Spline mimimizing rotation-invariant semi-norms in Sobolev spaces

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We define a family of semi-norms $\|u\|_{m,s} = \left(\int_{\mathbb{R}^n} |\tau|^{2s} |\mathfrak{F} D^m u(\tau)|^2 \, d\tau\right)^{1/2}$. Minimizing such semi-norms, siubject to some interpolating conditions, leads to function of very simple forms, providing interpolation methods that: 1) preserve polynomials of degree $\leq m-1$; 2) commute with similarities as well as translations and rotations of $\mathbb{R}^n$; and 3) converge in Sobolev spaces $H^{m+s}(\Omega)$.
Typical examples of such splines are: "thin plate" functions ($\sum_{a \in A} \lambda_a |t-a|^2 \log |t-a|$ $+ \alpha t + \beta$ with $\sum \lambda_a = 0$, $\sum \lambda_a a = 0$), "multi-conic" functions ($\sum \lambda_a |t-a| + C$ with $\sum \lambda_a = 0$), pseudo-cubic splines ($\sum \lambda_a |t-a|^3 + \alpha t + \beta$ with $\sum \lambda_a = 0$, $\sum \lambda_a a = 0$), as well as usual polynomial splines in one dimension. In general, data functional are only supposed to be distributions with compact supports, belonging to $H^{-m-s}(\mathbb{R}^n)$; there may be infinitely many of them. Splines are then expressed as convolutions $\mu \odot |t|^{2m+2s-n}$ (or $\mu \odot |t|^{2m+2s-n} \log |t|$) + polynomials.

0 Introduction

Splines in more than one dimension are usually constructed from one-dimensional ones, via tensor products. We follow here another point of view (more analogous to the physical interpretation of elementary cubic splines as equilibrium positions of a beam) developed by M. Atteia [1, 2, 3]: his splines (minimizing $\int_\Omega |D^2 v|^2$, a functional similar to the bending energy of a thin plate) are uneasy to compute, because their characterization involves a kernel given by series. But things are much simpler if we replace Ω by the whole plane $\mathbb{R}^2$, as is shown in [5](where present extensions are announced). This leads to what we call "thin plate" functions in $\mathbb{R}^2$ (engineers say "surface splines" [6]). Of course, it is possible to minimize $\int |D^m v|^2$instead of $\int |D^2 v|^2$, and deal with $\mathbb{R}^n$ instead of $\mathbb{R}^2$ (with some restriction: $m > \frac{n}{2}$, if point values are used).

We notice that these functionals $\int_{\mathbb{R}^n} |D^m v|^2$are invariant through translations and rotations. Moreover, if a similarity $t \to \lambda t$ is applied to v, they are multiplied by some power of $|\lambda|$.

Thus, corresponding interpolation methods will *commute with similarities*: interpolating on a contracted set of points λA gives the same result as interpolating on A (with same values) and then applying contraction $t \to \lambda t$.

Now, since Fourier transform is isometric on $L^2(\mathbb{R}^n)$, we may write $\int_{\mathbb{R}^n} |D^m v(t)|^2 \, dt = \int_{\mathbb{R}^n} |\mathfrak{F} D^m v(\tau)|^2 \, d\tau$. A natural idea, to get other interpolation methods, would be to introduce a weighting function w and minimize $\int_{\mathbb{R}^n} w(\tau) \, |\mathfrak{F} D^m v(\tau)|^2 \, d\tau$. In view of the above invariance properties, it is natural to put $w(\tau) = |\tau|^\theta$ and try to minimize $\int_{\mathbb{R}^n} |\tau|^\theta \, |\mathfrak{F} D^m v(\tau)|^2 \, d\tau$, a functional which is invariant through translations and rotations, and is multiplied by a constant if the variable t is changed into λt. This is actually possible (at least if $-2m - n < 0 < n$ and $2m + \theta$ is sufficiently large, depending on which kind of data are used), in a precise sense, as we shall see.

We first introduce some "Sobolev-type" spaces such as $\tilde{H}^s$, $D^{-m}\tilde{H}^s$, H^{m+s}_{loc}, H^{-m-s}_{comp}, and compare them. Central space is $D^{-m}\tilde{H}^s/P_{m-1}$ with Hilbert structure. Its dual contains $H^{-m-s}_{comp} \cap P^o_{m-1}$ as a dense subset (1).

In 2 we prove existence and uniqueness for interpolation problems, and abstract characterization using reproducing kernels of semi-Hilbert spaces.

3 is the crucial one. We compute the reproducing kernel of $D^{-m}\tilde{H}^s/P_{m-1}$ in H^{m+s}_{loc}/P_{m+1}, i.e. the natural isometric mapping from $H^{-m-s}_{comp} \cap P^o_{m-1}$, embedded in $(D^{-m}\tilde{H}^s/P_{m-1})$ into $D^{-m}\tilde{H}^s/P_{m-1}$.

4 summarizes our main result (theorems 4 and 4 bis). In 5 we apply it to various examples.

6 is concerned with convergence in Sobolev space $H^{m+s}(\Omega)$ (Ω bounded) for an interpolated function $f \in H^{m+s}(\Omega)$.

For brevity, we use classical or natural notations without explicit statement. Moreover, some familiarity with distributions is assumed (convolution, derivation, Fourier transform, multiplication with C^∞ functions, and their relations; correspondence between locally summble functions and distributions, etc.; standard reference is [9]). $D^m v(t)$ is the n^m-tuple of partial derivatives $(D_{i_1} \ldots D_{i_m} v(t); i_1, \ldots, i_m = 1, \ldots, n)$, with Euclidian norm $\left(\sum |D^m v(t)| = |D_{i_1} \ldots D_{i_m} v(t)|^2\right)^{1/2}$. Pseudo-functions $Pf.\, |\tau|^\theta$ are to be found in [9, p. 44] and their Fourier transforms p. 257. Fourier transform is $\mathfrak{F}$ or $\hat{}$.

The letter c is used to denote various constants, to avoid useless complication.

1 Functional spaces

1.1 Sobolev spaces H^s, H^s_{loc}, H^{-s}_{comp}

For any real s, $H^s(\mathbb{R}^n)$ is the set of tempered distributions u on $\mathbb{R}^n$ whose Fourier transform $\hat{u}$ is a (locally summable) function such that $\int_{\mathbb{R}^n} \left(1 + |\tau|^2\right)^s |\hat{u}(\tau)|^2 \, d\tau < \infty$. In other words $H^s = \mathfrak{F}\left(1 + |\tau|^2\right)^{-s/2} L^2$. It is a Hilbert space. Its dual is naturally identified with $H^{-s}(\mathbb{R}^n)$. When s is a positive integer, $H^s(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n); D^\alpha u \in L^2(\mathbb{R}^n), \forall\, |\alpha| \le s\right\}$.

If K is a closed subset of $\mathbb{R}^n$, $H^s_K(\mathbb{R}^n)$ is the set of distributions $\in H^s(\mathbb{R}^n)$ whose support is contained in K. H^s_K is a closed linear subspace of H^s.

If Ω is an open subset of $\mathbb{R}^n$, $H^s(\Omega)$ is the set of restrictions, to Ω, of distributions$\in H^s(\mathbb{R}^n)$. It is isomorphic to the quotient space $H^s(\mathbb{R}^n)/H^s_{\Omega^c}(\mathbb{R}^n)$ hence a Hilbert space too. Its dual is naturally identified with $H^{-s}_{\bar{\Omega}}(\mathbb{R}^n)$.

If Ω is bounded and sufficiently regular (e.g. Ω satisfies some "uniform cone" condition) then (for non integer $s > 0$) $H^s(\Omega)$ is the set of distributions u on Ω whose derivatives of order $[s]$ (integralpart of s) are in $L^2(\Omega)$, with

$$\iint_{\Omega\times\Omega} \frac{\left|D^{[s]}u(t) - D^{[s]}u(t')\right|^2}{|t-t'|^{n+2(s-[s])}} dtdt' < \infty.$$

For integer $s \geq 0$ it suffices that $D^\alpha u \in L^2(\Omega), \forall\, |\alpha| = s$.

Now $H^s_{loc}(\mathbb{R}^n)$ is the set of distributiorson $\mathbb{R}^n$ whose restriction to any bounded open set Ω is in $H^s(\Omega)$. Of course $C^k \subset H^s_{loc}$ for integer $k \geq s$. On the other hand, Sobolev embedding theorems assert that $H^s_{loc}(\mathbb{R}^n) \subset C^k$ for $s > k + \frac{n}{2}$.

H^s_{loc} may be equipped with a Fréchet space structure: putting $B_N =$ the open ball $|t| < N$, one defines a countable family of semi-norms $u \mapsto$ norm of $u_{|B_N}$ in $H^s(B_N)$. It is reflexive, and its dual is naturally identified with $H^{-s}_{comp}(\mathbb{R}^n)$, the union of $H^{-s}_{\bar{B}_N}(\mathbb{R}^n)$, equipped with the topology of (strict) inductive limit of this countable family of Hilbert spaces.

All previous norms could have been replaced by equivalent ones, they do not play a role by themselves in our problem. We now come to definig spaces $\tilde{H}^s$ and $D^{-m}\tilde{H}^s$,whose semi-norms $\|.\|_{m,s}$ are fundamental.

1.2 $\tilde{H}^s(\mathbb{R}^n)$, $s < \frac{n}{2}$

We put $\tilde{H}^s(\mathbb{R}^n) = \left\{u \in \mathfrak{J}'(\mathbb{R}^n); \hat{u} \in L^1_{loc}, \int_{\mathbb{R}^n} |\tau|^{2s} |\hat{u}(\tau)|^2 d\tau < \infty\right\}$. We equip it with norm $\|u\|_{0,s} = \left(\int_{\mathbb{R}^n} |\tau|^{2s} |\hat{u}(\tau)|^2 d\tau\right)^{1/2}\}$. $\tilde{H}^s(\mathbb{R}^n)$ *is a Hilbert space* if (and only if) $s < \frac{n}{2}$: for $f \in L^2(\mathbb{R}^n)$, the function $|\tau|^{-s} f(\tau)$ is locally summable, since $\int_K |\tau|^{-s} |f(\tau)| d\tau \leq \left(\int_K |\tau|^{-2s}\right)^{1/2} \left(\int_K |f(\tau)|^2 d\tau\right)^{1/2}$ and $|\tau|^\lambda$ is locally sunnmble if (and only if) $\lambda > -n$; therefore $|\tau|^{-s} f$ defines a distribution, easily seen to be tempered, so that $f \mapsto \mathfrak{F}\left(|\tau|^{-s} f\right)$ is an isometry from $L^2(\mathbb{R}^n)$ onto $\tilde{H}^s(\mathbb{R}^n)$. Moreover if $f_j \to 0$ in L^2, one can prove that $|\tau|^{-s} f_j \to 0$ in $\mathfrak{J}'$ (it suffices to show that $\int |\tau|^{-s} f_j(\tau)\varphi(\tau) d\tau \to 0$ for any C^∞ function φ rapidly decreasing at ∞). This implies that inclusion $\tilde{H}^s \curvearrowleft \mathfrak{J}'$ (a fortiori $\tilde{H}^s \curvearrowleft \mathfrak{D}'$) is continuous, or that $\tilde{H}^s$ *is a Hilbert subspace of* $\mathfrak{D}'$.

We might even prove that, for $-\frac{n}{2} < s < \frac{n}{2}$, $\tilde{H}^s$ is a *normal* subspace of $\mathfrak{D}'$ (i.e. contains $\mathfrak{J}$ as a dense subspace) with dual $\tilde{H}^s$.

(Some of these spaces are known and used in other fields. For example, $\tilde{H}^{-1}\left(\mathbb{R}^3\right)$ and $\tilde{H}^1\left(\mathbb{R}^3\right)$ are respectively spaces of charges and potentials of finite energy, in the theory of Newtonian potentials in $\mathbb{R}^3$. See [10, §11]).

From definitions, it is obvious that $H^s(\mathbb{R}^n) \subset \tilde{H}^s(\mathbb{R}^n)$ if $s > 0$, the converse if $s < 0$ (and $\tilde{H}^o = H^o = L^2$). In general $H^s(\mathbb{R}^n) \neq \tilde{H}^s(\mathbb{R}^n)$ ($\tilde{H}^s$ is not contained in L^2, for $s > 0$, while $H^s \subset L^2$). But (at least for $-\frac{n}{2} < s < \frac{n}{2}$) *equality holds on bounded subsets, as we now see.*

1.3 $H^s(\Omega) = \tilde{H}^s(\Omega)$, if $-\frac{n}{2} < s < \frac{n}{2}$

Of course, $H^s(\Omega)$ is the set of restrictions, to Ω (bounded), of elements of $\tilde{H}^s(\Omega)$, with corresponding Hilbert structure. In fact, we can prove that any $u \in \tilde{H}^s(\Omega)$ $(s > 0)$ is sum of $u_1 \in H^s(\mathbb{R}^n)$ and a C^∞ function u_2. We first note that $|\tau|^{2s} \leq \left(1+|\tau|^2\right)^s \leq 2\,|\tau|^{2s}$ except for $\tau \in$ some ball B. Now for $u \in \tilde{H}^s$, $\hat{u}$ is a function. We write it as $v_1 + v_2$ where v_2 coincides with $\hat{u}$ a.e. on B and is zero outside, v_i satisfies $\int_{\mathbb{R}^n} \left(1+|\tau|^2\right)^s |v_1(\tau)|^2 \, d\tau = \int_{B^C} \left(1+|\tau|^2\right)^s |v_1(\tau)|^2 \, d\tau \leq \int_{B^C} |\tau|^{2s} |v_1(\tau)|^2 \, d\tau < \infty$, so that its inverse Fourier transform $u_1 \in H^s(\mathbb{R}^n)$. On the other hand, v_2 has compact support, hence its (inverse) Fourier transform u_2 is a C^∞ function (in fact, an "entire function of exponential type", moreover $\to 0$ at ∞ since $v_2 \in L^1$). Case $s < 0$ is exactly similar: we prove $H^s \subset \tilde{H}^s + C^\infty$.

We have already seen that $C^\infty \subset H^s_{loc}$, so that $\tilde{H}^s(\Omega) = H^s(\Omega)$ if $s > 0$. For $s < 0$ we must prove that $C^\infty \subset \tilde{H}^s_{loc}$; *but for* $s > \frac{n}{2}$ we have $\mathfrak{D} \subset \tilde{H}^s$, which gives the result. So in general $\tilde{H}^s(\Omega) = H^s(\Omega)$ for Ω bounded, whenever $-\frac{n}{2} < s < \frac{n}{2}$.

Inclusion $H^s(\Omega) \curvearrowleft \tilde{H}^s(\Omega)$ for $s > 0$ ($\tilde{H}^s(\Omega) \curvearrowleft H^s(\Omega)$ for $s < 0$) is continuous, so that the two Hilbert structures induced on $\tilde{H}^s(\Omega) = H^s(\Omega)$ are comparable, hence equivalent, from one of Banach's theorems.

1.4 Semi-Hilbert spaces $D^{-m}\tilde{H}^s$

We put $D^{-m}\tilde{H}^s = \left\{u \in \mathfrak{D}'(\mathbb{R}^n); D^\alpha u \in \tilde{H}^s(\mathbb{R}^n), \forall\, |\alpha| = m\right\}$, equipped with natural semi-norm $\|u\|_{m,s} = \left(\int_{\mathbb{R}^n} |\tau|^{2s} |\mathfrak{F} D^m u(\tau)|^2 \, d\tau\right)^{1/2}$. We also consider the quotient space $D^{-m}\tilde{H}^s / P_{m-1}$ with corresponding norm. These are *spaces of Beppo Levi type* (see [4], where these spaces would be denoted $BL_m\left(\tilde{H}^s\right)$ and $BL_{\dot{m}}\left(\tilde{H}^s\right)$ respectively): since $\tilde{H}^s$ is a Hilbert subspace of $\mathfrak{D}'$, $D^{-m}\tilde{H}^s/P_{m-1}$ is a Hilbert subspace of $\mathfrak{D}'/P_{m-1}$ (i.e. $D^{-m}\tilde{H}^s$ is a semi-Hilbert subspace of $\mathfrak{D}'$, see 2). In fact, since $\tilde{H}^s \subset \mathfrak{I}'$, and a distribution whose derivatives are tempered is tempered, we have $D^{-m}\tilde{H}^s \subset \mathfrak{I}'$, hence $D^{-m}\tilde{H}^s/P_{m-1} \subset \mathfrak{I}'/P_{m-1}$ and the closed graph theorem shows that this inclusion is continuous. So$D^{-m}\tilde{H}^s$ is even a semi-Hilbert subspace of $\mathfrak{I}'$.

We now come to comparing $D^{-m}\tilde{H}^s$ withH^{m+s} . For Ω bounded we have :

1.5 $\left(D^{-m}\tilde{H}^s\right)(\Omega) = H^{m+s}(\Omega)$, if $-m - \frac{n}{2} < s < \frac{n}{2}$

We first prove that any $u \in H^{m+s}(\Omega)$ extends to an element of $D^{-m}\tilde{H}^s(\mathbb{R}^n)$. We know that u extends to some $Pu \in H^{m+s}(\mathbb{R}^n)$ with compact support. Then $\hat{Pu}$ is C^∞ and we can easily see that $\int |\tau|^{2s} \left|\tau^\alpha \hat{Pu}(\tau)\right|^2 d\tau < \infty$, $|\alpha| = m$, since $|\tau^\alpha|^2 \leq |\tau|^{2m}$ and $|\tau|^{2m+2s}$ is locally summble (since $2m + 2s > -n$) ,and $|\tau|^{2m+2s} \leq 2\left(1+|\tau|^2\right)^{m+s}$ outside some ball. Thus $H^{m+s}(\Omega) \subset \left(D^{-m}\tilde{H}^s\right)(\Omega)$.

Conversely, we may write $D^{-m}\tilde{H}^s(\Omega) \subset D^{-m}\left(\tilde{H}^s\right)(\Omega)$, with obvious meaning; on the other hand, $\tilde{H}^s(\Omega)$ is always contained in $H^s(\Omega)$ for $s < \frac{n}{2}$; and it is known that $D^{-m}(H^s(\Omega)) = H^{m+s}(\Omega)$.

This proves that $D^{-m}\tilde{H}^s(\mathbb{R}^n) \subset H^{m+s}_{loc}(\mathbb{R}^n)$, hence $D^{-m}\tilde{H}^s(\mathbb{R}^n)/P_{m-1} \subset H^{m+s}_{loc}(\mathbb{R}^n)/P_{m-1}$, and the closed graph theorem tells us this inclusion is continuous. Thus $D^{-m}\tilde{H}^s(\mathbb{R}^n)$ *is a (dense) semi-Hilbert subspace of* $H^{m+s}_{loc}(\mathbb{R}^n)$.

Another point is that, on $\left(D^{-m}\tilde{H}^s\right)(\Omega)/P_{m-1} \subset H^{m+s}(\Omega)/P_{m-1}$, the two Hilbert structures are equivalent (both of them are Hilbert subspaces of $\mathfrak{D}'(\Omega)/P_{m-1}$, so apply the closed graph theorem). From this one can easily deduce that $H^{-m-s}_{\bar{\Omega}}(\mathbb{R}^n) \cap P^o_{m-1}$, the dual of $H^{m+s}(\Omega)/P^o_{m-1}$, is closed in $\left(D^{-m}\tilde{H}^s/P_{m-1}\right)'$.

2 Existence and uniqueness

We begin with stating an abstract frame: let E be a locally convex topological vector space, E' its dual, N a finite-dimensional (for simplicity) subspace of E, N^o its orthogonal in E'. N^o is naturally identified with the dual space of E/N.

A linear subspace X of E is called a semi-Hilbert subspace of E with nullspace N if X is equipped with a semi-norm $\|.\|$ (with nullspace N) deriving from a "semi-inner product" (nonnegative bilinear form) $((.,.))$ such that X/N equipped with the natural norm $\|x+N\| = \|x\|$ is a Hilbert space, and inclusion $X/N \hookrightarrow E/N$ is continuous. Equivalently, X/N is a Hilbert subspace of E/N in the sense of Schwartz [10]. We know that X/N has a (unique) reproducing kernel in E/N, which is a linear mapping L from $(E/N)'$ into X/N, satisfying $((y,x)) = \langle e', x+N\rangle, \forall x \in X, \forall y \in Le'$. To be slightly simpler, we will say that a linear mapping $\theta : N^o \to E$ is reproducing kernel of X in E, if $\theta e' \in Le' \forall e' \in N^o \simeq (E/N)'$. Equivalently, a mapping θ from N^o into X is a reproducing kernel of X in E, if and only if $((\theta e', x)) = \langle e', x\rangle, \forall x \in X$.

Semi-Hilbert spaces and reproducing kernels provide a simple and convenient language for splines. However, following theorems 2.1. and 2.2. could be easily deduced from [8, chapter 4].

Theorem 2.1. *Let M be a linear subspace of E' such that, if $x \in N$ and $\langle e', x\rangle = 0 \forall e' \in M$, then $x = 0$. Let $f^M \in X$. There exists a unique element f^M in X satisfying $\langle e', f^M\rangle = \langle e', f\rangle \forall e' \in M$, with $\|f^M\|$ minimum.*

Proof. The set $f + M^o \cap X + N$ is a nonempty closed affine subspace of X/N, it has an element of minimum norm which is exactly $f^M + N$, which in turn contains one element $f^M \in f + M^o \cap X$, i.e. f^M satisfies $\langle e', f^M\rangle = \langle e', f\rangle \forall e' \in M$. □

Theorem 2.2. *Let us suppose that X is dense is E, so that $N^o \subset (X/N)'$; and assume that $M \cap N^o$ is closed in $(X/N)'$. Let θ be a reproducing kernel of X in E. Then f^M coincides with the only $g \in \theta(M \cap N^o) + N$ satisfying $\langle e', g\rangle = \langle e', f\rangle \forall e' \in M$.*

Proof. $f^M + N$ is the orthogonal projection of O ($\in X/N$) onto $f + M^o \cap X + N$ and therefore is the only element of $f + M^o \cap X + N$ belonging to the orthogonal of $M^o \cap X + N$ (in X/N for Hilbert structure), which is the image of $(M^o \cap X + N)^o$ (orthogonal in $(X/N)'$) by the canonical isometry from $(X/N)'$ onto X/N. But if $M \cap N^o$ is closed in $(X/N)'$, $(M^o \cap X + N)^o$ is exactly $M \cap N^o$, whose image is $\theta\,(M \cap N^o) + N$. □

Remark: Frequently M is finite-dimensional (finitely many data to interpolate) so that $M \cap N^o$ is automatically closed in $(X/N)'$.

Applying this to $X = D^{-m}\tilde{H}^s$ with $N = P_{m-1}$ and $E = H^{m+s}_{loc}$, $E' = H^{-m-s}_{comp}$, we get:

Theorem 2.3. *Let M be a closed linear subspace of $H^{-m-s}_{\bar{\Omega}}(\mathbb{R}^n)$ (Ω bounded), such that if $p \in P_{m-1}$ and $\langle \mu, p\rangle = 0 \forall \mu \in M$ then $p = 0$. Let $f \in H^{m+s}_{\bar{\Omega}}(\Omega)$. There exists exactly one element $f^M \in D^{-m}\tilde{H}^s$ satisfying $\langle \mu, f^M\rangle = \langle \mu, f\rangle$, $\forall \mu \in M$, with minimum semi-norm $\left\| f^M \right\|_{m,s}$. Let θ be a reproducing kernel of $D^{-m}\tilde{H}^s$ in H^{m+s}_{loc}. Then the only g of the form $\theta\mu + p$ with $\mu \in M \cap P^o_{m-1}$ $p \in P_{m-1}$ satisfying $\langle \mu, g\rangle = \langle \mu, f\rangle$, $\forall \mu \in M$, is f^M.*

Proof. We just have to check that $M \cap P^o_{m-1}$ is closed in $\left(D^{-m}\tilde{H}^s/P_{m-1}\right)'$. But M is closed in $H^{-m-s}_{\bar{\Omega}}$, hence $M \cap N^o$ is closed in $H^{-m-s}_{\bar{\Omega}} \cap N^o$, which is exactly the orthogonal, in $\left(D^{-m}\tilde{H}^s/P_{m-1}\right)'$, of the subset of (equivalence classes $\mod P_{m-1}$ of) elements in $D^{-m}\tilde{H}^s$ which are zero on Ω. Thus $M \cap N^o$ is closed in a closed subset of $\left(D^{-m}\tilde{H}^s/P_{m-1}\right)'$, hence closed itself. All this is useless if M is finite-dimensional. □

3 Reproducing kernel of $D^{-m}\tilde{H}^s$

Theorem 3.1. *$\theta : \mu \mapsto (2\pi)^{-2m}\,\mu \odot \mathfrak{F}Pf.\,|\tau|^{-2m-2s}$ maps $H^{-m-s}_{comp} \cap P^o_{m-1}$ into $D^{-m}\tilde{H}^s$ and is a reproducing kernel of $D^{-m}\tilde{H}^s$ as a semi-Hilbert subspace of $H^{m+s}_{loc}(\mathbb{R}^n)$.*

Proof. Since $D^{-m}\tilde{H}^s/P_{m-1}$ is a Hilbert subspace of H^{m+s}_{loc}/P_{m-1} it has a reproducing kernel, say L, which maps $H^{-m-s}_{comp} \cap P^o_{m-1}$ (dual space of $D^{-m}\tilde{H}^s/P_{m-1}$) into $D^{-m}\tilde{H}^s/P_{m-1}$ and is characterized by: $L\mu$ is the set of distributions $u \in D^{-m}\tilde{H}^s$satisfying:

$$(2\pi)^{2m}\int_{\mathbb{R}^n} |\tau|^{2s}\,(\tau^m\hat{u})\,(\tau)\,.(\tau^m\,\bar{\hat{w}})\,(\tau)\,d\tau = \langle \mu, w\rangle\,, \forall w \in D^{-m}\tilde{H}^s \tag{3.1}$$

We first deduce that $|2\pi\tau|^{2m}\,\hat{u} = \hat{\mu}\,|\tau|^{-2s}$ (1). Now it is easily seen that $v = (2\pi)^{-2m}\,\mu \odot \mathfrak{F}Pf.\,|\tau|^{-2m-2s}$ satisfies $|2\pi\tau|^{-2m}\,\hat{v} = \hat{\mu}\,|\tau|^{-2s}$, since $\hat{v} = (2\pi)^{2m}\,\hat{\mu}Pf.\,|\tau|^{-2m-2s}$ and $|2\pi|^{2m}\,Pf.\,|\tau|^{-2m-2s} = |\tau|^{-2s}$. On the other hand, lemma 2 proves $\tau^\alpha\hat{v} \in L^1_{loc}, \forall\,|\alpha| = m$. So that for any $u \in L\mu$ we have $|\alpha|^{2m}\,(\hat{u} - \hat{v}) = 0$ and $\tau^\alpha\,(\hat{u} - \hat{v}) \in L^1_{loc}, \forall\,|\alpha| = m$. *Lemma*3 then shows that $u - v \in P_{m-1}$, i.e. $v \in L\mu$ since $L\mu$ is an equivalence class modulo P_{m-1}. In other words $L\mu = (2\pi)^{2m}\,\mu \odot \mathfrak{F}Pf.\,|\tau|^{-2m-2s} + P_{m-1}$, i.e. θ is a reproducing kernel of $D^{-m}\tilde{H}^s$. □

Lemma 1. $|2\pi\tau|^{2m}\,\hat{\mu} = \hat{\mu}\,|\tau|^{-2s}$ *if* $u \in L\mu$.

Proof. Conditions $s > -m - \frac{n}{2}$ implies $\mathfrak{I} \subset D^{-m}\tilde{H}^s$, so that we may apply (3.1) whith $w = \hat{\varphi}, \varphi \in \mathfrak{D}$, and get $(2\pi)^{2m}\int |\tau|^{2s}\,(\tau^m\hat{u})\,(\tau)\,.\tau^m\varphi\,(\tau)\,d\tau = \langle \mu, \hat{\varphi}\rangle = \langle \hat{\mu}, \varphi\rangle\,, \forall \varphi \in \mathfrak{D}$. This implies $(2\pi)^{2m}\,|\tau|^{2s}\,\tau^m.\,(\tau^m\hat{u})\,(\tau)\,. = \hat{\mu}\,(\tau)$ a.e., hence $(2\pi)^{2m}\,\tau^m.\ (\tau^m\hat{u})\,(\tau) = |\tau|^{-2s}\,\hat{\mu}\,(\tau)$ a.e. and $(2\pi\tau)^{2m}\,\hat{u} = \widehat{\mu\,|\tau|}^{-2s}$ as distributions. □

Lemma 2. *If μ is a distribution with compact support, orthogonal to P_{m-1} i.e. $\mu \in \mathfrak{E}' \cap P^o_{m-1}$, then $\tau^\alpha \hat{\mu} Pf. |\tau|^{-2m-2s} \in L^1_{loc}, \forall |\alpha| = m$.*

Proof. It suffices to show that the *function* (in usual sense) $\tau^\alpha \hat{\mu}(\tau) |\tau|^{-2m-2s}$ is locally sunmable. But, since μ is orthogonal to P_{m-1}, the C^∞ function $\hat{\mu}$ has derivatives of order $\leq m-1$ vanishing at 0, so that $|\hat{\mu}(\tau)| \leq c |\tau|^m$ on a neighbourhood of 0. Then $\tau^\alpha \hat{\mu}(\tau) |\tau|^{-2m-2s} \leq c |\tau|^{-2s}$ on that neighbourhood of 0, and is C^∞ elsewhere, so is locally summable since $s < \frac{n}{2}$. □

Lemma 3. *Any tempered distribution T such that $|\tau|^{2m} \hat{T} = 0$ and $\tau^\alpha \hat{T} \in L^1_{loc}$ is in P_{m-1}.*

Proof. $T$is supported by 0, since $|\tau|^{2m}$ $\tau = 0$. Then $\tau^\alpha \hat{T}$ is also supported by 0 and should be in L^1_{loc} which is possible only if $\tau^\alpha \hat{T} = 0$, and then $D^\alpha T = 0, \forall |\tau| = m$, i.e. $T \in P_{m-1}$. □

4 A general characterization result

To be more explicit, we now use formulas giving Fourier transforms of pseudo-functions $Pf. |\tau|^\lambda$. In general $Pf. |\tau|^\lambda = cPf. |t|^{-n-\lambda}$ except if λ or $-n-\lambda$ is an even positive integer $2k : \mathfrak{F} |\tau|^{2k} = c\Delta^k \delta$, $\mathfrak{F} Pf. |\tau|^{-n-2k} = c |t|^{2k} \log |t| + |t|^{2k}$.

For simplicity, we assume $s > -m$. Then $\mathfrak{F} Pf. |\tau|^{-2m-2s}$ is $c |t|^{2m+2s-n} \log |t| + c |t|^{2m+2s-n}$ if $2m+2s-n$ is an even positive integer, $c |t|^{2m+2s-n}$ if not. It is easyly seen that, in the first case $(2m+2s-n = 2k)$, if $\mu \in P^o_{m-1}$ then $\mu \odot |t|^{2m+2s-n} \in P_{m-1}$. So that, *putting $K_\lambda(t) = |t|^\lambda \log |t|$ if λ is an even positive integer $K_\lambda(t) = |t|^\lambda$ otherwise, the mapping $\mu \mapsto \mu \odot K_{2m+2s-n}$ is proportional to a reproducing kernel of $D^{-m} \tilde{H}^s$ when $m+s > 0$.*

Now we are able to explicit theorem 2.3 (in the case $m+s > 0$, that is, H^{m+s}_{loc} is a space of (classes of locally summable) functions):

Theorem 4. *Let M be a closed linear, subspace of some $H^{-m-s}_{\bar{\Omega}}(\mathbb{R}^n)$ (Ω bounded), satisfying: if $p \in P_{m-1}$ and $\langle \mu, p \rangle = 0, \forall \mu \in M$, then $p = 0$. Let $f \in H^{m+s}(\Omega)$. Then there exists a unique function $f \in D^{-m} \tilde{H}^s(\mathbb{R}^n)$ satisfying $\langle \mu, f^M \rangle = \langle \mu, f \rangle \forall \mu \in M$, with minimum semi-norm $\|f^M\|_{m,s}$. Moreover, if $g = \nu \odot K_{2m+2s-n} + p$ (with $\nu \in M \cap P^o_{m-1}$ and $p \in P_{m-1}$) satisfies $\langle \mu, g \rangle = \langle \mu, f \rangle \forall \mu \in M$,then $g = f^M$.*

Let us now restrict ourselves to the important case where $m+s > \frac{n}{2}$, so that H^{m+s}_{loc} is a space of continuous functions (Sobolev theorem), and *data are finitely many point values.*

Theorem 4. ***bis*** *Let A be a finite subset of $\mathbb{R}^n$, containing a P_{m-1} - unisolvent subset. Then there exists exactly one function of the form $\sigma(t) = \sum_{a \in A} \lambda_a K_{2m+2s-n}(t-a) + p(t)$ with $p \in P_{m-1}$ and $\sum_{a \in A} \lambda_a q(a) = 0, \forall q \in P_{m-1}$, taking prescribed values on A. Moreover,. if f is another function taking the same values on A, one has $\|f\|_{m,s} \geq \|\sigma\|_{m,s}$.*

Actually, $\sum_{a \in A} \lambda_a K_{2m+2s-n}(t-a)$ is $\left(\sum_{a \in A} \lambda_a \delta_a \odot K_{2m+2s-n}\right)(t)$ Existence of a function $f \in D^{-m} \tilde{H}^s$ taking prescribed values on A (finite) is obvious: f nmay even be chosen in $\mathfrak{D}$.

5 Examples

5.1 Pseudo-polynomial splines

We put $s = \frac{n-1}{2}$ and consider a finite set $A \subset \mathbb{R}^n$ containing some P_{m-1} - unisolvent subset. Then there exists exactly one function of the form $\sigma(t) = \sum_{a\in A} \lambda_a |t-a|^{2m-1} + p(t)$ where $p \in P_{m-1}$ and $\sum_{a\in A} \lambda_a q(a) = 0, \forall q \in P_{m-1}$, taking prescribed values on A. For all f taking the same values on A one has $\|f\|_{m,\frac{n-1}{2}} \geq \|\sigma\|_{m,\frac{n-1}{2}}$.

For $m = 1$ we get *multi-conic functions* $\sum \lambda_a |t-a| + C$ with $\sum \lambda_a = 0$, and the set A must only contain two distinct points. The functional minimized is $\int_{\mathbb{R}^n} |\tau|^{n-1} |\mathfrak{F}Dv(\tau)|^2 \, d\tau$.

For $m = 2$ we get *pseudo-cubic splines*, if A is not contained in a hyperplane (a line if $n = 2$): functions of the form $\sum \lambda_a |t-a|^3 + \alpha t + \beta$ with $\sum \lambda_a = 0$ and $\sum \lambda_a a = 0$. Coefficients $(\lambda_a; a \in A)$ and $\alpha_{1,}\alpha_2, \beta$ may be computed from the linear system:

$$\begin{cases} \sum_{a\in A} |a-b|^3 \lambda_a + b_1\alpha_1 + b_2\alpha_2 + \beta = f(b) \quad (b \in A) \\ \sum_{a\in A} \lambda_a a_1 = 0 \\ \sum_{a\in A} \lambda_a a_2 = 0 \\ \sum_{a\in A} \lambda_a = 0 \end{cases}$$

We notice that, for $n = 1$, we get simply polynomial splines: polynomials of degree $\leq 2m-1$ on intervals, C^2, and degenerating to polynoraials of degree $\leq m-1$ at both ends (thanks to conditions $\sum \lambda_a a^k = 0, \ldots m-1$).

5.2 Thin plate functions

Putting $s = 0$ as an example, $n = 2$ and $m = 2$, we get functions of the form $\sigma(t) = \sum_{a\in A} \lambda_a |t-a|^2 \log|t-a| + \alpha t + \beta$ with $\sum_{a\in A} \lambda_a = 0$ and $\sum_{a\in A} \lambda_a a = 0$ (function $|t|^2 \log|t|$ is extended to 0 at 0, so as to be continuous). In this case we have $\int_{\mathbb{R}^2} |D^2\sigma|^2 \leq \int_{\mathbb{R}^2} |D^2 f|^2$for all f that coincides with σ on A. The set A must not be contained in a line.

5.3 Hermite polynomials

Since $H_{loc}^{2+\frac{n-1}{2}}(\mathbb{R}^n) \subset C^1$, we may minimize semi-norm $\|.\|_{2,\frac{n-1}{2}}$ subject to Hermite conditions: values and gradients prescribed on a finite set A. We get functions of the form $\sigma(t) = \sum_{a\in A} \lambda_a |t-a|^3 + \sum \lambda'_a (t-a)|t-a| + \alpha t + \beta$, with $\sum \lambda_a = 0$, $\sum \lambda_a a + \frac{1}{3} \sum \lambda'_a = 0$. In one dimension this corresponds to ordinary piecewise cubic Hermite interpolation.

6 Convergence in $H^{m+s}(\Omega)$

Let $f \in H^{m+s}(\Omega)$, and let (M_k) be a sequence of closed linear subspaees of $H_{\bar{\Omega}}^{-m-s}(\mathbb{R}^n)$. We suppose that for any $\mu \in H_{\bar{\Omega}}^{-m-s}(\mathbb{R}^n)$, the distance from μ to M_k converges to 0. Then:

1. For k sufficiently large, M_k is such that: if $p \in P_{m-1}$ satisfies $\langle \mu, p \rangle = 0, \forall \mu \in M_k$, then $p = 0$. So that there exists a unique $f_k \in D^{-m} \tilde{H}^s$ satisfying $\langle \mu, f_k \rangle = \langle \mu, f \rangle, \forall \mu \in M_k$, with $\|f_k\|_{m,s}$ minimum.

2. $f_k \to f$ in $H^{m+s}(\Omega)$.

This is a straightforward consequence of a general result of J.L. Joly [7], putting X=$H^{m+s}(\Omega)$, $Y = H^{m+s}(\Omega) / P_{m-1}$ with a norm derived from $\|.\|_{m,s}$. Another way to see it (partially) is the following: put f^Ω = the minimal extension of f, relatively to $\|.\|_{m,s}$, i.e. the unique element in $D^{-m} \tilde{H}^3$ that coincides with f on Ω with minimum semi-norm $\|.\|_{m,s}$. It is uniquely written $\mu \odot K + p$ with $p \in P_{m-1}$ and $\mu \in H_{\bar{\Omega}}^{-m-s}(\mathbb{R}^n) \cap P_{m-1}^o (K = \mathfrak{F} Pf. |\tau|^{-2m-2s})$ Same thing for $f_k = \mu_k \odot A + p_k$ with $p_k \in P_{m-1}$ and $\mu_k \in M_k \cap P_{m-1}^o$. And μ_k *is simply the orthogonal projection of* μ *onto* $M_k \cap P_{m-1}^o$, in Hilbert space $H_{\bar{\Omega}}^{-m-s}(\mathbb{R}^n) \cap P_{m-1}^o$ equipped with norm induced by $\left(D^{-m} \tilde{H}^s / P_{m-1} \right)'$ (equivalent to that induced by $H_{\bar{\Omega}}^{-m-s}(\mathbb{R}^n)$. So that $\|\mu_k - \mu\| \leq cd(\mu, M_k) \to 0$, hence $\|\mu_k \odot K - \mu \odot K\|_{m,s} \to 0$ and this proves that $f_k + P_{m-1} \to f + P_{m-1}$in $H^{m+s}(\Omega) / P_{m-1}$.

Let us now specialize to the case where $m + s > \frac{n}{2}$ and M_k is spanned by Dirac masses $(\delta_a; a \in A_k)$ where (A_k) is a sequence of subsets of $\bar{\Omega}$. Then the condition d$d(\mu, M_k) \to 0, \forall \mu \in H_{\bar{\Omega}}^{-m-s}$ is equivalent to saying that any point in Ω is limit of a sequence $(a_k \in A_k)$, or that Hausdorff distance from A_k to Ω tends to zero. This results from complete continuity of inclusion $H_{\bar{\Omega}}^{m+s}(\Omega) \backsim C(\Omega)$ (a bounded subset of $H_{\bar{\Omega}}^{m+s}(\Omega)$ is an equicontinuous set of functions on $\bar{\Omega}$). We then get:

Theorem 5. *If (A_k) is a sequence of subsets of $\bar{\Omega}$ (Ω bounded open subset of $\mathbb{R}^n$) such that $d(t, A_k) \to 0, \forall t \in \Omega$, and $f \in H^{m+s}(\Omega)$ with $m + s > \frac{n}{2}$, then the sequence (f_k) of functions coinciding with f on A_k with minimum semi-norm $\|.\|_{m,s}$ (uniquely determined for sufficiently large k) satisfies $f_k \to f$ in $H^{m+s}(\Omega)$.*

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