

# Spline mimimizing rotation-invariant semi-norms in Sobolev spaces

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We define a family of semi-norms  $\|u\|_{m,s} = \left( \int_{\mathbb{R}^n} |\tau|^{2s} |\mathfrak{F} D^m u(\tau)|^2 d\tau \right)^{1/2}$ . Minimizing such semi-norms, subject to some interpolating conditions, leads to function of very simple forms, providing interpolation methods that: 1) preserve polynomials of degree  $\leq m-1$ ; 2) commute with similarities as well as translations and rotations of  $\mathbb{R}^n$ ; and 3) converge in Sobolev spaces  $H^{m+s}(\Omega)$ .

Typical examples of such splines are: “thin plate” functions ( $\sum_{a \in A} \lambda_a |t-a|^2 \log |t-a| + \alpha t + \beta$  with  $\sum \lambda_a = 0$ ,  $\sum \lambda_a a = 0$ ), “multi-conic” functions ( $\sum \lambda_a |t-a| + C$  with  $\sum \lambda_a = 0$ ), pseudo-cubic splines ( $\sum \lambda_a |t-a|^3 + \alpha t + \beta$  with  $\sum \lambda_a = 0$ ,  $\sum \lambda_a a = 0$ ), as well as usual polynomial splines in one dimension. In general, data functional are only supposed to be distributions with compact supports, belonging to  $H^{-m-s}(\mathbb{R}^n)$ ; there may be infinitely many of them. Splines are then expressed as convolutions  $\mu \odot |t|^{2m+2s-n}$  (or  $\mu \odot |t|^{2m+2s-n} \log |t|$ ) + polynomials.

## 0 Introduction

Splines in more than one dimension are usually constructed from one-dimensional ones, via tensor products. We follow here another point of view (more analogous to the physical interpretation of elementary cubic splines as equilibrium positions of a beam) developed by M. Atteia [1, 2, 3]: his splines (minimizing  $\int_{\Omega} |D^2 v|^2$ , a functional similar to the bending energy of a thin plate) are uneasy to compute, because their characterization involves a kernel given by series. But things are much simpler if we replace  $\Omega$  by the whole plane  $\mathbb{R}^2$ , as is shown in [5] (where present extensions are announced). This leads to what we call “thin plate” functions in  $\mathbb{R}^2$  (engineers say “surface splines” [6]). Of course, it is possible to minimize  $\int |D^m v|^2$  instead of  $\int |D^2 v|^2$ , and deal with  $\mathbb{R}^n$  instead of  $\mathbb{R}^2$  (with some restriction:  $m > \frac{n}{2}$ , if point values are used).

We notice that these functionals  $\int_{\mathbb{R}^n} |D^m v|^2$  are invariant through translations and rotations. Moreover, if a similarity  $t \rightarrow \lambda t$  is applied to  $v$ , they are multiplied by some power of  $|\lambda|$ .

Thus, corresponding interpolation methods will *commute with similarities*: interpolating on a contracted set of points  $\lambda A$  gives the same result as interpolating on  $A$  (with same values) and then applying contraction  $t \rightarrow \lambda t$ .

Now, since Fourier transform is isometric on  $L^2(\mathbb{R}^n)$ , we may write  $\int_{\mathbb{R}^n} |D^m v(t)|^2 dt = \int_{\mathbb{R}^n} |\mathfrak{F} D^m v(\tau)|^2 d\tau$ . A natural idea, to get other interpolation methods, would be to introduce a weighting function  $w$  and minimize  $\int_{\mathbb{R}^n} w(\tau) |\mathfrak{F} D^m v(\tau)|^2 d\tau$ . In view of the above invariance properties, it is natural to put  $w(\tau) = |\tau|^\theta$  and try to minimize  $\int_{\mathbb{R}^n} |\tau|^\theta |\mathfrak{F} D^m v(\tau)|^2 d\tau$ , a functional which is invariant through translations and rotations, and is multiplied by a constant if the variable  $t$  is changed into  $\lambda t$ . This is actually possible (at least if  $-2m - n < 0 < n$  and  $2m + \theta$  is sufficiently large, depending on which kind of data are used), in a precise sense, as we shall see.

We first introduce some ‘‘Sobolev-type’’ spaces such as  $\tilde{H}^s$ ,  $D^{-m}\tilde{H}^s$ ,  $H_{loc}^{m+s}$ ,  $H_{comp}^{-m-s}$ , and compare them. Central space is  $D^{-m}\tilde{H}^s/P_{m-1}$  with Hilbert structure. Its dual contains  $H_{comp}^{-m-s} \cap P_{m-1}^o$  as a dense subset (1).

In 2 we prove existence and uniqueness for interpolation problems, and abstract characterization using reproducing kernels of semi-Hilbert spaces.

3 is the crucial one. We compute the reproducing kernel of  $D^{-m}\tilde{H}^s/P_{m-1}$  in  $H_{loc}^{m+s}/P_{m+1}$ , i.e. the natural isometric mapping from  $H_{comp}^{-m-s} \cap P_{m-1}^o$ , embedded in  $(D^{-m}\tilde{H}^s/P_{m-1})$  into  $D^{-m}\tilde{H}^s/P_{m-1}$ .

4 summarizes our main result (theorems 4 and 4 bis). In 5 we apply it to various examples.

6 is concerned with convergence in Sobolev space  $H^{m+s}(\Omega)$  ( $\Omega$  bounded) for an interpolated function  $f \in H^{m+s}(\Omega)$ .

For brevity, we use classical or natural notations without explicit statement. Moreover, some familiarity with distributions is assumed (convolution, derivation, Fourier transform, multiplication with  $C^\infty$  functions, and their relations; correspondence between locally summable functions and distributions, etc.; standard reference is [9]).  $D^m v(t)$  is the  $n^m$ -tuple of partial derivatives  $(D_{i_1} \dots D_{i_m} v(t); i_1, \dots, i_m = 1, \dots, n)$ , with Euclidian norm  $(\sum |D^m v(t)|^2 = |D_{i_1} \dots D_{i_m} v(t)|^2)^{1/2}$ . Pseudo-functions  $Pf \cdot |\tau|^\theta$  are to be found in [9, p. 44] and their Fourier transforms p. 257. Fourier transform is  $\mathfrak{F}$  or  $\hat{\cdot}$ .

The letter  $c$  is used to denote various constants, to avoid useless complication.

## 1 Functional spaces

### 1.1 Sobolev spaces $H^s$ , $H_{loc}^s$ , $H_{comp}^{-s}$

For any real  $s$ ,  $H^s(\mathbb{R}^n)$  is the set of tempered distributions  $u$  on  $\mathbb{R}^n$  whose Fourier transform  $\hat{u}$  is a (locally summable) function such that  $\int_{\mathbb{R}^n} (1 + |\tau|^2)^s |\hat{u}(\tau)|^2 d\tau < \infty$ . In other words

$H^s = \mathfrak{F} \left( (1 + |\tau|^2)^{-s/2} L^2 \right)$ . It is a Hilbert space. Its dual is naturally identified with  $H^{-s}(\mathbb{R}^n)$ .

When  $s$  is a positive integer,  $H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); D^\alpha u \in L^2(\mathbb{R}^n), \forall |\alpha| \leq s\}$ .

If  $K$  is a closed subset of  $\mathbb{R}^n$ ,  $H_K^s(\mathbb{R}^n)$  is the set of distributions  $\in H^s(\mathbb{R}^n)$  whose support is contained in  $K$ .  $H_K^s$  is a closed linear subspace of  $H^s$ .

If  $\Omega$  is an open subset of  $\mathbb{R}^n$ ,  $H^s(\Omega)$  is the set of restrictions, to  $\Omega$ , of distributions  $\in H^s(\mathbb{R}^n)$ . It is isomorphic to the quotient space  $H^s(\mathbb{R}^n)/H_{\Omega^c}^s(\mathbb{R}^n)$  hence a Hilbert space too. Its dual is naturally identified with  $H_{\bar{\Omega}}^{-s}(\mathbb{R}^n)$ .

If  $\Omega$  is bounded and sufficiently regular (e.g.  $\Omega$  satisfies some “uniform cone” condition) then (for non integer  $s > 0$ )  $H^s(\Omega)$  is the set of distributions  $u$  on  $\Omega$  whose derivatives of order  $[s]$  (integral part of  $s$ ) are in  $L^2(\Omega)$ , with

$$\iint_{\Omega \times \Omega} \frac{|D^{[s]}u(t) - D^{[s]}u(t')|^2}{|t - t'|^{n+2(s-[s])}} dt dt' < \infty.$$

For integer  $s \geq 0$  it suffices that  $D^\alpha u \in L^2(\Omega)$ ,  $\forall |\alpha| = s$ .

Now  $H_{loc}^s(\mathbb{R}^n)$  is the set of distributions on  $\mathbb{R}^n$  whose restriction to any bounded open set  $\Omega$  is in  $H^s(\Omega)$ . Of course  $C^k \subset H_{loc}^s$  for integer  $k \geq s$ . On the other hand, Sobolev embedding theorems assert that  $H_{loc}^s(\mathbb{R}^n) \subset C^k$  for  $s > k + \frac{n}{2}$ .

$H_{loc}^s$  may be equipped with a Fréchet space structure: putting  $B_N =$  the open ball  $|t| < N$ , one defines a countable family of semi-norms  $u \mapsto$  norm of  $u|_{B_N}$  in  $H^s(B_N)$ . It is reflexive, and its dual is naturally identified with  $H_{comp}^{-s}(\mathbb{R}^n)$ , the union of  $H_{\bar{B}_N}^{-s}(\mathbb{R}^n)$ , equipped with the topology of (strict) inductive limit of this countable family of Hilbert spaces.

All previous norms could have been replaced by equivalent ones, they do not play a role by themselves in our problem. We now come to defining spaces  $\tilde{H}^s$  and  $D^{-m}\tilde{H}^s$ , whose semi-norms  $\|\cdot\|_{m,s}$  are fundamental.

## 1.2 $\tilde{H}^s(\mathbb{R}^n)$ , $s < \frac{n}{2}$

We put  $\tilde{H}^s(\mathbb{R}^n) = \left\{ u \in \mathcal{J}'(\mathbb{R}^n); \hat{u} \in L_{loc}^1, \int_{\mathbb{R}^n} |\tau|^{2s} |\hat{u}(\tau)|^2 d\tau < \infty \right\}$ . We equip it with norm  $\|u\|_{0,s} = \left( \int_{\mathbb{R}^n} |\tau|^{2s} |\hat{u}(\tau)|^2 d\tau \right)^{1/2}$ .  $\tilde{H}^s(\mathbb{R}^n)$  is a Hilbert space if (and only if)  $s < \frac{n}{2}$ : for  $f \in L^2(\mathbb{R}^n)$ , the function  $|\tau|^{-s} f(\tau)$  is locally summable, since  $\int_K |\tau|^{-s} |f(\tau)| d\tau \leq \left( \int_K |\tau|^{-2s} \right)^{1/2} \left( \int_K |f(\tau)|^2 d\tau \right)^{1/2}$  and  $|\tau|^\lambda$  is locally summable if (and only if)  $\lambda > -n$ ; therefore  $|\tau|^{-s} f$  defines a distribution, easily seen to be tempered, so that  $f \mapsto \mathfrak{F}(|\tau|^{-s} f)$  is an isometry from  $L^2(\mathbb{R}^n)$  onto  $\tilde{H}^s(\mathbb{R}^n)$ . Moreover if  $f_j \rightarrow 0$  in  $L^2$ , one can prove that  $|\tau|^{-s} f_j \rightarrow 0$  in  $\mathcal{J}'$  (it suffices to show that  $\int |\tau|^{-s} f_j(\tau) \varphi(\tau) d\tau \rightarrow 0$  for any  $C^\infty$  function  $\varphi$  rapidly decreasing at  $\infty$ ). This implies that inclusion  $\tilde{H}^s \hookrightarrow \mathcal{J}'$  (a fortiori  $\tilde{H}^s \hookrightarrow \mathcal{D}'$ ) is continuous, or that  $\tilde{H}^s$  is a Hilbert subspace of  $\mathcal{D}'$ .

We might even prove that, for  $-\frac{n}{2} < s < \frac{n}{2}$ ,  $\tilde{H}^s$  is a *normal* subspace of  $\mathcal{D}'$  (i.e. contains  $\mathcal{I}$  as a dense subspace) with dual  $\tilde{H}^s$ .

(Some of these spaces are known and used in other fields. For example,  $\tilde{H}^{-1}(\mathbb{R}^3)$  and  $\tilde{H}^1(\mathbb{R}^3)$  are respectively spaces of charges and potentials of finite energy, in the theory of Newtonian potentials in  $\mathbb{R}^3$ . See [10, §11]).

From definitions, it is obvious that  $H^s(\mathbb{R}^n) \subset \tilde{H}^s(\mathbb{R}^n)$  if  $s > 0$ , the converse if  $s < 0$  (and  $\tilde{H}^0 = H^0 = L^2$ ). In general  $H^s(\mathbb{R}^n) \neq \tilde{H}^s(\mathbb{R}^n)$  ( $\tilde{H}^s$  is not contained in  $L^2$ , for  $s > 0$ , while  $H^s \subset L^2$ ). But (at least for  $-\frac{n}{2} < s < \frac{n}{2}$ ) *equality holds on bounded subsets, as we now see.*

### 1.3 $H^s(\Omega) = \tilde{H}^s(\Omega)$ , if $-\frac{n}{2} < s < \frac{n}{2}$

Of course,  $H^s(\Omega)$  is the set of restrictions, to  $\Omega$  (bounded), of elements of  $\tilde{H}^s(\Omega)$ , with corresponding Hilbert structure. In fact, we can prove that any  $u \in \tilde{H}^s(\Omega)$  ( $s > 0$ ) is sum of  $u_1 \in H^s(\mathbb{R}^n)$  and a  $C^\infty$  function  $u_2$ . We first note that  $|\tau|^{2s} \leq (1 + |\tau|^2)^s \leq 2|\tau|^{2s}$  except for  $\tau \in$  some ball  $B$ . Now for  $u \in \tilde{H}^s$ ,  $\hat{u}$  is a function. We write it as  $v_1 + v_2$  where  $v_2$  coincides with  $\hat{u}$  a.e. on  $B$  and is zero outside,  $v_i$  satisfies  $\int_{\mathbb{R}^n} (1 + |\tau|^2)^s |v_1(\tau)|^2 d\tau = \int_{B^c} (1 + |\tau|^2)^s |v_1(\tau)|^2 d\tau \leq \int_{B^c} |\tau|^{2s} |v_1(\tau)|^2 d\tau < \infty$ , so that its inverse Fourier transform  $u_1 \in H^s(\mathbb{R}^n)$ . On the other hand,  $v_2$  has compact support, hence its (inverse) Fourier transform  $u_2$  is a  $C^\infty$  function (in fact, an “entire function of exponential type”, moreover  $\rightarrow 0$  at  $\infty$  since  $v_2 \in L^1$ ). Case  $s < 0$  is exactly similar: we prove  $H^s \subset \tilde{H}^s + C^\infty$ .

We have already seen that  $C^\infty \subset H_{loc}^s$ , so that  $\tilde{H}^s(\Omega) = H^s(\Omega)$  if  $s > 0$ . For  $s < 0$  we must prove that  $C^\infty \subset \tilde{H}_{loc}^s$ ; but for  $s > \frac{n}{2}$  we have  $\mathcal{D} \subset \tilde{H}^s$ , which gives the result. So in general  $\tilde{H}^s(\Omega) = H^s(\Omega)$  for  $\Omega$  bounded, whenever  $-\frac{n}{2} < s < \frac{n}{2}$ .

Inclusion  $H^s(\Omega) \hookrightarrow \tilde{H}^s(\Omega)$  for  $s > 0$  ( $\tilde{H}^s(\Omega) \hookrightarrow H^s(\Omega)$  for  $s < 0$ ) is continuous, so that the two Hilbert structures induced on  $\tilde{H}^s(\Omega) = H^s(\Omega)$  are comparable, hence equivalent, from one of Banach’s theorems.

### 1.4 Semi-Hilbert spaces $D^{-m}\tilde{H}^s$

We put  $D^{-m}\tilde{H}^s = \left\{ u \in \mathcal{D}'(\mathbb{R}^n); D^\alpha u \in \tilde{H}^s(\mathbb{R}^n), \forall |\alpha| = m \right\}$ , equipped with natural semi-norm  $\|u\|_{m,s} = \left( \int_{\mathbb{R}^n} |\tau|^{2s} |\mathfrak{F} D^m u(\tau)|^2 d\tau \right)^{1/2}$ . We also consider the quotient space  $D^{-m}\tilde{H}^s / P_{m-1}$  with corresponding norm. These are *spaces of Beppo Levi type* (see [4], where these spaces would be denoted  $BL_m(\tilde{H}^s)$  and  $BL_m(\tilde{H}^s)$  respectively): since  $\tilde{H}^s$  is a Hilbert subspace of  $\mathcal{D}'$ ,  $D^{-m}\tilde{H}^s / P_{m-1}$  is a Hilbert subspace of  $\mathcal{D}' / P_{m-1}$  (i.e.  $D^{-m}\tilde{H}^s$  is a semi-Hilbert subspace of  $\mathcal{D}'$ , see 2). In fact, since  $\tilde{H}^s \subset \mathcal{T}'$ , and a distribution whose derivatives are tempered is tempered, we have  $D^{-m}\tilde{H}^s \subset \mathcal{T}'$ , hence  $D^{-m}\tilde{H}^s / P_{m-1} \subset \mathcal{T}' / P_{m-1}$  and the closed graph theorem shows that this inclusion is continuous. So  $D^{-m}\tilde{H}^s$  is even a semi-Hilbert subspace of  $\mathcal{T}'$ .

We now come to comparing  $D^{-m}\tilde{H}^s$  with  $H^{m+s}$ . For  $\Omega$  bounded we have :

### 1.5 $(D^{-m}\tilde{H}^s)(\Omega) = H^{m+s}(\Omega)$ , if $-m - \frac{n}{2} < s < \frac{n}{2}$

We first prove that any  $u \in H^{m+s}(\Omega)$  extends to an element of  $D^{-m}\tilde{H}^s(\mathbb{R}^n)$ . We know that  $u$  extends to some  $Pu \in H^{m+s}(\mathbb{R}^n)$  with compact support. Then  $\hat{P}u$  is  $C^\infty$  and we can easily see that  $\int |\tau|^{2s} |\tau^\alpha \hat{P}u(\tau)|^2 d\tau < \infty$ ,  $|\alpha| = m$ , since  $|\tau^\alpha|^2 \leq |\tau|^{2m}$  and  $|\tau|^{2m+2s}$  is locally summable (since  $2m + 2s > -n$ ), and  $|\tau|^{2m+2s} \leq 2(1 + |\tau|^2)^{m+s}$  outside some ball. Thus  $H^{m+s}(\Omega) \subset (D^{-m}\tilde{H}^s)(\Omega)$ .

Conversely, we may write  $D^{-m}\tilde{H}^s(\Omega) \subset D^{-m}(\tilde{H}^s)(\Omega)$ , with obvious meaning; on the other

hand,  $\tilde{H}^s(\Omega)$  is always contained in  $H^s(\Omega)$  for  $s < \frac{n}{2}$ ; and it is known that  $D^{-m}(H^s(\Omega)) = H^{m+s}(\Omega)$ .

This proves that  $D^{-m}\tilde{H}^s(\mathbb{R}^n) \subset H_{loc}^{m+s}(\mathbb{R}^n)$ , hence  $D^{-m}\tilde{H}^s(\mathbb{R}^n)/P_{m-1} \subset H_{loc}^{m+s}(\mathbb{R}^n)/P_{m-1}$ , and the closed graph theorem tells us this inclusion is continuous. Thus  $D^{-m}\tilde{H}^s(\mathbb{R}^n)$  is a (dense) semi-Hilbert subspace of  $H_{loc}^{m+s}(\mathbb{R}^n)$ .

Another point is that, on  $(D^{-m}\tilde{H}^s)(\Omega)/P_{m-1} \subset H^{m+s}(\Omega)/P_{m-1}$ , the two Hilbert structures are equivalent (both of them are Hilbert subspaces of  $\mathfrak{D}'(\Omega)/P_{m-1}$ , so apply the closed graph theorem). From this one can easily deduce that  $H_{\Omega}^{-m-s}(\mathbb{R}^n) \cap P_{m-1}^o$ , the dual of  $H^{m+s}(\Omega)/P_{m-1}^o$ , is closed in  $(D^{-m}\tilde{H}^s/P_{m-1})'$ .

## 2 Existence and uniqueness

We begin with stating an abstract frame: let  $E$  be a locally convex topological vector space,  $E'$  its dual,  $N$  a finite-dimensional (for simplicity) subspace of  $E$ ,  $N^o$  its orthogonal in  $E'$ .  $N^o$  is naturally identified with the dual space of  $E/N$ .

A linear subspace  $X$  of  $E$  is called a semi-Hilbert subspace of  $E$  with nullspace  $N$  if  $X$  is equipped with a semi-norm  $\|\cdot\|$  (with nullspace  $N$ ) deriving from a “semi-inner product” (nonnegative bilinear form)  $((\cdot, \cdot))$  such that  $X/N$  equipped with the natural norm  $\|x + N\| = \|x\|$  is a Hilbert space, and inclusion  $X/N \hookrightarrow E/N$  is continuous. Equivalently,  $X/N$  is a Hilbert subspace of  $E/N$  in the sense of Schwartz [10]. We know that  $X/N$  has a (unique) reproducing kernel in  $E/N$ , which is a linear mapping  $L$  from  $(E/N)'$  into  $X/N$ , satisfying  $((y, x)) = \langle e', x + N \rangle, \forall x \in X, \forall y \in Le'$ . To be slightly simpler, we will say that a linear mapping  $\theta : N^o \rightarrow E$  is reproducing kernel of  $X$  in  $E$ , if  $\theta e' \in Le' \forall e' \in N^o \simeq (E/N)'$ . Equivalently, a mapping  $\theta$  from  $N^o$  into  $X$  is a reproducing kernel of  $X$  in  $E$ , if and only if  $((\theta e', x)) = \langle e', x \rangle, \forall x \in X$ .

Semi-Hilbert spaces and reproducing kernels provide a simple and convenient language for splines. However, following theorems 2.1. and 2.2. could be easily deduced from [8, chapter 4].

**Theorem 2.1.** *Let  $M$  be a linear subspace of  $E'$  such that, if  $x \in N$  and  $\langle e', x \rangle = 0 \forall e' \in M$ , then  $x = 0$ . Let  $f^M \in X$ . There exists a unique element  $f^M$  in  $X$  satisfying  $\langle e', f^M \rangle = \langle e', f \rangle \forall e' \in M$ , with  $\|f^M\|$  minimum.*

*Proof.* The set  $f + M^o \cap X + N$  is a nonempty closed affine subspace of  $X/N$ , it has an element of minimum norm which is exactly  $f^M + N$ , which in turn contains one element  $f^M \in f + M^o \cap X$ , i.e.  $f^M$  satisfies  $\langle e', f^M \rangle = \langle e', f \rangle \forall e' \in M$ .  $\square$

**Theorem 2.2.** *Let us suppose that  $X$  is dense in  $E$ , so that  $N^o \subset (X/N)'$ ; and assume that  $M \cap N^o$  is closed in  $(X/N)'$ . Let  $\theta$  be a reproducing kernel of  $X$  in  $E$ . Then  $f^M$  coincides with the only  $g \in \theta(M \cap N^o) + N$  satisfying  $\langle e', g \rangle = \langle e', f \rangle \forall e' \in M$ .*

*Proof.*  $f^M + N$  is the orthogonal projection of  $O (\in X/N)$  onto  $f + M^o \cap X + N$  and therefore is the only element of  $f + M^o \cap X + N$  belonging to the orthogonal of  $M^o \cap X + N$  (in  $X/N$  for Hilbert structure), which is the image of  $(M^o \cap X + N)^o$  (orthogonal in  $(X/N)'$ ) by the

canonical isometry from  $(X/N)'$  onto  $X/N$ . But if  $M \cap N^o$  is closed in  $(X/N)'$ ,  $(M^o \cap X + N)^o$  is exactly  $M \cap N^o$ , whose image is  $\theta(M \cap N^o) + N$ .  $\square$

**Remark:** Frequently  $M$  is finite-dimensional (finitely many data to interpolate) so that  $M \cap N^o$  is automatically closed in  $(X/N)'$ .

Applying this to  $X = D^{-m}\tilde{H}^s$  with  $N = P_{m-1}$  and  $E = H_{loc}^{m+s}$ ,  $E' = H_{comp}^{-m-s}$ , we get:

**Theorem 2.3.** *Let  $M$  be a closed linear subspace of  $H_{\Omega}^{-m-s}(\mathbb{R}^n)$  ( $\Omega$  bounded), such that if  $p \in P_{m-1}$  and  $\langle \mu, p \rangle = 0 \forall \mu \in M$  then  $p = 0$ . Let  $f \in H_{\Omega}^{m+s}(\Omega)$ . There exists exactly one element  $f^M \in D^{-m}\tilde{H}^s$  satisfying  $\langle \mu, f^M \rangle = \langle \mu, f \rangle$ ,  $\forall \mu \in M$ , with minimum semi-norm  $\|f^M\|_{m,s}$ . Let  $\theta$  be a reproducing kernel of  $D^{-m}\tilde{H}^s$  in  $H_{loc}^{m+s}$ . Then the only  $g$  of the form  $\theta\mu + p$  with  $\mu \in M \cap P_{m-1}^o$ ,  $p \in P_{m-1}$  satisfying  $\langle \mu, g \rangle = \langle \mu, f \rangle$ ,  $\forall \mu \in M$ , is  $f^M$ .*

*Proof.* We just have to check that  $M \cap P_{m-1}^o$  is closed in  $(D^{-m}\tilde{H}^s/P_{m-1})'$ . But  $M$  is closed in  $H_{\Omega}^{-m-s}$ , hence  $M \cap N^o$  is closed in  $H_{\Omega}^{-m-s} \cap N^o$ , which is exactly the orthogonal, in  $(D^{-m}\tilde{H}^s/P_{m-1})'$ , of the subset of (equivalence classes mod  $P_{m-1}$  of) elements in  $D^{-m}\tilde{H}^s$  which are zero on  $\Omega$ . Thus  $M \cap N^o$  is closed in a closed subset of  $(D^{-m}\tilde{H}^s/P_{m-1})'$ , hence closed itself. All this is useless if  $M$  is finite-dimensional.  $\square$

### 3 Reproducing kernel of $D^{-m}\tilde{H}^s$

**Theorem 3.1.**  $\theta : \mu \mapsto (2\pi)^{-2m} \mu \odot \mathfrak{F}Pf. |\tau|^{-2m-2s}$  maps  $H_{comp}^{-m-s} \cap P_{m-1}^o$  into  $D^{-m}\tilde{H}^s$  and is a reproducing kernel of  $D^{-m}\tilde{H}^s$  as a semi-Hilbert subspace of  $H_{loc}^{m+s}(\mathbb{R}^n)$ .

*Proof.* Since  $D^{-m}\tilde{H}^s/P_{m-1}$  is a Hilbert subspace of  $H_{loc}^{m+s}/P_{m-1}$  it has a reproducing kernel, say  $L$ , which maps  $H_{comp}^{-m-s} \cap P_{m-1}^o$  (dual space of  $D^{-m}\tilde{H}^s/P_{m-1}$ ) into  $D^{-m}\tilde{H}^s/P_{m-1}$  and is characterized by:  $L\mu$  is the set of distributions  $u \in D^{-m}\tilde{H}^s$  satisfying:

$$(2\pi)^{2m} \int_{\mathbb{R}^n} |\tau|^{2s} (\tau^m \hat{u})(\tau) \cdot (\tau^m \hat{v})(\tau) d\tau = \langle \mu, v \rangle, \forall v \in D^{-m}\tilde{H}^s \quad (3.1)$$

We first deduce that  $|2\pi\tau|^{2m} \hat{u} = \hat{\mu} |\tau|^{-2s}$  (1). Now it is easily seen that  $v = (2\pi)^{-2m} \mu \odot \mathfrak{F}Pf. |\tau|^{-2m-2s}$  satisfies  $|2\pi\tau|^{-2m} \hat{v} = \hat{\mu} |\tau|^{-2s}$ , since  $\hat{v} = (2\pi)^{2m} \hat{\mu} Pf. |\tau|^{-2m-2s}$  and  $|2\pi|^{2m} Pf. |\tau|^{-2m-2s} = |\tau|^{-2s}$ . On the other hand, lemma 2 proves  $\tau^\alpha \hat{v} \in L_{loc}^1, \forall |\alpha| = m$ . So that for any  $u \in L\mu$  we have  $|\alpha|^{2m} (\hat{u} - \hat{v}) = 0$  and  $\tau^\alpha (\hat{u} - \hat{v}) \in L_{loc}^1, \forall |\alpha| = m$ . Lemma 3 then shows that  $u - v \in P_{m-1}$ , i.e.  $v \in L\mu$  since  $L\mu$  is an equivalence class modulo  $P_{m-1}$ . In other words  $L\mu = (2\pi)^{2m} \mu \odot \mathfrak{F}Pf. |\tau|^{-2m-2s} + P_{m-1}$ , i.e.  $\theta$  is a reproducing kernel of  $D^{-m}\tilde{H}^s$ .  $\square$

**Lemma 1.**  $|2\pi\tau|^{2m} \hat{\mu} = \hat{\mu} |\tau|^{-2s}$  if  $u \in L\mu$ .

*Proof.* Conditions  $s > -m - \frac{n}{2}$  implies  $\mathfrak{I} \subset D^{-m}\tilde{H}^s$ , so that we may apply (3.1) with  $w = \hat{\varphi}, \varphi \in \mathfrak{D}$ , and get  $(2\pi)^{2m} \int |\tau|^{2s} (\tau^m \hat{u})(\tau) \cdot \tau^m \varphi(\tau) d\tau = \langle \mu, \hat{\varphi} \rangle = \langle \hat{\mu}, \varphi \rangle, \forall \varphi \in \mathfrak{D}$ . This implies  $(2\pi)^{2m} |\tau|^{2s} \tau^m \cdot (\tau^m \hat{u})(\tau) = \hat{\mu}(\tau)$  a.e., hence  $(2\pi)^{2m} \tau^m \cdot (\tau^m \hat{u})(\tau) = |\tau|^{-2s} \hat{\mu}(\tau)$  a.e. and  $(2\pi\tau)^{2m} \hat{u} = \mu |\tau|^{-2s}$  as distributions.  $\square$

**Lemma 2.** *If  $\mu$  is a distribution with compact support, orthogonal to  $P_{m-1}$  i.e.  $\mu \in \mathfrak{E}' \cap P_{m-1}^o$ , then  $\tau^\alpha \hat{\mu} Pf. |\tau|^{-2m-2s} \in L_{loc}^1, \forall |\alpha| = m$ .*

*Proof.* It suffices to show that the function (in usual sense)  $\tau^\alpha \hat{\mu}(\tau) |\tau|^{-2m-2s}$  is locally summable. But, since  $\mu$  is orthogonal to  $P_{m-1}$ , the  $C^\infty$  function  $\hat{\mu}$  has derivatives of order  $\leq m-1$  vanishing at 0, so that  $|\hat{\mu}(\tau)| \leq c|\tau|^m$  on a neighbourhood of 0. Then  $\tau^\alpha \hat{\mu}(\tau) |\tau|^{-2m-2s} \leq c|\tau|^{-2s}$  on that neighbourhood of 0, and is  $C^\infty$  elsewhere, so is locally summable since  $s < \frac{n}{2}$ .  $\square$

**Lemma 3.** *Any tempered distribution  $T$  such that  $|\tau|^{2m} \hat{T} = 0$  and  $\tau^\alpha \hat{T} \in L_{loc}^1$  is in  $P_{m-1}$ .*

*Proof.*  $T$  is supported by 0, since  $|\tau|^{2m} \tau = 0$ . Then  $\tau^\alpha \hat{T}$  is also supported by 0 and should be in  $L_{loc}^1$  which is possible only if  $\tau^\alpha \hat{T} = 0$ , and then  $D^\alpha T = 0, \forall |\alpha| = m$ , i.e.  $T \in P_{m-1}$ .  $\square$

## 4 A general characterization result

To be more explicit, we now use formulas giving Fourier transforms of pseudo-functions  $Pf. |\tau|^\lambda$ . In general  $Pf. |\tau|^\lambda = c Pf. |t|^{-n-\lambda}$  except if  $\lambda$  or  $-n-\lambda$  is an even positive integer  $2k$ :  $\mathfrak{F} |\tau|^{2k} = c \Delta^k \delta$ ,  $\mathfrak{F} Pf. |\tau|^{-n-2k} = c |t|^{2k} \log |t| + |t|^{2k}$ .

For simplicity, we assume  $s > -m$ . Then  $\mathfrak{F} Pf. |\tau|^{-2m-2s}$  is  $c |t|^{2m+2s-n} \log |t| + c |t|^{2m+2s-n}$  if  $2m+2s-n$  is an even positive integer,  $c |t|^{2m+2s-n}$  if not. It is easily seen that, in the first case ( $2m+2s-n = 2k$ ), if  $\mu \in P_{m-1}^o$  then  $\mu \odot |t|^{2m+2s-n} \in P_{m-1}$ . So that, putting  $K_\lambda(t) = |t|^\lambda \log |t|$  if  $\lambda$  is an even positive integer  $K_\lambda(t) = |t|^\lambda$  otherwise, the mapping  $\mu \mapsto \mu \odot K_{2m+2s-n}$  is proportional to a reproducing kernel of  $D^{-m} \tilde{H}^s$  when  $m+s > 0$ .

Now we are able to explicit theorem 2.3 (in the case  $m+s > 0$ , that is,  $H_{loc}^{m+s}$  is a space of (classes of locally summable) functions):

**Theorem 4.** *Let  $M$  be a closed linear, subspace of some  $H_{\Omega}^{-m-s}(\mathbb{R}^n)$  ( $\Omega$  bounded), satisfying: if  $p \in P_{m-1}$  and  $\langle \mu, p \rangle = 0, \forall \mu \in M$ , then  $p = 0$ . Let  $f \in H^{m+s}(\Omega)$ . Then there exists a unique function  $f \in D^{-m} \tilde{H}^s(\mathbb{R}^n)$  satisfying  $\langle \mu, f^M \rangle = \langle \mu, f \rangle \forall \mu \in M$ , with minimum seminorm  $\|f^M\|_{m,s}$ . Moreover, if  $g = \nu \odot K_{2m+2s-n} + p$  (with  $\nu \in M \cap P_{m-1}^o$  and  $p \in P_{m-1}$ ) satisfies  $\langle \mu, g \rangle = \langle \mu, f \rangle \forall \mu \in M$ , then  $g = f^M$ .*

Let us now restrict ourselves to the important case where  $m+s > \frac{n}{2}$ , so that  $H_{loc}^{m+s}$  is a space of continuous functions (Sobolev theorem), and data are finitely many point values.

**Theorem 4. bis** *Let  $A$  be a finite subset of  $\mathbb{R}^n$ , containing a  $P_{m-1}$  - unisolvent subset. Then there exists exactly one function of the form  $\sigma(t) = \sum_{a \in A} \lambda_a K_{2m+2s-n}(t-a) + p(t)$  with  $p \in P_{m-1}$  and  $\sum_{a \in A} \lambda_a q(a) = 0, \forall q \in P_{m-1}$ , taking prescribed values on  $A$ . Moreover, if  $f$  is another function taking the same values on  $A$ , one has  $\|f\|_{m,s} \geq \|\sigma\|_{m,s}$ .*

Actually,  $\sum_{a \in A} \lambda_a K_{2m+2s-n}(t-a)$  is  $(\sum_{a \in A} \lambda_a \delta_a \odot K_{2m+2s-n})(t)$  Existence of a function  $f \in D^{-m} \tilde{H}^s$  taking prescribed values on  $A$  (finite) is obvious:  $f$  may even be chosen in  $\mathfrak{D}$ .

## 5 Examples

### 5.1 Pseudo-polynomial splines

We put  $s = \frac{n-1}{2}$  and consider a finite set  $A \subset \mathbb{R}^n$  containing some  $P_{m-1}$  - unisolvent subset. Then there exists exactly one function of the form  $\sigma(t) = \sum_{a \in A} \lambda_a |t - a|^{2m-1} + p(t)$  where  $p \in P_{m-1}$  and  $\sum_{a \in A} \lambda_a q(a) = 0, \forall q \in P_{m-1}$ , taking prescribed values on  $A$ . For all  $f$  taking the same values on  $A$  one has  $\|f\|_{m, \frac{n-1}{2}} \geq \|\sigma\|_{m, \frac{n-1}{2}}$ .

For  $m = 1$  we get *multi-conic functions*  $\sum \lambda_a |t - a| + C$  with  $\sum \lambda_a = 0$ , and the set  $A$  must only contain two distinct points. The functional minimized is  $\int_{\mathbb{R}^n} |\tau|^{n-1} |\mathfrak{F} Dv(\tau)|^2 d\tau$ .

For  $m = 2$  we get *pseudo-cubic splines*, if  $A$  is not contained in a hyperplane (a line if  $n = 2$ ): functions of the form  $\sum \lambda_a |t - a|^3 + \alpha t + \beta$  with  $\sum \lambda_a = 0$  and  $\sum \lambda_a a = 0$ . Coefficients  $(\lambda_a; a \in A)$  and  $\alpha_1, \alpha_2, \beta$  may be computed from the linear system:

$$\begin{cases} \sum_{a \in A} |a - b|^3 \lambda_a + b_1 \alpha_1 + b_2 \alpha_2 + \beta = f(b) & (b \in A) \\ \sum_{a \in A} \lambda_a a_1 = 0 \\ \sum_{a \in A} \lambda_a a_2 = 0 \\ \sum_{a \in A} \lambda_a = 0 \end{cases}$$

We notice that, for  $n = 1$ , we get simply polynomial splines: polynomials of degree  $\leq 2m - 1$  on intervals,  $C^2$ , and degenerating to polynomials of degree  $\leq m - 1$  at both ends (thanks to conditions  $\sum \lambda_a a^k = 0, \dots, m - 1$ ).

### 5.2 Thin plate functions

Putting  $s = 0$  as an example,  $n = 2$  and  $m = 2$ , we get functions of the form  $\sigma(t) = \sum_{a \in A} \lambda_a |t - a|^2 \log |t - a| + \alpha t + \beta$  with  $\sum_{a \in A} \lambda_a = 0$  and  $\sum_{a \in A} \lambda_a a = 0$  (function  $|t|^2 \log |t|$  is extended to 0 at 0, so as to be continuous). In this case we have  $\int_{\mathbb{R}^2} |D^2 \sigma|^2 \leq \int_{\mathbb{R}^2} |D^2 f|^2$  for all  $f$  that coincides with  $\sigma$  on  $A$ . The set  $A$  must not be contained in a line.

### 5.3 Hermite polynomials

Since  $H_{loc}^{2+\frac{n-1}{2}}(\mathbb{R}^n) \subset C^1$ , we may minimize semi-norm  $\|\cdot\|_{2, \frac{n-1}{2}}$  subject to Hermite conditions: values and gradients prescribed on a finite set  $A$ . We get functions of the form  $\sigma(t) = \sum_{a \in A} \lambda_a |t - a|^3 + \sum \lambda'_a (t - a) |t - a| + \alpha t + \beta$ , with  $\sum \lambda_a = 0$ ,  $\sum \lambda_a a + \frac{1}{3} \sum \lambda'_a = 0$ . In one dimension this corresponds to ordinary piecewise cubic Hermite interpolation.

## 6 Convergence in $H^{m+s}(\Omega)$

Let  $f \in H^{m+s}(\Omega)$ , and let  $(M_k)$  be a sequence of closed linear subspaces of  $H_{\Omega}^{-m-s}(\mathbb{R}^n)$ . We suppose that for any  $\mu \in H_{\Omega}^{-m-s}(\mathbb{R}^n)$ , the distance from  $\mu$  to  $M_k$  converges to 0. Then:



1. For  $k$  sufficiently large,  $M_k$  is such that: if  $p \in P_{m-1}$  satisfies  $\langle \mu, p \rangle = 0, \forall \mu \in M_k$ , then  $p = 0$ . So that there exists a unique  $f_k \in D^{-m}\tilde{H}^s$  satisfying  $\langle \mu, f_k \rangle = \langle \mu, f \rangle, \forall \mu \in M_k$ , with  $\|f_k\|_{m,s}$  minimum.
2.  $f_k \rightarrow f$  in  $H^{m+s}(\Omega)$ .

This is a straightforward consequence of a general result of J.L. Joly [7], putting  $X = H^{m+s}(\Omega)$ ,  $Y = H^{m+s}(\Omega)/P_{m-1}$  with a norm derived from  $\|\cdot\|_{m,s}$ . Another way to see it (partially) is the following: put  $f^\Omega =$  the minimal extension of  $f$ , relatively to  $\|\cdot\|_{m,s}$ , i.e. the unique element in  $D^{-m}\tilde{H}^3$  that coincides with  $f$  on  $\Omega$  with minimum semi-norm  $\|\cdot\|_{m,s}$ . It is uniquely written  $\mu \odot K + p$  with  $p \in P_{m-1}$  and  $\mu \in H_{\bar{\Omega}}^{-m-s}(\mathbb{R}^n) \cap P_{m-1}^o$  ( $K = \mathfrak{F}Pf. |\tau|^{-2m-2s}$ ) Same thing for  $f_k = \mu_k \odot A + p_k$  with  $p_k \in P_{m-1}$  and  $\mu_k \in M_k \cap P_{m-1}^o$ . And  $\mu_k$  is simply the orthogonal projection of  $\mu$  onto  $M_k \cap P_{m-1}^o$ , in Hilbert space  $H_{\bar{\Omega}}^{-m-s}(\mathbb{R}^n) \cap P_{m-1}^o$  equipped with norm induced by  $\left(D^{-m}\tilde{H}^s/P_{m-1}\right)'$  (equivalent to that induced by  $H_{\bar{\Omega}}^{-m-s}(\mathbb{R}^n)$ ). So that  $\|\mu_k - \mu\| \leq cd(\mu, M_k) \rightarrow 0$ , hence  $\|\mu_k \odot K - \mu \odot K\|_{m,s} \rightarrow 0$  and this proves that  $f_k + P_{m-1} \rightarrow f + P_{m-1}$  in  $H^{m+s}(\Omega)/P_{m-1}$ .

Let us now specialize to the case where  $m+s > \frac{n}{2}$  and  $M_k$  is spanned by Dirac masses  $(\delta_a; a \in A_k)$  where  $(A_k)$  is a sequence of subsets of  $\bar{\Omega}$ . Then the condition  $cd(\mu, M_k) \rightarrow 0, \forall \mu \in H_{\bar{\Omega}}^{-m-s}$  is equivalent to saying that any point in  $\Omega$  is limit of a sequence  $(a_k \in A_k)$ , or that Hausdorff distance from  $A_k$  to  $\Omega$  tends to zero. This results from complete continuity of inclusion  $H_{\bar{\Omega}}^{m+s}(\Omega) \hookrightarrow C(\Omega)$  (a bounded subset of  $H_{\bar{\Omega}}^{m+s}(\Omega)$  is an equicontinuous set of functions on  $\bar{\Omega}$ ). We then get:

**Theorem 5.** *If  $(A_k)$  is a sequence of subsets of  $\bar{\Omega}$  ( $\Omega$  bounded open subset of  $\mathbb{R}^n$ ) such that  $d(t, A_k) \rightarrow 0, \forall t \in \Omega$ , and  $f \in H^{m+s}(\Omega)$  with  $m+s > \frac{n}{2}$ , then the sequence  $(f_k)$  of functions coinciding with  $f$  on  $A_k$  with minimum semi-norm  $\|\cdot\|_{m,s}$  (uniquely determined for sufficiently large  $k$ ) satisfies  $f_k \rightarrow f$  in  $H^{m+s}(\Omega)$ .*

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