FRACTAL CLASSES OF MATROIDS

DILLON MAYHEW, MIKE NEWMAN, AND GEOFF WHITTLE

Abstract. A minor-closed class of matroids is (strongly) fractal if the number of n-element matroids in the class is dominated by the number of n-element excluded minors. We conjecture that when $\mathbb{K}$ is an infinite field, the class of $\mathbb{K}$-representable matroids is strongly fractal. We prove that the class of sparse paving matroids with at most k circuit-hyperplanes is a strongly fractal class when k is at least three. The minor-closure of the class of spikes with at most k circuit-hyperplanes (with $k \geq 5$) satisfies a strictly weaker condition: the number of $2t$-element matroids in the class is dominated by the number of $2t$-element excluded minors. However, there are only finitely many excluded minors with ground sets of odd size.

Dedicated to Joseph Kung, in gratitude for all his contributions to the matroid community.

1. Introduction

In [5] we proved that every real-representable matroid is contained as a minor in an excluded minor for the class of real-representable matroids. (The same phenomenon holds for any infinite field.) Geelen and Campbell strengthened this by showing that every real-representable matroid is a minor of a complex-representable excluded minor for the class of real-representable matroids [1]. In contrast to these results, the resolution of Rota's conjecture [2] implies that there are only finitely many excluded minors for $\mathbb{F}$-representability when $\mathbb{F}$ is a finite field.

In this article we consider another possible dichotomy between finite fields and infinite fields in matroid representation theory. Let $\mathcal{M}$ be a minor-closed class of matroids, and let $\mathcal{EX}$ be the class of excluded minors for $\mathcal{M}$. For any non-negative integer n, let m_n be the number of non-isomorphic n-element matroids in $\mathcal{M}$. Thus m_n counts the n-element members of $\mathcal{M}$ modulo the equivalence relation of isomorphism. Similarly, let x_n be the number of non-isomorphic n-element matroids in $\mathcal{EX}$. (Henceforth, when we refer to a matroid, we usually mean an isomorphism class of matroids.) We consider the probability that a matroid chosen randomly from the n-element members of $\mathcal{M} \cup \mathcal{EX}$ is an excluded minor. In other words, we consider the ratio

$$\frac{x_n}{m_n + x_n}.$$

Date: November 21, 2019.

We will denote this fraction by $\Gamma_{\mathcal{M}}(n)$. We consider only the case that $\mathcal{M}$ contains infinitely many matroids, so that $\Gamma_{\mathcal{M}}(n)$ is defined for all n. If $\mathcal{M}$ has only finitely many excluded minors, then $\Gamma_{\mathcal{M}}(n) = 0$ for all large enough values of n. It is impossible for $\Gamma_{\mathcal{M}}(n)$ to be equal to one, but it could tend to one in the limit. In this case our random choice is asymptotically certain to be an excluded minor, so in some sense, the class $\mathcal{M}$ is eventually overwhelmed by its 'boundary': the set of excluded minors. This leads us to the following terminology.

Definition 1.1. Let $\mathcal{M}$ be a minor-closed class of matroids. If

$$\lim_{n\to\infty} \Gamma_{\mathcal{M}}(n) = 1,$$

then $\mathcal{M}$ is a *strongly fractal* class.

If $\mathcal{M}$ is the class of $\mathbb{F}$-representable matroids where $\mathbb{F}$ is a finite field, then $\Gamma_{\mathcal{M}}(n) = 0$ for all large enough values of n, since Rota's conjecture is true. We believe that this fails for infinite fields in the strongest possible way.

Conjecture 1.2. *Let $\mathbb{K}$ be an infinite field. The class of $\mathbb{K}$-representable matroids is strongly fractal.*

The class of gammoids is like the class of real-representable matroids, in that the excluded minors form a maximal antichain [4]. We conjecture that the class of gammoids is strongly fractal.

In the present article we are content merely to establish the non-obvious fact that strongly fractal classes exist. A matroid is *sparse paving* if every non-spanning circuit is a hyperplane.

Theorem 1.3. *Let $k \geq 3$ be a positive integer. Let $\mathcal{P}_k$ be the class of sparse paving matroids with at most k circuit-hyperplanes. Then $\mathcal{P}_k$ is strongly fractal.*

A rank-r *spike* has a ground set of size $2r$, say $\{a_1, b_1, \ldots, a_r, b_r\}$. Assume that $r > 3$. Then the non-spanning circuits are exactly the sets of the form $\{a_i, b_i, a_j, b_j\}$, along with (possibly) some sets that intersect each $\{a_i, b_i\}$ in either a_i or b_i. Any circuit of the latter type is also a hyperplane. Sometimes such a matroid is called a *tipless* spike. The class of spikes with a bounded number of circuit-hyperplanes is not minor-closed, but if we close it under minors, we obtain a class that has a weaker fractal property.

Definition 1.4. Let $\mathcal{M}$ be a minor-closed class of matroids. If $\Gamma_{\mathcal{M}}(1), \Gamma_{\mathcal{M}}(2), \Gamma_{\mathcal{M}}(3), \ldots$ contains an infinite subsequence that converges to one, then $\mathcal{M}$ is *weakly fractal.*

Obviously a strongly fractal class is weakly fractal.

Theorem 1.5. *Let $k \geq 5$ be an integer. Let $\mathcal{S}_k$ be the class produced by taking minors of the spikes with at most k circuit-hyperplanes. Then $\mathcal{S}_k$ is a weakly fractal class, but is not strongly fractal.*

The reason the class in Theorem 1.5 is not strongly fractal is that eventually there are no excluded-minors with odd-cardinality ground sets (Lemma 3.10). In other words, $\Gamma_{\mathcal{S}_k}(2t+1) = 0$ for all large-enough values of t. However $\Gamma_{\mathcal{S}_k}(2t)$ converges to one.

Definition 1.4 does not require the subsequence that converges to one to have any particular structure. However, it is inconceivable that the following conjecture fails.

Conjecture 1.6. *Let $\mathcal{M}$ be a weakly fractal class of matroids. There exist integers a and b such that the sequence*

$$\Gamma_{\mathcal{M}}(a+b),\ \Gamma_{\mathcal{M}}(2a+b),\ \Gamma_{\mathcal{M}}(3a+b),\ldots$$

converges to one.

In the case of the class $\mathcal{S}_k$, the integers $a = 2$ and $b = 0$ satisfy Conjecture 1.6.

The only fractal classes we know of contain infinite antichains, and we think this exemplifies a general pattern.

Conjecture 1.7. *Any weakly fractal class of matroids contains an infinite antichain.*

We close this introduction with a consequence of the existence of strongly fractal classes.

Proposition 1.8. *Let γ be a real number in $[0,1]$. Let $\mathcal{M}_0$ be a strongly fractal class of matroids. There exists a minor-closed class of matroids, $\mathcal{M} \supseteq \mathcal{M}_0$, such that $\lim_{n\to\infty} \Gamma_{\mathcal{M}}(n) = \gamma$.*

Proof. For any positive integer i, let $N(i)$ be the smallest integer such that $\Gamma_{\mathcal{M}_0}(n) \geq (i-1)/i$ for every $n \geq N(i)$. The existence of $N(i)$ follows from $\lim_{n\to\infty} \Gamma_{\mathcal{M}_0}(n) = 1$. Note that $N(1), N(2), N(3), \ldots$ is a non-decreasing sequence.

We construct a series of classes, $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$ with the following properties. Let n be a positive integer, and choose i so that $N(i) \leq n \leq N(i+1)$. Then the only matroids in $\mathcal{M}_n - \mathcal{M}_{n-1}$ are n-element excluded minors for $\mathcal{M}_{n-1}$, and $\gamma - 1/i \leq \Gamma_{\mathcal{M}_n}(n) \leq \gamma$. Note that the first condition implies that $\mathcal{M}_n$ is a minor-closed class.

Assume that we have successfully constructed $\mathcal{M}_{n-1}$. Choose i so that $N(i) \leq n \leq N(i+1)$. Let m_n and x_n respectively denote the number of n-element matroids in $\mathcal{M}_0$, and the number of n-element excluded minors for $\mathcal{M}_0$. Because $\Gamma_{\mathcal{M}_0}(n) \geq (i-1)/i$, it follows that $m_n + x_n \geq i$, for otherwise the only way to satisfy $\Gamma_{\mathcal{M}_0}(n) \geq (i-1)/i$ is for m_n to be zero. This is impossible because $\mathcal{M}_0$ must contain infinitely many matroids. From $m_n + x_n \geq i$ and $\Gamma_{\mathcal{M}_0}(n) \geq (i-1)/i$, we in turn derive $x_n \geq i-1$. Note that an n-element excluded minor for $\mathcal{M}_0$ is not in $\mathcal{M}_{n-1}$, but all of its proper minors are in $\mathcal{M}_0$. Therefore any such matroid is an excluded minor for $\mathcal{M}_{n-1}$. Let x'_n be the number of n-element excluded minors for $\mathcal{M}_{n-1}$, where we have just demonstrated that $x'_n \geq x_n \geq i-1$.

We observe that

$$\frac{x'_n}{m_n + x'_n} \geq \frac{x_n}{m_n + x_n} \geq \frac{i-1}{i}.$$

Let z be the smallest integer such that

$$\frac{x'_n - z}{m_n + x'_n} \leq \gamma,$$

and note that z is in $\{0, 1, \ldots, x'_n\}$. Now we choose z matroids from the n-element excluded minors for $\mathcal{M}_{n-1}$, and construct $\mathcal{M}_n$ by adding these z matroids to $\mathcal{M}_{n-1}$.

The n-element members of $\mathcal{M}_n$ are exactly the n-element members of $\mathcal{M}_0$ along with the z matroids we have added. The n-element excluded minors for $\mathcal{M}_n$ are the $x'_n - z$ excluded minors that we did not add to $\mathcal{M}_{n-1}$. Therefore

$$\Gamma_{\mathcal{M}_n}(n) = \frac{x'_n - z}{(m_n + z) + (x'_n - z)} \leq \gamma.$$

Furthermore, since $m_n + x'_n \geq i$, our choice of z means that $\Gamma_{\mathcal{M}_n}(n)$ is at least $\gamma - 1/i$. In this way we can construct the entire sequence $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$.

Now we define $\mathcal{M}$ to be the union $\cup_{n\geq 0}\mathcal{M}_n$. The n-element members of $\mathcal{M}$ are the n-element members of $\mathcal{M}_n$, and the n-element excluded minors for $\mathcal{M}$ are the n-element excluded minors for $\mathcal{M}_n$. Therefore $\Gamma_{\mathcal{M}}(n) = \Gamma_{\mathcal{M}_n}(n)$. Since this value is between $\gamma - 1/i$ and γ, where $N(i) \leq n \leq N(i+1)$, it follows that $\Gamma_{\mathcal{M}}(n)$ converges to γ, as desired. $\square$

We use $\mathcal{P}(X)$ to denote the power set of the set X. The symbol $\mathbb{Z}_{\geq 0}$ stands for the set of non-negative integers. Our reference for matroid terms and concepts is [6]. Recall that a *triangle* is a circuit of size three, and a *triad* is a 3-element cocircuit. A *parallel pair* is a circuit of size two, and a *parallel class* is a maximal set such that every 2-element subset is a parallel pair. A *series pair* is a 2-element cocircuit, and a *series class* is a maximal set with every 2-element subset being a series pair. A *thin* edge of a graph is a non-loop edge that is not in any parallel pair. Let $f(n)$ and $g(n)$ be functions taking integers as input and returning real numbers as output. If we say that $f(n)$ is bounded by $O(g(n))$, we mean that there exist a positive constant c and an integer N such that $f(n) \leq cg(n)$ whenever $n > N$. Similarly, if f is at least $\Omega(g(n))$, then there exist a positive constant c and an integer N such that $f(n) \geq cg(n)$ whenever $n > N$.

2. Sparse paving matroids

A matroid is *sparse paving* if every non-spanning circuit is also a hyperplane. Every matroid with rank or corank equal to zero is vacuously sparse paving. Let E be an n-element set, and let r be an integer satisfying $1 \leq r \leq n-1$. Rank-r sparse paving matroids on the ground set E are in bijective correspondence with families, $\mathcal{C}$, of r-element subsets of E, such that if C_1 and C_2 are distinct members of $\mathcal{C}$, then $|C_1 - C_2| > 1$. If $\mathcal{C}$ is such a collection, then we use $M(\mathcal{C})$ to denote the rank-r sparse paving matroid on the ground set E with $\mathcal{C}$ as its family of circuit-hyperplanes.

For k a non-negative integer, let $\mathcal{P}_k$ denote the class of sparse paving matroids with at most k circuit-hyperplanes. Note that $\mathcal{P}_0$ is the class of uniform matroids. Let $\mathcal{P}$ be the set of all sparse paving matroids. Thus $\mathcal{P} = \cup_{k\geq 0}\mathcal{P}_k$. Let $M = M(\mathcal{C})$ be a sparse paving matroid on the ground set E, and let e be an element of E. If e is not a coloop in M then

$$M\backslash e = M(\{C \colon C \in \mathcal{C},\ e \notin C\})$$

and if e is not a loop, then

$$M/e = M(\{C - e \colon C \in \mathcal{C},\ e \in C\}).$$

This demonstrates that $\mathcal{P}_k$ is a minor-closed class for any $k \geq 0$, and so is $\mathcal{P}$.

Definition 2.1. Let k and n be positive integers. Let $\mathcal{C} = (C_1, \ldots, C_k)$ be a sequence of subsets of E, where E is a set of cardinality n. For every subset $I \subseteq \{1, \ldots, k\}$, define $\mathcal{C}(I)$ to be

$$\left(\bigcap_{i\in I} C_i\right) \cap \left(\bigcap_{i\in\{1,\ldots,k\}-I} (E - C_i)\right).$$

Thus $\mathcal{C}(I)$ contains those elements of E that are in C_i for every $i \in I$, and in the complement of C_i for every $i \notin I$. This means that $(\mathcal{C}(I))_{I\subseteq\{1,\ldots,k\}}$ is a partition of E (possibly containing empty blocks). We think of $\mathcal{C}(I)$ as being a cell in a Venn diagram with k sets.

Let $\psi_{\mathcal{C}}$ be the function that takes I to $|\mathcal{C}(I)|$ for each $I \subseteq \{1, \ldots, k\}$. Note that

$$\sum_{I\subseteq\{1,\ldots,k\}} \psi_{\mathcal{C}}(I) = n.$$

Let k and n be positive integers. We describe a relation, $\mathcal{R}_k^n$. The domain of $\mathcal{R}_k^n$ is the set of n-element sparse paving matroids with exactly k circuit-hyperplanes. The codomain is the set of functions from $\mathcal{P}(\{1, \ldots, k\})$ to $\mathbb{Z}_{\geq 0}$. The ordered pair (M, ψ) is in $\mathcal{R}_k^n$ if there is some ordering $\mathcal{C} = (C_1, \ldots, C_k)$ of the circuit-hyperplanes of M such that $\psi = \psi_{\mathcal{C}}$. We use $\mathcal{R}_k^n(M)$ to denote $\{\psi \colon (M, \psi) \in \mathcal{R}_k^n\}$, the image of M under $\mathcal{R}_k^n$. Thus $\mathcal{R}_k^n(M)$ contains at most $k!$ functions.

Proposition 2.2. *Let k and n be positive integers. Let M and N be n-element sparse paving matroids with exactly k circuit-hyperplanes. Then M and N are isomorphic if and only if $\mathcal{R}_k^n(M) \cap \mathcal{R}_k^n(N) \neq \emptyset$.*

Proof. Let ρ be an isomorphism from M to N. Let $\mathcal{C} = (C_1, \ldots, C_k)$ be an ordering of the circuit-hyperplanes in M. Then $\rho(\mathcal{C}) = (\rho(C_1), \ldots, \rho(C_k))$ is an ordering of the circuit-hyperplanes in N. Because ρ is a bijection, it is clear that $|\mathcal{C}(I)| = |\rho(\mathcal{C})(I)|$ for every $I \subseteq \{1, \ldots, k\}$. Thus $\psi_{\mathcal{C}} = \psi_{\rho(\mathcal{C})}$, and it follows that $\mathcal{R}_k^n(M) \cap \mathcal{R}_k^n(N)$ is not empty.

For the converse, we assume that $\mathcal{R}_k^n(M) \cap \mathcal{R}_k^n(N)$ contains at least one function. This means that there must be orderings, $\mathcal{C}_M$ and $\mathcal{C}_N$, of the circuit-hyperplanes in M and N, respectively, such that $\psi_{\mathcal{C}_M} = \psi_{\mathcal{C}_N}$. For each $I \subseteq \{1, \ldots, k\}$, let π_I be an arbitrary bijection from $\mathcal{C}_M(I)$ to $\mathcal{C}_M(I)$. Note that these sets have the same cardinality since $\psi_{\mathcal{C}_M}(I) = \psi_{\mathcal{C}_N}(I)$, so π_I exists. We consider each π_I to be a set of ordered pairs. Now we can define π to be

$$\bigcup_{I \subseteq \{1, \ldots, k\}} \pi_I.$$

Thus π is a bijection from $E(M)$ to $E(N)$. It is clear that $\pi(X)$ is a circuit-hyperplane of N if and only if X is a circuit-hyperplane of M. Therefore π is the desired isomorphism between M and N. $\square$

Lemma 2.3. *Let k be a non-negative integer. The number of n-element matroids in $\mathcal{P}_k$ is at most $O(n^{2^k-1})$.*

Proof. We observe that $\mathcal{P}_k$ can be partitioned into the classes

$$\mathcal{P}_k - \mathcal{P}_{k-1},\ \mathcal{P}_{k-1} - \mathcal{P}_{k-2}, \ldots,\ \mathcal{P}_1 - \mathcal{P}_0, \text{ and } \mathcal{P}_0.$$

Because $\mathcal{P}_0$ is the class of uniform matroids, it follows that the number of n-element matroids in $\mathcal{P}_0$ is at most $O(n)$. We will show that the number of n-element matroids in $\mathcal{P}_m - \mathcal{P}_{m-1}$ is at most $O(n^{2^m-1})$, and then the result will follow, since the number of classes above is constant relative to n.

From Proposition 2.2, it follows that for any positive m, the number of n-element matroids in $\mathcal{P}_m - \mathcal{P}_{m-1}$ is at most the number of functions $\psi\colon \mathcal{P}(\{1, \ldots, m\}) \to \mathbb{Z}_{\geq 0}$ such that $\sum_{I \subseteq \{1, \ldots, m\}} \psi(I) = n$. By standard enumeration techniques, the number of such functions is

$$\binom{n + 2^m - 1}{n} = \binom{n + 2^m - 1}{2^m - 1}.$$

Since m is a constant, this binomial coefficient is bounded by $O(n^{2^m-1})$, and we are done. $\square$

Lemma 2.4. *Let $k \geq 3$ be an integer. The number of n-element excluded minors for $\mathcal{P}_k$ is at least $\Omega(n^{2^{k+1}-k-4})$.*

Proof. We prove the lemma by considering sparse paving matroids with $k + 1$ circuit-hyperplanes. We insist that each element is in at least one of the circuit-hyperplanes, and that no element is in all of them. This is enough to ensure that the sparse paving matroids in question are excluded minors for $\mathcal{P}_k$. The technical details of the proof involve estimating the number of these excluded minors.

Let $\mathcal{I}$ be the collection $\{I \subseteq \{1, \ldots, k + 1\} \colon 2 \leq |I| \leq k\}$. Observe that $|\mathcal{I}| = 2^{k+1} - k - 3$. For $s \in \{2, \ldots, k\}$, let $\mathcal{I}^s$ denote the collection of sets in $\mathcal{I}$ with cardinality s. For each $I \in \mathcal{I}$ we introduce a variable x_I. We are going to consider non-negative integer solutions to the equation

$$k \sum_{I \in \mathcal{I}^2} x_I + (k-1) \sum_{I \in \mathcal{I}^3} x_I + \cdots + 2 \sum_{I \in \mathcal{I}^k} x_I = n - 2(k+1). \tag{1}$$

Claim 2.4.1. The number of non-negative integer solutions to (1) is at least $\Omega(n^{2^{k+1}-k-4})$.

Proof. The proof of this claim is essentially the same as the proof of Schur's Theorem given in [9, Theorem 3.15.2]. By standard techniques, we see that the number of non-negative integer solutions is equal to the coefficient of $z^{n-2(k+1)}$ in the generating function

$$f(z) = \left(\frac{1}{1-z^k}\right)^{\binom{k+1}{2}} \left(\frac{1}{1-z^{k-1}}\right)^{\binom{k+1}{3}} \cdots \left(\frac{1}{1-z^2}\right)^{\binom{k+1}{k}}.$$

Every pole of $f(z)$ is a root of unity. In particular, the denominator of $f(z)$ has as a factor

$$(1-z)^{\binom{k+1}{2}+\binom{k+1}{3}+\cdots+\binom{k+1}{k}} = (1-z)^{2^{k+1}-k-3},$$

which shows that $z = 1$ is a pole with multiplicity $2^{k+1} - k - 3$. If s is an integer greater than one, then s does not divide all the values in $2, \ldots, k$. In this case, if ω is an s-th root of unity and $z = \omega$ is a pole of $f(z)$, then its multiplicity is less than $2^{k+1} - k - 3$. So $f(z)$ has a pole of multiplicity $2^{k+1}-k-3$ at $z = 1$, and the multiplicity of every other pole is less than this value. Now the arguments in [9, Theorem 3.15.2] shows that the number of non-negative integer solutions is asymptotically equal to $n^{2^{k+1}-k-4}$, and this gives us the desired result. $\square$

Let ϕ be an arbitrary solution to (1). Thus ϕ takes the variables $\{x_I\}_{I\in\mathcal{I}}$ to non-negative integers, and

$$k \sum_{I \in \mathcal{I}^2} \phi(x_I) + (k-1) \sum_{I \in \mathcal{I}^3} \phi(x_I) + \cdots + 2 \sum_{I \in \mathcal{I}^k} \phi(x_I) = n - 2(k+1).$$

We will construct a sequence $\mathcal{C} = (C_1, \ldots, C_{k+1})$, of subsets of $\{1, \ldots, n\}$ such that:

(i) $C_1, \ldots, C_{k+1}$ are equicardinal,
(ii) $|C_i - C_j| > 1$ when i and j are distinct,
(iii) $\psi_{\mathcal{C}}(I) = \phi(x_I)$ for every $I \in \mathcal{I}$, and
(iv) the sparse paving matroid $M(\mathcal{C})$ is an n-element excluded minor for $\mathcal{P}_k$.

We construct $\mathcal{C} = (C_1, \ldots, C_{k+1})$ by allocating each element in $\{1, \ldots, n\}$ to a unique set of the form $\mathcal{C}(I)$ for some $I \subseteq \{1, \ldots, k+1\}$. We start by allocating two elements to each set of the form $\mathcal{C}(\{i\})$, for $i \in \{1, \ldots, k+1\}$. This ensures that statement (ii) holds. We now have $n - 2(k+1)$ elements left to allocate. We will allocate no elements to $\mathcal{C}(\emptyset)$ or $\mathcal{C}(\{1, \ldots, k+1\})$, so every element is in at least one of the sets $(C_1, \ldots, C_{k+1})$, and no element is in all of them.

We process each subset $I \in \mathcal{I}$ in turn. We allocate $\phi(x_I)$ elements to $\mathcal{C}(I)$, and then for each $i \in \{1, \ldots, k+1\} - I$, we allocate a further $\phi(x_I)$ elements to $C(\{i\})$. We have thus allocated an additional $\phi(x_I)$ elements to each set in $(C_1, \ldots, C_{k+1})$, ensuring the sets remain equicardinal during this process. Note that the number of elements we have allocated while processing I is $\phi(x_I) + ((k+1) - |I|)\phi(x_I)$. After processing every subset in $\mathcal{I}$, the number of elements we have allocated is therefore

$$k \sum_{I \in \mathcal{I}^2} \phi(x_I) + (k-1) \sum_{I \in \mathcal{I}^3} \phi(x_I) + \cdots + 2 \sum_{I \in \mathcal{I}^k} \phi(x_I) = n - 2(k+1).$$

Hence all n elements have now been allocated, and the sets $(C_1, \ldots, C_{k+1})$ are equicardinal, and satisfy $|C_i - C_j| > 1$ when i and j are distinct. Furthermore, our method of construction ensures that $\psi_{\mathcal{C}}(I) = \phi(x_I)$ for every $I \in \mathcal{I}$. Since each element $e \in \{1, \ldots, n\}$ is in at least one of the sets in $\mathcal{C}$, but not all of them, it follows that $M(\mathcal{C})\backslash e$ and $M(\mathcal{C})/e$ both have at most k circuit-hyperplanes, while $M(\mathcal{C})$ itself has $k+1$. Thus $M(\mathcal{C})$ is an excluded minor for $\mathcal{P}_k$, as desired.

The number of excluded minors we have constructed in this way is $\Omega(n^{2^{k+1}-k-4})$ by Claim 2.4.1. Some of these excluded minors may be isomorphic copies of the same matroid. But because $\mathcal{R}^n_{k+1}(M)$ is no larger than $(k+1)!$ for any excluded minor M, Proposition 2.2 implies that any isomorphism class of excluded minors corresponds to no more than $(k+1)!$ solutions to (1). As k is fixed with respect to n, dividing a function that is at least $\Omega(n^{2^{k+1}-k-4})$ by $(k+1)!$ produces another such function, so the proof of Lemma 2.4 is complete. □

Proof of Theorem 1.3. From Lemmas 2.3 and 2.4, it follows that there are positive constants c_1 and c_2 such that for sufficiently large values of n we have

$$\Gamma_{\mathcal{P}_k}(n) \geq \frac{c_1 n^{2^{k+1}-k-4}}{c_2 n^{2^k-1} + c_1 n^{2^{k+1}-k-4}} = \frac{1}{(c_2/c_1) n^{-2^k+k+3} + 1}.$$

Since $k \geq 3$, it follows that $-2^k + k + 3$ is negative, and hence $\Gamma_{\mathcal{P}_k}(n)$ tends to one as n tends to infinity. □

3. Spikes

We describe spikes and their minors using biased graphs. Let G be an undirected graph, which may contain loops and parallel edges. A *theta-subgraph* of G consists of two distinct vertices, u and v, and three paths from u to v that do not share any vertices other than u and v. A *linear class* is a collection, $\mathcal{B}$, of cycles of G satisfying the constraint that no theta-subgraph of G contains exactly two cycles in $\mathcal{B}$. In this case, we say that the pair $(G, \mathcal{B})$ is a *biased graph*. The cycles in $\mathcal{B}$ are *balanced* and any other cycle is *unbalanced*. A subgraph is *unbalanced* if it contains an unbalanced cycle, and otherwise it is *balanced*. Let E be the edge-set of G. We similarly say that $X \subseteq E$ is balanced if the subgraph $G[X]$ is balanced, and otherwise X is unbalanced.

Lift matroids were introduced by Zaslavsky [10]. The lift matroid, $L(G, \mathcal{B})$, has E as its ground set. The set $X \subseteq E$ is a circuit of $L(G, \mathcal{B})$ if and only if $G[X]$ is either: (i) a balanced cycle, (ii) a theta-subgraph containing no balanced cycles, or (iii) a pair of unbalanced cycles with at most one vertex in common. The rank of $L(G, \mathcal{B})$ is equal to the number of vertices in G, minus the number of balanced connected components. A set, $X \subseteq E$, is a hyperplane of $L(G, \mathcal{B})$ if and only if X is a maximal balanced set, or is unbalanced and is a hyperplane of the graphic matroid $M(G)$ [10, Theorem 3.1]. Naturally, this characterises the cocircuits of $L(G, \mathcal{B})$.

The next result is well known, but we include the proof for completeness.

Proposition 3.1. *Let e be an element of the matroid M, and assume that $M/e = M(G)$ for some graph G. Let G^e be the graph obtained from G by adding the loop e incident with an arbitrary vertex. Let $\mathcal{B}$ be the collection of cycles in G such that $C \in \mathcal{B}$ if and only if the edge-set of C is a circuit of M. Then $\mathcal{B}$ is a linear class of G^e and $M = L(G^e, \mathcal{B})$.*

Proof. Note that if C is the edge-set of a cycle in G, then either C or $C \cup e$ is a circuit in M.

Let X be a set of edges such that $G[X]$ is a theta-subgraph. Assume $G[X]$ contains two cycles in $\mathcal{B}$. Hence there are distinct circuits C_1 and C_2 of M that are contained in X. Two cycles in a theta-subgraph must contain a common edge, so we assume that x is in $C_1 \cap C_2$, and that therefore $(C_1 \cup C_2) - x$ contains a circuit, C_3, of M. Note that $C_3 \subseteq X$, and C_3 is a union of circuits in $M/e = M(G)$. But $G[X]$ contains exactly one cycle that does not contain x: the third cycle in the theta-subgraph. Therefore C_3 is the edge-set of this cycle, which implies that $G[X]$ contains three cycles in $\mathcal{B}$. This shows that $\mathcal{B}$ is a linear class.

Next we will show that every circuit of $L(G^e, \mathcal{B})$ is a circuit in M, and vice versa. Let C be a circuit of $L(G^e, \mathcal{B})$. If $G[C]$ is a balanced cycle, then C is also a circuit of M. Therefore we assume that $G[C]$ is either a theta-subgraph with no balanced cycles, or consists of two unbalanced cycles that share at most one vertex. In the latter case, if $G[C]$ contains the loop e, then C consists of e and an unbalanced cycle, so C is also a circuit in M. Therefore we assume that e is not in C. Hence there are disjoint circuits C_1 and C_2 in M/e such that $C = C_1 \cup C_2$ and both $C_1 \cup e$ and $C_2 \cup e$ are circuits of M. By circuit elimination, $C_1 \cup C_2$ contains a circuit, C_3, of M. Hence C_3 is a union of circuits in M/e. But any circuit of $M/e = M(G)$ contained in C_3 is either C_1 or C_2, so C_3 is either equal to one of these circuits, or to their union. Since C_1 and C_2 are not circuits of M it cannot be the case that $C = C_1$ or $C = C_2$. Thus $C = C_1 \cup C_2 = C_3$ is also a circuit in M.

Now assume that $G[C]$ is a theta-subgraph containing no balanced cycles. Let C_1 and C_2 be the edge-sets of two distinct cycles in $G[C]$. As in the previous paragraph, $C_1 \cup C_2$ contains a circuit, C_3, of M, and C_3 is a union of circuits in $M/e = M(G)$. Since C_1 and C_2 are not circuits of M, there are only two possibilities: C_3 is the edge-set of the third cycle in the theta-subgraph or C_3 is the entire theta-subgraph. The first case is impossible, since $G[C]$ contains no balanced cycles. Therefore $C_3 = C_1 \cup C_2$, and C is again a circuit in M.

Now we know that every circuit of $L(G^e, \mathcal{B})$ is also a circuit in M, so to complete the proof it suffices to show that every circuit of M contains a circuit of $L(G^e, \mathcal{B})$. Let C be a circuit of M. If e is in C, then $C - e$ is a circuit of $M/e = M(G)$, so $C - e$ is the edge-set of an unbalanced cycle. In this case C is the union of two unbalanced cycles, one of them being the loop e, so C is a circuit in $L(G^e, \mathcal{B})$. Hence we assume $e \notin C$. Now C is the edge-set of a union of cycles in G. If any of these is a balanced cycle, then C contains a circuit in $L(G^e, \mathcal{B})$, so we assume that $G[C]$ contains no balanced cycle. If the union contains only one cycle (with that cycle being unbalanced), then C is independent in M, which is not true. Thus the union contains at least two unbalanced cycles. It is now easy to see that it therefore contains a theta-subgraph, or two cycles that share at most one vertex. Thus C contains a circuit of $L(G^e, \mathcal{B})$ and we are done. □

Definition 3.2. Let $r \geq 3$ be an integer, and let Δ_r be the graph obtained from a cycle with r edges by replacing each edge with a parallel pair. A *(tipless) spike* is a matroid of the form $L(\Delta_r, \mathcal{B})$, where $\mathcal{B}$ is a linear class of Hamiltonian cycles. Let $\mathcal{S}$ denote the class of matroids that are isomorphic to minors of spikes. Let k be a non-negative integer. We use $\mathcal{S}_k$ to denote the class of matroids that are isomorphic to minors of spikes of the form $L(\Delta_r, \mathcal{B})$, where $\mathcal{B}$ contains at most k Hamiltonian cycles. Therefore $\mathcal{S} = \cup_{k \geq 0} \mathcal{S}_k$.

Recall that a *cyclic flat* is a flat that is a (possibly empty) union of circuits. A set X is dependent if and only if $|X \cap Z| > r(Z)$ for some cyclic flat Z, so any matroid is determined by its cyclic flats and their ranks. It is an easy exercise to prove the following result.

Proposition 3.3. *Let $r \geq 3$ be an integer, and let $\mathcal{B}$ be a linear class of Hamiltonian cycles in Δ_r. The cyclic flats of $L(\Delta_r, \mathcal{B})$ are as follows:*

(i) *the entire ground set is a cyclic flat of rank r,*
(ii) *the empty set is a cyclic flat of rank zero,*
(iii) *the edge-set of each cycle in $\mathcal{B}$ is a cyclic flat of rank $r - 1$,*
(iv) *Any set of p parallel pairs is a cyclic flat of rank $p+1$ when $2 \leq p \leq r - 2$.*

Let $r \geq 3$ be an integer, and let C be a Hamiltonian cycle of Δ_r. Let C^* be the Hamiltonian cycle that contains no edges in common with C. If $\mathcal{B}$ is a linear class of Hamiltonian cycles, then $\mathcal{B}^*$ is the linear class $\{C^* \colon C \in \mathcal{B}\}$. It is well-known that the cyclic flats of M^* are exactly the complements of cyclic flats of M. Now the next result is easy to check.

Proposition 3.4. *Let $r \geq 3$ be an integer, and let $\mathcal{B}$ be a linear class of Hamiltonian cycles of Δ_r. Then $(L(\Delta_r, \mathcal{B}))^* = L(\Delta_r, \mathcal{B}^*)$. Consequently, $\mathcal{S}_k$ is closed under duality for each $k \geq 0$, and so is $\mathcal{S}$.*

Although the set of spikes is not closed under taking minors, we are able to give an explicit description of all the matroids in $\mathcal{S}_k$. This is accomplished in Proposition 3.5, which we now move towards proving.

The following description of minor operations on lift matroids follows from [10, Theorem 3.6]. Let $\mathcal{B}$ be a linear class of cycles in the graph G, and let e be an edge of G. We define $\mathcal{B}\backslash e$ to be the collection of cycles in $\mathcal{B}$ that do not contain e. Then $L(G, \mathcal{B})\backslash e = L(G\backslash e, \mathcal{B}\backslash e)$. If e is not a loop, then we define $\mathcal{B}/e$ to be the collection of cycles in G/e with edges sets of the form $E(C) - e$, where C is a cycle in $\mathcal{B}$ that may or may not contain e. With this definition, the equality $L(G, \mathcal{B})/e = L(G/e, \mathcal{B}/e)$ holds. If e is a balanced loop, then $L(G, \mathcal{B})/e = L(G, \mathcal{B})\backslash e$, and if e is an unbalanced loop, then $L(G, \mathcal{B})/e$ is equal to $M(G\backslash e)$, the cycle matroid of $G\backslash e$. Since any cycle matroid can be expressed as a lift matroid (by making every cycle balanced), these observations show that the class of lift matroids is minor-closed.

We recall that graphs may contain loops and parallel edges. Let $\mathcal{G}$ be the class of graphs containing:

(i) any graph with a single vertex,
(ii) any connected graph with exactly two vertices, and at most four edges joining them,
(iii) any graph whose underlying simple graph is a cycle of at least three vertices, where each parallel class contains at most two edges.

We note that if two graphs in $\mathcal{G}$ with equal edge-sets have the same parallel pairs, loops, and thin edges, then their Hamiltonian cycles have the same edge-sets. Furthermore the lift matroids corresponding to equal linear classes are equal. In other words, the cyclic order in which the parallel pairs and thin edges appear is immaterial to the lift matroid. Nor are the incidences of loops important.

Proposition 3.5. *Let k be a non-negative integer. A matroid belongs to $\mathcal{S}_k$ if and only if it satisfies at least one of the following statements.*

(A) *$M = L(G, \mathcal{B})$, where $G \in \mathcal{G}$ has at least three vertices, and $\mathcal{B}$ is a linear class of at most k Hamiltonian cycles,*
(B) *$M = L(G, \mathcal{B})$, where $G \in \mathcal{G}$ has exactly two vertices, and $\mathcal{B}$ is a linear class of at most k edge-disjoint Hamiltonian cycles,*
(C) *$M = L(G, \mathcal{B})$, where $G \in \mathcal{G}$ has a single vertex, and $\mathcal{B}$ contains at most* $\min\{k, 1\}$ *loops,*
(D) *$M = M(G)$ for a graph $G \in \mathcal{G}$,*
(E) *$M = M^*(G)$ for a graph $G \in \mathcal{G}$, or*
(F) *every connected component of M has size at most two.*

Definition 3.6. We refer to matroids satisfying the statements in Proposition 3.5 as being *Category*-(A), (B), (C), (D), (E), or (F), respectively.

Proof of Proposition 3.5. We start by proving that any matroid in $\mathcal{S}_k$ satisfies one of the statements in Proposition 3.5. Assume this fails for $M \in \mathcal{S}_k$. Now M can be expressed as $L(\Delta_r, \mathcal{B})/I\backslash J$ for disjoint sets I and J, where $\mathcal{B}$ contains at most k Hamiltonian cycles. We assume that we have chosen M so that $|I \cup J|$ is as small as possible. If $|I \cup J| = 0$, then $M = L(\Delta_r, \mathcal{B})$ is a Category-(A) matroid, contrary to our hypothesis. Therefore we let e be an element in $I \cup J$, and we define M^e to be $L(\Delta_r, \mathcal{B})/(I - e)\backslash(J - e)$. Note that M^e is in $\mathcal{S}_k$, and M is either M^e/e or $M^e\backslash e$. Our choice of M means that M^e is not a counterexample to the proposition.

If M^e is Category-(F), then so is M, which is impossible. Assume that $M^e = M(G)$ is Category-(D). Then M is also Category-(D) unless $M = M^e\backslash e$ where e is a thin edge in G. But in this case any circuit of M is either a loop, or a parallel pair in $G\backslash e$. Hence M is Category-(F). The case when M^e is Category-(E) leads to a dual contradiction. It is easy to see that any minor of a Category-(C) matroid belongs to Category-(C) or (F). Assume that $M^e = L(G, \mathcal{B})$ is Category-(B). If e is a thin edge in G and $M = M^e\backslash e$, then M is a rank-one matroid with no loops, and is therefore Category-(C). In any other case $M^e\backslash e$ is Category-(B), so we assume that $M = M^e/e$. If e is a loop then it is unbalanced, and $M = M(G\backslash e)$. In this case M is Category-(D). So e is a non-loop edge. Since e is in at most one balanced cycle, $\mathcal{B}/e$ contains at most one balanced loop. Therefore $M = L(G/e, \mathcal{B}/e)$ is Category-(C).

Now we must assume that $M^e = L(G, \mathcal{B})$ is Category-(A). Assume $M = M^e\backslash e$. If e is not a thin edge, then M is also Category-(A). In the case that e is a thin edge, there are no cycles in $\mathcal{B}\backslash e$. The thin edges of $G\backslash e$ are coloops in $M = L(G\backslash e, \mathcal{B}\backslash e) = L(G\backslash e, \emptyset)$. The only circuits of M consist of a pair of loops in G, a loop and a parallel pair, or a pair of parallel pairs. Now it is easy to see that M is $M^*(H)$, where H is in $\mathcal{G}$, and has the same parallel pairs as $G\backslash e$. The loops of H are the thin edges of $G\backslash e$, and the thin edges of H are the loops of $G\backslash e$. Therefore M is Category-(E). Thus we assume that $M = M^e/e$. If e is a loop of G, then $M = M(G\backslash e)$ is Category-(D), so we assume e is not a loop. If G has more than three vertices, then M is certainly Category-(A), so we assume G has exactly three vertices. To show that M is Category-(B), we assume for a contradiction that two cycles in $\mathcal{B}/e$ share an edge. This means that two cycles in $\mathcal{B}$ have two common edges, one of which is e. Given that G has three vertices, these two cycles differ in only one edge, which is a contradiction as $\mathcal{B}$ is a linear class and contains no parallel pairs. We have now shown that matroids in $\mathcal{S}_k$ satisfy at least one of the statements in the proposition.

Next we prove the converse. Let M be a Category-(F) matroid with l loops, p parallel pairs, and c coloops. Then M is isomorphic to $M(G)\backslash e$, where G is a graph in $\mathcal{G}$ with l loops, p parallel pairs, and $c+1$ thin edges, one of which is e. This shows that every Category-(F) matroid is a minor of a Category-(D) matroid. Let $M = M(G)$ be a Category-(D) matroid. Then $M = L(G^e, \emptyset)/e$, where G^e is obtained from G by adding e as a loop. Since G is in $\mathcal{G}$, it follows that G^e is also. Thus every Category-(D) matroid is a minor of a matroid in Category-(A), (B), or (C). Similarly, let $M = M^*(G)$ be a Category-(E) matroid, where G has l loops, p parallel pairs, and c thin edges. Then M is isomorphic to $L(G^e, \emptyset)\backslash e$, where $G^e \in \mathcal{G}$ has p parallel pairs, c loops, and $l+1$ thin edges, one of which is e. Because of these arguments, it now suffices to show that Category-(A), (B), and (C) matroids are in $\mathcal{S}_k$.

Let M be an n-element Category-(C) matroid. Note that M has at most one matroid loop. Construct the two-vertex graph G^e with $n-1$ loops and two non-loop edges, one of which is e. If M has a loop, then set $\mathcal{B}^e$ to contain the unique Hamiltonian cycle of G^e, and otherwise make $\mathcal{B}^e$ empty. Then $M = L(G^e, \mathcal{B}^e)/e$. Next we let $M = L(G, \mathcal{B})$ be a Category-(B) matroid. If G has at most one non-loop, then M is the union of a coloop and a parallel class. In this case it is easy to see that M is a minor of a Category-(A) matroid. Therefore we assume that G has at least two non-loop edges. Since G has at most four such edges, $\mathcal{B}$ contains at most two Hamiltonian cycles. We construct G^e, a three-vertex graph in $\mathcal{G}$. We set the number of non-loop edges in G^e to be one more than the number of non-loops in G, and we make e a thin edge of G^e. Let the number of loops in G^e be equal to the number of loops in G. We can find a linear class $\mathcal{B}^e$ of Hamiltonian cycles in G^e so that $|\mathcal{B}^e| = |\mathcal{B}|$. Now M is isomorphic to $L(G^e, \mathcal{B}^e)/e$, so we have reduced the proof to showing that every Category-(A) matroid is in $\mathcal{S}_k$.

Let $M = L(G, \mathcal{B})$ be a Category-(A) matroid. Let the parallel pairs of G be $\{a_1, b_1\}, \dots, \{a_t, b_t\}$, let the loops be $\{c_1, \dots, c_p\}$, and let the thin edges be $\{d_1, \dots, d_s\}$. We construct G^+ isomorphic to Δ_{t+p+s} with parallel pairs $\{a_1, b_1\}, \dots, \{a_t, b_t\}$, $\{c_1, x_1\}, \dots, \{c_p, x_p\}$, and $\{d_1, y_1\}, \dots, \{d_s, y_s\}$. Let $C_1, \dots, C_q$ be the cycles in $\mathcal{B}$. For each C_i, let C_i^+ be the Hamiltonian cycle of G^+ containing all of $x_1, \dots, x_p$ and $d_1, \dots, d_s$, and such that C_i^+ intersects $\{a_j, b_j\}$ in the same edge as C_i for each j. It is clear that $\mathcal{B}^+$ is a linear class. Furthermore M is isomorphic to

$$L(G^+, \mathcal{B}^+)/\{x_1, \dots, x_p\}\backslash\{y_1, \dots, y_s\},$$

so M is in $\mathcal{S}_k$, and the proof of the proposition is complete. □

Proposition 3.7. *Let $M = L(G, \mathcal{B})$ be a Category-*(A) *matroid, where G has at least five vertices. If C is a circuit-hyperplane in M, then $G[C]$ is a cycle in $\mathcal{B}$.*

Proof. This follows very easily from Proposition 3.3. We note that the constraint $|V(G)| \geq 5$ is necessary, for if G has four vertices, then a pair of parallel pairs in G may form a circuit-hyperplane of M. □

We use the symbol $\triangle$ to denote the graph obtained from a three-vertex cycle by adding a single parallel edge. (This graph is sometimes also known as the wheel with two spokes.)

Proposition 3.8. *The following matroids are not in $\mathcal{S}$.*

(i) $U_{0,1} \oplus U_{1,1} \oplus U_{1,3}$,
(ii) $U_{0,1} \oplus U_{1,1} \oplus U_{2,3}$,
(iii) $U_{0,1} \oplus U_{2,4}$,
(iv) $U_{1,1} \oplus U_{2,4}$,
(v) $U_{1,2} \oplus M(\triangle)$.

Proof. By virtue of Proposition 3.4, we need only prove that the matroids in (i), (iii), and (v) are not in $\mathcal{S}$. If a matroid belongs to $\mathcal{S}$, then it belongs to $\mathcal{S}_k$ for some value of k. Therefore we can apply Proposition 3.5. Note that if a matroid in $\mathcal{S}_k$ has a loop, then it is Category-(C), (D), (E), or (F). If a Category-(C), (D), or (E) matroid has a loop and a coloop, then every element is a loop or a coloop. (For example, the only way a Category-(D) matroid can have a coloop is if it is the cycle matroid of a graph with a thin edge joining two vertices.) Category-(F) matroids have no components of size three. In any case, we see that $U_{0,1} \oplus U_{1,1} \oplus U_{1,3}$ is not in $\mathcal{S}_k$. Category-(C) matroids do not have rank two, Category-(D) and (E) matroids obviously have no $U_{2,4}$-minor as they are graphic or cographic, and nor do Category-(F) matroids. Therefore $U_{0,1} \oplus U_{2,4}$ is not in $\mathcal{S}_k$.

Finally, Category-(A), (B), or (C) matroids have at most one non-trivial parallel class, so they cannot be isomorphic to $U_{1,2} \oplus M(\triangle)$. If $U_{1,2} \oplus M(\triangle)$ is Category-(D), then it is isomorphic to $M(G)$ for some $G \in \mathcal{G}$ with four vertices. But any such matroid is connected up to loops, so $U_{1,2} \oplus M(\triangle)$ is not Category-(D). Duality tells us it is not Category-(E) either, and it certainly has a connected component with more than two elements so it is not Category-(F). □

Lemma 3.9. *Let M be an excluded minor for $\mathcal{S}$ such that $r(M), r^*(M) > 2$ and M is not isomorphic to $U_{1,2} \oplus M(\triangle)$. Then M is simple.*

Proof. We start with the following claim.

Claim 3.9.1. Let M be an excluded minor for $\mathcal{S}$ such that $r(M), r^*(M) > 2$. Then M is loopless.

Proof. Assume the contrary, and let e be a loop of M. If every connected component of $M \backslash e$ has size at most two, then the same statement applies to M, so Proposition 3.5 now implies that M is in $\mathcal{S}$, which is a contradiction. Therefore we let N be a component of $M \backslash e$ with at least three elements. It is an easy exercise to see that N has a minor isomorphic to either $U_{1,3}$ or $U_{2,3}$ (see [6, Chapter 4 Exercise 9]). If there is another component of $M \backslash e$ with rank at least one, then M has a minor isomorphic to $U_{0,1} \oplus U_{1,1} \oplus U_{1,3}$ or $U_{0,1} \oplus U_{1,1} \oplus U_{2,3}$. In this case, Proposition 3.8 implies that M must be isomorphic to one of these two matroids, so M has rank or corank equal to two, a contradiction to the hypotheses. Thus every component of $M \backslash e$ other than N is a loop. Since $r(M) \geq 3$, we deduce that $r(N) \geq 3$.

Proposition 3.8 implies that if N has an $U_{2,4}$-minor, then M is isomorphic to $U_{0,1} \oplus U_{2,4}$, which is not possible. Therefore N is binary. Furthermore, N

cannot have a minor isomorphic to $M(K_4)$, or else M has a minor isomorphic to $U_{0,1} \oplus U_{1,1} \oplus U_{2,3}$. Therefore N is graphic (see [6, Theorem 10.4.8]). We let G be a graph such that $N = M(G)$. As $r(N) \geq 3$, it follows that G has at least four vertices. If every cycle of G contains at most two vertices, then G is not 2-connected, and this implies that N is not connected. Since this is not the case, G has a cycle with at least three vertices. Let C be such a cycle with a minimum number of vertices. Thus C has no chords, meaning that any edge joining two vertices in C is in C, or is parallel to an edge in C. Assume that there is an edge, x, of G such that x is neither in C, nor parallel to an edge of C. Then x is not in the span of C in N. This means that N has a minor isomorphic to $U_{1,1} \oplus U_{2,3}$, and hence M has a minor isomorphic to $U_{0,1} \oplus U_{1,1} \oplus U_{2,3}$. This leads to a contradiction, so every edge of G is either in C, or parallel to an edge in C. If G contains a parallel class of size at least three, then N has a minor isomorphic to $U_{1,1} \oplus U_{1,3}$, which again leads to a contradiction. Therefore G is obtained from a cycle of at least three vertices by adding parallel edges in such a way that any parallel class has size one or two. Now it follows that M is the cycle matroid of a graph in $\mathcal{G}$, and hence Proposition 3.5 implies that M is in $\mathcal{S}$, which is a contradiction. $\square$

Let M be an excluded minor $\mathcal{S}$ satisfying $r(M), r^*(M) > 2$ and assume that M is not isomorphic to $U_{1,2} \oplus M(\triangle)$. Then M has no loops by Claim 3.9.1. Since Claim 3.9.1 also applies to M^*, we deduce that M has no coloops. Assume that M has at least one parallel pair, and let $\{x, y\}$ be such a pair.

Claim 3.9.2. *M/x has a connected component containing at least three elements.*

Proof. Assume for a contradiction that every connected component of M/x has size one or two. Then every connected component of M/x is a loop, or a 2-element circuit, since M has no coloops. Let L be the set of loops of M/x and note that L is not empty as it contains y. Then $L \cup x$ is a parallel class of M, since M is loopless. Let $C_1, \ldots, C_s$ be the 2-element components of M/x that are not circuits in M, and let $D_1, \ldots, D_t$ be the 2-element components that are circuits of M. Note that $s + t \geq 2$, since $r(M) > 2$ implies $r(M/x) \geq 2$.

If $t = 0$, then M is isomorphic to $M^*(G)$, where G is obtained from a cycle of length $s + |L| + 1$ by replacing s of the edges with parallel pairs. Thus G is in $\mathcal{G}$, and Proposition 3.5 implies that M is in $\mathcal{S}$, so we have reached a contradiction. Therefore $t > 0$. Assume that $s = 0$, so that $t \geq 2$. The connected components of M are now $D_1, \ldots, D_t$ and $L \cup x$. Thus $|L \cup x| > 2$, or else every component of M has size at most two, which is a contradiction as M is not in $\mathcal{S}$. We contract an element from D_1, so that the other element of D_1 is now a loop. We choose a single element from D_2, and three elements from $L \cup x$. Now we see that M has a proper minor isomorphic to $U_{0,1} \oplus U_{1,1} \oplus U_{1,3}$, which contradicts Proposition 3.8.

Therefore s and t are both positive. The restriction of M to $C_1 \cup D_1 \cup \{x, y\}$ is isomorphic to $U_{1,2} \oplus M(\triangle)$. So M is isomorphic to this matroid, which is not possible. This contradiction completes the proof of Claim 3.9.2. $\square$

As M is an excluded minor, we know that M/x is a member of $\mathcal{S}$. Therefore it satisfies one of the statements in Proposition 3.5. Claim 3.9.2 shows that M/x is not Category-(F). Category-(A) and (B) matroids do not have loops, and M/x contains the loop y, so it belongs to neither of these categories. As the rank of M/x is at least two, it is not Category-(C). If $M/x = M^*(G)$ for some $G \in \mathcal{G}$, then G has an isthmus, since M/x has a loop. This is only possible if G is obtained from K_2 by adding loops. In this case M/x has at least two coloops (since $r(M/x) \geq 2$). But this is impossible, as M has no coloops. Therefore M/x is not Category-(E), so it must be Category-(D). Let $G \in \mathcal{G}$ be chosen so that $M/x = M(G)$, and let L be the set of loops of G. Note that y is in L, and that G has at least three vertices, since $r(M) > 2$.

We let G^x be the graph obtained from G by adding x as a loop. Let $\mathcal{B}$ be the collection of cycles of G that correspond to circuits of M. Since M is loopless, $\mathcal{B}$ contains no loop. Proposition 3.1 tells us that $M = L(G^x, \mathcal{B})$. If $\mathcal{B}$ contains only Hamiltonian cycles, then M is in $\mathcal{S}$, a contradiction. Therefore $\mathcal{B}$ must contain a cycle with two edges, a and b. Hence $\{a, b\}$ is a circuit of M. Assume that there is a two-edge cycle that is not in $\mathcal{B}$ and let those edges be c and d. Then the restriction of M to $\{a, b, c, d, x, y\}$ is isomorphic to $U_{1,2} \oplus M(\triangle)$. This implies that M is isomorphic to $U_{1,2} \oplus M(\triangle)$, which is a contradiction. We conclude that $\mathcal{B}$ contains every two-edge cycle of G^x.

Assume that $\mathcal{B}$ contains no Hamiltonian cycles. The only circuits of M are pairs in $L \cup x$, parallel pairs in G, and the union of an element in $L \cup x$ with the edge-set of a Hamiltonian cycle. In other words, M is obtained from a circuit by adding parallel elements. If $|L \cap x| \leq 2$, then M is the cycle matroid of a graph in $\mathcal{G}$, which is not possible. Therefore $|L \cup x| \geq 3$. If G has no parallel pairs, then M is the lift matroid of a graph obtained from a cycle by adding loops (where no cycle is balanced). In this case M is a Category-(A) matroid, which is impossible. Therefore G contains a parallel pair of edges, a and b. We contract a, so that b is a loop, and then select b, three elements from $L \cup x$, and a single element not in $L \cup \{x, a, b\}$ (this element exists because $r(M) \geq 3$). This shows that M has a proper minor isomorphic to $U_{0,1} \oplus U_{1,1} \oplus U_{1,3}$, which is a contradiction.

Now we know that $\mathcal{B}$ contains a Hamiltonian cycle, C_1. Assume there is a Hamiltonian cycle, C_2 that is not in $\mathcal{B}$, and assume that we have chosen C_2 so that it has as many edges in common with C_1 as possible. Let $\{e, f\}$ be a parallel pair of G^x such that e is in C_1 and f is in C_2. Let C_3 be the Hamiltonian cycle obtained from C_2 by removing f and replacing it with e. Then C_3 is in $\mathcal{B}$ by our choice of C_2. The theta-subgraph obtained from C_3 by adding f contains C_2, C_3, and $\{e, f\}$, and exactly two of these cycles are in $\mathcal{B}$, which contradicts the fact that $\mathcal{B}$ is a linear class. We conclude that every Hamiltonian cycle is in $\mathcal{B}$. This means that the only cycles of G^x not in $\mathcal{B}$ are the loops. It follows that M is the direct sum of the cycle matroid $M(G\backslash L)$ and the parallel class $L \cup x$. But $G\backslash L$ has a minor isomorphic to $\triangle$, and hence M has a minor isomorphic to $U_{1,2} \oplus M(\triangle)$, and this leads to a contradiction that completes the proof of Lemma 3.9. $\square$

Lemma 3.10. *Let k be a non-negative integer. There exists an integer, N_k, with the following property: if M is an excluded minor for $\mathcal{S}_k$ with $|E(M)| > N_k$, then $|E(M)|$ is even.*

Proof. We start by noting that matroids of rank at most two are well-quasi-ordered. This is not difficult to prove directly, but it also follows from [3], since $U_{3,3}$ is the sole excluded minor for the class of matroids with rank at most two, and the main theorem in [3] implies that the class produced by excluding $U_{3,3}$ does not contain any infinite antichains. So there are only finitely many excluded minors for $\mathcal{S}_k$ with rank (or corank, by duality) at most two.

We let $\mathcal{S}'$ stand for the class of Category-(D), (E), or (F) matroids. By referring to the proof of Proposition 3.5, we can easily verify that $\mathcal{S}'$ is a minor-closed class. Note that all matroids in $\mathcal{S}'$ are graphic (since graphs in $\mathcal{G}$ are planar). Therefore any excluded minor for $\mathcal{S}'$ is either an excluded minor for the class of graphic matroids, or it is itself graphic. There are only five excluded minors for the class of graphic matroids [8]. The class of graphic matroids is well-quasi-ordered by the famous result of Robertson and Seymour [7], so there are only finitely many graphic excluded minors for $\mathcal{S}'$. Thus $\mathcal{S}'$ has finitely many excluded minors. (With only a small amount of extra effort we could find the excluded minors for $\mathcal{S}'$ directly, in which case we would not have to rely on Robertson and Seymour's result.)

These arguments show that we can choose N_k so that it satisfies $N_k \geq 12$, and also $N_k \geq |E(M)|$ whenever M is an excluded minor for $\mathcal{S}'$ or an excluded minor for $\mathcal{S}_k$ with rank or corank at most two. Now, if M is an excluded minor for $\mathcal{S}_k$ such that $|E(M)| > N_k$, then $r(M), r^*(M) \geq 3$, and M is not an excluded minor for $\mathcal{S}'$.

Claim 3.10.1. Let M be an excluded minor for $\mathcal{S}_k$. Assume $|E(M)|$ is odd and larger than N_k. Then M is not in $\mathcal{S}$.

Proof. Assume otherwise, so that M belongs to $\mathcal{S}_{k'}$ for some $k' > k$. We apply Proposition 3.5 to M. Since M is not in $\mathcal{S}'$, it is not Category-(D), (E), or (F). It is also not Category-(B) or (C), as $r(M) \geq 3$. Therefore M is Category-(A), so $M = L(G, \mathcal{B})$, where $G \in \mathcal{G}$ has at least three vertices, and $\mathcal{B}$ is a linear class of at most k' Hamiltonian cycles. As M is not in $\mathcal{S}_k$, it follows that $k < |\mathcal{B}| \leq k'$. Furthermore, $|E(M)|$ is odd, so the edge-set of G is not a union of parallel pairs. Hence G has either a loop or a thin edge. Note that M has no loops or coloops.

Assume that G has at most four vertices. Since $|E(M)| > N_k \geq 12$, it follows that G has at least five loops. Thus we can let P be a parallel class of M satisfying $|P| \geq 5$. Let x and y be distinct elements in P. We apply Proposition 3.5 to $M\backslash x$, which is in $\mathcal{S}_k$. Since $M\backslash x$ has a parallel class of size at least four, it is not Category-(D) or (F). Furthermore, $r(M) \geq 3$ implies $r(M\backslash x) \geq 3$, so it is not Category-(B) or (C). Hence $M\backslash x$ is Category-(A) or (E). If $M\backslash x$ is Category-(E), then $M\backslash x = M^*(G_x)$ for some graph $G_x \in \mathcal{G}$, and the elements in $P - x$ are thin edges of G_x. Let G_x^+ be obtained from G_x by subdividing y, and naming the two new edges x and y. Then $M^*(G_x^+)$ is obtained from $M^*(G_x)$ by placing x parallel to y. It follows that $M = M^*(G_x^+)$, and hence M is in $\mathcal{S}'$, a contradiction.

Therefore $M\backslash x$ is Category-(A), so it is equal to $L(G_x, \mathcal{B}_x)$, where $G_x \in \mathcal{G}$ contains at least three vertices, and $\mathcal{B}_x$ contains at most k Hamiltonian cycles of G_x. The only parallel pairs in $L(G_x, \mathcal{B}_x)$ arise from loops of G_x. We deduce that the elements of $P - x$ are loops in G_x. We obtain G_x^+ from G_x by adding x as a loop incident with an arbitrary vertex. Then $L(G_x^+, \mathcal{B}_x)$ is obtained by adding x parallel to y, so $L(G_x^+, \mathcal{B}_x) = M$. This implies M is in $\mathcal{S}_k$, which is not true. Therefore G has at least five vertices and hence $r(M) \geq 5$.

Recall that G has either a loop or a thin edge. Assume that G has a loop, x. Note that $M\backslash x = L(G\backslash x, \mathcal{B})$. Since $M\backslash x$ is in $\mathcal{S}_k$, we apply Proposition 3.5. Because $r(M\backslash x) \geq 5$, it follows that $M\backslash x$ is not Category-(B) or (C). Note that $\mathcal{B}$ is not empty, since it contains more than k Hamiltonian cycles. Any cycle in $\mathcal{B}$ corresponds to a circuit-hyperplane in $M\backslash x$, which necessarily has at least five elements. Therefore $M\backslash x$ is not Category-(F). The only Category-(D) matroids with a circuit-hyperplane have rank at most two, and $r(M\backslash x) \geq 3$, so $M\backslash x$ is not Category-(D). A simple analysis shows that a Category-(E) matroid with rank at least five has no circuit-hyperplane, so $M\backslash x$ is not Category-(E). The only remaining possibility is that $M\backslash x$ is Category-(A). Therefore $M\backslash x = L(G_x, \mathcal{B}_x)$, where $G_x \in \mathcal{G}$ has at least three vertices, and $\mathcal{B}_x$ contains at most k Hamiltonian cycles. Note that G_x has at least five vertices, as $r(M\backslash x) \geq 5$. Proposition 3.7 implies that $M\backslash x$ has at most k circuit-hyperplanes, which is impossible because $\mathcal{B}$ contains at least $k+1$ cycles, and thus M has at least $k+1$ circuit-hyperplanes that avoid x. Thus G has no loop. By an earlier conclusion, we can let x be a thin edge.

As G has no loops and at least 13 edges, we can now see that G has at least six vertices, so $r(M) \geq 6$. We consider the matroid M/x, which is in $\mathcal{S}_k$. As in the previous paragraph, we can argue that $M/x = L(G_x, \mathcal{B}_x)$, where $G_x \in \mathcal{G}$ has at least five vertices, and $\mathcal{B}_x$ contains at most k cycles. This implies that M/x has at most k circuit-hyperplanes. But this is impossible, as $\mathcal{B}$ contains at least $k+1$ cycles, and each corresponds to a circuit-hyperplane of M that contains x. $\square$

Now, whenever M is an excluded minor for $\mathcal{S}_k$ such that $|E(M)|$ is odd and larger than N_k, Claim 3.10.1 tells us that it is an excluded minor for $\mathcal{S}$. Since $r(M), r^*(M) \geq 3$ and $|E(M)| > 12$, Lemma 3.9 implies that M is simple. As M^* is also an excluded minor for $\mathcal{S}$, we can deduce that M is cosimple.

Claim 3.10.2. Let M be an excluded minor for $\mathcal{S}_k$. Assume $|E(M)|$ is odd and larger than N_k. If, for some $e \in E(M)$, we have $M\backslash e = L(G_e, \mathcal{B}_e)$, where $G_e \in \mathcal{G}$ has at least three vertices and $\mathcal{B}_e$ is a linear class of at most k Hamiltonian cycles, then G_e has no loops, and at least six vertices.

Proof. Assume x is a loop in G_e. Note that $M\backslash e/x = M(G_e\backslash x)$, so that $M\backslash e/x$ is Category-(D). As M is simple, $M\backslash e/x$ has no matroid loops. Therefore x is the unique loop in G_e. We will apply Proposition 3.5 to M/x, which is in $\mathcal{S}_k$. Our aim is to deduce that M/x too is Category-(D).

Since G_e has exactly one loop, and at least twelve edges, it follows that it has more than five vertices. Therefore $r(M) > 5$. This immediately rules out the cases where M/x is Category-(B) or (C). Furthermore, if we let C be the edge-set of any Hamiltonian cycle in G_e, then either C is a circuit-hyperplane of $M\backslash e$, or $C \cup x$ is a circuit. In any case, M/x has a circuit of more than five elements, so it is not Category-(F).

We note that Category-(A) and (E) matroids have at most one parallel class. So if M/x belongs to either of these categories, then $M/x\backslash e = M(G_e\backslash x)$ too has at most one parallel class. This implies that G_e has at most one parallel pair. But in this case, G_e comprises a cycle, a loop, and at most one parallel edge. This implies that $r^*(M\backslash e) \leq 1$, so $r^*(M) \leq 2$, a contradiction. Therefore we can conclude that M/x is Category-(D), exactly as we wanted.

Now let $G_x \in \mathcal{G}$ be chosen so that $M/x = M(G_x)$. Let G_x^+ be constructed from G_x by adding the loop x to an arbitrary vertex. Proposition 3.1 asserts that $M = L(G_x^+, \mathcal{B})$, for some linear class $\mathcal{B}$ of cycles in G_x^+. But $\mathcal{B}$ cannot contain a cycle with one or two edges, for M is simple. Therefore $\mathcal{B}$ contains only Hamiltonian cycles of G_x^+. This demonstrates that M is in $\mathcal{S}$, which is impossible, according to Claim 3.10.1. Therefore G_e has no loops, and as it has at least twelve edges, it has at least six vertices. □

Claim 3.10.3. Let M be an excluded minor for $\mathcal{S}_k$. Assume $|E(M)|$ is odd and larger than N_k. Assume also that $M\backslash e = L(G_e, \mathcal{B}_e)$ for some $e \in E(M)$, where $G_e \in \mathcal{G}$ has at least three vertices and $\mathcal{B}_e$ is a linear class of at most k Hamiltonian cycles. Then G_e contains at least three parallel pairs, and if $\{a, b\}$ is a parallel pair in G_e, then $\{e, a, b\}$ is a circuit of M.

Proof. Let the parallel pairs in G_e be $\{a_1, b_1\}, \ldots, \{a_t, b_t\}$. Assume that $t \leq 2$. Since G_e has no loops by Claim 3.10.2, it follows that $r^*(M\backslash e) \leq 1$, which is impossible. Hence $t \geq 3$. Since the numbering of the parallel pairs is arbitrary, we can finish the proof by showing that $\{e, a_1, b_1\}$ is a circuit of M.

Note that $M\backslash e/a_1 = L(G_e/a_1, \mathcal{B}_e/a_1)$, where $\mathcal{B}_e/a_1$ is obtained from $\mathcal{B}_e$ by removing the Hamiltonian cycles that do not contain a_1, and then contracting a_1 from each of the remaining cycles. As b_1 is a loop in G_e/a_1, it follows that $\{b_1, a_i, b_i\}$ is a triangle in $M\backslash e/a_1$ for each $i \geq 1$, and is thus a triangle in M/a_1 and a triad in $M^*\backslash a_1$.

We apply Proposition 3.5 to $M^*\backslash a_1$. Because it has triads, it is not Category-(F). Since $r(M^*\backslash a_1) = r^*(M) \geq 3$, it follows that $M^*\backslash a_1$ is not Category-(B) or (C). Note that $M^*\backslash a_1$ has no parallel pairs and no loops, as M is cosimple. So if $M^*\backslash a_1$ is Category-(D), then it is a circuit, which is impossible as its corank is at least two. Now assume that $M^*\backslash a_1$ is Category-(E), so that $M^*\backslash a_1 = M^*(G_{a_1})$ for some $G_{a_1} \in \mathcal{G}$. Then G_{a_1} has no loops, since $M^*\backslash a_1$ has no coloops. Furthermore, $M^*\backslash a_1$ is simple, so G_{a_1} has at most one thin edge. But G_{a_1} also has an even number of edges, so it follows that G_{a_1} is isomorphic to Δ_r where $r = \frac{1}{2}(|E(M)| - 1)$. But then $M^*\backslash a_1$ has no triads, which is impossible. We are forced to conclude that $M^*\backslash a_1$ is Category-(A).

Now we choose $G_{a_1} \in \mathcal{G}$ and a linear class $\mathcal{B}_{a_1}$ of at most k Hamiltonian cycles so that $M^*\backslash a_1 = L(G_{a_1}, \mathcal{B}_{a_1})$. By Claim 3.10.2 we see that G_{a_1} has no loops and at least six vertices. We observe that the only triads of $M^*\backslash a_1$ consist of a parallel pair in G_{a_1} along with a thin edge. Since $\{b_1, a_i, b_i\}$ is a triad in $M^*\backslash a_1$ for each $i \in \{2, \ldots, t\}$, we see that b_1 must be a thin edge of G_{a_1}, and each $\{a_i, b_i\}$ is a parallel pair. As G_{a_1} has an even number of edges, we let x be another thin edge, distinct from b_1. Then $\{b_1, x\}$ is a cocircuit of $M^*\backslash a_1$, and hence a circuit in M/a_1. Since M is simple, this implies that $\{x, a_1, b_1\}$ is a circuit. If $x = e$, then there is nothing left to prove, so we assume that $x \neq e$. Therefore $\{x, a_1, b_1\}$ is a circuit of $M\backslash e$. But this is impossible, as G_e has no loops, and since G_e has at least six vertices, it follows that $M\backslash e = L(G_e, \mathcal{B}_e)$ has no triangles. This completes the proof of the claim. $\square$

Now we let M be an excluded minor for $\mathcal{S}_k$ such that $|E(M)|$ is larger than N_k and odd. We recall that M is simple and cosimple. As M is not an excluded minor for $\mathcal{S}'$, there is an element $e \in E(M)$ such that either $M\backslash e$ or M/e is in $\mathcal{S}_k$ but not in $\mathcal{S}'$. By duality, we can assume that $M\backslash e$ is in $\mathcal{S}_k - \mathcal{S}'$. We apply Proposition 3.5. From $r(M) \geq 3$ we deduce that $M\backslash e$ is not Category-(B) or (C), and as it is not in $\mathcal{S}'$, $M\backslash e$ must be Category-(A). Choose $G_e \in \mathcal{G}$ and $\mathcal{B}_e$, a linear class of at most k Hamiltonian cycles in G_e, such that $M\backslash e = L(G_e, \mathcal{B}_e)$. From Claims 3.10.2 and 3.10.3, we know that G_e has no loops, at least six vertices, and least three parallel pairs. Thus $r(M) \geq 6$. Let $\{a_1, b_1\}, \ldots, \{a_t, b_t\}$ be the parallel pairs of edges. Then $\{e, a_i, b_i\}$ is a triangle of M for each i.

Now we apply Proposition 3.5 to $M\backslash a_1$. As this matroid contains triangles it is not Category-(F). The inequality $r(M) \geq 6$ rules out Categories-(B) and (C). If $M\backslash a_1$ is Category-(D), then it must be a circuit, for these are the only simple Category-(D) matroids. But this would contradict $r^*(M) \geq 3$. If

$M\backslash a_1$ is Category-(E), then $M\backslash a_1 = M^*(G_{a_1})$ for some $G_{a_1} \in \mathcal{G}$. Because $M\backslash a_1$ has no coloops, G_{a_1} has no loops. If it contains a thin edge, then it contains two thin edges, as the number of edges in G_{a_1} is even. This implies $M\backslash a_1$ contains a parallel pair, which is impossible. So G_{a_1} is Δ_r, where $r = \frac{1}{2}(|E(M)| - 1)$. But then $M^*(G_{a_1})$ contains no triangles, and this is impossible since $\{e, a_2, b_2\}$ is a triangle of $M\backslash a_1$. Therefore $M\backslash a_1$ is Category-(A). Now we can apply Claim 3.10.3 to $M\backslash a_1$. It tells us that a_1 is in at least three triangles of M, and that the intersection of any pair of these triangles is $\{a_1\}$. From this it follows that a_1 is in a triangle of $M\backslash e = L(G_e, \mathcal{B}_e)$, which is impossible as G_e has at least six vertices, and no loops. Now the proof of Lemma 3.10 is complete. □

We are positive that there are only finitely many excluded minors for $\mathcal{S}$. But we do not require this for our main results, so we leave it as an open problem.

Problem 3.11. *Prove that $\mathcal{S}$ has only finitely many excluded minors, and describe all of them.*

From this point onwards our strategy in proving Theorem 1.5 is similar to that used in the proof of Theorem 1.3. We start by recalling Definition 2.1: if $\mathcal{C} = (C_1, \ldots, C_k)$ is a sequence of subsets of the set E, then for any $I \subseteq \{1, \ldots, k\}$, we use $\mathcal{C}(I)$ to denote the set $\{e \in E\colon e \in C_i \Leftrightarrow i \in I\}$. The function $\psi_{\mathcal{C}}$ takes each I to $|\mathcal{C}(I)|$. When dealing with spikes, we can fix one of the circuit-hyperplanes, and then consider the pattern of its intersections with the other circuit-hyperplanes. To facilitate this approach, we introduce the following notation: if $\mathcal{C} = (C_1, \ldots, C_k)$ is a sequence of sets, then we define $\mathrm{trun}(\mathcal{C})$ to be the derived sequence $(C_1 \cap C_k, \ldots, C_{k-1} \cap C_k)$ of $k-1$ sets.

Let k and r be integers satisfying $k \geq 2$ and $r \geq 5$. The relation, $\mathcal{R}_k^r$, has as its domain the set of rank-r Category-(A) matroids with exactly k circuit-hyperplanes. The codomain is the set of functions from $\mathcal{P}(\{1, \ldots, k-1\})$ to $\mathbb{Z}_{\geq 0}$. Let M be a matroid in the domain, and let ψ be a function in the codomain. The ordered pair (M, ψ) belongs to $\mathcal{R}_k^r$ if and only if there is an ordering $\mathcal{C} = (C_1, \ldots, C_k)$ of the circuit-hyperplanes in M such that ψ is equal to $\psi_{\mathrm{trun}(\mathcal{C})}$. In this case, ψ takes $I \subseteq \{1, \ldots, k-1\}$ to the number of elements in C_k that are in every C_i for $i \in I$, and in no C_i for $i \notin I$. Furthermore,

$$\sum_{I \subseteq \{1, \ldots, k-1\}} \psi(I) = r.$$

Note that the image of M under $\mathcal{R}_k^r$ has cardinality at most $k!$.

A Category-(A) matroid $M = L(G, \mathcal{B})$ with rank at least five has at most one non-trivial parallel class (comprising the loops of G), and at most one non-trivial series class (comprising the thin edges). Any triangle of M comprises a loop of G and a parallel pair, and any triad comprises a parallel pair along with a thin edge. Moreover, the circuit-hyperplanes of M correspond exactly to the cycles in $\mathcal{B}$, as we noted in Proposition 3.7.

Let M be a Category-(A) matroid. If M has no triangle, we set $p(M)$ to be zero, and otherwise we set it to be the maximum size of a parallel class in M. Similarly, if M has no triad, we set $s(M)$ to be zero, and otherwise we set it to be the largest size of a series class.

Proposition 3.12. *Let k and r be integers satisfying $k \geq 2$ and $r \geq 5$. Let M and N be rank-r Category-*(A) *matroids with exactly k circuit-hyperplanes. Then M and N are isomorphic if and only if*

(i) $p(M) = p(N)$,
(ii) $s(M) = s(N)$, *and*
(iii) $\mathcal{R}_k^r(M) \cap \mathcal{R}_k^r(N) \neq \emptyset$.

Proof. Let ρ be an isomorphism from M to N. The existence of ρ clearly means that $p(M) = p(N)$ and $s(M) = s(N)$. Let $(C_1, \ldots, C_k)$ be an ordering of the circuit-hyperplanes in M. Then $\rho(\mathcal{C}) = (\rho(C_1), \ldots, \rho(C_k))$ is an ordering of the circuit-hyperplanes in N. It is now clear that $\psi_{\mathrm{trun}(\mathcal{C})} = \psi_{\mathrm{trun}(\rho(\mathcal{C}))}$, so $\mathcal{R}_k^r(M) \cap \mathcal{R}_k^r(N)$ contains at least one function.

For the converse, we assume $p(M) = p(N)$ and $s(M) = s(N)$, and that $\mathcal{R}_k^r(M) \cap \mathcal{R}_k^r(N)$ contains a function. This means that we can let $\mathcal{C}_M = (C_1^M, \ldots, C_k^M)$ and $\mathcal{C}_N = (C_1^N, \ldots, C_k^N)$ be orderings of the circuit-hyperplanes in M and N such that the functions $\psi_{\mathrm{trun}(\mathcal{C}_M)}$ and $\psi_{\mathrm{trun}(\mathcal{C}_N)}$ are equal.

Assume that $M = L(G_M, \mathcal{B}_M)$ and $N = L(G_N, \mathcal{B}_N)$, where $G_M, G_N \in \mathcal{G}$ have at least five vertices, and $\mathcal{B}_M$ and $\mathcal{B}_N$ contain exactly k Hamiltonian cycles. Both G_M and G_N contain $p := p(M) = p(N)$ loops, and we can assume these loops are labelled $c_1, \ldots, c_p$ in both graphs. Similarly, G_M and G_N have $s := s(M) = s(N)$ thin edges, and we assume these edges are labelled $d_1, \ldots, d_s$. Now G_M and G_N have $t := r - s$ parallel pairs, and we assume that these pairs are labelled $\{a_1, b_1\}, \ldots, \{a_t, b_t\}$.

We will construct a permutation π of the ground set

$$E(M) = E(N) = \{a_i, b_i\}_{i=1}^t \cup \{c_i\}_{i=1}^p \cup \{d_i\}_{i=1}^s$$

such that:

(i) π acts as the identity on $c_1, \ldots, c_p, d_1, \ldots, d_s$,
(ii) π takes any parallel pair $\{a_i, b_i\}$ to another such pair, and
(iii) π takes any circuit-hyperplane in M to a circuit-hyperplane of N.

Since the non-spanning circuits of M and N are exactly the circuit-hyperplanes, along with sets of the form $\{a_i, b_i, a_j, b_j\}$, the existence of π will show that M and N are isomorphic.

Let $\mathcal{I}$ be the collection of proper subsets of $\{1, \ldots, k-1\}$. For each $I \in \mathcal{I}$ let π_I be an arbitrary bijection from $\mathrm{trun}(\mathcal{C}_M)(I)$ to $\mathrm{trun}(\mathcal{C}_N)(I)$. This bijection exists because $\psi_{\mathrm{trun}(\mathcal{C}_M)}(I) = \psi_{\mathrm{trun}(\mathcal{C}_N)}(I)$, and hence

$$|\,\mathrm{trun}(\mathcal{C}_M)(I)| = |\,\mathrm{trun}(\mathcal{C}_N)(I)|.$$

Next we note that the thin edges $d_1, \ldots, d_s$ are contained in every circuit-hyperplane of M and N. That is, the elements $d_1, \ldots, d_s$ are contained in both $\mathrm{trun}(\mathcal{C}_M)(\{1, \ldots, k-1\})$ and $\mathrm{trun}(\mathcal{C}_N)(\{1, \ldots, k-1\})$. We let $\cap(M)$ stand for the intersection $\cap_{i=1}^{k} C_i^M$ and let $\cap(N)$ stand for $\cap_{i=1}^{k} C_i^N$. Let σ be an arbitrary bijection from

$$\cap(M) - \{d_1, \ldots, d_s\} \quad \text{to} \quad \cap(N) - \{d_1, \ldots, d_s\}.$$

Let id_d be the identity function on $\{d_1, \ldots, d_s\}$. Now let π_0 be the union

$$\mathrm{id}_d \ \cup \sigma \ \cup \ \bigcup_{I \in \mathcal{I}} \pi_I.$$

Observe that π_0 is a bijection from C_k^M to C_k^N.

We extend π_0 to a permutation of $E(M) = E(N)$ by insisting that it preserves parallel pairs. To this end, we note that C_k^M contains the thin edges $d_1, \ldots, d_s$, along with exactly one element from each of the parallel pairs $\{a_1, b_1\}, \ldots, \{a_t, b_t\}$. We construct π_1, a bijection from $\{a_1, b_1, \ldots, a_t, b_t\} - C_k^M$ to $\{a_1, b_1, \ldots, a_t, b_t\} - C_k^N$. If x is in the domain of π_1, then x is in a parallel pair with an edge y in G_M. Moreover, y is in C_k^M, so $\pi_0(y)$ is defined, and is in C_k^N. We note that $\pi_0(y)$ is in a parallel pair with an edge x' in G_N, and we set the image $\pi_1(x)$ to be x'. Now we set π to be $\pi_0 \cup \pi_1 \cup \mathrm{id}_c$, where id_c is the identity function on $\{c_1, \ldots, c_p\}$. Thus π is indeed a permutation of $E(M) = E(N)$, it acts as the identity on $\{c_1, \ldots, c_p, d_1, \ldots, d_s\}$, and it takes any pair $\{a_i, b_i\}$ to another such pair.

To complete the proof it suffices to show that π takes any circuit-hyperplane of M to a circuit-hyperplane of N. This in turn will follow if we can show that when x is in $\mathcal{C}_M(I)$ for some $I \subseteq \{1, \ldots, k\}$, the image $\pi(x)$ is in $\mathcal{C}_N(I)$. This is true when x is in $\{c_1, \ldots, c_p\}$, for then $x \in \mathcal{C}_M(I)$ implies $I = \emptyset$, and $x = \pi(x)$ is also in $\mathcal{C}_N(\emptyset)$. Similarly, if x is in $\{d_1, \ldots, d_s\}$, then $x \in \mathcal{C}_M(I)$ implies $I = \{1, \ldots, k\}$, and $x = \pi(x)$ is in $\mathcal{C}_N(\{1, \ldots, k\})$. So we assume that x is not equal to any element c_i or d_i. If I contains k, then x is in C_k^M, which means it is in the domain of π_0. In this case $\pi(x) = \pi_0(x)$ is in $\mathcal{C}_N(I)$, by construction of π_0. Therefore we assume that k is not in I, so x is not in C_k^M. This means that x is a non-loop edge that is contained in a parallel pair $\{x, y\}$ in G_M, and furthermore y is in C_k^M. Now y is in exactly the circuit-hyperplanes that x is not in. In other words, y is in $\mathcal{C}_M(\{1, \ldots, k\} - I)$. As y is in the domain of π_0, it now follows that $\pi_0(y)$ is in $\mathcal{C}_N(\{1, \ldots, k\} - I)$. But $\pi(x)$ is parallel to $\pi_0(y)$ in G_N, meaning that it is in exactly the circuit-hyperplanes of N that $\pi_0(y)$ is not in. Thus it follows that $\pi(x)$ is in $\mathcal{C}_M(I)$, exactly as desired. This completes the proof. □

Lemma 3.13. *Let k be a positive integer. The number of $2t$-element matroids in $\mathcal{S}_k$ is at most $O(t^{2^{k-1}+2})$.*

Proof. We refer to Proposition 3.5. A Category-(F) matroid with $2t$ elements is determined by giving the number of loops and the number of coloops. This argument shows that there are no more than $O((2t)^2)$ such matroids, so we will henceforth disregard them. Up to isomorphism, a $2t$-element matroid of the form $M(G)$ or $M^*(G)$ can be determined by the number of loops and thin edges in G. Therefore the number of $2t$-element Category-(D) or (E) matroids is at most $O((2t)^2)$, so we also disregard these classes. There is a constant number of loopless graphs in $\mathcal{G}$ with a bounded number of vertices. Thus there is a constant number of graphs with $\mathcal{G}$ with $2t$ edges and a bounded number of vertices. So we disregard any matroids of the form $L(G, \mathcal{B})$ when $G \in \mathcal{G}$ has fewer than five vertices. Thus we have disregarded Category-(B) and (C) matroids, and we now need only consider Category-(A) matroids with rank at least five. A Category-(A) matroid $L(G, \mathcal{B})$ with at most one circuit-hyperplane is determined up to isomorphism by the number of loops and thin edges in G, so there are at most $O((2t)^2)$ such matroids. Therefore we may as well assume that k is at least two, and we will consider only matroids with at least two circuit-hyperplanes.

Our arguments have shown that we need only consider $2t$-element Category-(A) matroids with rank at least five and at least two circuit-hyperplanes. We categorise these matroids as having rank r, where r satisfies $5 \leq r \leq 2t$, and having exactly m circuit-hyperplanes, where m satisfies $2 \leq m \leq k$. The number of pairs (r, m) is $O(t)$, so we will be done if we can show that the number of matroids corresponding to the pair (r, m) is at most $O(t^{2^{k-1}+1})$. By Proposition 3.12, these matroids can be determined by a pair of numbers from $\{0, \ldots, 2t\}$, and a function $\psi\colon \mathcal{P}(\{1, \ldots, m-1\}) \to \mathbb{Z}_{\geq 0}$ such that $\sum_{I \subseteq \{1,\ldots,m-1\}} \psi(I) = r$. The number of such functions is exactly

$$\binom{r + 2^{m-1} - 1}{r} = \binom{r + 2^{m-1} - 1}{2^{m-1} - 1},$$

which is at most $O(r^{2^{m-1}-1})$. This is in turn bounded by $O(t^{2^{k-1}-1})$. There are at most $O((2t)^2) = O(t^2)$ ways of selecting the two numbers in $\{0, \ldots, 2t\}$, so this leads to a bound of $O(t^{2^{k-1}+1})$ matroids corresponding to the pair (r, m), as we wanted. $\square$

Lemma 3.14. *Let $k \geq 2$ be an integer. The number of $2t$-element excluded minors for $\mathcal{S}_k$ is at least $\Omega(t^{2^k-k-3})$.*

Proof. As in the proof of Lemma 2.4, we construct spikes with $2t$ elements and $k+1$ circuit-hyperplanes. Again we insist that every element is in at least one circuit-hyperplane, and that no element is in all of them. We will assume that t is at least five.

Let $\mathcal{I}$ be the collection $\{I \subseteq \{1, \ldots, k\} \colon 1 \leq |I| \leq k-2\}$ and note that $|\mathcal{I}| = 2^k - k - 2$. For each $I \in \mathcal{I}$ we introduce a variable x_I. We consider non-negative integer solutions to the equation

$$\sum_{I \in \mathcal{I}} x_I = t - 2(k+1). \tag{2}$$

The number of such solutions is exactly

$$\binom{t+2^k-3k-5}{t-2k-2} = \binom{t+2^k-3k-5}{2^k-k-3},$$

which is at least $\Omega(t^{2^k-k-3})$.

Let ϕ be a solution to (2), so that ϕ is a function taking $\{x_I\}_{I\in\mathcal{I}}$ to non-negative values, and summing over the image of ϕ produces a total of $t-2(k+1)$. We are going to construct a sequence $\mathcal{D} = (D_1,\ldots,D_k)$ of subsets of $\{a_1,\ldots,a_t\}$ in such a way that $|\mathcal{D}(I)| = \phi(x_I)$ for each $I \in \mathcal{I}$. We do this by allocating each element in $\{a_1,\ldots,a_t\}$ to $\mathcal{D}(I)$ for a unique subset $I \subseteq \{1,\ldots,k\}$. We start by allocating two elements to $\mathcal{D}(\emptyset)$. These two elements are in none of the sets $D_1,\ldots,D_k$. Next, for each $i \in \{1,\ldots,k\}$, we allocate two elements to $\mathcal{D}(\{1,\ldots,k\}-i)$. These two elements will be in all of the sets $D_1,\ldots,D_k$ except for D_i. Now there are $t-2(k+1)$ elements left to allocate. We allocate no elements to $\mathcal{D}(\{1,\ldots,k\})$, so that no element of $\{a_1,\ldots,a_t\}$ is contained in all of the sets. The remaining $n-2(k+1)$ elements in $\{a_1,\ldots,a_t\}$ are allocated to the sets $\mathcal{D}(I)$ for $I \in \mathcal{I}$ according to the function ϕ, so that $|\mathcal{D}(I)| = \phi(x_I)$.

Next we construct subsets $\mathcal{C} = (C_1,\ldots,C_{k+1})$ of $\{a_1,\ldots,a_t,b_1,\ldots,b_t\}$. We set C_{k+1} to be $\{a_1,\ldots,a_t\}$. For $i \in \{1,\ldots,k\}$, we define C_i to be the union of D_i and $\{b_j : a_j \notin D_i, 1 \le j \le k\}$. Thus each set C_i contains exactly one element from each pair $\{a_j,b_j\}$. Furthermore, $\mathrm{trun}(\mathcal{C}) = (D_1,\ldots,D_k)$, so $\psi_{\mathrm{trun}(\mathcal{C})} = \phi$. It follows from $\mathcal{D}(\{1,\ldots,k\}) = \emptyset$ that no element of $\{a_1,\ldots,a_t,b_1,\ldots,b_t\}$ is in all of the sets $C_1,\ldots,C_{k+1}$, and that every element is in at least one of $C_1,\ldots,C_{k+1}$.

Let G be a graph obtained from a cycle of length t by replacing each edge with a parallel pair. Let $\{a_1,b_1\},\ldots,\{a_t,b_t\}$ be the parallel pairs in G. Let $\mathcal{B}$ be the class of Hamiltonian cycles in G with edge-sets $C_1,\ldots,C_{k+1}$. We claim that $\mathcal{B}$ is a linear class. Let us assume otherwise. Any theta-subgraph in G consists of a Hamiltonian cycle with one additional edge. So if $\mathcal{B}$ is not a linear class, then there are two sets C_i and C_j such that $C_i - C_j$ contains a single element. But two elements of $\{a_1,\ldots,a_t\}$ are in none of the sets $D_1,\ldots,D_k$, so these two elements are in C_{k+1} but none of $C_1,\ldots,C_k$. So i is not $k+1$, and hence i is in $\{1,\ldots,k\}$. There are two elements of $\{a_1,\ldots,a_t\}$ that are in all of the sets $D_1,\ldots,D_k$ other than D_i. This means that two elements in $\{b_1,\ldots,b_t\}$ are in none of the sets $C_1,\ldots,C_{k+1}$ except for C_i, and this is a contradiction. Therefore $\mathcal{B}$ is a linear class, as we claimed.

We let M be the spike $L(G,\mathcal{B})$. Since M has $k+1$ circuit-hyperplanes, and $t \ge 5$, it follows without difficulty from Proposition 3.7 that M is not in $\mathcal{S}_k$. However, since every element of M is in at least one circuit-hyperplane, and avoids at least one circuit-hyperplane, deleting or contracting any element from M produces a minor $L(G',\mathcal{B}')$, where G' is in $\mathcal{G}$, and $\mathcal{B}'$ contains at most k Hamiltonian cycles. So M is indeed an excluded minor for $\mathcal{S}_k$.

For each solution to (2) we construct an excluded minor for $\mathcal{S}_k$, as detailed above. Some of these excluded minors may be isomorphic. But all excluded minors constructed in this way have no triangles and no triads, so the functions p and s return zero. Now Proposition 3.12 implies that the excluded minors are isomorphic if and only if they have the same images under $\mathcal{R}^t_{k+1}$. So any isomorphism class amongst the constructed excluded minors is no larger than the image of a matroid under $\mathcal{R}^t_{k+1}$, which is at most $(k+1)!$. Since there are $\Omega(t^{2^k-k-3})$ solutions to (2), and k is constant with respect to t, it follows that the number of excluded minors is at least $\Omega(t^{2^k-k-3})$, as claimed. □

Proof of Theorem 1.5. Lemma 3.10 implies that $\Gamma_{\mathcal{S}_k}(2t+1) = 0$ for all sufficiently large values of t. So $\Gamma_{\mathcal{S}_k}(n)$ does not tend to one, and hence $\mathcal{S}_k$ is certainly not strongly fractal. However, by Lemmas 3.13 and 3.14 we see that for sufficiently large values of t we have

$$\Gamma_{\mathcal{S}_k}(2t) \geq \frac{c_1 t^{2^k-k-3}}{c_2 t^{2^{k-1}+2} + c_1 t^{2^k-k-3}} = \frac{1}{(c_2/c_1)t^{-2^{k-1}+k+5}+1}$$

for some positive constants c_1 and c_2. Since $k \geq 5$, we see that $-2^{k-1}+k+5$ is negative. Thus $\Gamma_{\mathcal{S}_k}(2t)$ tends to one as t tends to infinity, meaning that $\mathcal{S}_k$ is weakly fractal. □

4. Acknowledgements

We thank the referee for a careful report and helpful remarks. The first author was supported by a Rutherford Discovery Fellowship, managed by Royal Society Te Apārangi.

References

[1] R. Campbell and J. Geelen. On the complex-representable excluded minors for real-representability. *arXiv e-prints* (2018), arXiv:1801.08656.
[2] J. Geelen, B. Gerards, and G. Whittle. Solving Rota's conjecture. *Notices Amer. Math. Soc.* **61** (2014), no. 7, 736–743.
[3] N. Hine and J. Oxley. When excluding one matroid prevents infinite antichains. *Adv. in Appl. Math.* **45** (2010), no. 1, 74–76.
[4] D. Mayhew. The antichain of excluded minors for the class of gammoids is maximal. *arXiv e-prints* (2016), arXiv:1601.04220.
[5] D. Mayhew, M. Newman, and G. Whittle. On excluded minors for real-representability. *J. Combin. Theory Ser. B* **99** (2009), no. 4, 685–689.
[6] J. Oxley. *Matroid theory.* Oxford University Press, New York, second edition (2011).
[7] N. Robertson and P. D. Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B* **92** (2004), no. 2, 325–357.
[8] W. T. Tutte. A homotopy theorem for matroids. I, II. *Trans. Amer. Math. Soc.* **88** (1958), 144–174.
[9] H. S. Wilf. *generatingfunctionology.* Academic Press, Inc., Boston, MA, second edition (1994).
[10] T. Zaslavsky. Biased graphs. II. The three matroids. *J. Combin. Theory Ser. B* **51** (1991), no. 1, 46–72.

School of Mathematics and Statistics, Victoria University of Wellington, New Zealand
Email address: dillon.mayhew@vuw.ac.nz

Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada
Email address: mnewman@uottawa.ca

School of Mathematics and Statistics, Victoria University of Wellington, New Zealand
Email address: geoff.whittle@vuw.ac.nz