

# Understanding the Stochastic Partial Differential Equation Approach to Smoothing

---

**David L. Miller**   Richard Glennie   Andrew Seaton

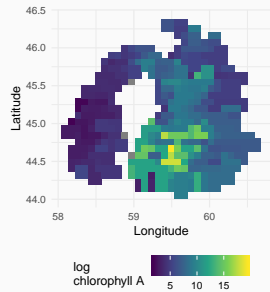
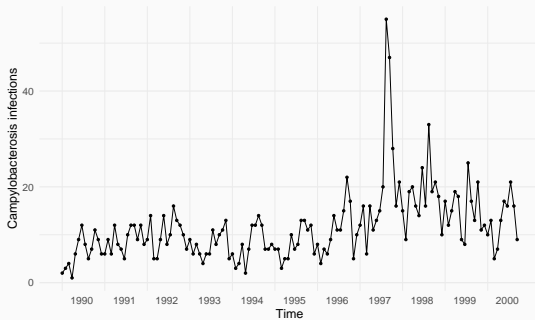
17 June 2021

International Biometric Society Journal Club

# Introduction

---

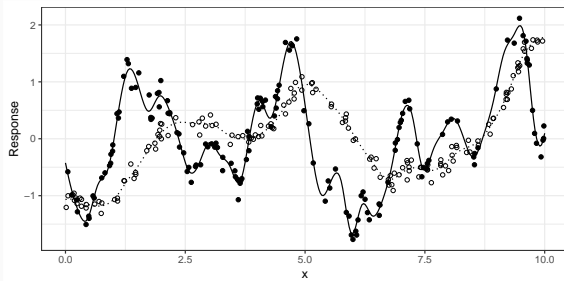
# The kinds of data we'll talk about



# How to model dependence?

**Tobler's law:** “everything is related to everything else, but near things are more related than distant things.”

Two ways to think about this: *Smoothness* and *Correlation*.

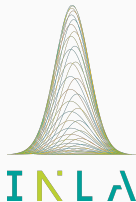


# How to model dependence?

Two ways to think about this: Smoothness and Correlation.

Two popular approaches to modelling it: GAMs and SPDEs.

- GAMs  $\approx$  “smoothing using spline-type things”
- SPDEs  $\approx$  “Matérn covariance approximation using SPDEs/MRFs”





**Jenny Bryan**

@JennyBryan

All models are wrong, so why not start with one you actually understand?

Refresh: what are GAMs made of?

Explain: how does the SPDE method work?

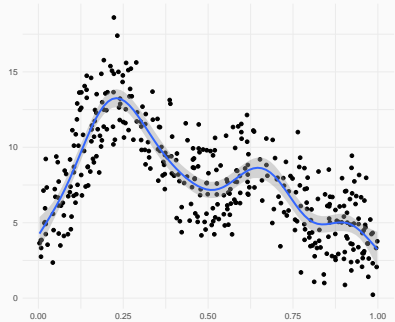
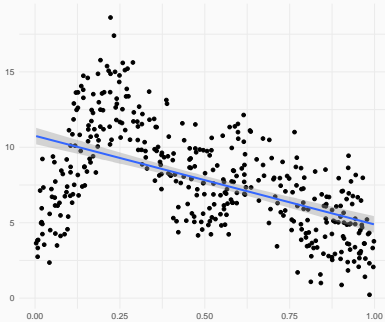
Apply: how can we use these results in practice?

# The GAM approach

---

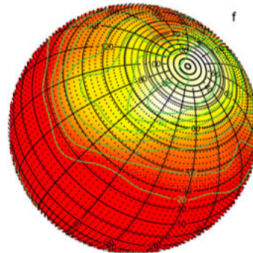
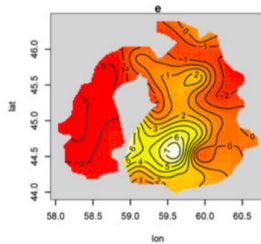
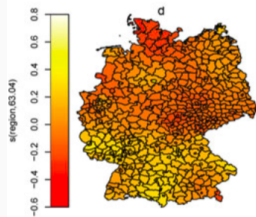
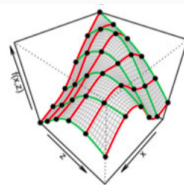
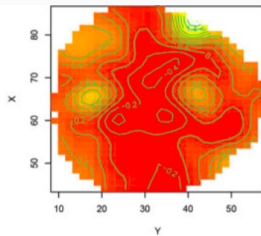
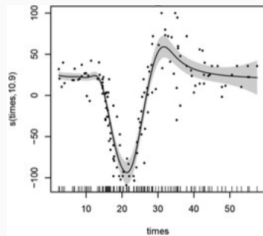
# From GLMs to GAMs

- Want flexible, structured models
- GAM provides this via “smooths”
- $y \sim s(x)$ , where  $s$  is a smooth function of  $x$





# Examples of smooths



# Smooths come in 2 bits...

## basis functions

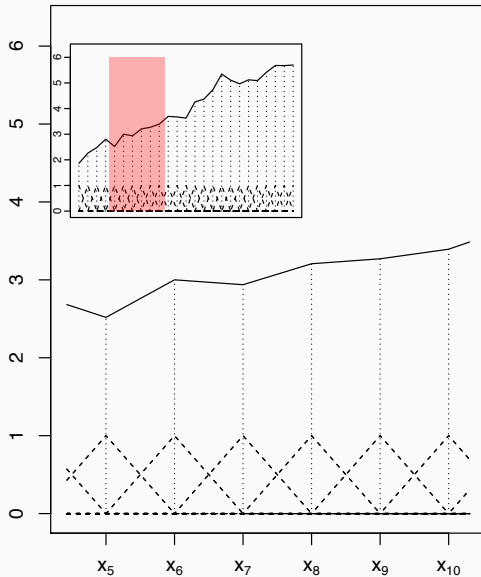
- join together to make up the overall curve,  $s(x)$
- go in the design matrix,  $\mathbf{X}$

## penalties

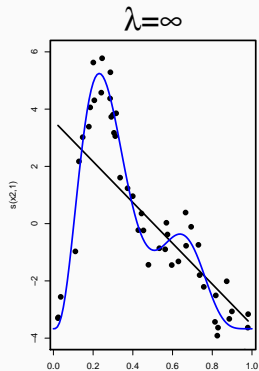
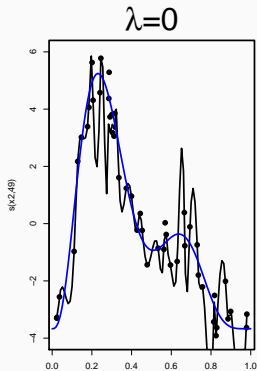
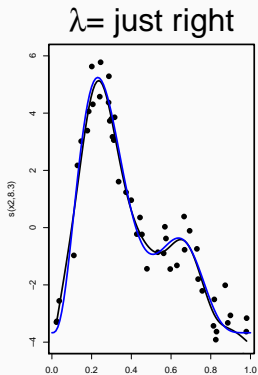
- controls how the line behaves
- penalty matrix  $\mathbf{S}$  (structure)
- smoothing parameter  $\lambda$  (how wiggly)

“basis-penalty smoothers”

# Basis functions



# Penalty and smoothing parameter



## SPDE approach

---

Journal of the  
Royal Statistical Society

SERIES B  
Statistical  
Methodology



*J. R. Statist. Soc. B* (2011)  
**73**, Part 4, pp. 423–498

## **An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach**

Finn Lindgren and Håvard Rue

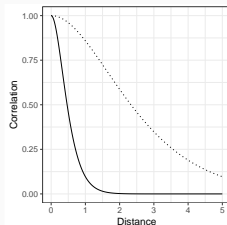
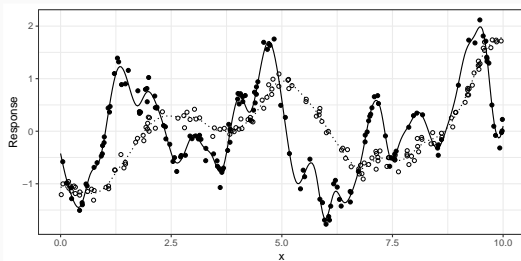
*Norwegian University of Science and Technology, Trondheim, Norway*

and Johan Lindström

*Lund University, Sweden*

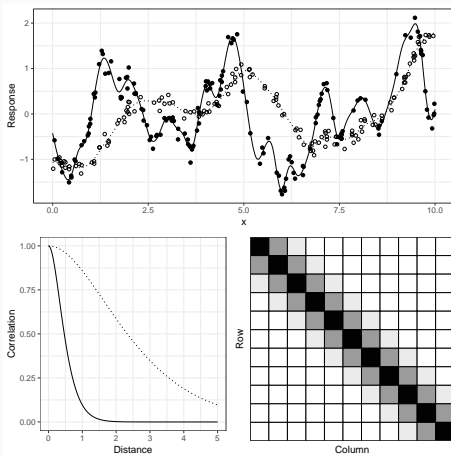
# Two Selling points of SPDE approach

1. A certain kind of SPDE produces curves that have Matérn correlation functions, a popular choice of correlation function.



## Two Selling points of SPDE approach

2. The precision matrix  $Q$  can be computed very efficiently, a bottleneck in many other correlation approaches.





# SPDE method comes in 2 bits...

## finite elements

- join together to make up the overall curve
- go in the predictor matrix, **A**

## precision

- controls how the curve behaves
- precision matrix **Q** (structure)
- correlation parameter  $\kappa$  (how wiggly)
- variance parameter  $\tau$

Wait a minute...

# SPDE method comes in 2 bits...

## ~~finite elements~~

### basis functions

- join together to make up the overall curve
- go in the predictor design matrix,  $\mathbf{A} = \mathbf{X}$

## ~~precision~~

### penalty

- controls how the curve behaves
- precision matrix  $\mathbf{Q} = \mathbf{S}$  (structure)
- correlation parameter  $\kappa$  (how wiggly)
- variance parameter  $\tau$
- $\kappa, \tau \rightsquigarrow \lambda$

## SPDE is a basis-penalty smoother

The SPDE has a corresponding smoothness penalty.

Pick same basis, then

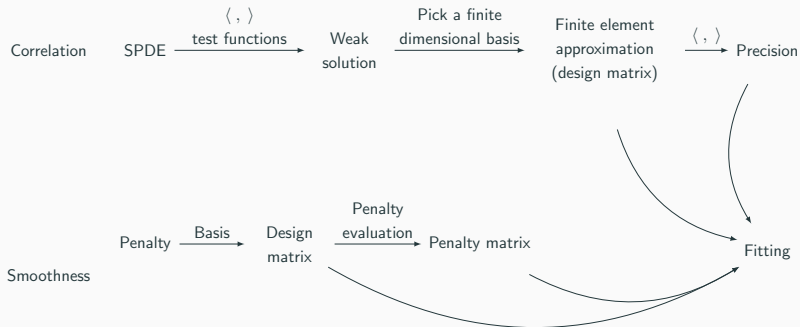
$$\mathbf{A} = \mathbf{X}$$

Use SPDE and corresponding penalty, then

$$\mathbf{Q} = \mathbf{S}$$

See our paper for more mathematical details.

# The SPDE method is a basis-penalty smoother



## **Practical implications**

---

## What about INLA and mgcv?

- Both are ways to fit this same model.
- Both R-INLA and mgcv do approximations:
  - R-INLA does a Laplace approximation to get around MCMC.
  - mgcv does REML using Laplace approximation too.
- mgcv does empirical Bayes, so uses point estimates for hyperparameters (smoothing parameters, variance components).
- (R-INLA can do this too!)

## What can we do with this?

- Writing SPDE models as basis-penalty smoothers means we can use them elsewhere
- Anywhere structured random effects can be used!
- `mgcv`, `TMB`, `Stan`, etc
- We can combine them with other approaches!

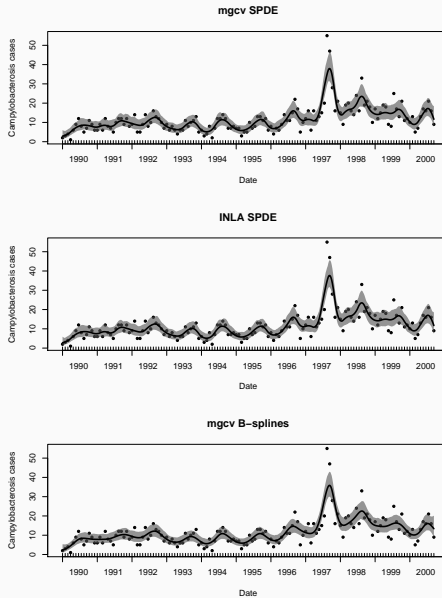


## mgcv code

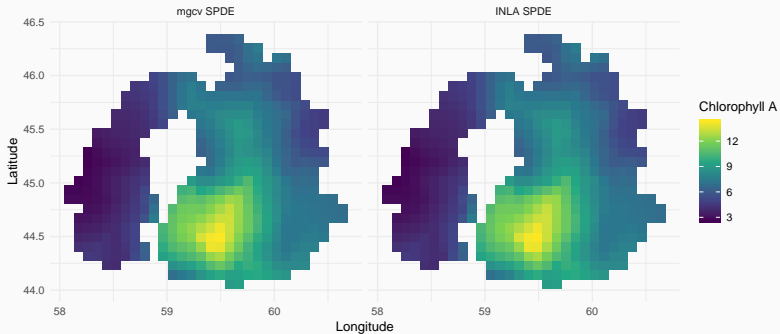
Setup using R-INLA, fit with mgcv

```
1  ## Simplified code-snippets
2  ## Full code in supplementary materials
3
4  # get finite element matrices
5  inlamats <- inla.mesh.fem(mesh)
6  c1 <- as.matrix(inlamats$c1)
7  g1 <- as.matrix(inlamats$g1)
8  g2 <- as.matrix(inlamats$g2)
9
10 # construct the design matrix
11 X <- as.matrix(inla.spde.make.A(mesh, observed_locs))
12
13 # build the penalty / precision matrix
14 S <- kappa^4 * c1 + 2*kappa^2 * g1 + g2
15
16 # fit
17 b <- gam(y ~ X - 1,
18          paraPen = list(X = list(S)),
19          method = "REML")
```

# 1D comparison



# Spatial smoothing comparison



# Take-home

---

## A modelling framework emerges

Use basis-penalty or basis-precision as a common currency.

1. **Choose a covariance model:** explicitly (as in kriging, etc), via smoothness penalty (e.g., splines), or with an SPDE;
2. **Approximate the precision matrix  $Q$ :** reduce dimension or induce sparsity in  $Q$ ;
3. **Draw approximate inference using a software implementation:** e.g., with `mgcv`, MCMC (e.g., Stan; JAGS), R-INLA, `lme4` or TMB.

**More info?**

---

# Understanding the Stochastic Partial Differential Equation Approach to Smoothing

David L. MILLER , Richard GLENNIE , and Andrew E. SEATON 

Correlation and smoothness are terms used to describe a wide variety of random quantities. In time, space, and many other domains, they both imply the same idea: quantities that occur closer together are more similar than those further apart. Two popular statistical models that represent this idea are basis-penalty smoothers (Wood in *Texts in statistical science*, CRC Press, Boca Raton, 2017) and stochastic partial differential equations (SPDEs) (Lindgren et al. in *J R Stat Soc Series B (Stat Methodol)* 73(4):423–498, 2011). In this paper, we discuss how the SPDE can be interpreted as a smoothing penalty and can be fitted using the R package `mgcv`, allowing practitioners with existing knowledge of smoothing penalties to better understand the implementation and theory behind the SPDE approach.

Supplementary materials accompanying this paper appear online.

**Key Words:** Smoothing; Stochastic partial differential equations; Generalized additive model; Spatial modelling; Basis-penalty smoothing.

Slides: <https://github.com/dill/SPDE-talk-IBS>