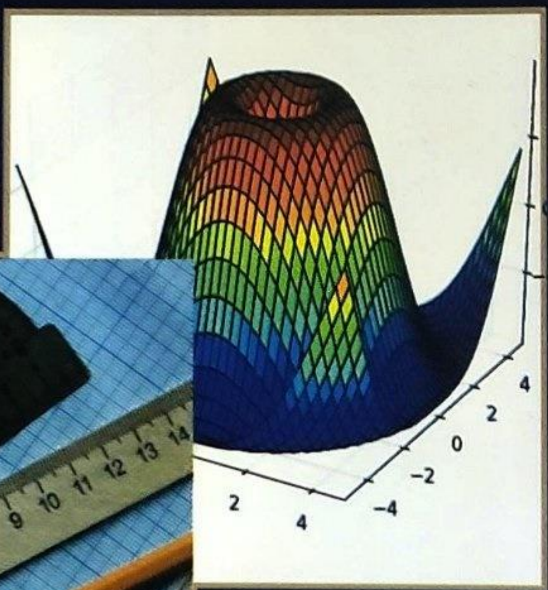
Based on New syllabus of Diploma in Engineering

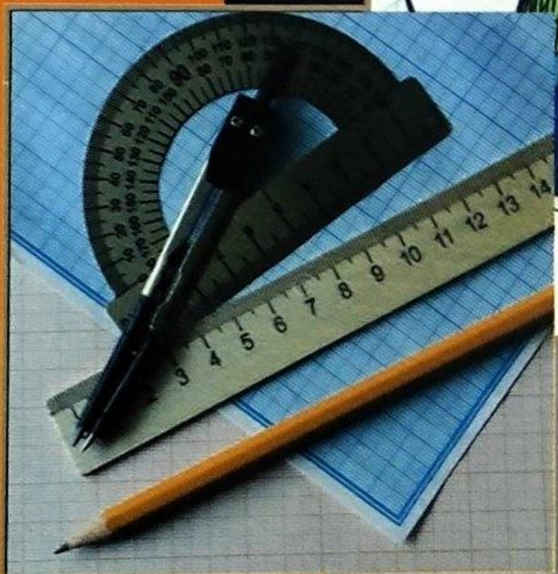
BOOK

CTEVT

ENGINEERING MATHEMATICS III

For Diploma in Engineering all Programme 2nd year I Part





Engineering Mathematics III
EG2101SH

Year: II | **Total: 4 Hrs./week**
Part: I | **Lecture: 3 Hrs./week**
Tutorial: 1 Hrs./week
Practical: Hrs./week
Lab: Hrs./week

Course Description:
This course consists of five units namely: Applications of derivatives, Partial derivatives, application of Anti-derivatives, Differential equations and Fourier series; which are basically necessary to develop mathematical knowledge and helpful for understanding as well as practicing their skills in the related engineering fields.

Course Objectives:
On completion of this course, students will be able to understand the concept of the following topics and apply them in the related fields of different engineering areas: Applications of derivatives and anti-derivatives, Partial derivatives, differential equations and Fourier series.

Course Contents:

Theory

Unit 1. Applications of Derivatives **[12 Hrs.]**
1.1. Derivatives of inverse circular functions and hyperbolic functions
1.2. Differentials, tangent and normal
1.3. Maxima and minima, concavity, increasing and decreasing functions
1.4. Rate measures
1.5. Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$ and $\infty - \infty$, L'Hospital's Rule (without proof)

Unit 2. Partial Derivatives **[6 Hrs.]**
2.1. Functions of more than two variables
2.2. Partial derivative from First principles
2.3. Partial derivatives of First and higher orders
2.4. Euler's theorem for function of two variables
2.5. Partial derivatives of composite functions

Unit 3. Applications of Anti-derivatives **[8 Hrs.]**
3.1. Standard Integrals, related numerical problems
3.2. Basic idea of curve sketching**:** odd and even functions, periodicity of a function, symmetry (about x-axis, y-axis and origin), monotonicity of a function, sketching graphs of polynomial, trigonometric, exponential, and logarithmic functions (simple cases only)
3.3. Area under a curve using limit of sum (without proof)
3.4. Area between two curves (without proof)
3.5. Area of closed a curve (circle and ellipse only)

Unit 4. Differential Equations **[14 Hrs.]**
4.1. Ordinary Differential Equations (ODEs)
   4.1.1. Definitions, order and degree of differential equation
   4.1.2. Differential equation of First order and First degree
   4.1.3. Variable separation and variable change methods

4.1.4. Homogeneous and linear differential equation of First order
4.1.5. Exact differential equation, condition of exactness
4.1.6. Simple applications of First order differential equations

4.2. Partial Differential Equations (PDEs)
4.2.1. Basic concepts, definition and formation
4.2.2. General solution of linear PDEs of first order (Pp + Qq = R form)

Unit 5. Fourier Series **[5 Hrs.]**

5.1. Periodic functions and fundamental period of periodic functions
5.2. Odd and even functions with their properties
5.3. Trigonometric series
5.4. Fourie's series in an interval of period 2π (arbitrary range is not required)

Tutorial: **[15 Hrs.]**

1. Applications of Derivatives [4 Hrs.]
2. Partial Derivatives [2 Hrs.]
3. Applications of Anti-derivatives [3 Hrs.]
4. Differential Equations [5 Hrs.]
5. Fourier Series [1 Hrs.]

Evaluation Scheme:

Unit wise Marks division for Final

S. No.	Units	Short questions (2 marks)	Long questions (4 marks)	Total Marks
1	Applications of Derivatives	4 x 2 = 8	3 x 4 = 12	20
2	Partial Derivatives	2 x 2 = 4	2 x 4 = 8	12
3	Applications of Anti-derivatives	3 x 2 = 6	3 x 4 = 12	18
4	Differential Equations	4 x 2 = 8	4 x 4 = 16	24
5	Fourier Series	1 x 2 = 2	1 x 4 = 4	6
		14 x 2 = 28	**13 x 4 = 52**	**80**

UNIT 1

Applications of Derivatives

Course Contents

- Derivatives of inverse circular functions and hyperbolic functions
- Differentials, tangent and normal
- Maxima and minima, concavity, increasing and decreasing functions
- Rate measures
- Indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}$ and $\infty - \infty$, L'Hospital's Rule (without proof)

1.1 Introduction

Derivatives have wide range of applications in a large number of disciplines such as engineering, computer science, physics, economics, social science, etc. The derivative as the rate of change has many applications in the field of science and technology as well as management and social sciences. We can use the derivative to obtain the information about the shape of graph of a function.

The derivative is a fundamental concept in calculus which has many applications in different field of engineering. For instances, the civil engineers are using derivatives to analyze the stability of structures, design of foundations, etc. and the derivative is used to design and optimize the algorithms by computer engineers. In fact, the derivative is powerful tool in different field of engineering that helps to model, optimize and analyze the engineering systems. Beginning with recalling the definitions and notations of derivatives, we proceed to discuss the applications of derivatives.

The process of finding the derivative is called differentiation. A formula of finding the derivative of a function was initiated independently by Sir Isaac Newton (1642 – 1727) and Gottfried W. Leibnitz (1646 – 1716). The differential calculus deals with the problem of finding instantaneous rate of change of a function. To determine the slope of a tangent line at an arbitrary point on a curve is the origin of derivative in geometrical sense. The derivative has great importance in calculus.

Notation

There are many ways to denote the derivative of a function $y = f(x)$. The most common notations are

$$f'(x),\ y',\ \frac{dy}{dx},\ \frac{df}{dx},\ \frac{d}{dx}f(x),\ D_x f,\ y'$$

NOTE i. Read y' as 'y prime'.

ii. Read $\frac{dy}{dx}$ as 'the derivative of y with respect to x'.

The prime notations come from notations that Newton used for derivatives. The $\frac{d}{dx}$ notations are similar to those used by Leibnitz.

Input: Function $y = f(x)$ $\longrightarrow$ Operator $\frac{d}{dx}$ $\longrightarrow$ **Output:** Derivative $y' = \frac{dy}{dx}$

Increments

When the independent variable changes from one value to another, then the difference obtained by subtracting the initial value from the new value is called increment. It is also simply known as difference. An increment in x is usually denoted by Δx and is read as "delta x". **It should be noted that Δx is a symbol representing the increment not the product of Δ and x.**

Let $y = f(x)$ be a given continuous function. Then the value of y depends upon the value of x. Hence x is an **independent variable** and y is a **dependent variable**. Let Δy be an increment in y corresponding to an increment Δx in x then,

$$y + \Delta y = f(x + \Delta x). \text{ Then, } \Delta y = f(x + \Delta x) - f(x).$$

Incremental Ratio

If y changes to $y + \Delta y$ corresponding to change of x to $x + \Delta x$, then $\frac{\Delta y}{\Delta x}$ is called incremental ratio or quotient of differences.

Definition of Derivative

Let $y = f(x)$ be a function. If Δy be an increment in y corresponding to an increment Δx in x, then $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, if it exists, is called derivative of y with respect to x and is denoted by $\frac{dy}{dx}$. It may be denoted by $\frac{df(x)}{dx}$ or $f'(x)$.

The process of finding the differential coefficient or derivative from this method is known as the method of differentiation from the **definition** or the **first principle**.

Thus, we can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Replacing the increment Δx by h, we write

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

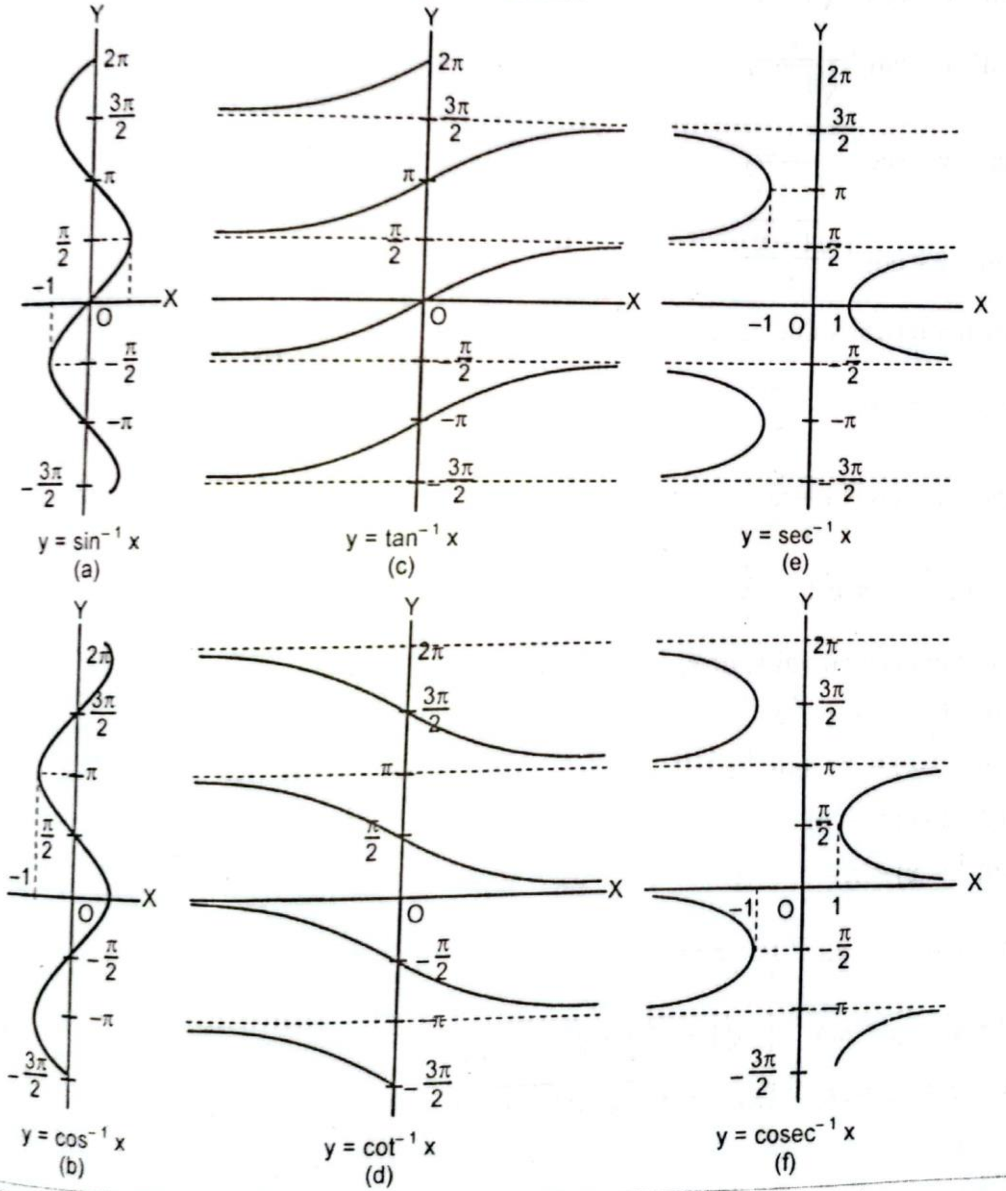
1.2 Derivatives of Inverse Circular Functions

We have studied inverse circular functions in first semester. Now, we recall the defined domain and range, graphs and some important results of inverse circular functions.

The domain and range of inverse circular functions are tabulated as follows:

Functions	Domain (x)	Range (y)
$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1}x$	$(-\infty, \infty)$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \cot^{-1}x$	$(-\infty, \infty)$	$0 < y < \pi$
$y = \sec^{-1}x$	$x \ge 1$ or $x \le -1$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$
$y = \operatorname{cosec}^{-1}x$	$x \ge 1$ or $x \le -1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$

The graphs of inverse circular functions are given below.



NOTE The inverse trigonometric function $y = \sin x$ is also written as $y = \text{arc} \sin x$.

Some Important Results on Inverse Circular Functions

1. Self adjust property
   a. $\sin \sin^{-1} x = x = \sin^{-1} \sin x$
   b. $\cos \cos^{-1} x = x = \cos^{-1} \cos x$
   c. $\tan \tan^{-1} x = x = \tan^{-1} \tan x$
2. Reciprocal property
   a. $\sin^{-1} x = \operatorname{cosec}^{-1} \frac{1}{x}$
   b. $\cos^{-1} x = \sec^{-1} \frac{1}{x}$
   c. $\tan^{-1} x = \cot^{-1} \frac{1}{x}$
3. Conversion property
   a. $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$
   b. $\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$
   c. $\sin^{-1} x = \sec^{-1} \frac{1}{\sqrt{1-x^2}}$
   d. $\sin^{-1} x = \cot^{-1} \frac{\sqrt{1-x^2}}{x}$
4. For any numerical value of x,
   a. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$
   b. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
   c. $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$
5. For given numerical value of x.
   a. $\sin^{-1}(-x) = -\sin^{-1} x$
   b. $\cos^{-1}(-x) = \pi - \cos^{-1} x$
   c. $\tan^{-1}(-x) = -\tan^{-1} x$
   d. $\cot^{-1}(-x) = \pi - \cot^{-1} x$
6. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $xy < 1$
7. $\sin^{-1} x + \sin^{-1} y = \sin^{-1} \left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$
8. $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

Now, we study about the derivative of inverse circular functions.

1. **Derivative of $\sin^{-1} x$ by first principles**

Let $f(x) = \sin^{-1}x$

$\therefore\ f(x+h) = \sin^{-1}(x+h)$

We have,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\therefore\ \frac{d}{dx}(\sin^{-1}x) = \lim_{h\to 0}\frac{\sin^{-1}(x+h)-\sin^{-1}x}{h} \quad \text{...(i)}$$

Put $y = \sin^{-1}x$ and $y + k = \sin^{-1}(x+h)$

so that $x = \sin y$ and $x + h = \sin(y+k)$

Then, $h = \sin(y+k) - \sin y$. As $h \to 0$, $k \to 0$

Then from (i)

$$\frac{d}{dx}(\sin^{-1}x) = \lim_{h\to 0}\frac{y+k-y}{h}$$

$$= \lim_{k\to 0}\left\{\frac{k}{\sin(y+k)-\sin y}\right\}$$

$$= \lim_{k\to 0}\left\{\frac{k}{2\cos\left(\frac{y+k+y}{2}\right)\sin\left(\frac{y+k-y}{2}\right)}\right\}$$

$$= \lim_{k\to 0}\left\{\frac{k}{2\cos\left(y+\frac{k}{2}\right)\cdot\sin\frac{k}{2}}\right\}$$

$$= \lim_{k\to 0}\left\{\frac{1}{2\cos\left(y+\frac{k}{2}\right)\cdot\frac{\sin\frac{k}{2}}{\frac{k}{2}\cdot 2}}\right\}$$

$$= \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Alternatively by formula

Let $y = \sin^{-1}x$; $x \in [-1, 1]$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Then $x = \sin y$.

Differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{d}{dy}(\sin y) = \cos y$$

or, $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

$\therefore \quad \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}, \; -1 < x < 1.$

2. Derivative of $\cos^{-1} x$

Let $y = \cos^{-1} x, x \in [-1, 1], y \in [0, \pi]$.

Then $x = \cos y$.

Differentiating both sides w.r.t. y, we get

$\frac{dx}{dy} = \frac{d}{dy} (\cos y) = -\sin y$

or, $\frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$

$\therefore \quad \frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$

3. Derivative of $\tan^{-1} x$

Let $y = \tan^{-1} x, \; x \in \mathbb{R}, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Then $x = \tan y$.

Differentiating both sides w.r.t. y, we get

$\frac{dx}{dy} = \frac{d}{dy} (\tan y) = \sec^2 y$

or, $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

$\therefore \quad \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$

4. Derivative of $\cot^{-1} x$

Let $y = \cot^{-1} x, x \in \mathbb{R}, y \in [0, 2\pi]$.

Then $x = \cot y$

Differentiating both sides w.r.t. y, we get

$\frac{dx}{dy} = \frac{d}{dy} (\cot y) = -\operatorname{cosec}^2 y$

or, $\frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}$

$\therefore \quad \frac{d(\cot^{-1} x)}{dx} = -\frac{1}{1+x^2}; \quad x \in \mathbb{R}$

5. **Derivative of $\sec^{-1}x$**

Let $y = \sec^{-1}x$; $|x| > 1$, $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

Then $x = \sec y$.

Differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{d}{dy}(\sec y) = \sec y \tan y$$

or, $$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{|x|\sqrt{x^2 - 1}}, x \neq 1.$$

For $0 < y < \frac{\pi}{2}$, $\tan y > 0$, and $\sec y > 0$ and for $\frac{\pi}{2} < y < \pi$, $\tan y < 0$ and $\sec y < 0$.

In both cases, the product of $\tan y$ and $\sec y$ is positive.

$$\therefore \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}, |x| > 1.$$

6. **Derivative of $\operatorname{cosec}^{-1}x$**

Let $y = \operatorname{cosec}^{-1}x$, $|x| > 1$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Then $x = \operatorname{cosec} y$.

Differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{d}{dy}(\operatorname{cosec} y) = -\operatorname{cosec} y \cot y$$

or, $$\frac{dy}{dx} = \frac{-1}{|\operatorname{cosec} y \cot y|}, \quad y \neq \pm\frac{\pi}{2}$$

or, $$\frac{dy}{dx} = \frac{-1}{\left|\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y - 1}\right|}, \quad y \neq \pm\frac{\pi}{2}$$

$$= \frac{-1}{|x|\sqrt{x^2 - 1}}.$$

For $0 < y < \frac{\pi}{2}$, $\cot y > 0$ and $\operatorname{cosec} y > 0$ and for $-\frac{\pi}{2} < y < 0$, $\cot y < 0$ and $\operatorname{cosec} y < 0$. In both cases, the product is always positive in $-\frac{\pi}{2} < y < \frac{\pi}{2}, y \neq 0$.

$$\therefore \frac{d(\operatorname{cosec}^{-1}x)}{dx} = \frac{-1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1.$$

Summary

1. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1.$
2. $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1.$

3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$

4. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \quad x \in \mathbb{R}$

5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$

6. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$

We also recall the important differentiation formulae that we have studied in first semester.

1. Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

2. Derivative of a Constant

$$\frac{d(c)}{dx} = 0, c \in \mathbb{R}$$

3. Derivative of a Constant Times a Function

$$\frac{d}{dx}[k \cdot f(x)] = k\frac{d}{dx}f(x)$$

4. Derivative of a Sum of Functions

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

5. Derivative of a Product of Functions

$$\frac{d}{dx}[u \cdot v] = u\frac{d(v)}{dx} + v\frac{d(u)}{dx}$$

6. Derivative of a Quotient of Functions

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

7. Derivative of Composite Function (Chain Rule)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

8. General Power Rule

$$\frac{du^n}{dx} = nu^{n-1} \cdot \frac{du}{dx}$$

9. $\frac{d}{dx}(\sin x) = \cos x$

10. $\frac{d}{dx}(\cos x) = -\sin x.$

11. $\frac{d}{dx}(\tan x) = \sec^2 x.$

12. $\frac{d}{dx}(\sec x) = \sec x \tan x.$

13. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

14. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

15. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

16. $\frac{d}{dx}(e^x) = e^x$

17. $\frac{d}{dx}(e^{ax}) = ae^{ax}$

Illustrative Examples

Example 1. Find from definition the derivative of $\cot^{-1} x$.

Solution

Let $f(x) = \cot^{-1} x$.

Then, $f(x+h) = \cot^{-1}(x+h)$

We have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cot^{-1}(x+h) - \cot^{-1} x}{h} \quad \ldots \text{(i)}$$

Put $y = \cot^{-1} x$ and $y + k = \cot^{-1}(x+h)$.

Then, $x = \cot y$ and $x + h = \cot(y+k)$

As, $h \to 0, k \to 0$.

From (i)

$$f'(x) = \lim_{h \to 0} \frac{y + k - y}{h}$$

$$= \lim_{h \to 0} \frac{k}{h}$$

$$= \lim_{k \to 0} \frac{k}{\cot(y+k) - \cot y}$$

$$= \lim_{k \to 0} \frac{k}{\frac{\cos(y+k)}{\sin(y+k)} - \frac{\cos y}{\sin y}}$$

$$= \lim_{k \to 0} \frac{k}{\frac{\sin y \cos(y+k) - \cos y \sin(y+k)}{\sin(y+k) \sin y}}$$

$$= \lim_{k \to 0} \frac{k \cdot \sin(y+k) \sin y}{\sin(y - y - k)}$$

$$= \lim_{k \to 0} \frac{k \cdot \sin(y+k)\sin y}{-\sin k}$$

$$= \lim_{k \to 0} \frac{-\sin(y+k)\sin y}{\frac{\sin k}{k}}$$

$$= -\sin y \cdot \sin y$$

$$= -\frac{1}{\operatorname{cosec}^2 y}$$

$$= -\frac{1}{\sqrt{1+\cot^2 y}}$$

$$= -\frac{1}{\sqrt{1+x^2}}$$

Example 2. Find the derivative of $\sin^{-1} x^2$.

Solution

Let $y = \sin^{-1} x^2$.

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x^2)$$

$$= \frac{d(\sin^{-1} x^2)}{d(x^2)} \cdot \frac{d(x^2)}{dx}$$

$$= \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

Example 3. Find the derivative of $\tan^{-1} x^3$.

Solution

Let $y = \tan^{-1} x^3$.

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x^3)$$

$$= \frac{d(\tan^{-1} x^3)}{d(x^3)} \cdot \frac{d(x^3)}{dx}$$

$$= \frac{1}{\sqrt{1+(x^3)^2}} \cdot 3x^2$$

$$= \frac{3x^2}{1+x^6}$$

Example 4. Find $\frac{dy}{dx}$ when $y = \sin^{-1}(3x - 4x^3)$.

Solution

Given,

$$y = \sin^{-1}(3x - 4x^3)$$

Put,

$$x = \sin\theta \quad \text{... (i)}$$

Then,

$$y = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$$

or, $y = \sin^{-1}(\sin 3\theta)$

$\therefore \quad y = 3\theta$ ... (ii)

From (i),

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin\theta) = \cos\theta$$

From (ii),

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(3\theta) = 3$$

Now, by Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{3}{\cos\theta}$$

$$= \frac{3}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{3}{\sqrt{1-x^2}} \quad [\because \text{Using (i)}]$$

$$\therefore \quad \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$$

Example 5. Find $\frac{dy}{dx}$ where $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

Solution

Given,

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Put

$$x = \tan\theta \quad \text{... (i)}$$

Then,

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

or, $y = \tan^{-1}(\tan 2\theta)$

$\therefore \quad y = 2\theta$ ... (ii)

From (i)

$$x = \tan\theta$$

$\therefore \quad \frac{dx}{d\theta} = \frac{d}{d\theta}(\tan\theta) = \sec^2\theta$

and from (ii) $y = 2\theta$

$\therefore \quad \frac{dy}{d\theta} = \frac{d}{d\theta}(2\theta) = 2$

Now,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{2}{\sec^2\theta}$$

$$= \frac{2}{1+\tan^2\theta}$$

$$= \frac{2}{1+x^2}$$

Alternatively,

We have,

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad \ldots \text{(i)}$$

Suppose, $x = \tan\theta \qquad \ldots$ (ii)

Now, (i) becomes

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$$

or, $y = 2\theta$

or, $y = 2\tan^{-1}x \qquad [\because$ from (ii)]

Differentiating both sides with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$= 2\frac{d}{dx}(\tan^{-1}x) = 2\left(\frac{1}{1+x^2}\right)$$

Hence, $\frac{dy}{dx} = \frac{2}{1+x^2}$

Example 6. Find the derivative of $\sin^{-1}(1-2x^2)$.

Solution

Let $y = \sin^{-1}(1-2x^2)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\sin^{-1}(1-2x^2)\}$$

$$= \frac{d\{\sin^{-1}(1-2x^2)\}}{d(1-2x^2)} \cdot \frac{d(}{d}$$

$$= \frac{1}{\sqrt{1-(1-2x^2)^2}} \cdot \left\{\frac{d}{dx}(1) - 2 \cdot \frac{d}{x}\right.$$

$$= \frac{1}{\sqrt{1-(1-2x^2)^2}}(-4x)$$

$$= \frac{1}{\sqrt{1-1+4x^2-4x^4}}(-4x)$$

$$= \frac{1}{\sqrt{4x^2(1-x^2)}}(-4x)$$

$$= \frac{1}{2x\sqrt{1-x^2}}(-4x)$$

$$= -\frac{2}{\sqrt{1-x^2}}$$

Example 7. Find the derivative of $\tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)$

Solution

Let $y = \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)$

$$= \tan^{-1}\left(\frac{2\sin x\cos x}{1+2\cos^2 x-1}\right)$$

$$= \tan^{-1}\left(\frac{2\sin x\cos x}{2\cos^2 x}\right)$$

$$= \tan^{-1}\left(\frac{\sin x}{\cos x}\right)$$

$$= \tan^{-1}(\tan x)$$

$$= x$$

Now, differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

Example 8. Find the derivative of $\tan^{-1}(\sec x + \tan x)$.

Solution

Let

$$y = \tan^{-1}(\sec x + \tan x)$$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right)$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}+\frac{x}{2}$$

Thus, $y = \frac{\pi}{4}+\frac{x}{2}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$= \frac{1}{2}$$

Exercise 1.1

1. Find from definition, the derivative of

   a. $\cos^{-1} x$ b. $\tan^{-1} x$ c. $\ln \cos^{-1} x$

2. Find the derivatives of the following functions:

   a. $\sin^{-1} x^3$ b. $\cos^{-1}\sqrt{x}$ c. $\sin^{-1}(3x-4x^3)$

   d. $\cos^{-1}(4x^3-3x)$ e. $\tan^{-1}\left(\frac{2x}{\ldots}\right)$ f. $\tan^{-1}\left(\frac{\sin x}{\ldots\cos x}\right)$

3. Find the derivative of

a. $\ln(\sin^{-1} x)$ b. $\sin^{-1}(\cos x)$ c. $\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

4. a. Find the derivative of $\tan^{-1} x$ with respect to $\cot^{-1} x$.

b. Find the derivative of $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Answers

1. a. $\frac{-1}{\sqrt{1-x^2}}$ b. $\frac{1}{1+x^2}$ c. $\frac{-1}{\cos^{-1} x\sqrt{1-x^2}}$

5. a. $\frac{3x^2}{\sqrt{1-x^6}}$ b. $-\frac{1}{2\sqrt{x}\sqrt{1-x}}$ c. $\frac{3}{\sqrt{1-x^2}}$ d. $-\frac{3}{\sqrt{1-x^2}}$ e. $\frac{2}{1+x^2}$

f. $\frac{1}{2}$ g. $\frac{2}{1+x^2}$ h. $\frac{1}{2(1+x^2)}$ i. $\frac{1}{2}$

3. a. $\frac{1}{\sin^{-1} x}\cdot\frac{1}{\sqrt{1-x^2}}$ b. -1 c. $\frac{3}{1+x^2}$

4. a. -1 b. $\frac{2}{3}$

1.3 Hyperbolic Functions

The hyperbolic functions are a set of functions which are closely related to the trigonometric functions. The hyperbolic functions are related to the hyperbola.

The hyperbolic sine is defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \ldots (i)$$

The hyperbolic cosine is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \ldots (ii)$$

The functions defined in (i) and (ii) have properties similar to the trigonometric functions $\sin x$ and $\cos x$.

Therefore

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbb{R}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad x \in \mathbb{R}$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \in \mathbb{R} - \{0\}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad x \in \mathbb{R}$$

Also,

$$\cosh x + \sinh x = e^x$$

$$\cosh x - \sinh x = e^{-x}$$

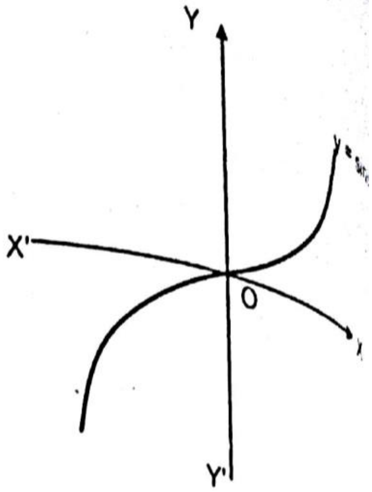
Also, we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$\therefore \quad \cosh x = \cos ix \qquad (\because i^2 = -1)$

$\sinh x = -i \sin ix$

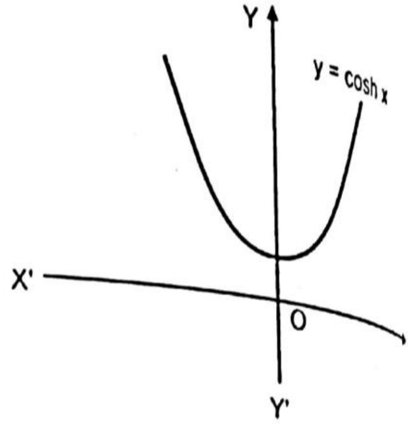
r, $\sinh 0 = 0$ and $\cosh 0 = 1$



$\sinh(-x) = -\sinh x,$

$\cosh(-x) = \cosh x$

and $\cosh^2 x - \sinh^2 x = 1$



We also have

$$\lim_{x \to \infty} \sinh x = \lim_{x \to \infty} \frac{e^x - e^{-x}}{2} = \infty$$

$$\lim_{x \to -\infty} \sinh x = \lim_{x \to -\infty} \frac{e^x - e^{-x}}{2} = -\infty$$

Similarly, $\lim_{x \to \pm\infty} \cosh x = \infty$. The graph of $\sinh x$ and $\cosh x$ are shown in the above figures.

Also, $\tanh 0 = 0$ and $\tanh(-x) = -\tanh x$

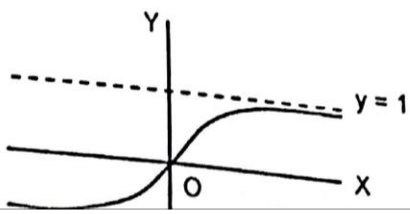
Now

$$\lim_{x \to \infty} \tanh x = \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to \infty} \frac{1 - e^{-2x}}{1 + e^{2x}} = 1.$$

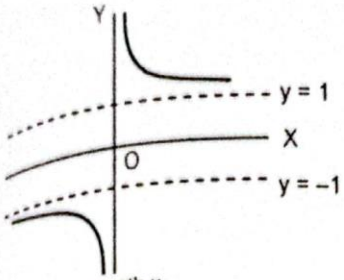
and

$$\lim_{x \to -\infty} \tanh x = \lim_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to -\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = -1.$$

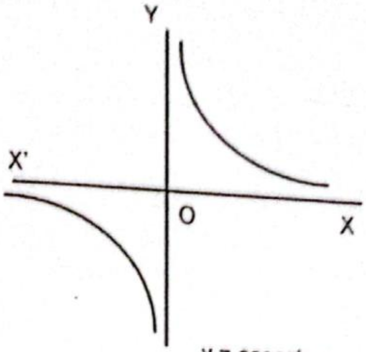
With these properties the graph of $y = \tanh x$ is as follows:



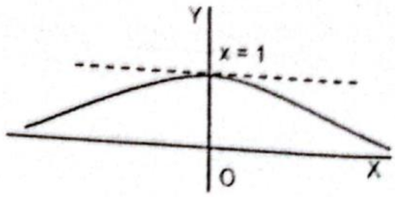
The graphs of $y = \coth x$, $y = \text{cosech } x$ and $y = \text{sech } x$ are shown below.



y = coth x



y = cosech x



y = sech x

From above, we can obtain the following relations:

1. $\cosh^2 x + \sinh^2 x = \cosh 2x$
2. $\cosh^2 x - \sinh^2 x = 1$
3. $\sinh 2x = 2 \sinh x \cosh x$
4. $\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y$
5. $\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y$

1.4 Derivative of Hyperbolic Functions

Derivative of sinh x

Let $y = \sinh x$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh x)$$

$$= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\frac{d}{dx}(e^x - e^{-x})$$

$$= \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

Hence, $\frac{d}{dx}(\sinh x) = \cosh x$

Similarly, $\frac{d}{dx}(\cosh x) = \sinh x$.

Derivative of tanh x

Let $y = \tanh x = \frac{\sinh x}{\cosh x}$. Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$

$$= \frac{\cosh x \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{(\cosh x)^2}$$ [Using quotient rule]

$$= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

Hence, $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

Similarly,

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \cdot \coth x.$$

Derivative of Inverse Hyperbolic Functions

Derivative of $\sinh^{-1} x$

Let $y = \sinh^{-1} x$

Then $x = \sinh y$

Differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{d}{dy}(\sinh y)$$

$$= \cosh y$$

$$= \sqrt{1 + \sinh^2 y}$$

$$= \sqrt{1 + x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

Similarly, we can obtain

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}, (x > 1)$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}, (-1 < x < 1)$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{-1}{x^2 - 1}, (|x| > 1)$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1 - x^2}}, (0 < x < 1)$$

and $\frac{d}{dx}(\operatorname{cosech}^{-1} x) = \frac{-1}{|x|\sqrt{1 + x^2}}, (x \in \mathbb{R} - \{0\}).$

Illustrative Examples

Example 1. Find the derivative of ln sinh x.

Solution

Let $y = \ln \sinh x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\ln \sinh x)$$

$$= \frac{d(\ln \sinh x)}{d(\sinh x)} \cdot \frac{d(\sinh x)}{dx}$$

$$= \frac{1}{\sinh x} \cdot \cosh x$$

$$= \coth x$$

Example 2. Find the derivative of Arc tan sin hx.

Solution

Let y = Arc tan sinh x

$= \tan^{-1}(\sinh x)$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} \sinh x)$$

$$= \frac{d(\tan^{-1} \sinh x)}{d(\sinh x)} \cdot \frac{d(\sinh x)}{dx}$$

$$= \frac{1}{1 + \sinh^2 x} \cdot \cosh x$$

$$= \frac{\cosh x}{\cosh^2 x}$$

$$= \operatorname{sech} x$$

Example 3. Find the derivative of $\tanh^{-1}(\sin x)$.

Solution

Let, $y = \tanh^{-1}(\sin x)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\tanh^{-1}(\sin x)\}$$

$$= \frac{d\{\tanh^{-1}(\sin x)\}}{d(\sin x)} \cdot \frac{d}{dx}(\sin x)$$

$$= \left(\frac{1}{1 - \sin^2 x}\right) \cos x$$

$$= \frac{\cos x}{\cos^2 x}$$

$$= \frac{1}{\cos x}$$

$$= \sec x$$

Example 4. Find the derivative of $2\tanh^{-1}\left(\tan\frac{x}{2}\right)$.

Solution

Let $y = 2\tanh^{-1}\left(\tan\frac{x}{2}\right)$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{2\tanh^{-1}\left(\tan\frac{x}{2}\right)\right\}$$

$$= 2\cdot\frac{d\left\{\tanh^{-1}\left(\tan\frac{x}{2}\right)\right\}}{d\left(\tan\frac{x}{2}\right)}\cdot\frac{d\left(\tan\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)}\cdot\frac{d\left(\frac{x}{2}\right)}{dx}$$

$$= 2\cdot\frac{1}{1-\tan^2\frac{x}{2}}\cdot\sec^2\frac{x}{2}\cdot\frac{1}{2}$$

$$= \frac{\sec^2\frac{x}{2}}{1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}$$

$$= \frac{\sec^2\frac{x}{2}\cdot\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$$

$$= \frac{1}{\cos x}$$

$$= \sec x$$

Example 5. Find the derivative of $e^{\sinh x}$.

Solution

Let $y = e^{\sinh x}$. Then

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sinh x})$$

$$= \frac{d(e^{\sinh x})}{d(\sinh x)}\cdot\frac{d}{dx}(\sinh x)$$

$$= e^{\sinh x}\cdot\cosh x$$

Example 6. Find the derivative of $x^{\cosh x}$.

Solution

Let $y = x^{\cosh x}$

Taking logarithm on both sides, we get

$$\ln y = \cosh x \ln x$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\cosh x \ln x)$$

or, $\frac{d}{dy}(\ln y) \cdot \frac{dy}{dx} = \cosh x \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(\cosh x)$

or, $\frac{1}{y} \cdot \frac{dy}{dx} = \cosh x \cdot \frac{1}{x} + \ln x \cdot \sinh x$

or, $\frac{dy}{dx} = y\left(\frac{1}{x}\cosh x + \ln x \sinh x\right)$

$\therefore \frac{dy}{dx} = x^{\cosh x}\left(\frac{1}{x}\cosh x + \ln x \sinh x\right)$

Example 7. Find the derivative of $\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)^{nx}$

Solution

Let $y = \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)^{nx}$

Taking logarithm on both sides, we get

$$\ln y = nx \ln\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left\{nx \ln\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)\right\}$$

or, $\frac{d}{dy}(\ln y) \cdot \frac{dy}{dx} = nx \frac{d}{dx}\left\{\ln\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)\right\} + \ln\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)\frac{d}{dx}(nx)$

or, $\frac{1}{y} \cdot \frac{dy}{dx} = nx \frac{d\left\{\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\}}{d\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)} \cdot \frac{d\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)} \cdot \frac{d\left(\frac{x}{a}\right)}{dx} + \ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right) . n$

or, $\frac{1}{y} \cdot \frac{dy}{dx} = \left[nx \left\{\frac{1}{\left(\sinh\frac{x}{a}+\cosh\frac{x}{a}\right)} \cdot \left(\cosh\frac{x}{a}+\sinh\frac{x}{a}\right) \cdot \frac{1}{a}\right\} + \ln\left(\sinh\frac{x}{a}+\cosh\frac{x}{a}\right) \cdot n\right]$

or, $\frac{dy}{dx} = y\left[\frac{nx}{a} + n \ln\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)\right]$

$\therefore \frac{dy}{dx} = n\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}\left\{\frac{x}{a} + \log\left(\sinh\frac{x}{a} + \cos\frac{x}{a}\right)\right\}$

Exercise 1.2

Find the derivatives:

1. $\ln(\sinh x)$
2. $\ln(\cosh 2x)$
3. $e^{\cosh x}$
4. $e^{\cosh^{-1} x}$
5. $\sinh^{-1} x - \text{cosech}^{-1} x$
6. Arc cosh (sec x)
7. $\tan^{-1}(\sinh x)$
8. $\cosh^{-1}(\sinh x)$
9. $x \tanh^{-1}\sqrt{x}$
10. $x^{\cosh\frac{x}{a}}$
11. $\left(\cosh\frac{x}{a}\right)^{\ln x}$
12. $x^{\tanh\frac{x}{a}}$
13. $(\cosh x)^{\sinh^{-1} x}$
14. $(\ln x)^{\sinh x}$

Answers

1. $\coth x$
2. $2\tanh 2x$
3. $e^{\cosh x} \cdot \sinh x$
4. $e^{\cosh^{-1} x}\frac{1}{\sqrt{x^2-1}}$
5. $\frac{x+1}{x\sqrt{x^2+1}}$
6. $\sec x$
7. $\text{sech}\, x$
8. $\frac{\cosh x}{\sqrt{\sinh^2 x - 1}}$
9. $\frac{\sqrt{x}}{2(1-x)} + \tanh^{-1}\sqrt{x}$
10. $x^{\cosh\frac{x}{a}}\left[\frac{\cosh\frac{x}{a}}{x} + \frac{\ln x \sinh\frac{x}{a}}{a}\right]$
11. $\left[\frac{1}{a}\ln x \tanh\frac{x}{a} + \frac{1}{x}\ln\cosh\frac{x}{a}\right]\left(\cosh\frac{x}{a}\right)^{\ln x}$
12. $x^{\tanh\frac{x}{a}}\left(\frac{1}{x}\tanh\frac{x}{a} + \frac{1}{a}\ln x\,\text{sech}^2\frac{x}{a}\right)$
13. $(\cosh x)^{\sinh^{-1} x}\left\{\sinh^{-1} x \cdot \tanh x + \frac{1}{\sqrt{1+x^2}}\log\cosh x\right\}$
14. $(\ln x)^{\sinh x}\left\{\frac{\sinh x}{x \ln x} + \cosh x \cdot \ln(\ln x)\right\}$

1.5 Differentials and Approximations

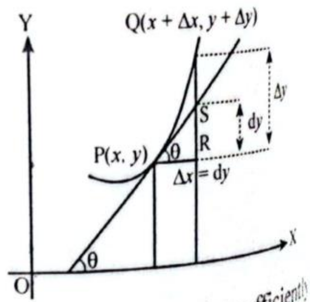
If $y = f(x)$ is a function then the derivative of y with respect to x is $\frac{dy}{dx} = f'(x)$ where $\frac{dy}{dx}$ is a fraction. The quantities dx and dy are called differentials.

If Δx be the small increment in x and Δy be the corresponding small increment in y.

Then, $\Delta y = f(x + \Delta x) - f(x)$

Let P(x, y) and Q$(x + \Delta x, y + \Delta y)$ be two neighbouring points on the curve $y = f(x)$.

Now,



$$\tan\theta = f'(x)$$

or, $\frac{SR}{PR} = f'(x)$

or, $SR = f'(x)\,dx$

or, $dy = f'(x)\,dx$.

The difference between Δy and dy can be made as small as we please by taking Δx sufficiently small. Then dy will approximate Δy.

NOTE
i. $\Delta y = f(x + \Delta x) - f(x)$ is the actual change in dependent variable y.
ii. The differential of independent variable x, denoted by dx, is defined by $dx = \Delta x$.
iii. The differential of dependent variable y, denoted by dy, is defined by $dy = f'(x)\, dx$; which is the approximate change in y.
iv. Error = |Actual change – Approximate change|.
v. Percentage error = $\left|\frac{\Delta y - dy}{y}\right| \times 100$.

Illustrative Examples

Example 1. If $f(x) = x^2 + 2x + 1$, find Δy and dy.

Solution

Here,

$$y = x^2 + 2x + 1$$

Then,

$$y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x) + 1$$

or, $\Delta y = x^2\ 2 \cdot x \cdot \Delta x + (\Delta x)^2 + 2x + 2\Delta x + 1 - x^2 - 2x - 1$

or, $\Delta y = 2x\,\Delta x + (\Delta x)^2 + 2\Delta x$

$\therefore\ \Delta y = (2x + 2 + \Delta x)\,\Delta x$

and $dy = f'(x)\, dx$

$\therefore\ dy = (2x + 2)\, dx.$

Example 2. A circular copper plate is heated so that its radius increases from 5 cm to 5.06 cm. Find the approximate increase in area.

Solution

Let 'r' be the radius and 'A' be the area of the circular plate. Given,

$$r = 5 \text{ cm},\ \Delta r = dr = 5.06 - 5 = 0.06$$

We have,

$$\begin{aligned} A &= \pi r^2 \\ dA &= 2\pi r\, dr \\ &= 2\pi \times 5 \times 0.06 \\ &= 0.6\pi \text{ cm}^2 \end{aligned}$$

$\therefore$ Approximate increase in area $= 0.6\pi \text{ cm}^2$.

Exercise 1.3

Find Δy, dy and $\Delta y - dy$ when $y = x^2 + 5x$ when $x = 2$ and $dx = 0.1$.

What is the exact change in the value of $y = x^2$ when x changes from 10 to 10.1? What is the approximate change in y?

If the radius of sphere changes from 2 cm to 2.01 cm, find the approximate increase in its volume.

If the radius of a circle is increased from 5 to 5.1 cm, find the approximate increase in area.

5. The radius of a circle increases from 10 m to 10.1 m. Estimate the increase in the cir... Also find true change ΔA.
6. Find the approximate increase in the area of a cube if the edge increases from 10 cm to ... Calculate the percentage error in the use of differential approximation.

Answers

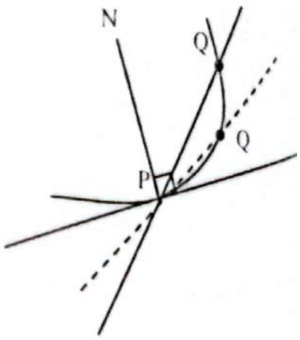
1. 0.91, 0.9, 0.01 2. 2.01, 2 3. $0.16\pi \text{ cm}^3$ 4. $\pi \text{ cm}^2$ 5. $2\pi \text{ m}^2, 2.01\pi$ ...
6. $24 \text{ cm}^2, 0.04\%$

1.6 Tangents and Normals

One of the problems which developed differential calculus was that of finding the slope of tangent to a curve. We first define a tangent and normal to any curve at a given point on it.

Definition

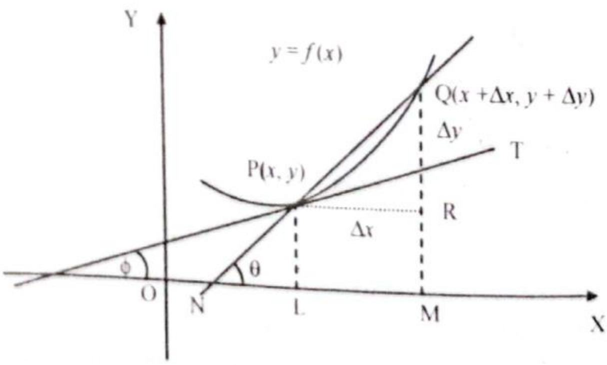
Let P be any fixed point on a given curve and Q be a variable neighboring point on the curve as shown in the figure. Join P and Q. Let Q move along the curve and approaches P; the secant line PQ will rotate about the point P, and PQ may approach a limiting position PT. The limiting position PT of PQ as Q approaches P along the curve is called the **tangent line** to the curve at the point P. The line which is perpendicular to the tangent at the point P is called a **normal** at the point P.



Let $y = f(x)$ be a curve and $P(x, y)$ be a fixed point and $Q(x + \Delta x, y + \Delta y)$ be a neighboring p... the curve. From P and Q, draw PL and QM perpendiculars to the x-axis. Also draw perpendic... on QM. Now, $QR = \Delta y$ and $PR = LM = \Delta x$.

Slope of the secant PQ $= \tan\theta = \dfrac{QR}{PR} = \dfrac{\Delta y}{\Delta x}$.

The ratio $\dfrac{\Delta y}{\Delta x}$ is called difference quotient, where $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$.



Now, when $Q \to P$, $\Delta x \to 0$, $\Delta y \to 0$, $\theta \to \phi$, then PT will be tangent at P. Thus, the slope of t... PT is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \tan\phi,$$ provided that the limit exists.

By the definition of derivative, $f'(x) = \tan\phi$ is the slope of the tangent at P.

Hence, the slope of the tangent to a curve $y = f(x)$ at a point (x, y) is given by the value of the derivative of the function $f(x)$ with respect to x at that point.

NOTE

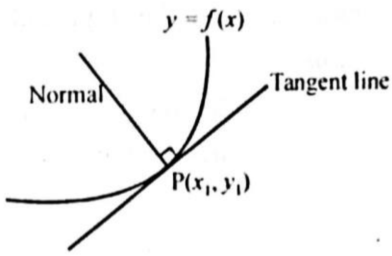
1. The derivative of $f(x)$ at $x = a$ is written as
$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$
2. The slope of the tangent to a curve is sometimes called the slope of the curve.
3. If the tangent is parallel to x axis, then $\phi = 0°$, so $\frac{dy}{dx} = 0$.
4. If the tangent is perpendicular to x axis, then $\phi = 90°$.
$\therefore \frac{dy}{dx} = \tan\phi = \tan 90° = \infty$, i.e., $\frac{dx}{dy} = 0$.

Equation of Tangents and Normals

Let $y = f(x)$ be a given curve and $P(x_1, y_1)$ be a point on the curve.

The equation of any straight line passing through the point $P(x_1, y_1)$ and having slope m is

$$y - y_1 = m(x - x_1)$$



The line becomes the tangent at $P(x_1, y_1)$ if

$$m = \left(\frac{dy}{dx}\right) \text{ at } P(x_1, y_1)$$

Hence, the equation of tangent to the curve

$y = f(x)$ at $P(x_1, y_1)$ is

$$y - y_1 = -\left(\frac{dy}{dx}\right)_{(x,y)=(x_1,y_1)} (x - x_1) \quad \ldots \text{(i)}$$

The normal to the curve at $P(x_1, y_1)$ is perpendicular to the tangent to the curve at $P(x_1, y_1)$

$$\therefore \text{ Slope of normal to the curve at } P(x_1, y_1) = -\frac{1}{\text{Slope of tangnet of the curve}} \text{ at } P(x_1, y_1)$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{(x,y)=(x_1,y_1)}}$$

$$= -\left(\frac{dx}{dy}\right)_{(x,y)=(x_1,y_1)}$$

Equation of normal to the curve

$y = f(x)$ at $P(x_1 y_1)$ is

$$y - y_1 = -\left(\frac{dx}{dy}\right)_{(x,y)=(x_1,y_1)} (x - x_1)$$

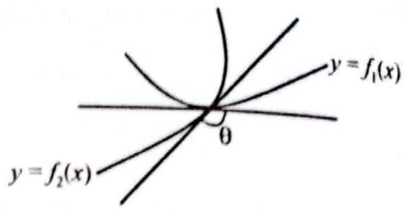
NOTE

1. If a tangent to the curve $y = f(x)$ at P makes angle θ with positive x - axis, then $\frac{dy}{dx}$ = slope of tangent at P = $\tan\theta$
2. The tangent to $y = f(x)$ at P is horizontal if and only if $\frac{dy}{dx} = 0$ at P.
3. The tangent to $y = f(x)$ at P is vertical if and only if $\frac{dx}{dy} = 0$ at P.

Angle between Two Curves

The angle of intersection of two curves at a point of intersection is defined as the angle between the tangents to the two curves at that point.

Let m_1 and m_2 be the slopes of the tangents to the curves at the point of intersection then the angle of intersection of curves, say θ, is given by



$$\tan\theta = \pm\frac{m_1 - m_2}{m_1 + m_2}$$

If $m_1.m_2 = -1$ then the two curves cut orthogonally.

Illustrative Examples

Example 1. Find the slope and inclination with x–axis of the tangent of the curve $2y = 2 - x^2$ at $x = 1$.

Solution

Given curve is $2y = 2 - x^2$

Differentiating both sides with respect to 'x'

$$2\frac{dy}{dx} = 0 - 2x$$

$$\frac{dy}{dx} = -x$$

At $x = 1$,

$$\frac{dy}{dx} = -1$$

i.e. Slope (m) -1

If θ be the inclination of the tangent with x–axis, then $\tan\theta = -1 = \tan\frac{3\pi}{4}$

$$\therefore \quad \theta = \frac{3\pi}{4}.$$

Example 2. Find the slope of the parabola $y = x^2$ at the point $p(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution:

Given $y = x^2$

$$\frac{dy}{dx} = 2x$$

Now,

$\frac{dy}{dx}$ at $(x, y) = (2, 4) = 2 \times 2 = 4.$

$\therefore$ Slope of the given parabola at (2, 4) is 4.

The equation of tangent to the parabola at $p(2, 4)$ is

$y = 4 = 4(x - 2)$

or, $y - 4 = 4d - 8$

or, $0 = 4x - y - 8 + 4$

$\therefore$ $4x - y - 4 = 0.$

Example 3. a. Find the slope of the curve $y = \frac{1}{x}$ at $x = a$.

b. Where does the slope equal to $-\frac{1}{4}$?

Solution:

Given $y = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-1})$$
$$= -1 \cdot x^{-1-1}$$
$$= \frac{1}{x^2}.$$

a. Slope of curve at $x = a$ is $\left(\frac{dy}{dx}\right)_{\text{at } x = a}$

$$= \left(\frac{-1}{x^2}\right)_{\text{at } x = a}$$
$$= \frac{-1}{a^2}$$

b. Slope of curve $= -\frac{1}{4}$

or, $-\frac{1}{x^2} = -\frac{1}{4}$

or, $x^2 = 4$

or, $x^2 = (\pm 2)^2$

$\therefore$ $x = \pm 2$

When $x = 2, y = \frac{1}{2}$

When $x = -2, y = -\frac{1}{2}$

Thus, the curve has slope $-\frac{1}{4}$ at the two points $\left(2, \frac{1}{2}\right)$ and $\left(-2, -\frac{1}{2}\right)$.

Example 4. Find the equations of the tangent and normal to the curve $y = 2x^2 - 5x + 2$ at $(1, -1)$.

Solution

Given curve is $y = 2x^2 - 5x + 2$

$$\frac{dy}{dx} = 4x - 5$$

Slope of tangent at $(1, -1)$ $= \left(\frac{dy}{dx}\right)$ at $(x, y) = (1, -1)$

$$= 4 \times 1 - 5$$
$$= -1.$$

Slope of normal at $(1, -1)$ $= -\frac{1}{\text{Slope of tangnet}}$

$$= 1$$

The equation of tangent at $(x_1, y_1) = (1, -1)$ is

$y - y_1 = m(x - x_1)$, where m = slope of tangent

or, $y + 1 = -1(x - 1)$

or, $y + 1 = -x + 1$

$\therefore$ $x + y = 0.$

The equation of normal at $(x_1, y_1) = (1, -1)$ is

$y - y_1 = m\,(x - x_1)$, where m = slope of normal

or, $y + 1 = 1\,(x - 1)$

or, $y + 1 = x - 1$

or, $0 = x - y - 1 - 1$

$\therefore$ $x - y = 2.$

Example 5. Find where the tangent is parallel to the x–axis for the curve $y = x^3 - 3x^2 - 9x + 15$.

Solution

Given,

$$y = x^3 - 3x^2 - 9x + 15$$
$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

For the tangent to be parallel to the x-axis, we have

$$\frac{dy}{dx} = 0$$

i.e. $3x^2 - 6x - 9 = 0$

or, $x^2 - 2x - 3 = 0$

or, $(x - 3)\,(x + 1) = 0$

$\therefore$ $x = -1, 3$

When $x = -1$,

$$\begin{aligned} y &= (-1)^3 - 3 \times (-1)^2 - 9 \times (-1) + 15 \\ &= -1 - 3 + 9 + 15 \\ &= 20 \end{aligned}$$

When $x = 3$,

$$y = 3^3 - 3 \times 3^2 - 9 \times 3 + 15$$
$$= -12$$

$\therefore$ Required points are $(-1, 20)$ and $(3, -12)$.

Example 6. Find the points on the curve $x^2 + y^2 = 36$ at which the tangents are parallel to the y–axis.

Solution

Given, curve is

$$x^2 + y^2 = 36 \quad \ldots \text{(i)}$$

Differentiating both sides w.r.to x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(36)$$

or, $2x + 2y\frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = \frac{-x}{y}$

For the tangent parallel to y–axis, we have,

$$\frac{dx}{dy} = 0$$

or, $\frac{-y}{x} = 0$

$\therefore$ $y = 0$

Putting the value of y in (i), we get,

$$x^2 = 36$$

or, $x = \pm 6$.

Hence, the required points are $(6, 0)$ and $(-6, 0)$.

Example 7. Find the angle of intersection between the curves $y = x^2$ and $6y = 7 - x^3$ at $(1, 1)$.

Solution

Given curves are

$$y = x^2 \quad \ldots \text{(i)}$$

and $6y = 7 - x^3$ ...(ii)

Differentiating (i) with respect to 'x'

$$\frac{dy}{dx} = 2x$$

At $(1, 1)$;

$$\frac{dy}{dx} = 2 \times 1 = 2$$

i.e. $m_1 = 2$

Again, differentiating (ii) both sides with respect to 'x'

$$6\frac{dy}{dx} = -3x^2$$

or, $\frac{dy}{dx} = \frac{-x^2}{2}$

At (1, 1);

$$\frac{dy}{dx} = \frac{-1}{2}$$

i.e. $m_2 = \frac{-1}{2}$

θ be the angle of intersection, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{2 + \frac{1}{2}}{1 + 2\cdot\left(\frac{-1}{2}\right)}$$

$$= \left(\frac{\frac{5}{2}}{0}\right)$$

$$= \infty$$

$$= \tan 90°$$

$\therefore\ \theta = 90°.$

Exercise 1.4

1. Find the slope and inclination with x–axis of the tangent of the following curves
   a. $3y = x^3 + 1$ at $x = 1$
   b. $y = -x^4 - 3x$ at $(-1, 2)$
   c. $x^2 + y^2 = 25$ at $(4, -3)$
2. At what angle does the curve $y(1 + x) = x$ cut x–axis?
3. Find the equations of the tangents and normals to the following curves.
   a. $y = 2x^3 - 5x^2 + 8$ at $(2, 4)$
   b. $x^2 + y^2 = 25$ at $(3, 4)$
4. Find the points on the following curves where the tangents are parallel to the x–axis
   a. $y = x^2 + 4x + 1$
   b. $y = x^3 - 2x^2 + 1$
5. Find the points on the curve $x^2 + y^2 = 25$ at which the tangents are parallel to the (a) x–axis (b) y–axis.
6. a. Find the equation of the tangent to the curve $y = 2x^2 - 3x + 1$ which is parallel to the line $x - y + 5 = 0$.

b. Find the equation of the tangent to the curv ... the line $x + 3y + 5 = 0$.

7. Find the angle of intersection of the following curves.
   a. $y^2 = x^3$ and $y = 2x$ at (0, 0)
   b. $xy = 6$ and $x^2y = 12$

Answers

1. a. $1, \frac{\pi}{4}$ b. $1, \frac{\pi}{4}$ c. $\frac{4}{3}, \tan^{-1}\left(\frac{4}{3}\right)$
2. $\frac{\pi}{4}$
3. a. $4x - y = 4, x + 4y = 18$ b. $3x + 4y = 25, 4x - 3y = 0$
4. a. $(-2, -3)$ b. $(0, 1)$ and $\left(\frac{4}{3}, -\frac{5}{27}\right)$
5. a. $(0, \pm 5)$ b. $(\pm 5, 0)$
6. a. $x - y - 1 = 0$ b. $3x - y + 2 = 0$
7. a. $\tan^{-1}(2)$ b. $\tan^{-1}\left(\frac{3}{11}\right)$

1.7 The Derivatives and Slope of the Curve

Let $y = f(x)$ be a given function. Let Δx be the small increment in x and Δy be the corresponding small increment in y. Then,

$$y + \Delta y = f(x + \Delta x)$$

or, $\Delta y = f(x + \Delta x) - f(x)$

Then, $\frac{\Delta y}{\Delta x}$ is called the **average rate of change** of y with respect to x. The **instantaneous rate of change** of a function $y = f(x)$ with respect to x is defined to be

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

It is the definition of derivative of $y = f(x)$ with respect to x. Thus,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

On the other hand, the derivative of $y = f(x)$ with respect to x denoted by $\frac{\Delta y}{\Delta x}$ or $f'(x)$ is the **slope of curve** or **slope of tangent** at any point of the curve $y = f(x)$. Thus, the slope of the curve at $x = a$ is the derivative of function (curve) at $x = a$.

1.8 Increasing and Decreasing Function

The derivative can be used in describing the behaviour of function. Now, we discuss about increasing and decreasing functions. These concepts are useful for determining where a function achieves maximum or minimum values.

Increasing Function

A function $f(x)$ is said to be increasing function on (a, b) if for all $x_1, x_2 \in (a, b)$,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

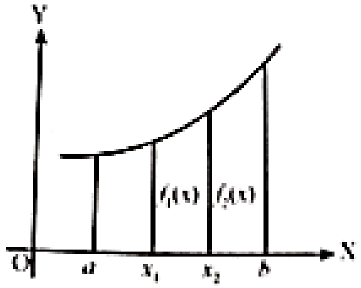


Fig. f is increasing on (a, b)

Decreasing Function

A function $f(x)$ is said to be decreasing function on (a, b) if for all $x_1, x_2 \in (a, b)$,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

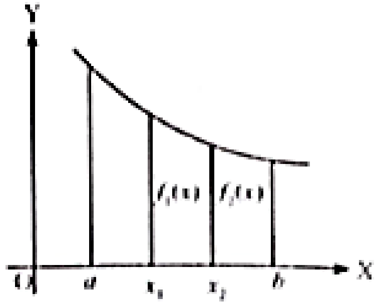


Fig. f is decreasing on (a, b)

The First Derivative Test for Increasing and Decreasing

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

a. If $f' > 0$ at each point of (a, b) then f increases on $[a, b]$.

b. If $f' < 0$ at each point of (a, b) then f decreases on $[a, b]$.

Thus functions with positive derivatives are increasing functions and the functions with negative derivatives are decreasing functions.

Example: Let us consider the function $f(x) = x^2$.

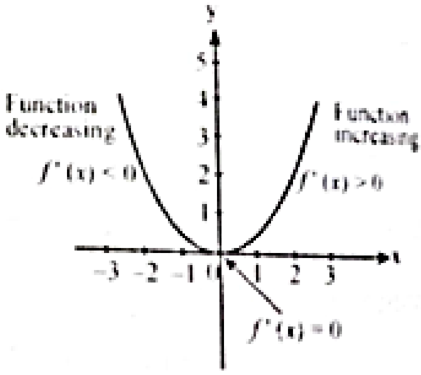
Then, $f'(x) = 2x$

Clearly $f'(x) = 2x > 0$ for all $x \in (0, \infty)$

$\therefore$ f is increasing on $(0, \infty)$.

Again, $f'(x) = 2x < 0$ for all $x \in (-\infty, 0)$

$\therefore$ f is decreasing on $(-\infty, 0)$.



NOTE i. A function that is increasing or decreasing on an interval I is called monotonic on I.

ii. A function is increasing (decreasing) on an interval then the function is increasing (decreasing) at every point within the interval.

1.9 Convexity of Curves

Geometrically, a function is said to be **concave up** (or **convex down**) on an interval if its curve bends upward and **concave down** (or **convex up**) on an interval if its curve bends downward.

Definition

The graph of a differentiable function $y = f(x)$ is **concave up** (or **convex down**) on an interval where $f'(x)$ is increasing and **concave down** (or **convex up**) on an interval where $f'(x)$ is decreasing.

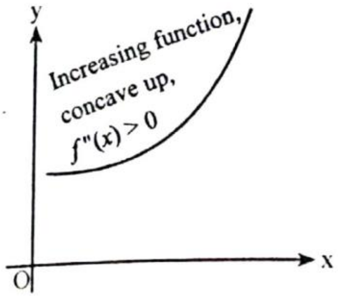
The Second Derivative Test for Concavity

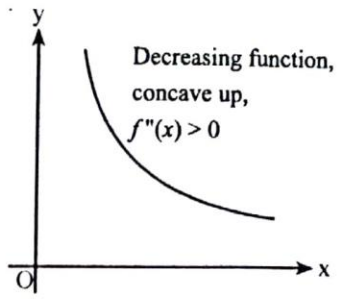
Let $y = f(x)$ be twice differentiable on an interval I.

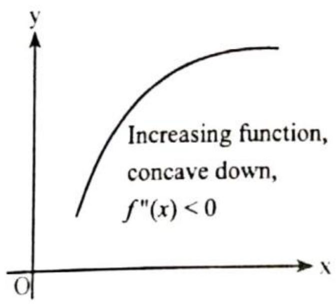
a. If $f''(x) > 0$ on I, the graph of f over I is concave up.

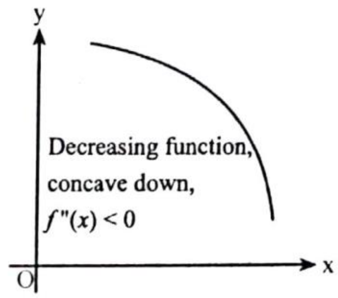
b. If $f''(x) < 0$ on I, the graph of f over I is concave down.

Note that the curve lies above the tangent lines are concave up whereas the curve that lies below the tangent lines are concave down.



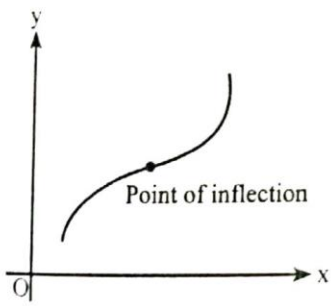


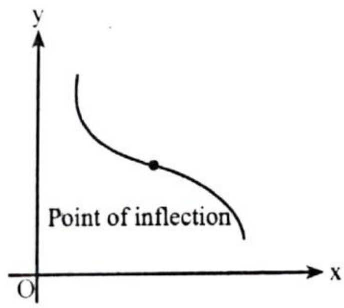




Point of Inflection

A point on the graph of $y = f(x)$ at which the curve changes from concave downward to concave upward or from concave upward to concave downward is called **point of inflection** or **inflection point**. The inflection points are illustrated in the following figures.





To be precise, a point where the graph of a function has a tangent line and where the concavity changes is called a point of inflection.

Thus, a point of inflection on a curve $y = f(x)$ is a point where y'' is positive on one side and negative on the other. At such point y'' is either zero or undefined.

Now, we discuss the method for determining points of inflection.

Method I

Step 1: Find $f''(x)$

Step 2: Set $f''(x) = 0$ and solve for x. Also find the points where y'' doesnot exist. Let $x = a$ be one point.

Step 3: If $f''(x)$ changes sign as x passes through $x = a$ then $x = a$ is a point of inflection. Otherwise it is not a point of inflection. Repeat this process for the other values (if any) in step 2.

Method II

Step 1: Find $f''(x)$.

Step 2: Set $f''(x) = 0$ and solve for x. Also find the points where y'' does not exist. Let $x =$ point.

Step 3: If $f'''(a)$. If $f'''(a) \neq 0$ then $x = a$ is a point of inflection of f. Repeat this process values of x (if any) in step 2.

If $f'''(a) = 0$, the above test fails. We use the following rule in such case. Let $f''(a)$ $\ldots = f^{n-1}(a) = 0$ but $f^{n}(a) \neq 0$. If n is odd, f has a point of inflection at $x = a$. If n is concave up or down according as $f^{n}(a) > 0$ or < 0.

Example: An inflection point where y'' does not exist.

The curve $y = x^{\frac{1}{3}}$ has a point of inflection at $x = 0$; but y'' does not exist there.

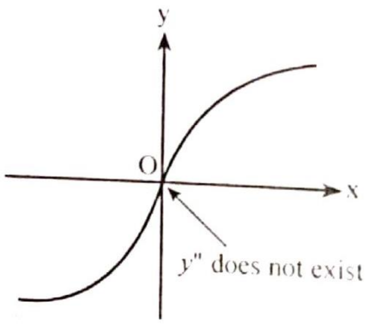


Fig. A point where y'' fails to exist can be a point of inflection

$$y' = \frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}}$$

$$y'' = \frac{d}{dx}\left(\frac{1}{3}x^{-\frac{2}{3}}\right) = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1} = -\frac{2}{9}x^{-\frac{5}{3}} = -\frac{2}{9x^{\frac{5}{3}}}$$

This shows that y'' does not exist at $x = 0$. But from figure, we see that $x = 0$ is a point of inflection of $y = x^{\frac{1}{3}}$.

Example: No point of inflection where $y'' = 0$.

The curve $y = x^4$ has no inflection point at $x = 0$. Although $y'' = 12x^2$ is zero there, it does not change sign.

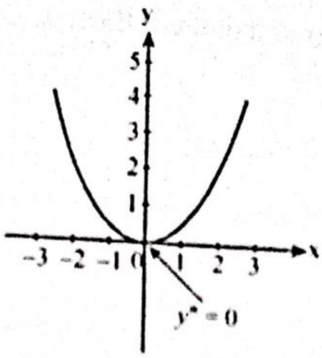


Fig. The graph of $y = x^4$ has no inflection point at $x = 0$ even though $y'' = 0$ there

Stationary Point

A point on the curve $y = f(x)$ where the tangent is parallel to the x-axis is known as the stationary point. At the stationary point, $y' = 0$.

Critical Point

An interior point of the domain of a function f where f' is zero or undefined is called a critical point of f.

Example 1. Find the critical points of the function $f(x) = x^{\frac{1}{3}}(x-4)$.

Solution

Here,

$$
\begin{aligned}
f(x) &= x^{\frac{1}{3}}(x-4) \\
&= x^{\frac{4}{3}} - 4x^{\frac{1}{3}} \\
f'(x) &= \frac{d}{dx}\left(x^{\frac{4}{3}} - 4x^{\frac{1}{3}}\right) \\
&= \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} \\
&= \frac{4}{3}x^{-\frac{2}{3}}(x-1) \\
&= \frac{4(x-1)}{3x^{\frac{2}{3}}}
\end{aligned}
$$

Clearly, $f'(x)$ is zero at $x = 1$ and is undefined at $x = 0$.

$\therefore$ $x = 0$ and $x = 1$ are the critical points of $f(x)$.

1.10 Maximization and Minimization of a Function

A function that is continuous at every point of a closed interval has an absolute maximum and an absolute minimum value on that interval. Now, we give a theorem about absolute extreme values.

Theorem (The Max. Min. Theorem for Continuous Functions)

If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is, there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in I.

The concepts of this theorem are illustrated in the following figures.

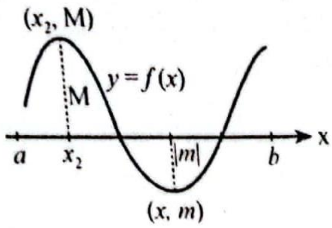


Fig.: Maximum and minimum at interior points

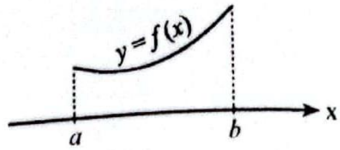


Fig.: Maximum and minimum at end points

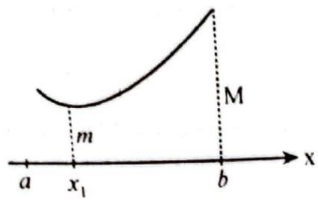


Fig.: Maximum at end point and minimum at interior point

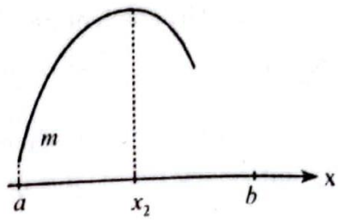


Fig.: Maximum at interior point and minimum at end point

Definition

Absolute Extreme Values

Let f be a function defined on the domain D. Then the function f has an **absolute maximum value** on D at a point c if $f(x) \le f(c)$ for all x in D;

and an **absolute minimum value** on D at c if $f(x) \ge f(c)$ for all x in D.

Absolute extreme values are also called **global extreme values.**

Steps for finding absolute extreme values:

Step 1: Find all critical points.

Step 2: Evaluate f at all critical points and end points.

Step 3: Take the largest and smallest of these values.

Example 2. Find the absolute maximum and absolute minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution

Since $f(x) = x^2$ is a polynomial function, so it is differentiable everywhere in $\mathbb{R}$. Thus, is differentiable at every points in $[-2, 1]$. So, the only critical point is where $f'(x) = 0$.

or, $2x = 0$

$\therefore$ $x = 0$.

Now, we have to evaluate f at critical point $x = 0$ and end points $x = -2$ and $x = 1$.

$$f(0) = 0^2 = 0$$
$$f(-2) = (-2)^2 = 4$$
$$f(1) = 1^2 = 1$$

$\therefore$ Absolute maximum value = 4 at $x = 2$.

Absolute minimum value = 0 at $x = 0$.

Local Extreme Values

Definition

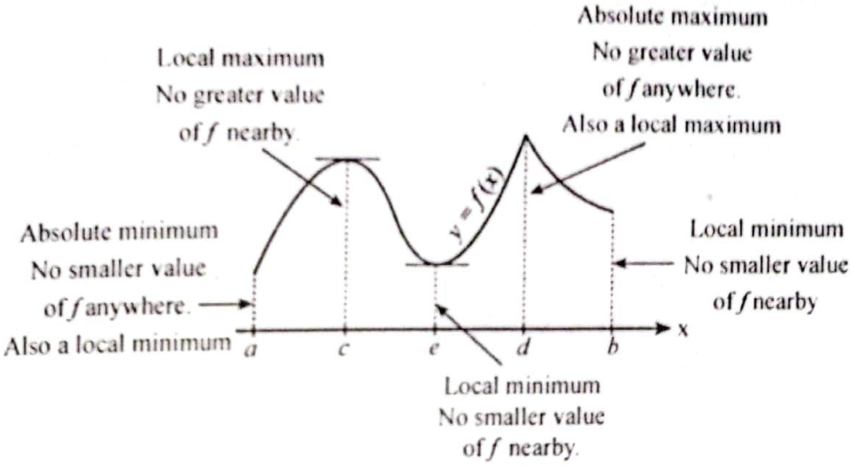
A function f has a local maximum value at an interior point c of its domain if

$f(x) \le f(c)$ for all x in some interval containing c and a local minimum value at an interior point c of its domain if

$f(x) \ge f(c)$ for all x in some open interval containing c.

NOTE Local extreme values are sometimes called relative extreme values.

Local vs. Absolute Extreme



This figure shows a graph with five different extreme points and values.

The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

First Derivative Test

a. Find the critical points.

b. Locate the critical points on the number line.

c. Determine the sign of $f'(x)$ for each interval.

d. Now let x increase through each critical value $x = c$.

i. If $f'(x)$ changes from + to −, then $f(x)$ has a relative (local) maximum value at $x = c$.

ii. If $f'(x)$ changes from − to +, then $f(x)$ has a relative (local) minimum value at $x = c$.

iii. If $f'(x)$ does not change sign, then $f(x)$ has neither maximum nor minimum value.

Example 3. Find local extreme values for $f(x) = x^4 - 6x^2 + 8x$.

Solution

Given,

$$f(x) = x^4 - 6x^2 + 8x.$$

Then,

$$f'(x) = 4x^3 - 12x + 8.$$

For the critical points, we have,

$f'(x) = 0.$

i.e., $4(x^3 - 3x + 2) = 0$

or, $4(x-1)^2(x+2) = 0$

Thus, the critical points are $x = 1$ and $x = -2$.

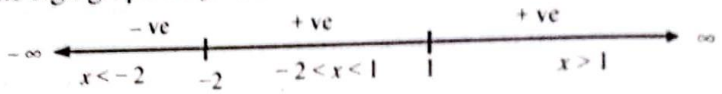
These critical points divide the number line into three intervals:

For $x < -2$: Taking $x = -3$ (test point), $f'(-3) = -64 < 0$.

For $-2 < x < 1$: Taking $x = 0$ (test point), $f'(0) = 4(-1)^2(2) = 8 > 0$.

For $x > 1$: Taking $x = 2$ (test point), $f'(2) = 16 > 0$.

We can draw the sign graph for $f'(x)$ as:



As x increases through the critical value of $x = -2$, $f'(x)$ changes sign from – to +, S $x = -2$ gives us a local minimum at $x = -2$. As x increases through the critical valu $x = 1$, $f'(x)$ does not change sign. Thus, there is neither maximum nor minimum ; $x = 1$. Hence, the given function has only minimum value and the minimum value is

$$f(-2) = (-2)^4 - 6(-2)^2 + 8(-2)$$
$$= 16 - 24 - 16$$
$$= -24.$$

The Second Derivative Theorem for Local Extreme Values

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

Producers for finding local extreme values

Step 1: Find $f'(x)$.

Step 2: Solve $f'(x) = 0$ for x. Let c be one such value of x.

Step 3: i. If $f''(x) < 0$ for $x = c$, then $f(x)$ has a maximum value at $x = c$.

ii. If $f''(x) > 0$ for $x = c$, then $f(x)$ has minimum value at $x = c$.

iii. $f''(x) = 0$ and $f'''(x) \neq 0$ at $x = c$ then f has neither maximum nor minimum value at $x = c$

Illustrative Examples

Example 1. Examine whether the function $f(x) = 15x^2 - 14x + 1$ is increasing or decreasing $x = \frac{2}{5}$ and $x = \frac{5}{2}$.

Solution

Here, $f(x) = 15x^2 - 14x + 1$

$f'(x) = 30x - 14$

At $x = \frac{2}{5}$, $f'\left(\frac{2}{5}\right) = 30 \times \frac{2}{5} - 14 = -2 < 0$

$\therefore$ $f(x)$ is decreasing at $x = \frac{2}{5}$

At $x = \frac{5}{2}, f'\left(\frac{5}{2}\right) = 30 \times \frac{5}{2} - 14 = 61 > 0$

$\therefore$ $f(x)$ is increasing at $x = \frac{5}{2}$.

Example 2. Find the interval in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing. Also, find the point of inflection.

Solution

Here, $f(x) = 2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36$

$f''(x) = 12x - 30$

For critical points, we have $f'(x) = 0$

or, $6x^2 - 30x + 36 = 0$

or, $x^2 - 5x + 6 = 0$

or, $(x - 2)(x - 3) = 0$

$\therefore$ $x = 2, 3$

Intervals	Sign of $f'(x)$	Remarks
$(-\infty, 2)$	+ ve	increasing
$(2, 3)$	– ve	decreasing
$(3, \infty)$	+ ve	increasing

$\therefore$ $f(x)$ is increasing on $(-\infty, 2) \cup (3, \infty)$ and decreasing on $(2, 3)$.

For point of inflection, $f''(x) = 0$

or, $12x - 30 = 0$

or, $12x = 30$

$\therefore$ $x = \frac{5}{2}$.

Example 3. Determine where the graph is concave upwards and where it is concave downwards. Also find the point of inflection $f(x) = (x^2 - 1)(x^2 - 5)$.

Solution

Here, $f(x) = (x^2 - 1)(x^2 - 5)$

$= x^4 - 5x^2 - x^2 + 5$

$= x^4 - 6x + 5$

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

For points of inflection

$f''(x) = 0$

or, $12x^2 - 12 = 0$

or, $x^2 - 1 = 0$,

$\therefore$ $x = \pm 1$

Intervals	Sign of $f''(x)$	Remarks
$(-\infty, -1)$	+ ve	concave upwards
$(-1, 1)$	– ve	concave downwards
$(1, \infty)$	+ ve	concave upwards

$\therefore$ The graph of $f(x)$ is concave upwards on the interval $(-\infty, -1) \cup (1, \infty)$ a concave downwards on the interval $(-1, 1)$.

Example 4. Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution

The function $g(t)$ is differentiable on its entire domain, so the only critical points occ where $g'(t) = 0$.

i.e. $8 - 4t^3 = 0$

or, $t^3 = 2$

$\therefore$ $t = 2^{\frac{1}{3}}$

This point does not belong to $[-2, 1]$.

Therefore, the absolute extreme values occur at the end points.

Now,

$$g(-2) = 8(-2) - (-2)^4 = -16 - 16 = -32$$

$$g(1) = 8 \times 1 - 1^4 = 7.$$

$\therefore$ Absolute maximum value = 7 at $t = 1$.

Absolute minimum value = –32 at $t = -2$.

Example 5. Find the maximum and minimum values of the function $f(x) = x^3 - 6x^2 + 9x - 2$. Als find the point of inflection, if any.

Solution

Here,

$$f(x) = x^3 - 6x^2 + 9x - 2$$

$$f'(x) = 3x^2 - 12x + 9$$

$$f''(x) = 6x - 12$$

For stationary points, we have

$f'(x) = 0$

or, $3x^2 - 12x + 9 = 0$

or, $x^2 - 4x + 3 = 0$

or, $(x - 1)(x - 3) = 0$

$\therefore$ $x = 1, 3$

At $x = 1$

$f''(1) = 6 \times 1 - 12 = -6 < 0$

So, $f(x)$ has maximum value at $x = 1$

Maximum value $= f(1) = 1^3 - 6 \times 1^2 + 9 \times 1 - 2 = 2$

At $x = 3$

$f''(3) = 6 \times 3 - 12 = 6 > 0$

So, $f(x)$ has minimum value at $x = 3$

Minimum value $= f(3)$

$= 3^3 - 6 \times 3^2 + 9 \times 3 - 2$

$= 27 - 54 + 27 - 2 = -2$

For point of inflection $f''(x) = 0$

$6x - 12 = 0$

$\therefore \quad x = 2.$

Example 6. Let $f(x)$ be a function whose derivative $f'(x)$ be given by $f'(x) = x(x - 1)$.

a. What are the critical points of f?

b. On what intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

Solution

a. Here $f'(x)$ exists for all real values of x.

So, for critical points, we have to solve $f'(x) = 0$.

i.e. $x(x - 1) = 0$.

$\therefore \quad x = 0, 1.$

b. The points $x = 0$ and $x = 1$ divide the whole real line into 3 intervals.

Now,

Intervals	Sign of $f'(x)$	Nature of f
$(-\infty, 0)$	+ ve	Increasing
$(0, 1)$	– ve	Decreasing
$(1, \infty)$	+ ve	Increasing

$\therefore$ $f(x)$ is decreasing on $(0, 1)$ and increasing on $(-\infty, 0) \cup (1, \infty)$.

c. Here, $f'(x) = x(x - 1)$

$= x^2 - x$

$f''(x) = 2x - 1$

The critical points are $x = 0$ and $x = 1$.

At $x = 0$, $f''(x) = 2 \times 0 - 1 = -1 < 0$.

So, f has local maximum value at $x = 0$.

At $x = 1$, $f''(x) = 2 \times 1 - 1 = 1 > 0$.

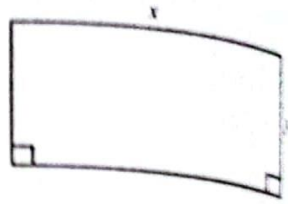
So, f has local minimum value at $x = 1$.

Example 7. Show that the rectangle of largest possible area for a given perimeter is a square.

Solution

Let x and y be the length and breadth, A be the area and P be the given perimeter of the given rectangle.

We have,



$$P = 2(x + y)$$

or, $\frac{P}{2} = x + y$

or, $y = \left(\frac{P}{2} - x\right)$ ... (i)

Now,

$$A = xy$$

or, $A = x\left(\frac{P}{2} - x\right)$ [$\because$ Using (i)]

or, $A = \frac{Px}{2} - x^2$ ... (ii)

Differentiate both sides w.r. to x, we get

$$\frac{dA}{dx} = \frac{d}{dx}\left(\frac{Px}{2} - x^2\right) = \frac{P}{2} - 2x$$

Again, diff. w.r. to x, we get

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{P}{2} - 2x\right) = 0 - 2 = -2$$

For maxima or minima, we have,

$$\frac{dA}{dx} = 0$$

i.e. $\frac{P}{2} - 2x = 0$

or, $2x = \frac{P}{2}$

$\therefore$ $x = \frac{P}{4}$

When $x = \frac{P}{4}$, $\frac{d^2A}{dx^2} = -2 < 0$

So, the area will be largest (maximum) when $x = \frac{P}{4}$

And, when $x = \frac{P}{4}$, from (i)

$$y = \frac{P}{2} - x = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$

$\therefore$ $x = y = \frac{P}{4}$

This shows that for the largest area of rectangle, length and breadth must be equal. Hence, the rectangle of largest possible area of the given perimeter is a square.

Example 8. Using derivatives, find two numbers whose sum is 10 and sum of whose squares is minimum.

Solution

Let x and y be two numbers such that $x + y = 10$

$\therefore \quad y = 10 - x \qquad \ldots \text{(i)}$

Again, let

$S = x^2 + y^2$

or, $S = x^2 + (10 - x)^2 \qquad$ [using (i)]

or, $S = x^2 + 100 - 20x + x^2$

or, $S = 2x^2 - 20x + 100$

$\therefore \quad \frac{dS}{dx} = 4x - 20$

and $\frac{d^2S}{dx^2} = 4$

For maxima or minima,

$\frac{ds}{dx} = 0$

or, $4x - 20 = 0$

or, $x = \frac{20}{4} = 5$

When $x = 5$, $\frac{d^2S}{dx^2} = 4 > 0$

So, when $x = 5$, S is minimum.

So, from (i)

$y = 10 - x$

$= 10 - 5$

$= 5$

$\therefore$ Hence the required two numbers are 5 and 5.

Exercise 1.5

1. a. Show that the function $f(x) = 3x^3 - 24x + 1$ is increasing at $x = 4$ and decreasing at $x = \frac{1}{2}$.

   b. Examine whether the function $f(x) = 2x^3 - x^2 + 4$ at $x = 1$ and at $x = \frac{1}{4}$.

   c. Show that $f(x) = x - \frac{1}{x}$ is increasing for all $x \in \mathbb{R}$ except at $x = 0$.

   d. Show that the function $f(x) = x^2 - 6x + 3$ is decreasing on the interval (0, 2).

Answers

1. b. Increasing at $x = 1$; Decreasing at $x = \frac{1}{4}$
2. a. Increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$
   b. Increasing on $\left(-\infty, -\frac{3}{2}\right)$ and decreasing on $\left(-\frac{3}{2}, \infty\right)$
   c. Increasing on $(-\infty, -1) \cup (3, \infty)$ and decreasing on $(-1, 3)$
   d. Increasing on $(-2, 2)$ and decreasing on $[-3, -2) \cup (2, 3]$
3. a. Absolute max. value $= -3$ at $x = 3$, Absolute min. value $= -\frac{19}{3}$ at $x = -2$
   b. Absolute max. value $= 3$ at $x = 2$, Absolute min. value $= -1$ at $x = 0$
   c. Absolute max. value $= 5$ at $x = 0$, Absolute min. value $= -15$ at $x = -2$
   d. Absolute max. value $= \sqrt[3]{9}$ at $x = 3$, Absolute min. value $= 0$ at $x = 0$
4. a. Min. value $= 1$ at $x = 1$, no point of inflection
   b. Max. value $= 3$ at $x = -1$, Min value $= -1$ at $x = 1$; point of inflection is at $x = 0$
   c. Max. value $= 32$ at $x = -1$; Min value $= 0$ at $x = 3$; Point of inflection is at $x = 1$
   d. Max. value $= 33$ at $x = 2$; Min value $= 32$ at $x = 3$; Point of inflection is at $x = \frac{5}{2}$
   e. Max. value $= \frac{7}{2}$ at $x = -\frac{1}{2}$; Min value $= -\frac{25}{2}$ at $x = \frac{3}{2}$; Point of inflection is at $x = \frac{1}{2}$
   f. Max. value $= -10$ at $x = 5$; Min. value $= 10$ at $x = -5$, no point of inflection
6. a. Concave upwards on $(-\infty, \infty)$
   b. Concave upwards on $(1, \infty)$ and downwards on $(-\infty, 1)$
   c. Concave upwards on $(-\infty, 1) \cup (3, \infty)$; Concave downwards on $(1, 3)$
7. 1296 m^2 8. 600 m^2 9. 10, 10 10. $\frac{9}{4+\pi}$ m
11. $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}, h = 2\left(\frac{13}{\pi}\right)^{\frac{1}{3}}$

1.11 Rate Measures

The Derivative as Rate of Change

Let $y = f(x)$ be a continuous function. If Δx and Δy be the small increments in x and y respectively, then

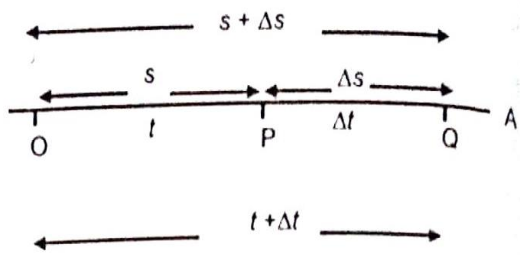
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

represents the slope of the secant line to the curve $y = f(x)$. This can be regarded as the average rate of change of y with respect to x in the interval from x to $x + \Delta x$. At the point $x = a$,

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \frac{dy}{dx}$$

is called the instantaneous rate of change of y with respect to x at $x = a$, if this limit exists.

Physical Interpretation of Derivative



Let us suppose that a particle be moving along a straight line OA and pass the points P and Q at the times t and $t + \Delta t$ respectively. Let s and $s + \Delta s$ be the distances described in times t and $t + \Delta t$ respectively, from the fixed point O. Now $s = f(t)$ is the equation of the law of motion.

If the ratio $\frac{\Delta s}{\Delta t}$ is constant, i.e., the equal distances are described in equal intervals of time, then the motion is called uniform, and the ratio is called the velocity at any instant. If the motion is not uniform, the ratio $\frac{\Delta s}{\Delta t}$ varies as Δt varies, and the ratio can be taken as the velocity at an instant. In such case, the ratio is defined as the average velocity of the particle during the interval Δt. Thus,

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t},$$

and $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$ is the instantaneous velocity of the particle at the time t. Similarly, if Δv is the change in velocity of the particle as it moves from P to Q during the time interval Δt, then $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$ is the instantaneous acceleration of the particle at time t.

Illustrative Examples

Example 1. Find the velocity of a rectilinear motion given by the equation $s = 3t^2 - 2t + 1$ at the instant when $t = 2$ seconds.

Solution

We have $s = 3t^2 - 2t + 1$. Then

$$\frac{ds}{dt} = 6t - 2$$

At $t = 3$,

$$\frac{ds}{dt} = 6 \times 3 - 2$$

$$= 16 \text{ m/sec.}$$

Example 2. The side of a square sheet is increasing at the rate of 5 cm/min. At what rate is the area increasing when the side is 12 cm long?

Solution

Let x be the length of side and A be the area of the square at time t. By question,

$$\frac{dx}{dt} = 5\text{cm/min}$$

When $x = 12$ cm, $\frac{dA}{dt} = ?$

We have,

$$A = x^2$$

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

When $x = 12$ cm,

$$\frac{dA}{dt} = 2\times12\times5$$

$$\therefore \quad \frac{dA}{dt} = 120 \text{ cm}^2/\text{min}$$

Example 3. A stone thrown into a pond produces circular ripples which expand from the point of impact. If the radius of the ripple increases at the rate of 3.5 cm/sec, how fast is the area growing when the radius is 15cm? $\left(\pi = \frac{22}{7}\right)$

Solution

Let r be the radius and A be the area of a circular ripple in time t. By given,

$$\frac{dr}{dt} = 3.5 \text{ cm/sec}$$

We have,

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

When $r = 15$ cm, then

$$\therefore \quad \frac{dA}{dt} = \pi \times 2 \times 15 \times 3.5$$

$$= \frac{22}{7} \times 30 \times 3.5$$

$$= 330 \text{ cm}^2/\text{sec}$$

Example 4: A spherical ballon is inflated at the rate of 10 cubic cm/sec. At what rate is the radius increasing when the radius is 10 cm.

Solution

Let r be the radius and V be the volume of the spherical balloon at time t. Then,

$$\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec}$$

When $r = 10$ cm, $\frac{dr}{dt} = ?$

We have,

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3} \cdot \pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

or, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

When $r = 10$ cm.

$$10 = 4\pi \times 10^2 \times \frac{dr}{dt}$$

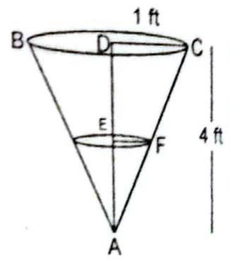
or, $\frac{1}{40\pi} = \frac{dr}{dt}$

$\therefore$ $\frac{dr}{dt} = \frac{1}{40\,\pi}$ cm/sec.

Example 5. Water flows into an inverted conical tank at the rate of 27ft^3/min. When the depth of water is 2ft, how fast is the level rising? Assume that the height of the tank is 4ft and the radius of the top is 1 ft.

Solution

Let ABC be the conical tank into which water is flowing at the rate of 27 ft^3/min. Let h be the height AE of the water surface and r be the radius EF of the water surface at time t. Now ΔACD and ΔAFE are similar. So,



$$\frac{AE}{AD} = \frac{EF}{DC}$$

or, $\frac{h}{4} = \frac{r}{1}$

$\therefore$ $r = \frac{h}{4}$

Let V be the volume of water in the tank. We have,

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h$$

$$= \frac{1}{48}\pi h^3$$

$$\therefore \frac{dV}{dt} = \frac{1}{48}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$= \frac{\pi}{16} \cdot h^2 \cdot \frac{dh}{dt}$$

When $h = 2$ft,

$$27 = \frac{\pi}{16} \times 2^2 \times \frac{dh}{dt}$$

$\therefore$ $\frac{dh}{dt} = \frac{108}{\pi}$ ft/min.

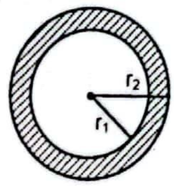
Example 6. Two concentric circles are expanding in such a way that the radius of the inner circle is increasing at the rate of 6 cm/sec and that of the outer circle at the rate of 2.5 cm/sec. At a certain time the radius of the inner and the outer circles are respectively 20 cm and 32 cm. At that time how fast is the area between the circle increasing or decreasing?

Solution

Let r_1 be the radius of inner circle and r_2 be the radius of the outer circles at time t. By question,

$$\frac{dr_1}{dt} = 6 \text{ cm/sec and}$$

$$\frac{dr_2}{dt} = 2.5 \text{ cm/sec}$$



Let A be the area between the circles at time t. Then,

$$A = \text{Area of outer circle} - \text{Area of inner circle}$$

$$= \pi r_2^2 - \pi r_1^2$$

$$\frac{dA}{dt} = \pi \cdot 2r_2 \cdot \frac{dr_2}{dt} - \pi \cdot 2r_1 \cdot \frac{dr_1}{dt}$$

or, $$\frac{dA}{dt} = 2\pi r_2 \times 2.5 - 2\pi r_1 \times 6$$

$$= 5\pi r_2 - 12\pi r_1$$

When $r_1 = 20$ cm and $r_2 = 32$ cm

$$\frac{dA}{dt} = 5\pi \times 32 - 12\pi \times 20$$

$$= 160\pi - 240\pi$$

$$= -80\pi$$

Hence, the area between the circles is decreasing at the rate of 80π cm^2/sec.

Example 7. A point is moving along the curve $y = 2x^3 - 3x^2$ in such a way that its x–coordinate is increasing at the rate of 4 ft/sec. Find the rate at which the distance of the point from the origin is increasing when the point is at (2, 4).

Solution

Let P(x, y) be a point moving along the curve $= 2x^3 - 3x^2$. By given,

$$\frac{dx}{dt} = 4\text{ft/sec.}$$

Again,

$$\frac{dy}{dt} = \frac{d}{dt}(2x^3 - 3x^2)$$

$$= (6x^2 - 6x)\frac{dx}{dt}$$

$$= (6x^2 - 6x) \times 4$$

$$= 24x(x - 1)$$

When $(x, y) = (2, 4)$,

$$\frac{dy}{dt} = 24 \times 2\,(2 - 1)$$

$$= 48$$

Let S be the distance between the origin O(0,0) and the point $P(x,y)$. Then,

$$S = \sqrt{x^2 + y^2}$$

$$\frac{dS}{dt} = \frac{d}{dt}(x^2 + y^2)^{1/2}$$

$$= \frac{1}{2\sqrt{x^2 + y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

When $(x, y) = (2, 4)$

$$\frac{dS}{dt} = \frac{1}{\sqrt{2^2 + 4^2}}\,(2 \times 4 + 4 \times 48)$$

$$= \frac{1}{\sqrt{20}}(200)$$

$$= \frac{1}{2\sqrt{5}} \times 200$$

$$\therefore \frac{dS}{dt} = 20\sqrt{5} \text{ ft/sec.}$$

Exercise 1.6

1. The distance s, in meters, travelled in t seconds by a particle moving in a straight line is giv by $s = t^3 - 2t^2$. Find the velocity and acceleration of the particle when $t = 2$ seconds.
2. Find the rate of change of the volume of a cylinder of radius r and height h will respect change in the radius.
3. The radius of a circular plate is increasing at the rate of 0.20 cm/ increasing when the radius of the plate is 25

L' Hospital's rule help us to get success with derivatives to evaluate limits that lead to indeterminate forms.

L Hospital's Rule

If $f(x)$ and $g(x)$ and their derivatives $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $f(a) = g(a) = 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)}$$

$$= \frac{f'(a)}{g'(a)}$$

provided that $g'(a) \neq 0$.

If $f'(a) = g'(a) = 0$ then above rule can be extended as

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \to a} f''(x)}{\lim_{x \to a} g''(x)}$$

$$= \frac{f''(a)}{g''(a)}$$

provided that $f''(x)$ and $g''(x)$ are continuous at $x = a$ and $g''(a) \neq 0$.

The above rule can further be used when $f''(a) = g''(a) = 0$

Example: It tooks a lot of work to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ in first semester. But it is easy with the help of this rule.

$$\lim_{x \to 0} \frac{\sin x}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)}$$

$$= \lim_{x \to 0} \frac{\cos x}{1}$$

$$= \frac{\cos 0}{1}$$

$$= \frac{1}{1}$$

$$= 1.$$

Illustrative Examples

Example 1. Using L Hospital's rule, evaluate $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \text{ still } \frac{0}{0} \quad \text{apply the rule again}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \text{ still } \frac{0}{0}; \quad \text{apply the rule again}$$

$$= \lim_{x \to 0} \frac{\cos}{6}$$

$$= \frac{1}{6} \text{ Not } \frac{0}{0}; \quad \text{limit is found.}$$

Example 2. Use L Hospital's rule to evaluate $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$.

Solution

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 4} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 2} \frac{1}{2x}$$

$$= \frac{1}{2} \times 2$$

$$= \frac{1}{4}.$$

Example 3. Use L Hospital's rule to evaluate $\lim_{x \to -3} \frac{t^3 - 4t + 15}{t^2 - t - 12}$.

Solution

$$\lim_{x \to -3} \frac{t^3 - 4t + 15}{t^2 - t - 12} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to -3} \frac{3t^2 - 4}{2t - 1}$$

$$= \frac{3(-3)^2 - 4}{2(-3) - 1}$$

$$= -\frac{23}{7}.$$

Example 4. Using L Hospital's rule, evaluate: $\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2}$.

Solution

$$\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{2x - 2\sin x \cos x}{2x} \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{2x - \sin 2x}{2x}$$

$$= \lim_{x \to 0} \frac{2 - 2\cos 2x}{2}$$

$$= \frac{2 - 2\cos 0}{2}$$

$$= \frac{2 - 2 \times 1}{2}$$

$$= \frac{2 - 2}{2}$$

$$= \frac{0}{2}$$

$$= 0.$$

Example 5. Using L Hospital's rule, evaluate: $\lim_{x \to 0} \frac{e^x - x - 1}{x^2}$.

Solution

$$\lim_{x \to 0} \frac{e^x - x - 1}{x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{e^x - 1}{2x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{e^x}{2}$$

$$= \frac{e^0}{2}$$

$$= \frac{1}{2}.$$

Example 6. Using L Hospital's rule, evaluate: $\lim_{x \to 0} \frac{xe^x - \log(1 + x)}{x^2}$.

Solution

$$\lim_{x \to 0} \frac{xe^x - \log(1 + x)}{x^2} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{xe^x + e^x - \frac{1}{1 + x}}{2x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{xe^x + e^x + e^x + \frac{1}{(x+1)^2}}{2}$$

$$= \frac{0+1+1+1}{2}$$

$$= \frac{3}{2}.$$

xample 7. Using L Hospital's rule, evaluate: $\lim\limits_{x \to \infty} \frac{3x^2 + 2x - 5}{2x^2 + 7x + 8}$.

olution

$$\lim_{x \to \infty} \frac{3x^2 + 2x - 5}{2x^2 + 7x + 8} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{6x + 2}{4x + 7} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{6}{4}$$

$$= \frac{3}{2}.$$

xample 8. Evaluate, using L Hospital's rule: $\lim\limits_{x \to 0} \frac{\ell n (\tan x)}{\ell n\, x}$.

olution

$$\lim_{x \to 0} \frac{\ell n (\tan x)}{\ell n\, x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{\tan x} . \sec^2 x}{\frac{1}{x}}$$

$$= \lim_{x \to 0} \frac{x \sec^2 x}{\tan x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{x.2\sec x. \sec x \tan x + \sec^2 x}{\sec^2 x}$$

$$= \frac{0+1}{1}$$

$$= 1.$$

Exercise 1.7

1. Evaluate the following using L Hospital's rule.

a. $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$

b. $\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$

c. $\lim_{x \to -5} \frac{x^2 - 25}{x + 5}$

d. $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - x - 2}$

e. $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$

f. $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - \frac{x}{2}}{x^2}$

g. $\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$

h. $\lim_{x \to 0} \frac{e^x - x - 1}{x^2}$

i. $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

j. $\lim_{x \to 0} \frac{x - \sin x \cos x}{x^3}$

k. $\lim_{x \to 0} \frac{(e^x - 1) \tan x}{x^2}$

l. $\lim_{t \to 0} \frac{t \sin t}{1 - \cos t}$

2. Evaluate the following limits using L Hospital's rule.

a. $\lim_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1}$

b. $\lim_{x \to \infty} \frac{2x^2 + 3x}{1 + 5x^2}$

c. $\lim_{x \to \infty} \frac{x^5}{e^x}$

d. $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$

Answers

1. a. 8 b. $\frac{9}{2}$ c. -10 d. $-\frac{1}{3}$ e. $\frac{1}{2}$

f. $-\frac{1}{8}$ g. 0 h. $\frac{1}{2}$ i. 2 j. $\frac{2}{3}$

k. 1 l. 2

2. a. $\frac{5}{7}$ b. $\frac{2}{5}$ c. 0 d. 3

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UNIT 2

Partial Derivatives

⌘ Course Contents

- Functions of more than two variables
- Partial derivative from First principles
- Partial derivatives of First and higher orders
- Euler's theorem for function of two variables
- Partial derivatives of composite functions

2.1 Introduction

In first semester, we have studied a function of single variable. But in the real world, a physical problem may depend on two or more than two variables. For example, the function $V = \frac{1}{3}\pi r^2 h$ calculates the volume of a cone depending on its two variables radius (r) and height (h). The function $f(x, y, z) = x^2 + y^2 + z^2$ depends on three independent variables x, y, z. In this unit, we discuss on the functions of two or more variables and their partial derivatives.

2.2 Functions of Two or More Variables

Let D be the set of all ordered pairs (x, y) of real numbers. A function f of two variables is a rule which assigns to each ordered pairs (x, y) in D, a unique real number z in R such that $z = f(x, y)$, is called a function of two variables x and y.

The variables x and y are called **independent variables** and z is called **dependent variable.**

The set D is called the domain and the set of values of z for every (x, y) in D is called the range of f.

Thus, domain (D) = $\{(x, y): x, y \in R\}$

range (R) = $\{z = f(x, y): (x, y) \in R\}$

In the same way, the function of three independent variables is defined and is denoted by $w = f(x, y, z)$.

We can define a function of n variables as follows:

Let D be a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A real valued function f on D is a rule that assigns a real number $z = f(x_1, x_2, \ldots, x_n)$ to each element in D. Functions of several variables are also called multivariable functions.

Examples

1. Let us define $z = \frac{x^2 + y^2}{y - 5}$. This defines a function of x and y. Moreover, the denominator is zero when $y = 5$. Thus the domain of the function is all ordered pairs (x, y) such that $y \neq 5$.
2. Let us define $z^2 = x^2 + y^2$. Then for $(x, y) = (4, 3)$, we have $z^2 = 25$. This gives $z = \pm 5$. Hence the ordered pair (4, 3) cannot associate exactly one value of z. So, z is not a function of x and y.

2.3 Limit of a Function of Two Variables

A function $z = f(x, y)$ of two variables x and y is said to have the limit l as (x, y) approaches (a, b) and write $\lim_{(x, y) \to (a, b)} f(x, y) = l$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x, y) in the domain of f,

$$|(x, y) - (a, b)| < \delta \Rightarrow |f(x, y) - l| < \varepsilon.$$

As for a function of single variable, it can be shown that

$$\lim_{(x, y) \to (a, b)} x = a$$

$$\lim_{(x, y) \to (a, b)} y = b$$

$$\lim_{(x, y) \to (a, b)} k = k, \text{ where } k \text{ is any constant.}$$

Properties of Limits of Functions of Two Variables

Let $\lim_{(x, y) \to (a, b)} f(x, y) = l_1$ and $\lim_{(x, y) \to (a, b)} g(x, y) = l_2$. Then,

1. **Sum rule:** $\lim_{(x, y) \to (a, b)} [f(x, y) + g(x, y)] = l_1 + l_2$
2. **Difference rule:** $\lim_{(x, y) \to (a, b)} [f(x, y) - g(x, y)] = l_1 - l_2$
3. **Product rule:** $\lim_{(x, y) \to (a, b)} [f(x, y) \cdot g(x, y)] = l_1 \cdot l_2$
4. **Quotient rule:** $\lim_{(x, y) \to (a, b)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{l_1}{l_2}, l_2 \neq 0$
5. **Constant multiple rule:** $\lim_{(x, y) \to (a, b)} [k f(x, y)] = k \cdot l_1$, where k is any constant.
6. **Power rule:** $\lim_{(x, y) \to (a, b)} [f(x, y)]^{\frac{m}{n}} = l_1^{\frac{m}{n}}$; where m and n are integers and provided that $l_1^{\frac{m}{n}}$ is a real number.

Example 1. Evaluate:

a. $\lim\limits_{(x, y) \to (4, 3)} \sqrt{x^2 + y^2}$

b. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

Solution

a. $\lim\limits_{(x, y) \to (4, 3)} \sqrt{x^2 + y^2}$

$= \sqrt{4^2 + 3^2}$

$= 5$

b. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$= \lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \times \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

$= \lim\limits_{(x, y) \to (0, 0)} \dfrac{x(x - y)(\sqrt{x} + \sqrt{y})}{(x - y)}$

$= \lim\limits_{(x, y) \to (0, 0)} x(\sqrt{x} + \sqrt{y})$

$= 0(\sqrt{0} + \sqrt{0})$

$= 0$

2.4 Continuity of a Function of Two Variables

As in the case of continuity of function of single independent variable, the continuity of the function of two independent variables is defined as equality of limiting value and functional value.

A function $z = f(x, y)$ is said to be continuous at the point (a, b) if

i. $f(a, b)$ exists [Functional value]

ii. $\lim\limits_{(x, y) \to (a, b)} f(x, y)$ exists [Limiting value]

iii. $\lim\limits_{(x, y) \to (a, b)} f(x, y) = f(a, b)$ [Equality]

More precisely, a function $f(x, y)$ is said to be continuous at (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b).$$

If a function is not continuous then it is said to be discontinuous at the point.

In the similar way, the limit and continuity of a function of three or more independent variables can be defined.

Example 2. Examine whether or not the function $f(x, y) = \begin{cases} \dfrac{xy}{x+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ is continuous at the point (0, 0).

Solution

Let $y = mx$, where m is any finite constant.

Then,

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{xy}{x + y^2}$$

$$= \lim_{x \to 0} \frac{x \cdot mx}{x + m^2x^2}$$

$$= \lim_{x \to 0} \frac{mx}{1 + mx}$$

$$= 0$$

and

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{xy}{x + y^2}$$

$$= \lim_{y \to 0} \frac{\frac{y}{m} \cdot y}{\frac{y}{m} + y^2}$$

$$= \lim_{y \to 0} \frac{y}{1 + my}$$

$$= 0$$

Hence, for any value of m, the limiting value is 0. That is, the limiting value under the different paths is 0. Also, $f(0, 0) = 0$.

$$\therefore \lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0) = 0$$

Hence $f(x, y)$ is continuous at the point (0, 0).

2.5 Partial Derivatives

In previous semester, we have studied the derivative of a function of a single independent variable. In this unit, we deal on the derivatives of a function of several independent variables. While doing partial derivative, we differentiate a function with respect to one of the independent variable and we treat the remaining variables as constants and do the derivative as in the case of single variable.

Let $z = f(x, y)$. The partial derivative of z or f with respect to x is the rate at which z changes as x changes if y is held constant. In the same way, the partial derivative of z or f with respect to y is the rate at which z changes as y changes if x is held constant. Now we give the definition of partial derivative.

Let $z = f(x, y)$ be a function of two independent variables x and y. The partial derivative of z or f with respect to x, denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ of $f_x(x, y)$ or z_x or f_x, is defined as

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$ provided that the limit exists.

It is the ordinary derivative of f with respect to x keeping the other variable y as a constant.

In the same way, the partial derivative of $z = f(x, y)$ with respect to y is defined by

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}$$ provided that the limit exists.

Similarly, if $w = f(x, y, z)$ be a function of three independent variables x, y and z then the partial derivative of $w = f(x, y, z)$ with respect to x, y and z are respectively defined by

$$f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

These are called the first order partial derivatives. These methods of calculating partial derivatives are called **first principles** or **definition method.**

Note that the symbol ∂ (read as 'del') is used to denote partial derivative instead of d, the symbol used in ordinary derivative.

Example 3. Use definition of partial derivatives to find $f_x(x, y)$ and $f_y(x, y)$ for

a. $f(x, y) = x^2y$

b. $f(x, y) = xy + y^2$

Solution

a. $f(x, y) = x^2y$

Then, $f(x + h, y) = (x + h)^2y$

and $f(x, y + k) = x^2(y + k)$

By definition, we have

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{(x + h)^2 y - x^2y}{h}$$

$$= \lim_{h \to 0} \frac{y\{(x + h)^2 - x^2\}}{h}$$

$$= \lim_{h \to 0} \frac{y(x^2 + 2xh + h^2 - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{y(2xh + h^2)}{h}$$

$$= \lim_{h \to 0} \frac{yh(2x + h)}{h}$$

$$= \lim_{h \to 0} y(2x + h)$$

$$= y(2x + 0)$$

$$= 2xy$$

Also, we have,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{x^2(y+k) - x^2 y}{k}$$

$$= \lim_{k \to 0} \frac{x^2(y+k-y)}{k}$$

$$= \lim_{k \to 0} \frac{x^2 k}{k}$$

$$= \lim_{k \to 0} (x^2)$$

$$= x^2$$

b. $f(x, y) = xy + y^2$

Then, $f(x+h, y) = (x+h)y + y^2$

and $f(x, y+k) = x(y+k) + (y+k)^2$

By definition, we have

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)y + y^2 - (xy + y^2)}{h}$$

$$= \lim_{h \to 0} \frac{xy + hy + y^2 - xy - y^2}{h}$$

$$= \lim_{h \to 0} \frac{hy}{h}$$

$$= \lim_{h \to 0} (y)$$

$$= y$$

Also, we have,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{x(y+k) + (y+k)^2 - (xy + y^2)}{k}$$

$$= \lim_{k \to 0} \frac{xy + xk + y^2 + 2yk + k^2 - xy - y^2}{k}$$

$$= \lim_{k \to 0} \frac{xk + 2yk + k^2}{k}$$

$$= \lim_{k \to 0} \frac{k(x + 2y + k)}{k}$$

$$= \lim_{k \to 0} (x + 2y + k)$$

$$= x + 2y + 0$$

$$= x + 2y$$

Also, we have,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{x^2 (y+k) - x^2 y}{k}$$

$$= \lim_{k \to 0} \frac{x^2 (y+k-y)}{k}$$

$$= \lim_{k \to 0} \frac{x^2 k}{k}$$

$$= \lim_{k \to 0} (x^2)$$

$$= x^2$$

b. $f(x, y) = xy + y^2$

Then, $f(x+h, y) = (x+h)y + y^2$

and $f(x, y+k) = x(y+k) + (y+k)^2$

By definition, we have

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)y + y^2 - (xy + y^2)}{h}$$

$$= \lim_{h \to 0} \frac{xy + hy + y^2 - xy - y^2}{h}$$

$$= \lim_{h \to 0} \frac{hy}{h}$$

$$= \lim_{h \to 0} (y)$$

$$= y$$

Also, we have,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{x(y+k) + (y+k)^2 - (xy + y^2)}{k}$$

$$= \lim_{k \to 0} \frac{xy + xk + y^2 + 2yk + k^2 - xy - y^2}{k}$$

$$= \lim_{k \to 0} \frac{xk + 2yk + k^2}{k}$$

$$= \lim_{k \to 0} \frac{k(x + 2y + k)}{k}$$

$$= \lim_{k \to 0} (x + 2y + k)$$

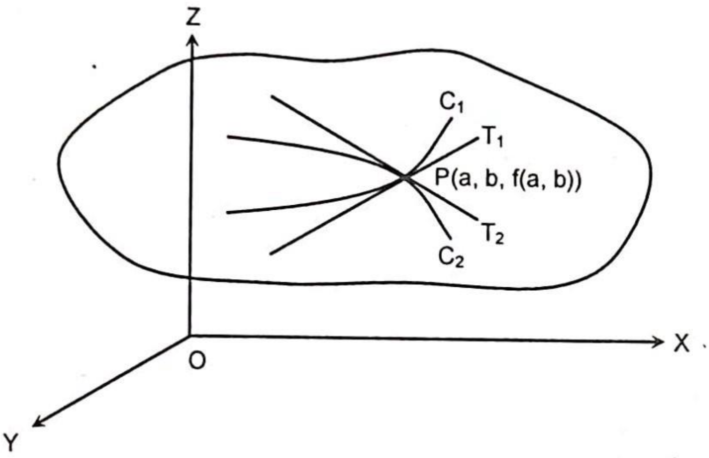
$$= x + 2y + 0$$

$$= x + 2y$$

Geometrical Interpretation of Partial Derivatives

Let $z = f(x, y)$ be a function of two independent variables x and y. The partial derivative $\frac{\partial z}{\partial x}$ at $(x, y) = (a, b)$ gives the slope of the tangent line drawn at (a, b) along the pla. $y = b$ to the graph of $z = f(x, y)$. In the same way, the partial derivative $\frac{\partial z}{\partial y}$ at $(x, y) = (a, b)$ gives the slope of the tangent line drawn at (a, b) along the plane $x = a$ to the graph of $z = f(x, y)$.

Hence the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are interpreted geometrically as the slopes of curves of intersection of surface $z = f(x, y)$ by the planes $y = b$ and $x = a$ respectively.



Partial Derivatives at a Point

The partial derivative of $z = f(x, y)$ with respect to x at the point (a, b) is

$$\left.\frac{\partial f}{\partial x}\right|_{(a, b)} = \left.\frac{d}{dx} f(x, b)\right|_{x=a}$$

$$= \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

provided that the limit exists.

It is denoted by

$$\left(\frac{\partial z}{\partial x}\right)_{(a, b)} \text{ or } f_x(a, b) \text{ or } z_x(a, b)$$

Similarly, the partial derivative of $z = f(x, y)$ with respect to y at the point (a, b) is

$$\left.\frac{\partial f}{\partial y}\right|_{(a, b)} = \left.\frac{d}{dy} f(a, y)\right|_{y=b}$$

$$= \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

provided that the limit exists.

It is denoted by

$$\left(\frac{\partial z}{\partial y}\right)_{(a, b)} \text{ or } f_y(a, b) \text{ or } z_y(a, b)$$

Example 4. If $f(x, y) = x^2 + 3xy + 2y^2$ then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution

To find $\frac{\partial f}{\partial x}$, we treat y as a constant and differentiate f with respect to x.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + 2y^2)$$

$$= 2x + 3y \cdot 1 + 0 \qquad \text{[Treat } y \text{ as a constant]}$$

$$= 2x + 3y$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + 2y^2)$$

$$= 0 + 3x \cdot 1 + 2 \cdot 2y$$

$$= 3x + 4y$$

Example 5. If $z = x^3 + 3xy^2 + y^3$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at (2, 1).

Solution

Given,

$$z = x^3 + 3xy^2 + y^3$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3 + 3xy^2 + y^3)$$

$$= 3x^2 + 3y^2 \text{ (keeping } y \text{ fixed)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3 + 3xy^2 + y^3)$$

$$= 0 + 3x \cdot 2y + 3y^2 \text{ (keeping } x \text{ fixed)}$$

$$= 6xy + 3y^2$$

Now,

$$\left(\frac{\partial z}{\partial x}\right) \text{ at } (2, 1) = \left(\frac{\partial z}{\partial x}\right)_{(2,1)}$$

$$= 3 \times 2^2 + 3 \times 1^2$$

$$= 15$$

$$\left(\frac{\partial z}{\partial y}\right) \text{ at } (2, 1) = \left(\frac{\partial z}{\partial y}\right)_{(2,1)}$$

$$= 6 \times 2 \times 1 + 3 \times 1^2$$

$$= 15$$

2.6 Second Order Partial Derivatives

If $z = f(x, y)$ is given function then the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y. Then their partial derivatives are called second order partial derivatives. They are

- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}

ii. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, denoted by $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}

iii. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, denoted by $\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx}

iv. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$, denoted by $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

The partial derivative f_{xx} and f_{yy} are called pure second order partial derivatives and f_{xy} and f_{yx} are called mixed second order partial derivatives.

In general, $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ i.e. $f_{yx} \neq f_{xy}$.

The following theorem gives the condition for equality of f_{yx} and f_{xy}.

The Mixed Derivative Theorem

If $z = f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) then

$f_{xy}(a, b) = f_{yx}(a, b)$.

NOTE $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ but not f_{xy}.

For, $f_{yx} = (f_y)_x$

$$= \frac{\partial}{\partial x}(f_y) \qquad \left[\because f_x = \frac{\partial f}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Similarly, $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ bot not f_{yx}.

Example 6. Find f_{xx}, f_{yy}, f_{xy} and f_{yx} for the function $f(x, y) = 3x^3 y^2$.

Solution

Given

$$f(x, y) = 3x^3 y^2$$

$$f_x = \frac{\partial}{\partial x}(3x^3 y^2) = 9x^2 y^2$$

$$f_y = \frac{\partial}{\partial y}(3x^3 y^2) = 6x^3 y$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(9x^2 y^2) = 18xy^2$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(6x^3 y) = 6x^3$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(9x^2 y^2) = 18x^2 y$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(6x^3 y) = 6x^3$$

Example 7. If $f(x, y) = x^3 e^{2y}$ then verify that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Solution

Given,

$f(x, y) = x^3 e^{2y}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 e^{2y}) = 3x^2 e^{2y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 e^{2y}) = x^3 2e^{2y} = 2x^3 e^{2y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 e^{2y}) = 3x^2 \cdot 2e^{2y} = 6x^2 e^{2y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x^3 e^{2y}) = 6x^2 e^{2y}$$

$$\therefore \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Example 8. If $z = \ln(x^2 + y^2)$ then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Solution

Given,

$$z = \ln(x^2 + y^2)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\{\ln(x^2 + y^2)\}$$

$$= \frac{\partial\{\ln(x^2 + y^2)\}}{\partial(x^2 + y^2)} \cdot \frac{\partial(x^2 + y^2)}{\partial x}$$

$$= \frac{1}{x^2 + y^2} \cdot 2x$$

$$= \frac{2x}{x^2 + y^2}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\{\ln(x^2 + y^2)\}$$

$$= \frac{\partial\{\ln(x^2 + y^2)\}}{\partial(x^2 + y^2)} \cdot \frac{\partial(x^2 + y^2)}{\partial y}$$

$$= \frac{1}{x^2 + y^2} \cdot 2y$$

$$= \frac{2y}{x^2 + y^2}$$

Now,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2x}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2)\frac{\partial}{\partial x}(2x) - 2x\frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2+y^2)\cdot 2 - 2x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

Again,

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\cdot 2 - 2y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

Now,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2}$$

$$= 0.$$

Example 9. If $w = \sqrt{x^2+y^2+z^2}$ then show that $w_{xx} + w_{yy} + w_{zz} = \frac{2}{w}$.

Solution

Given,

$$w = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$

$$w_x = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}$$

$$= \frac{\partial(x^2+y^2+z^2)^{1/2}}{\partial(x^2+y^2+z^2)}\cdot\frac{\partial(x^2+y^2+z^2)}{\partial x}$$

$$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x$$

$$= x(x^2+y^2+z^2)^{-1/2}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x}{w} \qquad [\because w = \sqrt{x^2+y^2+z^2}\,]$$

$$w_{xx} = \frac{\partial}{\partial x}(w_x)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{w}\right)$$

$$= \frac{w \cdot \frac{\partial}{\partial x}(x) - x \cdot \frac{\partial}{\partial x}(w)}{w^2}$$

$$= \frac{w \cdot 1 - x \cdot \frac{x}{w}}{w^2} \qquad \left[\because \frac{\partial}{\partial x}(w) = w_x = \frac{x}{w}\right]$$

$$= \frac{\frac{w^2 - x^2}{w}}{w^2} = \frac{w^2 - x^2}{w^3}$$

Similarly,

$$w_{yy} = \frac{w^2 - y^2}{w^3} \text{ and } w_{zz} = \frac{w^2 - z^2}{w^3}$$

Now,

$$\text{LHS} = w_{xx} + w_{yy} + w_{zz}$$

$$= \frac{w^2 - x^2}{w^3} + \frac{w^2 - y^2}{w^3} + \frac{w^2 - z^2}{w^3}$$

$$= \frac{w^2 - x^2 + w^2 - y^2 + w^2 - z^2}{w^3}$$

$$= \frac{3w^2 - (x^2 + y^2 + z^2)}{w^3}$$

$$= \frac{3w^2 - w^2}{w^3} \qquad [\because w = \sqrt{x^2 + y^2 + z^2}\ ]$$

$$= \frac{2w^2}{w^3} = \frac{2}{w}$$

$$= \text{RHS}$$

Exercise 2.1

1. Use definition (first principles) of partial derivatives to find $f_x(x, y)$ and $f_y(x, y)$ fo following functions.
   a. $f(x, y) = xy$ b. $f(x, y) = 3x^2y^2$
   c. $f(x, y) = xy + y^2$ d. $f(x, y) = x^2 - xy$
2. Using definition, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ from $f(x, y, z) = xyz$.
3. Find the first order partial derivatives of the following functions.
   a. $f(x, y) = 5x^4y^5$ b. $f(x, y) = ax^2 + 2hxy + by^2$
   c. $f(x, y) = x^3 + x^2y^2 - 2y^3$ d. $f(x, y) = \sqrt{x^2 + y^2}$
   e. $f(x, y) = x \sin xy$ f. $f(x, y) = \frac{3xy}{x + \cos y}$
   g. $f(x, y) = \sin^{-1}(3x + 4y)$ h. $f(x, y) = \ln(2x + 5y)$
   i. $f(x, y, z) = x^3y^5z^2$ j. $f(x, y, z) = \ln(3x + y + 2z)$
4. Find the second order partial derivatives of the following functions.
   a. $f(x, y) = x^3y^4 + 3x^2y^2 - 1$ b. $f(x, y) = \sin(2x + 3y)$
   c. $f(x, y) = x^5 e^{3y}$
5. Calculate all the first order and second order partial derivatives of $f(x, y) = e^{-x} \sin(x + y)$.

6. a. If $u = \ln(x^3 + y^3 + z^3 - 3xyz)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

b. If $z = \frac{x^2 + y^2}{x + y}$, then show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

7. a. If $z = \ln(x^2 + y^2)$, then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

b. If $z = \cos(3x + 4y)$, then show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

8. a. If $f(x, y) = x^3y^2 + x^5 + y^3$, then find f_x and f_y at the point (1, 2).

b. If $f(x, y) = x^4y^2 - x^3y^3$, then find f_{xx} and f_{xxx} at the point (2, 1).

b. If $f(x, y, z) = \frac{x}{x + y + z}$, then find $f_x(-1, 1, 2)$ and $f_y(-1,1, 2)$.

Answers

1. a. $f_x(x, y) = y, f_y(x, y) = x$ b. $f_x(x, y) = 6xy^2, f_y(x, y) = 6x^2y$
   c. $f_x(x, y) = y, f_y(x, y) = x + 2y$ d. $f_x(x, y) = 2x - y, f_y(x, y) = -x$
2. $\frac{\partial f}{\partial x} = yz, \frac{\partial f}{\partial y} = zx$ and $\frac{\partial f}{\partial z} = xy$.
3. a. $f_x = 20x^3y^5, f_y = 25x^4y^4$ b. $f_x = 2(2x + hy), f_y = 2(hx + by)$
   c. $f_x = 3x^2 + 2xy^2, f_y = 2x^2y - 6y^2$ d. $f_x = \frac{x}{\sqrt{x^2 + y^2}}, f_y = \frac{y}{\sqrt{x^2 + y^2}}$
   e. $f_x = xy\cos xy + \sin xy, f_y = x^2 \cos xy$ f. $f_x = \frac{3y\cos y}{(x + \cos y)^2}, f_y = \frac{3x^2 + 3x\cos y + 3xy\sin y}{(x + \cos y)^2}$
   g. $f_x = \frac{3}{\sqrt{1 - (3x + 4y)^2}}, f_y = \frac{4}{\sqrt{1 - (3x + 4y)^2}}$ h. $f_x = \frac{2}{2x + 5y}, f_y = \frac{5}{2x + 5y}$
   i. $f_x = 3x^2y^5z^2, f_y = 5x^3y^4z^2, f_z = 2x^3y^5z$ j. $f_x = \frac{3}{3x + y + 2z}, f_y = \frac{1}{3x + y + 2z}, f_z = \frac{2}{2x + y + 2z}$
4. a. $f_{xx} = 6xy^4 + 6y^2, f_{xy} = 12x^2y^3 + 12xy, f_{yx} = 12x^2y^3 + 12xy, f_{yy} = 12x^3y^2 + 6x^2$
   b. $f_{xx} = -4\sin(2x + 3y), f_{xy} = -2\sin(2x + 3y), f_{yx} = -6\sin(2x + 3y), f_{yy} = -9\sin(2x + 3y)$
   c. $f_{xx} = 20x^3e^{3y}, f_{xy} = 15x^4e^{3y}, f_{yx} = 15x^4e^{3y}, f_{yy} = 9x^5e^{3y}$
5. $f_x = e^{-x}\{\cos(x + y) - \sin(x + y)\}$ $f_y = e^{-x}\cos(x + y)$
   $f_{xx} = -2e^{-x}\cos(x + y)$ $f_{yy} = -e^{-x}\sin(x + y)$
   $f_{xy} = -e^{-x}\{\sin(x + y) + \cos(x + y)\}$ $f_{yx} = -e^{-x}\{\sin(x + y) + \cos(x + y)\}$
8. a. 8, 5 b. 36, 48 c. $\frac{3}{4}, \frac{1}{4}$

2.7 Homogeneous Functions

A function $z = f(x, y)$ of two independent variables x and y is said to be homogeneous of degree n if it can be expressed as

$$x^n \phi\left(\frac{y}{x}\right) \text{ or } y^n \left(\frac{x}{y}\right).$$

Alternatively, a function $z = f(x, y)$ of two independent variables x and y is said to be homogeneous of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for all values of λ, where λ is independent of x and y.

On the other words, a function $z = f(x, y)$ of two independent variables x and y is said to be homogeneous of degree n if the sum of the powers of the variables in each term is n.

The definition of homogeneous functions can be extended to the case of function of more than two variables.

The functions $w = f(x, y, z)$ of three independent variables x, y, z is said to homogeneous of degree if $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$ for all values of λ or it can be expressed in the form $x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right)$, $y^n \phi\left(\frac{x}{y}, \frac{z}{y}\right)$ or $z^n \phi\left(\frac{x}{z}, \frac{y}{z}\right)$.

Example 1. Show that the function $z = f(x, y) = x^2 + xy + y^2$ is a homogeneous function of degree 2.

Solution

$$
\begin{aligned}
f(x, y) &= x^2 + xy + y^2 \\
&= x^2\left(1 + \frac{y}{x} + \frac{y^2}{x^2}\right) \\
&= x^2\left\{\left(\frac{y}{x}\right)^0 + \left(\frac{y}{x}\right)^1 + \left(\frac{y}{x}\right)^2\right\} \\
&= x^2\, \phi\left(\frac{y}{x}\right)
\end{aligned}
$$

This shows that $f(x, y)$ is a homogeneous function of degree 2.

Alternatively method

$$
\begin{aligned}
f(x, y) &= x^2 + xy + y^2 \\
&= y^2 + xy + x^2 \\
&= y^2\left(1 + \frac{x}{y} + \frac{x^2}{y^2}\right) \\
&= y^2\left\{\left(\frac{x}{y}\right)^0 + \left(\frac{x}{y}\right)^1 + \left(\frac{x}{y}\right)^2\right\} \\
&= y^2\, \phi\left(\frac{x}{y}\right)
\end{aligned}
$$

This shows that $f(x, y)$ is a homogeneous function of degree 2.

Alternatively method

$$f(x, y) = x^2 + xy + y^2$$

Now,

$$
\begin{aligned}
f(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda x)(\lambda y) + (\lambda y)^2 \\
&= \lambda^2 x^2 + \lambda^2 xy + \lambda^2 y^2 \\
&= \lambda^2(x^2 + xy + y^2) \\
&= \lambda^2 f(x, y)
\end{aligned}
$$

Hence $f(x, y)$ is a homogeneous function of degree 2.

Example 2. Show that the function $f(x, y) = \frac{x - y}{x + y}$ is a homogeneous function of degree 0.

Solution

Given,

$$f(x, y) = \frac{x - y}{x + y}$$

Now,

$$
\begin{aligned}
f(\lambda x, \lambda y) &= \frac{\lambda x - \lambda y}{\lambda x + \lambda y} = \frac{x - y}{x + y} \\
&= f(x, y) \\
&= \lambda^0 f(x, y)
\end{aligned}
$$

Hence $f(x, y)$ is a homogeneous function of degree 0.

Example 3. Show that the function $z = f(x, y) = \dfrac{x - y}{x^3 + y^3}$ is a homogeneous function of degree -2.

Solution

Given,

$$f(x, y) = \frac{x - y}{x^3 + y^3}$$

Now,

$$\begin{aligned} f(\lambda x, \lambda y) &= \frac{\lambda x - \lambda y}{(\lambda x)^3 + (\lambda y)^3} \\ &= \frac{\lambda(x - y)}{\lambda^3(x^3 + y^3)} \\ &= \lambda^{-2}\left(\frac{x - y}{x^3 + y^3}\right) \\ &= \lambda^{-2} f(x, y) \end{aligned}$$

This shows that it is a homogeneous function of degree -2.

Note:

i. A polynomial function is homogeneous if the degree of each term in the polynomial is same.

ii. A rational function is homogeneous if both numerator and denominator are separately homogeneous.

2.8 Euler's Theorem on Homogeneous Function

f $v = f(x, y)$ be a homogeneous function of two independent variables x and y of degree n, then

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

Proof

Since v is a homogeneous function of degree n, it can be written as

$$v = x^n \phi\left(\frac{y}{x}\right) \qquad \ldots(i)$$

Differentiating both sides of (i) partially w.r.t. x, we get

$$\begin{aligned} \frac{\partial v}{\partial x} &= nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) \\ &= -x^{n-2} y \phi'\left(\frac{y}{x}\right) + nx^{n-1}\phi\left(\frac{y}{x}\right) \end{aligned}$$

Again, differentiating both sides of (i) partially w.r.t. y, we get

$$\begin{aligned} \frac{\partial v}{\partial y} &= x^n \phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \\ &= x^{n-1}\phi'\left(\frac{y}{x}\right) \end{aligned}$$

Now,

$$\begin{aligned} x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} &= x\left[-x^{n-2} y \phi'\left(\frac{y}{x}\right) + nx^{n-1}\phi\left(\frac{y}{x}\right)\right] + y\left[x^{n-1}\phi'\left(\frac{y}{x}\right)\right] \\ &= -x^{n-1} y \phi'\left(\frac{y}{x}\right) + nx^n \phi\left(\frac{y}{x}\right) + x^{n-1} y \phi'\left(\frac{y}{x}\right) \end{aligned}$$

$$= nx^n \phi\left(\frac{y}{x}\right)$$

$$= nv \qquad \text{[Using (i)]}$$

Thus, $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$. This completes the proof.

If $w = f(x, y, z)$ is a homogeneous function of degree n then

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = nw$$

Example 4. Show that the function $u = f(x, y) = x^2 + y^2$ is homogeneous function and verify Eu theorem for it.

Solution

We have,

$$u = f(x, y) = x^2 + y^2$$

Now,

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2$$
$$= \lambda^2(x^2 + y^2)$$
$$= \lambda^2 f(x, y)$$

This shows that $u = f(x, y)$ is a homogeneous function of degree 2.

Thus $n = 2$.

Next, to verify Euler's theorem we first calculate $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ then verify

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Now,

$$u = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(2x) + y(2y)$$
$$= 2x^2 + 2y^2$$
$$= 2(x^2 + y^2)$$
$$= 2u \qquad [\because u = x^2 + y^2]$$

Hence, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$

Thus, Euler's theorem is verified.

Total Differential

Let $z = f(x, y)$ be a function of two independent variables x and y respectively. The differential of tl dependent variable z is defined by

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

i.e. $$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

It can be also written as

$$df = f_x\, dx + f_y\, dy$$

This expression is often called total differential of the function $z = f(x, y)$.

In the same way $w = f(x, y, z)$ then the differential of the dependent variable w is defined by

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial y} dz$$

i.e. $$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial y} dz$$

It can also be written as

$$df = f_x\, dx + f_y\, dy + f_z\, dz$$

2.9 Partial Derivative of Composite Function

Let $z = f(x, y)$ be a differentiable function of two variables x and y. Let $x = g(t)$ and $y = h(t)$ be both differentiable functions of independent variable t. Then

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

If $z = f(x, y)$ be a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Again, if $w = f(x, y, z)$ is a differentiable function of x, y and z, where $x = g(r, s, t)$, $y = h(r, s, t)$ and $z = k(r, s, t)$ are differentiable functions of s and t, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$

These rules are called chain rules.

If $z = f(x, y)$ be a differentiable function of two variables x and y; where $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then $z = f(x, y)$ is a function of single variable t. Now,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

The chain rule can be extended for the function of more than two variables.

2.10 Implicit Differentiation

With the help of total derivative formula, we can easily calculate the derivative of implicit function. Let $f(x, y) = 0$ define y implicitly as a differentiable function of x. Then using total derivative formula, we have

$$f_x \frac{dx}{dx} + f_y \frac{dy}{dx} = 0$$

or, $f_x + f_y \dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

This is called implicit differentiation formula.

Illustrative Examples

Example 1. If $u = \sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

Solution

Given,

$$u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

or, $\sin u = \dfrac{x^2+y^2}{x+y}$

$$= \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)}$$

$$= x\frac{\left\{1+\left(\frac{y}{x}\right)^2\right\}}{\left(1+\frac{y}{x}\right)} = x\,\phi\left(\frac{y}{x}\right)$$

Thus, $\sin u$ is a homogeneous function of degree 1.

Let $\sin u = v$.

By Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \cdot v$$

or, $x\dfrac{\partial}{\partial x}(\sin u) + y\dfrac{\partial}{\partial y}(\sin u) = \sin u$

or, $x\dfrac{\partial}{\partial u}(\sin u)\cdot\dfrac{\partial u}{\partial x} + y\dfrac{\partial}{\partial u}(\sin u)\cdot\dfrac{\partial u}{\partial y} = \sin u$

or, $x\cos u\dfrac{\partial u}{\partial x} + y\cos u\dfrac{\partial u}{\partial y} = \sin u$

or, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{\sin u}{\cos u}$

$\therefore \quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u.$

Example 2. If $\tan u = \frac{y^2}{x}$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$.

Solution

Given,

$$\tan u = \frac{y^2}{x} = y\left(\frac{y}{x}\right)$$
$$= y\,\phi\left(\frac{y}{x}\right)$$

Thus, tan u is a homogeneous function of degree 1.

Let $v = \tan u$.

By Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \cdot v$$

or, $x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u$

or, $x\frac{\partial}{\partial u}(\tan u)\cdot\frac{\partial u}{\partial x} + y\frac{\partial}{\partial u}(\tan u)\cdot\frac{\partial u}{\partial y} = \tan u$

or, $x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \tan u$

or, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u}$

or, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin u\cos u$

or, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u\cos u}{2}$

$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u.$

Example 3. Verify Euler's theorem for the given function $u = f(x, y, z) = 3x^2y + xyz + 2y^2z$.

Solution

Given,

$$u = f(x, y, z) = 3x^2y + xyz + 2y^2z$$

Now,

$$f(\lambda x, \lambda y, \lambda z) = 3(\lambda x)^2(\lambda y) + (\lambda x)(\lambda y)(\lambda z) + 2(\lambda y)^2(\lambda z)$$
$$= \lambda^3(3x^2y + xyz + y^2z)$$
$$= \lambda^3 f(x, y, z)$$

This shows that $u = f(x, y, z)$ is a homogeneous function of degree 3. Thus, we have to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$ to verify Euler's theorem.

Now, differentiating (i) partially with respect to x, y and z respectively, we get

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^2y + xyz + 2y^2z) = 6xy + yz$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x^2y + xyz + 2y^2z) = 3x^2 + xz + 4yz$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(3x^2y + xyz + 2y^2z) = xy + 2y^2$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x(6xy + yz) + y(3x^2 + xz + 4yz) + z(xy + 2y^2)$$
$$= 6x^2y + xyz + 3x^2y + xyz + 4y^2z + xyz + 2y^2z$$
$$= 9x^2y + 3xyz + 6y^2z$$
$$= 3(3x^2y + xyz + 2y^2z)$$
$$= 3u.$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$$

Example 4. Find the total differential for the function $z = f(x, y) = x^3 + y^3$.

Solution

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3) = 3x^2$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3 + y^3) = 3y^2$$

The total differential of z is $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (3x^2)\, dx + (3y^2)\, dy$.

Example 5. If $x^3 + y^3 - 3xy = 0$ then find $\frac{dy}{dx}$ using partial derivatives.

Solution

Let $f(x, y) = x^3 + y^3 - 3xy$

$f_x = 3x^2 - 3y$

$f_y = 3y^2 - 3x$

We have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$
$$= -\frac{3x^2 - 3y}{3y^2 - 3x}$$
$$= -\frac{x^2 - y}{y^2 - x}$$
$$= \frac{y - x^2}{y^2 - x}$$

Example 6. Find $\frac{du}{dt}$ if $u = x^2 + y^2$, $x = 2t + 1$, $y = t^2 + 2$

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

Also,

$$\frac{dx}{dt} = \frac{d}{dt}(2t+1) = 2$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^2+2) = 2t$$

Now,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= 2x \cdot 2 + 2y \cdot 2t$$

$$= 4x + 4yt$$

$$= 4(x + yt)$$

$$= 4\{2t + 1 + (t^2+2)t\}$$

$$= 4(2t + 1 + t^3 + 2t)$$

$$= 4(t^3 + 4t + 1)$$

Example 7. If $z = x^2y + 3y^4$, where $x = \sin t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

Solution

Given,

$$z = x^2y + 3y^4$$

Here,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y + 3y^4) = 2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2y + 3y^4) = x^2 + 12y^3$$

Also,

$$\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cot t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

Now,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy)(\cos t) + (x^2 + 12y^3)(-\sin t)$$

When $t = 0$,

$$x = \sin 0 = 0$$

$$y = \cos 0 = 1$$

$$\therefore \left.\frac{dz}{dt}\right|_{t=0} = (2 \times 0 \times 1)\cos 0 - (0^2 + 12 \times 1^3)(-\sin 0)$$

$$= 0 - 0$$

$$= 0$$

Example 8. Find $\frac{du}{dt}$ if $u = e^{xyz}$, $x = t^3$, $y = \frac{1}{t}$, $z = e^t$.

Solution

Given,

$$u = e^{xyz}$$

$$x = t^3$$

$$y = \frac{1}{t}$$

$$z = e^t$$

We have,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{\partial}{\partial x}(e^{xyz})\cdot\frac{d}{dt}(t^3) + \frac{\partial}{\partial y}(e^{xyz})\cdot\frac{d}{dt}\left(\frac{1}{t}\right) + \frac{\partial}{\partial z}(e^{xyz})\cdot\frac{d}{dt}(e^t)$$

$$= yz\cdot e^{xyz}\cdot 3t^2 + xz\cdot e^{xyz}\cdot\left(-\frac{1}{t^2}\right) + xy\cdot e^{xyz}\cdot e^t$$

$$= e^{xyz}\left[3t^2 yz - \frac{xz}{t^2} + xye^t\right]$$

$$= e^{\left(t^3\cdot\frac{1}{t}\cdot e^t\right)}\left[3t^2\cdot\frac{1}{t}\cdot e^t - \frac{t^3\cdot e^t}{t^2} + t^3\cdot\frac{1}{t}\cdot e^t\right]$$

$$= e^{t^2e^t}(3te^t - te^t + t^2e^t)$$

$$= e^{t^2e^t}\cdot e^t(3t - t + t^2)$$

$$= e^{t^2e^t+t}(2t + t^2)$$

$$= te^{t(te^t+1)}(t+2)$$

Example 9. If $z = 2x + 3y$, $x = r + s$ and $y = r - s$ then find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$.

Solution

Given,

$$z = 2x + 3y$$

$$x = r + s$$

$$y = r - s$$

We have,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial}{\partial x}(2x+3y)\cdot\frac{\partial}{\partial r}(r+s) + \frac{\partial}{\partial y}(2x+3y)\cdot\frac{\partial}{\partial r}(r-s)$$

$$= 2\cdot 1 + 3\cdot 1$$

$$= 5$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}(2x+3y)\cdot\frac{\partial}{\partial s}(r+s) + \frac{\partial}{\partial y}(2x+3y)\cdot\frac{\partial}{\partial s}(r-s)$$

$$= 2\cdot 1 + 3\cdot(-1)$$

$$= -1$$

Example 10. Find $\frac{\partial w}{\partial t}$ if $w = xyz$, $x = \sin t$, $y = \cos t$, $z = t$.

Solution

Given,

$$w = xyz$$
$$x = \sin t$$
$$y = \cos t$$
$$z = t$$

We have,

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t} \\
&= \frac{\partial}{\partial x}(xyz)\cdot\frac{\partial}{\partial t}(\sin t) + \frac{\partial}{\partial y}(xyz)\cdot\frac{\partial}{\partial t}(\cos t) + \frac{\partial}{\partial z}(xyz)\cdot\frac{\partial}{\partial t}(t) \\
&= yz\cdot\cos t + xz\cdot(-\sin t) + xy\cdot 1 \\
&= \cos t\cdot t\cdot\cos t + \sin t\cdot t\,(-\sin t) + \sin t\cdot\cos t \\
&= t(\cos^2 t - \sin^2 t) + \sin t\cos t \\
&= t\cos 2t + \sin t\cos t.
\end{aligned}$$

Exercise 2.2

1. Verify Euler's theorem for the following functions.
   a. $u = x^2 + y^2$
   b. $u = ax^2 + 2hxy + by^2$
   c. $u = \frac{xy}{x+y}$
   d. $u = x^2 e^{\frac{y}{x}}$.
   e. $u = x^3 + y^3 + z^3 - 3xyz$
   f. $u = \frac{x^2 + z^2}{xy + yz}$
   g. $u = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

2. a. Check $f(x, y) = x^n \cdot \tan^{-1}\left(\frac{y}{x}\right)$ for homogeneity and verify Euler's theorem if homogenous.
   b. If $u = \frac{x^4 + y^4}{x+y}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$.
   c. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.
   d. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.
   e. If $u = x^n \ln\left(\frac{y}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

3. Using total derivative, find $\frac{du}{dt}$ if $u = x^2 + y^2$, $x = at^2$ and $y = 2at$.

4. Find the total derivative $\frac{du}{dt}$ if $u = (x+y)e^{xy}$, $x = t$, $y = \frac{1}{t^2}$.

5. Find $\frac{du}{dt}$ if

   a. $u = x^2 + y^2, x = \cos t + \sin t, y = \cos t - \sin t$ at $t = 0$.

   b. $u = e^{xyz}, x = t^3, y = \frac{1}{t}, z = e^t$.

6. a. Find $\frac{dz}{dt}$ if $z = f(x, y) = xy, x = \cos t, y = \sin t$, at $t = 0$.

   b. Find $\frac{dw}{dt}$ if $w = f(x, y, z) = xy + z, x = \cos t, y = \sin t, z = t$, at $t = \frac{\pi}{2}$.

   c. Find $\frac{dw}{dt}$ if $w = f(x, y, z) = \ln(x^2 + y^2 + z^2), x = \cos t, y = \sin t, z = 4\sqrt{t}$, at $t = 1$.

7. Using chain rule, find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s.

   a. $w = x^2 + y^2, x = 3r + 2s, y = 2r + 3s$

   b. $w = xy + \ln z, x = \frac{s^2}{r}, y = r + s, z = \cos r$.

8. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ when $w = x^3 + y^3, x = r + s$ and $y = r - s$ at $r = 1$ and $s = 2$.

9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $y^3 + z^3 + yz - xy = 2$ at $(x, y, z) = (1, 1, 1)$

10. Using partial derivatives, find $\frac{dy}{dx}$

    a. $x^2 + y^2 = 25$

    b. $ax^2 + 2hyx + by^2 = 0$

    c. $x + y = \sin(x + y)$

Answers

3. $4a^2t(t^2 + 2)$

4. $e^{\frac{1}{t}}\left(\frac{t^4 - t^3 - 2t - 1}{t^2}\right)$

5. a. 0 b. $te^{t(te^t - 1)}(t + 2)$

6. a. 1 b. 0 c. $\frac{16}{17}$

7. a. $26r + 24s, 24r + 26s$ b. $-\frac{s^3}{r^2} - \tan r, \frac{s(3s + 2r)}{r}$

8. $30, 24$

9. $\frac{1}{4}, -\frac{3}{4}$

10. a. $-\frac{y}{x}$ b. $-\frac{(ax + hy)}{hx + by}$ c. -1

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Applications of Anti-derivatives

3

⌘ Course Contents

- Standard Integrals, related numerical problems
- Basic idea of curve sketching: odd and even functions, periodicity of a function, symmetry (about *x*-axis, *y*-axis and origin), monotonicity of a function, sketching graphs of polynomial, trigonometric, exponential, and logarithmic functions (simple cases only)
- Area under a curve using limit of sum (without proof)
- Area between two curves (without proof)
- Area of closed a curve (circle and ellipse only)

3.1 Introduction

The reverse process of differentiation is antiderivative or integration. To find the area between the curves and coordinate axes, the integral calculus was developed. Here, we have to find a function whose derivative is known. This inverse process is integration.

$$f \xrightarrow{\text{differentation}} f' \text{(derivative)}$$

$$\text{(integral) } f \xleftarrow{\text{integration}} f'$$

Indefinite Integral

Consider the expression $\frac{d}{dx}(3x^2 + 5) = 6x$.

Thus the derivative of $3x^2 + 5$ with respect to x is $6x$ or $3x^2 + 5$ is a function whose derivative is $6x$. Therefore, we can say that integration of $6x$ is $3x^2 + 5$.

Again, consider the followings:

$$\frac{d}{dx}(3x^2 - 10) = 6x$$

$$\frac{d}{dx}\left(3x^2 + \frac{7}{2}\right) = 6x$$

$$\frac{d}{dx}\left(3x^2 - \frac{1}{2}\right) = 6x$$

Thus, we can see that the function $f(x) = 6x$ has several antiderivatives. It can be sh antiderivative of $6x$ must have the form $3x^2 + c$. This leads to following definition o integral.

Definition

A function $F(x)$ is antiderivative of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain o of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx$$

The symbol $\int$ is an **integral sign.**

The function f is the **integrand** of the integral and x is the **variable of integration.**

From above example and definition, we can write.

$$\int f(x)\, dx = F(x) + c$$

Integration sign

Integrand

an antiderivative of $f(x)$

constant of integration (Arbitrary constant)

So, we can write the above example in notation form as

$$\int 6x\, dx = 3x^2 + c.$$

Many of indefinite integrals needed in scientific work are found by reversing the formulae. Some important integral formulas are given below.

Indefinite Integral	Reversed derivative formul
1. $\int 1\, dx = x + c$	$\frac{d}{dx}(x + c) = 1$
2. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
3. $\int \frac{1}{x} dx = \ln x + c$	$\frac{d}{dx}(\ln x + c) = \frac{1}{x}$
4. $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{(n+1).a} + c, n \neq -1$	$\frac{d}{dx}\left(\frac{(ax+b)^{n+1}}{(n+1)\cdot a} + c\right) = (ax+b)^n$
5. $\int \frac{1}{ax+b} dx = \frac{1}{a}\ln(ax+b) + c$	$\frac{d}{dx}\left(\frac{1}{a}\ln(ax+b) + c\right) = \frac{1}{ax+b}$
6. $\int e^x\, dx = e^x + c$	$\frac{d}{dx}(e^x + c) = e^x$
7. $\int e^{ax}\, dx = \frac{e^{ax}}{a} + c$	$\frac{d}{dx}\left(\frac{e^{ax}}{a} + c\right) = e^{ax}$
8. $\int a^x\, dx = \frac{a^x}{\ln a} + c$	$\frac{d}{dx}\left(\frac{a^x}{\ln a} + c\right) = a^x$

Integrals of Trigonometric Functions

Indefinite integral	Reversed derivative formula
1. $\int \sin x \, dx = -\cos x + c$	$\frac{d}{dx}(-\cos x + c) = \sin x$
2. $\int \cos x \, dx = \sin x + c$	$\frac{d}{dx}(\sin x + c) = \cos x$
3. $\int \sec^2 x \, dx = \tan x + c$	$\frac{d}{dx}(\tan x + c) = \sec^2 x$
4. $\int \text{cosec}^2 x \, dx = -\cot x + c$	$\frac{d}{dx}(-\cot x + c) = \text{cosec}^2 x$
5. $\int \sec x \tan x \, dx = \sec x + c$	$\frac{d}{dx}(\sec x + c) = \sec x \tan x$
6. $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + c$	$\frac{d}{dx}(-\text{cosec}\, x + c) = \text{cosec}\, x \cot x$
7. $\int \sin ax \, dx = -\frac{\cos ax}{a} + c$	$\frac{d}{dx}\left(-\frac{\cos ax}{a} + c\right) = \sin ax$
8. $\int \cos ax \, dx = \frac{\sin ax}{a} + c$	$\frac{d}{dx}\left(\frac{\sin ax}{a} + c\right) = \cos ax$

Examples

1. $\int x^4 dx = \frac{x^{4+1}}{4+1} + c$

$$= \frac{x^5}{5} + c$$

2. $\int \frac{1}{\sqrt{x}} dx = \int \frac{1}{x^{\frac{1}{2}}} dx = \int x^{-\frac{1}{2}} dx$

$$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2x^{\frac{1}{2}} + c$$

$$= 2\sqrt{x} + c.$$

3. $\int \sin 5x \, dx = -\frac{\cos 5x}{5} + c$

$$\cos\frac{x}{2}\,dx = \int\cos\frac{1}{2}x\,dx$$

$$= \frac{\sin\left(\frac{1}{2}\right)x}{\frac{1}{2}} + c$$

$$= 2\sin\frac{x}{2} + c.$$

Rules for Indefinite Integration

1. **Common Multiple Rule**

   $\int kf(x)\,dx = k\int f(x) + c$ where k is constant.

2. **Rules of Negatives:**

   $\int -f(x)\,dx = -\int f(x)\,dx$

3. **Sum and difference Rule:**

   $\int(f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$

3.2 Standard Integrals I

1. $\displaystyle\int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Proof:

Put $x = a\tan\theta$. Then $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

and $\dfrac{dx}{d\theta} = a\dfrac{d}{d\theta}(\tan\theta) = a\sec^2\theta$

$\Rightarrow\quad dx = a\sec^2\theta\,d\theta$

$$\therefore\quad \int\frac{1}{x^2+a^2}\,dx = \int\frac{a\sec^2\theta}{a^2\tan^2\theta + a^2}\,d\theta$$

$$= \int\frac{a\sec^2\theta}{a^2(1+\tan^2\theta)}\,d\theta$$

$$= \int\frac{a\sec^2\theta}{a^2\sec^2\theta}\,d\theta$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\,\theta + c$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

2. $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c, (x > a)$

Proof:

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x-a)(x+a)}\,dx$$

$$= \frac{1}{2a}\int\left[\frac{1}{x-a} - \frac{1}{x+a}\right]dx$$

$$= \frac{1}{2a}\int \frac{1}{x-a}\,dx - \frac{1}{2a}\int \frac{1}{x+a}\,dx$$

$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + c$$

$$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + c$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c.$$

3. $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c, (x < a)$

Proof:

$$\int \frac{1}{a^2 - x^2}\,dx = \int \frac{1}{(a-x)(a+x)}\,dx$$

$$= \int\left[\frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right)\right]dx$$

$$= \frac{1}{2a}\int \frac{1}{a+x}\,dx + \frac{1}{2a}\int \frac{1}{a-x}\,dx$$

$$= \frac{1}{2a}\ln|a+x| + \frac{1}{2a}\ln|a-x|(-1) + c$$

$$= \frac{1}{2a}[\ln|a+x| - \ln|a-x|] + c$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c.$$

4. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Proof:

Put $x = a\sin\theta$. Then $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

and $dx = a\cos\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{a \cos\theta}{\sqrt{a^2 - a^2 \sin^2\theta}}\, d\theta$$

$$= \int \frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}}\, d\theta$$

$$= \int \frac{a\cos\theta}{a\sqrt{\cos^2\theta}}\, d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + c.$$

5. $$\int \frac{1}{\sqrt{a^2+x^2}}\, dx = \ln\left|x + \sqrt{a^2+x^2}\right| + c = \sinh^{-1}\left(\frac{x}{a}\right) + c$$

Proof:

Put $x = a\tan\theta$. Then $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

and $dx = a\sec^2\theta\, d\theta$

Now,

$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = a\sec\theta.$

$$\therefore \int \frac{1}{\sqrt{a^2+x^2}}\, dx = \int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c_1$$

$$= \ln\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + c_1$$

$$= \ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| + c_1$$

$$= \ln\left|\sqrt{a^2+x^2}+x\right| - \ln a + c_1$$

$$= \ln\left|\sqrt{a^2+x^2}+x\right| + c \qquad [c_1 - \ln a = c]$$

Again,

Put $x = a\sinh\theta$. Then $dx = a\cosh\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2+x^2}}\, dx = \int \frac{a\cosh\theta}{\sqrt{a^2+a^2\sinh^2\theta}}\, d\theta$$

$$= \int \frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}\, d\theta$$

$$= \int \frac{\cosh\theta}{\sqrt{\cosh^2\theta}}\, d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + c$$

6. $$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + c = \cosh^{-1}\left(\frac{x}{a}\right) + c$$

Proof:

Put $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta\, d\theta$

and $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta$

$$\therefore \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln(\sec\theta + \tan\theta) + c_1$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + c_1$$

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + c_1$$

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| + c.$$

Again,

Put $x = a\cosh\theta$. Then $dx = a\sinh\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \int \frac{a\sinh\theta}{\sqrt{a^2\cosh^2\theta - a^2}}\, d\theta$$

$$= \int \frac{a\sinh\theta}{a\sinh\theta}\, d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

$$= \cosh^{-1}\left(\frac{x}{a}\right) + c$$

Integrals Reducible to Standard Integrals

The integrals of types $\int \frac{1}{ax^2 + bx + c}\, dx$, $\int \frac{px + q}{ax^2 + bx + c}\, dx$, $\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$ and $\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx$ can be evaluated by reducing them to the standard integrals.

The ideas of reducing them in the form of standard integrals can be seen in illustrative examples.

Illustrative Examples

Example 1. Evaluate: $\int \frac{1}{x^2+4}\,dx$

Solution

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{x^2+2^2}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c. \qquad \left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

Example 2. Evaluate: $\int \frac{1}{9-x^2}\,dx$

Solution

$$\int \frac{1}{9-x^2}\,dx = \int \frac{1}{3^2-x^2}\,dx$$

$$= \frac{1}{2\cdot 3}\ln\left|\frac{3+x}{3-x}\right| + c \qquad \left[\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + c.$$

Example 3. Evaluate: $\int \frac{1}{x^2+2x+10}\,dx$

Solution

$$\int \frac{1}{x^2+2x+10}\,dx = \int \frac{1}{x^2+2x+1+9}\,dx$$

$$= \int \frac{1}{(x+1)^2+3^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + c$$

Example 4. Evaluate: $\int \frac{dx}{3-2x-x^2}$.

Solution

$$\int \frac{dx}{3-2x-x^2} = \int \frac{dx}{3-2x-x^2}$$

$$= \int \frac{dx}{3-(x^2+2x)}$$

$$= \int \frac{dx}{3-(x^2+2x+1-1)}$$

$$= \int \frac{dx}{3-(x+1)^2+1}$$

$$= \int \frac{dx}{2^2 - (x+1)^2}$$

$$= \frac{1}{2 \cdot 2} \ln \left(\frac{2+x+1}{2-x-1}\right) + c$$

$$= \frac{1}{4} \ln \left(\frac{3+x}{1-x}\right) + c$$

Example 5. Evaluate: $\int \frac{2x+3}{4x^2+1} dx$.

Solution

Let $I = \int \frac{2x+3}{4x^2+1} dx$

$$= \int \frac{2x}{4x^2+1} dx + \int \frac{3}{4x^2+1} dx$$

$$= \frac{1}{4} \int \frac{8x}{4x^2+1} dx + \frac{3}{4} \int \frac{1}{x^2 + \frac{1}{4}} dx$$

$$= \frac{1}{4} \int \frac{8x}{4x^2+1} dx + \frac{3}{4} \int \frac{1}{x^2 + \left(\frac{1}{2}\right)^2} dx$$

$$= \frac{1}{4} \ln (4x^2+1) + \frac{3}{4} \times \frac{1}{\frac{1}{2}} \tan^{-1} \left(\frac{x}{\frac{1}{2}}\right) + c$$

$$= \frac{1}{4} \ln (4x^2+1) + \frac{3}{2} \tan^{-1} (2x) + c$$

Example 6. Evaluate: $\int \frac{7x-9}{x^2-2x+35} dx$

Solution

$$\int \frac{7x-9}{x^2-2x+35} dx = \int \frac{\frac{7}{2}(2x-2) - 2}{x^2-2x+35} dx$$

$$= \frac{7}{2} \int \frac{2x-2}{x^2-2x+35} dx - 2 \int \frac{1}{x^2-2x+35} dx$$

$$= \frac{7}{2} \ln |x^2-2x+35| - 2 \int \frac{1}{(x-1)^2+34} dx$$

$$\left[\because \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c\right]$$

$$= \frac{7}{2} \ln |x^2-2x+35| - \frac{2}{\sqrt{34}} \tan^{-1} \left(\frac{x-1}{\sqrt{34}}\right) + c.$$

Alternatively

Let

$$\int \frac{7x-9}{x^2-2x+35}\,dx = p\int \frac{2x-2}{x^2-2x+35}\,dx + q\int \frac{dx}{x^2-2x+35}$$

Then, by equating like terms, we have

$$2p = 7 \text{ and } -2p + q = -9$$

$$\therefore \quad p = \frac{7}{2} \text{ and } q = -2$$

Thus

$$\int \frac{7x-9}{x^2-2x+35}\,dx = \frac{7}{2}\int \frac{2x-2}{x^2-2x+35}\,dx - 2\int \frac{1}{x^2-2x+35}\,dx$$

$$= \frac{7}{2}\ln|x^2-2x+35| - 2\int \frac{1}{(x-1)^2+34}\,dx$$

$$= \frac{7}{2}\ln|x^2-2x+35| - 2\int \frac{1}{(x-1)^2+(\sqrt{34})^2}\,dx$$

$$= \frac{7}{2}\ln|x^2-2x+35| - \frac{2}{\sqrt{34}}\tan^{-1}\left(\frac{x-1}{\sqrt{34}}\right) + c.$$

Example 7. Evaluate: $\int \frac{6x+1}{x^2+9}\,dx$.

Solution

$$\text{Let } I = \int \frac{6x+1}{x^2+9}\,dx$$

$$= \int \frac{6x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx$$

$$= I_1 + I_2 \text{ (say)}$$

$$I_1 = \int \frac{6x+1}{x^2+9}\,dx$$

Put $y = x^2 + 9$. Then

$dy = 2x\,dx$

$$\therefore \quad I_1 = \int \frac{3\,dy}{y} = 3\ln y + c_1 = 3\ln(x^2+9) + c_1$$

$$I_2 = \int \frac{dx}{x^2+3^2} = \frac{1}{3}\tan^{-1}\frac{x}{3} + c_2$$

$$\therefore \quad I = I_1 + I_2$$

$$= 3\ln(x^2+9) + c_1 + \frac{1}{3}\tan^{-1}\frac{x}{3} + c_2$$

$$= 3\ln(x^2+9) + \frac{1}{3}\tan^{-1}\frac{x}{3} + c, \quad \text{where } c = c_1 + c_2.$$

Example 8. Evaluate: $\int \frac{1}{\sqrt{x^2+5}}\,dx$

Solution

$$\int \frac{1}{\sqrt{x^2+5}}\,dx = \int \frac{1}{\sqrt{x^2+(\sqrt{5})^2}}\,dx$$
$$= \ln|x+\sqrt{x^2+(\sqrt{5})^2}|+c$$
$$= \ln|x+\sqrt{x^2+5}|+c.$$

Example 9. Evaluate: $\int \frac{x}{\sqrt{x^2+4x+5}}\,dx$

Solution

$$\int \frac{x}{\sqrt{x^2+4x+5}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+4x+5}}\,dx$$
$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+4x+5}}\,dx - \frac{1}{2}\int \frac{4}{\sqrt{x^2+4x+5}}\,dx$$

Let $y = x^2+4x+5$. Then

$$\frac{dy}{dx} = 2x+4$$

or, $dy = (2x+4)\,dx$

$$\therefore \int \frac{x}{\sqrt{x^2+4x+5}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{y}}\,dy - 2\int \frac{1}{\sqrt{(x+2)^2+1^2}}\,dx$$
$$= \frac{1}{2}\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2\ln|x+2+\sqrt{x^2+4x+5}|+c$$
$$= \sqrt{y} - 2\ln|x+2+\sqrt{x^2+4x+5}|+c$$
$$= \sqrt{x^2+4x+5} - 2\ln|x+2+\sqrt{x^2+4x+5}|+c.$$

Example 10. Evaluate: $\int \frac{1}{\sqrt{(x-\alpha)(x-\beta)}}\,dx$

Solution

Put $x-\alpha = z^2$. Then $dx = 2z\,dz$

and $x-\beta = \alpha-\beta+z^2$

Now,

$$\int \frac{1}{\sqrt{(x-\alpha)(x-\beta)}} = \int \frac{2z}{\sqrt{z^2(\alpha-\beta+z^2)}}\,dz$$
$$= \int \frac{2}{\sqrt{(\sqrt{\alpha-\beta})^2+z^2}}\,dz$$

$$= 2 \ln \left| \frac{z + \sqrt{z^2 + (\sqrt{\alpha - \beta})^2}}{\sqrt{\alpha - \beta}} \right| + c_1$$

$$= 2 \ln \left|\sqrt{x - \alpha} + \sqrt{x - \alpha + \alpha - \beta}\right| - 2 \ln \sqrt{\alpha - \beta} + c_1$$

$$= 2 \ln \left|\sqrt{x - \alpha} + \sqrt{x - \beta}\right| + c.$$

[Where $c = -2 \ln \sqrt{\alpha - \beta} + c_1$]

Example ~~11~~. Evaluate: $\displaystyle\int \frac{1}{(x+2)\sqrt{1+x}}\, dx$

Solution

Put $1 + x = t^2 \Rightarrow x = t^2 - 1$

Then $dx = 2t\, dt$

$$\therefore \int \frac{1}{(x+2)\sqrt{1+x}}\, dx = \int \frac{2t}{(t^2 - 1 + 2)\sqrt{t^2}}\, dt$$

$$= 2 \int \frac{1}{t^2 + 1}\, dt$$

$$= 2 \tan^{-1}(t)$$

$$= 2 \tan^{-1}(\sqrt{1 + x}) + c$$

Example 12. Evaluate: $\displaystyle\int \frac{\cos x}{\sin^2 x + 3 \sin x + 2}\, dx$

Solution

Let $I = \displaystyle\int \frac{\cos x}{\sin^2 x + 3 \sin x + 2}\, dx$

Put $\sin x = t$

Then $\cos x\, dx = dt$

$$\therefore \quad I = \int \frac{1}{t^2 + 3t + 2}\, dt$$

$$= \int \frac{1}{t^2 + 2 \cdot \frac{3}{2} t + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2}\, dt$$

$$= \int \frac{1}{\left(t + \frac{3}{2}\right)^2 - \frac{1}{4}}\, dt$$

$$= \int \frac{1}{\left(t + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\, dt$$

$$= \frac{1}{2 \cdot \frac{1}{2}} \ln \left| \frac{t + \frac{3}{2} - \frac{1}{2}}{t + \frac{3}{2} + \frac{1}{2}} \right| + c$$

$$= \ln \left| \frac{2 \sin x + 2}{2\sin x + 4} \right| + c$$

$$= \ln \left| \frac{\sin x + 1}{\sin x + 2} \right| + c$$

Example 13. Integrate: $\int \sqrt{\frac{1-x}{1+x}}\,dx$.

Solution

Let $I = \int \sqrt{\frac{1-x}{1+x}}\,dx$

$$= \int \sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}}\,dx$$

$$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{-x\,dx}{\sqrt{1-x^2}}$$

$= I_1 + I_2$ (say)

$$I_1 = \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\frac{x}{1} + c_1 = \sin^{-1} x + c_1$$

& $I_2 = \int \frac{-x\,dx}{\sqrt{1-x^2}}$

Put $y = 1 - x^2$

$dy = -2x\,dx$

$\frac{dy}{2} = -x dx$

$$\therefore \quad I_2 = \int \frac{\frac{dy}{2}}{\sqrt{y}}$$

$$= \frac{1}{2} \int y^{\frac{-1}{2}}\,dy$$

$$= \frac{1}{2} \times \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c_2$$

$$= \sqrt{y} + c_2$$

$$= \sqrt{1-x^2} + c_2$$

Answers

1. a. $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$ b. $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+c$ $\tan^{-1}(e^x)+c$

d. $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2}{\sqrt{3}}\right)+c$

2. a. $\frac{1}{12}\ln\left|\frac{x-1}{x+11}\right|+c$ b. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$ c. $\frac{1}{6}\tan^{-1}\left(\frac{2x+6}{3}\right)+c$

d. $\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}\right|+c$ e. $^{-1}(\sin x+2)+c$ f. $\ln|x^2+4x-12|-\frac{1}{2}\ln\left|\frac{x-2}{x+6}\right|+c$

g. $\ln|x^2+4x+5|-\tan^{-1}(x+2)+c$ h. $x+\ln\left|\frac{x-2}{x+2}\right|+c$

3. a. $\ln\left|x+\frac{1}{2}+\sqrt{x^2+x-2}\right|+c$ b. $\ln\left|x+a+\sqrt{2ax+x^2}\right|+c$ c. $\ln\left|2x+3+2\sqrt{3+3x+x^2}\right|+c$

d. $\sin^{-1}\left(\frac{x-a}{a}\right)+c$ e. $\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+c$

f. $\sqrt{x^2+2x-1}+\ln\left|x+1+\sqrt{x^2+2x-1}\right|+c$ g. $2\sqrt{x^2+x+1}+2\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c$

h. $2\sqrt{6+x-x^2}+4\sin^{-1}\left(\frac{2x-1}{5}\right)+c$

4. a. $\frac{1}{2}\ln\left|\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right|+c$ b. $\sin^{-1}x-\sqrt{1-x^2}+c$ c. $\ln\left(e^x+\sqrt{e^{2x}+1}\right)+c$

3.3 Standard Integrals II

Before proceeding to the standard integrals II, we state the formula of integration by parts.

$$\int (u\,v)\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx.$$

1. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Proof

Let $I = \int e^{ax}\cos bx\,dx$. Then, integrating by parts, we obtain

$$I = e^{ax}\int\cos bx\,dx - \int\left(\frac{d}{dx}(e^{ax})\int\cos bx\,dx\right)dx$$

or, $$I = e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

or, $$I = e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\left[e^{ax}\int\sin bx\,dx - \int\left(\frac{d}{dx}(e^{ax})\right)\int\sin bx\,dx\,dx\right]$$

or, $$I = e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\left[e^{ax}\left(\frac{-\cos bx}{b}\right) + \frac{a}{b}\int\cos bx\,e^{ax}\,dx\right]$$

or, $$I = \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

or, $$\left(1+\frac{a^2}{b^2}\right)I = e^{ax}\left(\frac{a\cos bx + b\sin bx}{b^2}\right)$$

$$\therefore \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + c$$

2. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + c$

Proof

Let $I = \int e^{ax} \sin bx \, dx$. Then, integrating by parts, we obtain

$$I = e^{ax} \int \sin bx \, dx - \int \left(\frac{d}{dx}(e^{ax}) \int \sin bx \, dx \right) dx$$

$$\text{or,} \quad e^{ax} \int \sin bx \, dx + \int \frac{\cos bx}{b} ae^{ax} \, dx$$

$$\text{or,} \quad I = -e^{ax} \frac{\cos bx}{b} + \frac{a}{b} \left\{ e^{ax} \int \cos bx \, dx - \int \left(\frac{d}{dx}(e^{ax}) \int \cos bx \, dx \right) dx \right\}$$

$$\text{or,} \quad I = -e^{ax} \frac{\cos bx}{b} + \frac{a}{b} \left\{ \frac{e^{ax} \sin bx}{b} - \int \frac{\sin bx}{b} ae^{ax} \, dx \right\}$$

$$\text{or,} \quad I = -e^{ax} \frac{\cos bx}{b} + \frac{a}{b} \left\{ \frac{e^{ax} \sin bx}{b} - \int \frac{\sin bx}{b} ae^{ax} \, dx \right\}$$

$$\text{or,} \quad I = -e^{ax} \frac{\cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx \, dx$$

$$\text{or,} \quad I = \frac{a}{b^2} e^{ax} \sin bx - \frac{e^{ax} \cos bx}{b} - \frac{a^2}{b^2} I \qquad [\because I = \int e^{ax} \sin bx \, dx]$$

$$\text{or,} \quad I + \frac{a^2}{b^2} I = \frac{a}{b^2} e^{ax} \sin bx - \frac{e^{ax} \cos bx}{b}$$

$$\text{or,} \quad I \left(\frac{a^2 + b^2}{b^2} \right) = e^{ax} \left(\frac{a \sin bx - b \cos bx}{b^2} \right) + c$$

$$\text{or,} \quad I = \frac{b^2 e^{ax} (a \sin bx - b \cos bx)}{b^2 (a^2 + b^2)} + c$$

$$\therefore \int e^{ax} \sin bx \, dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{(a^2 + b^2)} + c.$$

3. $\int \sqrt{x^2 + a^2} \, dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \ln \left(x + \sqrt{x^2 + a^2} \right) + c$

$$= \boxed{\frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + c.}$$

Proof

$$\int \sqrt{x^2 + a^2} \, dx = \int 1 \cdot \sqrt{x^2 + a^2} \, dx$$

$$= x\sqrt{x^2 + a^2} - \int x \cdot \frac{1}{2\sqrt{x^2 + a^2}} 2x \, dx \qquad \text{[Using integration by parts]}$$

$$= x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2 + a^2}} \, dx \qquad \ldots \text{(i)}$$

Also

$$\int\sqrt{x^2+a^2}\,dx = \int\frac{x^2+a^2}{\sqrt{x^2+a^2}}\,dx$$

$$= \int\frac{x^2}{\sqrt{x^2+a^2}}\,dx + a^2\int\frac{dx}{\sqrt{x^2+a^2}} \qquad \ldots\text{(ii)}$$

Adding (i) and (ii) and dividing by 2 gives

$$\int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\int\frac{dx}{\sqrt{x^2+a^2}}$$

$$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$$

4. $$\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$= \sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c.$$

The proof of (4) can be done as (3).

5. $$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

Proof

Let $I = \int\sqrt{a^2-x^2}\,dx$.

Now,

$$I = \int 1\cdot\sqrt{a^2-x^2}\,dx$$

$$= \sqrt{a^2-x^2}\int 1\,dx - \int\left[\frac{d}{dx}(\sqrt{a^2-x^2}\int 1\cdot dx)\right]dx$$

$$= \sqrt{a^2-x^2}\cdot x - \int\frac{1}{\sqrt{a^2-x^2}}\cdot(-x)\cdot x\,dx$$

$$= x\sqrt{a^2-x^2} - \int\frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx$$

or, $$I = x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\left(\frac{x}{a}\right)$$

or, $$2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)$$

or, $$I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c.$$

Alternatively

Here, $I = \int \sqrt{a^2 - x^2}\ dx$

Put $x = a \sin\theta$. Then $dx = a\cos\theta\, d\theta$

$$\therefore \int \sqrt{a^2 - x^2}\ dx = a^2 \int \cos^2\theta\, d\theta$$

$$= a^2 \frac{1}{2} \int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2} [\int d\theta + \int \cos 2\theta\, d\theta]$$

$$= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + c$$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + c$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\right] + c$$

$$= \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c,\ a \neq 0.$$

Integrals Reducible to Standard Integrals

1. $\int \sqrt{ax^2 + bx + c}\, dx$

   We express $ax^2 + bx + c$ in the form $a[(x + \alpha)^2 \pm \beta^2]$ by the method of completing and use standard integrals to solve it.

2. $\int (px + q)\sqrt{ax^2 + bx + c}\ dx$

   Let $\int (px + q)\sqrt{ax^2 + bx + c}\ dx = m\int (2ax + b)\sqrt{ax^2 + bx + c}\ dx + n\int \sqrt{ax^2 + \ldots}$ Then, by comparing $2am = p$ and $mb + n = q$. We find the values of m and n. Substitu n and using standard integrals, we get the required result.

Illustrative Examples

Example 1. Find the integral $\int (2x - 5)\sqrt{x^2 - 5x + 1} \cdot dx$

Solution

Put $x^2 - 5x + 1 = y$

$(2x - 5)\, dx = dy$

$$\therefore \int (2x - 5)\sqrt{x^2 - 5x + 1}\ dx = \int \sqrt{y}\, dy$$

$$= \int y^{1/2}\, dy$$

$$= \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}(x^2 - 5x + 1)^{\frac{3}{2}} + c$$

Example 2. Evaluate: $\int e^x \cos x \, dx$.

Solution

Here, $I = \int e^x \cos x \, dx$

Integrating by parts, we get

$$I = \cos x \int e^x dx - \int \left[\frac{d}{dx}(\cos x) \int e^x dx\right] dx$$

$$= e^x \cos x - \int (-\sin x) \, e^x dx$$

$$= e^x \cos x + \int \sin x \, e^x dx$$

$$= e^x \cos x + \left\{\sin x \int e^x dx - \int \left[\frac{d}{dx}(\sin x) \int e^x dx\right] dx\right\}$$

$$= e^x \cos x + e^x \sin x - \int \cos x \, e^x dx$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

or, $I = e^x \cos x + e^x \sin x - I$ $\quad \left[\because I = \int e^x \cos x \, dx\right]$

or, $2I = e^x \cos x + e^x \sin x + c$

$\therefore \quad I = \dfrac{e^x}{2}(\cos x + \sin x) + c$

Example 3. Evaluate: $\int \sqrt{2ax - x^2} \, dx$

Solution

Let $I = \int \sqrt{2ax - x^2} \, dx$

$$= \int \sqrt{a^2 - (x - a)^2} \, dx$$

Put $x - a = y$. Then $dx = dy$

$\therefore \quad I = \int \sqrt{a^2 - y^2} \, dy$

$$= \frac{1}{2} y \sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} + c$$

$$= \frac{1}{2}(x - a)\sqrt{a^2 - (x - a)^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x - a}{a}\right) + c$$

$$= \frac{1}{2}(x - a)\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x - a}{a}\right) + c.$$

Alternatively

Let $I = \int \sqrt{2ax - x^2} \, dx$

$$= \int \sqrt{a^2 - (x - a)^2} \, dx$$

$$= \frac{x - a}{2}\sqrt{a^2 - (x - a)^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x - a}{a}\right) + c$$

$$= \frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x - a}{a}\right) + c.$$

Example 2. Evaluate: $\int e^x \cos x\, dx$.

Solution

Here, $I = \int e^x \cos x\, dx$

Integrating by parts, we get

$$I = \cos x \int e^x\, dx - \int \left[\frac{d}{dx}(\cos x) \int e^x\, dx\right] dx$$
$$= e^x \cos x - \int (-\sin x)\, e^x\, dx$$
$$= e^x \cos x + \int \sin x\, e^x\, dx$$
$$= e^x \cos x + \left\{\sin x \int e^x\, dx - \int \left[\frac{d}{dx}(\sin x) \int e^x\, dx\right] dx\right\}$$
$$= e^x \cos x + e^x \sin x - \int \cos x\, e^x\, dx$$
$$= e^x \cos x + e^x \sin x - \int e^x \cos x\, dx$$

or, $I = e^x \cos x + e^x \sin x - I$ $\quad \left[\because I = \int e^x \cos x\, dx\right]$

or, $2I = e^x \cos x + e^x \sin x + c$

$\therefore \quad I = \dfrac{e^x}{2}(\cos x + \sin x) + c$

Example 3. Evaluate: $\int \sqrt{2ax - x^2}\, dx$

Solution

Let $I = \int \sqrt{2ax - x^2}\, dx$

$$= \int \sqrt{a^2 - (x-a)^2}\, dx$$

Put $x - a = y$. Then $dx = dy$

$\therefore \quad I = \int \sqrt{a^2 - y^2}\, dy$

$$= \frac{1}{2} y \sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} + c$$
$$= \frac{1}{2}(x-a)\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) + c$$
$$= \frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) + c.$$

Alternatively

Let $I = \int \sqrt{2ax - x^2}\, dx$

$$= \int \sqrt{a^2 - (x-a)^2}\, dx$$
$$= \frac{x-a}{2}\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) + c$$
$$= \frac{x-a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) + c.$$

Example 4. Evaluate: $\int \sqrt{18x - 65 - x^2}\, dx$

Solution

$$\int \sqrt{18x - 65 - x^2}\, dx = \int \sqrt{-(x^2 - 18x + 65)}\, dx$$
$$= \int \sqrt{-\{x^2 - 2 \cdot x \cdot 9 + 9^2 - 16\}}\, dx$$
$$= \int \sqrt{16 - (x-9)^2}\, dx$$
$$= \int \sqrt{4^2 - (x-9)^2}\, dx$$
$$= \left(\frac{x-9}{2}\right)\sqrt{4 - (x-9)^2} + \frac{4^2}{2}\sin^{-1}\left(\frac{x-9}{4}\right) + c$$
$$= \frac{1}{2}(x-9)\sqrt{18x - 65 - x^2} + 8\sin^{-1}\frac{1}{4}(x-9) + c$$

Example 5. Evaluate: $\int (2x+3)\sqrt{x^2 - 2x - 3}\, dx.$

Solution

Let

$$I = \int (2x+3)\sqrt{x^2 - 2x - 3}\, dx$$
$$= \int (2x - 2 + 2 + 3)\sqrt{x^2 - 2x - 3}\, dx$$
$$= \int \{(2x-2) + 5\}\sqrt{x^2 - 2x - 3}\, dx$$
$$= \int (2x-2)\sqrt{x^2 - 2x - 3}\, dx + \int 5\sqrt{x^2 - 2x - 3}\, dx$$
$$= I_1 + I_2 \text{ (say)}$$
$$I_1 = \int (2x-2)\sqrt{x^2 - 2x - 3}\, dx$$

Put $y = x^2 - 2x - 3$

$$dy = (2x-2)dx$$
$$\therefore \quad I_1 = \int \sqrt{y}\, dy$$
$$= \int y^{1/2}\, dy$$
$$= \frac{y^{3/2}}{3/2} + c_1$$
$$= \frac{2}{3} y^{3/2} + c_1$$
$$= \frac{2}{3}(x^2 - 2x - 3)^{3/2} + c_1$$

And,

$$I_2 = 5\int \sqrt{x^2 - 2x - 3}\, dx$$
$$= 5\int \sqrt{x^2 - 2 \cdot x \cdot 1 + 1^2 - 1 - 3}\, dx$$
$$= 5\int \sqrt{(x-1)^2 - 4}\, dx$$
$$= 5\int \sqrt{(x-1)^2 - 2^2}\, dx$$
$$= 5\left[\frac{1}{2}(x-1)\sqrt{(x-1)^2 - 4} - \frac{1}{2} \cdot 2^2 \ln\left(x - 1 + \sqrt{(x-1)^2 - 4}\right)\right] + c$$
$$= \frac{5}{2}(x-1)\sqrt{x^2 - 2x - 3} - 10\ln\left(x - 1 + \sqrt{x^2 - 2x - 3}\right) + c_2$$

$$\therefore \quad I = I_1 + I_2$$

$$= \frac{2}{3}(x^2-2x-3)^{3/2} + c_1 + \frac{5}{2}(x-1)\sqrt{x^2-2x-3} - 10\ln(x-1+\sqrt{x^2-2x-3}) + c_2$$

$$= \frac{2}{3}(x^2-2x-3)^{3/2} + \frac{5}{2}(x-1)\sqrt{x^2-2x-3} - 10\ln(x-1+\sqrt{x^2-2x-3}) + c,$$

where $c = c_1 + c_2$.

Example 6. Evaluate: $\int (2-x)\sqrt{16-6x-x^2}\,dx$

Solution

$$\text{Let } I = \int (2-x)\sqrt{16-6x-x^2}\,dx$$

$$= \frac{1}{2}\int (4-2x)\sqrt{16-6x-x^2}\,dx$$

$$= \frac{1}{2}\int \{10+(-6-2x)\}\sqrt{16-6x-x^2}\,dx$$

$$= 5\int\sqrt{16-6x-x^2}\,dx + \frac{1}{2}\int(-6-2x)\sqrt{16-6x-x^2}\,dx$$

$$= I_1 + I_2 \text{ (say)}$$

Now,

$$I_1 = 5\int\sqrt{16-6x-x^2}\,dx$$

$$= 5\int\sqrt{5^2-(x+3)^2}\,dx$$

$$= 5\left\{\frac{1}{2}(x+3)\sqrt{5^2-(x+3)^2} + \frac{5^2}{2}\sin^{-1}\left(\frac{x+3}{5}\right)\right\} + c_1$$

$$= \frac{5}{2}(x+3)\sqrt{16-6x-x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + c_1$$

Again,

$$I_2 = \frac{1}{2}\int(-6-2x)\sqrt{16-6x-x^2}\,dx$$

Put $y = 16-6x-x^2$

or, $dy = (-6-2x)\,dx$

Then,

$$I_2 = \frac{1}{2}\int\sqrt{y}\,dy$$

$$= \frac{1}{2}\cdot\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c_2$$

$$= \frac{1}{3}(16-6x-x^2)^{\frac{3}{2}} + c_2$$

$\therefore \quad I = I_1 + I_2$

$= \frac{5}{2}(x+3)\sqrt{16-6x-x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + c_1 + \frac{1}{3}(16-6x-x^2)^{\frac{3}{2}}$

$= \frac{5}{2}\ (x+3)\ \sqrt{16-6x-x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + \frac{1}{3}\ (16-6x-x^2)^{\frac{3}{2}} + c$

where $c = c_1 + c_2$

Example 7. Evaluate: $\int \frac{1}{x+\sqrt{x^2-1}}\,dx$

Solution

Let $I = \int \frac{1}{x+\sqrt{x^2-1}}\,dx$

$= \int \frac{x-\sqrt{x^2-1}}{x^2-(x^2-1)}\,dx$

$= \int [x-\sqrt{x^2-1}]\,dx$

$= \int x\,dx - \int\sqrt{x^2-1}\,dx$

$= \frac{x^2}{2} - \left[\frac{x\sqrt{x^2-1}}{2} + \frac{1}{2}\ln|x+\sqrt{x^2-1}|\right] + c$

$= \frac{x^2}{2} - \frac{1}{2}\left[x\sqrt{x^2-1} + \ln|x+\sqrt{x^2-1}|\right] + c.$

Exercise 3.2

1. Evaluate the following integrals.

a. $\int\sqrt{x^2-9}\,dx$

b. $\int\sqrt{x^2-2x+5}\,dx$

c. $\int\sqrt{4+8x-5x^2}\,dx$

d. $\int\sqrt{2ax-x^2}\,dx$

e. $\int\sqrt{4x^2-4x+5}\,dx$

f. $\int\sqrt{(x-\alpha)(\beta-x)}\,dx$

2. Evaluate the following integrals.

a. $\int(x-3)\sqrt{x^2-1}\,dx$

b. $\int(x+2)\sqrt{x^2-10x-11}\,dx$

c. $\int(x-1)\sqrt{x^2-x+1}\,dx$

d. $\int(3x-2)\sqrt{x^2-x+1}\,dx$

e. $\int(2x+1)\sqrt{4x^2+20x+21}\,dx$

f. $\frac{1}{x-\sqrt{x^2-1}}\,dx$

Answers

1. a. $\frac{x}{2}\sqrt{x^2-9} - \frac{9}{2}\ln|x+\sqrt{x^2-9}| + c$

b. $\frac{1}{2}(x-1)\sqrt{5-2x+x^2} + 2\ln|x-1+\sqrt{5-2x+x^2}| + c$

c. $\frac{1}{10}(5x-4)\sqrt{4+8x-5x^2} + \frac{18}{5\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{6}\right) + c$

d. $\frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x-a}{a}\right) + c$

e. $\frac{1}{4}(2x-1)\left(\sqrt{4x^2-4x+5}+\ln\left(2x-1+\sqrt{4x^2-4x+5}\right)\right)+c$

f. $\frac{2x-\alpha-\beta}{4}\sqrt{(x-\alpha)(x-\beta)}+\frac{(\beta-\alpha)^2}{8}\sin^{-1}\left(\frac{2x-\alpha-\beta}{\beta-\alpha}\right)+c$

2. a. $\frac{1}{3}(x^2-1)^{3/2}-\frac{3x}{2}\sqrt{x^2-1}+\frac{3}{2}\ln|x+\sqrt{x^2-1}|$

b. $\frac{1}{3}(x^2-10x-11)^{3/2}+\frac{7}{5}(x-5)\sqrt{x^2-10x-11}-126\ln\left|x-5+\sqrt{x^2-10x-11}\right|+c$

c. $\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{8}(2x-1)\sqrt{x^2-x+1}-\frac{3}{16}\ln\left|2x-1+2\sqrt{x^2-x+1}\right|+c$

d. $(x^2-x+1)^{3/2}-\frac{1}{8}(2x-1)\sqrt{x^2-x+1}-\frac{3}{16}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x+1}\right|+c$

e. $\frac{1}{6}(4x^2+20x+21)^{3/2}-(2x+5)\sqrt{4x^2+20x+21}+4\ln|(2x+5)+\sqrt{4x^2+20x+21}|+c$

f. $\frac{x^2}{2}+\frac{1}{2}\left(x\sqrt{x^2-1}+\ln|x+\sqrt{x^2-1}|\right)+c$

3.4 Standard Integrals of Trigonometric Functions

1. $\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}{(\operatorname{cosec} x-\cot x)}\,dx$

Put $\operatorname{cosec} x-\cot x = y$. Then

$(-\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x)\,dx = dy$

$$\therefore \int \operatorname{cosec} x\,dx = \int\frac{1}{y}\,dy = \ln y + c = \ln|\operatorname{cosec} x-\cot x|+c.$$

Alternatively

$$\int \operatorname{cosec} x\,dx = \int\frac{1}{\sin x}\,dx = \int\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \frac{1}{2}\int\frac{\sec^2\frac{x}{2}}{\tan\frac{x}{2}}\,dx = \ln\left|\tan\frac{x}{2}\right|+c.$$

. $\int \sec x\,dx = \int\frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx$

Put $\sec x+\tan x = y$. Then

$(\sec x\tan x+\sec^2 x)\,dx = dy$

$$\therefore \int \sec x\,dx = \int\frac{1}{y}\,dy = \ln|y|+c = \ln|\sec x+\tan x|+c.$$

Alternatively

$$\int \sec x\, dx = \int \operatorname{cosec}\left(\frac{\pi}{2}+x\right) dx$$

$$= \int \operatorname{cosec} y\, dy \qquad \left(\text{Put } \frac{\pi}{2}+x = y, \text{ then } dx = dy\right)$$

$$= \ln\left|\tan\frac{y}{2}\right| + c$$

$$= \ln\left|\tan\frac{1}{2}\left(\frac{\pi}{2}+x\right)\right| + c$$

$$= \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c.$$

3. $\displaystyle\int \frac{1}{a\sin x + b\cos x}\, dx$ (IMP)·(V.V. IMP)

Put $a = r\cos\theta$ and $b = r\sin\theta$. Then

$a\sin x + b\cos x = r\cos\theta\sin x + r\sin\theta\cos x = r\sin(x+\theta)$.

Also $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

$$\therefore \int \frac{1}{a\sin x + b\cos x}\, dx = \int \frac{1}{r\sin(x+\theta)}\, dx$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta)\, dx$$

$$= \frac{1}{r}\int \operatorname{cosec} y\, dy \qquad [\text{where } y = x+\theta \text{ and } dx = dy]$$

$$= \frac{1}{r}\ln\left|\tan\frac{y}{2}\right| + c$$

$$= \frac{1}{r}\ln\left|\tan\frac{x+\theta}{2}\right| + c$$

$$= \frac{1}{\sqrt{a^2+b^2}}\ln\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right| + c.$$

4. $\displaystyle\int \frac{1}{a+b\cos x}\, dx$

$$\int \frac{1}{a+b\cos x}\, dx = \int \frac{1}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}\, dx$$

$$= \int \frac{1}{(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}\, dx$$

Put $y = \tan\frac{x}{2}$. Then $dy = \frac{1}{2}\sec^2\frac{x}{2}\,dx$.

$$\therefore \int\frac{1}{a+b\cos x}\,dx = 2\int\frac{1}{(a+b)+(a-b)y^2}\,dy = \frac{2}{(a-b)}\int\frac{1}{\frac{(a+b)}{(a-b)}+y^2}\,dy.$$

Case 1: $a > b$

$$\int\frac{1}{a+b\cos x}\,dx = \frac{2}{(a-b)}\int\frac{1}{\left(\sqrt{\frac{(a+b)}{(a-b)}}\right)^2+y^2}\,dy$$

$$= \frac{2}{(a-b)}\frac{1}{\sqrt{\frac{(a+b)}{(a-b)}}}\tan^{-1}\left(\frac{y}{\sqrt{\frac{(a+b)}{(a-b)}}}\right)+c$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{(a-b)}{(a+b)}}\tan\frac{x}{2}\right)+c.$$

Case 2: $a < b$

$$\int\frac{1}{a+b\cos x}\,dx = 2\int\frac{1}{(a+b)-(b-a)y^2}\,dy$$

$$= \frac{2}{(b-a)}\int\frac{1}{\frac{(a+b)}{(b-a)}-y^2}\,dy$$

$$= \frac{2}{b-a}\cdot\frac{1}{2\sqrt{\frac{(a+b)}{(b-a)}}}\ln\left|\frac{\sqrt{\frac{(a+b)}{(b-a)}}+y}{\sqrt{\frac{a+b}{b-a}}-y}\right|+c$$

$$= \frac{1}{\sqrt{(b^2-a^2)}}\ln\left|\frac{\sqrt{a+b}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{a+b}-\sqrt{b-a}\tan\frac{x}{2}}\right|+c$$

Case 3: $a^2 = b^2$, then either $b = a$ or $b = -a$

a. When $b = a$

$$\int\frac{1}{a+b\cos x}\,dx = \frac{1}{a}\int\frac{1}{1+\cos x}\,dx$$

$$= \frac{1}{a}\int\frac{1}{2\cos^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2a}\int\sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{a}\tan\frac{x}{2}+c.$$

b. When $b = -a$

$$\int \frac{1}{a + b\cos x}\,dx = \frac{1}{a}\int \frac{1}{1 - \cos x}\,dx$$

$$= \frac{1}{a}\int \frac{1}{2\sin^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2a}\int \operatorname{cosec}^2\frac{x}{2}\,dx$$

$$= -\frac{1}{a}\cot\frac{x}{2} + c.$$

5. $\displaystyle\int \frac{1}{a + b\sin x}\,dx$

This integral can be evaluated using the similar process as done above.

Integrals of Hyperbolic Functions

1. $\displaystyle\int \sinh x\,dx = \int\left(\frac{e^x - e^{-x}}{2}\right)dx$

$$= \frac{e^x + e^{-x}}{2} + c$$

$$= \cosh x + c$$

2. $\displaystyle\int \cosh x\,dx = \int\left(\frac{e^x + e^{-x}}{2}\right)dx$

$$= \frac{e^x - e^{-x}}{2} + c$$

$$= \sinh x + c$$

3. $\displaystyle\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx$

Put $y = \cosh x$. Then

$dy = \sinh x\,dx$

$\therefore$ $\displaystyle\int \tanh x\,dx = \int \frac{dy}{y}$

$$= \ln|y| + c$$

$$= \ln|\cosh x| + c.$$

4. $\displaystyle\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx$

Put $\sinh x = y$. Then

$\cosh x\,dx = dy$

$$\therefore \int \frac{\cosh x}{\sinh x} dx = \int \frac{dy}{y}$$

$$= \ln|y| + c$$

$$= \ln|\sinh x| + c.$$

5. $\int \operatorname{sech} x \, dx = 2 \tan^{-1}\left(\tanh \frac{x}{2}\right) + c.$

6. $\int \operatorname{cosech} x \, dx = \ln \left|\tanh \frac{x}{2}\right| + c.$

7. $\int \operatorname{sech}^2 x \, dx = \tanh x + c.$

8. $\int \operatorname{cosech}^2 x \, dx = -\coth x + c.$

9. $\int \operatorname{sech} x \cdot \tanh x \, dx = -\operatorname{sech} x + c.$

10. $\int \operatorname{cosech} x \cdot \coth x \, dx = -\operatorname{cosech} x + c.$

Illustrative Examples

Example 1. Evaluate: $\int \frac{1}{2 + \cos x} dx$

Solution

Let I $= \int \frac{1}{2 + \cos x} dx$

$$= \int \frac{1}{2\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right) + \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)} dx$$

$$= \int \frac{1}{3\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} dx$$

$$= \int \frac{dx}{3\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} \times \frac{\sec^2 \frac{x}{2}}{\sec^2 \frac{x}{2}}$$

$$= \int \frac{\sec^2 \frac{x}{2} dx}{\tan^2 \frac{x}{2} + 3}$$

Put $\tan \frac{x}{2} = y$

$$\frac{1}{2} \sec^2 \frac{x}{2} dx = dy$$

$$\sec^2 \frac{x}{2} dx = 2dy$$

Then,

$$I = \int \frac{2dy}{y^2+3}$$

$$= 2\int \frac{dy}{y^2+(\sqrt{3})^2}$$

$$= 2\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}} + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + c$$

Example 2. Find the integral $\int \frac{1}{1+2\sin x}\,dx$

Solution

$$\text{Let } I = \int \frac{1}{1+2\sin x}\,dx$$

$$= \int \frac{dx}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{dx}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1}$$

Put $\tan\frac{x}{2} = y$

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy$$

$$\sec^2\frac{x}{2}\,dx = 2dy$$

$$\therefore\quad I = \int \frac{2dy}{y^2+4y+1}$$

$$= 2\int \frac{dy}{(y+2)^2-3}$$

$$= 2\int \frac{dy}{(y+2)^2-(\sqrt{3})^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{3}} \ln\left(\frac{y+2-\sqrt{3}}{y+2+\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}} \ln\left(\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}\right) + c$$

Example 3. Integrate: $\int \frac{\cos x - \sin x}{\sqrt{\sin 2x}}\, dx$

Solution

Let $I = \int \frac{\cos x - \sin x}{\sqrt{\sin 2x}}\, dx$

$$= \int \frac{\cos x - \sin x}{\sqrt{2 \sin x \cos x}}$$

$$= \int \frac{\cos x - \sin x}{\sqrt{1 + 2 \sin x \cos x - 1}}\, dx$$

$$= \int \frac{(\cos x - \sin x)\, dx}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x - 1}}$$

$$= \int \frac{(\cos x - \sin x)\, dx}{(\sin x + \cos x)^2 - 1}$$

Put $\sin x + \cos y = y$. Then

$(\cos x - \sin x)\, dx = dy$

$$\therefore \quad I = \int \frac{dy}{\sqrt{y^2-1}}$$

$$= \ln(y + \sqrt{y^2-1}) + c$$

$$= \ln(\sin x + \cos x + \sqrt{(\sin x + \cos x)^2 - 1}) + c$$

$$= \ln(\sin x + \cos x + \sqrt{\sin 2x}) + c$$

Example 4. Evaluate: $\int \frac{1}{3\sin x + 4\cos x}\, dx.$

Solution

$$\int \frac{1}{3\sin x + 4\cos x}\, dx$$

Put $3 = r\cos\alpha$ and $4 = r\sin\alpha$. Then,

$r = \sqrt{3^2+4^2} = 5$ and $\alpha = \tan^{-1}\frac{4}{3}$

$$\therefore \int \frac{1}{3\sin x + 4\cos x}\,dx = \int \frac{1}{r\cos\alpha\sin x + r\sin\alpha\cos x}\,dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(x+\alpha)}\,dx$$

$$= \frac{1}{5}\int \operatorname{cosec}(x+\alpha)\,dxyh$$

$$= \frac{1}{5}\ln\left\{\tan\frac{1}{2}(x+\alpha)\right\} + c$$

$$= \frac{1}{5}\ln\left\{\tan\frac{1}{2}\left(x + \tan^{-1}\frac{4}{3}\right)\right\} + c$$

$$= \frac{1}{5}\ln\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{4}{3}\right) + c$$

Alternatively

$$\text{Let I} = \int \frac{1}{3\sin x + 4\cos x}\,dx$$

$$= \int \frac{1}{3\cdot 2\sin\frac{x}{2}\cos\frac{x}{2} + 4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}\,dx$$

$$= \int \frac{dx}{4\cos^2\frac{x}{2} + 6\sin\frac{x}{2}\cos\frac{x}{2} - 4\sin^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{4\left(1 + \frac{3}{2}\tan\frac{x}{2} - \tan^2\frac{x}{2}\right)}$$

$$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}\,dx}{1 - \left(\tan^2\frac{x}{2} - \frac{3}{2}\tan\frac{x}{2}\right)}$$

$$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}\,dx}{1 - \left\{\tan^2\frac{x}{2} - 2\cdot\tan\frac{x}{2}\cdot\frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right\}}$$

$$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}\,dx}{1 - \left(\tan\frac{x}{2} - \frac{3}{4}\right)^2 + \frac{9}{16}}$$

$$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}\,dx}{\frac{25}{16} - \left(\tan\frac{x}{2} - \frac{3}{4}\right)^2}$$

Put $y = \tan\frac{x}{2} - \frac{3}{4}$. Then

$$dy = \frac{1}{2}\sec^2\frac{x}{2}\ dx$$

$$2dy = \sec^2\frac{x}{2}\, dy$$

$$\therefore \quad I = \frac{1}{4}\int \frac{2dy}{\left(\frac{5}{4}\right)^2 - y^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{5}{4}}\ \ln\left(\frac{\frac{5}{4}+y}{\frac{5}{4}-y}\right) + c$$

$$= \frac{1}{5}\ln\left(\frac{\frac{5}{4} + \tan\frac{x}{2} - \frac{3}{4}}{\frac{5}{4} - \tan\frac{x}{2} + \frac{3}{4}}\right) + c$$

$$= \frac{1}{5}\ln\left(\frac{\frac{1}{2} + \tan\frac{x}{2}}{2 - \tan\frac{x}{2}}\right) + c$$

$$= \frac{1}{5}\ln\left(\frac{1 + 2\tan\frac{x}{2}}{2 - \tan\frac{x}{2}}\right) + c$$

Example 5. Evaluate: $\int \frac{1}{\sin x + \cos x}\ dx$

Solution

Let $I = \int \frac{1}{\sin x + \cos x}\ dx$

Put $1 = r\cos\theta$, $1 = r\sin\theta$. Then, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}(1) = \frac{\pi}{4}$

$$\therefore \quad I = \int \frac{1}{\sin x\cdot r\cos\theta + \cos x\cdot r\sin\theta}\, dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(x+\theta)}\, dx$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta)\, dx$$

$$= \frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{x+\theta}{2}\right)\right| + c$$

$$= \frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| + c.$$

Example 6. Integrate: $\int \frac{\cos x - \sin x}{\sqrt{\sin 2x}} dx$

Solution

$$\text{Let I} = \int \frac{\cos x - \sin x}{\sqrt{\sin 2x}} dx$$

$$= \int \frac{\cos x - \sin x}{\sqrt{2 \sin x \cos x}}$$

$$= \int \frac{\cos x - \sin x}{\sqrt{1 + 2 \sin x \cos x - 1}} dx$$

$$= \int \frac{(\cos x - \sin x)\, dx}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x - 1}}$$

$$= \int \frac{(\cos x - \sin x)\, dx}{(\sin x + \cos x)^2 - 1}$$

Put $\sin x + \cos y = y$. Then

$(\cos x - \sin x)\, dx = dy$

$$\therefore \quad I = \int \frac{dy}{\sqrt{y^2 - 1}}$$

$$= \ln (y + \sqrt{y^2 - 1}) + c$$

$$= \ln (\sin x + \cos x + \sqrt{(\sin x + \cos x)^2 - 1}) + c$$

$$= \ln (\sin x + \cos x + \sqrt{\sin 2x}) + c$$

Example 7. Evaluate: $\int \frac{1}{3 + 4 \cosh x} dx$

Solution

$$\text{Let I} = \int \frac{1}{3\left(\cosh^2 \frac{x}{2} - \sinh^2 \frac{x}{2}\right) + 4 \left(\cosh^2 \frac{x}{2} + \sinh^2 \frac{x}{2}\right)} dx$$

$$= \int \frac{1}{7\cosh^2 \frac{x}{2} + \sinh^2 \frac{x}{2}} dx$$

$$= \int \frac{\operatorname{sech}^2 \frac{x}{2}}{7 + \tanh^2 \frac{x}{2}} dx \qquad \left[\begin{array}{l}\text{Multiplying both the numerator} \\ \text{and denominator by } \operatorname{sech}^2 \frac{x}{2}\end{array}\right]$$

Put $\tan h \frac{x}{2} = y$. Then $\frac{1}{2} \operatorname{sech}^2 \frac{x}{2}\, dx = dy$

$$\therefore \quad I = 2\int \frac{1}{7+z^2}\, dy$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \frac{y}{\sqrt{7}} + c$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{1}{\sqrt{7}} \tanh \frac{x}{2}\right) + c.$$

Example 8. Evaluate: $\int \frac{1}{3 + 5\cosh x}\, dx.$

Solution

Let $I = \int \frac{1}{3 + 5\cosh x}\, dx$

$$= \int \frac{1}{3\left(\cosh^2 \frac{x}{2} - \sin h^2 \frac{x}{2}\right) + 5\left(\cosh^2 \frac{x}{2} + \sinh^2 \frac{x}{2}\right)}\, dx$$

$$= \int \frac{1}{8\cosh^2 \frac{x}{2} + 2\sinh^2 \frac{x}{2}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{4\cos h^2 \frac{x}{2} + \sin h^2 \frac{x}{2}}\, dx \times \frac{\operatorname{sech}^2 \frac{x}{2}}{\sec h^2 \frac{x}{2}}$$

$$= \frac{1}{2} \int \frac{\operatorname{sech}^2 \frac{x}{2}}{4 + \tanh^2 \frac{x}{2}} dx$$

Put $y = \tanh \frac{x}{2}$.

$$\frac{dy}{dx} = \sec h^2 \frac{x}{2} \cdot \frac{1}{2}$$

$$\Rightarrow \quad 2dy = \sec h^2 \frac{x}{2}\, dx$$

Then,

$$I = \frac{1}{2} \int \frac{2dy}{4 + y^2}$$

$$= \int \frac{dy}{y^2 + 2^2}$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{y}{2}\right) + c$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{\tan h \frac{x}{2}}{2}\right) + c$$

...ion by Partial Fractions

So...nes, we express the rational fractions into partial fractions. Then we inte... function.

Proper fractions	Partial fractions
$\frac{\text{numerator}}{(x+a)(x+b)}$	$\frac{A}{x+a}+\frac{B}{x+b}$
$\frac{\text{numerator}}{(x+a)^2}$	$\frac{A}{x+a}+\frac{B}{(x+a)^2}$
$\frac{\text{numerator}}{(x^2+a)(x+b)}$	$\frac{Ax+B}{x^2+a}+\frac{C}{x+b}$

Example 9. Evaluate: $\int \frac{2x-11}{x^2+x-2}\,dx$.

Solution

Let

$$\frac{2x-11}{x^2-x-2}=\frac{2x-11}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1} \quad \ldots \text{(i)}$$

or, $\frac{2x-11}{(x+2)(x-1)}=\frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

or, $2x-11=A(x-1)+B(x+2)$ ... (ii)

Put $x=1$ in (ii), we get,

$$2-11=B(1+2)$$

$$B=-3$$

Again, put $x=-2$ in (ii), we get,

$$-4-11=A(-2-1)$$

$$A=5$$

Then from (i)

$$\frac{2x-11}{x^2+x-2}=\frac{5}{x+2}-\frac{3}{x-1}$$

Now,

$$\int\frac{2x-11}{x^2+x-2}\,dx=\int\left(\frac{5}{x+2}-\frac{3}{x-1}\right)dx$$

$$=5\int\frac{1}{x+2}\,dx-3\int\frac{1}{x-1}\,dx$$

$$=5\ln(x+2)-3\ln(x-1)+c.$$

Example 10. Evaluate: $\int \frac{6x+7}{(x+2)^2}\,dx$.

Solution

Let $\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$ ... (i)

or, $\frac{6x+7}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$

or, $6x + 7 = A(x+2) + B$ ... (ii)

Put $x = -2$ in (ii), we get,

$-12 + 7 = B$

or, $B = -5$

Again, put $x = 0$ in (ii), we get,

$6 \times 0 + 7 = 2A + B$

or, $7 = 2A - 5$

or, $2A = 12$

$\therefore$ $A = 6$

Hence, from (i),

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}$$

Now,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \frac{6}{x+2}\,dx - \int \frac{5}{(x+2)^2}\,dx$$

$$= 6\int \frac{1}{x+2}\,dx - 5\int (x+2)^{-2}\,dx$$

$$= 6 \cdot \frac{1}{1} \cdot \ln(x+2) - 5\,\frac{(x+2)^{-2+1}}{(-2+1)\cdot 1} + c$$

$$= 6\ln(x+2) + \frac{5}{x+2} + c.$$

Example 11. Integrate: $\int \frac{x^2}{(x+2)(x+3)^2}\,dx$.

Solution

Let $\frac{x^2}{(x+2)(x+3)^2} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

or, $\frac{x^2}{(x+2)(x+3)^2} = \frac{A(x+3)^2 + B(x+2)(x+3) + C(x+2)}{(x+2)(x+3)^2}$

or, $x^2 = A(x+3)^2 + B(x+2)(x+3) + C(x+2)$ ... (i)

Put $x = -2$ in (i), we get $4 = A$

Again, put $x = -3$ in (i), we get

$9 \quad = C(-3+2)$

$C \quad = -9$

And, put $x = 0$ in (i), we get

$0 \quad = 9A + 6B + 2C$

or, $0 \quad = 9 \times 4 + 6B + 2 \times (-9)$

$\therefore \quad B \quad = -3$

$$\therefore \quad \frac{x^2}{(x+2)(x+3)^2} = \frac{4}{x+2} - \frac{3}{x+3} - \frac{9}{(x+3)^2}$$

Now,

$$\int \frac{x^2}{(x+2)(x+3)^2} = 4\int \frac{1}{x+2}\,dx - 3\int \frac{1}{x+3}\,dx - 9\int \frac{1}{(x+3)^2}\,dx$$

$$= 4\ln(x+2) - 3\ln(x+3) + \frac{9}{x+3} + c$$

Exercise 3.3

Evaluate the following integrals.

1. $\int \frac{1}{2+\sin x}\,dx$
2. $\int \frac{1}{1+2\sin x}\,dx$
3. $\int \frac{1}{5-4\sin x}\,dx$
4. $\int \frac{1}{2+\cos x}\,dx$
5. $\int \frac{1}{4+5\cos x}\,dx$
6. $\int \frac{1}{1-2\cos x}\,dx$
7. $\int \frac{1}{1+\sin x+\cos x}\,dx$
8. $\int \frac{1}{4-5\sin^2 x}\,dx$
9. $\int \frac{\sin 2x}{(\sin x+\cos x)^2}\,dx$
10. $\int \frac{1}{3\sin x+4\cos x}\,dx$
11. $\int \frac{dx}{3\sin x-5\cos x}$
12. $\int \frac{1}{4+3\sinh x}\,dx$
13. $\int \frac{1}{3+5\cosh x}\,dx$
14. $\int \frac{\tanh x}{\cosh x+9\,\text{sech}\,x}\,dx$

Answers

1. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right)+c$
2. $\frac{1}{\sqrt{3}}\ln\left|\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}\right|+c$
3. $\frac{2}{3}\tan^{-1}\left[\frac{5\tan\frac{x}{2}\ldots}{3}\right.$
4. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)+c$

7. $\ln\left|\tan\frac{x}{2}+1\right| + c$

8. $\frac{1}{4}\ln\left|\frac{2+\tan x}{2-\tan x}\right| + c$

9. $x + \frac{1}{\tan x + 1} + c$

10. $\frac{1}{5}\ln\left|\tan\frac{x+\theta}{2}\right| + c$, where $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

11. $\frac{1}{\sqrt{34}}\ln\left(\frac{5\tan\frac{x}{2}+3-\sqrt{34}}{5\tan\frac{x}{2}+3\sqrt{34}}\right)+c$, $\frac{1}{\sqrt{34}}\ln\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{5}{3}\right)+c$

12. $\frac{1}{5}\ln\left|\frac{1+2\tanh\frac{x}{2}}{4-2\tanh\frac{x}{2}}\right| + c$

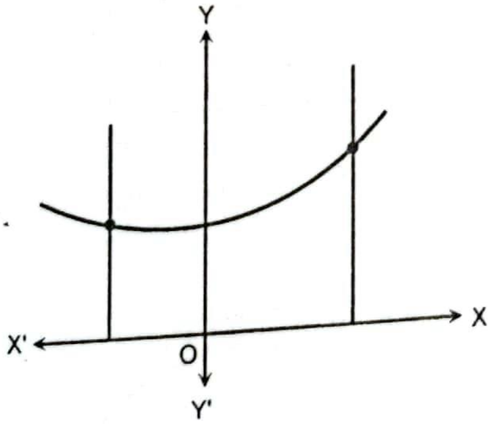
13. $\frac{1}{2}\tan^{-1}\left(\frac{\tanh\frac{x}{2}}{2}\right)+c$

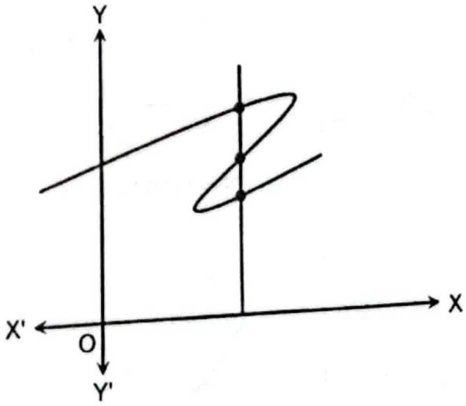
14. $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\cosh x\right)+c$

3.5 Curve Sketching

In this section, we discuss the various techniques of curve sketching and then use these techniques to sketch the graph of basic functions. We begin with the definition graph of a function.

A graph is a pictorial way of presenting the relationship between the two variables. All graphs may not represent function. When a vertical line intersects a graph in more than one point, the same number x corresponds to more than one values of y and hence, the graph is not the graph of a function as shown in the following diagram. This is called vertical line test. In the following figure, the graph in first figure represents a function but in second the graph is not the graph of function.





3.6 Even and Odd Functions

A function $f(x)$ is said to be even when $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$

For example, $f(x) = x^2$ is an even function as

$$f(-x) = (-x)^2 = x^2 = f(x)$$

and $g(x) = \sin x$ is an odd function as

$$g(-x) = \sin(-x) = -\sin x = -g(x)$$

We note that a function could be neither odd nor even.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + \sin x + 5$. Then

$$f(-x) = (-x)^2 + \sin(-x) + 5$$
$$= x^2 - \sin x + 5 \text{ which is neither } f(x) \text{ nor } -f(x)$$

Thus $f(x)$ is neither even nor odd function.

Reflections

1. The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ about x-axis.
2. The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ about y-axis.

Vertical and Horizontal Shift

1. The graph of $y = f(x) + a$ is obtained by shifting the graph of $y = f(x)$, a units upward.
2. The graph of $y = f(x) - a$ is obtained by shifting the graph of $y = f(x)$, a units downward.
3. The graph of $y = f(x - a)$ is obtained by shifting the graph of $y = f(x)$ horizontally to ... a units.
4. The graph of $y = f(x + a)$ is obtained by shifting the graph of $y = f(x)$ horizontally to ... by a units.

Vertical and Horizontal Shrinking

1. The graph of $y = a\,f(x)$ is obtained from the graph of $y = f(x)$ by stretching vertically ... $|a| > 1$ or shrinking vertically for $0 < |a| < 1$.
2. The graph of $y = f(ax)$ is obtained from the graph of $y = f(x)$ by stretching horizontally ... $|a| > 1$ or shrinking horizontally for $0 < |a| < 1$.

Guidelines for Sketching the Graph of Functions

We now discuss the following guidelines that we may need while sketching the graph of a fu...

1. **Domain and Range:** The domain is the set of all possible real numbers, x for which ... real number whereas the range of f is the set all real number $f(x)$ as x varies through ... domain. of graph begins with the determination of domain of the function and we a... range of the function.
2. **Intercepts:** Intercepts of a graph are essential while sketching a graph. We set $x = 0$ ... y-intercepts. To find the x intercepts, we set $y = 0$ and solve for x. Thus, x-intercepts ... where the graph intersects x-axis. We also check that whether the graph passes though ... not.
3. **Symmetry:** We check for various types of symmetry discussed earlier which help us ... the complete graph from the graph of half of the region. We check whether the function ... even or neither.
4. **Interval of increase or decrease:** We need to find the interval of increase and ... decrease of the given function. This will enable us to figure out rise and fall of graph ... domain.
5. **Periodicity:** If the given function is periodic, we find the fundamental period. We th... the graph on an interval of length equal to period, and then we use translation to ob... entire graph.
6. **Asymptotes:** In sketching the curve it is very useful to know exactly where the given c... asymptotes. We need to check whether the given curve has horizontal, vertical or ... asymptote or not.

With the above information, we can sketch the graph of function.

3.10 Sketching the Graphs of Functions

a. Quadratic Function

Consider a quadratic function

$y = f(x) = ax^2 + bx + c, a \neq 0.$

The various characteristics of the graph of the function $y = ax^2 + bx + c$ are as follows:

i. The graph opens up if $a > 0$ and opens down if $a < 0$.

ii. Domain = $(-\infty, \infty)$.

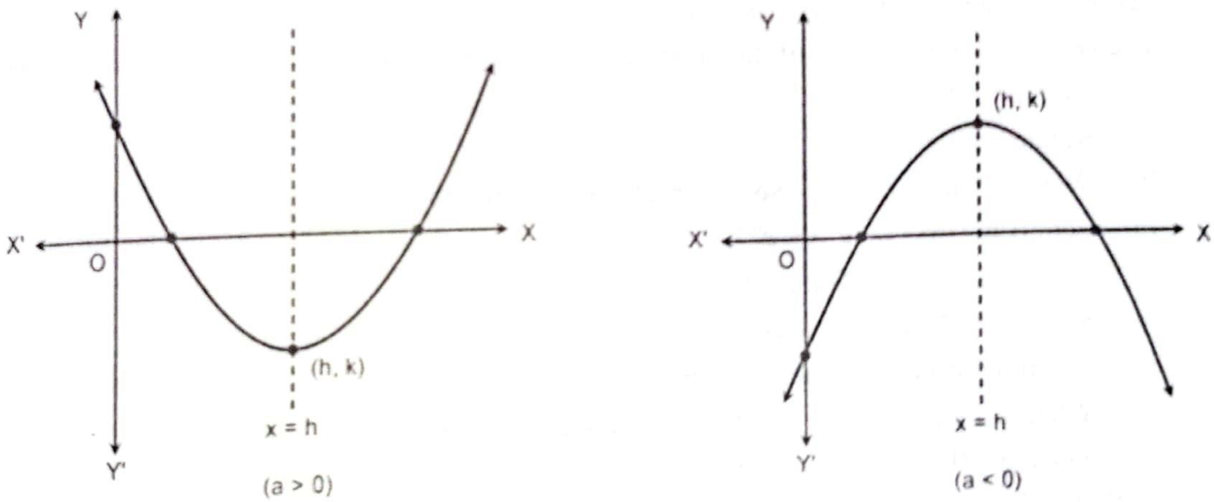
iii. Vertex $(h, k) = \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

iv. The points (h, k) is called the vertex or turning point of the graph (parabola) and hence, the graph is symmetrical about the line $x = h$.

v. If $a > 0$, the range is $[k, \infty)$ and if $a < 0$, then the range is $(-\infty, k]$.

vi. Put $x = 0$ to obtain y-intercepts, put $y = 0$ to obtain x-intercepts.

vii. We can take some additional points on the graph to get help in sketching.



b. Cubic Function

Consider a cubic function given by

$f(x) = ax^3 + bx^2 + cx + d, a \neq 0.$

It is not easy to sketch the graph of all types of cubic function. We discuss about the graph of $y = f(x) = x^3$

The various characteristics of the functions are as follows:

i. Domain = $(-\infty, \infty)$

Range = $(-\infty, \infty)$.

ii. It is an odd function and hence the graph is symmetrical about origin

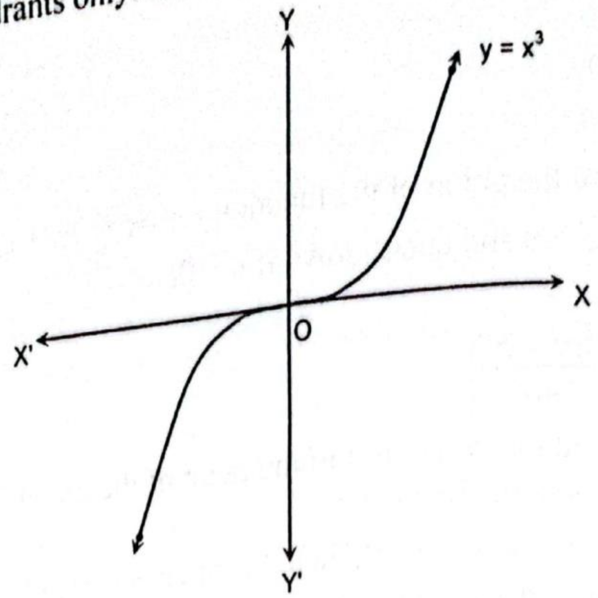
iii. The graph passes through origin.

iv. Since $x_1 < x_2 \Rightarrow (x_1)^3 < (x_2)^3$

$\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (-\infty, \infty)$.

The graph is always increasing.

v. When $x > 0$ then $y > 0$ and when $x < 0$ then $y < 0$. This shows that the graph lies in first and third quadrants only. With the help of above characteristics, the graph is as follows:



c. Exponential function

Let us consider $y = f(x) = e^{ax}$ $(a > 0)$

The various characteristics of the given function are as follows:

i. Domain = $(-\infty, \infty)$.

Range = $(0, \infty)$.

ii. When $x = 0$, $y = e^0 = 1$. So the graph passes through (0, 1).

iii. For all $x_1 < x_2 \Rightarrow e^{ax_1} < e^{ax_2}$

$\Rightarrow f(x_1) < f(x_2)$.

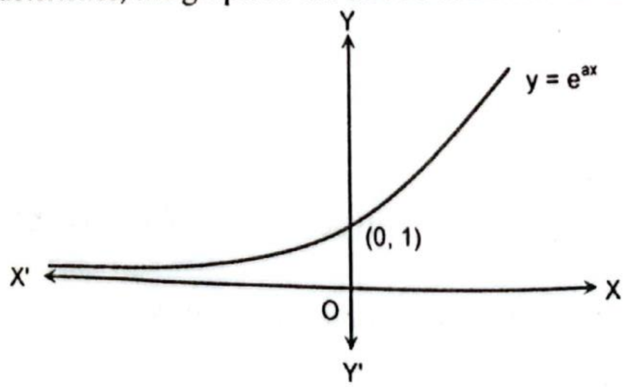
So $f(x) = e^{ax}$ is an increasing function on its domain.

iv. The function is not symmetrical.

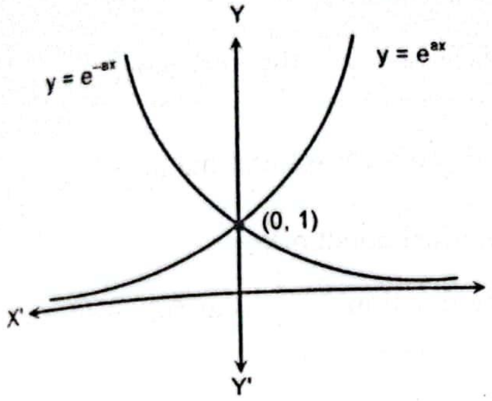
v. When x approaches to $-\infty$, $f(x) = e^{ax}$ approach to zero.

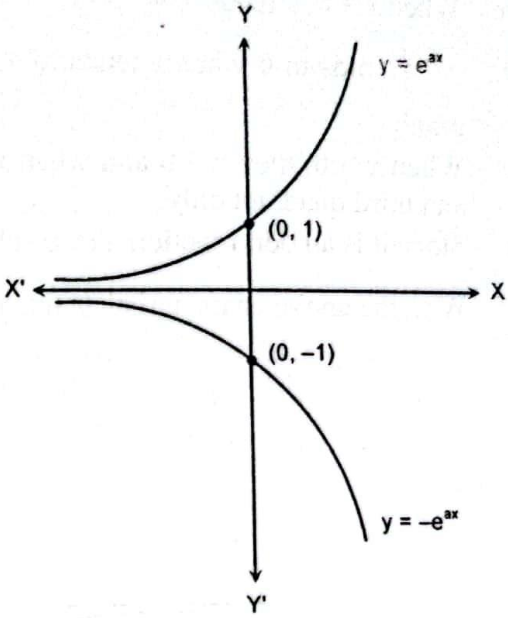
Thus, $y = 0$ is a horizontal asymptote.

With the above characteristics, the graph of the above function is as follows:



Now the graph of $y = e^{-ax}$ $(a > 0)$ is obtained by reflecting the graph of $y = e^{ax}$ about y-axis. And the graph of $y = -e^{ax}$ is obtained by reflecting the graph about x-axis.





The graph exponential function $y = a^x$ $(a > 0, a \neq 1)$ is similar to the graph of $y = e^x$ and hence can be obtained similarly.

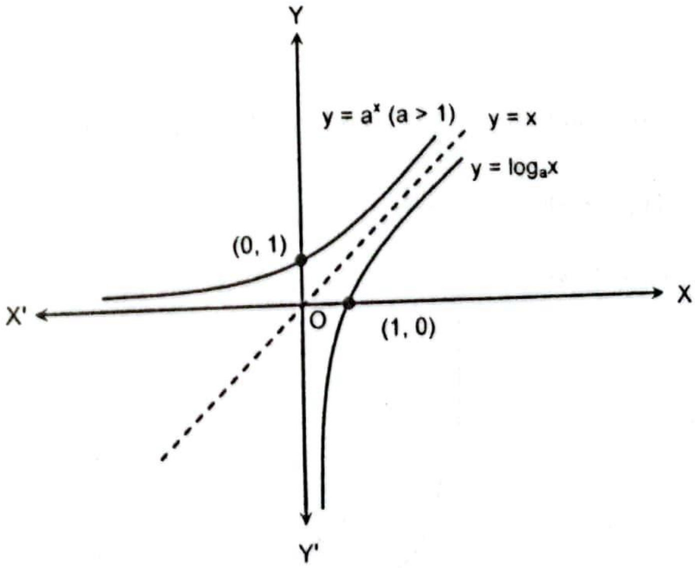
d. Logarithmic Function

Let $y = a^x$ $(a > 0, a \neq 1)$ be an exponential function. Then, its inverse function is called the logarithmic function with base a and it is denoted by $\log_a x$.

That is, $\log_a x = y \Leftrightarrow a^y = x$.

We note that $y = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. So logarithmic function has domain $(0, \infty)$ and range $\mathbb{R}$. Thus, the graph logarithmic function with any base can be obtained by reflecting the corresponding exponential function about the line $y = x$.

For example graph of $y = \log_a x$ $(a > 1)$ is obtained by reflecting the graph of $y = a^x$ about $y = x$ as shown in figure.



e. Rational Function

The transformation techniques is easy to obtain the graph of rational functions. We plot the simplest rational function $y = f(x) = \frac{1}{x}$ and transform this graph to obtain the other graphs of rational functions.

1. Graph of $y = f(x) = \frac{1}{x}$

The different characteristics of the function are as follows:

i. Domain $= (-\infty, \infty) - \{0\}$.

ii. Range $= (-\infty, \infty) - \{0\}$.

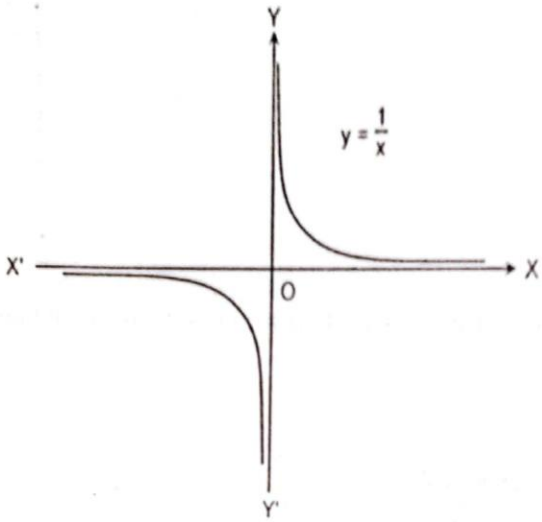
iii. When $x = 0$, y tends to ∞. So $x = 0$ is a vertical asymptote.

iv. $y = \frac{1}{x}$ tends to 0 when x tends to $+\infty$ or $-\infty$. So $y = 0$ is the horizontal asymptote to the graph.

v. When $x > 0$, then $y > 0$ and when $x < 0$, $y < 0$. This shows that the graph lies on the first and third quadrant only.

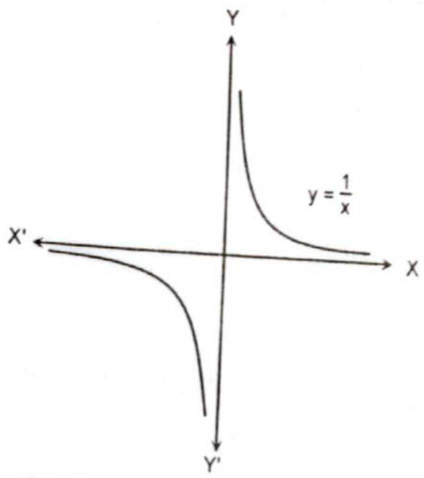
vi. Since it is an odd function, the graph is symmetrical about origin.

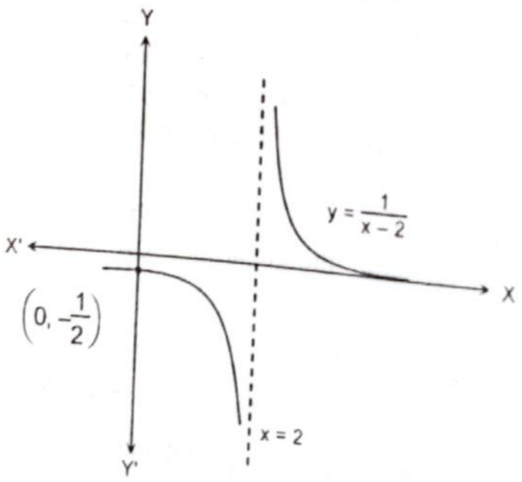
With the above characteristics, the graph of the function $y = \frac{1}{x}$ is as follows:



The graph of $y = \frac{1}{x}$ is a rectangular hyperbola.

2. The graph of $y = \frac{1}{x-2}$ is obtained from the graph of $y = \frac{1}{x}$ by shifting horizontally to the right by 2 units.





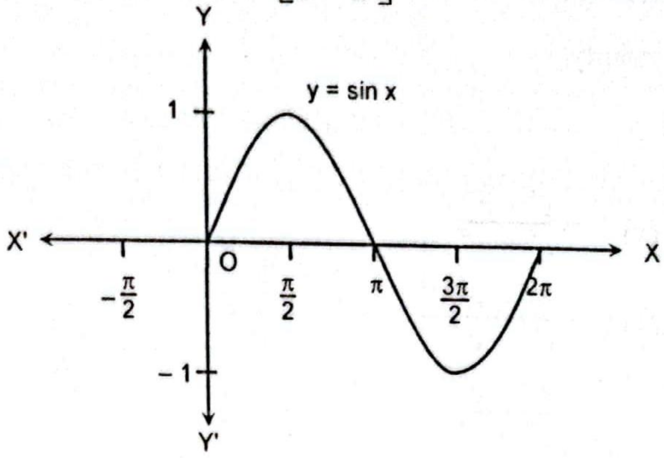
f. Trigonometric Function

The various characteristics of $y = \sin x$ are as follows

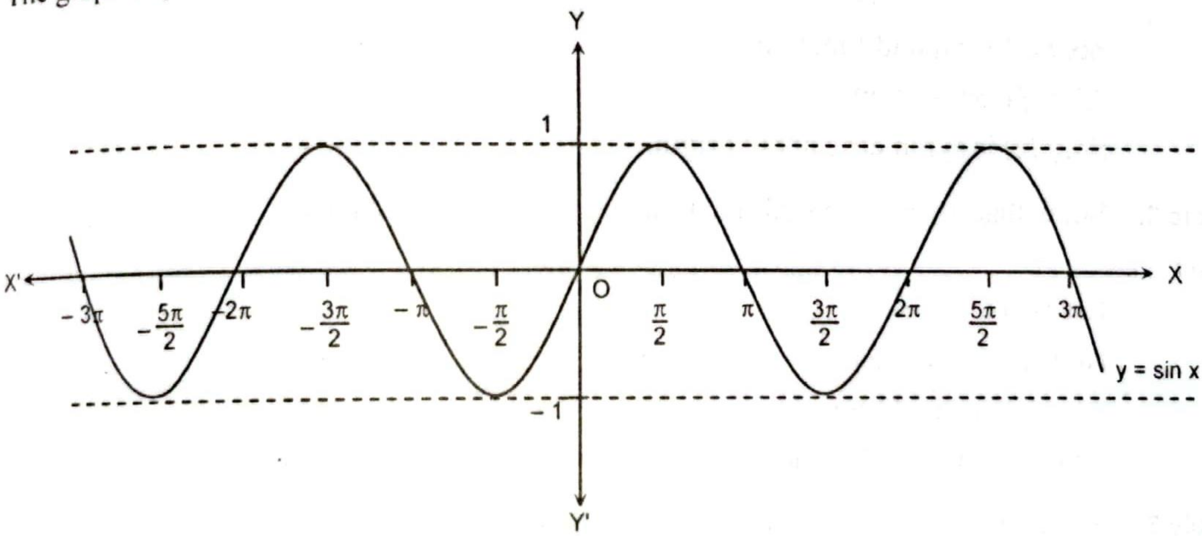
i. Domain = $(-\infty, \infty)$, Range = $[-1, 1]$.
ii. It passes through $(0, 0)$.
iii. It is symmetric about origin.
iv. It is periodic function having fundamental period 2π.

v. When $x = n\pi$, $\sin x = 0$, n is an integer.
$y = 0$, when $x = 0, x = \pi, x = 2\pi$ on $[0, 2\pi]$.

vi. Graph rises to 1 on $\left[0, \frac{\pi}{2}\right]$, falls down to -1 on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ an(...in rises to ze($\left[\frac{3\pi}{2}, 2\pi\right]$.

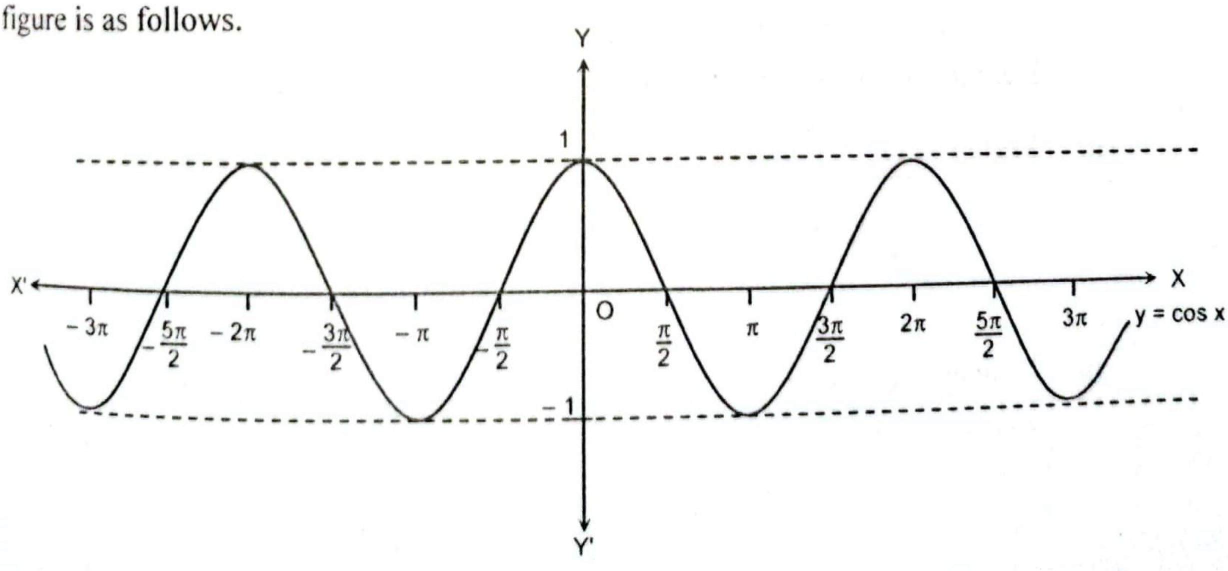


The graph of $y = \sin x$ is as follows:



The graph of $\cos x$ is obtained by shifting the graph of $\sin x$ by $\frac{\pi}{2}$ to the left because $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

The figure is as follows.



Illustrative Examples

Example 1. Examine whether the function: $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is even or odd. Also examine for its symmetricity.

Solution

Here,

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$f(-x) = \frac{e^{-x} - e^{-(-x)}}{e^{-x} + e^{-(-x)}}$$

$$= \frac{e^{-x} - e^{x}}{e^{-x} + e^{x}}$$

$$= -\frac{(e^x - e^{-x})}{e^x + e^{-x}}$$

$$= -f(x)$$

So, $f(x)$ is an odd function.

Also, $f(-x) = -f(x)$

So, $f(x)$ is symmetric about origin.

Example 2. Show that $y = x^3$ is an odd function and test its symmetricity.

Solution

Let $f(x) = y = x^3$

Now, $f(-x) = (-x)^3 = -x^3 = -f(x)$

So, $f(x)$ is an odd function.

Since $f(x)$ is an odd function, the curve is symmetric about origin.

Example 3. Stating main characteristics of the function $f(x) = x^2 + 4x + 3$, sketch the graph.

Solution

The characteristics of the function $y = f(x) = x^2 + 4x + 3$ are as follows:

i. Comparing $y = x^2 + 4x + 3$ with $y = ax^2 + bx + c$, we have $a = 1$, $b = 4$, $c = 3$

Here $a = 1 > 0$, so the parabola turns upwards.

ii. Vertex $= \left(\dfrac{-b}{2a}, \dfrac{4ac - b^2}{4a}\right)$

$$= \left(\frac{-4}{2 \cdot 1}, \frac{4 \cdot 1 \cdot 3 - 4^2}{4 \cdot 1}\right)$$

$$= (-2, -1)$$

iii. Domain $= (-\infty, \infty)$

Range $= [-1, \infty)$

iv. Axis of parabola: $x = -2$

v. When $x = 0$, $y = 3$, so the curve meets y – axis at (0, 3).

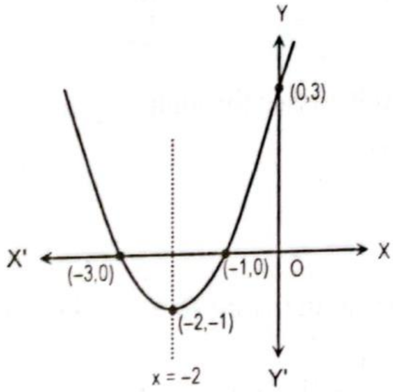
vi. When $y = 0$, $x^2 + 4x + 3 = 0$

or, $(x + 3)(x + 1) = 0$

$\therefore x = -3, -1$

So, the curve meets the x–axis at (–3, 0) and (–1, 0)

With these characteristics the sketch of the graph of given functions is as follows:



Example 4. Sketch the graph of the function $y = x - x^2$ stating its different characteristics.

Solution

Here, $y = x - x^2$

The given function represents a parabola. It has following characteristics.

i. Comparing $y = x - x^2$ with $y = ax^2 + bx + c$, we get

$a = -1$, $b = 1$, $c = 0$

Since $a < 0$, the parabola turns downwards.

ii. Vertex: $x = \frac{-b}{2a} = \frac{-1}{2 \times (-1)} = \frac{1}{2}$

When $x = \frac{1}{2}$, $y = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\therefore$ Vertex $= \left(\frac{1}{2}, \frac{1}{4}\right)$

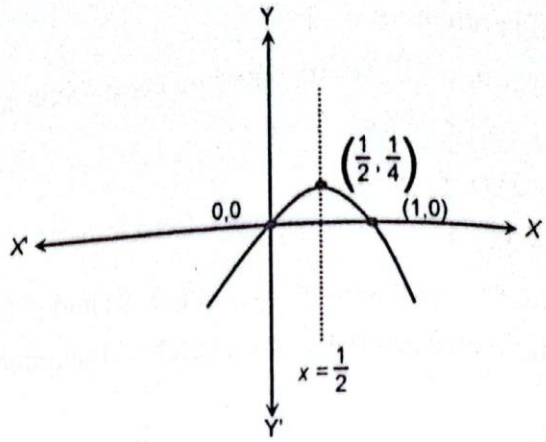
iii. Domain $= (-\infty, \infty)$

Range $= \left(-\infty, \frac{1}{4}\right]$

iv. The axis of the parabola is $x = \frac{1}{2}$

The parabola is symmetric about $x = \frac{1}{2}$.

The sketch of the graph of $y = x - x^2$ with above characteristics is



v. When $x = 0, y = 0$. So it passes through origin.
When $y = 0, x - x^2 = 0$

$\Rightarrow$ $x(1 - x) = 0$

$\Rightarrow$ $x = 0, 1$

i.e. the curve cuts the x-axis at the point (0, 0) and (1, 0)

Example 5. Sketch the graph of $y = \left(\frac{1}{3}\right)^x$ indicating its different characteristics.

Solution

First we sketch the graph of $y = 3^x$.

The characteristics of $y = 3^x$ are as follows:

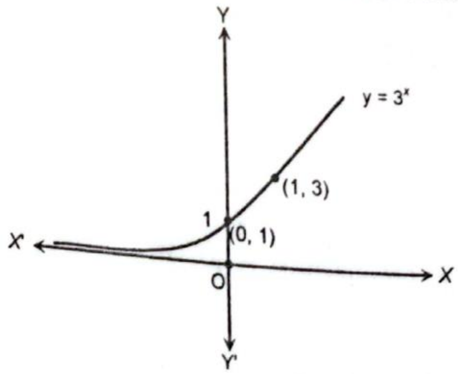
i. When $x = 0, y = 1$. So, the curve cuts y-axis at (0, 1)

ii. When $y = 0, x = -\infty$. So, the curve does not meet the x-axis at a finite distan approaches the x-axis on its negative side at an infinite distance. So, $y = 0$ (is the asymptote to the curve.

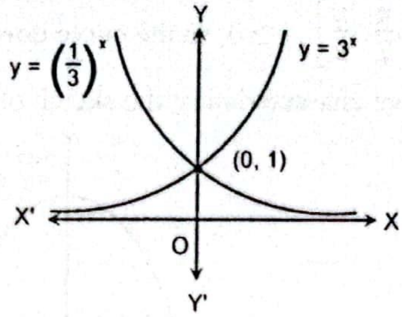
iii. When $x = 1, y = 3$. So, the curve passes through the point (1, 3)

iv. When $x > 1, y > 3$. So, when x increases y also increases. Similarly y dec when $x < 0$.

v. No value of x makes y negative, so no part of the curve lies below x-axis.



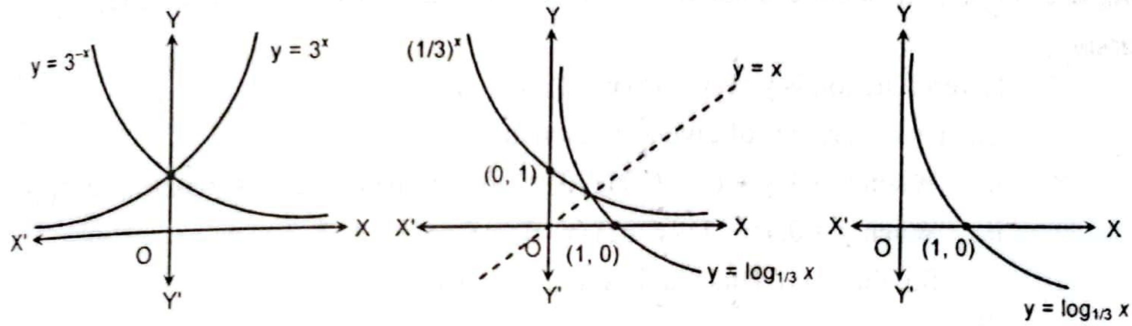
Now the graph of $y = f(x) = \left(\frac{1}{3}\right)^x$ is obtained by reflecting the graph of $f($ about y-axis as shown below.



Example 6. Sketch the graph of $y = f(x) = \log_{1/3} x$.

Solution

The functions $\left(\frac{1}{3}\right)^x$ and $\log_{1/3} x$ are inverse to each other, Thus, the graph if $\log_{1/3} x$ is obtained by reflecting the graph of $\left(\frac{1}{3}\right)^x$ about the line $y = x$.



Example 7. Draw the graph of $y = \cos x \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$ using its different characteristics.

Solution

Given function is $y = \cos x \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$

The characteristics of the above function are:

i. When $x = 0$, $y = 1$, so the curve cuts y-axis at $(0, 1)$

ii. When $y = 0$, $\cos x = 0$

$\therefore \quad \cos x = \cos\left(\pm\frac{\pi}{2}\right)$

$\therefore \quad x = \pm\frac{\pi}{2}$

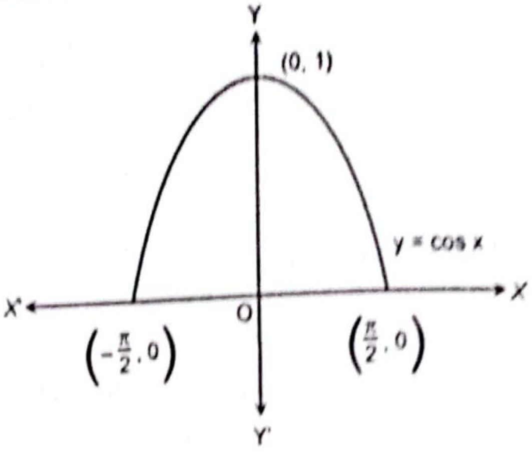
Hence, the curve meets the x-axis at $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$

iii. Since $\cos x$ is an even function, so it is symmetric about y-axis for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

iv. For $\frac{-\pi}{2} \le x \le 0$, y increases and for $0 \le x \le \frac{\pi}{2}$, y decreases.

v. For all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y \geq 0$, so the curve does not lie below x-axis.

With the above characteristics, the sketch of the graph of given function is



Example 8. Using different characteristics, sketch the graph of $y = (x - 1)(x - 2)(x - 3)$.

Solution

Given function is $y = (x - 1)(x - 2)(x - 3)$

The characteristics of given function are:

i. When $x = 0$, $y = (-1)(-2)(-3) = -6$. So, the curve cuts y-axis at $(0, -6)$

ii. When $y = 0$, $(x - 1)(x - 2)(x - 3) = 0 \Rightarrow x = 1, 2, 3$

So, the curve cuts x-axis at $(1, 0)$ $(2, 0)$ and $(3, 0)$

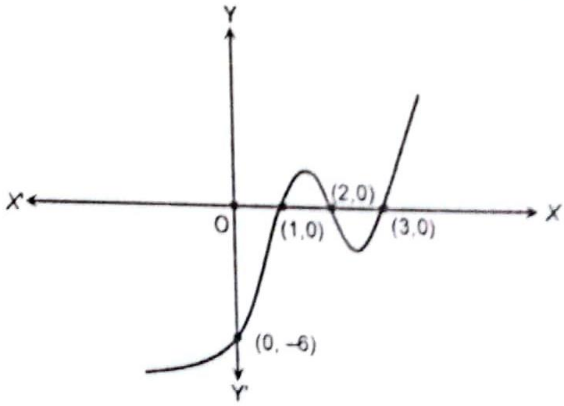
iii. When $x > 3$, $y > 0$. This shows that when x increases y also increases for $x > 3$.

Also when $x < 1$, $y < 0$. This shows that when x decreases, y also decreases.

iv. When $1 < x < 2$, $y > 0$. So the curve lies above x-axis when $1 < x < 2$.

Also, when $2 < x < 3$, $y < 0$. So the curve lies below the x-axis when $2 < x < 3$.

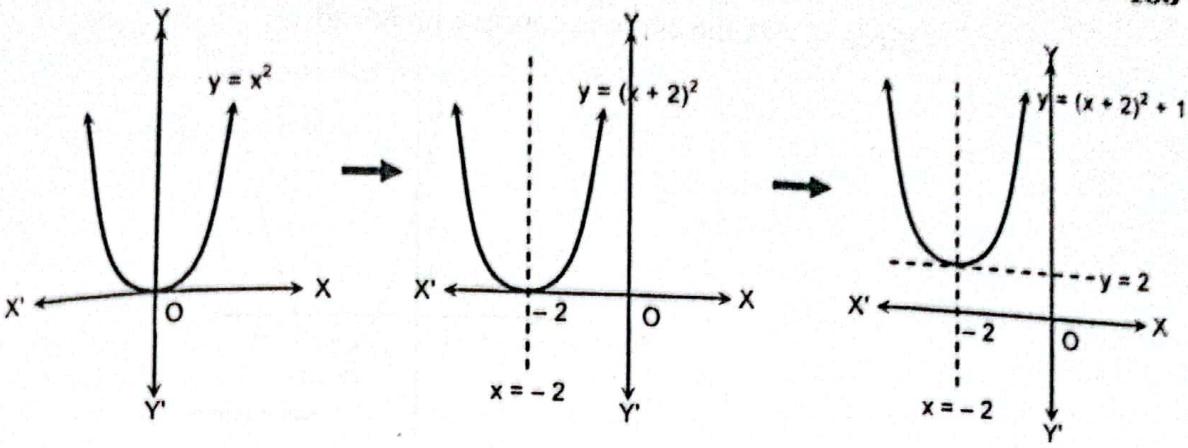
With the above characteristics, the sketch of the graph of given function is:



Example 9. Sketch the graph of $y = (x + 2)^2 + 1$ by suitable transformation of the graph of $y = x$

Solution

We first sketch the graph of $y = x^2$. We then obtain the graph of $y = (x + 2)^2$ by shi horizontally the graph of $y = x^2$ to the left by 2 units. Finally $y = (x + 1)^2 + 1$ is obt by upward vertical shifting of $y = (x + 2)^2$ by 1 unit.



Using calculus, we can sketch the graph of the functions.

Strategy for graphing $y = f(x)$

1. Find y' and y''.
2. Find the rise and fall of the curve.
3. Determine the concavity of the curve.
4. Make a summary and show the curve's general shape.
5. Plot specific points and sketch the curve.

Example 10. Sketch the graph the following functions.

a. $y = x^2 - 4x + 3$

b. $y = x^3 - 3x + 3$.

Solution

a. Here,

$$y = x^2 - 4x + 3$$
$$y' = 2x - 4$$
$$y'' = 2$$

For critical points, we have,

$$y' = 0$$

or, $2x - 4 = 0$

$\therefore \quad x = 2.$

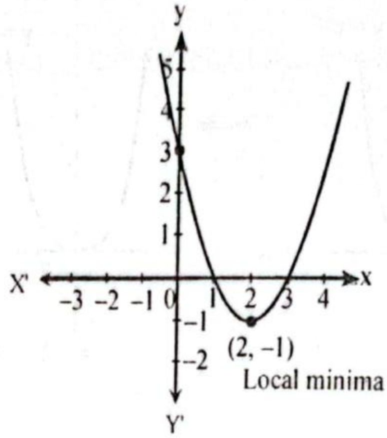
For $x > 2$, $y' = 2(x - 2) > 0$. So, the curve rises on $(2, \infty)$.

For $x < 2$, $y' = 2(x - 2) < 0$. So, the curve falls on $(-\infty, 2)$

At $x = 2$, $y'' = 2 > 0$. So y has minimum value at $x = 2$.

Min. value $= 2^2 - 4 \times 2 + 3 = -1$.

Since $y'' > 0$, the curve is concave up for all x.



b. Given,

$$y = x^3 - 3x + 3$$
$$y' = 3x^2 - 3 = 3(x-1)(x+1)$$
$$y'' = 6x$$

For critical points, we have,

$y' = 0$

or, $3x^2 - 3 = 0$

or, $3x^2 = 3$

or, $x^2 = 1$

or, $x^2 = (\pm 1)^2$

$\therefore$ $x = \pm 1$.

The increasing and decreasing interval of given function are

Intervals	Sign of		
	$(x-1)$	$(x+1)$	$y' = 3(x-1)(x+1)$
$(-\infty, -1)$	−	−	+
$(-1, 1)$	−	+	−
$(1, \infty)$	+	+	+

$\therefore$ The curve rises on $(-\infty, -1) \cup (1, \infty)$ and falls on $(-1, 1)$.

For point of inflection, we have,

$f''(x) = 0$

or, $6x = 0$

$\therefore$ $x = 0$.

For $x > 0$, $y'' > 0$ and for $x < 0$, $y'' < 0$. So, the curve is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

When $x = 1$, $y'' = 6 \times 1 = 6 > 0$.

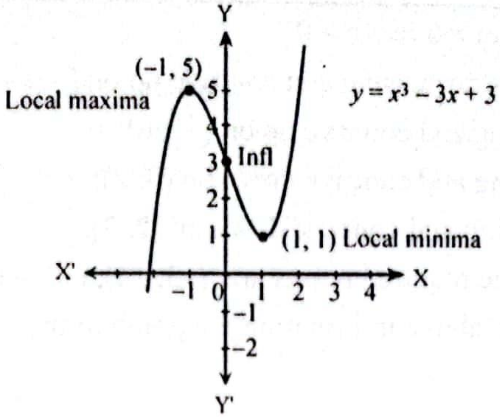
So, y has minimum value at $x = 1$.

Minimum value $= y_{min} = 1^3 - 3 \times 1 + 3 = 1$.

When $x = -1$, $y'' = 6 \times (-1) = -6 < 0$.

So, y has maximum value at $x = -1$.

Maximum value $= y_{max} = (-1)^3 - 3 \times (-1) + 3 = 5$



Example 11. Sketch the graph of the function $f(x) = -x^3 + 12x + 5$ for $-3 \le x \le 3$.

Solution

We have,

$$y = f(x) = -x^3 + 12x + 5$$

Then,

$$y' = -3x^2 + 12$$

$$y'' = -6x$$

For critical point, we have

$$f'(x) = 0$$

$$-3x^3 + 12 = 0$$

or, $x^2 = 4$

$\therefore$ $x = 2, -2$

Also for the point of inflection, we have

$$f''(x) = 0$$

$\Rightarrow$ $-6x = 0$

$\Rightarrow$ $x = 0$

Since $f''(2) = -12 < 0$, so graph of $f(x)$ is maximum at $x = 2$ i.e. at $(2, 21)$ and $f''(-2) = 12 > 0$, graph of $f(x)$ is minimum at $x = -2$ i.e. at $(-2, -11)$.

The increasing and decreasing intervals of $f(x)$ are shown below.

Intervals	Sign of		
	$(x-2)$	$(x+2)$	$-(x-2)(x+2)$
$[-3, -2)$	–	–	–
$(-2, 0)$	–	+	+
$(0, 2)$	–	+	+
$(2, 3]$	+	+	–

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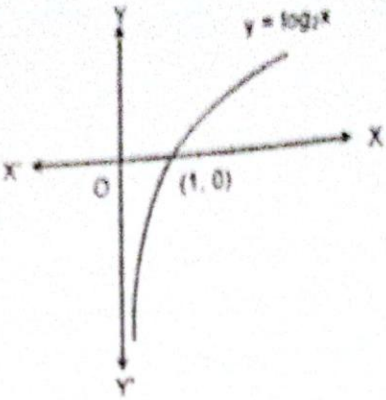

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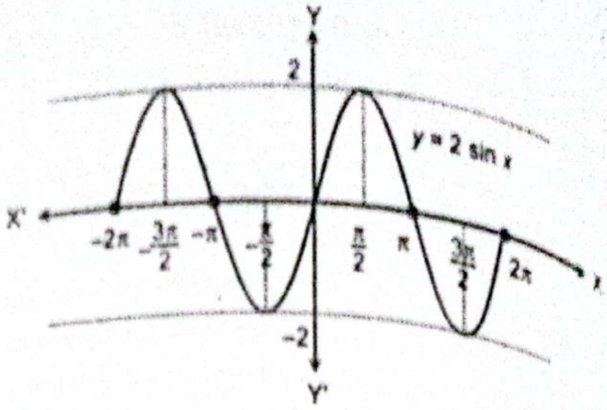

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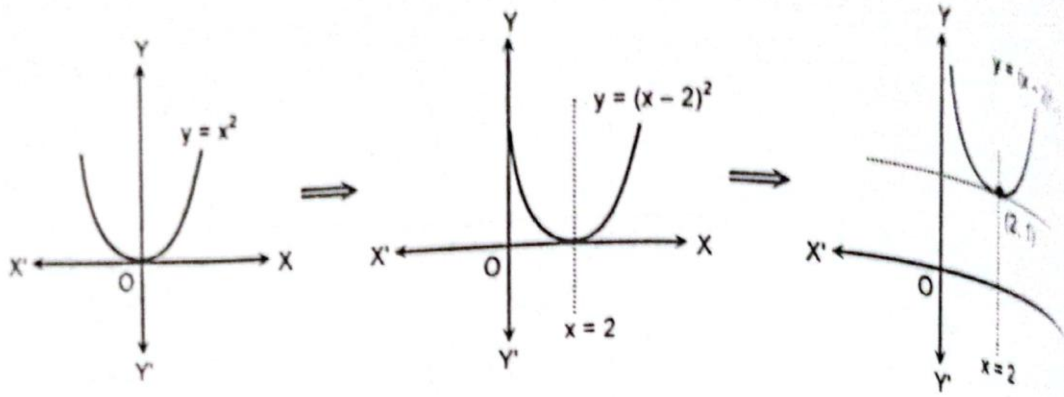
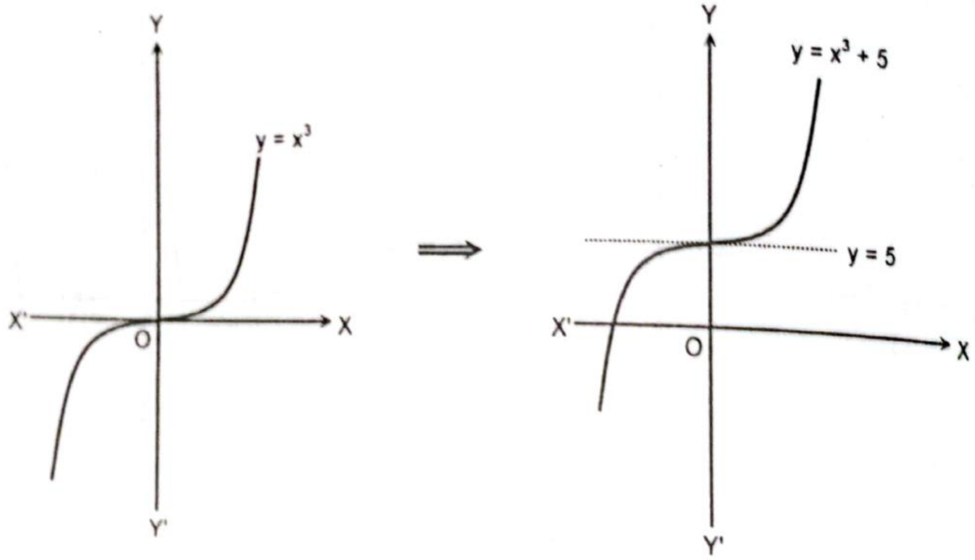


6.



3.11 Definite Integral

The Definite integral was initially developed to determine the areas of regions enclosed by different sizes and shapes without the reference to indefinite integral. Later, with the fundamental theorem of integral calculus, the definite integrals can be evaluated through integrals.

With the help of integrals we can calculate areas between the curves, the volumes and s of solids, the length of curves, etc. We define all of these as limits of Riemann sums of functions on closed intervals, that is, as integrals, and evaluate these limits with calculus the properties of definite integrals and fundamental theorem of calculus.

The Fundamental Theorem of Calculus

The fundamental theorem of calculus is the independent discovery by Leibniz and Newton. It gives the connection between differentiation and integration.

Theorem (The Fundamental Theorem of Calculus, Part 1)

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\, dt$ has a derivative at every point of $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\, dt = f(x);\ a \le x \le b.$$

Example 1. Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t\, dt$.

Solution

Here, the upper limit of integration is not x but x^2. To find $\frac{dy}{dx}$, we treat y as the composite of $y = \int_1^{u} \cos t\, dt$ and $u = x^2$

By chain rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}\left(\int_1^u \cos t\, dt\right)\cdot\frac{du}{dx}\\ &= \cos u.\frac{du}{dx}\\ &= \cos x^2\cdot 2x\\ &= 2x\cos x^2.\end{aligned}$$

Theorem (The Fundamental Theorem of Calculus, Part 2).

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Steps of finding $\int_a^b f(x)\, dx$

1. Find an antiderivative F of f.
2. Calculate $F(b) - F(a)$
3. The number obtained in step - 2 is the value of $\int_a^b f(x)\, dx$.

Example 2. Evaluate: $\int_{-2}^{0}(2x+5)\,dx$.

Solution

$$\int_{-2}^{0}(2x+5)dx$$
$$= \left[2\cdot\frac{x^2}{2}+5x\right]_{-2}^{0}$$
$$= [x^2+5x]_{-2}^{0}$$
$$= (0^2+5\times 0)-\{(-2)^2+5\times(-2)\}$$
$$= 0-(4-10)$$
$$= 0+6$$
$$= 6.$$

Evaluation of Definite Integrals

The quantity $\int_a^b f(x)\,dx$ is called the **definite integral** of $f(x)$ from a to b as we have already discussed. The definite integral $\int_a^b f(x)\,dx$ is a number and it does not depend on x. In fact, we can use any letter in place of x without changing the value of the integral. That is $\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt$. When it is very difficult or impossible to find its exact value of definite integral, we compute an approximate value for the definite integral rather than calculating its exact value. There are various methods for approximating definite integral.

While evaluating definite integrals, we also use substitution method, integration by parts, etc.

Properties of Definite Integrals

1. **Zero:** $\int_a^a f(x)\,dx = 0$
2. **Order of integration:** $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
3. **Constant multiple:** $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$ where k is a constant.
4. **Sums and Differences:** $\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. **Additivity:** $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

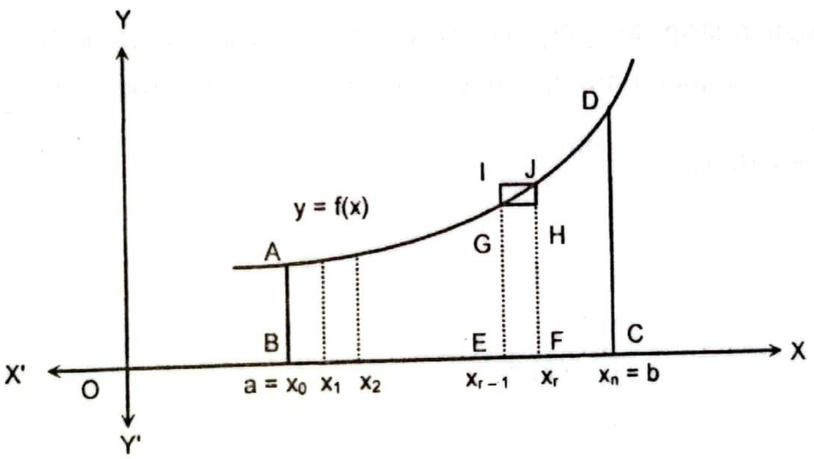
6. **Max - Min Inequality:** If max f and min f are maximum and minimum values of f on $[a, b]$, then
min $f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$

7. **Domination:** $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Also, $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$.

3.12 Definite Integral as a Limit of Sum

Let $f(x)$ be a continuous function defined in the closed interval $[a, b]$. Assume that $f(x) \ge 0$ for all x in $[a, b]$.



The integral of $f(x)$ gives the area of the region bounded by the curve $y = f(x)$. In the above figure, this area is represented by the region ABCD.

Let us divide the interval $[a, b]$ into n subintervals of equal width h given by

$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{r-1}, x_r], \ldots [x_{n-1}, x_n]$. Then

$$x_0 = a$$
$$x_1 = a + h$$
$$x_2 = a + 2h$$
$$\vdots$$
$$x_r = a + rh$$
$$\vdots$$
$$x_n = a + nh$$

and $n = \dfrac{b-a}{h}$

Here, we have choosen n and h in such a way that $n \to \infty$ and h $\to 0$.

From figure,

Area of rectangle EFGH < area of the region EFJGE < area of rectangle EFJI

Since $h \to 0$, $x_r - x_{r-1} \to 0$. We can define the following sums

$$s_n = h[f(x_0) + f(x_1) + \ldots + f(x_{n-1})]$$

$$= h \sum_{r=0}^{n-1} f(x_r)$$

$$S_n \quad h[f(x_1) + f(x_2) + \ldots + f(x_n)]$$

$$= h \sum_{r=1}^{n} f(x_r)$$

Then (i) can be written as

s_n < area of the region ABCD < S_n.

Since $n \to \infty$, the rectangular strips are very narrow and then we can assume that the limiting of s_n and S_n are equal. The common limiting value gives the area under the curve.

$$\text{Area of the region ABCD} = \lim_{n \to \infty} S_n$$

$$= \lim_{n \to \infty} s_n$$

$$= \int_a^b f(x)\, dx$$

Therefore,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a) + f(a+h) + \ldots + f(a+(n-1)h]$$

$$= (b-a) \lim_{n \to \infty} \frac{1}{n} [f(a) + f(a+h) + \ldots + f(a+(n-1)h]$$

Where, $h = \dfrac{b-a}{n}$

The above relation can also be written as

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+nh)]$$

Example 3. Using the limit of a sum, evaluate $\int_0^1 x^2\, dx$.

Solution

Here, $f(x) = x^2$

We have,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+nh)]$$

$$\therefore \int_0^1 x^2\, dx = \lim_{h \to 0} h[f(h) + f(2h) + f(3h) + \ldots + f(nh)]$$

$$= \lim_{h \to 0} h[h^2 + 2^2h^2 + 3^2h^2 + \ldots + n^2h^2]$$

$$= \lim_{h \to 0} [h \cdot h^2(1^2 + 2^2 + 3^2 + \ldots + n^2)]$$

$$= \lim_{h \to 0} h^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{h \to 0} \frac{nh(nh+h)(2nh+h)}{6}$$

$$= \frac{1(1+0)(2 \times 1 + 0)}{6} \qquad [\because nh = b - a = 1 - 0 = 1]$$

$$= \frac{1}{3}$$

Example 4. Using limit of a sum, find the area bounded by the curve $y = 4x^3$, the x-axis and the ordinates $x = 0$ and $x = 4$.

Solution

The area bounded by the curve $y = 4x^3$, the x-axis and the ordinates $x = 0$ and $x = 4$ is given by $\int_0^4 y\, dx = \int_0^4 4x^3\, dx$.

Here, $a = 0$, $b = 4$, $f(x) = 4x^3$

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+nh)]$$

$$= \lim_{h \to 0} h[f(h) + f(2h) + f(3h) + \ldots + f(nh)]$$

$$= \lim_{h \to 0} h[4h^3 + 4(2h)^3 + 4(3h)^3 + \ldots + 4(nh)^3]$$

$$= \lim_{h \to 0} h \cdot 4h^3[1^3 + 2^3 + 3^3 + \ldots + n^3]$$

$$= \lim_{h \to 0} 4h^4 \cdot \left[\frac{n(n+1)}{2}\right]^2 \quad \left[\because 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2\right]$$

$$= \lim_{h \to 0} h^4 n^2 (n+1)^2$$

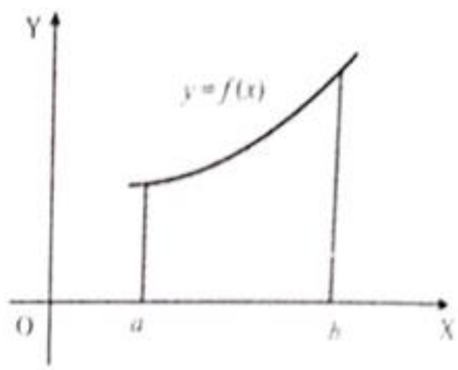
$$= \lim_{h \to 0} (nh)^2 (nh + h)^2$$

$$= 4^2 (4 + 0)^2 \qquad [\because nh = b - a = 4 - 0 = 4]$$

$$= 256$$

3.13 Area under a Curve

The definite integral was developed to determine the areas of regions enclosed by curves of different sizes and shapes without the reference to indefinite integral. Later, with the help of fundamental theorem of integral calculus, the definite integrals can be evaluated through indefinite integrals. In this section, we are going to calculate the areas using integrals. We now state the theorem that gives the existence of definite integral.



Theorem: If f is continuous in the interval $[a, b]$ then $\int_a^b f(x)\,dx$ exists.

Now we have the following formulas.

Formula 1: If $f(x)$ is continuous on $[a, b]$ then the area bounded by the curve $y = f(x)$, the x-axis and the two ordinates $x = a$, $x = b$ is equal to $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.

Example 5. Find the area bounded by the x-axis and the curve $y = 4x^3$ and the ordinates at $x = 2$ and $x = 4$.

Solution

Here, $y = 4x^3$, $x = 2$ and $x = 4$.

$\therefore$ The required area A $= \int_2^4 y\,dx$

$$= \int_2^4 4x^3\,dx$$

$$= 4\left[\frac{x^4}{4}\right]_2^4$$

$$= \frac{4}{4}\,[4^4 - 2^4]$$

$$= [256 - 16]$$

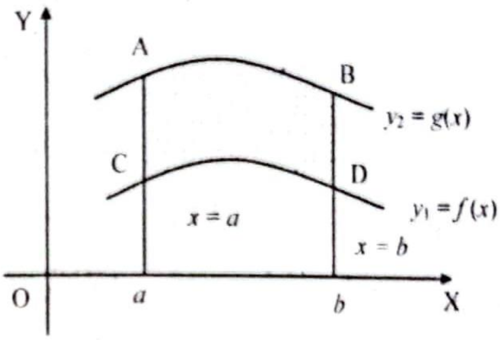
$$= 240 \text{ sq. units.}$$

Formula 2: The area bounded by the curve $x = f(y)$, the y-axis and the two abscissae $y = a$ and $y = b$ is equal to $\int_a^b f(y)\,dy$ or $\int_a^b x\,dy$.

3.14 Area between Two Curves

Here, we want to find the area between the curves $y_1 = f(x)$ and $y_2 = g(x)$, and the lines $x = a$ and $x = b$. The area is the difference between the areas under the two curves.

Thus, $A = \int_a^b y_2\,dx - \int_a^b y_1\,dx = \int_a^b (y_2 - y_1)\,dx.$



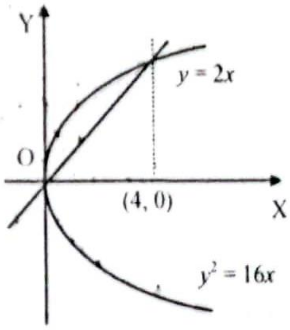
Hence, the area of the region bounded by the curves $y_1 = f(x)$, $y_2 = g(x)$, and between the lines $x = a$ and $x = b$ is given by

$$A = \int_a^b (y_2 - y_1)\,dx$$

$$= \int_a^b [g(x) - f(x)]\,dx.$$

Example 6. Find the area bounded by the curve $y^2 = 4x$ and the line $y = x$.

Solution



Here given curve and line are

$y^2 = 4x$ . . . (1)

and $y = x$ . . . (2)

Solving (1) and (2), we get

$x^2 = 4x$

$\Rightarrow\ x^2 - 4x = 0$

$\Rightarrow\ x(x - 4) = 0$

i.e., $x = 0$ and $x = 4$

i.e., $x = 0$ and $x = 4$ are the ordinates of the points at which the given curves intersect.

The required area $= \int_0^4 (y_1 - y_2)\,dx$

$$= \int_0^4 \left(\sqrt{4x} - x\right) dx$$

$$= 2\int_0^4 \sqrt{x}\,dx - \int_0^4 x\,dx$$

$$= 2.\frac{2}{3}[x^{3/2}]_0^4 - \frac{1}{2}[x^2]_0^4$$

$$= \frac{4}{3}[4^{3/2} - 0] - \frac{1}{2}[4^2 - 0^2]$$

$$= \frac{32}{3} - 8 = \frac{8}{3} \text{ sq. units.}$$

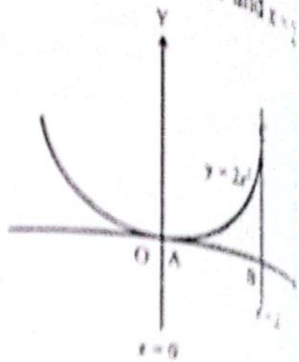
Illustrative Examples

Example 1. Find the area bounded by the curve $y = 2x^2$, the x-axis and the ordinates $x = 0$ and

Solution

The area bounded by the curve $y = 2x^2$, the x-axis and the ordinates $x = 0$ and x

$$\int_0^2 y\,dx = 2\int_0^2 x^2\,dx$$
$$= 2\left[\frac{x^3}{3}\right]_0^2$$
$$= \frac{2}{3}[2^3 - 0^3]$$
$$= \frac{16}{3}\text{ sq. units}$$



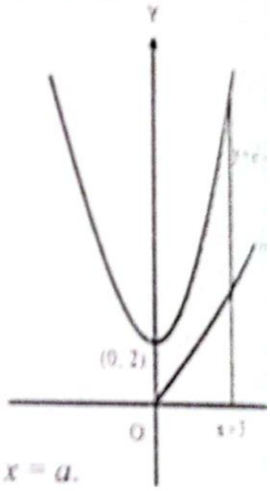
Example 2. Find the area of region bounded by the curves $y = x^2 + 2$, $y = x$, and ordinate x
$x = 3$.

Solution

Here $y = x^2 + 2$ is a parabola and $y = x$ is a line passing through the origin.

Required area = Area under the parabola – Area under the line $y = x$

$$= \int_0^3 (x^2 + 2)\,dx - \int_0^3 x\,dx$$
$$= \left[\frac{x^3}{3} + 2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3$$
$$= (9 + 6) - \left(\frac{9}{2}\right)$$
$$= \frac{21}{2}\text{ sq. units.}$$



Example 3. Find the area bounded by the curve $y^2 = 4ax$ and the line $x = a$.

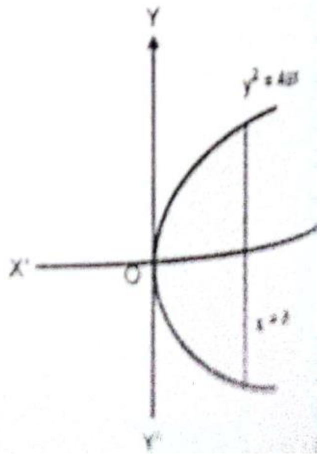
Solution

The given curve is $y^2 = 4ax$ . . . (i)

and the line is $x = a$ . . . (ii)

The curve (i) is symmetrical about x-axis. So, to find the whole area bounded by (ii), we first find the area lying in first quadrant and multiply it by 2.

Required area
$$= 2\int_0^a y\,dx$$
$$= 2\int_0^a \sqrt{4ax}\,dx$$
$$= 2\cdot 2\sqrt{a}\int_0^a x^{1/2}\,dx$$



$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= 4\sqrt{a}\cdot\frac{2}{3}[a^{3/2} - 0^{3/2}]$$

$$= \frac{8a^2}{3} \text{ sq. units}$$

Example 4. Find the area bounded by the y-axis, the curve $x^2 = 4by$ and $y = b$.

Solution

Here, the given parabola is

$x^2 = 4by$ . . . (1)

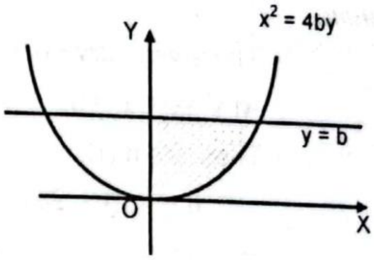
and the line is

$y = b.$ . . . (2)

The parabola passes through the origin (0, 0).

The area bounded by the parabola, the y-axis and abscissae $y = 0$, $y = b$ is given by

$$= \int_0^b x\,dy = \int_0^b \sqrt{4by}\,dy$$



$$= 2\sqrt{b}\int_0^b \sqrt{y}\,dy$$

$$= 2\sqrt{b}\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_0^b$$

$$= 2\sqrt{b}\,\frac{2}{3}\,[y^{3/2}]_0^b$$

$$= 2\sqrt{b}\,\frac{2}{3}\,b^{3/2}$$

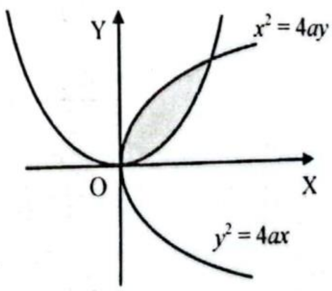
$$= \frac{4}{3}b^2 \text{ sq. units.}$$

Example 5. Find the area of the region between the curves $y^2 = 4ax$ and $x^2 = 4ay$.

Solution

Here, first curve is $y^2 = 4ax$ …(1)

and the second curve is $x^2 = 4ay$ …(2)



Solving (1) and (2), we get

$$\left(\frac{x^2}{4a}\right)^2 = 4ax \Rightarrow \frac{x^4}{16a^2} = 4ax$$

or, $x^4 - 64a^3x = 0$

or, $x(x^3 - 64a^3) = 0$

or, $x = 0, 4a$

Therefore, the required area of the bounded region between the curves (1) and (2) is given by,

$$A = \int_0^{4a} (y_1 - y_2)\, dx$$

$$= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right) dx$$

$$= 2\sqrt{a} \int_0^{4a} x^{1/2}\, dx - \frac{1}{4a} \int_0^{4a} x^2 dx$$

$$= 2\sqrt{a}\,\frac{2}{3}\,[x^{3/2}]_0^{4a} - \frac{1}{4a}\cdot\frac{1}{3}\,[x^3]_0^{4a}$$

$$= \frac{16}{3} a^2 (2 - 1)$$

$$= \frac{16}{3} a^2 \text{ sq. units.}$$

Example 6. Find the area bounded by y-axis, the curve $x^2 = 4a(y - 2a)$ and $y = 6a$.

Solution

The given curve is $x^2 = 4a(y - 2a)$ ...(i)

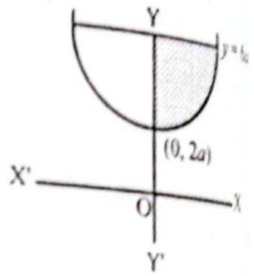
At y-axis, $x = 0$.

Then, from (i),

$$0 = 4a(y - 2a)$$

$\Rightarrow$ $y = 2a$

Also, given line is $y = 6a$ ...(ii)



$\therefore$ Hence, the required area bounded by y-axis, the curve (i) and (ii) is $\int_{2a}^{6a} x\, dy$

$$= \int_{2a}^{6a} \sqrt{4a(y - 2a)}\, dy$$

$$= 2\sqrt{a} \int_{2a}^{6a} (y-2a)^{1/2}\, dy$$

$$= 2\sqrt{a} \left[\frac{(y - 2a)^{3/2}}{3/2 \cdot 1}\right]_{2a}^{6a}$$

$$= \frac{4\sqrt{a}}{3}\,[(y - 2a)^{3/2}]_{2a}^{6a}$$

$$= \frac{4\sqrt{a}}{3}\,[(6a - 2a)^{3/2} - (2a - 2a)^{3/2}]$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2}$$

$$= \frac{4a^{1/2}}{3} \cdot 8a^{3/2}$$

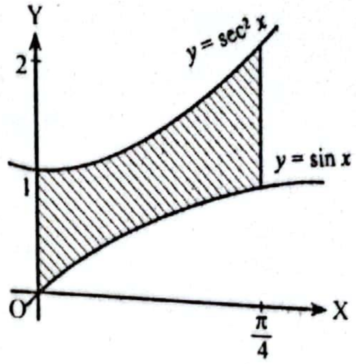
$$= \frac{32a^2}{3} \text{ sq. units}$$

Example 7. Find the area between two curves $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Solution

Here the upper curve is $f(x) = \sec^2 x$ and lower curve is $g(x) = \sin x$.

The limits of integration are $a = 0$; $b = \frac{\pi}{4}$



$$\text{Area} = \int_0^{\frac{\pi}{4}} [f(x) - g(x)]\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x - \sin x)\, dx$$

$$= [\tan x + \cos x]_0^{\frac{\pi}{4}}$$

$$= \left[\tan\frac{\pi}{4} + \cos\frac{\pi}{4}\right] - [\tan 0 + \cos 0]$$

$$= \left(1 + \frac{1}{\sqrt{2}}\right) - (0 + 1)$$

$$= \frac{1}{\sqrt{2}} \text{ sq. unit.}$$

Example 8. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

We first sketch the given curves. For limits of integration, we solve given curves, that is,

$$2 - x^2 = -x$$

or, $x^2 - x - 2 = 0$

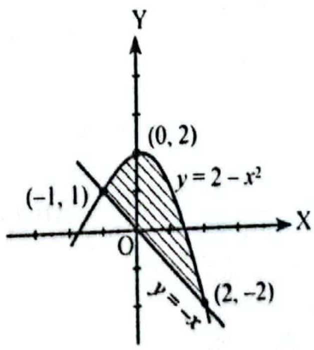
or, $(x + 1)(x - 2) = 0$

$\therefore$ $x = -1, 2.$

The limits of integration: are $a = -1$ and $b = 2$

Also the upper curve is $f(x) = 2 - x^2$ and lower curve is $g(x) = -x$

$$\text{Area (A)} = \int_a^b [f(x) - g(x)]dx$$



$$= \int_{-1}^{2} [(2 - x^2) - (-x)]dx$$

$$= \int_{-1}^{2} (2 - x^2 + x)\, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[2 \times 2 + \frac{2^2}{2} - \frac{2^3}{3}\right] - \left[2 \times (-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right]$$

$$= \left(4+2-\frac{8}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right)$$

$$= 6-\frac{8}{3}+2-\frac{1}{2}-\frac{1}{3}$$

$$= 8-\frac{9}{3}-\frac{1}{2}$$

$$= 8-3-\frac{1}{2}$$

$$= 5-\frac{1}{2}$$

$$= \frac{10-1}{1}$$

$$= \frac{9}{2} \text{ sq. units.}$$

Example 9. Find the area of the circle $x^2 + y^2 = 36$, using method of integration.

Solution

Equation of given circle is $x^2 + y^2 = 36$

$\therefore \quad y = \sqrt{36 - x^2} \qquad \ldots \text{(i)}$

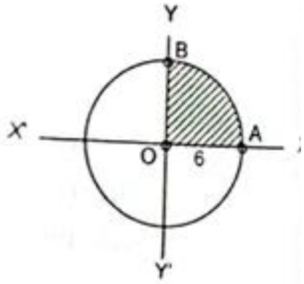
The centre of circle (i) is at (0, 0) and radius OA = 6 units

Since, circle (i) is symmetrical about both the axes, *x-axis* and *y*-axis divide it into fo equal parts.

OAB is the portion of the circle lying in the first quadrant, which is bounded by t axes and the curve (i).

Now,

$$\text{Area of OAB} = \int_0^6 y\,dx = \int_0^6 \sqrt{36-x^2}\,dx$$



Put $x = 6 \sin\theta$, then

$dx = 6\cos\theta\, d\theta$

When $x = 0$, then

$0 = 6\sin\theta$

or, $\sin\theta = 0 = \sin 0 \Rightarrow \theta = 0$

When $x = 6$, then

$6 = 6\sin\theta \Rightarrow \sin\theta = 1$

or, $\sin\theta = \sin \pi/2 \Rightarrow \theta = {}^{\pi}/_{2}$

Now,

$$\text{Area of OAB} = \int_0^{\pi/2} \sqrt{36 - 36\sin^2\theta} \times 6\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} 6\sqrt{36\cos^2\theta} \times \cos\theta\, d\theta$$

$$= 36\int_0^{\pi/2} \cos\theta \times \cos\theta \, d\theta$$

$$= \frac{36}{2}\int_0^{\pi/2} 2\cos^2\theta \, d\theta = 18\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= 18\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 18\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - (0 + 0)\right]$$

$$= 18\left(\frac{\pi}{2} + \frac{0}{2}\right) = 9\pi$$

If A be the area of circle, then

A = 4 × Area of OAB

= 4 × 9π = 36 π sq. units

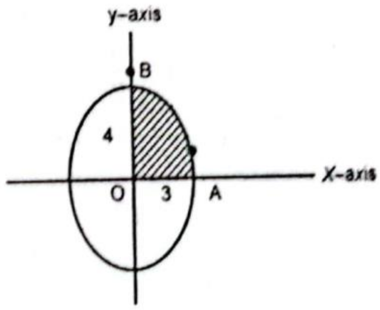
Hence, the area of given circle = 36 π sq. units.

Example 10. Using integration, find the area under the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Solution

The given curve is

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \quad \ldots (i)$$



or, $\frac{y^2}{16} = 1 - \frac{x^2}{9}$

or, $\frac{y^2}{16} = \frac{9 - x^2}{9}$

or, $y^2 = \frac{16}{9}(9 - x^2)$

$$\therefore \quad y = \frac{4}{3}\sqrt{9 - x^2} \quad \ldots (ii)$$

The given curve is symmetrical about axes. So the coordinate axes divide the area enclosed by the given curve in four equal parts as shown in the figure.

So, area of ellipse = 4 (Area of the portion OAB shaded on the figure lying in 1st quadrant).

Here, OA = 3, OB = 4

$\therefore$ Area of the portion OAB $= \int_0^3 y\, dx$

$= \int_0^3 \frac{4}{3}\sqrt{9 - x^2}\, dx$ [∵ From eq^n (ii)]

$= \frac{4}{3}\int_0^3 \sqrt{9 - x^2}\, dx \quad \ldots (iii)$

Put $x = 3 \sin \theta$

Differentiating both sides with respect to θ, we get,

$$\therefore \quad \frac{dx}{d\theta} = 3 \cos \theta$$

or, $dx = 3 \cos \theta \, d\theta$

And,

$$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 \theta}$$
$$= \sqrt{9 \cos^2 \theta}$$
$$= 3 \cos \theta$$

When $x = 0$,

$3 \sin \theta = 0$

or, $\sin \theta = 0$

or, $\sin \theta = \sin 0^c$

$\therefore \quad \theta = 0$

When $x = 3$,

$3 \sin \theta = 3$

or, $\sin \theta = 1$

or, $\sin \theta = \sin \frac{\pi}{2}^c$

$\therefore \quad \theta = \frac{\pi}{2}^c$

So form (3) Area of portion

$$OAB = \frac{4}{3} \int_0^{\pi/2} 3 \cos \theta \times 3 \cos \theta \, d\theta$$

$$= 12 \int_0^{\pi/2} \cos^2 \theta \, d\theta = 12 \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= 6 \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta$$

$$= 6 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= 6 \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right]$$

$$= 6 \left[\left(\frac{\pi}{2} + \frac{0}{2} \right) - 0 \right]$$

$$= 6 \times \frac{\pi}{2}$$

$= 3\pi$ sq. units

Hence, the area of ellipse $= 4 \times 3\pi = 12\pi$ sq. units

UNIT 4

Differential Equations

Course Contents

- Ordinary Differential Equations (ODEs)
  - Definitions, order and degree of differential equation
  - Differential equation of First order and First degree
  - Variable separation and variable change methods
  - Homogeneous and linear differential equation of First order
  - Exact differential equation, condition of exactness
  - Simple applications of First order differential equations
- Partial Differential Equations (PDEs)
  - Basic concepts, definition and formation
  - General solution of linear PDEs of first order (Pp + Qq = R form)

4.1 Introduction

There are various physical problems in our surroundings. We also call them engineering problems. To solve such type of problems, we have to formulate the problems in terms of mathematical expressions containing variables, functions, equations etc. This type of expression is called the mathematical model of the problem. The process of setting up a model, solving it and interpreting the result is called mathematical modeling.

Many physical problems are based on velocity and acceleration (i.e. on derivatives). Thus, a model is an equation containing derivatives of an unknown function. This type of equation is called a differential equation.

In this unit, we begin our study from differential equation. Our study will be focused to solve the ordinary differential equations of first order and first degree by some standard methods. We will discuss about simple applications of first order differential equation. Finally we will discuss about first order partial differential equation.

4.2 Differential Equation

An equation that involves independent and dependent variables together with the derivatives of dependent variable with respect to independent variable is called a differential equation.

If the function, whose derivatives are used in the equation, is of a single independent variable then it is called an **ordinary differential equation (ODE).**

Examples

i. $\frac{dy}{dx} + y = x$

ii. $y'' + 6y' + 5y = 6x$ etc.

When an equation contains the derivative of a function with respect to two or more than two independent variables then it is called a **partial differential equation (PDE).**

For examples

i. $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial t} = \lambda f$

ii. $\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$ etc.

4.3 Order and Degree of Differential Equation

Order of ODE

The order of the highest derivative of the function present in the differential equation is called order of the ODE.

For examples

i. $\frac{dy}{dx} = x^5$ is an ODE of order 1.

ii. $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 4 = 0$ is an ODE of order 2.

Degree of ODE

The degree of the highest derivative of the function present in the differential equation after making free from fractions and the radicals is called the degree of an ODE.

For examples

i. $\frac{dy}{dx} + \frac{y}{x} = e^x$ is an ODE of degree 1.

ii. $(y'')^5 + 2y' = e^x$ is an ODE of degree 5.

iii. $\left(\frac{dy}{dx}\right)^2 - 5\left(\frac{d^2y}{dx^2}\right)^3 + 4 = 0$ is an ODE of degree 3.

Formation of Differential Equation

A differential equation can be formed by eliminating the arbitrary constants, which can be done by successive differentiation.

Example 1. Form a differential equation from $y = a \cos x + b \sin x$.

Solution

Differentiating both sides w.r. to x, we get,

$$\frac{dy}{dx} = -a \sin x + b \cos x$$

Again, differentiating both sides w.r. to x, we get,

$$\frac{d^2y}{dx^2} = -a \cos x - b \sin x$$

$$= -(a \cos x + b \sin x)$$

$$= -y$$

$\therefore \quad \frac{d^2y}{dx^2} + y = 0$ which is the required differential equation.

Solution of a Differential Equation

Any relation between the variables which is free from the derivatives and satisfies the given differential equation identically is called the solution of the differential equation.

For example: $y = a \cos x + b \sin x$ is solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

General and Particular Solution

If a solution consists the number of arbitrary constants equal to the order of the differential equation then the solution is said to be **general solution.**

The solution obtained from the general solution by giving particular values to one or more of the arbitrary constants is called a **particular solution.**

For example: $y = cx$ is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$.

If we set the condition $y(1) = 1$ then the solution becomes $y = x$ which is the particular solution of $\frac{dy}{dx} = \frac{y}{x}$.

A particular solution is obtained from a general solution by an initial condition $y(x_0) = y_0$, with given values x_0 and y_0. It is used to determine a value of arbitrary constant. An ODE together with an initial condition is called **initial value problem** (IVP).

If the differential equation is given together with some additional conditions on the boundary of regions then it is called the **boundary value problem** (BVP).

4.4 Differential Equation of First Order and First Degree

A differential equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is called a differential equation of first order and first degree. The general solution of this type of equation contains only one arbitrary constants.

NOTE We can not solve all differential equations.

In this unit, we will discuss the following forms of ordinary differential equations of first order and first degree and their simple applications.

1. Variable separation and variable change form
2. Homogeneous form
3. Exact form
4. Linear form

4.5 Differential Equations with Separable Variables

If the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad \ldots(i)$$

can be written in the form

$$f(x)\,dx + f(y)\,dy = 0 \qquad \ldots(ii)$$

then the equation is in the **variable separation form**. To obtain the solution, we integrate both sides of (ii)

$$\int f(x)\,dx + \int f(y)\,dy = c \qquad \ldots(iii)$$

where c is an arbitrary constant of integration.

If we give particular value to the constant c then the solution is called particular solution.

4.6 Reduction to Variable Separation Form by Change of Variables

In some cases, the nonseparable ordinary differential equations can be transformed into variable separation form by making the suitable substitution. For example, let us consider the ODE in the form

$$\frac{dy}{dx} = f(ax + by + c) \qquad \ldots(i)$$

Put $v = ax + by + c$

Then,

$$\frac{dv}{dx} = a + b\frac{dy}{dx}$$

or, $$\frac{dv}{dx} - a = b\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

Then equation (i) can be written as

$$\frac{1}{b}\left(\frac{dv}{dx}-a\right)=f(v)$$

or, $\dfrac{dv}{dx}-a=bf(v)$

or, $\dfrac{dv}{dx}=a+bf(v)$

or, $\dfrac{dv}{a+bf(v)}=dx$

which is the variable separation form and the solution of this equation can be obtained by integrating on both sides.

Example 2. Solve: $\dfrac{dy}{dx}=\sin(x+y)$.

Solution

Given equation is

$$\frac{dy}{dx}=\sin(x+y) \qquad \ldots \text{(i)}$$

Put $v=x+y$. Then

$$\frac{dv}{dx}=1+\frac{dy}{dx}$$

$\therefore \quad \dfrac{dv}{dx}-1=\dfrac{dy}{dx}$

Now, equation (i) can be written as

$$\frac{dv}{dx}-1=\sin v$$

or, $\dfrac{dv}{dx}=1+\sin v$

or, $\dfrac{dv}{1+\sin v}=dx$

or, $\dfrac{1-\sin v}{(1+\sin v)(1-\sin v)}dv=dx$

or, $\left(\dfrac{1-\sin v}{\cos^2 v}\right)dv=dx$

or, $(\sec^2 v-\tan v\sec v)\,dv=dx$

Integrating, we get

$$\tan v-\sec v=x+c$$

$\therefore \quad \tan(x+y)-\sec(x+y)=x+c \quad [\because v=x+y]$

Illustrative Examples

Example 1. Verify that $y = e^{-x}$ is a solution of $\frac{d^2y}{dx^2} - y = 0$.

Solution

$$y = e^{-x}$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-x}) = -e^{-x}$$

Again, differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = -(-e^{-x}) = e^{-x}$$

$$\therefore \quad \frac{d^2y}{dx^2} - y = e^{-x} - e^{-x} = 0$$

Example 2. Solve: $\frac{dy}{dx} = -\frac{y}{x}$.

Solution

Given,

$$\frac{dy}{dx} = -\frac{y}{x}$$

or, $\frac{dy}{y} = -\frac{dx}{x}$

or, $\frac{dx}{x} + \frac{dy}{y} = 0$

Integrating both sides, we have,

$$\log x + \log y = A$$

or, $\log(xy) = \log c$

$\therefore \quad xy = c.$

Example 3. Solve: $\frac{dy}{dx} = \frac{x}{y}$.

Solution

The equation can be written as,

$$y\,dy = x\,dx$$

or, $x\,dx - y\,dy = 0$

Integrating both sides, we have,

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c}{2}$$

$\therefore \quad x^2 - y^2 = c.$

Example 4. Solve: $y' = 1 + y^2$.

Solution

$$\frac{dy}{dx} = 1 + y^2$$

or, $$\frac{dy}{1+y^2} = dx$$

Integrating both sides, we have,

or, $$\int \frac{dy}{1+y^2} = \int dx + c$$

or, $\tan^{-1} y = x + c$

or, $y = \tan(x + c)$

$\therefore$ $y = \tan(x + c)$.

Example 5. Solve: $\frac{dy}{dx} = \frac{xy+y}{xy+x}$.

Solution

$$\frac{dy}{dx} = \frac{xy+y}{xy+x}$$

or, $$\frac{dy}{dx} = \frac{y(x+1)}{x(y+1)}$$

or, $x(y+1)\,dy = y(x+1)\,dx$

or, $$\left(\frac{y+1}{y}\right)dy = \left(\frac{x+1}{x}\right)dx$$

or, $$\left(1+\frac{1}{y}\right)dy = \left(1+\frac{1}{x}\right)dx$$

Integrating both sides, we have,

$y + \log y = x + \log x + c$

or, $\log y - \log x + y - x = c$

$\therefore$ $\log\frac{y}{x} + y - x = c$.

Example 6. Solve: $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$.

Solution

Given,

$$\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$$

or, $$\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

Integrating, we have,

$\sin^{-1} x + \sin^{-1} y = \sin^{-1} c$

or, $\sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = \sin^{-1} c$

$\therefore$ $x\sqrt{1-y^2} + y\sqrt{1-x^2} = c$.

Example 7. Solve: $\frac{dy}{dx} = e^{x-y} + x^3 \cdot e^{-y}$.

Solution

Here,

$$\frac{dy}{dx} = e^{x-y} + x^3 \cdot e^{-y}$$

or, $\frac{dy}{dx} = e^x \cdot e^{-y} + x^3 e^{-y}$

or, $\frac{dy}{dx} = e^{-y}(e^x + x^3)$

or, $\frac{dy}{e^{-y}} = (e^x + x^3)\,dx$

or, $e^y\,dy = (e^x + x^3)\,dx$

Integrating, we get,

$$\int e^y\,dy = \int (e^x + x^3)\,dx + c$$

$\therefore \quad e^y = e^x + \frac{x^4}{4} + c.$

Example 8. Solve: $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0.$

Solution

Integrating both sides, we get,

$$\int \frac{dx}{1+x^2} + \int \frac{dy}{1+y^2} = \tan^{-1} c$$

or, $\tan^{-1} x + \tan^{-1} y = \tan^{-1} c$

or, $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} c$

or, $\frac{x+y}{1-xy} = c$

$\therefore \quad x + y = c\,(1 - xy).$

Example 9. Solve: $x\,dy - y\,dx = 0$, $y(0) = 1$.

Solution

$$x\,dy - y\,dx = 0$$

or, $x\,dy = y\,dx$

or, $\frac{dy}{y} = \frac{dx}{x}$

Integrating both sides, we have,

$$\log x = \log y + \log c$$

or, $\log x = \log cy$

$\therefore \quad x = cy \qquad \ldots(i)$

By given, $y(0) = 1$

i.e. when $x = 0, y = 1$

Then, from (i), we have,

$$0 = c \cdot 1$$

or, $c = 0$

Substituting the value of c in (i), we get, $x = y$, which is the required solution.

Example 10. Solve: $\frac{dy}{dx} = \frac{1}{(x+y)^2}$.

Solution Given equation is

$$\frac{dy}{dx} = \frac{1}{(x+y)^2} \quad \dots (i)$$

Put $v = x + y$. Then

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now, equation (i) can be written as

$$\frac{dv}{dx} - 1 = \frac{1}{v^2}$$

or, $\frac{dv}{dx} = \frac{1}{v^2} + 1$

or, $\frac{dv}{dx} = \frac{1+v^2}{v^2}$

or, $\frac{v^2}{1+v^2} dv = dx$

or, $\frac{1+v^2-1}{1+v^2} dv = dx$

or, $\left(1 - \frac{1}{1+v^2}\right) dv = dx$

Integrating, we get

$$v - \frac{1}{1}\tan^{-1}(v) = x + c$$

or, $x + y - \tan^{-1}(x+y) = x + c \quad [\because v = x + y]$

or, $y - c = \tan^{-1}(x+y)$

$\because \quad x + y = \tan(y - c)$

Exercise 4.1

1. Find the order and degree of the following differential equations.

   a. $\frac{dy}{dx} = 2$

   b. $\frac{d^2y}{dx^2} = \sin x$

   c. $x\frac{d^3y}{dx^3} + y + \left(\frac{dy}{dx}\right)^4 = 0$

   d. $(y'')^3 + 4y' = e^x$

   e. $\frac{d^2y}{dx^2} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3}$

2. Slve the following differential equations using separation of variables.

   a. $x\,dx - y\,dy = 0$

   b. $\frac{dy}{dx} = \frac{y}{x}$

   c. $\frac{dy}{dx} = \frac{x^3 + 1}{y^3 + 1}$

   d. $(1 + x^2)y' = 1$

   e. $y\,dx - x\,dy = xy\,dx$

   f. $(xy^2 + x)dx + (yx^2 + y)dy$

   g. $\frac{dy}{dx} = \frac{e^x + 1}{y}$

   h. $\frac{dy}{dx} = e^{x-y} + e^{-y}$

   i. $e^{x-y}dx + e^{y-x}dy = 0$

   j. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

   k. $\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2x} = 0$

3. Solve the initial value problems

   a. $\frac{dy}{dx} = 2x - 7, y(2) = 0$

   b. $\frac{dy}{dx} = 10 - x, y(0) = -1$

   c. $\frac{dy}{dx} = 9x^2 - 4x + 5, y(-1)$

4. Solve (Change of variables)

   a. $\frac{dy}{dx} = \frac{1}{x + y + 5}$

   b. $\frac{dy}{dx} = \cos(x + y)$

   c. $1 - \frac{dy}{dx} = e^{x-y}$

   d. $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

Answers

1. a. order = 1, degree = 1
   b. order = 2, degree = 1
   c. order = 3, degree = 1
   d. order = 2, degree = 3
   e. order = 2, degree = 2

2. a. $x^2 - y^2 = c$
   b. $x = cy$
   c. $\frac{x^4}{4} + x = \frac{y^4}{4} + y + c$
   d. $y = \tan^{-1}x + c$
   e. $\log\left(\frac{x}{y}\right) = x + c$
   f. $(x^2 + 1)(y^2 + 1) = c$
   g. $y^2 = 2e^x + 2x + c$
   h. $e^y = e^x + x + c$
   i. $e^{2x} + e^{2y} = c$
   j. $\tan x \tan y = c$
   k. $\cot x = \tan y + c$

3. a. $y = x^2 - 7x + 10$
   b. $y = 10x - \frac{x^2}{2} - 1$
   c. $y = 3x^3 - 2x^2 + 5x$

4. a. $y = \ln(x + y + 6) + c$
   b. $\tan\left(\frac{x + y}{2}\right) = x + c$
   c. $(x + c)e^{x-y} + 1 = 0$
   d. $\ln\left[1 + \tan\left(\frac{x + y}{2}\right)\right] = x + c$

4.7 Homogeneous Differential Equations

A differential equation of the form $\frac{dy}{dx} = \frac{\phi(x, y)}{\Psi(x, y)}$, where $\phi(x, y)$ and $\Psi(x, y)$ are the homogeneous functions of x and y of same degree, is called a homogeneous differential equation.

In order words, a differential equation of the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$ is called a homogeneous differential equation.

For example, the differential equation $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ is homogeneous differential equation because it is in the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$.

But the equation $\frac{dy}{dx} = \frac{y+1}{x}$ is not homogeneous differential equation because it can not be written in the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$ as R.H.S is of the form $\frac{y}{x} + \frac{1}{x}$.

A very easy way to check whether a first degree and first order differential equation is homogeneous or not, put $y = vx$ and see whether or not all x's cancel out or not. If all x's cancel out then it is homogeneous, otherwise it is not homogenous.

Procedures for Solving Homogeneous Equation

1. Put $y = vx$ in the equation. Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$
2. Substitute these values in the given equation $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$, we get

$$v + x\frac{dv}{dx} = \phi(v)$$

$$\Rightarrow \frac{dv}{\phi(v) - v} = \frac{dx}{x}$$

which is variable separated form and we solve it by integrating both sides as in previous section.

Illustrative Examples

Example 1. Solve: $\frac{dy}{dx} = \frac{2x + y}{x}$.

Solution

The equation can be written as $\frac{dy}{dx} = 2 + \frac{y}{x}$ which is of the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$.

So, it is a homogeneous differential equation.

Put, $y = vx$.

Differentiating both sides, we get,

$$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$$

$$= v + x\frac{dv}{dx}$$

Putting these values in given equation, we get,

$$v + x\frac{dv}{dx} = \frac{2x + vx}{x}$$

or, $v + x\dfrac{dv}{dx} = 2 + v$

or, $x\dfrac{dv}{dx} = 2$

or, $dv = 2\dfrac{dx}{x}$ (variables separated from)

Integrating both sides, we have,

$$v = 2\log x + \log c$$

or, $v = \log x^2 + \log c$

or, $v = \log cx^2$

or, $\dfrac{y}{x} = \log cx^2$

$\therefore$ $y = x\log cx^2$.

Example 2. Solve: $xy\dfrac{dy}{dx} = x^2 + y^2$.

Solution

Given differential equation is

$$xy\frac{dy}{dx} = x^2 + y^2$$

or, $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$ ...(i)

This is homogeneous differential equation.

Put $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then, from (i)

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x.\,vx}$$

or, $v + x\dfrac{dv}{dx} = \dfrac{1 + v^2}{v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 + v^2}{v} - v$

or, $x\frac{dv}{dx} = \frac{1 + v^2 - v^2}{v}$

or, $x\frac{dv}{dx} = \frac{1}{v}$

or, $v dv = \frac{dx}{x}$

Integrating, we have,

$$\frac{v^2}{2} = \log x + c$$

or, $\left(\frac{y}{x}\right)^2 = 2(\log x + c)$

$\therefore$ $y^2 = 2x^2(\log x + c)$.

Example 3. Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$.

Solution

Here, $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ ...(i)

The given equation is of the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$.

So, it is homogeneous differential equation.

Put $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now, equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{vx}{x} + \tan\frac{vx}{x}$$

or, $x\frac{dv}{dx} = \tan v$

or $\frac{dv}{\tan v} = \frac{dx}{x}$

or, $\frac{\cos v}{\sin v} dv = \frac{dx}{x}$

Integrating, we have,

$$\log \sin v = \log x + \log c$$

or, $\log \sin v = \log cx$

or, $\sin v = cx$

$\therefore$ $\sin\left(\frac{y}{x}\right) = cx$.

Example 4. Solve $x^2 y\,dx = (x^3 + y^3)\,dy$.

Solution

Here, $x^2 y\,dx = (x^3 + y^3)\,dy$

or, $\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$

This is homogeneous differential equation.

Put $y = vx$ then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Hence equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3 + v^3 x^3}$$

or, $v + x\frac{dv}{dx} = \frac{v}{1 + v^3}$

or, $x\frac{dv}{dx} = \frac{v}{1 + v^3} - v$

or, $x\frac{dv}{dx} = \frac{v - v - v^4}{1 + v^3}$

or, $x\frac{dv}{dx} = \frac{-v^4}{1 + v^3}$

or, $\frac{1 + v^3}{v^4}\,dv = -\frac{dx}{x}$

or, $\left(v^{-4} + \frac{1}{v}\right)dv = -\frac{dx}{x}$

Integrating, we get,

$$\frac{v^{-3}}{-3} + \log v = -\log x + \log c$$

or, $\frac{-1}{3} \cdot \frac{x^3}{y^3} = -\left(\log\frac{y}{x} + \log x - \log c\right)$

or, $\frac{x^3}{3y^3} = \log\left(\frac{y}{c}\right)$

$\therefore \quad x^3 = 3y^3 \log\left(\frac{y}{c}\right).$

Example 5. Solve: $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$.

Solution

The given equation is

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$$

Put $y = vx$.

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, the given equation is

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2}$$

or, $v + x\frac{dv}{dx} = \frac{1 + v^2}{2}$

or, $x\frac{dv}{dx} = \frac{1 + v^2}{2} - v$

or, $x\frac{dv}{dx} = \frac{v^2 - 2v + 1}{2}$

or, $x\frac{dv}{dx} = \frac{(v-1)^2}{2}$

or, $\frac{2dv}{(v-1)^2} = \frac{dx}{x}$

Integrating, we have,

$$\frac{-2}{v-1} = \log x + \log c$$

or, $\frac{-2}{\frac{y}{x} - 1} = \log cx$

or, $\frac{2x}{x-y} = \log cx$

$\therefore$ $2x = (x - y)\log cx.$

Example 6. Solve: $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}.$

Solution

Given, $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

Put $y = vx.$

Then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now, given equation can be written as

$$2\left(v + x\frac{dv}{dx}\right) = \frac{vx}{x} + \frac{v^2x^2}{x^2}$$

or, $2v + 2x\frac{dv}{dx} = v + v^2$

or, $2x\frac{dv}{dx} = v^2 - v$

or, $\dfrac{2}{v(v-1)}\,dv = \dfrac{dx}{x}$

or, $2\left(\dfrac{1}{v-1} - \dfrac{1}{v}\right) dv = \dfrac{dx}{x}$

Integrating, we get,

$$2\left[\int \frac{1}{v-1}\,dv - \int \frac{1}{v}\,dv\right] = \int \frac{dx}{x} + \log c$$

or, $2\{\log(v-1) - \log v\} = \log x + \log c$

or, $2\log\left(\dfrac{v-1}{v}\right) = \log cx$

or, $\log\left(\dfrac{v-1}{v}\right)^2 = \log cx$

or, $\left(\dfrac{v-1}{v}\right)^2 = cx$

or, $\left(\dfrac{\frac{y}{x} - 1}{\frac{y}{x}}\right)^2 = cx$

or, $\left(\dfrac{y-x}{y}\right)^2 = cx$

$\therefore\ (y-x)^2 = cxy^2.$

Exercise 4.2

Solve the following differential equations:

1. $\dfrac{dy}{dx} = \dfrac{x+y}{x}$
2. $\dfrac{dy}{dx} = \dfrac{2x+y}{x}$
3. $\dfrac{dy}{dx} = \dfrac{2y-x}{x}$
4. $\dfrac{dy}{dx} = \dfrac{xy}{x^2+y^2}$
5. $2xy\dfrac{dy}{dx} = x^2 + y^2$
6. $xy\dfrac{dy}{dx} = x^2 - y^2$
7. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$
8. $\dfrac{dy}{dx} = \dfrac{y}{x} - \sin^2\dfrac{y}{x}$

Answers

1. $y = x\log cx$
2. $y = x\log cx^2$
3. $y - x = cx^2$
4. $x^2 = 2y^2\log cy$
5. $y^2 - x^2 = cx$
6. $x^2(x^2 - 2y^2) + c = 0$
7. $\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{1}{2}\log(x^2+y^2) + c$
8. $\cot\left(\dfrac{y}{x}\right) = \log x + c$

4.8 Exact Differential Equations

A differential equation $M(x, y)dx + N(x, y)dy = 0$ is said to be exact if there exists a function $f(x, y)$ such that $M(x, y)dx + N(x, y)dy = df(x, y)$. That is, the given differential equation is exact if $M(x, y)dx + N(x, y)dy$ is exact or perfect differential.

The differential equation $Mdx + Ndy = 0$ will be exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ where $\frac{\partial}{\partial x}$ denotes the partial derivative.

NOTE Every differential equation $M(x, y)dx + N(x, y)dy = 0$ is not exact.

For example: $y\,dx + x\,dy = 0$ ... (i) is an exact differential equation since it can be written as $y\,dx + x\,dy = d(xy)$.

So, equation (i) can be written as

$$d(xy) = 0$$

Integrating, we have,

$$xy = c.$$

Sometimes, a differential equation which is not exact can be made exact by multiplying both sides by some suitable function of x and y. This suitable function of x and y is called an **integrating factor** of the differential equation.

For example: The differential equation $x\,dy - y\,dx = 0$ is not exact.

If we multiply both sides by $\frac{1}{y^2}$ then it becomes exact.

That is $\frac{x\,dy - y\,dx}{y^2} = d\left(\frac{x}{y}\right)$

So, given equation becomes

$$d\left(\frac{x}{y}\right) = 0$$

Integrating, we have,

$$\frac{x}{y} = c$$

$$\therefore \quad x = cy.$$

Here, $\frac{1}{y^2}$ is the integrating factor (I.F.)

NOTE A differential equation can have more than one integrating factor (I.F.)

Solution by Inspection

The following types of differentials are very important and are frequently used in solving equations.

1. $x\,dy + y\,dx \quad = d(xy)$
2. $\frac{x\,dy - y\,dx}{x^2} \quad = d\left(\frac{y}{x}\right)$

3. $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

4. $x\,dx + y\,dy = d\left(\dfrac{x^2}{2}\right) + d\left(\dfrac{y^2}{2}\right) = d\left(\dfrac{x^2+y^2}{2}\right)$

5. $\dfrac{x\,dy - y\,dx}{xy} = \dfrac{dy}{y} - \dfrac{dx}{x}$

$= d(\log y) - d(\log x)$

$= d(\log y - \log x)$

$= d\left(\log \dfrac{x}{y}\right)$

6. $\dfrac{2xy\,dx - x^2\,dy}{y^2} = d\left(\dfrac{x^2}{y}\right)$

7. $\dfrac{2xy\,dy - y^2\,dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

8. $\dfrac{y\,dx - x\,dy}{x^2+y^2} = \dfrac{\dfrac{y\,dx - x\,dy}{x^2}}{1+\left(\dfrac{x}{y}\right)^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

Illustrative Examples

Example 1. Solve: $x\,dx - y\,dy = 0$

Solution

$$x\,dx - y\,dy = 0$$

or, $d\left(\dfrac{x^2}{2}\right) - d\left(\dfrac{y^2}{2}\right) = 0$

or, $d\left(\dfrac{x^2-y^2}{2}\right) = 0$

Integrating, we get,

$$\frac{x^2-y^2}{2} = \frac{c}{2}$$

$\therefore\ x^2 - y^2 = c.$

Example 2. Solve: $x\,dy + (x+y)\,dx = 0$.

Solution

$$x\,dy + (x+y)\,dx = 0$$

or, $x\,dy + x\,dx + y\,dx = 0$

or, $(x\,dy + y\,dx) + x\,dx = 0$

or, $d(xy) + d\left(\dfrac{x^2}{2}\right) = 0$

or, $d\left(xy + \dfrac{x^2}{2}\right) = 0$

Integrating, we get,

$$xy + \frac{x^2}{2} = c.$$

Example 3. $(x + 2y - 3)\, dy - (2x - y + 1)\, dx = 0.$

Solution

Here,

$$(x + 2y - 3)\, dy - (2x - y + 1)\, dx = 0$$

or, $x\, dy + 2y\, dy - 3dy - 2x\, dx + y\, dx - dx = 0$

or, $(x\, dy + y\, dx) + d(y^2) - d(x^2) - d(3y) - d(x) = 0$

or, $d(xy) + d(y^2) - d(x^2) - d(3y) - d(x) = 0$

or, $d(xy + y^2 - x^2 - 3y - x) = 0$

Integrating, we get,

$$xy + y^2 - x^2 - 3y - x = c.$$

Example 4. Solve: $y\, dx - x\, dy = xy\, dy.$

Solution

Here, $y\, dx - x\, dy = xy\, dy$

Dividing both sides by y^2, we get,

$$\frac{y\, dx - x\, dy}{y^2} = \frac{x}{y} dy$$

or, $d\left(\frac{x}{y}\right) = \frac{x}{y} dy$

or, $\dfrac{d\left(\frac{x}{y}\right)}{\frac{x}{y}} = dy$

or, $d\left(\log\left(\frac{x}{y}\right)\right) = dy$

or, $d\left(\log \frac{x}{y} - y\right) = 0$

Integrating, we get,

$$\log\left(\frac{x}{y}\right) = y + c.$$

Example 5. Solve: $\sec^2 x\, dx + \sec^2 y\, dy = 0$

Solution

$$\sec^2 x\, dx + \sec^2 y\, dy = 0$$

or, $d(\tan x) + d(\tan y) = 0$

or, $d(\tan x + \tan y) = 0$

Integrating, we get,

$$\tan x + \tan y = c.$$

Example 6. Solve: $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Example 6. Solve: $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Solution

The given equation can be written as

$$\tan y(\sec^2 x\, dx) + \tan x(\sec^2 y\, dy) = 0$$

or, $\tan y\, d(\tan x) + \tan x\, d(\tan y) = 0$

or, $d(\tan x \tan y) = 0$

Integrating, we get,

$$\tan x \tan y = c.$$

Exercise 4.3

Solve the following differential equations by reducing to exact form.

1. $x\, dy + y\, dx = 0$
2. $2xy\, dy + y^2\, dx = 0$
3. $y\, dx - x\, dy = 0$
4. $2xy\, dx - x^2\, dy = 0$
5. $y\, dx + (x + y)\, dy = 0$
6. $(2xy + y^2)\, dy + (y^2 + x)\, dx = 0$
7. $\dfrac{dy}{dx} = \dfrac{y - x + 1}{y - x + 5}$
8. $(x^2 + 5xy^2)dx + (5x^2y + y^2)dy = 0$
9. $\sin x \cos x\, dx + \sin y \cos y\, dy = 0$

Answers

1. $xy = c$
2. $xy^2 = c$
3. $x = cy$
4. $x^2 = cy$
5. $2xy + y^2 = c$
6. $3x^2 + 6xy^2 + 2y^3 = c$
7. $x^2 + y^2 - 2xy - 2x + 10y = c$
8. $2x^3 + 2y^3 + 15x^2y^2 = c$
9. $\sin^2 x + \sin^2 y = c$

4.9 Linear Differential Equations

A differential equation is said to be a linear equation if it can be written in the form

$$\frac{dy}{dx} + Py = Q \qquad \ldots \text{(i)}$$

where P and Q are functions of x (not of y) or constants.

To solve linear differential equation, we multiply both sides by $e^{\int P\, dx}$.

Then (i) becomes,

$$e^{\int P\, dx} \cdot \frac{dy}{dx} + Py \cdot e^{\int P\, dx} = Q\, e^{\int P\, dx}$$

or, $\dfrac{d}{dx}(y \cdot e^{\int P\, dx}) = Q\, e^{\int P\, dx}$

Integrating both sides, we get,

$$y \cdot e^{\int P\, dx} = \int Q\, e^{\int P\, dx}\, dx + c$$

which gives the solution of the differential equation (i).

NOTE The factor $e^{\int P\, dx}$ is called integrating factor. In short, we write integrating factor as I.F.

Illustrative Examples

Example 1. Solve: $\frac{dy}{dx}+\frac{y}{x}=1.$

Solution

Here,

$$\frac{dy}{dx}+\frac{y}{x}=1 \qquad \ldots \text{(i)}$$

Comparing it with $\frac{dy}{dx}+Py=Q$, we get, $P=\frac{1}{x}$, $Q=1$

$$\text{I.F}=e^{\int P\,dx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$$

Multiplying both sides of (i) by x, we get,

$$x\frac{dy}{dx}+y=x$$

or $\quad d(y\cdot x)=x\,dx$

Integrating, we have,

$$xy=\frac{x^2}{2}+c.$$

Example 2. Solve: $\frac{dy}{dx}+y=e^x.$

Solution

Given equation is

$$\frac{dy}{dx}+y=e^x \qquad \ldots\text{(i)}$$

Comparing (i) with $\frac{dy}{dx}+Py=Q$, we get,

$$P=1,\ Q=e^x$$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int 1\,dx}=e^x$$

Multiplying both sides of (i) by equation, we have,

$$e^x\frac{dy}{dx}+e^x\cdot y=e^x\cdot e^x$$

or, $\quad d(y\cdot e^x)=e^{2x}\,dx$

Integrating both sides, we get,

$$y\cdot e^x=\frac{e^{2x}}{2}+c$$

or, $\quad y=\frac{e^{2x}}{2e^x}+\frac{c}{e^x}$

or, $\quad y=\frac{1}{2}e^x+ce^{-x}.$

Example 3. Solve: $\frac{dy}{dx} - \frac{y}{x} = 2x^2$.

Solution

Given differential equation is $\frac{dy}{dx} - \frac{y}{x} = 2x^2$ ... (i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$$P = -\frac{1}{x}, Q = 2x^2$$

Now,

$$\begin{aligned} \text{I.F.} &= e^{\int P\,dx} \\ &= e^{\int -\frac{1}{x}dx} \\ &= e^{-\log x} \\ &= e^{\log x^{-1}} \\ &= x^{-1} = \frac{1}{x} \end{aligned}$$

Multiplying both sides of (i) by $\frac{1}{x}$, we have,

$$\frac{1}{x}\cdot\frac{dy}{dx} - \frac{1}{x}\cdot\frac{y}{x} = \frac{1}{x}\cdot 2x^2$$

or, $\frac{d}{dx}\left(y\cdot\frac{1}{x}\right) = 2x$

or, $d\left(\frac{y}{x}\right) = 2x\,dx$

Integrating both sides, we get,

$$\frac{y}{x} = \int 2x\,dx$$

or, $\frac{y}{x} = x^2 + c$

$\therefore$ $y = x^3 + cx$.

Example 4. Solve: $\tan x\frac{dy}{dx} + y = \sec x$.

Solution

Here, $\tan x\frac{dy}{dx} + y = \sec x$

Dividing both sides by $\tan x$, we have,

$$\frac{dy}{dx} + \cot x . y = \operatorname{cosec} x \quad \ldots(i)$$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$$P = \cot x, \quad Q = \operatorname{cosec} x$$

$$\text{I.F} = e^{\int P dx} = e^{\int \frac{\cos x}{\sin x} dx} = e^{\log \sin x} = \sin x$$

Multiplying both sides of equation (i) by I.F, we get,

$$\sin x \frac{dy}{dx} + \cot x \cdot \sin x \cdot y = \operatorname{cosec} x \cdot \sin x$$

or, $\sin x \dfrac{dy}{dx} + \cos x \cdot y = dx$

or, $d(y \cdot \sin x) = dx$

Integrating, we have,

$$y \sin x = x + c.$$

Example 5. Solve: $\sin x \dfrac{dy}{dx} + (\cos x)y = \sin x \cos x.$

Solution

Here,

$$\sin x \frac{dy}{dx} + (\cos x)y = \sin x \cos x$$

or, $\dfrac{dy}{dx} + \dfrac{\cos x}{\sin x} y = \cos x$ ...(i)

Comparing equation (i) with $\dfrac{dy}{dx} + Py = Q$, we get,

$$P = \frac{\cos x}{\sin x}, \quad Q = \cos x$$

$$\text{I.F} = e^{\int P\, dx}$$

$$= e^{\int \frac{\cos x}{\sin x} dx}$$

$$= e^{\log \sin x}$$

$$= \sin x$$

Multiplying both sides of (i) by $\sin x$, we have,

$$\sin x \frac{dy}{dx} + \cos x . y = \sin x . \cos x$$

or, $d(y. \sin x) = \dfrac{1}{2} \sin 2x\, dx$

Integrating, we have,

$$y \sin x = \int \frac{1}{2} \sin 2x\, dx + c$$

or, $y \sin x = \dfrac{1}{2}\left(\dfrac{-\cos 2x}{2}\right) + c$

$\therefore\ y \sin x + \dfrac{1}{4} \cos 2x = c.$

Example 6. Solve: $(1+x)\dfrac{dy}{dx} - xy = 1 - x.$

Solution

Here,

$$(1+x)\frac{dy}{dx} = xy - x + 1$$

or, $$(1+x)\frac{dy}{dx} - xy = 1 - x$$

Dividing both sides by $(1+x)$, we get

$$\frac{dy}{dx} - \frac{x}{1+x}\cdot y = \frac{1-x}{1+x} \qquad \ldots\text{(i)}$$

Comparing equation (i) with $\dfrac{dy}{dx} + Py = Q$, we get

$$P = \frac{-x}{1+x},\ Q = \frac{1-x}{1+x}$$

$$\begin{aligned}
\text{I.F} &= e^{\int P dx} \\
&= e^{\int \frac{-x}{1+x} dx} \\
&= e^{\int \frac{1-(1+x)}{1+x} dx} \\
&= e^{\int \left(\frac{1}{1+x} - 1\right) dx} \\
&= e^{\ln(1+x) - x} \\
&= e^{\ln(x+1)} \cdot e^{-x} \\
&= (1+x)\, e^{-x}
\end{aligned}$$

Multiplying both sides of equation (i) by $(1+x)\, e^{-x}$, we have

$$(1+x)\, e^{-x}\left(\frac{dy}{dx} - \frac{x}{1+x}\cdot y\right) = e^{-x}(1-x)$$

or, $$d\{e^{-x}(1+x)\, y\} = e^{-x}(1-x)\, dx$$

Integrating, we get

$$\begin{aligned}
y(1+x)\, e^{-x} &= \int (1-x)\, e^{-x}\, dx + c \\
\text{or, } y(1+x)\, e^{-x} &= \int e^{-x}\, dx - \int x\, e^{-x}\, dx + c \\
&= \frac{e^{-x}}{(-1)} - \left[x \int e^{-x}\, dx - \int \left\{\left(\frac{dx}{dx}\right) \int e^{-x}\, dx\right\} dx\right] + c \\
&= -e^{-x} - \left[\frac{x\, e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1}\, dx\right] + c \\
&= -e^{-x} - \left[-xe^{-x} + \frac{e^{-x}}{(-1)}\right] + c \\
&= -e^{-x} + xe^{-x} + e^{-x} + c \\
&= xe^{-x} + c
\end{aligned}$$

Dividing both sides by e^{-x}, we get

$\therefore \quad y(1+x) = x + ce^{x}$

...ple 7. Solve $x\frac{dy}{dx} + 2y = x^2 \ln x$.

...ion Given,

$$x\frac{dy}{dx} + 2y = x^2 \ln x$$

Dividing both sides by x, we get

$$\frac{dy}{dx} + \frac{2}{x}\cdot y = x \ln x \qquad \ldots(i)$$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{2}{x},\ Q = x \ln x$$

$$\begin{aligned} \text{I.F} &= e^{\int P dx} \\ &= e^{\int \frac{2}{x} dx} \\ &= e^{2 \ln x} \\ &= e^{\ln x^2} \\ &= x^2 \end{aligned}$$

Multiplying both sides of (i) by x^2, we get

$$x^2\frac{dy}{dx} + x^2\cdot\frac{2}{x}\cdot y = x^2 \cdot x \ln x$$

$$d(x^2 \cdot y) = x^3 \ln x\, dx$$

Integrating, we get

$$x^2 y = \int \ln x \cdot x^3\, dx$$

or, $x^2 y = \ln x \int x^3\, dx - \int\left[\frac{d}{dx}(\ln x)\int x^3\, dx\right]dx$ [Using integration by parts in right side]

or, $x^2 y = \ln x \cdot \frac{x^4}{4} - \int \frac{1}{x}\cdot\frac{x^4}{4}\, dx$

or, $x^2 y = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx$

or, $x^2 y = \frac{x^4}{4}\ln x - \frac{x^4}{16} + c$

$\therefore\ y = \frac{1}{4}x^2 \ln x - \frac{1}{16}x^2 + \frac{c}{x^2}$.

...quation Reducible to Linear Form

...n equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone or constants, is called **...ernoulli's equation**. If we divide both sides by y^n, the equation is reduced to linear form.

Example 8. Reduce the equation $\frac{dy}{dx} + \frac{y}{x} = y^2$ in linear form hence solve it.

Solution

Given equation is

$$\frac{dy}{dx} + \frac{y}{x} = y^2$$

Dividing both sides by y^2, we have

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = 1 \qquad \ldots(i)$$

Put $y^{-1} = z$ then

$$-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

Then equation (i) can be written as

$$\frac{dz}{dx} - \frac{1}{x}z = -1 \qquad \ldots(ii)$$

which is a linear equation.

Here, $P = \frac{-1}{x}$, $Q = -1$

$$I.F = e^{\int P\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

Multiplying equation (i) by $\frac{1}{x}$, we get

$$\frac{1}{x}\frac{dz}{dx} - \frac{1}{x^2}z = -\frac{1}{x}$$

or, $d\left(\frac{1}{x}\cdot z\right) = -\frac{1}{x}dx$

Integrating

$$\frac{z}{x} = -\ln x + c$$

or, $\left(\frac{\frac{1}{y}}{x}\right) = c - \ln x$

$\therefore \quad \frac{1}{xy} = c - \ln x.$

Example 9. Solve: $x\frac{dy}{dx} + y = y^2 \ln x.$

Solution

Given,

$$x\frac{dy}{dx} + y = y^2 \ln x$$

or, $\frac{dy}{dx} + \frac{1}{x} \cdot y = \frac{\ln x}{x} \cdot y^2$ ... (i)

which is Bernoulli's differential equation.

Dividing both sides of (i) by y^2, we get

$$y^{-2}\frac{dy}{dx} + \frac{1}{x} \cdot y^{-1} = \frac{\ln x}{x} \quad \text{...(ii)}$$

Let $y^{-1} = z$. Then

$$-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

or, $y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$

Then, equation (ii) can be written as

$$-\frac{dz}{dx} + \frac{1}{x} \cdot z = \frac{\ln x}{x}$$

or, $\frac{dz}{dx} + \left(-\frac{1}{x}\right) \cdot z = -\frac{\ln x}{x}$ ...(iii)

The equation (iii) is a linear differential equation.

Here, $P = -\frac{1}{x}$, $Q = -\frac{\ln x}{x}$

Now,

$$\text{I.F} = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying both sides of equation (iii) by $\frac{1}{x}$, we get

$$\frac{1}{x}\frac{dz}{dx} - \frac{1}{x^2} z = -\frac{\ln x}{x^2}$$

or, $d\left(\frac{1}{x} \cdot z\right) = -\frac{\ln x}{x^2} dx$

Integrating, we get

$$\frac{z}{x} = -\int \frac{\ln x}{x^2} dx$$

or, $\frac{z}{x} = -\left[\ln x \int x^{-2} dx - \int \left\{\frac{d}{dx}(\ln x) \int x^{-2} dx\right\} dx\right]$

or, $\frac{z}{x} = -\left[\ln x \left(\frac{x^{-1}}{-1}\right) - \int \left\{\frac{1}{x} \cdot \frac{x^{-1}}{(-1)}\right\} dx\right]$

or, $\frac{z}{x} = -\left[-\frac{\ln x}{x} + \int x^{-2} dx\right]$

or, $\frac{z}{x} = -\left[-\frac{\ln x}{x} + \frac{x^{-1}}{-1}\right] + c$

or, $\frac{z}{x} = \frac{\ln x}{x} + \frac{1}{x} + c$

or, $\frac{\frac{1}{y}}{x} = \frac{\ln x}{x} + \frac{1}{x} + c$

or, $\frac{1}{yx} = \frac{\ln x + 1 + cx}{x}$

or, $\frac{1}{y} = \ln x + 1 + cx$

$\therefore$ $y(1 + \ln x) + cxy = 1$ which is the required solution.

Exercise 4.4

Solve the following linear differential equations.

1. $\frac{dy}{dx} + y = 1$
2. $\frac{dy}{dx} - y = e^x$
3. $\frac{dy}{dx} + \frac{y}{x} = x$
4. $x\frac{dy}{dx} + y = x^4$
5. $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$
6. $\frac{dy}{dx} + 2y \tan x = \sin x$
7. $\sin x \frac{dy}{dx} + y \cos x = x \sin x$
8. $\cos^2 x \frac{dy}{dx} + y = 1$
9. $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$
10. $\frac{dy}{dx} + y = xy^2$
11. $\frac{dy}{dx} + y \tan x = y^3 \sec x$

Answers

1. $y = 1 + ce^{-x}$
2. $y = e^x(x + c)$
3. $xy = \frac{x^3}{3} + c$
4. $xy = \frac{x^5}{5} + c$
5. $y(1 + x^2) = \frac{4x^3}{3} + c$
6. $y \sec^2 x = \sec x + c$
7. $y \sin x = \sin x - x \cos x + c$
8. $y = 1 + ce^{-\tan x}$
9. $y = \frac{1}{2}e^{\tan^{-1}x} + ce^{-\tan^{-1}x}$
10. $xy + y + cy\,e^x = 1$
11. $(c - 2 \sin x)y^2 = \cos^2 x$

4.10 Simple Applications of First Order Differential Equations

There are wide range of application of differential equations in engineering as well as in many other areas. In this unit, we have studied to solve certain types of first order differential equations. Next, we discuss about the simple applications of first order differential equation.

Law of Growth and Decay

A quantity $y = y(t)$ is said to have exponential growth if it increases at the rate which is proportional to the amount of the quantity in it. That is

$$\frac{dy}{dt} \propto y$$

i.e. $\frac{dy}{dt} = ky$, where $k > 0$.

In the same way, the exponential decay model is given by

$\frac{dy}{dt} = -ky$, where $k > 0$.

These both are homogeneous linear differential equation of first order and can be solved by variable separation method. The solutions of these equations are of the form $y = ce^{kt}$ and $y = ce^{-kt}$ respectively where c is constant of integration.

There are several problems that can be solved by using first order differential equation. The problem of mixture, problems related to compound interest and depreciation, problems related to Newton's law of cooling, population growth model, dynamics of market analysis, radioactive decay, etc. can be solved using first order differential equation. While solving these types of problems, we need to model the given problems in the form of differential equations and solve them. We now give some examples.

Illustrative Examples

Example 1. The population growth rate of a certain town is 8% per year. Model the situation using a differential equation. What will be the population after 10 years?

Solution

Let P_0 be the initial population and P be the population after t years. Then

$$\frac{dP}{dt} = 8\% \text{ of } P$$

or, $\frac{dP}{dt} = \frac{8}{100} \times P$

or, $\frac{dP}{dt} = \frac{2}{25} P$

or, $\frac{dP}{P} = \frac{2}{25} dt$

Integrating, we get

$$\int \frac{1}{P}\,dP = \frac{2}{25}\int dt$$

$$\ln P = \frac{2}{25}t + c \qquad \ldots(i)$$

At $t = 0$, $P = P_0$.

Thus,

$$\ln P_0 = \frac{2}{25} \times 0 = c.$$

$\therefore \quad c = \ln P_0$

From (i),

$$\ln P = \frac{2}{25}t + \ln P_0$$

When $t = 10$,

$$\ln P = \frac{2}{25} \times 10 + \ln P_0$$

or, $\ln P - \ln P_0 = \frac{4}{5}$

or, $\ln\left(\frac{P}{P_0}\right) = 0.8$

or, $\frac{P}{P_0} = e^{0.8}$

$\therefore \quad P = P_0 e^{0.8}$ which is the required population after 10 years.

Example 2. A culture of bacteria contained 5 million bacteria at 10 PM. At 2 PM, the number of bacteria had increased to 10 million. Assuming that the condition for growth had not changed over the four hour interval, how many bacteria were in the culture at 12 noon.

Solution

Let $y(t)$ be the number of bacteria in t hours.

Then, $y(t) = ce^{kt}$.

Initially, at 10 AM, $t = 0$,

$$y(0) = ce^{0}$$

or, $5 = c$

$\therefore \quad c = 5$

At 2 PM, $t = 4$,

$$y(4) = ce^{4k}$$

or, $10 = 5e^{4k}$

or, $e^{4k} = 2$

or, $4k = \ln 2$

or, $k = \frac{1}{4}\ln 2$

Hence, $y(t) = 5e^{\left(\frac{1}{4}\ln 2\right)t}$

At 12 noon, $t = 2$,

$$y(2) = 5e^{\left(\frac{1}{4}\ln 2\right)2}$$

$$= 5e^{\frac{1}{2}\ln 2}$$

$$= 5e^{\ln 2^{\frac{1}{2}}}$$

$$= 5e^{\ln\sqrt{2}}$$

$$= 5\sqrt{2} \text{ million.}$$

Example 3. If demand and supply functions in a competitive market are $Q_d = 32 - 0.5P$ and $Q_s = -8 + 0.3P$ and the rate of adjustment of price when the market is out of equilibrium is $\frac{dP}{dt} = 0.2(Q_d - Q_s)$. Derive and solve the obtained differential equation to get a function for P in terms of t given that price is 12 in the time period 0.

Solution

Here

$$Q_d = 32 - 0.5P$$

$$Q_s = -8 + 0.3P$$

$$\frac{dP}{dt} = 0.2(Q_d - Q_s)$$

or, $\frac{dP}{dt} = 0.2[(32 - 0.5P) - (-8 + 0.3P)]$

or, $\frac{dP}{dt} = 0.2(32 - 0.5P + 8 - 0.3P)$

or, $\frac{dP}{dt} = 0.2(40 - 0.8P)$

or, $\frac{dP}{dt} = 8 - 0.16P$

or, $\frac{dP}{dt} = -0.16(P - 50)$

or, $\frac{dP}{P - 50} = -0.16dt$

Integrating on both sides,

$$\ln(P - 50) = -0.16t + \ln c$$

or, $\ln(P - 50) - \ln c = -0.16t$

or, $\ln\left(\frac{P - 50}{c}\right) = -0.16t$

or, $P - 50 = c \cdot e^{-0.16t}$

or, $P = 50 + c \cdot e^{-0.16t}$ . . . (i)

By question, when $t = 0$, $P = 12$

Then from (i)

$$12 = 50 + c \cdot e^{-0.16 \times 0}$$

or, $12 - 50 = c$

$\therefore \quad c = -38$

Substituting the value of c in (i)

$$P = 50 - 38 \cdot e^{-0.16t}$$

which is required solution.

Example 4. The half-life of isotopic radium is 200 years. Find the time required to decay 5% initial amount.

Solution

The solution of decay is of the form $y(t) = ce^{kt}$

When $t = 0$,

$$y(0) = c \cdot e^0$$

$\Rightarrow \quad c = y(0)$

$\therefore \quad y(t) = y(0)\, e^{kt}$

When $t = 200$,

$$y(200) = y(0)\, e^{200k}$$

By question,

$$\frac{1}{2} \cdot y(0) = y(0)\, e^{200k}$$

or, $\frac{1}{2} = e^{200k}$

or, $0.5 = e^{200k}$

or, $200k = \ln (0.5)$

or, $k = \dfrac{\ln(0.5)}{200}$

Thus, $y(t) = y(0)\, e^{\left(\frac{\ln 0.5}{200}\right)t}$

Let the required time be T.

Then,

$$y(\text{T}) = y(0)\, e^{\left(\frac{\ln 0.5}{200}\right)\text{T}}$$

or, 95% of $y(0) = y(0)\, e^{\left(\frac{\ln 0.5}{200}\right)\text{T}}$

or, $\dfrac{95}{100} \times y(0) = y(0)\, e^{\left(\frac{\ln 0.5}{200}\right)\text{T}}$

or, $0.95 = e^{\left(\frac{\ln 0.5}{200}\right)T}$

or, $\left(\frac{\ln 0.5}{200}\right)T = \ln 0.95$

or, $T = \frac{200 \times \ln 0.95}{\ln 0.5}$

$\therefore$ T = 14.8 years

Required time = 14.8 years

Example 5. A tank initially contains 500 litres of water with 5 kg of salt. A mixture containing 0.2 kg of salt per litre enters the tank at a rate of 5 litres per minute. The mixture, kept uniform by stirring, is flowing out at the same rate. Find the amount of salt of the tank after 10 minutes.

Solution

Let $y(t)$ be the amount of salt in kgs after t minutes. Given $y(0) = 5$. We have to find $y(10)$.

The rate of change of salt in mixture in tank is

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}$$

$$= 0.2 \times 5 - \frac{y(t)}{500} \times 5$$

$$= 1 - \frac{y}{100}$$

or, $\frac{dy}{dt} + \frac{y}{100} = 1$ ...(i)

Thus the initial value problem can be written as

$$\frac{dy}{dt} + \frac{y}{100} = 1, \ y(0) = 5$$

$$\text{I.F.} = e^{\int P dt} = e^{\int \frac{1}{100} dt} = e^{\frac{t}{100}}$$

Multiplying (i) both sides by I.F. $= e^{\frac{t}{100}}$, we get

$$\frac{dy}{dt} \cdot e^{\frac{t}{100}} + \frac{y}{100} \cdot e^{\frac{t}{100}} = e^{\frac{t}{100}}$$

or, $d\left(y \cdot e^{\frac{t}{100}}\right) = e^{\frac{t}{100}} dt$

Integrating, we get

$$y \cdot e^{\frac{t}{100}} = \frac{e^{\frac{t}{100}}}{\frac{1}{100}} + c$$

$\therefore$ $y(t) = 100 + c\, e^{-\frac{t}{100}}$ ...(iii)

By given, when $t = 0$, $y(0) = 5$

Thus, from (iii),

$$5 = 100 + ce^0$$

$$c = -95$$

$$\therefore \quad y(t) = 100 - 95\, e^{-\frac{t}{100}} \qquad \ldots\text{(iv)}$$

When $t = 10$,

$$y(10) = 100 - 95\, e^{-\frac{t}{100}}$$
$$= 100 - 95\, e^{-0.1}$$
$$= 42.38 \text{ kg}$$

Exercise 4.5

1. Population of a city is increasing at the rate of 3% per year. How long does it take for the population to be doubled?
2. The per capita income of a country is increasing at a rate of 4% per year. When will it be doubled?
3. A model for the population $y(t)$ in millions of a city at time t is given by

   $$\frac{dy}{dt} = -0.08y + 12$$

   The population at time $t = 0$ is 200 million.

   a. Find $y(15)$ correct to 2 decimal places.

   b. Find the value of t for which $y(t) = 155$, correct to 2 decimal place.
4. A culture of bacteria contained 20 million bacteria at 1 PM. At 5 PM, the number of bacteria had increased to 40 million. Assuming that the condition for growth had not changed over the four hour interval, how many bacteria were in the culture at 3 PM.
5. The half-life of isotopic radium is 300 years. Find the time required to decay 10% of its initial amount.
6. The rate at which an infection spreads in poultry house is given as $\frac{dP}{dt} = 0.3(3000 - P)$, where t is time in days. Given $P = 0$, $t = 0$.

   a. Solve the differential equation to determine an expression for the number of Poultry infected at any time t.

   b. Calculate the time taken for 2000 poultry to become infected.
7. If the demand and supply functions in a competitive market are $Q_d = 32 - 0.5P$ and $Q_s = -8 + 0.3P$ and the rate of adjustment of price when the market is out of equilibrium is $\frac{dP}{dt} = 0.25(Q_d - Q_s)$. Determine and solve the obtained differential equation to get a function for P in terms of 't' given that price is 15 in time period '0'.

8. Suppose that a sum P is invested at an annual rate of return $r\%$ compounded continuously.
   a. Find the time t required for the sum P to double in value as a function of r.
   b. Find t if $r = 10\%$.
   c. Find r if the sum of money is to be doubled in 5 years.
9. A tank initially contains 500 litres of water with 5 kg of salt. A mixture containing 0.2 kg of salt per litre enters the tank at a rate of 5 litre per minute. The mixture, kept uniform by stirring, is flowing out at the same rate. Find the amount of salt of the tank after 20 minutes.
10. A water tank is being filled up with water which was empty initially. The rate of increase of volume of water at any time t is given by $\frac{dV}{dt} = 0.2(100 - t)$ where V is measured in litres and time is measured in seconds.
    a. Find the volume of water filled up in the tank in t seconds.
    b. Find the volume of water filled up in 10 seconds.
    c. How much time will it require to fill up the whole tank if the total capacity of the tank is 200 litres?

Answers

1. 23.1 years
2. 17.33 years
3. a. 165.06 b. 28.78
4. $20\sqrt{2}$ million
5. 45.6 years
6. a. $P = 3000 - 3000e^{-0.3t}$ b. 3.662 days
7. $P = 40 - 25e^{-0.25t}$
8. a. $\frac{\ln 2}{r}$ b. 6.93 years c. 13.86%
9. 22.22 kg
10. a. $V = 20t - 0.1t^2$ b. 190 c. 10.56 seconds

4.11 Partial Differential Equations

Many physical problems in science and engineering depend on two or more variables. They usually depend on time t and one or more space variables. In such situation, when they are formulated mathematically, the partial differential equations many arise whereas the simplest physical problems can be modeled by ordinary differential equations. In most problems related to heat transfer, quantum mechanics, electromagnetic theory, etc., we need to use PDE's to model and solve them. We now give the definition of partial differential equation (PDE).

Definition

An equation involving a dependent variable, several independent variables and partial derivatives of dependent variable with respect to independent variables is called a partial differential equation.

If z is a dependent variable depending on two independent variables x and y then the equation of the form $f(x, y, z, p, q, r, s, t) = 0$ is a partial differential equation, where

$$p = \frac{\partial z}{\partial x}$$

$q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial x \partial y}$

$\partial^2 z$

For example,

a. $p + q = z$

b. $r + 3s + 2t = x + y$

Order and Degree of PDE

The order of a PDE is defined as the order of the highest partial derivative occurring in the PDE.

The degree of a PDE is defined as the order of the highest derivative in the PDE.

These are same as in the case of ordinary differential equations.

For example,

Some examples of order and degree of PDE are

a. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$ First order and first degree

b. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ Second order and first degree

c. $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial y}$ First order and second degree

d. $\left(\frac{\partial^2 z}{\partial x^2}\right)^3 + \left(\frac{\partial^3 z}{\partial y^3}\right)^2 = \frac{\partial z}{\partial x}$ Third order and second degree

Linear and Non-linear PDE

If the dependent variable and its partial derivatives occur only in the first degree and they are no multiplied then it is called linear PDE. Otherwise it is called non-linear.

Partial Differential Equation of First Order

It z is a dependent variable depending on two independent variables x and y then the equation of the form $f(x, y, z, p, q) = 0$ is called the partial differential equation of first order, when $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$

For example,

a. $p + q = z$

b. $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

4.12 Formation of Partial Differential Equation

The PDE can be formed by any one of the following method.

i. Eliminating the arbitrary constants

ii. Eliminating the arbitrary functions

I. Derivation of PDE by eliminating arbitrary constants.

Let us consider an equation $f(x, y, z, a, b) = 0$ ... (i),

where, a and b are arbitrary constants. Let z be a function of two independent variables x and y.

Now, differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

i.e. $$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \qquad \text{... (ii)}$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

i.e. $$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \qquad \text{... (iii)}$$

By eliminating two arbitrary constants a and b from (i), (ii) and (iii), we obtain an equation of the form $f(x, y, z, p, q) = 0$, which is the partial differential equation of first order.

If we have more than two constants then the above procedure of elimination will give a PDE.

Example 1. Find the partial differential equation from the relation $z = ax + by + ab$.

Solution

Given,

$$z = ax + by + ab \qquad \text{... (i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x} = a$$

i.e. $p = a$ ... (ii)

and

$$\frac{\partial z}{\partial y} = b$$

i.e. $q = b$ ... (iii)

Eliminating a and b from (i), (ii) and (iii), we get

$$z = px + qy + pq$$

which is the required PDE.

4.12 Formation of Partial Differential Equation

The PDE can be formed by any one of the following method.

i. Eliminating the arbitrary constants

ii. Eliminating the arbitrary functions

1. Derivation of PDE by eliminating arbitrary constants.

Let us consider an equation $f(x, y, z, a, b) = 0$ ... (i),

where, a and b are arbitrary constants. Let z be a function of two independent variables x and y. Now, differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$

i.e. $$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0 \qquad \text{... (ii)}$$

and

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=0$$

i.e. $$\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0 \qquad \text{... (iii)}$$

By eliminating two arbitrary constants a and b from (i), (ii) and (iii), we obtain an equation of the form $f(x, y, z, p, q) = 0$, which is the partial differential equation of first order.

If we have more than two constants then the above procedure of elimination will give a PDE.

Example 1. Find the partial differential equation from the relation $z = ax + by + ab$.

Solution

Given,

$$z = ax + by + ab \qquad \text{... (i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x}=a$$

i.e. $p = a$ ... (ii)

and

$$\frac{\partial z}{\partial y}=b$$

i.e. $q = b$ ... (iii)

Eliminating a and b from (i), (ii) and (iii), we get

$$z = px + qy + pq$$

which is the required PDE.

Example 2. From the partial differential equation from the relation

$$(x-a)^2 + (y-b)^2 + z^2 = c^2$$

Solution

Given,

$$(x-a)^2 + (y-b)^2 + z^2 = c^2 \qquad \ldots \text{(i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0 \text{ and } 2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad (x-a) = -z\frac{\partial z}{\partial x} \text{ and } (y-b) = -z\frac{\partial z}{\partial y}$$

Substituting these values in equation (i), we obtain

$$\left(-z\frac{\partial z}{\partial x}\right)^2 + \left(-z\frac{\partial z}{\partial y}\right)^2 + z^2 = c^2$$

or, $$z^2\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\} = c^2$$

$\therefore \quad z^2(p^2 + q^2 + 1) = c^2$, which is the required PDE.

Example 3. Form a partial differential equation if $z = ke^{ax}\sin ay$.

Solution

Given,

$$z = ke^{ax}\sin ay \qquad \ldots \text{(i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x} = kae^{ax}\sin ay \qquad \ldots \text{(ii)}$$

and

$$\frac{\partial z}{\partial y} = kae^{ax}\cos ay \qquad \ldots \text{(iii)}$$

Again, differentiating equation (ii) partially with respect to x and equation (iii) partially with respect to y, we get,

$$\frac{\partial^2 z}{\partial x^2} = ka^2 e^{ax}\sin ay \qquad \ldots \text{(iv)}$$

and

$$\frac{\partial^2 z}{\partial y^2} = -ka^2 e^{ax}\sin ay \qquad \ldots \text{(v)}$$

Adding equation (iv) and equation (v), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = ka^2 e^{ax}\sin ay - ka^2 e^{ax}\sin ay$$

or, $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

i.e. $r + s = 0$ which is the required PDE.

Derivation of PDE by eliminating arbitrary functions.

Suppose u and v are any two given functions of x, y and z. Let f be an arbitrary function of u and v which is written as

$$f(u, v) = 0 \qquad \ldots \text{(i)}$$

We treat x and y as independent variables and z as a dependent variables.

Then,

$$\frac{\partial x}{\partial y} = 0 \text{ and } \frac{\partial y}{\partial x} = 0 \qquad \ldots \text{(ii)}$$

Differentiating (i) partially with respect to x, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

or, $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0 \qquad \text{[Using (ii)]}$$

or, $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \left[\because \frac{\partial z}{\partial x} = p\right]$$

or, $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = -\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)$$

or, $$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = -\frac{\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)} \qquad \ldots \text{(iii)}$$

In the same way, differentiating (i) partially with respect to y, we get

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = -\frac{\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)} \qquad \ldots \text{(iv)}$$

Eliminating the arbitrary function f from (iii) and (iv), we obtain

$$\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial u} + q\frac{\partial u}{\partial z}}$$

or, $$\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)$$

or, $$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} + q\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} + p\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} + pq\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z} = \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial x} + p\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z} + q\frac{\partial v}{\partial z}\cdot\frac{\partial u}{\partial x} + pq\frac{\partial v}{\partial z}\cdot\frac{\partial u}{\partial z}$$

or, $$p\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right) + q\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right) = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

i.e. Pp + ... $\qquad \ldots$(v)

where,

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

and $$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

Equation (v) is the required PDE.

Example 4. Form a partial differential equation by elimination arbitrary function from the relation $z = f(x^2 + y^2)$

Solution

Given,

$$z = f(x^2 + y^2) \quad \dots \text{(i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x} = f'(x^2 + y^2) \cdot 2x$$

$$\Rightarrow \quad f'(x^2 + y^2) = \frac{1}{2x} \frac{\partial z}{\partial x} \quad \dots \text{(ii)}$$

and

$$\frac{\partial z}{\partial y} = f'(x^2 + y^2) \cdot 2y$$

$$\Rightarrow \quad f'(x^2 + y^2) = \frac{1}{2y} \frac{\partial z}{\partial y} \quad \dots \text{(iii)}$$

From equation (ii) and equation (iii), we get

$$\frac{1}{2x} \frac{\partial z}{\partial x} = \frac{1}{2y} \frac{\partial z}{\partial y}$$

or, $$x \frac{\partial z}{\partial y} = y \frac{\partial z}{\partial x}$$

or, $$xq = yp$$

$$\therefore \quad yp - xq = 0$$

Example 5. From a PDE by eliminating f from $lx + my + nz = f(x^2 + y^2 + z^2)$

Solution

Given,

$$lx + my + nz = f(x^2 + y^2 + z^2) \quad \dots \text{(i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$l \cdot 1 + 0 + n \frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2) \cdot \left(2x + 2z \cdot \frac{\partial z}{\partial x}\right)$$

or, $$l + np = f'(x^2 + y^2 + z^2) \cdot (2x + 2zp)$$

$$\Rightarrow \quad f'(x^2 + y^2 + z^2) = \frac{l + np}{2x + 2zp} \quad \dots \text{(ii)}$$

and

$$0 + m \cdot 1 + n \cdot \frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2) \cdot \left(2y + 2z \cdot \frac{\partial z}{\partial y}\right)$$

or, $m + nq = f'(x^2 + y^2 + z^2) \cdot (2y + 2zq)$

$\Rightarrow$ $f'(x^2 + y^2 + z^2) = \dfrac{m + nq}{2y + 2zq}$ ... (iii)

From equation (ii) and equation (iii), we get

$$\frac{l + np}{2x + 2zp} = \frac{m + nq}{2y + 2zq}$$

or, $\dfrac{l + np}{x + zp} = \dfrac{m + nq}{y + zq}$

or, $(l + np)(y + zq) = (m + nq)(x + zp)$

or, $ly + lzq + npy + nzpq = mx + mzp + nqx + npqz$

$\therefore$ $(ny - mz)p + (lz - nx)q = mx - ly$

which is the required PDE.

xample 6. Form the PDE by eliminating the arbitrary functions from

$z = f_1(y + 2x) + f_2(y - 2x)$

lution

Given,

$z = f_1(y + 2x) + f_2(y - 2x)$ ... (i)

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x} = f_1'(y + 2x) \cdot 2 + f_2'(y - 2x) \cdot (-2)$$

$$= 2 f_1'(y + 2x) - 2 f_2'(y - 2x) \quad \text{... (ii)}$$

and

$$\frac{\partial z}{\partial y} = f_1'(y + 2x) \cdot 1 + f_2'(y - 2x) \cdot 1$$

$$= f_1'(y + 2x) + f_2'(y - 2x) \quad \text{... (iii)}$$

Differentiating equation (ii) partially with respect to x and differentiating equation (iii) partially with respect to y, we get

$$\frac{\partial^2 z}{\partial x^2} = 4 f_1''(y + 2x) + 4 f_2''(y - 2x)$$

$$= 4[f_1''(y + 2x) + f_2''(y - 2x)] \quad \text{... (iv)}$$

and

$$\frac{\partial^2 z}{\partial y^2} = f_1''(y + 2x) + f_2''(y - 2x) \quad \text{... (v)}$$

From equation (iv) and equation (v), we get

$$\frac{\partial^2 z}{\partial x^2} = 4 \cdot \frac{\partial^2 z}{\partial y^2}$$

i.e. $r = 4t$ which is the required PDE.

Exercise 4.6

1. Form the partial differential equations by eliminating arbitrary constants from the following relations.

   a. $z = a(x + y) + b$ b. $z = (x + a)(y + b)$

   c. $z = (x^2 + a)(y^2 + b)$ d. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

2. Form the partial differential equations by eliminating arbitrary functions from the following relations.

   a. $z = \phi(x^2 - y^2)$ b. $z = \phi(x^2 + y^2)$

   c. $z = f(xy)$ d. $z = f\left(\frac{x}{y}\right)$

   e. $z = y^2 + 2f\left(\frac{1}{x} + \ln y\right)$ f. $x + y + z = f(x^2 + y^2 + z^2)$

   g. $z = \phi(x + ay) + \psi(x - ay)$ h. $z = f_1(x + iy) + f_2(x - iy)$

3. Find the partial differential equation of a plane cutting off equal intercepts from the axis of x and y.

4. Form the partial differential equation of the set of all spheres whose centre line on the axis of z.

Answers

1. a. $p = q$ b. $z = pq$ c. $pq = 4xyz$ d. $xpz^2 - pz + xzr = 0$
2. a. $qx + py = 0$ b. $xq - yp = 0$ c. $px - qy = 0$ d. $px + qy = 0$ e. $x^2p + yq = 2y^2$
   f. $p(y - z) + q(z - x) = x - y$ g. $t = a^2r$ h. $r + t = 0$
3. $p - q = 0$
4. $xq - yp = 0$

4.13 Solutions of Linear PDEs of First Order

In this section, we discuss the solution of linear PDEs of first order. According to the syllabus, our study will be limited to study linear PDEs of first order and its solution.

The solution of a PDE means the relation obtained by reducing the equation into the relation free of derivatives. While making the relation free from the derivative terms, we use several constants and arbitrary functions according to the nature of PDE. If initial or boundary conditions are given then we evaluate the values of the constants and arbitrary functions.

Different types of solutions of PDEs are given below:

Let

$f(x, y, z, p, q) = 0$ … (i) be a given PDE.

1. **Complete integral:** The solution of equation (i) in the form F $(x, y, z, a, b) = 0$, where a and b are arbitrary constants is called complete integral of (i).
2. **General integral:** The solution of equation (i) in the form F $(x, y, z, a, \phi(a)) = 0$, where a and $\phi(a)$ are constants is called general integral of (i).

Particular integral: If we find the values of a and b in the complete integral $F(x, y, z, a, b) = 0$ by using the given conditions, the solution so obtained is called particular integral of (i).

Singular integral: If we eliminate a and b in the complete integral $F(x, y, z, a, b) = 0$ then the solution so obtained is called singular integral of (i).

Linear partial differential equation of first order is of the form

$$Pp + Qq = R \qquad \ldots(i)$$

where P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. The equation (i) is called **Lagrange's partial differential equation.**

From previous section, by eliminating the arbitrary function f from

$$f(u, v) = 0 \qquad \ldots(ii)$$

the PDE $Pp + Qq = R$ has been obtained.

Thus (ii) is the general integral of (i). Here u and v are functions of x, y and z.

So, let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$

where c_1 and c_2 are arbitrary constants.

By differentiation, we get

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \qquad \ldots(iii)$$

and

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \qquad \ldots(iv)$$

Solving equations (iii) and (iv) for dx, dy and dz, we get

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}}$$

or, $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots(v)$$

where

$$P = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

The equation (v) is called Lagrange's auxiliary or subsidiary or characteristics equations for $Pp + Qq = R$.

Hence, if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ then $f(u, v) = 0$ is the general solution.

Steps for Solving the PDE $Pp + Qq = R$ by Lagrange's Method

Step 1 Write the linear PDE of first order in the form $Pp + Qq = R$.

Step 2 Write Lagrange's auxiliary equations in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Step 3 Solve the equations given in step (2) for two independent solutions. Name two solution $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$.

Step 4 The general solution can be written in any one of the following three equivalent forms. $f(u, v) = 0$ or $u = f(v)$ or $v = f(u)$ where f is arbitrary function.

There are several methods of solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Briefly, we discuss the following methods.

i. **Method of grouping:** We take any two ratios so that it is easily integrable. The two integ obtained in this way form the complete solution.

ii. **Method of Lagrange's multipliers:** We write $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$.

Here, l, m, n are Lagrange's multipliers and they are so chosen that $lP + mQ + nR = 0$. The integrals obtained from this way or obtained one from method (i) and another from method (ii) the complete solution.

Geometrical Meaning

Let $f(x, y, z) = 0$ be an integral surface generated by the PDE $Pp + Qq = R$.

i.e. $Pp + Qq + R(-1) = 0$

The direction cosines of the normal at any point on the surface $f(x, y, z) = 0$ are proportion $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

Now,

$$\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} = \frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} : -1 = p : q : -1$$

Hence, the normal to the surface $f(x, y, z) = 0$ is perpendicular to a line whose direction ratios Q, R. Therefore, $Pp + Qq = R$ and $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ give the same set of surfaces. Thus these equ are equivalent.

Illustrative Examples

Example 1. Solve: $p + q = x$.

Solution

Given,

$$p + q = x \qquad \ldots(i)$$

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = 1, Q = 1, R = x$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

or, $$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{x}$$

Taking first two ratios, we get

$$\frac{dx}{1}=\frac{dy}{1}$$

Integrating, we get

$$x=y+c_1$$

or, $x-y=c_1$

Again, taking first and third ratios, we get

$$\frac{dx}{1}=\frac{dz}{x}$$

or, $x dx=dz$

Integrating, we get

$$\frac{x^2}{2}=z+c_2$$

or, $$\frac{x^2}{2}-z=c_2$$

The general solution is $\phi(c_1, c_2)=0$

i.e. $$\phi\left(x-y, \frac{x^2}{2}-z\right)=0$$

Example 2. Solve: $xz\frac{\partial z}{\partial x}+yz\frac{\partial z}{\partial y}=xy$

Solution

Given,

$$xz\frac{\partial z}{\partial x}+yz\frac{\partial z}{\partial y}=xy$$

or, $xz\cdot p+yz\cdot q=xy$ ...(i)

Comparing equation (i) with $Pp+Qq=R$, we get

$$P=xz, Q=yz, R=xy$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

or, $$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{xy} \quad \text{...(ii)}$$

Taking first two ratios, we get

$$\frac{dx}{xz} = \frac{dy}{yz}$$

or, $\frac{dx}{x} = \frac{dy}{y}$

Integrating, we get

$$\ln x = \ln y + \ln c_1$$

or, $\ln x - \ln y = \ln c_1$

or, $\ln\left(\frac{x}{y}\right) = \ln c_1$

$\therefore \quad \frac{x}{y} = c_1$ ...(iii)

Again, taking second and third ratios of (ii), we get

$$\frac{dy}{yz} = \frac{dz}{xy}$$

or, $\frac{dy}{yz} = \frac{dz}{c_1 y \cdot y}$ [Using (iii)]

or, $\frac{dy}{yz} = \frac{dz}{c_1 y^2}$

or, $c_1 y dy = z dz$

Integrating, we get

$$c_1 \frac{y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

or, $c_1 y^2 - z^2 = c_2$

or, $\frac{x}{y} \cdot y^2 - z^2 = c_2$ [Using (iii)]

$\therefore \quad xy - z^2 = c_2$ ...(iv)

From (iii) and (ii), the required general solution is $\phi(c_1, c_2) = 0$

i.e. $\phi\left(\frac{x}{y}, xy - z^2\right) = 0$ where ϕ is an arbitrary function.

Example 3. Solve: $(y - z)\frac{\partial z}{\partial x} + (x - y)\frac{\partial z}{\partial y} = z - x$

Solution

Given,

$$(y - z)\frac{\partial z}{\partial x} + (x - y)\frac{\partial z}{\partial y} = z - x$$

or, $(y - z)p + (x - y)q = z - x$ ...(i)

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = y - z,\ Q = x - y,\ R = z - x$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x} = k \text{ (suppose)} \qquad \text{...(ii)}$$

Using the multipliers 1, 1, 1 in (ii), we get

$$\frac{dx + dy + dz}{y - z + x - y + z - x} = k$$

or, $dx + dy + dz = 0$

Integrating, we get

$$x + y + z = c_1 \qquad \text{...(iii)}$$

Again, using the multipliers x, y, z in (ii), we get

$$\frac{xdx + zdy + ydz}{x(y-z) + z(x-y) + y(z-x)} = k$$

or, $xdx + zdy + ydz = 0$

Integrating, we get

$$\frac{x^2}{2} + yz = c_2 \qquad \text{...(iv)}$$

The required general solution is $\phi\left(x + y + z, \frac{x^2}{2} + yz\right) = 0$

Example 4. Solve: $\tan x \cdot \frac{\partial z}{\partial x} + \tan y \cdot \frac{\partial z}{\partial y} = \tan z.$

Solution

Given,

$$\tan x \cdot \frac{\partial z}{\partial x} + \tan y \cdot \frac{\partial z}{\partial y} = \tan z \qquad \text{...(i)}$$

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = \tan x, Q = \tan y, R = \tan z$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \qquad \text{...(ii)}$$

Taking first two ratios, we get

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Integrating, we get

$$\int \frac{\cos x}{\sin x} dx = \int \frac{\cos y}{\sin y} dy$$

or, $\ln \sin x = \ln \sin y + \ln c_1$

8. $(mz - ny)p + (nx - lz)q = ly - mx$

9. $x^2 \dfrac{\partial z}{\partial x} + y^2 \dfrac{\partial z}{\partial y} = (x + y)z$

10. $p + q = \sin x$

11. $(y + z) \dfrac{\partial z}{\partial x} + (x + z) \dfrac{\partial z}{\partial y} = x + y$

Answers

1. $\phi(x^2 + z^2, y) = 0$
2. $z = e^{y a} \phi(x - y)$
3. $\phi\left(\dfrac{1}{x} + y, \dfrac{1}{x} - \dfrac{1}{z}\right) = 0$
4. $\phi\left(\dfrac{1}{x} - \dfrac{1}{z}, \dfrac{1}{x} - \dfrac{1}{y}\right) = 0$
5. $\phi(x^3 - y^3, x^2 - z^2) = 0$
6. $\phi(x^2 + y^2 + 2z, xy) = 0$
7. $\phi(xyz, x + y + z) = 0.$
8. $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$
9. $\phi\left(\dfrac{xy}{2}, \dfrac{x - y}{z}\right) = 0$
10. $z + \cos x = \phi(y - x)$
11. $(x + y + z)(x - y)^2 = \phi\left(\dfrac{x - y}{x - z}\right)$

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UNIT 5

Fourier Series

⌘ Course Contents

- Periodic functions and fundamental period of periodic functions
- Odd and even functions with their properties
- Trigonometric series
- Fourie's series in an interval of period 2π (arbitrary range is not required)

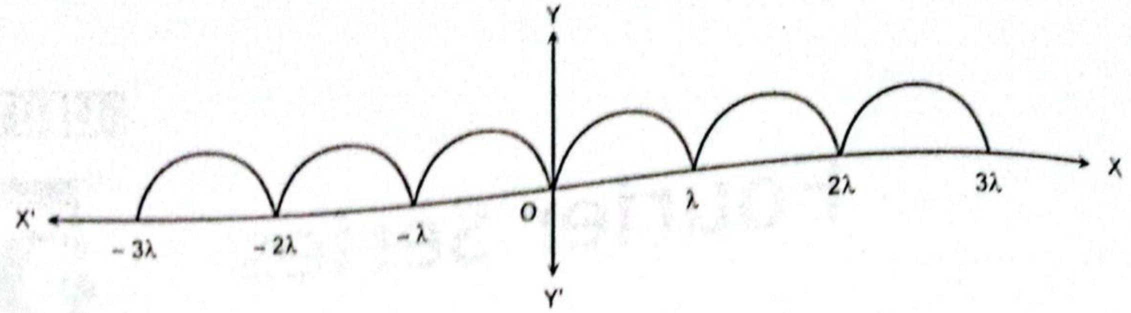
5.1 Introduction

Fourier series are the basic tool to represent the periodic functions in terms of sine and cosine series. Many functions having discontinuities but periodic can be expanded in series of sine and cosine known as Fourier series. It was first considered by French mathematician Fourier in 19^{th} century while solving a heat equation. Fourier series have wide range of applications in different branches of mathematics, engineering as well as other field of science and technology. More precisely, they are used to solve the problems that involve ODEs and PDEs. If the engineering problems are periodic in nature then they can be solved by a process of Fourier series.

Before defining Fourier series, we recall the concepts of periodic functions as well as odd and even functions. These concepts have been introduced and studied in Unit 3 of this book. We briefly recall them.

5.2 Periodic Functions and Fundamental Period

Let $f(x)$ be a function defined on the domain D. Then, $f(x)$ is said to be periodic if $f(x + \lambda) = f(x)$ for all x in D where λ is a positive constant and the smallest such number λ is called the **fundamental period** of the function. From the definition, we can say that the graph of a periodic function will be same on every interval with length equal to the period of the function. Thus, if we know the looks of the graph in an interval of length λ (period), then we can obtain the entire graph as shown in the following figure.



Trigonometric functions $\sin x$ and $\cos x$ are periodic with period 2π and $\tan x$ and $\cot x$ are periodic with period π.

By definition of periodicity, we have

i. Period of $f(x) = \sin(ax)$ is $\frac{2\pi}{|a|}$.

ii. Period of $f(x) = \cos(ax)$ is $\frac{2\pi}{|a|}$.

iii. Period of $f(x) = \tan(ax)$ is $\frac{\pi}{|a|}$.

iv. Period of $f(x) = \cot(ax)$ is $\frac{\pi}{|a|}$.

Example 1. Show that the function $f(x) = \sin 5x$ is periodic and find its fundamental period.

Solution

Here,

$$f(x) = \sin 5x$$

If λ be the period of $f(x)$, then

$$f(x + \lambda) = f(x)$$

or, $\sin 5(x + \lambda) = \sin 5x$

or, $\sin 5(x + \lambda) = \sin(2\pi + 5x)$

or, $5(x + \lambda) = 2\pi + 5x$

or, $5\lambda = 2\pi$

$\therefore \quad \lambda = \frac{2\pi}{5}$

This shows that $f(x)$ is a periodic function with fundamental period $\frac{2\pi}{5}$.

Properties of Periodic Functions

Let $f(x)$ be a periodic function of period 2π and a, b and constants. Then we have

1. $\int_{a}^{b} f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx$

2. $\int_{-\pi}^{\pi} f(x)\,dx = \int_{a}^{a+2\pi} f(x)\,dx$

3. $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} f(a+x)\,dx$

Proof:

1. Since $f(x)$ is periodic function with period 2π, $f(x+2\pi) = f(x)$

Let I $= \int_a^b f(x)\,dx$

$= \int_a^b f(x+2\pi)\,dx$

Put $x + 2\pi = t$

$dx = dt$

When $x = a$, $t = a + 2\pi$

When $x = b$, $t = b + 2\pi$

Then the integral becomes

$$I = \int_{t=a+2\pi}^{t=b+2\pi} f(t)\,dt = \int_{x=a+2\pi}^{x=b+2\pi} f(x)\,dx$$

$$= \int_{a+2\pi}^{b+2\pi} f(x)\,dx$$

Hence, $\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx.$

We can prove (2) and (3) in the similar way.

Also, we have the following properties of periodic functions.

i. Let f and g be any two periodic functions with a common period λ. Then the product fg and any linear combination $c_1f + c_2g$, are also periodic with the same period λ provided that c_1 and c_2 are constants.

ii. A constant function has any positive number as period but it has no fundamental period.

5.3 Odd and Even Functions with Their Properties

If a function f satisfies $f(-x) = f(x)$ for all x in its domain, then the function f is said to be an even function. Similarly, a function f is said to be odd function if $f(-x) = -f(x)$ for all x in its domain.

We now give some examples

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4$. Then, this function is even as $f(-x) = (-x)^4 = x^4 = f(x)$.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Then, this is an odd function as $f(-x) = (-x)^3 = -x^3 = -f(x)$.

We note that a function could be neither odd nor even.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + \sin x$. Then, $f(-x) = (-x)^2 + \sin(-x) = x^2 - \sin x$ which is neither $f(x)$ nor $-f(x)$.

So, the function $f(x) = x^2 + \sin x$ is neither odd nor even.

We now recall some important algebraic properties of even and odd functions.

a. Sum of even functions is even.

b. Sum of odd functions is odd.

c. Product of two odd functions is even.

d. Product of even and odd functions is odd.

e. Product of even functions is even.

f. Sum of even and odd function is neither odd nor even.

Example 2. Examine whether the function $f(x) = x \sin x + \cos x$ is even or odd or neither.

Solution

Given,

$$f(x) = x \sin x + \cos x$$

Now,

$$\begin{aligned} f(-x) &= (-x) \sin(-x) + \cos(-x) \\ &= (-x) \cdot (-\sin x) + \cos x \\ &= x \sin x + \cos x \\ &= f(x) \end{aligned}$$

Hence, the function $f(x)$ is even.

Properties of Even and Odd Functions

1. If $f(x)$ is even function then $\int_{-a}^{a} f(x)\, dx = 2 \int_{0}^{a} f(x)\, dx$

2. If $f(x)$ is odd function then $\int_{-a}^{a} f(x)\, dx = 0$

3. The zero function is the only function which is both even and odd.

4. The derivative of an even function is odd and the derivative of an odd function is even.

Proof:

1. Since $f(x)$ is even, $f(x) = f(-x)$.

$$\begin{aligned} \int_{-a}^{a} f(x)\, dx &= \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx \\ &= -\int_{0}^{-a} f(x)\, dx + \int_{0}^{a} f(x)\, dx \qquad \ldots\text{(i)} \end{aligned}$$

$$\begin{aligned} \text{Let } I &= -\int_{0}^{-a} f(x)\, dx \\ &= -\int_{0}^{-a} f(-x)\, dx \qquad [\because f(x) = f(-x)] \end{aligned}$$

Put $-x = t$. Then $-dx = dt$

When $x = 0, t = 0$

When $x = -a, -(-a) = t \Rightarrow t = a.$

Then,

$$I = -\int_0^{-a} f(-x)\,dx$$

$$= \int_0^{-a} f(-x)\,(-dx)$$

$$= \int_0^{a} f(t)\,dt$$

$$= \int_0^{a} f(x)\,dx$$

Now, from (i)

$$\int_{-a}^{a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx$$

$$= 2\int_0^{a} f(x)\,dx$$

2. Let $I = \int_{-a}^{a} f(x)\,dx$

Put $x = -t$. Then,

$dx = -dt$

When $x = -a$,

$-a = -t \Rightarrow t = a$

When $x = a$,

$a = -t \Rightarrow t = -a$

The integral becomes

$$I = \int_a^{-a} f(-t)\,(-dt)$$

$$= \int_a^{-a} -f(t)\,(-dt) \qquad [\because f \text{ is odd function}]$$

$$= \int_a^{-a} f(t)\,dt$$

$$= -\int_{-a}^{a} f(t)\, dt \qquad \text{[Property of definite integral]}$$

$$= -\int_{-a}^{a} f(x)\, dx$$

$$= -I$$

or, $I + I = 0$

or, $2I = 0$

$\therefore\ I = 0$

$$\therefore \int_{-a}^{a} f(x)\, dx = 0$$

The proof of (3) and (4) are left as exercise.

5.4 Orthogonal and Orthonormal Sets of Functions

The set of functions $f_1(x), f_2(x)$ ... defined in the interval $[a, b]$ is said to be orthogonal set of functions if

$$\int_{a}^{b} f_m(x) f_n(x)\, dx = 0 \text{ if } m \neq n.$$

The set of functions $f_1(x), f_2(x)$ ... defined in the interval $[a, b]$ is said to be orthonormal set of functions if

$$\int_{a}^{b} f_m(x) f_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}.$$

The trigonometric functions of sin mx and cos nx for $m, n = 1, 2, 3, \ldots$ form a mutually orthogonal set of functions on the interval $[-\pi, \pi]$ or $[0, 2\pi]$ or any other intervals of length 2π because they are 2π periodic functions. They satisfy the following orthogonality relations.

1. $$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$$

2. $$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$$

3. $$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0 \text{ for all integers } m \text{ and } n.$$

By integration, we can obtain the above results. For example, to show the result in (1), we convert the multiplication in terms of sum of two trigonometric functions as follows:

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} 2\cos mx \cos nx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left\{\cos\left(\frac{mx+nx}{2}\right) + \cos\left(\frac{mx-nx}{2}\right)\right\} dx$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{m+n}{2}\right)x}{\left(\frac{m+n}{2}\right)} + \frac{\sin\left(\frac{m-n}{2}\right)x}{\left(\frac{m-n}{2}\right)}\right]_{-\pi}^{\pi}$$

$$= 0 \text{ as } m \neq n.$$

The remaining statements can be obtained in the similar ways.

5.5 Trigonometric Series

A series of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, a_0, a_n and b_n are any real constants, is called trigonometric series.

If $a_n = 0$ then the above series is a pure sine series and if $b_n = 0$ then the above series is a pure cosine series.

5.6 Fourier Series in an Interval of Period 2π

If the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(i)$$

converges to a function $f(x)$ in an interval then the series (i) is called Fourier series and the real constants a_0, a_n, b_n are called Fourier coefficients. Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

In other words, we can say that a Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines.

A trigonometric series may or may not be a Fourier series. If a trigonometric series converges to a function in the interval then it becomes a Fourier series.

$$= -\int_{-a}^{a} f(t)\,dt \qquad \text{[Property of definite integral]}$$

$$= -\int_{-a}^{a} f(x)\,dx$$

$$= -I$$

or, $I + I = 0$

or, $2I = 0$

$\therefore\ I = 0$

$$\therefore \int_{-a}^{a} f(x)\,dx = 0$$

The proof of (3) and (4) are left as exercise.

5.4 Orthogonal and Orthonormal Sets of Functions

The set of functions $f_1(x), f_2(x)$ ... defined in the interval $[a, b]$ is said to be orthogonal set of functions if

$$\int_a^b f_m(x) f_n(x)\,dx = 0 \text{ if } m \neq n.$$

The set of functions $f_1(x), f_2(x)$ ... defined in the interval $[a, b]$ is said to be orthonormal set of functions if

$$\int_a^b f_m(x) f_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}.$$

The trigonometric functions of sin mx and cos nx for $m, n = 1, 2, 3, \ldots$ form a mutually orthogonal set of functions on the interval $[-\pi, \pi]$ or $[0, 2\pi]$ or any other intervals of length 2π because they are 2π periodic functions. They satisfy the following orthogonality relations.

1. $$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$$

2. $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$$

3. $$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0 \text{ for all integers } m \text{ and } n.$$

By integration, we can obtain the above results. For example, to show the result in (1), we convert the multiplication in terms of sum of two trigonometric functions as follows:

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} 2 \cos mx \cos nx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left\{\cos\left(\frac{mx+nx}{2}\right) + \cos\left(\frac{mx-nx}{2}\right)\right\} dx$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{m+n}{2}\right)x}{\left(\frac{m+n}{2}\right)} + \frac{\sin\left(\frac{m-n}{2}\right)x}{\left(\frac{m-n}{2}\right)}\right]_{-\pi}^{\pi}$$

$= 0$ as $m \neq n$.

The remaining statements can be obtained in the similar ways.

5.5 Trigonometric Series

A series of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, a_0, a_n and b_n are any real constants, is called trigonometric series.

If $a_n = 0$ then the above series is a pure sine series and if $b_n = 0$ then the above series is a pure cosine series.

5.6 Fourier Series in an Interval of Period 2π

If the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(i)$$

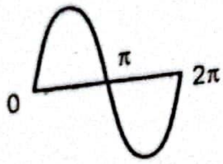
converges to a function $f(x)$ in an interval then the series (i) is called Fourier series and the real constants a_0, a_n, b_n are called Fourier coefficients. Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

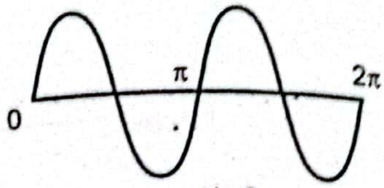
In other words, we can say that a Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines.

A trigonometric series may or may not be a Fourier series. If a trigonometric series converges to a function in the interval then it becomes a Fourier series.

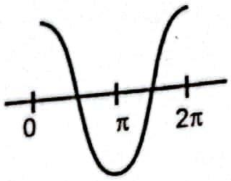
The periodic sine and cosines functions $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, etc. can be shown as follows:



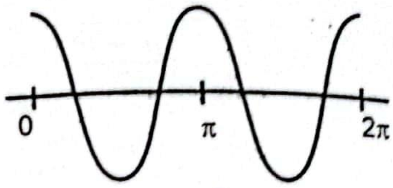
sin x



sin 2x



cos x



cos 2x

The Fourier series of periodic function $f(x)$ on $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the Fourier coefficients a_0, a_n and b_n are given by Euler's Formulae given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx, n = 1, 2, 3, \ldots$$

Determination of Fourier Coefficients

The Fourier series of a periodic function $f(x)$ in the interval $[-\pi, \pi]$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots \text{(i)}$$

We find the Fourier coefficients a_0, a_n and b_n. To find a_0, integrating (i) on both sides term by term from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} a_0\, dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx\, dx + b_n \int_{-\pi}^{\pi} \sin nx\, dx\right]$$

or, $$\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2}\cdot a_0 \cdot [\,x\,]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\left[a_n\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} + b_n\left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi}\right]$$

or, $$\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2}\cdot a_0 \cdot (\pi + \pi) + \sum_{n=1}^{\infty}\left[a_n\left(\frac{\sin n\pi}{n} + \frac{\sin n\pi}{n}\right) - b_n\left(\frac{\cos nx}{n} - \frac{\cos nx}{n}\right)\right]$$

or, $\int_{-\pi}^{\pi} f(x)\, dx = a_0\pi + \sum_{n=1}^{\infty} [a_n \times 0 - b_n \times 0]$ $[\because \sin n\pi = 0]$

or, $a_0\pi = \int_{-\pi}^{\pi} f(x)\, dx$

$\therefore$ $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

To find a_n, we first multiply (i) by cos nx and integrate both sides term by term from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{2}\cdot a_0 \cdot \int_{-\pi}^{\pi}\cos nx\, dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi}\cos nx \cos nx\, dx + b_n \int_{-\pi}^{\pi}\sin nx \cos nx\, dx\right]$$

or, $\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{2}\cdot a_0 \cdot \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} + a_n\pi + 0$ [From 5.2]

or, $\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0 + a_n\pi + 0$

$\therefore$ $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

To find b_n, multiply (i) by sin nx and integrate both sides term by term from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin nx\, dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi}\sin nx \cos nx\, dx + b_n \int_{-\pi}^{\pi}\sin nx \sin nx\, dx\right]$$

or, $\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{a_0}{2} \times 0 + a_n \times 0 + b_n \times \pi$ [From 5.2]

$\therefore$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

NOTE The Fourier series of $f(x)$ in the interval $[0, 2\pi]$ is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ and the Fourier coefficients are given by

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx,\ n = 1, 2, 3, \ldots$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx,\ n = 1, 2, 3, \ldots$

Fourier Series for Even and Odd Functions

Let the Fourier series of $f(x)$ in $[-\pi, \pi]$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots\text{(i)}$$

Case I

If $f(x)$ is an even function then $f(x)$ cos nx is even and $f(x)$ sin nx is odd. The Fourier coefficients in this case can be calculated as follows:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\cdot 2\int_{0}^{\pi} f(x)\,dx \qquad \text{[Using property of even function]}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot 2\cdot\int_{0}^{\pi} f(x)\cos nx\,dx \qquad \text{[Using property of even function]}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= 0 \qquad \text{[Using property of odd function]}$$

In this case, equation (i) reduces to $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

This is called Fourier cosine series.

Case II

Let $f(x)$ be an odd function. Then $f(x)$ cos nx is odd and $f(x)$ sin nx is even. The Fourier coefficients in this case can be calculated as follows:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= 0 \qquad \text{[Using property of odd function]}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= 0 \qquad \text{[Using property of odd function]}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\cdot 2\int_{0}^{\pi} f(x)\sin nx\,dx \qquad \text{[Using property of even function]}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

In this case, equation (i) reduces to $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

This is called Fourier sine series.

Illustrative Examples

Example 1. Find the Fourier series representing $f(x) = \pi - x$ for $0 < x < 2\pi$.

Solution

Given,

$$f(x) = \pi - x$$

The Fourier coefficients are calculated as follows:

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\pi\cdot 2\pi - \frac{(2\pi)^2}{2} - 0\right]$$

$$= \frac{1}{\pi}\left[2\pi^2 - \frac{4\pi^2}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\cdot\frac{\sin nx}{n} - (0-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$\left[\because \int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right.$
where dash is for derivative
$\left. \text{and suffices for antiderivative}\right]$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{(\pi - 2\pi)\frac{\sin 2n\pi}{n} - \left(\frac{\cos 2n\pi}{n^2}\right)\right\} - \left\{(\pi + 0)\frac{\sin 0}{n} - \frac{\cos 0}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[0 - \frac{1}{n^2} - 0 + \frac{1}{n^2}\right] \qquad [\because \sin 2n\pi = 0, \cos 2n\pi = 1]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\cdot\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(x - \pi)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{(2\pi - \pi)\frac{\cos 2n\pi}{n} - \frac{\sin 2n\pi}{n^2}\right\} - \left\{(0 - \pi)\frac{\cos 0}{n} - \frac{\sin 0}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\pi\cdot\frac{1}{n} - 0 + \pi\cdot\frac{1}{n} - 0\right]$$

$$= \frac{1}{\pi}\cdot\frac{2\pi}{n}$$

$$= \frac{2}{n}$$

The required Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

i.e. $$\pi - x = 0 + \sum_{n=1}^{\infty}\left[0\cdot\cos nx + \frac{2}{n}\sin nx\right]$$

$$= 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$= 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right)$$

Example 2. Expand $f(x) = x^2$ for $-\pi < x < \pi$ in a Fourier series.

Solution

Let the required Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\frac{2\pi^3}{3} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\left[\because \int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\pi^2\frac{\sin n\pi}{n} + 2\pi\frac{\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right\} - \left\{(-\pi)^2\frac{\sin(-n\pi)}{n} + 2(-\pi)\frac{\cos(-n\pi)}{n^2} - \frac{2\sin(-n\pi)}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left\{0 + 2\pi\cdot\frac{(-1)^n}{n^2} - 0\right\} - \left\{0 - 2\pi\cdot\frac{(-1)^n}{n^2} + 0\right\}\right]$$

$$[\because \sin 2n\pi = 0,\ \cos 2n\pi = (-1)^n]$$

$$= \frac{1}{\pi}\left[2\pi\cdot\frac{(-1)^n}{n^2} + 2\pi\cdot\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\cdot 4\pi\frac{(-1)^n}{n^2}$$

$$= (-1)^n\cdot\frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$$

$$= 0 \qquad \text{[Using property of odd function]}$$

Putting the values of a_0, a_n and b_n in (i), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

i.e. $$x^2 = \frac{1}{2}\cdot\frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\left[(-1)^n\cdot\frac{4}{n^2}\cos nx + 0\right]$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos nx\right]$$

Example 3. Find the Fourier series of given function on the given interval

$$f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \end{cases}$$

Solution

Given,

$$f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \end{cases}$$

The Fourier coefficients are calculated as follows:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(0)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(1)\,dx$$

$$= 0 + \frac{1}{\pi}[x]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}(2\pi - \pi)$$

$$= 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (0)\cdot\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (1)\cdot\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin 2\pi n}{n} - \frac{\sin n\pi}{n}\right] \qquad [\because \sin 2n\pi = 0, \sin n\pi = 0]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (0)\cdot\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (1)\cdot\sin nx\,dx$$

$$= 0 + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{\pi}^{2\pi}$$

$$= -\frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi}^{2\pi}$$

$$= -\frac{1}{n\pi}(\cos 2n\pi - \cos n\pi)$$

$$= -\frac{1}{n\pi}[1-(-1)^n] \qquad [\because \cos 2n\pi = 1, \cos n\pi = (-1)^n]$$

Putting the values of a_0, a_n and b_n in

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx), \text{ we get}$$

$$= \frac{1}{2}(1) + \sum_{n=1}^{\infty}\left[0\cdot\cos nx - \frac{1}{n\pi}[1-(-1)^n]\sin nx\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin nx$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\frac{2}{1}\sin x + 0 + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \dots\right]$$

$$= \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right]$$

Example 4. Find the Fourier series of the function $f(x) = |x|$ in the interval $(-\pi, \pi)$.

Solution

Given,

$$f(x) = |x|$$

We have,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[-\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}[x^2]_{-\pi}^{0} + \frac{1}{2\pi}[x^2]_0^{\pi}$$

$$= -\frac{1}{2\pi}[0-(-\pi)^2] + \frac{1}{2\pi}[\pi^2 - 0]$$

$$= \frac{\pi^2}{2\pi} + \frac{\pi^2}{2\pi}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\left(0+\frac{1}{n^2}\right)-\left\{-\pi\frac{\sin(-n\pi)}{n}+\frac{\cos(-n\pi)}{n^2}\right\}\right]+ \frac{1}{\pi}\left[\left\{\pi\frac{\sin n\pi}{n}+\frac{\cos n\pi}{n^2}\right\}-\left\{0+\frac{\cos 0}{n^2}\right\}\right]$$

$$= -\frac{1}{\pi}\left[\frac{1}{n^2}-0-\frac{\cos n\pi}{n^2}\right]+\frac{1}{\pi}\left[0+\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2}+\frac{\cos n\pi}{n^2}+\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\cos n\pi}{n^2}-\frac{2}{n^2}\right]$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1)$$

$$= \frac{2}{\pi n^2}((-1)^n - 1) \quad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx\, dx$$

$$= 0 \qquad \text{[Using property of odd function]}$$

The required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi}[(-1)^n - 1]\cos nx + 0\right]$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[-\frac{2}{1^2}\cos x + 0 - \frac{2}{3^2}\cos 3x + 0 - \frac{2}{5^2}\cos 5x + \dots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \dots\right]$$

Example 5. Test whether the function

$$f(x) = \begin{cases} -2x & \text{for } -\pi < x < 0 \\ 2x & \text{for } 0 < x < \pi \end{cases}$$

is odd or even and hence obtain the corresponding Fourier series.

Solution

Given,

$$f(x) = \begin{cases} -2x \text{ for } -\pi < x < 0 \\ 2x \text{ for } 0 < x < \pi \end{cases}$$

Replacing x by $-x$, we get

$$f(-x) = \begin{cases} -2(-x) \text{ for } -\pi < -x < 0 \\ 2(-x) \text{ for } 0 < -x < \pi \end{cases}$$

$$= \begin{cases} 2x \text{ for } \pi > x > 0 \\ -2x \text{ for } 0 > x > -\pi \end{cases}$$

$$= \begin{cases} 2x \text{ for } 0 < x < \pi \\ -2x \text{ for } -\pi < x < 0 \end{cases}$$

$$= \begin{cases} -2x \text{ for } -\pi < x < 0 \\ 2x \text{ for } 0 < x < \pi \end{cases}$$

$$= f(x)$$

Hence $f(x)$ is an even function.

Next, let the Fourier series of the function in the interval $(-\pi, \pi)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(i)$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\cdot 2\cdot\int_0^{\pi} f(x)\,dx \qquad [\because f(x) \text{ is even function}]$$

$$= \frac{2}{\pi}\int_0^{\pi} 2x\,dx = \frac{2}{\pi}\left[\frac{2x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\,[\pi^2 - 0]$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot 2\cdot\int_0^{\pi} \cos nx\,dx \qquad [\because f(x) \text{ is even function}]$$

$$= \frac{2}{\pi}\int_0^{\pi} 2x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[2x\cdot\left(\frac{\sin nx}{n}\right) - 2\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{4}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{4}{\pi}\left[\left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{4}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad [\because \sin n\pi = 0, \cos n\pi = (-1)^n]$$

$$= \frac{4}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin x \, dx$$

$$= 0 \qquad [\because f(x) \sin nx \text{ is odd function}]$$

Putting the values of a_0, a_n and b_n in equation (i), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2}(2\pi) + \sum_{n=1}^{\infty}\left[\frac{4}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx + 0\right]$$

$$= \pi + \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx\right]$$

Example 6. Find the Fourier series of the function

$$f(x) = \begin{cases} 0 \text{ for } -\pi < x < -\frac{\pi}{2} \\ 1 \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 \text{ for } \frac{\pi}{2} < x < \pi \end{cases}$$

Solution

Let the Fourier series be ...(i)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} f(x) \, dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x) \, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} f(x) \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} (0) \, dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (1) \, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (0) \, dx$$

$$= 0 + \frac{1}{\pi}[x]_{-\pi/2}^{\pi/2} + 0$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= \frac{1}{\pi} \cdot \pi = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} (0)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (1)\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (0)\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi/2}^{\pi/2} + 0$$

$$= \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{n\pi}\left[\frac{\sin n\pi}{2} + \frac{\sin n\pi}{2}\right]$$

$$= \frac{1}{n\pi}\cdot 2\sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} (0)\sin nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (0)\sin nx\,dx$$

$$= 0 + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi/2}^{\pi/2} + 0$$

$$= -\frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\left(-\frac{n\pi}{2}\right)\right]$$

$$= -\frac{1}{n\pi}\left[\frac{\cos n\pi}{2} - \frac{\cos n\pi}{2}\right] \qquad [\because \cos(-x) = \cos x]$$

$$= 0$$

Putting the values of a_0, a_n and b_n in (i), we get

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos nx + 0\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\sin\left(\frac{n\pi}{2}\right)\cos nx$$

5.7 Fourier Series of Function Having Arbitrary Range

The functions that we have discussed had period 2π. That is, we have studied the Fourier series of a function in the interval $[-\pi, \pi]$ or $[0, 2\pi]$. But many engineering problems related with periodic functions may have other periods. The Fourier series of the function $f(x)$ of period 2L is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \frac{n\pi x}{L}\right)$$

where the Fourier coefficients a_0, a_n and b_n of $f(x)$ are given by Euler's formulae

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,\ n = 1, 2, 3 \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx,\ n = 1, 2, 3 \ldots$$

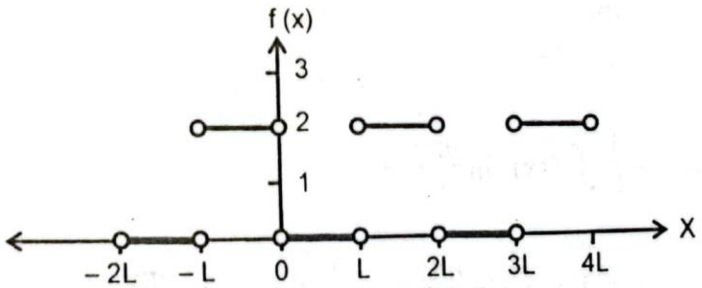
Example 7. Sketch the graph of the function

$$f(x) = \begin{cases} 2 & \text{for } -L < x < 0 \\ 0 & \text{for } \ 0 < x < L \end{cases}$$

and $f(x + 2L) = f(x)$ for three periods in the interval $(-L, L)$. Also find the Fourier series of $f(x)$.

Solution

The graph of given function $f(x)$ for three periods in the interval $(-L, L)$ is as follows:



Let the required Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad \ldots \text{(i)}$$

The Fourier coefficients are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} f(x)\,dx + \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} (2)\, dx + \frac{1}{L}\int_{0}^{L} (0)\, dx$$

$$= \frac{1}{L}[\,2x\,]_{-L}^{0} + 0$$

$$= \frac{1}{L}(0 + 2L)$$

$$= 2$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\, dx + \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{L}\int_{-L}^{0} (2)\cos\frac{n\pi x}{L}\, dx + \frac{1}{L}\int_{0}^{L} (0)\cos\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_{-L}^{0} + 0$$

$$= \frac{2}{L}\cdot\frac{L}{n\pi}\left[\sin 0 - \sin\left(-\frac{n\pi L}{L}\right)\right]$$

$$= \frac{2}{n\pi}\sin n\pi$$

$$= \frac{2}{n\pi}\cdot 0$$

$$= 0 \qquad [\because \sin n\pi = 0]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\, dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{L}\int_{-L}^{0} (2)\sin\frac{n\pi x}{L}\, dx + \frac{1}{L}\int_{0}^{L} (0)\sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\left[-\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_{-L}^{0} + 0$$

$$= -\frac{2}{L}\cdot\frac{L}{n\pi}\left[\cos 0 - \cos\left(-\frac{n\pi L}{L}\right)\right]$$

$$= -\frac{2}{n\pi}[1 - \cos n\pi]$$

$$= -\frac{2}{n\pi}[1 - (-1)^n] \qquad [\because \cos n\pi = (-1)^n]$$

Putting the values of a_0, a_n and b_n in equation (i), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \frac{n\pi x}{L}\right)$$

$$= \frac{2}{2} + \sum_{n=1}^{\infty}\left[0 - \frac{2}{n\pi}[1 - (-1)^n]\sin\frac{n\pi x}{L}\right]$$

$$= 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}[1 - (-1)^n]\sin\frac{n\pi x}{L}\right]$$

$$= 1 - \frac{2}{\pi}\left[\frac{2}{1}\sin\frac{\pi x}{L} + 0 + \frac{2}{3}\sin\frac{3\pi x}{L} + \frac{2}{5}\sin\frac{5\pi x}{L} + \dots\right]$$

$$= 1 - \frac{4}{\pi}\left[\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \dots\right]$$

For the convergence of Fourier series, we have an important theorem 'Fourier Convergence Theorem'. We first give the definition of piecewise continuous function.

A function $f(x)$ is called piecewise continuous on the interval $[a, b]$ if

i. f has at most finite points of discontinuity in (a, b)

ii. $f(x_0^-)$ exists if $a < x_0 \le b$.

iii. $f(x_0^+)$ exists if $a \le x_0 < b$.

Fourier Convergence Theorem

Let the function f and its derivative f' be piecewise continuous on the interval $[-L, L]$. Also, let f be defined outside the interval $[-L, L]$ so that it is periodic with period 2L. Then f has a Fourier seires

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

The Fourier series converges to $f(x)$ at all points where f is continuous and to the mean of left and right hand limits at all points where f is discontinuous.

Thus, Fourier series can be used to represent discontinuous functions where the derivatives of all orders may not exist. In such situation, Taylor series cannot be used. The Fourier expansion gives all modes of oscillations.

Half Range Fourier Series

If the function is even then its Fourier series has only consine terms and if the function is odd then the Fourier series has only sine terms. These are called half range Fourier series.

If a function $f(x)$ with period 2L is defined on $[0, L]$ then it can be extended to the next half $[-L, 0]$ in order to obtain either an odd or even function in the internal $[-L, L]$.

Now, we define half range Fourier cosine series and Fourier sine series.

1. The Fourier series of an even function of period 2L, known as Fourier cosine series, is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

with Fourier coefficients

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\ n = 1, 2, 3 \ldots$$

2. The Fourier series of an odd function of period 2L, known as Fourier sine series, is defined by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

with Fourier coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,\ n = 1, 2, 3 \ldots$$

Example 8. Obtain the half range Fourier cosine series for $f(x) = \frac{\pi}{2} - x$ in the interval $[0, \pi]$ and hence show that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Solution

Here, $f(x) = \frac{\pi}{2} - x$ and $L = \pi$

We know that the half range Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(i)$$

The Fourier coefficients are

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)dx$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}\cdot\pi - \frac{\pi^2}{2}\right) - 0\right]$$

$$= 0$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^\pi \left(\frac{\pi}{2} - x\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - \frac{\cos nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left\{\left(\frac{\pi}{2} - \pi\right)\frac{\sin n\pi}{n} - \frac{\cos n\pi}{n^2}\right\} - \left(0 - \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[0 - \frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right] \qquad [\because \cos n\pi = (-1)^n]$$

Putting the values of a_0 and a_n in equation (i), we get

$$f(x) = 0 + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

or, $f(x) = \frac{2}{\pi}\left[\frac{2}{1^2}\cos x + 0 + \frac{2}{3^2}\cos 3x + 0 + \frac{2}{5^2}\cos 5x + \dots\right]$

$$\therefore \quad \frac{\pi}{2} - x = \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right] \qquad \dots (ii)$$

Hence (ii) is the required half range Fourier cosines series.

Putting $x = 0$ in (ii), we get

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\cos 0 + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \dots\right]$$

or, $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

$$\therefore \quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$