1} = \frac{2}{5} \int \frac{dy}{y^2 + 2 \cdot y \cdot \frac{3}{5} + \left(\frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^2 - 1}$$

$$= \frac{2}{5} \int \frac{dy}{\left(y + \frac{3}{5}\right)^2 - \left(\frac{\sqrt{34}}{5}\right)^2}$$

$$= \frac{2}{5} \times \frac{1}{2 \times \frac{\sqrt{34}}{5}} \log \left(\frac{y + \frac{3}{5} - \frac{\sqrt{34}}{5}}{y + \frac{3}{5} + \frac{\sqrt{34}}{5}} \right) + C$$

$$= \frac{1}{\sqrt{34}} \log \left(\frac{5y + 3 - \sqrt{34}}{5y + 3 \sqrt{34}} \right) + C$$

$$= \frac{1}{\sqrt{34}} \log \left(\frac{5 \tan \frac{x}{2} + 3 - \sqrt{34}}{5 \tan \frac{x}{2} + 3 \sqrt{34}} \right) + C$$

Alternative Method

Put $3 = r \cos \theta$ and $5 = r \sin \theta$

so that $r^2 = 34$

$\therefore \quad r = \sqrt{34}$

Also, $\tan \theta = \frac{5}{3}$

$\therefore \quad \theta = \tan^{-1} \frac{5}{3}$

$$\therefore \quad I = \int \frac{dx}{r \cos \theta \sin x + r \sin \theta \cos x}$$

$$= \frac{1}{r} \int \frac{dx}{\sin (x + \theta)} = \frac{1}{r} \int \operatorname{cosec} (x + \theta)\ dx$$

$$= \frac{1}{r} \log \tan \frac{1}{2}(x + \theta) + c$$

$$= \frac{1}{\sqrt{34}} \log \tan \frac{1}{2}(x + \tan^{-1}\frac{5}{3}) + c$$

$$= \frac{1}{\sqrt{34}} \log \tan \left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{5}{3}\right) + c$$

12. $\int \frac{1}{4 + 3\sinh x}\, dx$

Solution

Let $I = \int \frac{1}{4 + 3\sinh x}\, dx$

$$= \int \frac{1}{4\left(\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2}\right) + 3 \cdot 2\sinh\frac{x}{2} \cdot \cosh\frac{x}{2}}\, dx$$

$$= \int \frac{\operatorname{sech}^2\frac{x}{2}}{4 - 4\tanh^2\frac{x}{2} + 6\tanh\frac{x}{2}}\, dx$$

$$= \int \frac{\operatorname{sech}^2\frac{x}{2}}{-4\left(\tanh^2\frac{x}{2} - \frac{3}{2}\tanh\frac{x}{2} - 1\right)}\, dx$$

$$= -\frac{1}{4}\int \frac{\operatorname{sech}^2\frac{x}{2}}{\tanh^2\frac{x}{2} - \frac{3}{2}\tanh\frac{x}{2} - 1}\, dx$$

Put $\tanh\frac{x}{2} = t$

Then, $\operatorname{sech}^2\frac{x}{2} \cdot \frac{1}{2}\, dx = dt$

or, $\operatorname{sech}^2\frac{x}{2}\, dx = 2\, dt$

$\therefore$ I $= -\frac{1}{4}\int \frac{2dt}{t^2 - \frac{3}{2}t - 1}$

$$= -\frac{1}{2}\int \frac{1}{t^2 - 2\frac{3}{4}t + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - 1}\, dt$$

$$= -\frac{1}{2}\int \frac{1}{\left(t - \frac{3}{4}\right)^2 - \frac{25}{16}}\, dt$$

$$= \frac{1}{2}\int \frac{1}{\frac{25}{16} - \left(t - \frac{3}{4}\right)^2}\, dt$$

$$=\frac{1}{2}\int\frac{1}{\left(\frac{5}{4}\right)^2-\left(t-\frac{3}{4}\right)^2}dt$$

$$=\frac{1}{2}\cdot\frac{1}{2\cdot\frac{5}{4}}\ln\left|\frac{\frac{5}{4}+t-\frac{3}{4}}{\frac{5}{4}-t+\frac{3}{4}}\right|+c$$

$$=\frac{1}{5}\ln\left|\frac{1+2t}{4-2t}\right|+c$$

$$=\frac{1}{5}\ln\left|\frac{1+2\tanh\frac{x}{2}}{4-2\tanh\frac{x}{2}}\right|+c$$

13. $\int\frac{1}{3+5\cosh x}dx$

Solution

$$\text{Let } I=\int\frac{1}{3+5\cosh x}dx$$

$$=\int\frac{dx}{3\left(\cos h^2\frac{x}{2}-\sin h^2\frac{x}{2}\right)+5\left(\cos h^2\frac{x}{2}+\sin h^2\frac{x}{2}\right)}$$

$$=\int\frac{dx}{8\cos h^2\frac{x}{2}+2\sin h^2\frac{x}{2}}$$

$$=\frac{1}{2}\int\frac{dx}{4\cos h^2\frac{x}{2}+\sin h^2\frac{x}{2}}\times\frac{\sec h^2\frac{x}{2}}{\sec h^2\frac{x}{2}}$$

$$=\frac{1}{2}\int\frac{\sec h^2\frac{x}{2}}{4+\tan h^2\frac{x}{2}}dx$$

Put $y=\tanh\frac{x}{2}$.

$$\frac{dy}{dx}=\sec h^2\frac{x}{2}\cdot\frac{1}{2}$$

$\Rightarrow$ $2dy=\sec h^2\frac{x}{2}dx$

Then,

$$I=\frac{1}{2}\int\frac{2dy}{4+y^2}=\int\frac{dy}{y^2+2^2}$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)+c$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{\tanh\frac{x}{2}}{2}\right)+c$$

14. $\int \frac{\tanh x}{\cosh x + 9 \operatorname{sech} x} dx$

Solution

Let $I = \int \frac{\tanh x}{\cosh x + 9 \operatorname{sech} x} dx$

$$= \int \frac{\frac{\sinh x}{\cosh x}}{\cosh x + 9 \frac{1}{\cosh x}} dx$$

$$= \int \frac{\sinh x}{\cosh^2 x + 9} dx$$

Put $\cosh x = t$

Then $\sinh x \, dx = dt$

$$\therefore I = \int \frac{1}{t^2 + 9} dt$$

$$= \int \frac{1}{t^2 + 3^3} dt$$

$$= \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + c$$

$$= \frac{1}{3} \tan^{-1}\left(\frac{1}{3} \cosh x\right) + c$$

Exercise 3.4

1. Determine whether the following functions are even, odd or neither:

a. $f(x) = x^5$
b. $f(x) = x^4 + \cos x$
c. $f(x) = 3x + \sin x$
d. $f(x) = x \sin x + x^3$
e. $f(x) = |x - 2|$
f. $f(x) = \sin x \tan x$

Solution

a. Here, $f(x) = x^5$

Then $f(-x) = (-x)^5$
$= -x^5$
$= -f(x)$

$\therefore$ $f(x)$ is an odd function.

b. Here, $f(x) = x^4 + \cos x$

Then $f(-x) = (-x)^4 + \cos(-x)$
$= x^4 + \cos x$
$= f(x)$

$\therefore$ $f(x)$ is an even function

c. Here, $f(x) = 3x + \sin x$

Then $f(-x) = 3(-x) + \sin(-x)$
$= -3x - \sin x$
$= -(3x + \sin x)$
$= -f(x)$

d. Here, $f(x) = x \sin x + x^3$

then $f(-x) = -x \sin(-x) + (-x)^3$
$= x \sin x - x^3$
$= x \sin x - x^3$
$= -(-x \sin x + x^3)$

So, $f(x) \neq -f(x)$
and $f(x) \neq f(-x)$

$\therefore$ $f(x)$ is neither even nor odd.

e. Here, $f(x) = |x - 2|$

Then $f(-x) = |-x - 2|$
$= |-(x + 2)|$
$= |x + 2|$

So, $f(-x) \neq f(x)$
and $f(-x) \neq -f(x)$

$\therefore f(x)$ is neither even nor odd.

f. Here, $f(x) = \sin x \tan x$

then $f(-x) = \sin(-x) \tan(-x)$
$= \sin x \tan x$
$= f(x)$

$\therefore f(x)$ is an even function.

2. Test the symmetry of the given functions:

a. $f(x) = x^2$
b. $f(x) = x^3$
c. $f(x) = x^3 - x$
d. $f(x) = x^4 + 3x^2 + 5$
e. $f(x) = \cos(5x)$
f. $f(x) = 10^x - 10^{-x}$
g. $f(x) = \sqrt{x^2 - 4}$
h. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

a. Here, $f(x) = x^2$

Replace x by $-x$, we get

$f(x) = (-x)^2$
$= x^2$
$= f(x)$

$\therefore$ It is symmetric about $y-$axis.

b. Here, $f(x) = x^2$

Replace x by $-x$, we get

$f(x) = (-x)^3$
$= -x^3$
$= -f(x)$

So, it is symmetric about the origin.

c. Here, $f(x) = x^3 - x$

Replace x by $-x$, we get

$f(x) = (-x)^3 - (-x)$
$= -x^3 + x$
$= -(x^3 - x)$
$= -f(x)$

$\therefore$ It is symmetric about the origin.

d. Here, $f(x) = x^4 + 3x^2 + 5$

Replace x by $-x$, then

$f(x) = (-x)^4 + 3(-x)^2 + 5$
$= x^4 + 3x^2 + 5$
$= f(x)$

$\therefore$ It is symmetric about $y-$axis.

Here, $f(x) = \cos(5x)$

Replace x by $-x$, we get

$f(x) = \cos\{5(-x)\}$
$= \cos 5x$
$= f(x)$

It is symmetric about y – axis.

Here, $f(x) = \sqrt{x^2 - 4}$

Replace x by $-x$, we get

$f(x) = \sqrt{(-x)^2 - 4}$
$= \sqrt{x^2 - 4}$
$= f(x)$

It is symmetric about y–axis.

f. Here, $f(x) = 10^x - 10^{-x}$

Replace x by $-x$, we get

$f(x) = 10^{-x} - 10^x$
$= -(10^x - 10^{-x})$
$= -f(x)$

$\therefore$ It is symmetric about the origin.

h. Here, $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Replace x by $-x$, we get

$f(x) = \dfrac{e^{-x} - e^x}{e^{-x} + e^x} = -\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)$
$= -f(x)$

$\therefore$ It is symmetric about the origin.

Test the periodicity of the given functions and find the period:

a. $f(x) = \sin(4x)$

b. $f(x) = \cos(\pi x)$

c. $f(x) = \cos(ax + b)$

d. $f(x) = \tan\left(\dfrac{2x}{5}\right)$

e. $f(x) = \sin x + \tan x$

f. $f(x) = \sin^2 x$

Solution

$f(x) = \sin(4x) = \sin(4\pi + 4x)$
$= \sin\left\{4\left(\dfrac{\pi}{2} + x\right)\right\}$
$= f\left(\dfrac{\pi}{2} + x\right)$

Period $= \dfrac{\pi}{2}$

b. $f(x) = \cos(\pi x)$
$= \cos(2\pi + \pi x)$
$= \cos\{\pi(2 + x)\}$
$= f(2 + x)$

$\therefore$ Period = 2

$f(x) = \cos(ax + b)$
$= \cos(2\pi + ax + b)$
$= \cos\left\{a\left(x + \dfrac{2\pi}{a}\right) + b\right\}$
$= f\left(x + \dfrac{2\pi}{a}\right)$

Period $= \dfrac{2\pi}{a}$

d. $f(x) = \tan\left(\dfrac{2x}{5}\right) = \tan\left(\pi + \dfrac{2x}{5}\right)$
$= \tan\left\{\dfrac{2}{5}\left(\dfrac{5\pi}{2} + x\right)\right\}$
$= f\left(\dfrac{5\pi}{2} + x\right)$

$\therefore$ Period $= \dfrac{5\pi}{2}$

$f(x) = \sin x + \tan x$
$= \sin(2\pi + x) + \tan(2\pi + x)$
$= f(2\pi + x)$

Period $= 2\pi$

f. $f(x) = \sin^2 x = \dfrac{1 - \cos 2x}{2}$
$= \dfrac{1}{2} - \dfrac{1}{2}\cos 2x$
$= \dfrac{1}{2} - \dfrac{1}{2}\cos(2x + 2\pi)$
$= \dfrac{1}{2} - \dfrac{1}{2}\cos 2(x + \pi)$
$= \dfrac{1 - \cos 2(\pi + x)}{2}$
$= f(\pi + x)$

$\therefore$ Period $= \pi$

4. Sketch the graph of the following functions:

a. $y = 3x + 5$

b. $y = x^2 - 3x$

c. $y = x^2 - 4x + 3$

d. $y = x - x^2$

e. $y = x(x-1)(x-2)$

f. $y = (x+1)(x-2)(x-3)$

g. $y = \frac{1}{x-3}$

h. $y = \frac{x^2-4}{x-2}$

i. $y = 5^x$

j. $y = \log_2 x$

k. $y = 2\sin x, (-2\pi \le x \le 2\pi)$

Solution

a. $f(x) = 3x + 5$

Let $y = 3x + 5$

The various characteristics of the function are as follows:

i. Domain = $\mathbb{R}$

ii. Range = $\mathbb{R}$

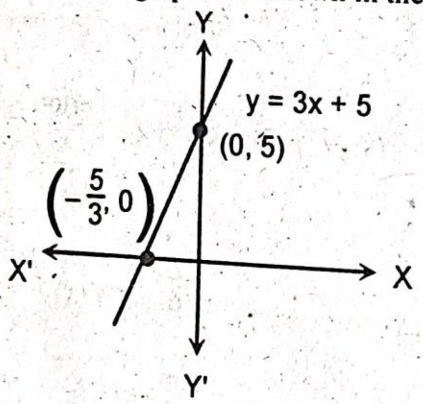
iii. x - intercept, $x = -\frac{5}{3}$

So, the graph meets x - axis at the point $\left(-\frac{5}{3}, 0\right)$

iv. y-intercept, $y = 5$

So, the graph meets y - axis at the point (0, 5)

With these informations the graph is as shown in the figure:



b. Let $y = f(x) = x^2 - 3x$

The characteristics of given function are as follows:

i. Since $y = x^2 - 3x$ is in the form of $y = ax^2 + bx + c$, so it represents a parabola.

Here, a = 1, b = –3, c = 0.

Since, a = 1 > 0, the parabola turns upwards.

ii. Vertex:

$$x = -\frac{b}{2a} = -\frac{(-3)}{2 \times 1} = \frac{3}{2}$$

When, $x = \frac{3}{2}$,

$$y = \left(\frac{3}{2}\right)^2 - 3 \times \frac{3}{2} = \frac{9}{4} - \frac{9}{2} = \frac{9-18}{4} = -\frac{9}{4}$$

$\therefore$ Vertex = $\left(\frac{3}{2}, -\frac{9}{4}\right)$

iii. Domain = $(-\infty, \infty)$

Range = $\left[-\frac{9}{4}, \infty\right)$

Axis of the parabola: $x = \frac{3}{2}$ is the axis of the parabola.

iv. When $x = 0$,

v. $y = 0^2 - 3 \times 0 = 0$

So the curve passes through the origin.

When $y = 0$,

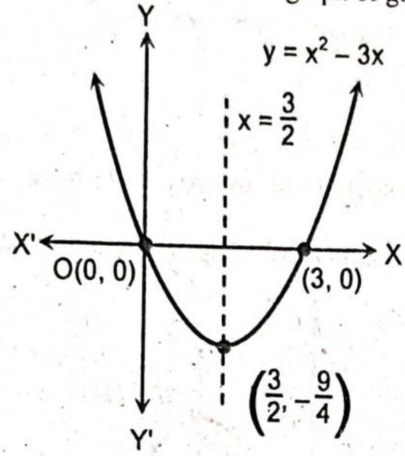
vi. $0 = x^2 - 3x$

$x(x - 3) = 0$

or, $x = 0, x = 3$.

∴ So the curve cuts x–axis at (0, 0) and (3, 0).

With the above characteristics, the sketch of the graph of given function is as follows:



c. $f(x) = x^2 - 4x + 3$

Let, $y = x^2 - 4x + 3$

The various characteristics of the function are as follows:-

i. Domain = $(-\infty, \infty)$

ii. $y = x^2 - 4x + 3$

or, $y = (x - 2)^2 - 1$

Compare it with $y = a(x - h)^2 + k$, we get

$a = 1, h = 2, k = 1$

iii. $a = 1 > 0$, so the curve of the function is a parabola open upwards

iv. The vertex of the parabola is (2, 1)

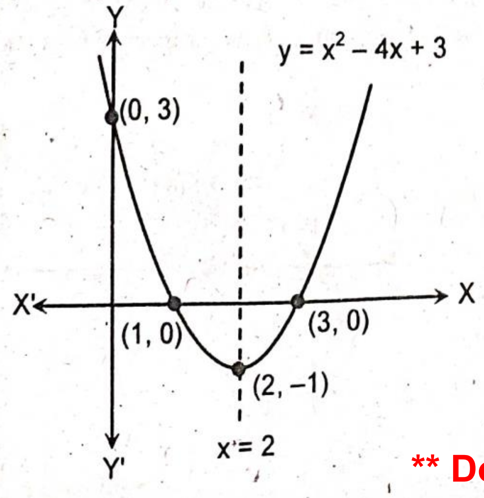
v. The axis of the parabola is $x = 2$

vi. x – intercept: $x^2 - 4x + 3 = 0$

$\Rightarrow x = 1, 3$

y – intercept: $y = 3$

So, the curve meets x - axis at the point (1, 0) and (3, 0) and meets y - axis at the point (0, 3). The curve of the function with above properties is as shown in the figure:



d. $y = x - x^2$

$= -(x^2 - x)$

$= -\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right] = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$

Compare with $y = a(x - h)^2 + k$ we get

$a = -1, h = \frac{1}{2}, k = \frac{1}{4}$

We observe that

i. $a = -1 < 0$, so the curve of the function is a parabola open downwards

ii. The vertex of the parabola is $\left(\frac{1}{2}, \frac{1}{4}\right)$.

iii. The axis of the parabola is $x = \frac{1}{2}$

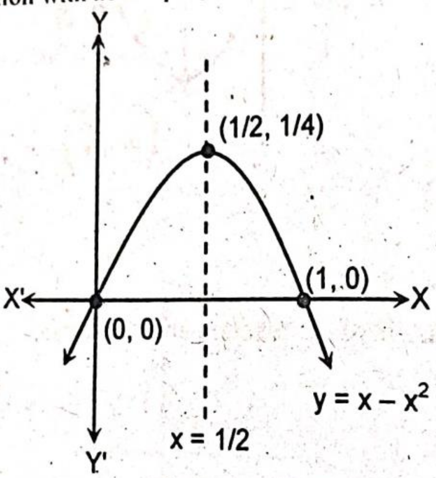
iv. x – intercept: $x - x^2 = 0$

$x = 0, 1$

y – intercept: $y = 0$

So, the curve of the function meets x – axis at the points (0, 0) and (1, 0) and meets y – axis at the point (0, 0)

The curve of the function with above properties is as shown in the figure:



e. $f(x) = x(x - 1)(x - 2)$

Let, $y = x(x - 1)(x - 2)$

We observe that:

i. When $y = 0$, then $x = 0, 1, 2$

So, the curve meets x - axis at the points (0, 0), (1, 0), and (2, 0).

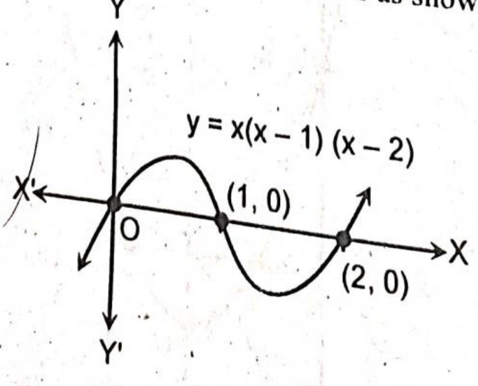
ii. When $x < 0$, then $y < 0$, so the curve lies below x – axis in this portion.

iii. When $0 < x < 1$, then $y > 0$, so the curve lies above x - axis in this portion

iv. When $1 < x < 2$, then $y < 0$, so the curve lies below x – axis in this portion.

v. When $x > 2$, then $y > 0$, so the curve lies above x – axis in this portion.

∴ The curve of the function with above properties is as shown in the figure:



f. Given, $y = (x+1)(x-2)(x-3)$

The given function has following characteristics.

i. When $x = 0$, $y = (1)(-2)(-3) = 6$. So, the curve cuts the y-axis at (0, 6)

ii. When $y = 0$, $(x+1)(x-2)(x-3) = 0$

$\therefore$ $x = -1, 2, 3$

Hence the curve cuts x-axis at (–1, 0) (2, 0) and (3, 0)

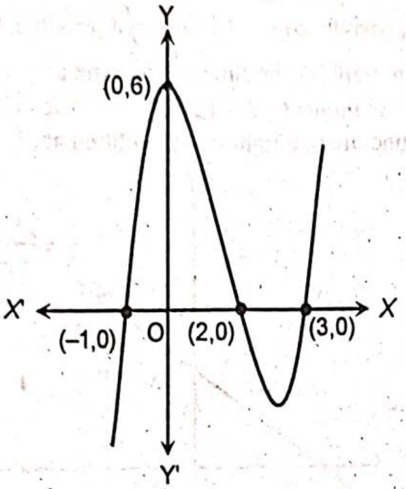
iii. When $x < -1$, $y < 0$ i.e. when x decreases y also decreases.

iv. When $-1 < x < 0$, y increases from 0 to 6 and $0 < x < 2$, y decreases from 6 to 0.

v. When $2 < x < 3$, $y < 0$. So, the curve lies below x-axis when $2 < x < 3$.

vi. When $x > 3$, then $y > 0$.

With the above characteristics the sketch of the given function is as follows.



g. $y = \dfrac{1}{x-3}$

We observe that

i. Domain = $\Re - \{3\}$

ii. Compare $y = \dfrac{1}{x-3}$ with $y = \dfrac{1}{ax+b} + c$

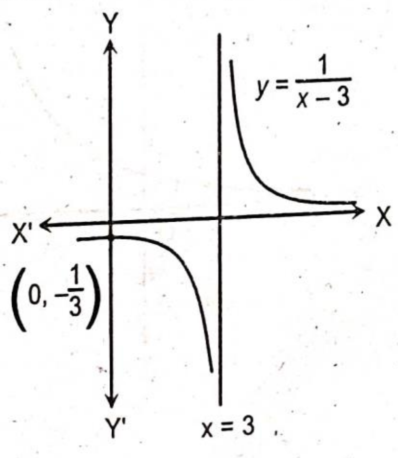
We get a = 1, b = – 3, c = 0

iii. The asymptotes to the curve are $x = 3$ and $y = 0$

iv. a = 1 > 0 so the curve of the function is a hyperbola lying in first and third quadrant with respect to the asymptotes.

v. When $x = 0$, then $y = \dfrac{-1}{3}$, so the curve meets y – axis at the point $\left(0, \dfrac{-1}{3}\right)$.

The curve of the function with above properties is as shown in the figure:



h. $f(x) = \frac{x^2 - 4}{x - 2}$

Let, $y = \frac{x^2 - 4}{x - 2}$

We observe that:

i. The domain of the function in $\Re - \{2\}$

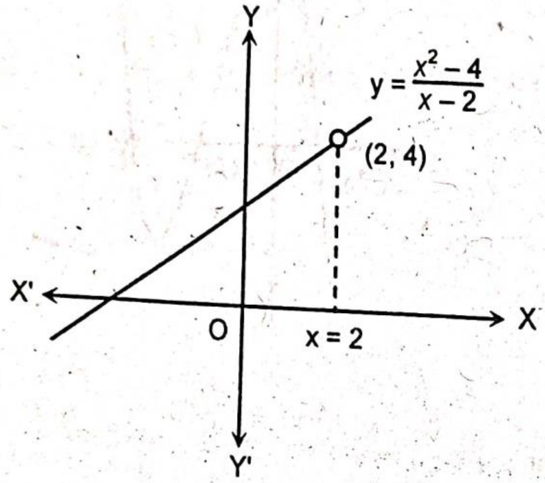
ii. When $x \neq 2$, then

$$y = \frac{(x-2)(x+2)}{x-2} = x + 2$$

If $x = 2$, then $y = 4$

But $x = 2$ is not possible, so $y = 4$ is also not possible. So, Range $= \Re - \{4\}$

iii. When $x \neq 2$, then graph of the function is same as $y = x + 2$, which is a straight line passing through the points $(-2, 0)$ and $(0, 2)$. The curve of the function with above properties is a punctured straight line punctured at $(2, 4)$ as shown in the figure:



i. $f(x) = 5^x$

Let, $y = 5^x$

We observe that

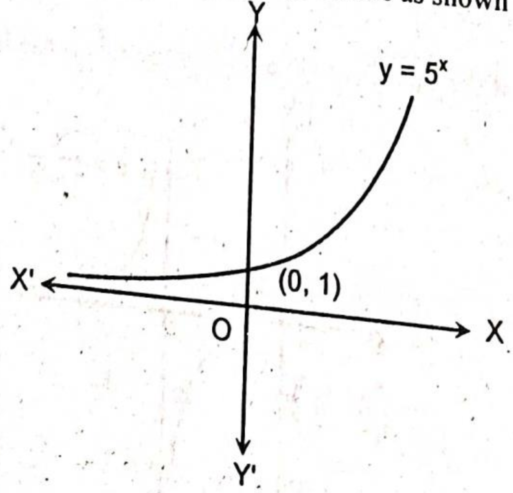
i. When $x = 0$, then $y = 1$, so the curve of the function passes through the point $(0, 1)$

ii. $y > 0$ for all x, so the curve lies entirely above the x - axis.

iii. When $x > 0$ and increases to $+\infty$, then y increases from 1 to $+\infty$

iv. When $x < 0$, and decreases to $-\infty$ then y decreases from 1 to 0.

$\therefore$ The curve of the function with above properties is as shown in the figure:



j. $f(x) = \log_2 x$

Let, $y = \log_2 x$

$\Rightarrow x = 2^y$

We observe that

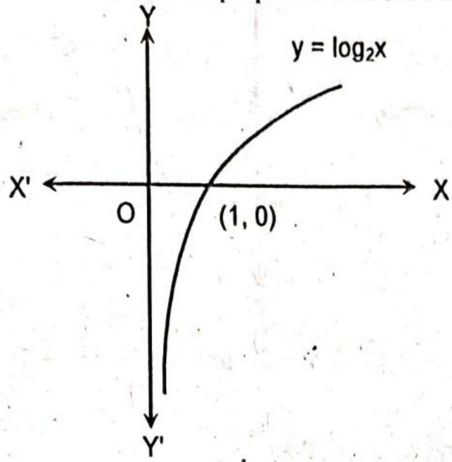
i. When $y = 0$, then $x = 1$, so the curve passes through the point (1, 0).

ii. $x > 0$, so the curve lies entirely right side of y – axis.

iii. When $x > 1$ and increases to $+\infty$, then y increases from 0 to $+\infty$.

iv. When $x < 1$ and decreases to 0, then y decreases from 0 to $-\infty$.

The curve of the function with above properties is as shown in the figure:



k. $f(x) = 2\sin x \quad (-2\pi \le x \le 2\pi)$

Let $y = 2\sin x$

We observe that:

i. When $x = 0$, then $y = 0$, So, the curve passes through (0, 0).

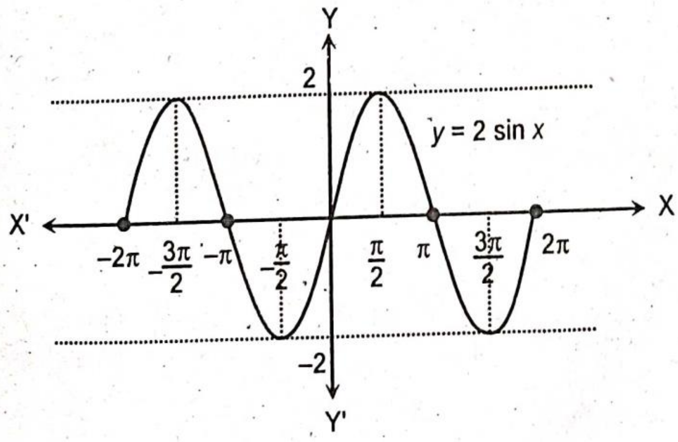
ii. When $y = 0$, then $x = -2\pi, -\pi, 0, \pi, 2\pi$, So the curve meets x– axis at the points $(-2\pi, 0)$ $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$ $(2\pi, 0)$.

iii. Since $-1 \le \sin x \le 1$, so the curve moves between $y = -2$ and $y = 2$

iv. When x increases from 0 to $\frac{\pi}{2}$, then y increases from 0 to 1, when x increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, then y decreases from 1 to -1, when x increases from $\frac{3\pi}{2}$ to 2π, then y increases from -1 to 0.

v. When x decreases from 0 to $-\frac{\pi}{2}$, then y decreases from 0 to -1, when x decreases from $-\frac{\pi}{2}$ to $-\frac{3\pi}{2}$, then y increases from -1 to 1, when x decreases from $-\frac{3\pi}{2}$ to -2π, then y decreases from 1 to 0.

$\therefore$ The curve of the function with above properties is as shown in the figure:

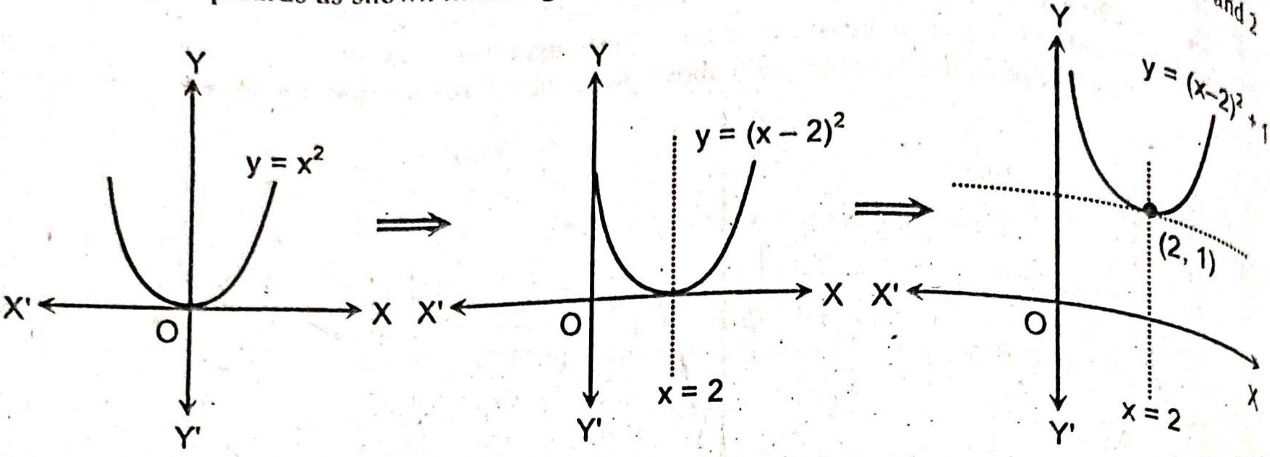


5. Using the graph of $y = x^2$, sketch the graph of $y = (x-2)^2 + 1$.

Solution

Sketch the curve of $y = x^2$

Then curve of $y = (x-1)^2 + 2$ is same as the curve $y = x^2$ shifting 1 unit to the right and 2 units upwards as shown in the figure.

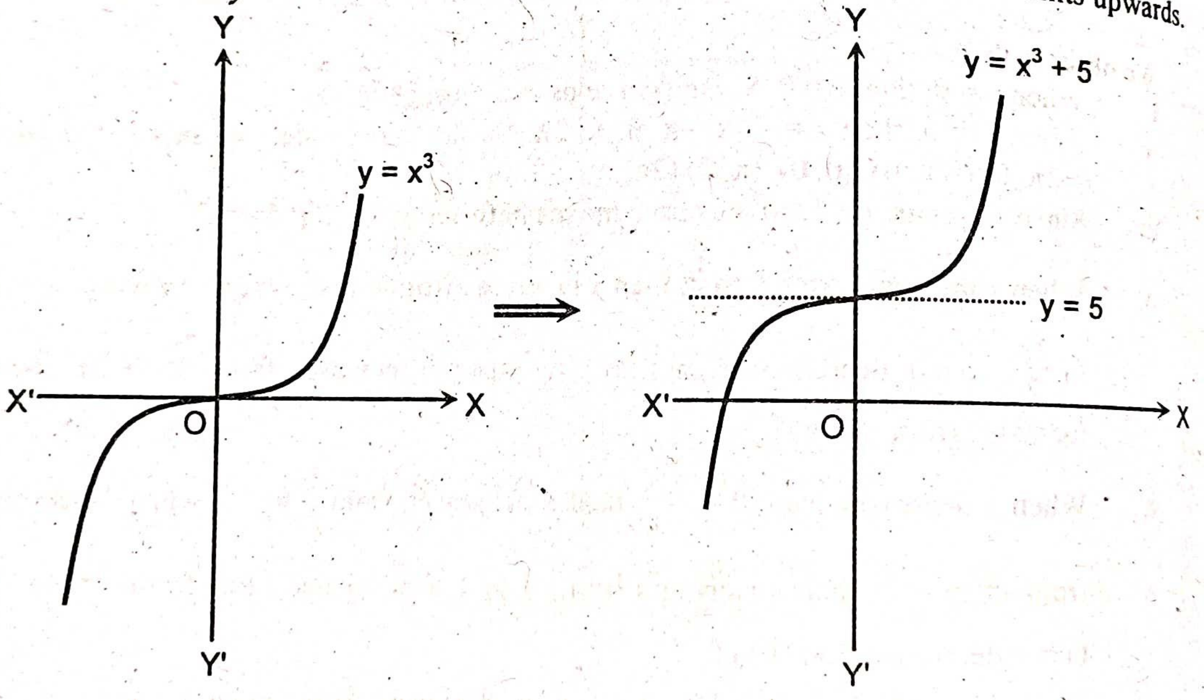


6. Using the graph of $y = x^3$, sketch the graph of $y = x^3 + 5$.

Solution

First sketch the curve of $y = x^3$, than

The curve of $y = x^3 + 5$ is same as the curve of $y = x^3$ shifting the curve 5 units upwards.



Exercise 3.5

1. a. Using the limit of a sum, evaluate $\int_0^1 4x^2\,dx$.

   b. Using the limit of a sum, find the area bounded by the curve $y = x^2 - 5$, x-axis, $x = 0$, and $x = a$.

   c. Using the limit of a sum, find the area enclosed by the curve $y = x^3$, x-axis and $x = 0$ and $x = 4$.

Solution

a. Here, $f(x) = 4x^2$

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a+h) + f(a+2h) + f(a+3h) + \dots + f(a+nh)]$$

$$\therefore \int_0^1 4x^2\,dx = \lim_{h\to 0} h[f(h) + f(2h) + \dots + f(nh)]$$

$$= \lim_{h\to 0} h[4\cdot h^2 + 4\cdot(2h)^2 + \dots + 4\cdot nh^2]$$

$$= \lim_{h\to 0} h\cdot 4h^2(1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= \lim_{h\to 0} 4h^3\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{h\to 0} \frac{4\cdot nh(nh+h)(2nh+h)}{6}$$

$$= \frac{4\times 1\times 1\times 2}{6} \qquad [\because nh = b - a = 1 - 0 = 1]$$

$$= \frac{4}{3}$$

b. Here, $f(x) = x^2 - 5$

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a+h) + f(a+2h) + \dots + f(a+nh)]$$

$$\therefore \int_0^1 (x^2 - 5)\,dx = \lim_{h\to 0} h[f(h) + f(2h) + \dots + f(nh)]$$

$$= \lim_{h\to 0} h[(h^2 - 5) + \{(2h)^2 - 5\} + \dots + \{(nh)^2 - 5\}]$$

$$= \lim_{h\to 0} h\,[h^2(1^2 + 2^2 + \dots + n^2) - 5n]$$

$$= \lim_{h\to 0} h\left[h^2\cdot\frac{n(n+1)(2n+1)}{6} - 5n\right]$$

$$= \lim_{h\to 0} \left[\frac{nh(nh+h)(2nh+h)}{6} - 5nh\right]$$

$$= \frac{a\cdot(a+0)(2a+0)}{6} - 5a \qquad [\because nh = b - a = a - 0 = a]$$

$$= \frac{a^3}{3} - 5a$$

c. The area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = 0$ and $x = 4$ is given by $\int_0^4 y\, dx = \int_0^4 x^3\, dx$.

Here, $a = 0$, $b = 4$, $f(x) = x^3$

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+nh)]$$

$$= \lim_{h \to 0} h[f(h) + f(2h) + f(3h) + \ldots + f(nh)]$$

$$= \lim_{h \to 0} h[h^3 + (2h)^3 + (3h)^3 + \ldots + (nh)^3]$$

$$= \lim_{h \to 0} h \cdot 4h^3[1^3 + 2^3 + 3^3 + \ldots + n^3]$$

$$= \lim_{h \to 0} h^4 \cdot \left[\frac{n(n+1)}{2}\right]^2 \quad \left[\because 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2\right]$$

$$= \lim_{h \to 0} \frac{h^4 n^2 (n+1)^2}{4}$$

$$= \lim_{h \to 0} \frac{(nh)^2 (nh+h)^2}{4}$$

$$= \frac{4^2 (4+0)^2}{4} \quad [\because nh = b - a = 4 - 0 = 4]$$

$$= 64$$

2. Find the area of curves bounded by the x-axis, and the given ordinates.

a. $y = 3x^2; x = 1, x = 3$

b. $y = 3x^2 + 2x + 1; x = 0, x = 4$

c. $y = e^{2x}; x = 1, x = 2$

d. $y = \ln x; x = 1, x = e$

e. $y = \sin x, x = 0, x = \pi$

Solution

a. $$A = \int_1^3 y\, dx = \int_1^3 3x^2\, dx$$

$$= \left[\frac{3x^3}{3}\right]_1^3$$

$$= [x^3]_1^3$$

$$= 3^3 - 1^3$$

$$= 26 \text{ sq. units}$$

b. $$A = \int_0^4 y\, dx = \int_0^4 (3x^2 + 2x + 1)\, dx$$

$$= \left[\frac{3x^3}{3} + \frac{2x^2}{2} + x\right]_0^4$$

$$= [x^3 + x^2 + x]_0^4$$

$$= (4^3 + 4^2 + 4) - 0$$

$$= 84 \text{ sq. units}$$

c. $$A = \int_1^2 y\, dx$$

$$= \int_1^2 e^{2x}\, dx$$

$$= \left[\frac{e^{2x}}{2}\right]_1^2$$

$$= \frac{e^4}{2} - \frac{e^2}{2}$$

$$= \frac{1}{2} e^2 (e^2 - 1) \text{ sq. units}$$

d. $$A = \int_1^e \ln x\, dx$$

$$= [x \ln x - x]_1^2$$

$$[\because \int \ln x\, dx = x \ln x - x + c]$$

$$= (e \ln e - e) - (\ln 1 - 1)$$

$$= (e \times 1 - e) - (0 - 1)$$

$$= 0 + 1$$

$$= 1 \text{ sq. units}$$

c. $\text{Area} = \int_0^{\pi} \sin x \, dx$

$= [-\cos x]_0^{\pi}$

$= -\cos \pi - (-\cos 0)$

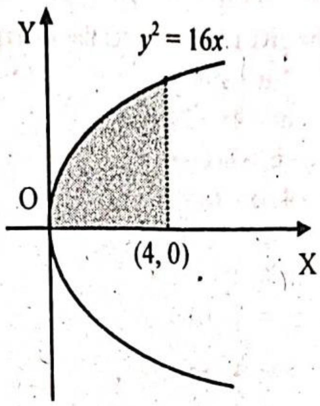
$= -(-1) - (-1)$

$= 1 + 1 = 2$

3. Find the area bounded by the x-axis:
   a. the curve $y^2 = 16x$ and the ordinate at the point (4, 0)
   b. the curve $x^2 = 2y$ and the ordinate at the point (2, 0)

Solution

a. $A = \int_0^4 y \, dx$



$= \int_0^4 \sqrt{16x} \, dx$

$= \int_0^4 4x^{1/2} \, dx$

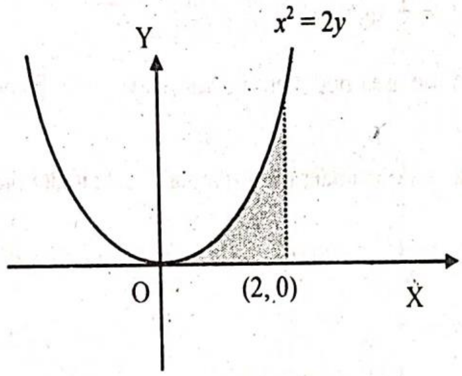
$= 4 \left[\frac{x^{3/2}}{3/2}\right]_0^4$

$= \frac{8}{3} [4^{3/2} - 0^{3/2}]$

$= \frac{8}{3} \times 8$

$= \frac{64}{3}$ sq. units

b. $A = \int_0^2 y \, dx$



$= \int_0^2 \frac{x^2}{2} \, dx$

$= \frac{1}{2} \left[\frac{x^3}{3}\right]_0^2$

$= \frac{1}{2} \left[\frac{2^3}{3} - 0\right]$

$= \frac{4}{3}$ sq. units

4. Find the area bounded by the axis of x and the following curves.
   a. $y = 3x - x^2$
   b. $y = x^2 - 10x + 24$

Solution

a. $y = 3x - x^2$

The given curve meets the x-axis at the point where $y = 0$

i.e. $3x - x^2 = 0$

or, $x(3 - x) = 0$

$\therefore$ $x = 0, 3$

$$A = \int_0^3 y\,dx = \int_0^3 (3x - x^2)\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= 3 \times \frac{3^2}{2} - \frac{3^3}{3}$$

$$= 27\left(\frac{1}{2} - \frac{1}{3}\right) = 27\left(\frac{3-2}{6}\right)$$

$$= \frac{27}{6} \text{ sq. units}$$

b. $y = x^2 - 10x + 24$

The given curve meets the x –axis at the points where $y = 0$

i.e. $x^2 - 10x + 24 = 0$

or, $x^2 - 6x - 4x + 24 = 0$

or, $x(x - 6) - 4(x - 6) = 0$

or, $(x - 4)(x - 6) = 0$

$\therefore$ $x = 4, 6$

$$A = \int_4^6 y\,dx$$

$$= \int_4^6 (x^2 - 10x + 24)\,dx$$

$$= \left[\frac{x^3}{3} - 10\frac{x^2}{2} + 24x\right]_4^6$$

$$= \left[\frac{6^3}{3} - 10 \times \frac{6^2}{2} + 24 \times 6\right] - \left[\frac{4^3}{3} - 10 \times \frac{4^2}{2} + 24 \times 4\right]$$

$$= \frac{4}{3} \text{ sq. units}$$

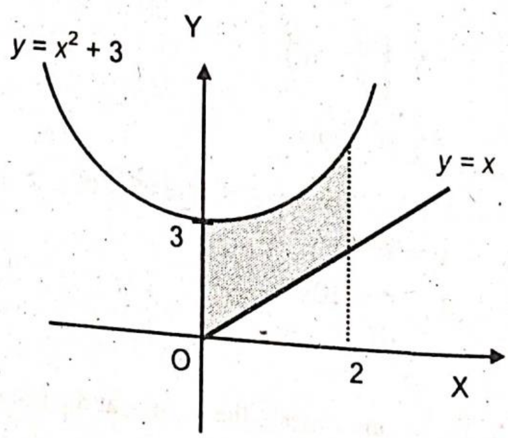
5. Find the area between the curves $y = x^2 + 3$ and $y = x$ for $0 \le x \le 2$.

Solution

Area = Area under the parabola – Area under the line $y = x$

$$= \int_0^2 (y_1 - y_2)\,dx \quad [\text{where } y_1 = x^2 + 3 \text{ and } y_2 = x]$$

$$= \int_0^2 (x^2 + 3 - x)\,dx$$



$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + 3x\right]_0^2$$

$$= \frac{2^3}{3} - \frac{2^2}{2} + 3 \times 2$$

$$= \frac{20}{3} \text{ sq. units}$$

Find the area bounded by the curves $y = x^2$ and $y = 2x$.
Find the area between the curve $y^2 = 16x$ and the line $y = 2x$.
Find the area bounded by the curve $y^2 = 16x$ and the line $x = 4$.
Find the area of the region between the curve $y^2 = x$ and $x^2 = y$.

Here, the give curve is: $y = x^2$ and $y = 2x$.
Eliminating y between the given equations, we get

$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ and $x = 2$ are the ordinates of the points at which the given curves intersects.
It cuts the y-axis at the point (0, 0) and (2, 4) when $x = 0, y = 0$ and $x - 2, y = 4$.

The required area (A) $= \int_0^2 (y_1 - y_2)\, dx$

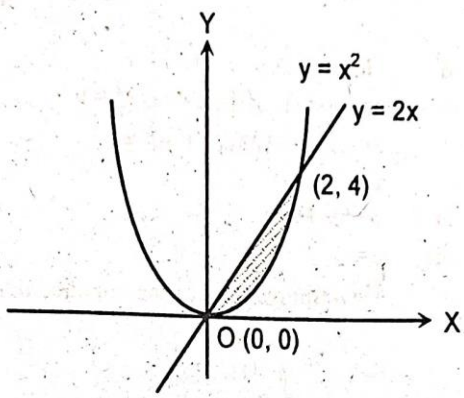
$= \int_0^2 (2x - x^2)\, dx$ [Considering $y_1 = 2x$ and $y_2 = x^2$]

$= \left[2 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$

$= \left[x^2 - \frac{x^3}{3}\right]_0^2$

$= 2^2 - \frac{2^3}{3} = 4 - \frac{8}{3}$

$= \frac{4}{3}$ sq. unit



Here, the given curve is: $y^2 = 16x$ and $y = 2x$.
Solving these equations:

$4x^2 = 16x$

$4x^2 - 16x = 0$

$4x(x-4) = 0$

Either $x = 0$, or $x = 4$

The required area (A) $= \int_0^4 (y_1 - y_2)\, dx$

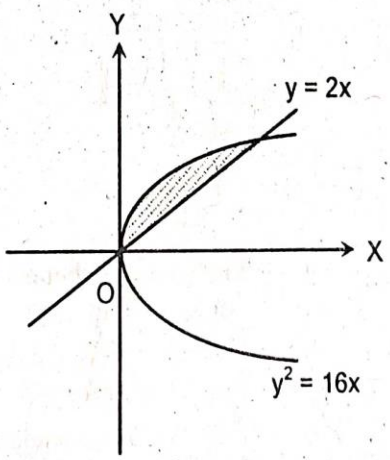
$= \int_0^4 \left(\sqrt{16x} - 2x\right)\, dx$

$[\because y_1 = \sqrt{16x}$ and $y_2 = 2x]$

$= \left[\frac{2}{3} 4x^{3/2} - 2 \cdot \frac{x^2}{2}\right]_0^4$

$= \frac{8}{3} \cdot 4^{3/2} - 2 \cdot \frac{4^2}{2}$

$= \frac{64}{3} - 16 = \frac{16}{3}$ sq. unit



c. Here, the given curve is: $y^2 = 16x$

When, $x = 0$, $y = 0$, so curve passes through the origin.

$\therefore$ The required area (A) $= 2 \times$ Area of Shaded region

$$= 2 \times \int_0^4 y\, dx$$

$$= 2 \times \int_0^4 \sqrt{16x}\, dx$$

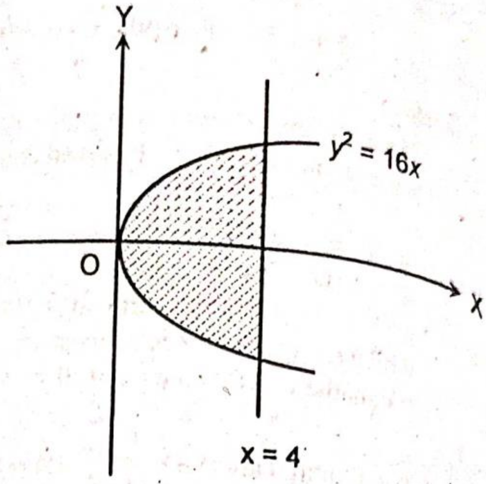
$$= 2 \times 4 \int_0^4 x^{1/2}\, dx$$

$$= 8 \cdot \left[\frac{2}{3} x^{3/2}\right]_0^4$$

$$= 8 \cdot \frac{2}{3} 4^{3/2}$$

$$= \frac{16}{3} 2^3$$

$$= \frac{128}{3} \text{ sq. units}$$



d. Here, first curve is $y^2 = x$ ...(1)

and the second curve is $x^2 = y$ ...(2)

Solving (1) and (2), we get

$x^4 = x$

or, $x(x^3 - 1) = 0$

or, $x = 0, 1$

$\therefore$ The required area of the bounded region between the curves (1) and (2) is given by,

$$A = \int_0^1 (y_1 - y_2)\, dx$$

$$= \int_0^1 (\sqrt{x} - x^2)\, dx$$

$$= \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{2}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3} \text{ sq. units}$$

7. a. Find the area bounded by the parabola $y^2 = 4x$ and the y-axis between the points $y = 0$ to $y = 2$.

b. Find the area of the region bounded by the curve, $x = y^2 + 2$, the y-axis, and the lines $y = 1$ and $y = 2$.

c. Find the area bounded by the y-axis, the curve $x^2 = 4a(y - 2a)$ and the line $y = 6a$.

d. Find the area bounded by the axis of coordinates, the curve $x^2 = 4a(y - 2a)$ and the ordinate of the point (h, k).

Solution

Here, the given parabola is: $y^2 = 4x$

The points are $y = 0$ to $y = 2$

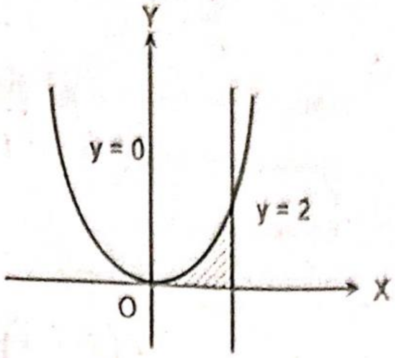
$\therefore$ The required area (A) $= \int_0^2 x\,dy$

$= \int_0^2 \frac{y^2}{4}\,dy$

$= \frac{1}{4}\int_0^2 y^2\,dy$

$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^2 = \frac{1}{12}[2^3 - 0]$

$= \frac{2}{3}$ sq. unit



b. Here, the given curve is: $x = y^2 + 2$

The lines $y = 1$ and $y = 2$

$\therefore$ The required area (A) $= \int_1^2 x\,dy$

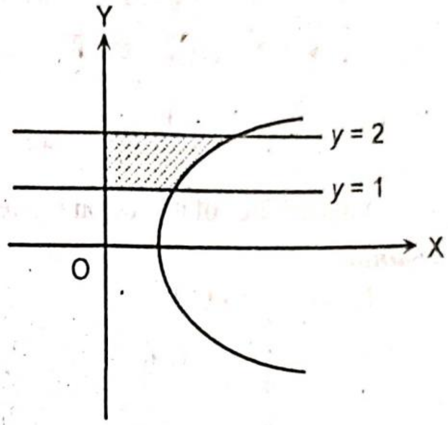
$= \int_1^2 (y^2 + 2)\,dy$

$= \left[\frac{y^3}{3} + 2y\right]_1^2$

$= \frac{2^3}{3} + 2\cdot 2 - \frac{1^3}{3} - 2\cdot 1$

$= \frac{8}{3} + 4 - \frac{1}{3} - 2 = \frac{7}{3} + 2$

$= \frac{13}{3}$ sq. units



c. Here, the given curve is: $x^2 = 4a(y - 2a)$

It cuts the y-axis at the point where $x = 0$.

i.e. $0 = 4a(y - 2a)$

or, $y = 2a$

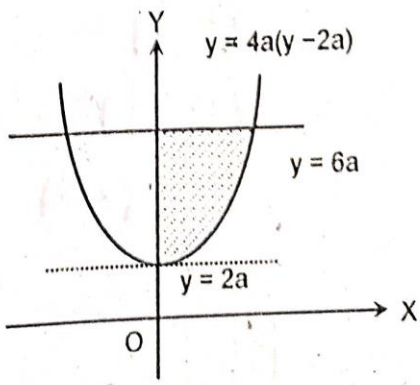
$\therefore$ The required area (A) $= \int_{2a}^{6a} x\,dy$

$= \int_{2a}^{6a} \sqrt{4a(y - 2a)}\,dy$

$= \left[\sqrt{4a}\frac{2}{3}(y - 2a)^{3/2}\right]_{2a}^{6a}$

$= \sqrt{4a}\cdot\frac{2}{3}(4a)^{3/2}$

$= \frac{32a^2}{3}$ sq. units



d. Given curve is $x^2 = 4a(y-2a)$ ... (i)

The equation (i) can be written as

$$y = \frac{x^2}{4a} + 2a$$

Thus, we have to find the area bounded by the curve $y = \frac{x^2}{4a} + 2a$, the x-axis and the ordinates $x = 0$ and $x = h$.

$$\text{Required area} = \int_0^h y\,dx$$

$$= \int_0^h \left(\frac{x^2}{4a} + 2a\right) dx$$

$$= \left[\frac{x^3}{4a \times 3} + 2ax\right]_0^h$$

$$= \left(\frac{h^3}{12a} + 2ah\right) - 0$$

$$= \frac{h^3 + 24a^2h}{12a}$$

$$= \frac{h}{12a}(h^2 + 24a^2)$$

8. Find the area of the region between the curve $y = 4 - x^2$ and $0 \le x \le 3$ and the x-axis.

Solution

Integral over [0, 2]

$$A_1 = \int_0^2 (4 - x^2)\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \left[4 \times 2 - \frac{2^3}{3}\right] - 0$$

$$= \frac{16}{3}$$

Integral over [2, 3]

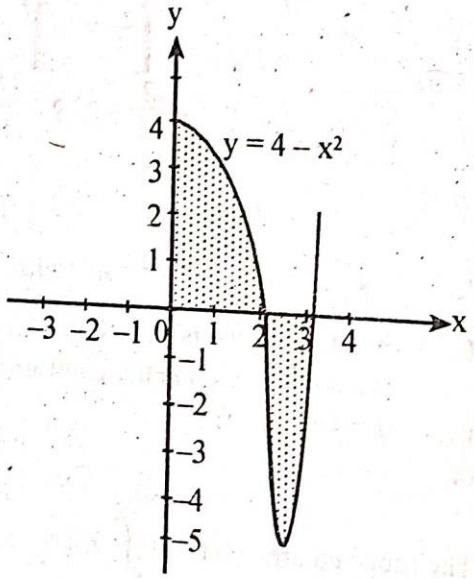
$$A_2 = \int_2^3 (4 - x^2)\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_2^3$$

$$= \left(4 \times 3 - \frac{3^3}{3}\right) - \left(4 \times 2 - \frac{2^3}{3}\right)$$

$$= (12 - 9) - \left(8 - \frac{8}{3}\right)$$

$$= 3 - \frac{16}{3} = -\frac{7}{3}$$



$\therefore \quad A_2 = \left|\frac{-7}{3}\right| = \frac{7}{3}$

$\therefore \quad \text{Total area} = A_1 + A_2 = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$

9. a. Find the area of the circle $x^2 + y^2 = 9$ using method of integration.
b. Using integration, find the area of the circle $x^2 + y^2 = a^2$.

Solution

The equation of circle is

$x^2 + y^2 = 9$

$y = \sqrt{9 - x^2}$ ... (i)

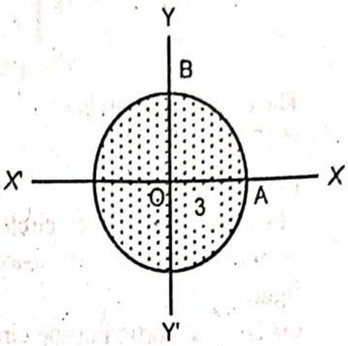
The centre of the given circle is at origin (0, 0) and its radius is 3. since the given circle is symmetrical about axes, x-axis and y-axis divide it into four equal parts as shown in the figure.

OAB is the portion of the circle lying in the first quadrant which is bounded by the given curve and the axes.

At A, $y = 0 \Rightarrow x^2 = 9$

$x = 3$

At O, $x = y = 0$



$\therefore$ Area of OAB $= \int_0^3 y\,dx$

Hence,

Area of the circle $= 4 \times$ Area of OAB

$$= 4\int_0^3 y\,dx$$

$$= 4\int_0^3 \sqrt{9 - x^2}\,dx$$

Put $x = 3\sin\theta$, then $dx = 3\cos\theta\,d\theta$

When $x = 0$, $\sin\theta = 0$

or, $\sin\theta = \sin 0$

$\therefore$ $\theta = 0$

When $x = 3$, $3\sin\theta = 3$

or, $\sin\theta = 1$

or, $\sin\theta = \sin\frac{\pi}{2}$

$\therefore$ $\theta = \frac{\pi}{2}$

$\therefore$ Area of the circle $= 4\int_0^{\pi/2} \sqrt{9 - 9\sin^2\theta} \times 3\cos\theta\,d\theta$

$$= 4\int_0^{\pi/2} 3\cos\theta \times 3\cos\theta\,d\theta$$

$$= 4 \times 9\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 36\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= 36 \times \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= 18 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 18\left[\left(\frac{\pi}{2} + \frac{\sin 2 \times \pi/2}{2}\right) - \left(0 + \frac{\sin 2 \times 0}{2}\right)\right]$$

$$= 18 \left[\frac{\pi}{2} + \frac{\sin \pi}{2} - \frac{\sin 0}{2}\right]$$

$$= 18 \left[\frac{\pi}{2} + \frac{0}{2} - \frac{0}{2}\right]$$

$$= 9\pi \text{ sq. units}$$

b. The equation of circle is

$$x^2 + y^2 = a^2$$

or, $y = \sqrt{a^2 - x^2}$ ... (i)

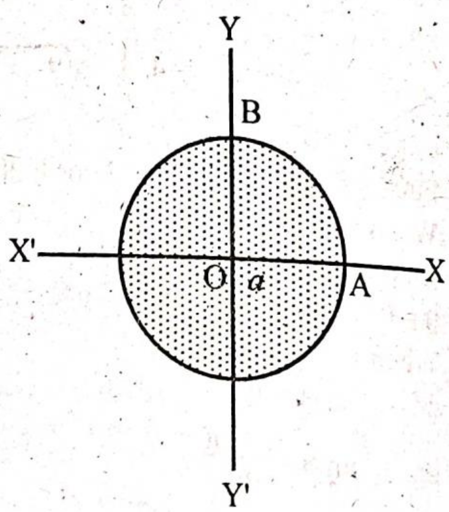
The centre of the given circle is at origin (0, 0) and its radius is a. since the given circle is symmetrical about axes, x-axis and y-axis divide it into four equal parts as shown in the figure.

OAB is the portion of the circle lying in the first quadrant which is bounded by the given curve and the axes.

At A, $y = 0 \Rightarrow x^2 = a^2$

∴ $x = a$

At O, $x = y = 0$



∴ Area of OAB $= \int_0^a y\, dx$

Hence,

Area of the circle $= 4 \times$ Area of OAB

$$= 4 \int_0^a y\, dx$$

$$= 4 \int_0^a \sqrt{a^2 - x^2}\, dx$$

Put $x = a \sin\theta$, then $dx = a \cos\theta\, d\theta$

When $x = 0$,

$\sin\theta = 0$

or, $\sin\theta = \sin 0$

∴ $\theta = 0$

When $x = a$,

$a \sin\theta = a$

or, $\sin\theta = 1$

or, $\sin\theta = \sin\frac{\pi}{2}$

∴ $\theta = \frac{\pi}{2}$

Area of the circle $= 4\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta \, d\theta$

$$= 4\int_0^{\pi/2} a\cos\theta \times a\cos\theta \, d\theta$$

$$= 4a^2\int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= 4a^2\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{4a^2}{2}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 2a^2\left[\left(\frac{\pi}{2} + \frac{\sin 2\times\pi/2}{2}\right) - \left(0 + \frac{\sin 2\times 0}{2}\right)\right]$$

$$= 2a^2\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - \frac{\sin 0}{2}\right]$$

$$= 2a^2\left[\frac{\pi}{2} + \frac{0}{2} - \frac{0}{2}\right] \quad \left[\because \begin{array}{l}\sin\pi = 0\\ \sin 0 = 0\end{array}\right]$$

$= \pi a^2$ sq. units.

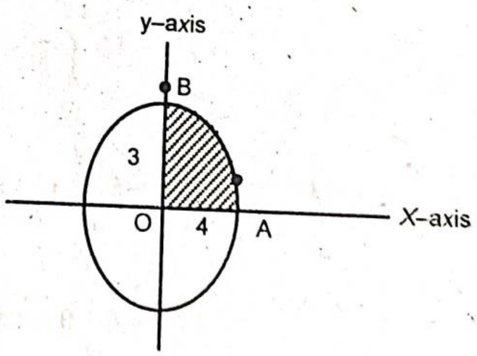
10. a. Find the area enclosed by ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, using method of integration.

b. Using integration, find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

a. The given curve is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \ldots (1)$$



or, $\frac{y^2}{9} = 1 - \frac{x^2}{16}$

or, $\frac{y^2}{9} = \frac{16-x^2}{16}$

or, $y^2 = \frac{9}{16}(16-x^2)$

$\therefore y = \frac{3}{4}\sqrt{16-x^2} \quad \ldots (2)$

The given curve is symmetrical about axes. So the coordinate axes divide the area enclosed by the given curve in four equal parts as shown in the figure.

So, area of ellipse = 4 (Area of the portion OAB shaded on the figure lying in 1st quadrant).

Here, OA = 4, OB = 3

$\therefore$ Area of the portion OAB $= \int_0^4 y\,dx$

$$= \int_0^4 \frac{3}{4}\sqrt{16-x^2}\,dx \qquad [\because \text{From eq}^n\ (2)]$$

$$= \frac{3}{4}\int_0^4 \sqrt{16-x^2}\,dx \quad \ldots (3)$$

Put $x = 4\sin\theta$

Differentiating both sides with respect to θ, we get,

$\therefore \quad \frac{dx}{d\theta} = 4\cos\theta$

or, $\quad dx = 4\cos\theta\, d\theta$

And, $\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\cos\theta$

When $x = 0$,

$4\sin\theta = 0$

or, $\quad \sin\theta = 0$

or, $\quad \sin\theta = \sin 0$

$\therefore \quad \theta = 0$

When $x = 4$,

$4\sin\theta = 4$

or, $\quad \sin\theta = 1$

or, $\quad \sin\theta = \sin\frac{\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{2}$

So form (3)

Area of portion OAB $= \frac{3}{4}\int_0^{\pi/2} 4\cos\theta \times 4\cos\theta\, d\theta$

$$= 12\int_0^{\pi/2} \cos^2\theta\, d\theta = 12\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 6\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= 6\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 6\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$

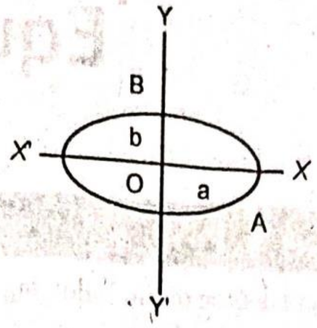
$$= 6\left[\left(\frac{\pi}{2} + \frac{0}{2}\right) - 0\right] = 6 \times \frac{\pi}{2}$$

$$= 3\pi \text{ sq. units}$$

Hence, the area of ellipse $= 4 \times 3\pi = 12\pi$ sq. units.

Given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The curve is symmetrical about *x-axis* and y-axis. So, to find the area of the whole ellipse, we first find the area of the portion lying in the first quadrant and then multiply it by 4. The area of the portion lying in the first quadrant is bounded by the curve, *x-axis* and the ordinates $x = 0$ and $x = a$. So its area is



$$A = \int_0^a y\, dx$$

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx \quad \ldots(i)$$

Put $x = a \sin\theta$

Then, $dx = a\cos\theta\, d\theta$

$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$

When $x = 0$, $\theta = 0$

When $x = a$, $\theta = \frac{\pi}{2}$

Then, from (i)

$$A = \frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= ab\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= ab\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{ab}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - 0\right]$$

$$= \frac{ab\pi}{4}$$

$\therefore$ The whole area of the ellipse $= 4$(area of the portion lying in the first quadrant)

$$= 4 \cdot \frac{ab\pi}{4}$$

$$= \pi ab \text{ sq. units}$$

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UNIT 4

Differential Equations

Exercise 4.1

1. Find the order and degree of the following differential equations.

a. $\frac{dy}{dx} = 2$

b. $\frac{d^2y}{dx^2} = \sin x$

c. $x\frac{d^3y}{dx^3} + y + \left(\frac{dy}{dx}\right)^4 = 0$

d. $(y'')^3 + 4y' = e^x$

e. $\frac{d^2y}{dx^2} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3}$

Solution

a. order = 1, degree = 1

b. order = 2, degree = 1

c. order = 3, degree = 1

d. order = 2, degree = 3

e. order = 2, degree = 2

2. Solve the following differential equations using separation of variables.

a. $x\,dx - y\,dy = 0$

b. $\frac{dy}{dx} = \frac{y}{x}$

c. $\frac{dy}{dx} = \frac{x^3+1}{y^3+1}$

d. $(1 + x^2)y' = 1$

e. $y\,dx - x\,dy = xy\,dx$

f. $(xy^2 + x)dx + (yx^2 + y)dy = 0$

g. $\frac{dy}{dx} = \frac{e^x+1}{y}$

h. $\frac{dy}{dx} = e^{x-y} + e^{-y}$

i. $e^{x-y}dx + e^{y-x}dy = 0$

j. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

k. $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

Solution

a. $x dx - y dy = 0$

or, $x dx = y dy$

Integrating both sides, we get,

$\int x dx = \int y\, dy + \frac{c}{2}$

or, $\frac{x^2}{2} = \frac{y^2}{2} + \frac{c}{2}$

$\therefore$ $x^2 - y^2 = c$

b. $\frac{dy}{dx} = \frac{y}{x}$

or. $\frac{dy}{y} = \frac{dx}{x}$

Integrating, we have,

$$\int \frac{dx}{x} = \int \frac{dy}{y} + \log c$$

$\log x = \log y + \log c$

or. $\log x = \log (cy)$

or. $\log x = \log (cy)$

∴ $x = cy$

∴ Given,

c. $\frac{dy}{dx} = \frac{x^3 + 1}{y^3 + 1}$

or. $(y^3 + 1)\, dy = (x^3 + 1)\, dx$

Integrating, we get

$\frac{y^4}{4} + y = \frac{x^4}{4} + x + c$

d. Given,

$(1 + x^2)\frac{dy}{dx} = 1$

or, $\frac{dy}{dx} = \frac{1}{1 + x^2}$

or, $dy = \frac{1}{1 + x^2} dx$

Integrating, we have

$$\int dy = \int \frac{1}{1 + x^2} dx + c$$

∴ $y = \tan^{-1} x + c$

e. $y\, dx - x\, dy = xy\, dx$

Diving both sides by xy

$\frac{dx}{x} - \frac{dy}{y} = dx$

Integrating, we have,

$$\int \frac{dx}{x} - \int \frac{dy}{y} = \int dx + c$$

or, $\log x - \log y = x + c$

or, $\log \left(\frac{x}{y}\right) = x + c$

f. Here,

$\frac{dy}{dx} = \frac{e^x + 1}{y}$

or, $y\, dy = (e^x + 1)\, dx$

Integrating, we have

$\frac{y^2}{2} = e^x + x + c$

or, $y^2 = 2e^x + 2x + 2c_1$

∴ $y^2 = 2e^x + 2x + c$, where $C = 2c_1$

g. Here,

$(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

or, $x(y^2 + 1)\,dx + y(x^2 + 1)\,dy = 0$

Dividing both sides by $(x^2 + 1)(y^2 + 1)$, we get

or, $\frac{x}{x^2+1}\,dx + \frac{y}{y^2+1}\,dy = 0$

or, $\frac{2x\,dx}{x^2+1} + \frac{2y\,dy}{y^2+1} = 0$

Integrating, we get

$\log(x^2 + 1) + \log(y^2 + 1) = \log c$

or, $\log(x^2 + 1)(y^2 + 1) = \log c$

$\therefore$ $(x^2 + 1)(y^2 + 1) = c$

h. $\frac{dy}{dx} = e^{x-y} + e^{-y}$

or, $\frac{dy}{dx} = e^x.e^{-y} + e^{-y}$

or, $\frac{dy}{dx} = e^{-y}(e^x + 1)$

or, $\frac{dy}{e^{-y}} = (e^x + 1)dx$

or, $e^y\,dy = (e^x + 1)\,dx$

Integrating, we have,

$\int e^y\,dy = \int(e^x + 1)\,dx + c$

$e^y = e^x + x + c$

i. Here,

$e^{x-y}.\,dx + e^{y-x}\,dy = 0$

or, $\frac{e^x}{e^y}\,dx + \frac{e^y}{e^x}\,dy = 0$

or, $e^{2x}\,dx + e^{2y}\,dy = 0$

Integrating we have

$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{c}{2}$

$\therefore$ $e^{2x} + e^{2y} = c$

j. Given,

$\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

Diving both sides by $\tan x \tan y$, we get

$\frac{\sec^2 x}{\tan x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$

Integrating, we have,

$\int \frac{\sec^2 x}{\tan x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = \log c$

or, $\log(\tan x) + \log(\tan y) = \log c$ $\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c\right]$

or, $\log(\tan x \tan y) = \log c$

$\therefore$ $\tan x \tan y = c$

k. $\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2x} = 0$

or, $\frac{dy}{dx} = -\frac{1 + \cos 2y}{1 - \cos 2x}$

$$\frac{dy}{dx} = -\frac{2\cos^2 y}{2\sin^2 x}$$

or, $\frac{dy}{\cos^2 y} = -\frac{dx}{\sin^2 x}$

or, $\sec^2 y\, dy = -\operatorname{cosec}^2 x\, dx$

or, Integrating, we get,

$\int(-\operatorname{cosec}^2 x) dx = \int \sec^2 y\, dy + c$

∴ $\cot x = \tan y + c$

3. Solve the initial value problems

a. $\frac{dy}{dx} = 2x - 7, y(2) = 0$

b. $\frac{dy}{dx} = 10 - x, y(0) = -1$

c. $\frac{dy}{dx} = 9x^2 - 4x + 5, y(-1) = 0$

Solution

a. $\frac{dy}{dx} = 2x - 7$

or, $dy = (2x - 7)dx$

Integrating, we have,

$\int dy = \int (2x - 7)\, dx + c$

$y = \frac{2x^2}{2} - 7x + c$

or, $y = x^2 - 7x + c \quad \ldots (i)$

Given, $y(2) = 0$

i.e. when $x = 2, y = 0$

Then, From (i)

$0 = 2^2 - 7 \times 2 + c$

or, $c = 10$

Putting the value of c in (i)

$y = x^2 - 7x + 10$

b. Given, $\frac{dy}{dx} = 10 - x$

$dy = (10 - x)\, dx$

Integrating, we have,

$\int dy = \int (10 - x)dx + c$

$y = 10x - \frac{x^2}{2} + c \quad \ldots (i)$

Given $y(0) = -1$

i.e. when $x = 0, y = -1$

Then, from (i)

$-1 = 10 \times 0 - \frac{0^2}{2} + c$

∴ $c = -1$

Putting the value of c in (i)

$y = 10x - \frac{x^2}{2} - 1$

c. $\frac{dy}{dx} = 9x^2 - 4x + 5$

$dy = (9x^2 - 4x + 5)\, dx$

Integrating,

$$y = \frac{9x^3}{3} - \frac{4x^2}{2} + 5x + c$$

or, $y = 3x^3 - 2x^2 + 5x + c$ ... (i)

By given, $y(-1) = 0$

i.e. when $x = -1, y = 0$

Then, from (i),

$0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + c$

or, $0 = -3 - 2 - 5 + c$

$\therefore$ $c = 10$

Putting the value of c in (i),

$y = 3x^3 - 2x^2 + 5x + 10.$

4. Solve (Change of variables)

a. $\frac{dy}{dx} = \frac{1}{x+y+5}$

b. $\frac{dy}{dx} = \cos(x+y)$

c. $1 - \frac{dy}{dx} = e^{x-y}$

d. $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Solution

a. Given equation is

$$\frac{dy}{dx} = \frac{1}{x+y+5} \quad \text{... (i)}$$

Put $v = x + y + 5$. Then

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

$\therefore$ $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Now, equation (i) can be written as

$$\frac{dv}{dx} - 1 = \frac{1}{v}$$

or, $\frac{dv}{dx} = \frac{1}{v} + 1$

or, $\frac{dv}{dx} = \frac{v+1}{v}$

or, $\left(\frac{v}{v+1}\right) dv = dx$

or, $\left(\frac{v+1-1}{v+1}\right) dv = dx$

or, $\left(\frac{v+1}{v+1} - \frac{1}{v+1}\right) dv = dx$

or, $\left(1 - \frac{1}{v+1}\right) dv = dx$

Integrating, we get

$v - \ln(v+1) = x + c'$ ...(ii)

Putting the value of v in equation (ii), we get

$(x+y+5) - \ln(x+y+5+1) = x + c'$

or, $y - \ln(x+y+6) = c' - 5$

$\therefore$ $y = \ln(x+y+6) + c$ where $c = c' - 5$

b. Given equation is

$$\frac{dy}{dx} = \cos(x+y) \quad \ldots \text{(i)}$$

Put $v = x + y$. Then

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

$\therefore$ $$\frac{dv}{dx} - 1 = \frac{dy}{dx}$$

Now, equation (i) can be written as

$$\frac{dv}{dx} - 1 = \cos v$$

or, $$\frac{dv}{dx} = 1 + \cos v$$

or, $$\frac{dv}{1+\cos v} = dx$$

or, $$\frac{dv}{2\cos^2\frac{v}{2}} = dx$$

or, $$\frac{1}{2}\sec^2\frac{v}{2}\, dv = dx$$

Integrating, we get

$$\tan\frac{v}{2} = x + c$$

$\therefore$ $$\tan\left(\frac{x+y}{2}\right) = x + c$$

c. Given ODE is

$$1 - \frac{dy}{dx} = e^{x-y} \quad \ldots \text{(i)}$$

Put $v = x - y$. Then

$$\frac{dv}{dx} = 1 - \frac{dy}{dx}$$

Now, equation (i) can be written as

$$\frac{dv}{dx} = e^{v}$$

or, $$\frac{dv}{e^{v}} = dx$$

or, $$e^{-v}\, dv = dx$$

Integrating, we get

$$\frac{e^{-v}}{-1} = x + c$$

or, $$-e^{-v} = x + c$$

or, $$-e^{-(-x-y)} = x + c \qquad [\because v = x - y]$$

or, $$-1 = \frac{x+c}{e^{-(x-y)}}$$

or, $$(x+c)e^{x-y} = -1$$

$\therefore$ $$(x+c)\, e^{x-y} + 1 = 0$$

d. Given equation is

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y) \quad \ldots(i)$$

Put $v = x + y$. Then

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

$$\therefore \quad \frac{dv}{dx} - 1 = \frac{dy}{dx}$$

Now, equation (i) can be written as

$$\frac{dv}{dx} - 1 = \sin v + \cos v$$

or, $$\frac{dv}{dx} = \sin v + (1 + \cos v)$$

or, $$\frac{dv}{dx} = 2\sin\frac{v}{2}\cos\frac{v}{2} + 2\cos^2\frac{v}{2}$$

or, $$\frac{dv}{2\sin\frac{v}{2}\cos\frac{v}{2} + 2\cos^2\frac{v}{2}} = dx$$

or, $$\frac{\frac{1}{2}\sec^2\frac{v}{2}\,dv}{1 + \tan\frac{v}{2}} = dx$$

Integrating, we get

$$\ln\left(1 + \tan\frac{v}{2}\right) = x + c \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \ln f(x) + c\right]$$

$$\therefore \quad \ln\left[1 + \tan\left(\frac{x+y}{2}\right)\right] = x + c \qquad [\because v = x + y]$$

Exercise 4.2

Solve the following differential equations:

1. $\frac{dy}{dx} = \frac{x+y}{x}$

Solution

$\frac{dy}{dx} = \frac{x+y}{x}$

$\frac{dy}{dx} = 1 + \frac{y}{x}$ ... (i)

The given equation is of the form $\frac{dy}{dx} \varphi\left(\frac{y}{x}\right)$

Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) can be written as

$v + x\frac{dv}{dx} = 1 + \frac{vx}{x}$

or, $v + \frac{xdv}{dx} = 1 + v$

or, $x\frac{dv}{dx} = 1$

or, $dv = \frac{dx}{x}$

Integrating, we get,

$\int dv = \int \frac{dx}{x} + \log c$

or, $v = \log x + \log c$

or, $v = \log cx$

or, $\frac{y}{x} = \log cx$

$\therefore$ $y = x \log cx$.

2. $\frac{dy}{dx} = \frac{2x+y}{x}$

Solution

Given,

$\frac{dy}{dx} = \frac{2x+y}{x}$ ...(i)

So, put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) can be written as

$v + x\frac{dv}{dx} = \frac{2x}{x} + \frac{vx}{x}$

or, $v + x\frac{dv}{dx} = 2 + v$

or, $x\frac{dv}{dx} = 2$

or, $dv = 2\frac{dx}{x}$

Integrating, we have,

$\int dv = 2\int \frac{dx}{x} + \log c$

or, $v = 2\log x + \log c$

or, $v = \log x^2 + \log c$

or, $\frac{y}{x} = \log cx^2$

$\therefore$ $y = x\log cx^2$

3. $\frac{dy}{dx} = \frac{2y - x}{x}$

Solution

$\frac{dy}{dx} = \frac{2y - x}{x}$ . . . (i)

This is a homogeneous equation.

So, put $y = vx$. Then

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) can be written as

$v + x\frac{dv}{dx} = \frac{2.vx}{x} - \frac{x}{x}$

or, $x\frac{dv}{dx} = 2v - v - 1$

or, $x\frac{dv}{dx} = v - 1$

or, $\frac{dv}{v-1} = \frac{dx}{x}$

Integrating, we have

$\log(v - 1) = \log x + \log c$

or, $\log(v - 1) = \log cx$

or, $v - 1 = cx$

or, $\frac{y}{x} - 1 = cx$

or, $\frac{y - x}{x} = cx$

$\therefore$ $y - x = cx^2$

4. $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

Solution

Here,

$(x^2 + y^2)\,dy = xy\,dx$

or, $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ ...(i)

Put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) can be written as

$v + x\frac{dv}{dx} = \frac{x.\,vx}{x^2 + v^2x^2}$

or, $x\frac{dv}{dx} = \frac{v}{1 + v^2} - v$

or, $x\frac{dv}{dx} = \frac{v - v - v^3}{1 + v^2}$

or, $x\dfrac{dv}{dx}=\dfrac{-v^3}{1+v^2}$

or, $\dfrac{1+v^2}{v^3}dv=\dfrac{-dv}{x}$

or, $\dfrac{1}{v^3}dv+\dfrac{1}{v}dv+\dfrac{dx}{x}=0$

or, $\dfrac{1}{v}dv+\dfrac{dx}{x}=-\dfrac{1}{v^3}dv$

Integrating, we get

$\log v+\log x+\log c=\dfrac{-v^{-3+1}}{(-3+1)}$

or, $\log(v.x.c)=\dfrac{1}{2v^2}$

or, $\log\left(\dfrac{y}{x}.x.c\right)=\dfrac{1}{2\left(\dfrac{y}{x}\right)^2}$

or, $\log(cy)=\dfrac{x^2}{2y^2}$

∴ $x^2=2y^2\log cy$

5. $2xy\dfrac{dy}{dx}=x^2+y^2$

Solution

Given,

$\dfrac{dy}{dx}=\dfrac{x^2+y^2}{2xy}$ . . . (i)

This is a homogeneous equation.

Put $y=vx$. Then, $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

Now, equation (i) can be written as

$v+x\dfrac{dv}{dx}=\dfrac{x^2+v^2x^2}{2x.vx}$

or, $v+x\dfrac{dv}{dx}=\dfrac{1+v^2}{2v}$

or, $x\dfrac{dv}{dx}=\dfrac{1+v^2}{2v}-v$

or, $x\dfrac{dv}{dx}=\dfrac{1+v^2-2v^2}{2v}$

or, $x\dfrac{dv}{dx}=\dfrac{1-v^2}{2v}$

or, $\dfrac{2v}{v^2-1}dv=-\dfrac{dx}{x}$

or, $\dfrac{2v}{v^2-1}dv+\dfrac{dx}{x}=0$

Integrating,

$\log(v^2-1)+\log x=\log c$

or, $\log((v^2-1).x)=\log c$

or, $x(v^2-1)=c$

or, $x\left(\dfrac{y^2}{x^2}-1\right)=c$

or, $\dfrac{x(y^2-x^2)}{x^2}=c$

∴ $y^2-x^2=cx$

6. $xy\dfrac{dy}{dx} = x^2 - y^2$

Solution

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy} \qquad \ldots (i)$$

This is a homogeneous equation.

Put $y = vx$. Then, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now, equation (i) can be written as

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{x.vx}$$

or, $v + x\dfrac{dv}{dx} = \dfrac{1 - v^2}{v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 - v^2}{v} - v$

or, $x\dfrac{dv}{dx} = \dfrac{1 - v^2 - v^2}{v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 - 2v^2}{v}$

or, $x\dfrac{dv}{dx} = -\dfrac{(2v^2 - 1)}{v}$

or, $\dfrac{vdv}{2v^2 - 1} = -\dfrac{dx}{x}$

or, $\dfrac{4vdv}{2v^2 - 1} = -4\dfrac{dx}{x}$

Integrating, we have,

$\log(2v^2 - 1) = -4\log x + \log c$

or, $\log(2v^2 - 1) + 4\log x = \log c$

or, $\log(2v^2 - 1) + \log x^4 = \log c$

or, $\log x^4(2v^2 - 1) = \log c$

or, $x^4(2v^2 - 1) = c$

or, $x^4\left\{2\left(\dfrac{y}{x}\right)^2 - 1\right\} = c$

or, $x^4\dfrac{(2y^2 - x^2)}{x^2} = c$

or, $x^2(2y^2 - x^2) = c$

$\therefore$ $x^2(x^2 - 2y^2) + c = 0$

7. $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$

Solution

Given, equation is $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$ $\qquad \ldots (i)$

This is a homogeneous equation.

So, put $y = vx$. Then, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Now, equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx}$$

or, $v + x\dfrac{dv}{dx} = \dfrac{1 + v}{1 - v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 + v}{1 - v} - v$

or, $x\frac{dv}{dx}=\frac{1+v-v(1-v)}{1-v}$

or, $x\frac{dv}{dx}=\frac{1+v-v+v^2}{1-v}$

or, $x\frac{dv}{dx}=\frac{1+v^2}{1-v}$

or, $\frac{1-v}{1+v^2}dv=\frac{dx}{x}$

or, $\left(\frac{1}{1+v^2}-\frac{v}{1+v^2}\right)dv=\frac{dx}{x}$

Integrating

$$\int\frac{1}{1+v^2}dv-\frac{1}{2}\int\frac{2v}{1+v^2}dv=\int\frac{dx}{x}+\log c$$

or, $\tan^{-1}v-\frac{1}{2}\log(1+v^2)=\log x+\log c$

or, $\tan^{-1}v-\log(1+v^2)^{\frac{1}{2}}=\log cx$

or, $\tan^{-1}v=\log\sqrt{1+v^2}+\log cx$

or, $\tan^{-1}v=\log cx\sqrt{1+v^2}$

or, $\tan^{-1}\left(\frac{y}{x}\right)=\log cx\sqrt{1+\frac{y^2}{x^2}}$

or, $\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}+c$

or, $\tan^{-1}\left(\frac{y}{x}\right)=\log(x^2+y^2)^{\frac{1}{2}}+c$

$\therefore$ $\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log(x^2+y^2)+c$

8. $\frac{dy}{dx}=\frac{y}{x}-\sin^2\frac{y}{x}$

Solution

Given equation is, $\frac{dy}{dx}=\frac{y}{x}-\sin^2\frac{y}{x}$

Put $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Now, the given equation can be written as

$v+x\frac{dv}{dx}=v-\sin^2v$

or, $x\frac{dv}{dx}=-\sin^2v$

or, $\frac{-dv}{\sin^2v}=\frac{dx}{x}$

Integrating, we have

$$\int-\operatorname{cosec}^2v\,dv=\int\frac{dx}{x}+c$$

or, $\cot v=\log x+c$

$\therefore$ $\cot\left(\frac{y}{x}\right)=\log x+c$

Exercise 4.3

Solve the following differential equations by reducing to exact form.

1. $x\,dy + y\,dx = 0$

Solution

$xdy + ydx = 0$

or, $d(xy) = 0$

Integrating, we have,

$xy = c$

2. $2xy\,dy + y^2\,dx = 0$

Solution

$ydx - xdy = 0$

Dividing both sides by y^2, we get,

$$\frac{ydx - xdy}{y^2} = 0$$

or, $d\left(\frac{x}{y}\right) = 0$

Integrating, we get,

$\frac{x}{y} = c$

$\therefore$ $x = cy$

3. $y\,dx - x\,dy = 0$

Solution

$2xydy + y^2dx = 0$

or, $x.2ydy + y^2\,dx = 0$

or, $xd(y^2) + y^2\,d(x) = 0$

or, $d(xy^2) = 0$

Integrating, we get

$\therefore$ $xy^2 = c$

4. $2xy\,dx - x^2\,dy = 0$

Solution

$2xydx - x^2dy = 0$

or, $y.d(x^2) - x^2d(y) = 0$

Dividing both sides by y^2

$$\frac{yd(x) - x^2d(y)}{y^2} = 0$$

or, $d\left(\frac{x^2}{y}\right) = 0$

Integrating, we get $\frac{x^2}{y} = c$

$\therefore$ $x^2 = cy$

$y\,dx + (x + y)\,dy = 0$

Solution

$y\,dx + (x + y)\,dy = 0$

$y\,dx + x\,dy + y\,dy = 0$

$(y\,dx + x\,dy) + y\,dy = 0$

$d(xy) + d\left(\frac{y^2}{2}\right) = 0$

$d\left(xy + \frac{y^2}{2}\right) = 0$

Integrating, we get, $xy + \frac{y^2}{2} = \frac{c}{2}$

$2xy + y^2 = c$

$(2xy + y^2)\,dy + (y^2 + x)\,dx = 0$

Solution

$2xy\,dy + y^2\,dy + y^2\,dx + x\,dx = 0$

$2xy\,dy + y^2\,dx + y^2\,dy + x\,dx = 0$

$x\,d(y^2) + y^2\,d(x) + d\left(\frac{y^3}{3}\right) + d\left(\frac{x^2}{2}\right) = 0$

$d(xy^2) + d\left(\frac{y^3}{3}\right) + d\left(\frac{x^2}{2}\right) = 0$

$d\left(xy^2 + \frac{y^3}{3} + \frac{x^2}{2}\right) = 0$

Integrating, we get,

$xy^2 + \frac{y^3}{3} + \frac{x^2}{2} = \frac{c}{6}$

$6xy^2 + 2y^3 + 3x^2 = c$

$3x^2 + 6xy^2 + 2y^3 = c$

$\frac{dy}{dx} = \frac{y - x + 1}{y - x + 5}$

Solution

Given,

$\frac{dy}{dx} = \frac{y - x + 1}{y - x + 5}$

$ydy - xdy + 5dy = ydx - xdx + dx$

$xdx + ydy - xdy - ydx - dx + 5dy = 0$

$xdx + ydy - (xdy + ydx) - dx + 5dy = 0$

$d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right) - d(xy) - d(x) + d(5y) = 0$

$d\left(\frac{x^2}{2} + \frac{y^2}{2} - xy - x + 5y\right) = 0$

Integrating, we get,

$\frac{x^2}{2} + \frac{y^2}{2} - xy - x + 5y = \frac{c}{2}$

$x^2 + y^2 - 2xy - 2x + 10y = c$

8. $(x^2 + 5xy^2)dx + (5x^2y + y^2)dy = 0$

Solution

$(x^2 + 5xy^2)dx + (5x^2y + y^2)dy = 0$

or, $x^2\,dx + 5xy^2\,dx + 5x^2\,y\,dy + y^2\,dy = 0$

or, $x^2\,dx + 5(xy^2\,dx + x^2\,y\,dy) + y^2\,dy = 0$

or, $d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + 5.\frac{1}{2}d(x^2y^2) = 0$

or, $d\left(\frac{x^3}{3} + \frac{y^3}{3} + \frac{5}{2}x^2y^2\right) = 0$

Integrating, we get,

$\frac{x^3}{3} + \frac{y^3}{3} + \frac{5x^2y^2}{2} = \frac{c}{6}$

$\therefore\ 2x^3 + 2y^3 + 15x^2y^2 = c$

9. $\sin x \cos x\,dx + \sin y \cos y\,dy = 0$

Solution

$\sin x \cos x\,dx + \sin y \cos y\,dy = 0$

or, $2\sin x \cos x\,dx + 2\sin y \cos y\,dy = 0$

or, $d(\sin^2 x) + d(\sin^2 y) = 0$

or, $d(\sin^2 x + \sin^2 y) = 0$

Integrating, we get,

$\sin^2 x + \sin^2 y = c$

Exercise 4.4

Solve the following linear differential equations.

1. $\frac{dy}{dx}+y=1$

Solution

$\frac{dy}{dx}+y=1$ ... (i)

Comparing (i) with $\frac{dy}{dx}+Py=Q$, we get,

$P=1, Q=1$

I.F. $e^{\int P dx}=e^{\int 1\, dx}=e^x$.

Multiplying both sides of (i) by e^x, we have

$e^x.\frac{dy}{dx}+e^x.y=e^x$

or, $d(y.e^x)=e^x\,dx$

Integrating, we get,

$ye^x=\int e^x\,dx+c$

or, $ye^x=e^x+c$

∴ $y=1+ce^{-x}$

2. $\frac{dy}{dx}-y=e^x$

Solution

$\frac{dy}{dx}-y=e^x$ ... (i)

Comparing (i) with $\frac{dy}{dx}+Py=Q$, we get,

$P=-1, Q=e^x$

I.F. $e^{\int P dx}\ e^{\int -dx}=e^{-x}$.

Multiplying both sides of (i) by e^{-x}, we have,

$e^{-x}\frac{dy}{dx}-e^{-x}.y=e^{-x}.e^x$.

or, $\frac{d}{dx}(y.e^{-x})=1$

or, $d(y.e^{-x})=dx$

Integrating, we have,

$y.e^{-x}=x+c$

∴ $y=e^x(x+c)$

3. $\frac{dy}{dx}+\frac{y}{x}=x$

Solution

$\frac{dy}{dx}+\frac{y}{x}=x$ ... (i)

Comparing (i) with $\frac{dy}{dx}+Py=Q$, we get,

$P=\frac{1}{x}, Q=x$

I.F. $= e^{\int P\,dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Multiplying both sides of (i) by x, we get

$$x.\frac{dy}{dx} + x\frac{y}{x} = x.x$$

or, $\frac{d}{dx}(x.y) = x^2$

or, $d(xy) = x^2\,dx$

Integrating,

$xy = \int x^2\,dx + c$

or, $xy = \frac{x^3}{3} + c$

4. $x\frac{dy}{dx} + y = x^4$

Solution

$x\frac{dy}{dx} + y = x^4$

or, $\frac{dy}{dx} + \frac{y}{x} = x^3 \qquad \ldots (i)$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$P = \frac{1}{x}, Q = x^3$

I.F. $= e^{\int P\,dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Multiplying both sides of (i) by x,

$x.\frac{dy}{dx} + x.\frac{y}{x} = x.x^3$

or, $\frac{d}{dx}(x.y) = x^4$

or, $d(xy) = x^4\,dx$

Integrating, $xy = \frac{x^5}{5} + c.$

5. $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Solution

Here, $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

or, $\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{4x^2}{1 + x^2} \qquad \ldots (i)$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{2x}{1 + x^2}$ and $Q = \frac{4x^2}{1 + x^2}$

Now, I.F $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1 + x^2)} = 1 + x^2$

Multiplying both sides of (i) by $1 + x^2$, we get,

$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

or, $\frac{d}{dx}\{y(1 + x^2)\} = 4x^2$

or, $d\{y(1 + x^2)\} = 4x^2\,dx$

Integrating, we get,

$y(1+x^2) = \int 4x^2\,dx + c$

or, $y(1+x^2) = 4 \cdot \frac{x^3}{3} + c$

$\therefore$ $y(1+x^2) = \frac{4x^3}{3} + c$

6. $\frac{dy}{dx} + 2y \tan x = \sin x$

Solution

Given equation is $\frac{dy}{dx} + 2y \tan x = \sin x$ ... (i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$P = 2\tan x$, $Q = \sin x$.

Now,

$$\begin{aligned} \text{I.F} &= e^{\int P\,dx} = e^{\int 2\tan x\,dx} \\ &= e^{2\int \tan x\,dx} \\ &= e^{2\log \sec x} \\ &= e^{\log \sec^2 x} \\ &= \sec^2 x \end{aligned}$$

Multiplying both sides of (i) by $\sec^2 x$

$\sec^2 x.\frac{dy}{dx} + \sec^2 x.\ 2y.\ \tan x = \sec^2 x.\sin x$

or, $d(y.\sec^2 x) = \sec x \tan x\,dx$

Integrating, we get

$y \sec^2 x = \int \sec x \tan x\,dx + c$

$\therefore$ $y \sec^2 x = \sec x + c$.

7. $\sin x \frac{dy}{dx} + y \cos x = x \sin x$

Solution

Given,

$\sin x \frac{dy}{dx} + \cos x.y = x \sin x$

Dividing both sides by $\sin x$,

$\frac{dy}{dx} + \frac{\cos x}{\sin x} y = x$ ... (i)

Comparing (i) with

$\frac{dy}{dx} + Py = Q$, we get

$P = \frac{\cos x}{\sin x}$, $Q = x$

$$\begin{aligned} \text{I.F} &= e^{\int P dx} \\ &= e^{\int \frac{\cos x}{\sin x} dx} \\ &= e^{\log \sin x} \\ &= \sin x \end{aligned}$$

Multiplying both sides of equation (i) by $\sin x$

$\sin x \frac{dy}{dx} + \cos x.\ y = x \sin x$

$d(y.\sin x) = x \sin x\,dx$

Integrating,

$y \sin x = \int x \sin x \, dx + c$

or, $y \sin x = x \int \sin x \, dx - \int \left[\frac{dx}{dx} \int \sin x \, dx \right] dx + c$

or, $y \sin x = -x \cos x - \int 1 \, (-\cos x) \, dx + c$

or, $y \sin x = -x \cos x + \sin x + c$

$\therefore$ $y \sin x = \sin x - x \cos x + c$

8. $\cos^2 x \frac{dy}{dx} + y = 1$

Solution

Here,

$\cos^2 x \frac{dy}{dx} + y = 1$

Dividing both sides by $\cos^2 x$, we have,

$\frac{dy}{dx} + \sec^2 x . y = \sec^2 x$ ...(i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$, we get,

P $= \sec^2 x$,

Q $= \sec^2 x$

I.F $= e^{\int P dx} = e^{\int \sec^2 x \, dx} = e^{\tan x}$

Multiplying equation (i) both sides by $e^{\tan x}$,

we get

$e^{\tan x} . \frac{dy}{dx} + e^{\tan x} . \sec^2 x . y = e^{\tan x} . \sec^2 x$

$d(y . e^{\tan x}) = e^{\tan x} . \sec^2 x \, dx$

Integrating, we have,

$y \cdot e^{\tan x} = \int d(e^{\tan x}) + c$

or, $y \cdot e^{\tan x} = e^{\tan x} + c$

$\therefore$ $y = 1 + c \, e^{-\tan x}$

9. $(1 + x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$

Solution

Given equation is:

$(1 + x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$

or, $\frac{dy}{dx} + \frac{1}{1 + x^2} . y = \frac{e^{\tan^{-1} x}}{1 + x^2}$ ...(i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$, we get

P $= \frac{1}{1 + x^2}$, $Q = \frac{e^{\tan^{-1} x}}{1 + x^2}$,

I.F $= e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$

Multiplying both sides of (i) by $e^{\tan^{-1} x}$, we have

$e^{\tan^{-1} x} . \frac{dy}{dx} + e^{\tan^{-1} x} \frac{1}{1 + x^2} \cdot y = \frac{(e^{\tan^{-1} x})^2}{1 + x^2}$

or, $d(y . e^{\tan^{-1} x}) = \frac{(e^{\tan^{-1} x})^2}{1 + x^2} \, dx$

Integrating, we get

$$y\,e^{\tan^{-1}x} = \int \frac{(e^{\tan^{-1}x})^2}{1+x^2}\,dx$$

Put $\tan^{-1}x = z$

Then,

$$\frac{1}{1+x^2}\,dx = dz$$

$$\int \frac{(e^{\tan^{-1}x})^2}{1+x^2}\,dx = \int e^{2z}\,dz$$

$$= \frac{1}{2}e^{2z} + c$$

$$= \frac{1}{2}(e^{\tan^{-1}x})^2 + c$$

$$y\,e^{\tan^{-1}x} = \frac{1}{2}(e^{\tan^{-1}x})^2 + c$$

$$y = \frac{1}{2}e^{\tan^{-1}x} + c\,e^{-\tan^{-1}x}$$

0. $\frac{dy}{dx} + y = xy^2$.

Solution

We have

$$\frac{dy}{dx} + y = xy^2$$

$$y^{-2}\frac{dy}{dx} + \frac{1}{y} = x \qquad \ldots (1)$$

Putting $\frac{1}{y} = z$, then

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$$

Hence, equation (1) becomes

$$-\frac{dz}{dx} + z = x$$

$$\frac{dz}{dx} + (-1)z = -x \qquad \ldots (2)$$

which is linear differential equation

Now

$$\int P\,dx = \int -1\,dx = -x$$

$$\text{I.F.} = e^{\int P\,dx} = e^{-x}$$

Multiplying both sides of (2) by e^{-x}, we get

$$e^{-x}\frac{dz}{dx} + z(-1)\,e^{-x} = -xe^{-x}$$

$$d[e^{-x}z] = -xe^{-x}\,dx \qquad \ldots (3)$$

Integrating (3), we get

$$e^{-x}z = -\int x\,e^{-x}\,dx$$

$$= -\left[x(-e^{-x}) - \int(-e^{-x})\,dx\right]$$

$$= xe^{-x} + e^{-x} + c$$

But $z = \frac{1}{y}$, so the required solution is

$$\frac{1}{y} = x + 1 + ce^x$$

$\therefore$ $xy + y + cy\,e^x = 1$.

11. $\frac{dy}{dx} + y \tan x = y^3 \sec x$

Solution

We have

$$\frac{dy}{dx} + y \tan x = y^3 \sec x$$

or, $y^{-3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \sec x$ . . . (i)

Put $\frac{1}{y^2} = z$, then

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

or, $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

or, $y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$

Equation (1) becomes

$$-\frac{1}{2}\frac{dz}{dx} + z\tan x = \sec x$$

or, $\frac{dz}{dx} - 2\tan x . z = -2\sec x$ . . . (i)

Which is the linear differential equation where $P = -2\tan x$, $Q = 2\sec x$.

So,

$$\int p dx = \int -2\tan x = -2\ln\sec x = \ln\cos^2 x$$

$\therefore$ I.F. $= e^{\int p dx} = e^{\ln\cos x^2} = \cos^2 x$

Multiplying (1) by I.F., we get

$$\cos^2 x\frac{dz}{dx} - 2\sin 2x\, z = -2\cos^2 x \times \sec x$$

or, $\frac{d}{dx}(z.\cos^2 x) = -2\cos x$ . . . (ii)

Integrating both sides of (ii), we get.

$$z.\cos^2 x = -\int 2\cos x$$

$$z\cos^2 x = -2\sin x + c$$

$$\frac{1}{y^2}\cos^2 x = y^2(c - 2\sin x) \quad [\because z = \frac{1}{y}]$$

$\therefore$ $(c - 2\sin x)y^2 = \cos^2 x$

Exercise 4.5

1. Population of a city is increasing at the rate of 3% per year. How long does it take for the population to be doubled?

Solution

Let $y(t)$ be the population at time t.

Then,

$\frac{dy}{dt} = 3\%$ of y

or, $\frac{dy}{dt} = 0.03y$

or, $\frac{dy}{y} = 0.03\, dt$

Integrating,

$\ln y = 0.03\, t + \ln c$

or, $\ln\left(\frac{y}{c}\right) = 0.03t$

or, $y = ce^{0.03t}$ . . . (i)

Let $y(t) = y_0$ at $t = 0$.

Then

$y_0 = ce^{0.03 \times 0}$

or, $c = y_0$

From (i)

$y = y_0 e^{0.03t}$

By question,

When $y = 2y_0$, $t = ?$

So,

$2y_0 = y_0\, e^{0.03t}$

or, $2 = e^{0.03t}$

Taking natural logarithm on both sides,

$\ln 2 = \ln e^{0.03}\, t$

or, $\ln 2 = 0.03\, t$ $[\because \ln e^x = x]$

or, $t = \frac{\ln 2}{0.03} = 23.1$

$\therefore$ $t = 23.1$ years

2. The per capita income of a country is increasing at a rate of 4% per year. When will it be doubled?

Solution

Let $y(t)$ be the per capita income at time t.

Then,

$\frac{dy}{dt} = 4\%$ of y

or, $\frac{dy}{dt} = 0.04y$

or, $\frac{dy}{y} = 0.04\, dt$

Integrating,

$\int \frac{dy}{y} = \int 0.04\, dt$

or, $\ln y = 0.04\, t + \ln c$

or, $\ln\left(\frac{y}{c}\right) = 0.04t$

or, $y = ce^{0.04t}$ . . . (i)

Let y_0 be the per capital at $t = 0$.
Then

$y_0 = ce^{\times 0}$

or, $c = y_0$

By question, when $y = 2y_0$, $t = ?$
Thus,

$2y_0 = y_0\, e^{0.04t}$

or, $2 = e^{0.04t}$

Taking natural logarithm on both sides,

$\ln 2 = \ln e^{0.04}\, t$

or, $\ln 2 = 0.043\, t$ $[\because \ln e^x = x]$

or, $t = \frac{\ln 2}{0.04} = 17.33$

$\therefore$ $t = 17.33$ years

3. A model for the population $y(t)$ in millions of a city at time t is given by

$$\frac{dy}{dt} = -0.08y + 12$$

The population at time $t = 0$ is 200 million.

a. Find $y(15)$ correct to 2 decimal places.

b. Find the value of t for which $y(t) = 155$, correct to 2 decimal place.

Solution

$$\frac{dy}{dt} = -0.08(y - 150)$$

or, $\frac{dy}{d(y-150)} = -0.08dt$

Integrating,

$$\int \frac{dy}{d(y-150)} = \int -0.08dt + \ln c$$

or, $\ln(y - 150) = -0.08t + \ln c$

or, $\ln(y - 150) - \ln c = -0.08t$

or, $\ln\left(\frac{y-150}{c}\right) = -0.08t$

or, $\frac{y-150}{c} = e^{-0.08t}$

or, $y - 150 = ce^{-0.08t}$

or, $y = 150 + ce^{-0.08t}$ . . . (i)

By given, when $t = 0$, $y = 200$
Thus,

$200 = 150 + ce^{-0}$

or, $c = 50$

From (i)

$y(t) = 150 + 50e^{-0.08t}$

a. When $t = 15$,

$y(15) = 150 + 50e^{-0.08 \times 15}$
$= 150 + 15.06$
$= 165.06$

$= 150 + 50e^{-0.08t}$

$y(t) = 150 + 50e^{-0.08t}$

$155 = 50e^{-0.08t}$

$5 = e^{-0.08t}$

0.1

Taking ln on both sides

$\ln 0.1 = -0.08t$

$t = -\frac{1}{0.08}\ln 0.1 = 28.79$

$t = 28.78$ (correct to the decimal place)

4. A culture of bacteria contained 20 million bacteria at 1 PM. At 5 PM, the number of bacteria had increased to 40 million. Assuming that the condition for growth had not changed over the four hour interval, how many bacteria were in the culture at 3 PM.

Solution

Let $y(t)$ be the number of bacteria in t hours.

Then, $y(t) = ce^{kt}$

Initially, at 1 PM, $t = 0$

$y(0) = ce^0$

$20 = c$

or, $c = 20$

∴ At 3 PM, $t = 4$

$y(4) = ce^{4k}$

$40 = 20\, ce^{4k}$

or, $e^{4k} = 2$

or, $4k = \ln 2$

or, $k = \frac{\ln 2}{4}$

∴ Hence, $y(t) = 20\, e^{\left(\frac{\ln 2}{4}\right)t}$

At 3 PM, $t = 2$

∴ $y(2) = 20\, e^{\left(\frac{\ln 2}{4}\right)2}$

$= 20\, e^{\left(\frac{1}{2}\right)\ln 2}$

$= 20\, e^{\ln\sqrt{2}}$

$= 20\sqrt{2}$ million

5. The half-life of isotopic radium is 300 years. Find the time required to decay 10% of its initial amount.

Solution

The solution of decay is of the form

$y(t) = ce^{kt}$

When $t = 0$,

$y(0) = ce^0$

⇒ $c = y(0)$

∴ $y(t) = y(0)\, e^{kt}$

When $t = 300$,

$y(300) = y(0)\, e^{300k}$

By question,

$\frac{1}{2}y(0) = y(0)\, e^{300k}$

or, $\frac{1}{2} = e^{300k}$

or, $300k = \ln(0.5)$

$\therefore$ $k = \frac{\ln(0.5)}{300}$

Thus, $y(t) = y(0)\, e^{\frac{\ln(0.5)}{300}t}$

Let the required time be T.

Then,

$\therefore$ $y(T) = y(0)\, e^{\frac{\ln(0.5)}{300}T}$

or, 90% of $y(0) = y(0)\, e^{\frac{\ln(0.5)}{300}T}$

or, $0.9 = e^{\frac{\ln(0.5)}{300}T}$

or, $\frac{\ln 0.5}{300} T = \ln 0.9$

or, $T = \frac{\ln 0.9 \times 300}{\ln 0.5} = 45.6$ years

6. The rate at which an infection spreads in poultry house is given as $\frac{dP}{dt} = 0.3(3000 - P)$, where t is time in days. Given $P = 0$, $t = 0$.

   a. Solve the differential equation to determine an expression for the number of Poultry P infected at any time t.

   b. Calculate the time taken for 2000 poultry to become infected.

Solution

a. $\frac{dP}{dt} = 0.3(3000 - P)$

or, $\frac{dP}{dt} = -0.3(P - 3000)$

or, $\frac{dP}{P - 3000} = -0.3dt$

Integrating,

$\ln(P - 3{,}000) = -0.3t + \ln C$

or, $\ln\left(\frac{P - 3000}{C}\right) = -0.3t$

or, $\frac{P - 3000}{C} = e^{-0.3t}$

$\therefore$ $P = 3000 + Ce^{-0.3t}$ ... (i)

By given,

$P = 0$, when $t = 0$

From (i),

$0 = 3000 + C \times 1$

$C = -3{,}000$

From (i)

$P = 3000 - 3000e^{-0.3t}$ ... (ii)

b. Given $P = 2000$

From (ii)

$2000 = 3000 - 3000e^{-0.3t}$

or, $e^{-0.3t} = \frac{1}{3}$

or, $t = 3.662$ days.

7. If the demand and supply functions in a competitive market are $Q_d = 32 - 0.5P$ and $Q_s = -8 + 0.3P$ and the rate of adjustment of price when the market is out of equilibrium is $\frac{dP}{dt} = 0.25(Q_d - Q_s)$. Determine and solve the obtained differential equation to get a function for 'P' in terms of 't' given that price is 15 in time period '0'.

Solution

Given,

$Q_d = 32 - 0.5P$

$Q_s = -8 + 0.3P$

$\frac{dP}{dt} = 0.25(Q_d - Q_s)$

or, $\frac{dP}{dt} = 0.25(32 - 0.5P + 8 - 0.3P)$

or, $\frac{dP}{dt} = 0.25(40 - 0.8P)$

or, $\frac{dP}{dt} = 10 - 0.25P$

or, $\frac{dP}{dt} = -0.25(P - 40)$

or, $\frac{dP}{P - 40} = -0.25dt$

Integrating both sides,

$\ln(P - 40) = -0.25t + \ln C$

or, $\ln\left(\frac{P - 40}{C}\right) = -0.25t$

or, $P - 40 = Ce^{-0.25t}$

or, $P = 40 + Ce^{-0.25t}$ ... (i)

By question, when $t = 0$, $P = 15$

From (i)

$15 = 40 + Ce^0$

$C = -25$

From (i)

$P = 40 - 25e^{-0.25t}$

8. Suppose that a sum P is invested at an annual rate of return r% compounded continuously.

   a. Find the time t required for the sum P to double in value as a function of r.

   b. Find t if $r = 10\%$.

   c. Find r if the sum of money is to be doubled in 5 years.

Solution

Let P_0 be the initial sum invested and P be the sum of money at any time t at the rate of r% per annum. The rate of change of investment is $\frac{dP}{dt}$. Then

$\frac{dP}{dt} = rp$

or, $\frac{dP}{P} = rdt$

Integrating,

$\ln P = rt + \ln c$

or, $\ln\left(\frac{P}{c}\right) = rt$

∴ $P = ce^{rt}$

When $t = 0$, $P = P_0$

$P_0 = ce^0$

∴ $c = P_0$

Thus, $P = P_0e^{rt}$

a. When $P = 2P_0$, we have

$2P_0 = P_0e^{rt}$

or, $e^{rt} = 2$

or, $rt = \ln 2$

∴ $t = \frac{\ln 2}{r}$

b. If $r = 10\%$ p.a., then

$t = \frac{\ln 2}{0.1} = 6.93$ years

c. If $t = 5$ years, then

$5 = \frac{\ln 2}{r}$

$r = \frac{\ln 2}{5}$

∴ $r = 13.86\%$

9. A tank initially contains 500 litres of water with 5 kg of salt. A mixture containing 0.2 kg of salt per litre enters the tank at a rate of 5 litre per minute. The mixture, kept uniform by stirring, is flowing out at the same rate. Find the amount of salt of the tank after 20 minutes.

Solution

Let $y(t)$ be the amount of salt in kgs after t minutes. Given $y(0) = 5$. We have to find $y(20)$.

The rate of change of salt in mixture in tank is

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}$$

$$= 0.2 \times 5 - \frac{y(t)}{500} \times 5$$

$$= 1 - \frac{y}{100}$$

or, $\frac{dy}{dt} + \frac{y}{100} = 1$ ...(i)

Thus the initial value problem can be written as

$$\frac{dy}{dt} + \frac{y}{100} = 1, y(0) = 5$$

$$\text{I.F.} = e^{\int P dt} = e^{\int \frac{1}{100} dt} = e^{\frac{t}{100}}$$

Multiplying (i) both sides by I.F. $= e^{\frac{t}{100}}$, we get

$$\frac{dy}{dt} \cdot e^{\frac{t}{100}} + \frac{y}{100} \cdot e^{\frac{t}{100}} = e^{\frac{t}{100}}$$

or, $d(y \cdot e^{\frac{t}{100}}) = e^{\frac{t}{100}} dt$

Integrating, we get

$$y \cdot e^{\frac{t}{100}} = \frac{e^{\frac{t}{100}}}{\frac{1}{100}} + c$$

$\therefore \quad y(t) = 100 + c\,e^{-\frac{t}{100}}$ ...(iii)

By given, when $t = 0$, $y(0) = 5$

Thus, from (iii),

$5 = 100 + ce^{0}$

$c = -95$

$\therefore \quad y(t) = 100 - 95\,e^{-\frac{t}{100}}$ ...(iv)

When, $t = 20$,

$$y(20) = 100 - 95\,e^{-\frac{20}{100}}$$
$$= 100 - 95\,e^{-0.2}$$
$$= 22.22 \text{ kg}$$

10. A water tank is being filled up with water which was empty initially. The rate of increase of volume of water at any time t is given by $\frac{dV}{dt} = 0.2(100 - t)$ where V is measured in litres and time is measured in seconds.

a. Find the volume of water filled up in the tank in t seconds.

b. Find the volume of water filled up in 10 seconds.

c. How much time will it require to fill up the whole tank if the total capacity of the tank is 200 litres?

Solution

a. Given

$$\frac{dV}{dt} = 0.2(100 - t)$$

or, $dV = 0.2(100 - t)dt$

or, $dV = (20 - 0.2t)\,dt$

Integrating,

$\int dV = \int (20 - 0.2t)\,dt$

or, $V = 20t - 0.2\frac{t^2}{2} + c$

or, $V = 20t - 0.1t^2 + c$ ...(1)

By question,

When $t = 0$, $V = 0$

From (1)

$0 = 0 - 0 + c$

$\Rightarrow \quad c = 0$

From (1)

$V = 20t - 0.1t^2$ ...(2)

b. When $t = 10$ seconds,

$$V = 20 \times 10 - 0.1 \times 10^2$$
$$= 200 - 10$$
$$= 190$$

c. $V = 200$,

Then, from (2)

$200 = 20t - 0.1t^2$

or, $0.1t^2 - 20t + 200 = 0$

or, $t^2 - 200t + 2000 = 0$

$\therefore \quad t = 10.56, 189.44$

Required time is 10.56 seconds.

Exercise 4.6

1. Form the partial differential equations by eliminating arbitrary constants from the following relations.

a. $z = a(x + y) + b$

b. $z = (x + a)(y + b)$

c. $z = (x^2 + a)(y^2 + b)$

d. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution

a. Given,

$z = a(x + y) + b$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$\frac{\partial z}{\partial x} = a$ ...(ii)

and

$\frac{\partial z}{\partial y} = a$ ...(iii)

From (ii) and (ii), we get

$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$

i.e. $p = q$

b. Given,

$z = (x + a)(y + b)$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$\frac{\partial z}{\partial x} = y + b$ ...(ii)

and

$\frac{\partial z}{\partial y} = x + a$ ...(iii)

From (i), (ii) and (ii), we get

$\therefore z = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$

i.e. $z = pq$

c. Given,

$z = (x^2 + a)(y^2 + b)$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$\frac{\partial z}{\partial x} = 2x(y^2 + b)$

or, $\frac{1}{2x}\frac{\partial z}{\partial x} = y^2 + b$ ...(ii)

and

$\frac{\partial z}{\partial y} = 2y(x^2 + a)$

or, $\frac{1}{2y}\frac{\partial z}{\partial y} = x^2 + a$ ...(iii)

From (i), (ii) and (iii), we get

$\therefore z = \frac{1}{2x}\frac{\partial z}{\partial x} \cdot \frac{1}{2y}\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = 4xyz$$

or, $pq = 4xyz$

2. Given,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(i)$$

Differential (i) partially w.r.t. x and y respectively, we get

$$\frac{2x}{a^2}+0+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$\frac{x}{a^2}+\frac{z}{c^2}\frac{\partial z}{\partial x}=0$$

or, $$c^2x+a^2z\frac{\partial z}{\partial x}=0 \qquad \ldots(ii)$$

And

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

or, $$c^2y+b^2z\frac{\partial z}{\partial y}=0 \qquad \ldots(iii)$$

Differentiating (ii) partially w.r.t. x respectively, we get

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \qquad \ldots(iv)$$

Putting the value of c^2 from equation (ii) in equation (iv)

$$-\frac{a^2z}{x}\frac{\partial z}{\partial x}+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \qquad \left[\text{From (ii) } c^2=-\frac{a^2z}{x}\frac{\partial z}{\partial x}\right]$$

or, $$-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0$$

or, $$-z\frac{\partial z}{\partial x}+x\left(\frac{\partial z}{\partial x}\right)^2+zx\frac{\partial^2 z}{\partial x^2}=0$$

$\therefore$ $xp^2-pz+zxr=0$ which is the required PDE.

Note: The answer (the obtained PDE) may be seen different while eliminating arbitrary constants differently.

2. Form the partial differential equations by eliminating arbitrary functions from the following relations.

a. $z=\phi(x^2-y^2)$

b. $z=\phi(x^2+y^2)$

c. $z=f(xy)$

d. $z=f\left(\frac{x}{y}\right)$

e. $z=y^2+2f\left(\frac{1}{x}+\ln y\right)$

f. $x+y+z=f(x^2+y^2+z^2)$

g. $z=\phi(x+ay)+\psi(x-ay)$

h. $z=f_1(x+iy)+f_2(x-iy)$

Solution

a. Given,

$$z=\phi(x^2-y^2) \qquad \ldots(i)$$

Differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x}=\phi'(x^2-y^2)\cdot 2x$$

or, $$\phi'(x^2-y^2)=\frac{1}{2x}\frac{\partial z}{\partial x} \qquad \ldots(ii)$$

and

$$\frac{\partial z}{\partial y} = \phi'(x^2 - y^2) \cdot (-2y)$$

or, $\phi'(x^2 - y^2) = -\frac{1}{2y}\frac{\partial z}{\partial y}$ ...(iii)

From (ii) and (iii), we get

$$\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y}$$

or, $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

$\therefore$ $qx + py = 0$ which is the required PDE.

b. Given,

$z = \phi(x^2 + y^2)$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x} = \phi'(x^2 + y^2) \cdot 2x$$

or, $\frac{1}{2x}\frac{\partial z}{\partial x} = \phi'(x^2 + y^2)$ ...(ii)

and

$$\frac{\partial z}{\partial y} = \phi'(x^2 + y^2) \cdot 2y$$

or, $\frac{1}{2y}\frac{\partial z}{\partial y} = \phi'(x^2 - y^2)$ ...(iii)

From (ii) and (iii), we get

$$\frac{1}{2x}\frac{\partial z}{\partial x} = \frac{1}{2y}\frac{\partial z}{\partial y}$$

$\therefore$ $xq - yp = 0$

c. Given,

$z = f(xy)$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x} = f'(xy) \cdot y$$

or, $\frac{1}{y}\frac{\partial z}{\partial x} = f'(xy)$ ...(ii)

and

$$\frac{\partial z}{\partial y} = f'(xy) \cdot y$$

or, $\frac{1}{x}\frac{\partial z}{\partial y} = f'(xy)$ ...(iii)

From (ii) and (iii), we get

$$\frac{1}{y}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$$

$\therefore$ $px - qy = 0$

d. Given,

$z = f\left(\frac{x}{y}\right)$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right) \cdot \frac{1}{y}$$

or, $y\dfrac{\partial z}{\partial x} = f'\left(\dfrac{x}{y}\right)$ ...(ii)

and

$\dfrac{\partial z}{\partial y} = f'\left(\dfrac{x}{y}\right) \cdot \left(-\dfrac{x}{y^2}\right)$

or, $\left(-\dfrac{y^2}{x}\right)\dfrac{\partial z}{\partial y} = f'\left(\dfrac{x}{y}\right)$ ...(iii)

From (ii) and (iii), we get

$y\dfrac{\partial z}{\partial x} = \left(-\dfrac{y^2}{x}\right)\dfrac{\partial z}{\partial y}$

$\therefore$ $px + qy = 0$

e. Given,

$z = y^2 + 2f\left(\dfrac{1}{x} + \ln y\right)$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$\dfrac{\partial z}{\partial x} = 0 + 2f'\left(\dfrac{1}{x} + \ln y\right) \cdot \left(-\dfrac{1}{x^2}\right)$

or, $-\dfrac{x^2}{2}\dfrac{\partial z}{\partial x} = f'\left(\dfrac{1}{x} + \ln y\right)$ ...(ii)

and

$\dfrac{\partial z}{\partial y} = 2y + 2f'\left(\dfrac{1}{x} + \ln y\right) \cdot \dfrac{1}{y}$

or, $\dfrac{\partial z}{\partial y} - 2y = \dfrac{2}{y} f'\left(\dfrac{1}{x} + \ln y\right)$

or, $\dfrac{y}{2}\dfrac{\partial z}{\partial y} - y^2 = f'\left(\dfrac{1}{x} + \ln y\right)$ ...(iii)

From (ii) and (ii), we get

$-\dfrac{x^2}{2}\dfrac{\partial z}{\partial x} = \dfrac{y}{2}\dfrac{\partial z}{\partial y} - y^2$

or, $-x^2\dfrac{\partial z}{\partial x} = y\dfrac{\partial z}{\partial y} - 2y^2$

or, $x^2\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2y^2$

$\therefore$ $x^2p + yq = 2y^2$

f. Given,

$x + y + z = f(x^2 + y^2 + z^2)$ ...(i)

Differentiating (i) partially w.r.t. x and y respectively, we get

$1 + \dfrac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2)\left(2x + 2z\dfrac{\partial z}{\partial x}\right)$

or, $\dfrac{1 + p}{2x + 2zp} = f'(x^2 + y^2 + z^2)$ ...(ii)

and

$1 + \dfrac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2)\left(2y + 2z\dfrac{\partial z}{\partial y}\right)$

or, $\dfrac{1 + q}{2y + 2zq} = f'(x^2 + y^2 + z^2)$ ...(iii)

From (ii) and (iii), we get

$$\frac{1+p}{2x+2zp}=\frac{1+q}{2y+2zq}$$

or, $(1+p)(y+zq)=(1+q)(x+zp)$

or, $y+zq+py+zpq=x+zp+qx+zpq$

$\therefore$ $p(y-z)+q(z-x)=x-y$

g. Given,

$z = \phi(x+ay)+\psi(x-ay)$ ...(i)

$\frac{\partial z}{\partial x} = \phi'(x+ay)\cdot 1+\psi'(x-ay)\cdot 1$

$\frac{\partial^2 z}{\partial x^2} = \phi''(x+ay)+\psi''(x-ay)$ ...(ii)

and

$\frac{\partial z}{\partial y} = \phi'(x+ay)\cdot a+\psi'(x-ay)\cdot(-a)$

$\frac{\partial^2 z}{\partial y^2} = \phi''(x+ay)\cdot a^2+\psi''(x-ay)\cdot a^2$

$= a^2\{\phi''(x+ay)+\psi''(x-ay)\}$

$= a^2\frac{\partial^2 z}{\partial x^2}$ [Using (ii)]

or, $\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial x^2}$

i.e. $t=a^2r$

h. Given,

$z=f_1(x+iy)+f_2(x-iy)$ ...(i)

$\frac{\partial z}{\partial x} = f_1'(x+iy)\cdot 1+f_2'(x-iy)\cdot 1$

$\frac{\partial^2 z}{\partial x^2} = f_1''(x+iy)+f_2''(x-iy)$ ...(ii)

and

$\frac{\partial z}{\partial y} = f_1'(x+iy)\cdot i+f_2'(x-iy)\cdot(-i)$

$\frac{\partial^2 z}{\partial y^2} = f_1''(x+iy)\cdot i^2+f_2''(x-iy)\cdot i^2$

$= i^2\{f_1''(x+y)+f_2''(x-iy)\}$

$= i^2\frac{\partial^2 z}{\partial x^2}$ [Using (ii)]

$= -\frac{\partial^2 z}{\partial x^2}$ [$\because i^2=-1$]

or, $\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial x^2}=0$

i.e. $r+t=0$

3. Find the partial differential equation of a plane cutting off equal intercepts from the axis of x and y.

Solution

The equation of plane is

$\frac{x}{a}+\frac{y}{a}+\frac{z}{c}=1$ ... (i) [$\because$ x and y intercepts are equal]

Differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial x}=0$$

or. $$\frac{1}{a}=-\frac{1}{c}\frac{\partial z}{\partial x} \quad \ldots\text{(ii)}$$

and

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0$$

or. $$\frac{1}{a}=-\frac{1}{c}\frac{\partial z}{\partial y} \quad \ldots\text{(iii)}$$

From (ii) and (iii), we get

$$-\frac{1}{c}\frac{\partial z}{\partial x}=-\frac{1}{c}\frac{\partial z}{\partial y}$$

∴ $p=q$

4. Form the partial differential equation of the set of all spheres whose centre lie on the axis of z.

Solution

The equation of sphere whose centres line on z-axis is of the form

$$x^2+y^2+(z-k)^2=r^2 \quad \ldots\text{(i)}$$

where k is any constant and r be the radius of the sphere.

Differentiating (i) partially w.r.t. x and y, we get

$$2x+2(z-k)\frac{\partial z}{\partial x}=0$$

or, $$(z-k)\frac{\partial z}{\partial x}=-x$$

or, $$-\frac{1}{x}\frac{\partial z}{\partial x}=\frac{1}{z-k} \quad \ldots\text{(ii)}$$

And,

$$2y+2(z-k)\frac{\partial z}{\partial y}=0$$

or, $$(z-k)\frac{\partial z}{\partial y}=-y$$

or, $$-\frac{1}{y}\frac{\partial z}{\partial y}=\frac{1}{z-k} \quad \ldots\text{(iii)}$$

From (ii) and (iii), we get

$$-\frac{1}{x}\frac{\partial z}{\partial x}=-\frac{1}{y}\frac{\partial z}{\partial y}$$

or, $yp=xq$

or, $xq-yp=0$

Exercise 4.7

Find the general solution (integral) of the following PDEs.

1. $pz + x = 0$

Solution

Given,

$pz + x = 0$

or, $pz + 0 \cdot q = -x$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = z, Q = 0, R = -x$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$

From first and last ratios, we get

$$\frac{dx}{z} = \frac{dz}{-x}$$

or, $x\,dx + z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

or, $x^2 + z^2 = c_1$ ...(ii)

Again, taking last two ratios, we get

$$\frac{dy}{0} = \frac{dz}{-x}$$

or, $dy = 0$

Integrating, we get

$y = c_2$ ...(iii)

The required solution is $\phi(c_1, c_2) = 0$

$\therefore$ $\phi(x^2 + z^2, y) = 0$

2. $z = a(p + q)$

Solution

Given,

$z = a(p + q)$

or, $p + q = \frac{z}{a}$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = 1, Q = 1, R = \frac{z}{a}$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Then,

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\frac{z}{a}}$$

From first two ratios, we get

$dx = dy$

Integrating, we get

$x = y + c_1$ ...(ii)

or, $x - y = c_1$

Again, taking last two ratios, we get

$dy = \frac{a\,dz}{z}$

or, $\frac{dz}{z} = \frac{dy}{a}$

Integrating, we get

$\ln z = \frac{1}{a}y + \ln c_2$

or, $\ln z - \ln c_2 = \frac{y}{a}$

or, $\ln\left(\frac{z}{c_2}\right) = \frac{y}{a}$

or, $\frac{z}{c_2} = e^{\frac{y}{a}}$

or, $z = c_2 e^{\frac{y}{a}}$

The required solution is $z = e^{y/a}\,\phi(c_1)$ [writing $c_2 = \phi(c_1)$]

$\therefore$ $z = e^{y/a}\,\phi(x - y)$

3. $x^2p + q = z^2$

Solution

Given,

$x^2p + q = z^2$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = x^2, Q = 1, R = z^2$

The Lagrange's auxiliary equations are

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, we get

or, $\frac{dx}{x^2} = \frac{dy}{1} = \frac{dz}{z^2}$

Taking first two ratios, we get

$\frac{dx}{x^2} = \frac{dy}{1}$

Integrating, we get

$\int x^{-2}\,dx + c_1 = \int dy$

or, $-\frac{1}{x} + c_1 = y$

$\therefore$ $c_1 = \frac{1}{x} + y$

Again, taking last two ratios, we get

$\frac{dx}{x^2} = \frac{dz}{z^2}$

Integrating, we get

$-\frac{1}{x} + c_2 = -\frac{1}{z}$

$c_2 = \frac{1}{x} - \frac{1}{z}$

The required solution is $\phi(c_1, c_2) = 0$

$\therefore$ $\phi\left(\frac{1}{x} + y, \frac{1}{x} - \frac{1}{z}\right) = 0$

4. $x^2p + y^2q = z^2$

Solution

Given,

$x^2p + y^2q = z^2$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = x^2, Q = y^2, R = z^2$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Taking first two ratios, we get

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating, we get

$$\int x^{-2}\,dx + c_1 = \int y^{-2}\,dy$$

or, $$-\frac{1}{x} + c_1 = -\frac{1}{y}$$

$\therefore$ $$c_1 = \frac{1}{x} - \frac{1}{y} \quad \text{...(ii)}$$

Again, taking first and last ratios, we get

$$\frac{dx}{x^2} = \frac{dz}{z^2}$$

Integrating, we get

$$\int x^{-2}\,dx + c_2 = \int z^{-2}\,dz$$

or, $$-\frac{1}{x} + c_2 = -\frac{1}{z}$$

$\therefore$ $$c_2 = \frac{1}{x} - \frac{1}{z} \quad \text{...(ii)}$$

The required solution is $\phi(c_1, c_2) = 0$

$\therefore$ $$\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{x} - \frac{1}{z}\right) = 0$$

5. $\dfrac{y^2z}{x}p + zxq = y^2$

Solution

Given,

$$\frac{y^2z}{x}p + zxq = y^2$$

or, $y^2zp + zx^2q = xy^2$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = y^2z, Q = zx^2, R = xy^2$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{y^2z} = \frac{dy}{zx^2} = \frac{dz}{xy^2}$$

Taking first two ratios, we get

$$\frac{dx}{y^2z} = \frac{dy}{zx^2}$$

or, $x^2\,dx = y^2\,dy$

Integrating, we get

$$\frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3}$$

or, $x^3 - y^3 = c_1$ ...(ii)

Again, taking first and last ratios, we get

$$\frac{dx}{y^2z} = \frac{dz}{xy^2}$$

or, $xdx = zdz$

Integrating, we get

$$\frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

∴ $x^2 - z^2 = c_2$ ...(ii)

The required solution is $\phi(c_1, c_2) = 0$

∴ $\phi(x^3 - y^3, x^2 - z^2) = 0$

6. $xp - yq + x^2 - y^2 = 0$

Solution

Given,

$$xp - yq + x^2 - y^2 = 0$$

or, $xp - yq = y^2 - x^2$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = x, Q = -y, R = y^2 - x^2$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$

Taking the Lagrange's multipliers as $x, y, 1$, we get

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2} = \frac{x\,dx + y\,dy + dz}{0}$$

Taking first and last ratios, we get

$$\frac{dx}{x} = \frac{x\,dx + y\,dy + dz}{0}$$

i.e. $x\,dx + y\,dy + dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + z = \frac{c_1}{2}$$

or, $x^2 + y^2 + 2z = c_1$ ....(ii)

Again, taking first two ratios, we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating, we get

$\ln x = -\ln y + \ln c_2$

∴ $\ln x + \ln y = \ln c_2$

$\ln(xy) = \ln c_2$

or, $c_2 = xy$

The required solution is $\phi(c_1, c_2) = 0$

∴ $\phi(x^2 + y^2 + 2z, xy) = 0$

7. $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$

Solution

Given,

$$\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$$

or, $x(y-z)p + y(z-x)q = (x-y)z$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = x(y-z),\ Q = y(z-x),\ R = (x-y)z$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{(x-y)z}$$

Then,

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{(x-y)z} = \frac{dx+dy+dz}{0}$$

i.e. $dx + dy + dz = 0.$

Integrating, we get

$x + y + z = c_1$ ...(ii)

Again,

$$\frac{\frac{dx}{x}}{y-z} = \frac{\frac{dy}{y}}{z-x} = \frac{\frac{dz}{z}}{x-y} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y-z+z-x+x-y}$$

i.e. $$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get

$\ln x + \ln y + \ln z = \ln c_2$

or, $\ln (xyz) = \ln c_2$

i.e. $xyz = c_2$ ...(iii)

The required solution is $\phi(c_1, c_2) = 0$

$\therefore$ $\phi(x + y + z, xyz) = 0.$

8. $(mz - ny)p + (nx - lz)q = ly - mx$

Solution

Given,

$(mz - ny)p + (nx - lz)q = ly - mx$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = mx - ny,\ Q = nx - lz,\ R = ly - mx$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{mx-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Using the multipliers, x, y, z, we get

$$\frac{dx}{mx-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{xdx+ydy+zdz}{0}$$

i.e. $xdx + ydy + zdz = 0.$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$\therefore$ $x^2 + y^2 + z^2 = c_1$ ...(ii)

Again, using the multipliers l, m, n, we get

$$\frac{dx}{mx-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

Then,

$l\,dx + m\,dy + n\,dz = 0$

Integrating, we get

$lx + my + nz = c_2$

The required solution is $\phi(c_1, c_2) = 0$

$\therefore$ $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$

9. $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z$

Solution

Given,

$x^2p+y^2q=(x+y)z$ ... (i)

Comparing (i) with $Pp+Qq=R$, we get

$P=x^2, Q=y^2, R=(x+y)z$

The Lagrange's auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

or, $$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z}$$ ...(ii)

From (ii), we can write

$$\frac{\frac{dx}{x}}{x}=\frac{\frac{dy}{y}}{y}=\frac{dz}{(x+y)z}$$

or, $$\frac{\frac{dx}{x}+\frac{dy}{y}}{x+y}=\frac{dz}{(x+y)z}$$

or, $$\frac{dx}{x}+\frac{dy}{y}=\frac{dz}{dz}$$

Integrating, we get

$\ln x+\ln y=\ln z+\ln c_1$

$$\ln\left(\frac{xy}{z}\right)=\ln c_1$$

$$c_1=\frac{xy}{z}$$

Again from (ii),

$$\frac{dx-dy}{x^2-y^2}=\frac{dz}{(x+y)z}$$

or, $$\frac{d(x-y)}{(x-y)(x+y)}=\frac{dz}{(x+y)z}$$

Integrating, we get

$\ln(x-y)=\ln z+\ln c_2$

or, $$\ln\left(\frac{x-y}{z}\right)=\ln c_2$$

$\therefore$ $$\frac{x-y}{z}=c_2$$ ...(iv)

The required solution is $\phi(c_1, c_2)=0$

$\therefore$ $$\phi\left(\frac{xy}{z},\frac{x-y}{z}\right)=0$$

10. $p+q=\sin x$

Solution

Given,

$p+q=\sin x$ ... (i)

Comparing (i) with $Pp+Qq=R$, we get

$P=1, Q=1, R=\sin x$

The Lagrange's auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

or, $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\sin x}$

Taking first two ratios, we get

$dx = dy$

Integrating, we get

$x + c_1 = y$

$\therefore$ $c_1 = y - x$

Again, taking first and last ratios, we get

$\frac{dx}{1} = \frac{dz}{\sin x}$

or, $\sin x \, dx = dz$

Integrating, we get

$-\cos x + c_2 = z$

$\therefore$ $c_2 = z + \cos x$

The required solution is $c_2 = \phi(c_1)$

$\therefore$ $z + \cos x = \phi(y - x)$

11. $(y + z)\frac{\partial z}{\partial x} + (x + z)\frac{\partial z}{\partial y} = x + y$

Solution

Given,

$(y + z)p + (x + z)q = x + y$ ... (i)

Comparing (i) with $Pp + Qq = R$, we get

$P = y + z$, $Q = x + z$, $R = x + y$

The Lagrange's auxiliary equations are

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

or, $\frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{x+y}$ ...(ii)

From (ii), we can write

$\frac{dx + dy + dz}{2x + 2y + 2z} = \frac{dx - dy}{-(x - y)} = \frac{dx - dz}{-(x - z)}$

Taking first two ratios, we get

$\frac{d(x + y + z)}{(x + y + z)} = \frac{2d(x - y)}{-(x - y)}$

Integrating, we get

$\ln(x + y + z) = -2\ln(x - y) + \ln c_1$

$\ln\{(x + y + z)(x - y)^2\} = \ln c_1$

$\therefore$ $c_1 = (x + y + z)(x - y)^2$

Again, taking last two ratios, we get

$\frac{d(x - y)}{x - y} = \frac{d(x - z)}{x - z}$

Integrating,

$\ln(x - y) = \ln(x - z) + \ln c_2$

or, $\ln\left(\frac{x - y}{x - z}\right) = \ln c_2$

$\therefore$ $c_2 = \frac{x - y}{x - z}$ ...(iii)

The required solution is $c_1 = \phi(c_2)$

$\therefore$ $(x + y + z)(x - y)^2 = \phi\left(\frac{x - y}{x - z}\right)$

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UNIT 5

Fourier Series

Exercise 5

1. Find the fundamental period of the following periodic functions.

a. $\sin 2x$

b. $\sin 2\pi x$

c. $\cos \frac{5x}{2}$

d. $\sin \frac{x}{2} + \cos \frac{x}{2}$

e. $\tan 2x$

f. $\sin x + \tan x$

Solution

a. $\sin 2x$

Here, $a = 2$

$$\text{Period} = \frac{2\pi}{a} = \frac{2\pi}{2} = \pi$$

b. $\sin 2\pi x$

Here, $a = 2\pi$

$$\text{Period} = \frac{2\pi}{a} = \frac{2\pi}{2\pi} = 1$$

c. $\cos \frac{5x}{2}$

Here, $a = \frac{5}{2}$

$$\text{Period} = \frac{2\pi}{a} = \frac{2\pi}{\frac{5}{2}} = \frac{4\pi}{5}$$

d. $\sin \frac{x}{2} + \cos \frac{x}{2}$

Here, $a = \frac{1}{2}$

$$\text{Period} = \frac{2\pi}{a} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

e. $\tan 2x$

Here, $a = 2$

$$\text{Period} = \frac{\pi}{a} = \frac{\pi}{2}$$

f. $\sin x + \tan x$

Period $= 2\pi$

2. Determine whether the following functions are even, odd or neither.

a. $f(x) = x^3 \sin x$

b. $f(x) = \sin x \cos x$

c. $f(x) = \sin x + \cos x$

d. $f(x) = |\sin x|$

e. $f(x) = \begin{cases} -4x & \text{for } -\pi < x < 0 \\ 4x & \text{for } 0 < x < \pi \end{cases}$

Solution

a. Here, $f(x) = x^3 \sin x$

Now,

$$f(-x) = (-x)^3 \sin(-x)$$
$$= -x^3 \cdot (-\sin x)$$
$$= x^3 \sin x$$
$$= f(x)$$

This shows that $f(x)$ is an even function.

b. Here, $f(x) = \sin x \cos x$

Now,

$$f(-x) = \sin x \cos x$$
$$= -\sin x \cos x$$
$$= -f(x)$$

This shows that $f(x)$ is an odd function.

c. Here, $f(x) = \sin x + \cos x$

Now,

$$f(-x) = \sin(-x) + \cos(-x)$$
$$= -\sin x + \cos x$$

which is neither $f(x)$ nor $-f(x)$.

This shows that $f(x)$ is neither even or odd function.

d. Here, $f(x) = |\sin x|$

Now,

$$f(-x) = |\sin(-x)|$$
$$= |-\sin x|$$
$$= |\sin x|$$
$$= f(x)$$

This shows that $f(x)$ is an even function.

e. Given,

$$f(x) = \begin{cases} -4x & \text{for } -\pi < x < 0 \\ 4x & \text{for } 0 < x < \pi \end{cases}$$

Replacing x by $-x$, we get

Now,

$$f(-x) = \begin{cases} -4(-x) \text{ for } -\pi < -x < 0 \\ 4(-x) \text{ for } 0 < -x < \pi \end{cases}$$
$$= \begin{cases} 4x \text{ for } \pi > x > 0 \\ -4x \text{ for } 0 > x > -\pi \end{cases}$$
$$= \begin{cases} 4x \text{ for } 0 < x < \pi \\ -4x \text{ for } -\pi < x < 0 \end{cases}$$
$$= \begin{cases} -4x \text{ for } -\pi < x < 0 \\ 4x \text{ for } 0 < x < \pi \end{cases}$$
$$= f(x)$$

This shows that $f(x)$ is an even function.

3. Find the Fourier series of the following functions.

a. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$

b. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \end{cases}$

c. $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 \le x < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$

Solution

a. Given,

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots(i)$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\,dx$$

$$= \frac{1}{\pi}[-x]_0^{-\pi} + \frac{1}{\pi}[x]_0^{\pi}$$

$$= \frac{1}{\pi}(0+\pi) + \frac{1}{\pi}(\pi - 0)$$

$$= 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{\sin(-n\pi)}{n}\right] + \frac{1}{\pi}\left[\frac{\sin n\pi}{n} + 0\right]$$

$$= 0 \qquad [\because \sin n\pi = 0]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n} - \frac{\cos n\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{\cos n\pi}{n} + \frac{1}{n}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n} - \frac{\cos n\pi}{n}\right]$$

$$= \frac{2}{\pi}\left\{\frac{1-(-1)^n}{n}\right\} \quad [\because \cos n\pi = (-1)^n]$$

From (i)

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\left[0 + \frac{2}{\pi}\left\{\frac{1-(-1)^n}{n}\right\}\sin nx\right]$$

$$= 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin nx$$

b. Given,

$$f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \end{cases}$$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \ldots(i)$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} (0)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(1)\,dx$$

$$= \frac{1}{\pi}[x]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}[2\pi - \pi]$$

$$= 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\, dx + \frac{1}{\pi}\int_\pi^{2\pi} f(x)\cos nx\, dx$$

$$= 0 + \frac{1}{\pi}\int_\pi^{2\pi} (1)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_\pi^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin 2n\pi}{n} - \frac{\sin nx}{n}\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\, dx + \frac{1}{\pi}\int_\pi^{2\pi} f(x)\sin nx\, dx$$

$$= 0 + \frac{1}{\pi}\int_\pi^{2\pi} (1)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_\pi^{2\pi}$$

$$= -\frac{1}{n\pi}[\cos 2n\pi - \cos n\pi]$$

$$= -\frac{1}{n\pi}[1 - (-1)^n]$$

$$= \frac{(-1)^n - 1}{n\pi}$$

From (i)

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} 0 + \left\{\frac{(-1)^n - 1}{n\pi}\right\} \sin nx$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right] \sin nx$$

c. Given,

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 \le x < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_0^{\pi/2} f(x)\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (0)\,dx + \frac{1}{\pi}\int_{0}^{\pi/2} (1)\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (0)\,dx$$

$$= 0 + \frac{1}{\pi}[x]_0^{\pi/2} + 0$$

$$= 0 + \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi/2} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} f(x)\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi/2} (1)\cos nx\,dx + 0$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi/2}$$

$$= \frac{1}{\pi}\,\frac{\sin\left(\frac{n\pi}{2}\right)}{n}$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi/2} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} f(x)\sin nx\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi/2} (1)\sin nx\,dx + 0$$

$$= \left[-\frac{\cos nx}{n}\right]_0^{\pi/2}$$

$$= -\frac{1}{n}\left\{\cos\left(\frac{n\pi}{2}\right) - \cos(0)\right\}$$

$$= -\frac{1}{n}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$= \frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right)$$

From (i)

$$f(x) = \frac{1}{2}\times\frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos nx + \frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right)\sin nx\right]$$

$$= \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos nx + \frac{1}{n}\left\{1 - \cos\left(\frac{n\pi}{2}\right)\right\}\sin nx\right]$$

4. Find the Fourier series of the following functions in the interval $(-\pi, \pi)$.

a. $f(x) = x$

b. $f(x) = \frac{x}{\pi}$

c. $f(x) = \pi + x$

d. $f(x) = x^2$

e. $f(x) = |x|$

Solution

a. Given,

$f(x) = x$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x)\,dx$$

$= 0$ [Using property of odd function]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx$$

$= 0$ [Using property of odd function]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\pi\frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - \left(\pi\frac{\cos n\pi}{n} - \frac{\sin n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + 0 - \frac{\pi\cos n\pi}{n} + 0\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n}\right]$$

$$= \frac{\pi}{\pi}\left[\frac{-2(-1)^n}{n}\right]$$

$$= -\frac{2}{n}(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}$$

Required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= 0 + \sum_{n=1}^{\infty} \left[0 + \frac{2}{n}(-1)^{n+1} \sin nx\right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

b. Given,

$$f(x) = \frac{x}{\pi}$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{\pi}\, dx$$

$= 0$ [Using property of odd function]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{\pi} \cos nx\, dx$$

$= 0$ [Using property of odd function]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{\pi} \sin nx\, dx$$

$$= \frac{1}{\pi^2}\left[x\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi^2}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi^2}\left[\left(-\pi\frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - \left(\pi\frac{\cos n\pi}{n} - \frac{\sin n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi^2}\left[-\frac{\pi \cos n\pi}{n} + 0 - \frac{\pi \cos n\pi}{n} + 0\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi \cos n\pi}{n}\right]$$

$$= \frac{\pi^2}{\pi}\left[\frac{-2(-1)^n}{n}\right]$$

$$= -\frac{2}{n\pi}(-1)^n$$

$$= \frac{2}{n\pi}(-1)^{n+1}$$

Required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= 0 + \sum_{n=1}^{\infty} \left[0 + \frac{2}{n\pi}(-1)^{n+1} \sin nx\right]$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

c. Given,

$f(x) = \pi + x$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi + x)\, dx$$

$$= \frac{1}{\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\pi \cdot \pi + \frac{\pi^2}{2}\right) - \left\{\pi(-\pi) + \frac{(-\pi)^2}{2}\right\}\right]$$

$$= \frac{1}{\pi}\left(\pi^2 + \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right)$$

$$= \frac{2\pi^2}{\pi}$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi + x) \cos nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi + x)\frac{\sin nx}{n} - 1 \cdot \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi + x)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos n\pi}{n^2}\right)\right] \quad [\because \sin n\pi = 0]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi + x) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi + x)\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi + x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{-(\pi + \pi)\frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right\} - \left\{-(\pi - \pi)\frac{\cos n\pi}{n} + \frac{\sin(-n\pi)}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left(-2\pi\cdot\frac{\cos n\pi}{n}\right)$$

$$= -\frac{2}{n}\cos n\pi$$

Required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2}\cdot 2\pi + \sum_{n=1}^{\infty}\left[0 + \left(-\frac{2}{n}\cos n\pi\right)\sin nx\right]$$

$$= \pi + 2\sum_{n=1}^{\infty}[-(-1)^n]$$

$$= \pi + 2\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin nx$$

d. Given,

$f(x) = x^2$

Let the required Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\frac{2\pi^3}{3}$$

$$= \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\left[\because \int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots\right]$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\pi^2\frac{\sin n\pi}{n} + 2\pi\frac{\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right\} - \left\{(-\pi)^2\frac{\sin(-n\pi)}{n} + 2(-\pi)\frac{\cos(-n\pi)}{n^2} - \frac{2\sin(-n\pi)}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left\{0 + 2\pi\cdot\frac{(-1)^n}{n^2} - 0\right\} - \left\{0 - 2\pi\cdot\frac{(-1)^n}{n^2} + 0\right\}\right]$$

$$[\because \sin 2n\pi = 0,\ \cos 2n\pi = (-1)^n]$$

$$= \frac{1}{\pi}\left[2\pi\cdot\frac{(-1)^n}{n^2} + 2\pi\cdot\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\cdot 4\pi\frac{(-1)^n}{n^2}$$

$$= (-1)^n\cdot\frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= 0 \qquad \text{[Using property of odd function]}$$

Putting the values of a_0, a_n and b_n in (i), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

i.e. $$x^2 = \frac{1}{2}\cdot\frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\left[(-1)^n\cdot\frac{4}{n^2}\cos nx + 0\right]$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos nx\right]$$

e. Given,

$f(x) = |x|$

We have,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[-\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}[x^2]_{-\pi}^{0} + \frac{1}{2\pi}[x^2]_0^{\pi}$$

$$= -\frac{1}{2\pi}[0-(-\pi)^2] + \frac{1}{2\pi}[\pi^2 - 0]$$

$$= \frac{\pi^2}{2\pi} + \frac{\pi^2}{2\pi}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= -\frac{1}{\pi}\left[\left(0+\frac{1}{n^2}\right) - \left\{-\pi\frac{\sin(-n\pi)}{n} + \frac{\cos(-n\pi)}{n^2}\right\}\right] + \frac{1}{\pi}\left[\left\{\pi\frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right\} - \left\{0 + \frac{\cos 0}{n^2}\right\}\right]$$

$$= -\frac{1}{\pi}\left[\frac{1}{n^2} - 0 - \frac{\cos n\pi}{n^2}\right] + \frac{1}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{\cos n\pi}{n^2} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\cos n\pi}{n^2} - \frac{2}{n^2}\right]$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1)$$

$$= \frac{2}{\pi n^2}((-1)^n - 1) \quad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx \, dx$$

[Using property of odd function]

$$= 0$$

The required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{n^2 \pi}[(-1)^n - 1] \cos nx + 0\right]$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2}\right] \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[-\frac{2}{1^2}\cos x + 0 - \frac{2}{3^2}\cos 3x + 0 - \frac{2}{5^2}\cos 5x + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots\right]$$

1. Obtain the Fourier series for the following functions in the interval $(0, 2\pi)$.

a. $f(x) = -x$

b. $f(x) = 3x$

c. $f(x) = \pi - x$

d. $f(x) = x^2$

e. $f(x) = e^x$

Solution

1. Given,

$f(x) = -x$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x) \, dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} (-x) \, dx$$

$$= \frac{1}{\pi}\left[-\frac{x^2}{2}\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi}[(2\pi)^2 - 0]$$

$$= -\frac{4\pi^2}{2\pi}$$

$$= -2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (-x)\cos nx\,dx$$

$$= -\frac{1}{\pi}\int_0^{2\pi} (x)\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= -\frac{1}{\pi}\left[\left\{2\pi\cdot\frac{\sin 2n\pi}{n} + \frac{\cos 2nx}{n^2}\right\} - \left\{0 + \frac{\cos 0}{n^2}\right\}\right]$$

$$= -\frac{1}{\pi}\left\{0 + \frac{1}{n^2} - \frac{1}{n^2}\right\}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (-x)\sin nx\,dx$$

$$= -\frac{1}{\pi}\int_0^{2\pi} (x)\sin nx\,dx$$

$$= -\frac{1}{\pi}\left[x\cdot\left(-\frac{\cos nx}{n}\right) - x\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{x\cos nx}{n} - \frac{x\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{2\pi\cos 2n\pi}{n} + \frac{x\sin\cos 2nx}{n^2}\right) - (0-0)\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n}\cdot 1 - 0\right]$$

$$= \frac{2}{n}$$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx)$$

$$= \frac{1}{2}\cdot(-2\pi) + \sum_{n=1}^{\infty}\left(0 + \frac{2}{n}\sin nx\right)$$

$$= -\pi + 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Given,

$f(x) = 3x$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (3x)\,dx$$

$$= \frac{3}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{3}{2\pi}[(2\pi)^2 - 0]$$

$$= \frac{12\pi^2}{2\pi}$$

$$= 6\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (3x)\cos nx\,dx$$

$$= \frac{3}{\pi}\int_0^{2\pi} (x)\cos nx\,dx$$

$$= \frac{3}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{3}{\pi}\left[\left\{2\pi\cdot\frac{\sin 2n\pi}{n} + \frac{\cos 2nx}{n^2}\right\} - \left\{0 + \frac{\cos 0}{n^2}\right\}\right]$$

$$= \frac{3}{\pi}\left(0 + \frac{1}{n^2} - \frac{1}{n^2}\right)$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (3x)\sin nx\,dx = \frac{3}{\pi}\int_0^{2\pi} (x)\sin nx\,dx$$

$$= \frac{3}{\pi}\left[x\cdot\left(-\frac{\cos nx}{n}\right) - x\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= -\frac{3}{\pi}\left[\frac{x\cos nx}{n} + \frac{x\sin nx}{n^2}\right]_0^{2\pi}$$

$$= -\frac{3}{\pi}\left[\left(\frac{2\pi\cos 2n\pi}{n} + \frac{x\sin\cos 2nx}{n^2}\right) - (0-0)\right]$$

$$= -\frac{3}{\pi}\left[\frac{2\pi}{n}\cdot 1 - 0\right] = -\frac{6}{n}$$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{6\pi}{2} + \sum_{n=1}^{\infty} \left(0 - \frac{6}{n} \sin nx\right)$$

$$= 3\pi - 6 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

c. Given,

$f(x) = \pi - x$

The Fourier coefficients are calculated as follows:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)\, dx$$

$$= \frac{1}{\pi} \left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\pi \cdot 2\pi - \frac{(2\pi)^2}{2} - 0\right]$$

$$= \frac{1}{\pi} \left[2\pi^2 - \frac{4\pi^2}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[(\pi - x) \cdot \frac{\sin nx}{n} - (0 - 1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$\left[\because \int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right.$ where dash is for derivative and suffices for antiderivative $\left.\right]$

$$= \frac{1}{\pi} \left[(\pi - x) \frac{\sin nx}{n} - \left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\left\{(\pi - 2\pi) \frac{\sin 2n\pi}{n} - \left(\frac{\cos 2n\pi}{n^2}\right)\right\} - \left\{(\pi + 0) \frac{\sin 0}{n} - \frac{\cos 0}{n^2}\right\}\right]$$

$$= \frac{1}{\pi} \left[0 - \frac{1}{n^2} - 0 + \frac{1}{n^2}\right] \qquad [\because \sin 2n\pi = 0,\ \cos 2n\pi = 1]$$

$$= 0$$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$

$= \frac{1}{\pi}\left[(\pi - x)\cdot\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[(x-\pi)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[\left\{(2\pi-\pi)\frac{\cos 2n\pi}{n} - \frac{\sin 2n\pi}{n^2}\right\} - \left\{(0-\pi)\frac{\cos 0}{n} - \frac{\sin 0}{n^2}\right\}\right]$

$= \frac{1}{\pi}\left[\pi\cdot\frac{1}{n} - 0 + \pi\cdot\frac{1}{n} - 0\right]$

$= \frac{1}{\pi}\cdot\frac{2\pi}{n}$

$= \frac{2}{n}$

The required Fourier series is

$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

i.e. $\pi - x = 0 + \sum_{n=1}^{\infty}\left[0\cdot\cos nx + \frac{2}{n}\sin nx\right]$

$= 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

$= 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$

d. Given,

$f(x) = x^2$

The Fourier coefficients are

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{2\pi}(x^2)\,dx$

$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left\{\frac{(2\pi)^3}{3} - 0\right\}$

$= \frac{1}{\pi}\frac{8\pi^3}{3}$

$= \frac{8\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - 2x \cdot \left(-\frac{\cos nx}{n^2}\right) + 2 \cdot \left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} + 2x \cdot \frac{\cos nx}{n^2} - 2 \cdot \frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{(2\pi)^2 \cdot \frac{\sin 2n\pi}{n} + 4\pi\frac{\cos 2nx}{n^2} - 2\frac{\sin 2n\pi}{n^3}\right\} - \left\{0 + 0 - 2\frac{\sin 0}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left(0 + \frac{4\pi}{n^2} - 0 - 0\right)$$

$$= \frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x^2 \cdot \left(-\frac{\cos nx}{n}\right) - 2x \cdot \left(-\frac{\sin nx}{n^2}\right) + 2 \cdot \left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2 \cdot \frac{\cos nx}{n} + 2x \cdot \frac{\sin nx}{n^2} + 2 \cdot \frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-(2\pi)^2 \cdot \frac{\cos 2n\pi}{n} + 2 \cdot 2\pi\frac{\sin 2n\pi}{n^2} - 2 \cdot \frac{\cos 2n\pi}{n^3}\right\} - \left\{-0 + 0 + 2\frac{\cos 0}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left(-\frac{4\pi^2}{n} + 0 + \frac{2}{n^3} - \frac{2}{n^2}\right)$$

$$= -\frac{4\pi}{n}$$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx + \left(-\frac{4\pi}{n}\right)\sin nx\right]$$

$$= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx - \frac{\pi}{n}\sin nx\right)$$

c. Given,

$f(x) = e^x$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (e^x)\,dx$$

$$= \frac{1}{\pi}[e^x]_0^{2\pi}$$

$$= \frac{1}{\pi}(e^{2\pi} - e^0)$$

$$= \frac{1}{\pi}(e^{2\pi} - 1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x(\cos nx + n\sin nx)}{1+n^2}\right]_0^{2\pi} \quad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + c\right]$$

$$= \frac{1}{\pi(n^2+1)}[\{e^{2\pi}(\cos 2n\pi + n\sin 2n\pi)\} - e^0(\cos 0 + n\sin 0)]$$

$$= \frac{1}{\pi(n^2+1)}[e^{2\pi}(1+0) - 1(1+0)]$$

$$= \frac{e^{2\pi}-1}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x(\sin nx - n\cos nx)}{1+n^2}\right]_0^{2\pi} \quad [\text{Using Formula of} \int e^{ax}\sin bx\,dx]$$

$$= \frac{1}{\pi(n^2+1)}[\{e^{2\pi}(\sin 2n\pi - n\cos 2n\pi)\} - e^0(\sin 0 - n\cos 0)]$$

$$= \frac{1}{\pi(n^2+1)}[e^{2\pi}(0 - n\cdot 1) - 1(0 - n\cdot 1)]$$

$$= \frac{1}{\pi(n^2+1)}(n - ne^{2\pi})$$

$$= \frac{n(1-e^{2\pi})}{\pi(n^2+1)}$$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2\pi} \cdot (e^{2\pi} - 1) + \sum_{n=1}^{\infty} \left[\frac{e^{2\pi} - 1}{\pi(n^2+1)} \cos nx + \frac{n(1-e^{2\pi})}{\pi(n^2+1)} \sin nx \right]$$

$$= \frac{e^{2\pi} - 1}{2\pi} + \frac{e^{2\pi} - 1}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n^2+1} \cos nx - \frac{n}{n^2+1} \sin nx \right]$$

$$= \frac{e^{2\pi} - 1}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\cos nx}{1+n^2} - \frac{n \sin nx}{1+n^2} \right) \right]$$

6. Find the Fourier series of the following functions.

a. $f(x) = \begin{cases} -4x & \text{for } -\pi < x < 0 \\ 4x & \text{for } 0 < x < \pi \end{cases}$

b. $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 2x & \text{for } 0 \le x < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$

c. $f(x) = \begin{cases} \pi + x & \text{for } -\pi < x < 0 \\ \pi - x & \text{for } 0 < x < \pi \end{cases}$

Solution

a. Given,

$$f(x) = \begin{cases} -4x & \text{for } -\pi < x < 0 \\ 4x & \text{for } 0 < x < \pi \end{cases}$$

Replacing x by $-x$, we get

$$f(-x) = \begin{cases} -4(-x) \text{ for } -\pi < -x < 0 \\ 4(-x) \text{ for } 0 < -x < \pi \end{cases}$$

$$= \begin{cases} 4x \text{ for } \pi > x > 0 \\ -4x \text{ for } 0 > x > -\pi \end{cases}$$

$$= \begin{cases} 4x \text{ for } 0 < x < \pi \\ -4x \text{ for } -\pi < x < 0 \end{cases}$$

$$= \begin{cases} -4x \text{ for } -\pi < x < 0 \\ 4x \text{ for } 0 < x < \pi \end{cases}$$

$$= f(x)$$

Hence $f(x)$ is an even function.

Next, let the Fourier series of the function in the interval $(-\pi, \pi)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots(i)$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\cdot 2\cdot\int_{0}^{\pi} f(x)\,dx \qquad [\because f(x) \text{ is even function}]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} 4x\,dx = \frac{2}{\pi}\left[\frac{4x^2}{2}\right]_0^{\pi}$$

$$= \frac{4}{\pi}[\pi^2 - 0]$$

$$= 4\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot 2\cdot\int_{0}^{\pi} f(x)\cos nx\,dx \qquad [\because f(x) \text{ is even function}]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} 4x\cos nx\,dx$$

$$= \frac{8}{\pi}\left[4x\cdot\left(\frac{\sin nx}{n}\right) - 4\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{8}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{8}{\pi}\left[\left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{8}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad [\because \sin n\pi = 0,\ \cos n\pi = (-1)^n]$$

$$= \frac{8}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$$

$$= 0 \qquad [\because f(x)\sin nx \text{ is odd function}]$$

Putting the values of a_0, a_n and b_n in equation (i), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{1}{2}(4\pi) + \sum_{n=1}^{\infty}\left[\frac{8}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx + 0\right]$$

$$= 2\pi + \frac{8}{\pi} \sum_{n=1}^{\infty} \left[\left(\frac{(-1)^n - 1}{n^2} \right) \cos nx \right]$$

$$= 2\pi + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$$

b. Given,

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 2x & \text{for } 0 \le x < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi} \int_0^{\pi/2} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi/2} 2x\, dx$$

$$= \frac{1}{\pi} \left[\frac{2x^2}{2} \right]_0^{\pi/2}$$

$$= \frac{1}{\pi} \left[\left(\frac{\pi}{2} \right)^2 - 0 \right]$$

$$= \frac{1}{\pi} \times \frac{\pi^2}{2}$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi/2} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi/2} 2x \cos nx\, dx$$

$$= \frac{2}{\pi} \left[x \cdot \frac{\sin nx}{n} - 1 \cdot \left(-\frac{\cos nx}{n^2} \right) \right]_0^{\pi/2}$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2} \frac{\sin\left(\frac{n\pi}{2}\right)}{n} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n} - 0 - \frac{1}{n^2} \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2n} \sin \frac{n\pi}{2} + \frac{1}{n^2} \cos \frac{n\pi}{2} - \frac{1}{n^2} \right]$$

$$b_n = \frac{1}{\pi} \int_0^{\pi/2} f(x) \sin x\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi/2} 2x \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x \cdot \left(-\frac{\cos nx}{n}\right) - 1 \cdot \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{n} + \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right]$$

The required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\left\{\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\right\}\cos nx + \frac{2}{\pi}\left\{-\frac{\pi}{2}\frac{\cos\left(\frac{n\pi}{2}\right)}{n} + \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right\}\sin nx\right]$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\left\{\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\right\}\cos nx + \left\{\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)\right\}\sin nx\right]$$

c. Given,

$$f(x) = \begin{cases} \pi + x & \text{for } -\pi < x < 0 \\ \pi - x & \text{for } 0 < x < \pi \end{cases}$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\, dx$$

$$= \frac{1}{\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{0 - \left(-\pi^2 + \frac{\pi^2}{2}\right)\right\} + \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right)$$

$$= \frac{1}{\pi} \times \frac{\pi^2}{2} + \frac{1}{\pi} \times \frac{\pi^2}{2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi + x)\frac{\sin nx}{n} - 1 \cdot \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1) \cdot \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2} + \frac{\cos n\pi}{n^2}\right] + \frac{1}{\pi}\left[0 - \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x) \sin nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi+x)\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\right] + \frac{1}{\pi}\left[0 - \left(-\frac{\pi}{n} - 0\right)\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\pi}{n}\right)$$

$$= -\frac{2}{n}$$

Required Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{2} - 2\sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

7. Find the Fourier series of the following functions in the given interval.

a. $f(x) = \begin{cases} 0 & \text{for } -2 < x < 0 \\ 2 & \text{for } 0 < x < 2 \end{cases}$ and $f(x + 4) = f(x)$

b. $f(x) = \begin{cases} 1, & -L < x < 0 \\ 0, & 0 \le x < L \end{cases}$ and $f(x + 2L) = f(x)$

Solution

a. Given,

$$f(x) = \begin{cases} 0 & \text{for } -2 < x < 0 \\ 2 & \text{for } 0 < x < 2 \end{cases} \text{ and } f(x+4) = f(x)$$

The Fourier coefficients are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x) \, dx$$

$$= \frac{1}{2}\int_{-2}^{0} f(x) \, dx + \frac{1}{2}\int_{0}^{2} f(x) \, dx$$

$$= \frac{1}{2}\int_{-2}^{0} (0) \, dx + \frac{1}{2}\int_{0}^{2} (2) \, dx$$

$$= 0 + [x]_0^2$$

$$= 2$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2}\int_{-2}^{0} f(x)\cos\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2}\int_{0}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2}\int_{-2}^{0} (0)\cos\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2}\int_{0}^{2} (2)\cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_{0}^{2} \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{2}{n\pi}[\sin n\pi - \sin 0]$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2}\int_{-2}^{0} f(x)\sin\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2}\int_{0}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right) dx$$

$$= 0 + \frac{1}{2}\int_{0}^{2} (2)\sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_0^2$$

$$= -\frac{2}{n\pi}[\cos n\pi - \cos 0]$$

$$= -\frac{2}{n\pi}[(-1)^n - 1]$$

$$= \frac{2}{n\pi}[1 - (-1)^n]$$

Required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \frac{n\pi x}{2}\right)$$

$$= \frac{2}{2} + \sum_{n=1}^{\infty}\left[0 + \frac{2}{n\pi}[1-(-1)^n]\sin\left(\frac{n\pi x}{2}\right)\right]$$

$$= 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}[1-(-1)^n]\sin\left(\frac{n\pi x}{2}\right)$$

$$= 1 + \frac{2}{\pi}\left[2\cdot\sin\left(\frac{\pi x}{2}\right) + 2\cdot\sin\left(\frac{3\pi x}{2}\right) + 2\cdot\sin\left(\frac{5\pi x}{2}\right) + \ldots\right]$$

$$= 1 + \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{2}\right) + \sin\left(\frac{3\pi x}{2}\right) + \sin\left(\frac{5\pi x}{2}\right) + \ldots\right]$$

b. Given,

$$f(x) = \begin{cases} 1, & -L < x < 0 \\ 0, & 0 \le x < L \end{cases} \text{ and } f(x+2L) = f(x)$$

Solution

Let the required Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad \ldots \text{(i)}$$

The Fourier coefficients are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} f(x)\,dx + \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} (1)\,dx + \frac{1}{L}\int_{0}^{L} (0)\,dx$$

$$= \frac{1}{L}[x]_{-L}^{0} + 0$$

$$= \frac{1}{L}(0 + L)$$

$$= 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} (1)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} (0)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_{-L}^{0} + 0$$

$$= \frac{1}{L}\cdot\frac{L}{n\pi}\left[\sin 0 - \sin\left(-\frac{n\pi L}{L}\right)\right]$$

$$= \frac{1}{n\pi}\sin n\pi$$

$$= \frac{1}{n\pi}\cdot 0$$

$$= 0 \qquad [\because \sin n\pi = 0]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} (1)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} (0)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[-\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_{-L}^{0} + 0$$

$$= -\frac{1}{L}\cdot\frac{L}{n\pi}\left[\cos 0 - \cos\left(-\frac{n\pi L}{L}\right)\right]$$

$$= -\frac{1}{n\pi}[1 - \cos n\pi]$$

$$= -\frac{1}{n\pi}[1-(-1)^n] \quad [\because \cos n\pi = (-1)^n]$$

Putting the values of a_0, a_n and b_n in equation (i), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\frac{n\pi x}{L}\right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left[0 - \frac{1}{n\pi}[1-(-1)^n]\sin\frac{n\pi x}{L}\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}[1-(-1)^n]\sin\frac{n\pi x}{L}\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\frac{2}{1}\sin\frac{\pi x}{L} + 0 + \frac{2}{3}\sin\frac{3\pi x}{L} + \frac{2}{5}\sin\frac{5\pi x}{L} + \dots\right]$$

$$= \frac{1}{2} - \frac{2}{\pi}\left[\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \dots\right]$$

8. Expand $f(x) = x,\ 0 < x < \pi$ in a half range (a) sine series (b) cosine series.

Solution

Here, $f(x) = x$ and $L = \pi$

a. The half range Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin(nx) \qquad \ldots(i)$$

The Fourier coefficients are

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{\pi}\int_0^\pi x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left(-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - 0\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cdot(-1)^n}{n} + 0\right] \qquad [\because \sin n\pi = 0,\ \cos n\pi = (-1)^n]$$

$$= \frac{2}{\pi}\frac{(-1)^{n+1}\cdot\pi}{n}$$

$$= \frac{2}{n}(-1)^{n+1}$$

Required Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

b. The half range Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ldots \text{(i) as } L = \pi$$

The Fourier coefficients are

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$= \pi$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\pi\cdot 0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

Required half range Fourier cosine series from (i) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$$

9. Expand $f(x) = \pi - x,\ 0 < x < \pi$ in a half range cosine series and hence deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Solution

Given,

$f(x) = \pi - x$

Here, $L = \pi$

Required half range Fourier cosine series from (i) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \dots \text{(i)}$$

Now,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]$$

$$= \frac{2}{\pi}\cdot\frac{\pi^2}{2}$$

$$= \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{(\pi - \pi)\frac{\sin n\pi}{n} - \frac{\cos n\pi}{n^2}\right\} - \left(0 - \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{\pi n^2}[1 - (-1)^n]$$

Required half range Fourier cosine series from (i) is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[1 - (-1)^n]\cos nx$$

or, $$\pi - x = \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} + \ldots\right]$$

or, $$\pi - x = \frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

Put $x = 0$ in (ii), we get

$$\pi - 0 = \frac{\pi}{2} + \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

or, $$\frac{\pi}{2}\times\frac{\pi}{4} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$\therefore$ $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

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