New Syllabus - Model SET

Engineering Mathematics-III

(for Diploma II Yrs. I Part)

Third Semester

Diploma in Engineering

By

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S.No	Exam Year, Month
1.	2021 New Model Set

Engg. Mathematics-III (Engg. All) 3rd Sem

(New Model Set Solution)

Compile by Arjun Chy

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1.(a) Find the derivative of $sin^{-1}(3x-4)$.

Solution:-

Let $y = sin^{-1}(3x-4)$.

Differentiating both Side w.r.t 'x'

$$\frac{dy}{dx} = \frac{d\sin^{-1}(3x-4)}{d(3x-4)} \times \frac{d(3x-4)}{dx}$$

$$= \frac{1}{\sqrt{1-(3x-4)^2}} \times 3$$

$$= \frac{3}{\sqrt{1-(9x^2-24x+16)}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{24x-9x^2-15)}}$$

b) Find the slope of the curve $y = 2x^2 - x$ at $(1, 0)$.

Solution:-

$y = 2x^2 - x$ at $(1, 0)$.

Differentiating both Side w.r.t 'x'

$$\frac{dy}{dx} = \frac{d(2x^2 - x)}{dx}$$

$$\frac{dy}{dx} = 2 . \frac{dx^2}{dx} - \frac{dx}{dx}$$

$$\frac{dy}{dx} = 2 \ . \ 2x - 1$$

$$\frac{dy}{dx} = 4x - 1$$

At $(1, 0)$, $\frac{dy}{dx} = 4 \times 1 - 0 = 4 - 1$

$$\frac{dy}{dx} = 3$$

Hence,

Slope of curve = 3 at $(1, 0)$.

2.(a) Find the points of stationary points $f(x) = x^3 - 3x^2 + 9$.

Solution:-

$$f(x) = x^3 - 3x^2 + 9$$

$$Let\ y = f(x) = x^3 - 3x^2 + 9$$

$$For\ Stationary\ points, \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{d(x^3 - 3x^2 + 9)}{dx}$$

$$= \frac{dx^3}{dx} - \frac{d(3x^2)}{dx} + \frac{d(9)}{dx}$$

$$\frac{dy}{dx} = 3x^2 - 6x + 0$$

$$For\ Stationary\ points,$$

$$\frac{dy}{dx} = 0$$

$$3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

$$x = 0, x = 2$$

Hence,

$$x = 0, x = 2\ \ are\ Stationary\ points.$$

(b) State L' Hospitals's Theorem and use it to evaluate

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

➢ L' Hospital's Rule

$If\ f(x)\ and\ g(x) are\ two\ functions\ such\ that\ their\ derivatives$

$f'(x)\ and\ g'(x)\ are\ continuous\ at\ x = a\ and$

$f(a) = g(a) = 0, then,$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}\ provided\ g'(x) \neq 0.$$

Second Part Answer

➢ Solution:-

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} \left(\frac{0}{0}\right)$$

$$\lim_{x \to 4} \frac{2x}{1}$$

$$= 2 \times 4$$

$$= 8$$

3. (a) Find first order partial derivatives of

$$\boldsymbol{f(x,y) = ax^2 + 2hxy + by^2.}$$

Solution:-

$$f(x,y) = ax^2 + 2hxy + by^2.$$

Differentiating both Side w.r.t 'x' Partially,

$$f_x = \frac{\partial}{\partial x}(\,ax^2 + 2hxy + by^2)$$

$$= \frac{\partial}{\partial x}(ax^2) + \frac{\partial}{\partial x}(2hxy) + \frac{\partial}{\partial x}\,(by^2)$$

$$\boldsymbol{f_x = 2ax + 2hy}$$

Differentiating both Side w.r.t 'y' Partially,

$$f_y = \frac{\partial}{\partial y}\;ax^2 + 2hxy + by^2$$

$$= \frac{\partial(ax^2)}{\partial y} + \frac{\partial(2hxy)}{\partial y} + \frac{\partial(by^2)}{\partial y}$$

$$= 0 + 2xh + 2by$$

$$\boldsymbol{fy = 2by + 2hx}$$

(b) Find $\frac{\partial y}{\partial x}$ if $y = x^2 - 5x + 7$ and $x + 9rs + 2r^2s^2$.

Solution:-

$$y = x^2 - 5x + 7$$

$$\frac{\partial y}{\partial x} = \frac{\partial(x^2 - 5x + 7)}{\partial x}$$

$$= \frac{\partial x^2}{\partial x} - \frac{\partial(5x)}{\partial x} + \frac{\partial 7}{\partial x}$$

$$= 2x - 5 + 0$$

$$\therefore \boldsymbol{\frac{\partial y}{\partial x} = 2x - 5}$$

$$Let\ y = x + 9rs + 2r^2s^2$$

$$\frac{\partial y}{\partial x} = \frac{\partial(x + 9rs + 2r^2s^2)}{\partial x}$$

$$= \frac{\partial x}{\partial x} + \frac{\partial(9rs)}{\partial x} + \frac{\partial(2r^2s^2)}{\partial x}$$

$$= 1 + 0 + 0$$

$$\therefore \boldsymbol{\frac{\partial y}{\partial x} = 1}$$

4. (a) Test whether the given function is even, odd or neither where,

$$\boldsymbol{f(x) = \sqrt{1+x^2} - \sqrt{1-x^2}}$$

Solution:-

$$f(x) = \sqrt{1+x^2} - \sqrt{1-x^2}$$

For even function, $f(x) = f(-x)$

$$f(-x) = \sqrt{1+(-x)^2} - \sqrt{1-(-x)^2}$$

$$f(-x) = \sqrt{1+x} - \sqrt{1-x^2}$$

$$f(-x) = f(x)$$

Hence,

Given function is even function.

(b) Find the smallest period of $\boldsymbol{f(x) = \sin 2x}$.

Solution:-

Let, $f(x) = \sin 2x$.

If 'p' is a period of $f(x)$; then,

$$f(x+\mathrm{p}) = f(x)$$

or, $sin2(x+\mathrm{p}) = f(x)$

or, $sin2(x+\mathrm{p}) = \sin(2x+2\pi)$ **[∵The fundamental Period of $\boldsymbol{sin\ x}$ is $\boldsymbol{\pi}$.]**

or, $2(x+\mathrm{p}) = 2x+2\pi$

or, $2x+2\mathrm{p} = 2x+2\pi$

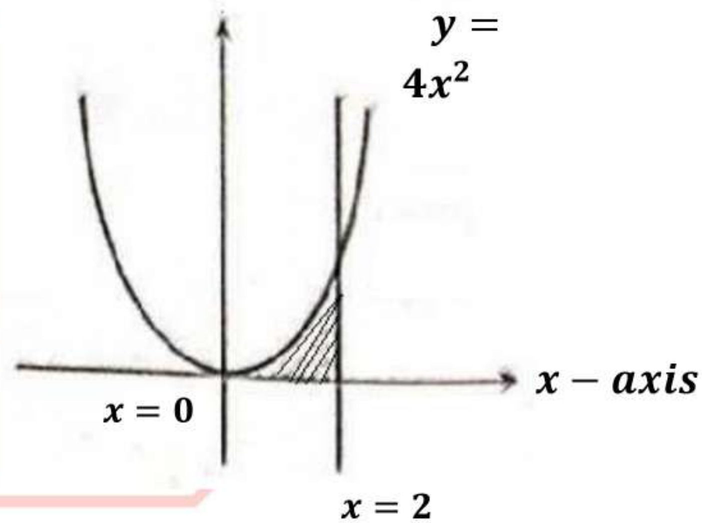
Or, $2\mathrm{p} = 2\pi$

Or, $\mathrm{p} = \pi$

Thus, the smallest period of $sin2x$ is π.

5.a)Find the area bounded by the curve $y = 4x^2, x - axis$ and the ordinates $x = 0, x = 2$.

Solution:-



$$y = 4x^2$$

$$x = 0 \; to \; x = 2$$

Area is given by,

$$Area = \int_{x_1}^{x_2} ydx$$

$$= \int_0^2 4x^2 dx$$

$$= 4\int_0^2 x^2 dx$$

$$= 4\left[\frac{x^3}{3}\right]_0^2$$

$$= 4\frac{(2)^3}{3}$$

$$= 4 \times \frac{8}{3}$$

$$\therefore \boldsymbol{Area} = \frac{\mathbf{32}}{\mathbf{3}} \boldsymbol{sq.units}.$$

(b) Determine the order and degree of the differential equation:

$$\boldsymbol{\frac{dy}{dx} = (x + y + 1)^2}$$

Solution:-

$$\frac{dy}{dx} = (x + y + 1)^2$$

Since, $highest\ derivative\ of\ y\ w.r.t\ 'x'\ is\ one\ i.e.,$

$\left(\frac{dy}{dx}\right)$.

So, order is 1.

And, The whole power raise to $\left(\frac{dy}{dx}\right)^1$ is one so,

Degree is one.

Order and degree of differential equation are 1 and 1.

Another Example :-

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = xy$$

Highest derivative of y w.r.t $'x'$ is $\frac{d^2y}{dx^2}$ $i.e., 2$ $so,$

Order is 2 and the power raise to [highest derivative of y w.r.t x is 1. $\left(\frac{d^2y}{dx^2}\right)^1$]

So, degree 1.

Note :- If $\left(\frac{d^2y}{dx^2}\right)^2$ given then degree 2 , order 2.

6. (a) Solve by separation of variable method of

$$(1 + cosx)dy = (1 - cosx)dx \; .$$

Solution:-

$$(1 + cosx)dy = (1 - cosx)dx$$

$$dy = \frac{(1-cosx)}{(1+cosx)}dx$$

$$\because \left\{\begin{array}{r} since\ 1 - cosx = 2\sin^2\left(\frac{x}{2}\right) \\ 1 + cosx = 2\cos^2\left(\frac{x}{2}\right) \end{array}\right\}$$

$$dy = \frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}\,dx$$

$$dy = tan^2\left(\frac{x}{2}\right)dx$$

$$dy = \left(sec^2\left(\frac{x}{2}\right) - 1\right)dx$$

$$\left[\because sec^2\left(\frac{x}{2}\right) - tan^2\left(\frac{x}{2}\right) = 1\right]$$

Integrating ; Both side,

$$\int dy = \int sec^2\left(\frac{x}{2}\right)dx - \int dx$$

$$y = \frac{tan^2\left(\frac{x}{2}\right)}{\frac{1}{2}} - x + c$$

$$\therefore \boldsymbol{y = 2tan\left(\frac{x}{2}\right) - x + c}.$$

(b) Text the exactness of $(x + y^2)\, dx - 2xy\, dy = 0$.

Solution:-

$$(x + y^2)\, dx - 2xy\, dy = 0$$

$$xdx + y^2 dx - 2xy\, dy = 0$$

Dividing both side by x^2

$$\frac{xdx - (2xy\, dy - y^2 dx)}{x^2} = \frac{0}{x^2}$$

$$\frac{xdx}{x^2} - \frac{(xy\,(y)^2 - y^2 dx)}{x^2} = 0$$

$$\left[\begin{array}{l} \because d(y^2) = 2ydy \\ \frac{Vdu - udV}{V^2} = d\left(\frac{u}{V}\right) \end{array}\right]$$

$$\frac{dx}{x} - d\left(\frac{y^2}{x}\right) = 0$$

Differentiating both side,

$$\int \frac{dx}{x} - \int d\left(\frac{y^2}{x}\right) = \int 0$$

$$\boldsymbol{ln\, x - \frac{y^2}{x} = C.} \qquad [\because \int dx = x]$$

7. (a) State Langrange's Linear Differential Equation with an example.

- **Lagrange's linear** differential equation is a type of linear second-order ordinary differential equation with variable coefficients. It has the form:-

$$\boldsymbol{y'' + P(x)y' + Q(x)y = 0}$$

Where y'' represents the second derivative of y with respect to x, $P(x)$ and $Q(x)$ are functions of x and y is the unknown function we trying to solve for.

(b) Define Orthogonality of two function with example.

- **Orthogonality** of two functions means that their inner product or dot product is zero. In other words, they have no overlap with respect to a given weight function.

8. Find the equation of tangent and normal to the curve $\boldsymbol{f(x) = x^2 + 3x + 1}$ at $\boldsymbol{(0, 1)}$.

Solution:-

$$\boldsymbol{Let\ f(x) = x^2 + 3x + 1 \text{ at } (0, 1).}$$

As, derivative gives the eq^n of tangent.

$$f'(x) = 2x + 3$$

Slope of tangent => $f'(x)\ \ at\ (0,1) = 2 \times 0 + 3 = 3$

Eq^n of tangent at $(0, 1)$

$$y - y_1 = f'(x) = (x - x_1)$$

$$y - 1 = 3\ (x - 0)$$

$$y - 1 = 3x$$

$\boldsymbol{y = 3x + 1}$ **is eq^n of tangent**

Since, Normal and tangent are perpendicular So,

Slope of Normal $\times$ Slope of tangent $= -1$

Slope of Normal $= \dfrac{-1}{f'(x)}$

$$\mathrm{S_T} = \frac{-1}{3}$$

Eq^n of Normal at (0, 1)

$$y - y_1 = \mathrm{S_T}(x - x_1)$$

$$y - 1 = \frac{-1}{3}\ (x - 0)$$

$$y - 1 = \frac{-x}{3}$$

$$3y - 3 = -x$$

$\boldsymbol{x + 3y - 3 = 0}$ **is Eq^{n} of Normal.**

9. Find the local maxima and local minima of the function $\boldsymbol{f(x) = 2x^3 - 15x^2 + 36x + 5}$. Also, find the point of inflection.

Solution:-

$$Let\ y = f(x) = 2x^3 - 15x^2 + 36x + 5.$$

$$f'(x) = 6x^2 - 30x + 36$$

$$f''(x) = 12x - 30$$

Where, $f'(x)$ is $\frac{dy}{dx}$

$f''(x)$ is $\frac{d^2y}{dx^2}$

For Stationary points, $f'(x) = 0$

$$6x^2 - 30x + 36 = 0$$

$$6\,(x^2 - 5x + 6) = 0$$

$$x^2 - 5x + 6 = 0$$
$$x^2 - 2x - 3x + 6 = 0$$
$$x(x-2) - 3(x-2) = 0$$
$$(x-3)(x-2) = 0$$

$x = 2,\ 3$ are Stationary Point.

At $x = 2$,

$$f''(x) = 12 \times 2 - 30 = -6$$

$i.e.,\ f''(x) < 0$ So, there is local maxima

Maximum Value at 2

$$f(2) = 2 \times 2^3 - 15 \times 2^2 + 36 \times 2 + 5$$
$$= \mathbf{33}$$

Maximum Value = 33

At $x = 3$,

$$f''(x) = 12 \times 3 - 30 = 6 > 0$$

$i.e.,\ f''(x) > 0$ So, there is local maxima

Maximum Value at 3

$$f(3) = 2 \times 3^3 - 15 \times 3^2 + 36 \times 3 + 5$$
$$= \mathbf{32}$$

Hence,

Local Minima at $x = 3$ at minimum Value = 32

Local Maxima at $x = 2$ at minimum Value = 33

For point of inflection, $f''(x) = 0$

$$12x - 30 = 0$$

$$x = \frac{30}{12} = \frac{5}{2}$$

At $x = \frac{5}{2}$ there is Point of inflection.

10. Evaluate: $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$

Solution:-

$$\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$$

Let $k = \lim_{x\to 0} \dfrac{\tan x - x}{x^2 \tan x}$

When the limit $x \to 0$ is applied, the given function is in the form $\frac{0}{0}$.
To change the given function in the standard form multiplying and dividing the denominator by $'x'$, we get

$$k = \lim_{x\to 0} \frac{\tan x - x}{x^2 \frac{\tan x}{x} . x} \text{ .... } (1)$$

We know that $\lim_{x\to 0} \dfrac{\tan x}{x} = 1$ (Standard limit) .... (2)

Applying the standard limit in equation (1), we get

$$k = \lim_{x\to 0} \frac{\tan x - x}{x^3} . 1$$

After applying limit, the above function is the form $\frac{0}{0}$, hence, applying L' Hospital's rule, we get

$$\frac{f'(x)}{g'(x)} = k = \lim_{x\to 0} \frac{sec^2 x - 1}{3 x^2} = \lim_{x\to 0} \frac{tan^2 x}{3 x^2} = \frac{1}{3}$$

$$\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2 \text{ .... } (3)$$

Applying the standard limit from equation (2) in equation (3), we get

$$k = \frac{1}{3} . 1 = \frac{1}{3}$$

$$\therefore \boldsymbol{k = \frac{1}{3}}$$

11. If $u = \log(x^2+y^2)$ then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution:-

$u = \log(x^2+y^2)$ then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Differentiating both Side Partially w.r.t $'x'$

$$\frac{\partial u}{\partial x} = \frac{\partial \log(x^2+y^2)}{\partial(x^2+y^2)} \times \frac{\partial(x^2+y^2)}{\partial x}$$

$$= \frac{1}{x^2+y^2} \times 2x$$

$$\frac{\partial u}{\partial x} = \frac{2x}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial\left[\frac{2x}{(x^2+y^2)}\right]}{\partial x}$$

$$= \frac{(x^2+y^2)\ \partial\frac{(2x)}{\partial x} - 2x\ \partial\frac{(x^2+y^2)}{\partial x}}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)2 - 2x\ \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2\ - 4x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2} \dots\dots\dots (ii)$$

Differentiating both Side Partially w.r.t $'y'$

$$\frac{\partial u}{\partial y} = \frac{\partial \log(x^2+y^2)}{\partial y}$$

$$= \frac{\partial \log(x^2+y^2)}{\partial (x^2+y^2)} \times \frac{\partial (x^2+y^2)}{\partial y}$$

$$= \frac{1}{x^2+y^2} \times 2y$$

$$\frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial \left[\frac{2y}{(x^2+y^2)}\right]}{\partial y}$$

$$= \frac{(x^2+y^2)\ \partial\frac{(2y)}{\partial y} - 2y\ \partial\frac{(x^2+y^2)}{\partial y}}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)2 - 2y \cdot 2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2 - 2y^2}{(x^2+y^2)^2} \dots\dots\dots (iii)$$

Adding both eq^n (ii) & (iii)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2}+\frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$=\frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.\ \text{Proved.}$$

12. State the Euler's theorem of homogenous function and show that

$$\mathbf{u=\sin^{-1}\frac{x^2+y^2}{x+y}},\ \textbf{Prove that}\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

- **If $f(x,y)$ be homogenous function in x,y of degree n having Continuous partial derivatives Then,**

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

For the Second Part

$$u=\sin^{-1}\frac{x^2+y^2}{x+y},\ \text{Prove that}\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

Now,

$$u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

$$\sin u = \frac{x^2+y^2}{x+y}$$

$$= \frac{x^2}{x}\frac{\left[1+\frac{y^2}{x^2}\right]}{\left[1+\left(\frac{y}{x}\right)\right]}$$

$$\sin u = x^1\,\phi\left(\frac{y}{x}\right)$$

Let $Z = sinu = \ x^1\,\phi\left(\frac{y}{x}\right)$

Where, $\phi\left(\frac{y}{x}\right)$ is homogenous function in x, y

$i.e, x = 1$, as comparing with $\left[z = x^n\phi\left(\frac{y}{x}\right)\right]$

By Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nu$$

$$x\frac{\partial(sin\,u)}{\partial x} + y\frac{\partial(sin\,u)}{\partial y} + 1.\,tan\,u$$

$$x\,cos\,u\,\frac{\partial u}{\partial x} + y\,cos\,u\frac{\partial u}{\partial y} = 1.\,tan\,u$$

$$cos\,u\,\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = tan\,u$$

$$\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{\tan u}{\cos u}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$ **Proved**

13. Integrate the standard integral: $\int\frac{dx}{a\sin x+b\cos x}$.

Solution:-

$$\int\frac{dx}{a\sin x+b\cos x}.$$

Put $a=r\cos\alpha$ and $b=r\sin\alpha$

so that $r=\sqrt{a^2+b^2}$ and $\alpha=\tan^{-1}\frac{b}{a}$

$$\therefore\ \int\frac{dx}{a\sin x+b\cos x}=\frac{1}{r}\int\frac{dx}{\cos\alpha\sin x+\sin\alpha\cos x}$$

$$=\frac{1}{r}\int\frac{dx}{\sin(x+\alpha)}$$

$$=\frac{1}{r}\int\operatorname{cosec}(x+\alpha)\,dx$$

$$=\frac{1}{r}\log\left(\tan\frac{1}{2}(x+\alpha)\right)+c$$

$$=\frac{1}{\sqrt{a^2+b^2}}\log\left(\tan\frac{1}{2}\left(x+\tan^{-1}\frac{b}{a}\right)\right)+c$$

14. Find the area of the plane region bounded by the x-axis the curve $y = e^x$ and the ordinates $x = 0, x = b$ using the limit of sum.

Solution:-

$$y = e^x, x = 0, x = b$$

Let $f(x) = e^x$

We have, Limit of sum

$$Area = \int_a^b f(x)dx = \lim_{x\to 0} h\,[f(h) + f(2h) + f(3h) + \cdots f(nh)]$$

Where, $n = \frac{b-a}{h}$

Now,

Area using limit of sum,

$$A = \int_0^b e^x dx$$

$$\lim_{x\to 0} h\,[f(h) + f(2h) + f(3h) + \cdots f(nh)]$$

$$A = \lim_{x\to 0} h\,[e^h + e^{2h} + e^{3h} + \cdots e^{nh}] \ldots \ldots (i)$$

And, $n = \frac{b-0}{h}$

From (i) $\implies \boldsymbol{nh = b}$

$$A = \lim_{x\to 0} h\left[e^h + \left(e^h\right)^2 + \left(e^h\right)^3 + \cdots \left(e^h\right)^n\right] \ldots \ldots (ii)$$

Let $e^h = y$

$$A = \lim_{x \to 0} h\,[y + y^2 + y^3 + \cdots y^n] \ldots\ldots (ii)$$

Since,

$$y + y^2 + y^3 + \cdots y^n$$

Common ratio, $\frac{y^2}{y} = \frac{y^3}{y^2} = y.$

So, Sum to n terms $= \frac{a(r^n - 1)}{r - 1}$ For $[x + x^2 + x^3 + \cdots x^n]$

So,

From eq^n (ii)

$$A = \lim_{x \to 0} h\left[\frac{y(y^n - 1)}{y - 1}\right]$$

$$A = \lim_{x \to 0} h\left[\frac{e^h (e^h)^n - 1}{e^h - 1}\right]$$

$$A = \lim_{x \to 0} h\,\frac{e^h (e^h)^n - 1}{e^h - 1}$$

$$A = \lim_{x \to 0} e^h (e^{nh} - 1) \times \frac{h}{e^h - 1}$$

$$= \lim_{x \to 0} e^h (e^b - 1) \times \left(\lim_{x \to 0} \frac{h}{e^h - 1}\right) \left[\because \boldsymbol{\lim_{x \to 0} \frac{h}{e^h - 1} = 1}\right]$$

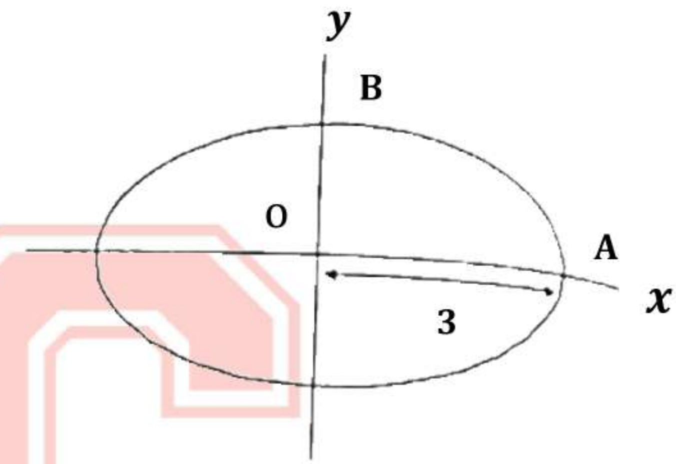
$$= e^0 \left(e^b - 1\right) \times 1$$

$$\therefore \boldsymbol{Area = e^b - 1}$$

15. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution:-

- The curve is symmetrical about $x-axis$ and $y-axis$. So to find the area of the whole ellipse, first we find the area of portion lying in the first quadrant and then multiply it by 4. Here OA = a and OB = b. The area of the portion lying in the first quadrant is bounded by the curve, $x-axis$ and the ordinates $x = 0, x = a$. So its area is



$$A=\int_0^a y\,dx=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

Put $x = a\sin\theta$, then $dx = a\cos\theta\, d\theta$

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}$$

$$=a\sqrt{1-\sin^2\theta}=a\cos\theta$$

When $x = 0$, $\theta = 0$ and when $x = a$, $\theta=\frac{\pi}{2}$

$$\therefore\ A=\frac{b}{a}\int_0^{\pi/2} a\cos\theta\, a\cos\theta\, d\theta$$

$$= ab\int_0^{\pi/2} \cos^2\theta \; d\theta$$

$$= ab\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2} d\theta + \frac{ab}{2}\int_0^{\pi/2} \cos 2\theta \; d\theta$$

$$= \frac{ab}{2}[\theta]_0^{\pi/2} + \frac{ab}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{ab}{2}\left(\frac{\pi}{2} - 0\right) + \frac{ab}{4}(\sin\pi - \sin 0) = \frac{\pi ab}{4}$$

$\therefore$ Therefore, the whole area $= \dfrac{4\pi ab}{4} = \pi ab$

16. Define linear equation and solve $2\cos x\frac{dy}{dx} + 4y\sin x = \sin 2x$.

- An equation of the form $\frac{dy}{dx} + Py = Q$ in which P and Q are function of x along or constants is called a **linear equation** of the first order

For the Second Part

Solution:-

$$\boldsymbol{2\cos x\frac{dy}{dx}+4y\sin x=\sin 2x.}$$

$$\frac{dy}{dx}+\frac{4y}{2}\ \frac{\sin x}{\cos x}=\frac{\sin 2x}{2\cos x}$$

$$\frac{dy}{dx}+2y\tan x=\frac{2\sin x\cos x}{2\cos x}$$

$$\boldsymbol{\frac{dy}{dx}+2y\tan x=\sin x\cdots\cdots\cdots(i)}$$

Eq^n (i) Comparing with.

$$\frac{dy}{dx}+Py=Q$$

Where, P= Function of x

Q= Function of x or Constant

$$P\ =\ 2\,tan\,x\ \ ,Q=\ sin\ x$$

$$\mathrm{I.F}=\ e^{\int p dx}=e^{\int 2tanx\, dx}$$

$$=e^{\left(2\int\frac{sin\,x}{cos\,x}dx\right)}$$

$$= e^{2\log(sec\, x)}$$

$$= e^{\log(sec\, x)^2}$$

$$\mathbf{I.F} = \boldsymbol{sec^2\, x}$$

The Solution is given by

$$\mathbf{I.F.}\,\boldsymbol{y} = \int \mathbf{I.F}\,\boldsymbol{Qdx}$$

$$sec^2\, x.\, y = \int sec^2 x.\, sin\, x\, dx$$

$$sec^2\, x.\, y = \int tan\, x.\, sec\, x\, dx$$

$$sec^2\, x.\, y = sec\, x + C$$

$$y = \frac{sec\, x}{sec^2\, x} + \frac{C}{sec^2\, x}$$

$$\boldsymbol{y = cos\, x + C.\, cos^2\, x.}$$

17. Show that the differential equation exact and solve:

$$(x+y-1)\,dx+(x-y-2)dy=0$$

Solution:-

Comparing with $Mdx + Ndy = 0$,

$$M = x + y - 1, N = x - y - 2$$

$$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 1$$

Thus,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, the given equation is an exact equation. To solve it, we proceed as follows.

$$\int M\,dx \text{ (taking } y \text{ constt)}$$

$$= \int (x+y-1)\,dx \text{ (taking } y \text{ constant)}$$

$$= \frac{x^2}{2} + xy - x + c_1$$

$$\int N\,dy \text{ (terms not containing } x\text{)}$$

$$= \int (-y-2)\,dy = \frac{-y^2}{2} - 2y + c_2$$

$\therefore$ The solution is

$$\frac{x^2}{2} + xy - x - \frac{y^2}{2} - 2y = c.$$

18. If the normal at every point of a curve passes through a fixed point,using first order differential equation so that the curve is circle.

Solu:- Let fixed point be (h, k)

$$y = f(x)$$

$$\frac{dy}{dx} = M_T$$

$$M_T \,|_{p(x,y)} = \frac{dy}{dx}$$

$$= \frac{-dx}{dy}$$

Equantion of N at $P(x, y)$

$$Y - y = -\frac{dx}{dy}(X - x)$$

$$k - y = \frac{-dx}{dy}(h - x)$$

$$(k - y)dy = (x - h)dx$$

$$\int(k - y)dy = \int(x - h)dx$$

$$ky - \frac{y^2}{2} = \frac{x^2}{2} - hx + C$$

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 - k^2 = -2C$$

$(x - h)^2 + (y - k)^2 = A$ where $-2C + h^2 + k^2 = A$

Hence, it is a circle.

19. Solve: $(mz - ny)p + (nx - lz)q = ly - mx$.

Solution:-

The given equation is

$$(mz - ny)p + (nx - lz)q = ly - mx$$

$$\text{Comparing with } \mathrm{Pp + Qq = R}$$

$$\mathrm{P = mz - ny}$$

$$\mathrm{Q = nx - lz}$$

$$\mathrm{R} = l\mathrm{y - mx}$$

The auxillary equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Choosing multipliers x, y and z

$$= \frac{xdx+ydy+zdz}{mxz-nxy+nxy-lzy+lyz-mxz}$$

$$= \frac{xdx + ydy + zdz}{0}$$

Hence,

$$xdx + ydy + zdz = 0$$

Integrating we get,

$$\int xdx + \int ydy + \int zdz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{C_1}{2}$$

$$\boldsymbol{x^2 + y^2 + z^2 = C_1}$$

On choosing multipliers $\boldsymbol{l, m}$ and $\boldsymbol{n}$

$$= \frac{\mathrm{ld}\, x + mdy + zdz}{lmz - \ln y + mnx - lmz + \ln y - mnx}$$

$$ldx + mdy + zdz = 0$$

On integrating,

$$\int ldx + \int mdy + \int ndz = 0$$

$$lx + my + nz = C_2$$

Hence, the general solution is

$$\boldsymbol{\left(x^2 + y^2 + z^2\right) = f(lx + my + nz).}$$

20. Find the Fourier series of $f(x)$ which is defined by

$$f(x) = \begin{cases} 0 & for -\pi \le x \le 0 \\ \frac{\pi}{4}x & for\ 0 < x \le \pi \end{cases}$$

The Fourier series of is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + bn\ sinx) \cdots\cdots (i)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{-\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\left(\frac{\pi}{4}x\right)dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{4}\left[\frac{x^2}{2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{4}\ \frac{x^2}{2}\right)$$

$$a_0 = \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{-\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx dx + \int_{0}^{\pi} f(x)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\, cosnxdx \; + \int_{0}^{\pi} \frac{\pi}{4} x \cos nx dx\right]$$

$$= \frac{1}{\pi} \times \frac{\pi}{4}\int_{0}^{\pi} x cos\, nx dx$$

$$= \frac{1}{4}\left[\frac{x\, sin\, nx}{n} - \frac{(-cos\, nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{x\, sin\, nx}{n} - \frac{(-\cos nx)}{n^2}\right]_0^{\pi}$$

$$\left[\because \int uvdx = uv_1 - u' v_2 + \cdots\cdots\right]$$

$v_1, v_2 =$ Successive integration

$u', u'' =$ Successive derivatives

$$= \frac{1}{4}\left[\left(\frac{\pi\, sin\, \pi x}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos(0)}{n^2}\right)\right]$$

$$\boldsymbol{Sin\, n\pi = 0\,,\; Cos\, n\pi = (-1)^n}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\boldsymbol{a_n = \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right].}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_{-\pi}^{-\pi} f(x)\sin nx dx}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx dx + \int_{0}^{\pi} f(x)\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\ sin\ nxdx\ + \int_{0}^{\pi}\frac{\pi}{4}x\ sin\ nxdx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{4}\int_{0}^{\pi} x\ sin\ nx\ dx\right]$$

$$= \frac{1}{\pi} \times \frac{\pi}{4}\left[\frac{x(-cos\ nx)}{n} - \frac{(-sin\ nx)}{n^2}\right]_0^{\pi}$$

Note: $-\left\{\begin{matrix} [\because \int uvdx = uv_1 - u'v_2 + u''v_3 \cdots\cdots\cdots] \\ \left[\begin{matrix} e.g \int x\, e^x dx = xe^x - 1.e^x + 0.e^x \\ = xe^x - e^x. \end{matrix}\right] \end{matrix}\right\}$

$$b_n = \frac{1}{4}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{1}{4}\left[\left(\frac{-\pi\cos(n\pi)}{n} + \frac{\sin n\pi}{n^2}\right) - (0+0)\right]$$

$$b_n = \frac{1}{4}\left[\left(\frac{-\pi(-1)^n}{n}\right)\right] \qquad \left[\because \begin{array}{c} \sin nx = 0 \\ \cos n\pi = (-1)^n \end{array}\right]$$

$$\boldsymbol{b_n = \frac{\pi}{4}\frac{(-1)^{n+1}}{n}}$$

Hence,

Fourier series is ,

$$f(x) = \frac{\pi^2}{8.2} + \sum_{n=1}^{\infty}\left[\frac{1}{4}\left(\frac{(-1)^2}{n^2}\frac{1}{n^2}\right)\cos nx + \frac{\pi}{4}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$\boldsymbol{f(x) = \frac{\pi^2}{16}} + \sum_{n=1}^{\infty}\left[\frac{1}{4n^2}\left((-1)^n - 1\right)\cos nx + \frac{\pi}{4}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

-The End-