THE INTERNALLY 4-CONNECTED BINARY MATROIDS WITH NO $M(K_5 \backslash e)$-MINOR

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ABSTRACT. Let $\mathrm{AG}(3,2) \boxtimes U_{1,1}$ denote the binary matroid obtained from $\mathrm{AG}(3,2) \oplus U_{1,1}$ by completing the 3-point lines between every element in $\mathrm{AG}(3,2)$ and the element of $U_{1,1}$. We prove that every internally 4-connected binary matroid that does not have a minor isomorphic to $M(K_5 \backslash e)$ is isomorphic to a minor of $(\mathrm{AG}(3,2) \boxtimes U_{1,1})^*$.

1. INTRODUCTION

The study of minor-closed classes of graphs has a long history, starting with Wagner's characterization [16] of planar graphs as those graphs with no K_5 or $K_{3,3}$ minor, and Hall's study [3] of the minor-closed class that arises by excluding $K_{3,3}$.

Numerous authors have studied minor-closed classes of graphs defined by some natural property, such as embeddability on a particular surface, aiming to determine the excluded minors for that class of graphs; the standard example (other than planarity) being the determination of the 35 excluded minors for the class of projective planar graphs (Archdeacon [1]). Other authors have studied the "dual problem" of finding a structural description of the minor-closed classes of graphs obtained by excluding a particular minor or set of minors. Examples of classes investigated in this way include those obtained by excluding $K_5 \backslash e$, K_5, $K_{3,3}$, V_8, the cube, or the octahedron (see [5] and the references therein).

Analogous questions arise in the study of matroids, particularly binary matroids, and a number of excluded minor theorems and structural characterisations of classes of matroids defined by excluded minors are known. For example, Oxley has determined the binary matroids with no 4-wheel minor, the ternary matroids with no $M(K_4)$ minor, and the regular matroids with no 5-wheel minor, among others (see [10, 11, 12]). For graphs, the Kuratowski graphs $K_{3,3}$ and K_5 are especially important, and for binary matroids, the Kuratowski graphs and their duals play an analogous role (see Kung [6] for an early study of these classes). For graphs, decomposition theorems exist for all classes produced by excluding either or both of the Kuratowski graphs, but for binary matroids our knowledge is incomplete. There are 15 classes of binary matroids obtained by excluding some non-empty subset of

$$\{M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}.$$

Mayhew, Royle and Whittle [8] determined the internally 4-connected binary matroids with no $M(K_{3,3})$-minor, and this leads directly to characterisations (and associated decomposition theorems) for 12 of these classes [7]. This subsumes earlier results of Qin and Zhou [13], who described the binary matroids obtained by excluding all the cycle and bond matroids of the Kuratowski graphs.

Our failure to characterize the remaining three classes is due to the fact that we do not have a decomposition theorem for the class of binary matroids with no $M(K_5)$-minor. Unfortunately, while characterizing the internally 4-connected binary matroids with no $M(K_{3,3})$-minor is difficult, and requires a long and technical proof, it seems that it will be even more challenging to characterize binary matroids with no $M(K_5)$-minor.

In this article, we prove a partial result by determining the internally 4-connected binary matroids with no $M(K_5\backslash e)$-minor. Our hope is that this will assist us in future exploration, since we now know that an internally 4-connected binary matroid with no $M(K_5)$-minor either has an $M(K_5\backslash e)$-minor, or is one of the finite number of matroids we describe in this article. As in the characterization of the binary matroids with no $M(K_{3,3})$-minor, the proof involves a great deal of case-checking, enough so that completing the argument by hand is not feasible. Instead, we use a computer verification for these portions of the proof; however we emphasize that the computations, which were independently performed by two different means, are only used to verify routine assertions about specific matroids.

Robertson & Seymour [14] considered the *graphs* with no $M(K_5\backslash e)$-minor, obtaining the result that a 3-connected graph with no $M(K_5\backslash e)$-minor is either a wheel, isomorphic to $K_{3,3}$, or isomorphic to the triangular prism $(K_5\backslash e)^*$. (Figure 1 shows the graphs $K_5\backslash e$ and the triangular prism.)

For a number of reasons, we find it more convenient to consider binary matroids without an $M^*(K_5\backslash e)$-minor. Therefore we will henceforth state all our results in terms of binary matroids with no minor isomorphic to the cycle matroid of the triangular prism. (We abbreviate this to "no prism-minor" or "prism-free".)
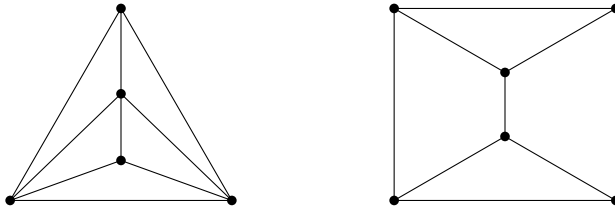

FIGURE 1. $K_5\backslash e$, and its geometric dual, the triangular prism.

Suppose that M_1 and M_2 are binary matroids on disjoint ground sets. Let S be a set of cardinality $|E(M_1)| \times |E(M_2)|$, disjoint from $E(M_1) \cup E(M_2)$, where every element $e_{a,b} \in S$ corresponds to a pair $(a, b) \in E(M_1) \times E(M_2)$.

Then $M_1 \boxtimes M_2$ is the unique binary matroid on the ground set $E(M_1) \cup E(M_2) \cup S$ satisfying

$$(M_1 \boxtimes M_2)|(E(M_1) \cup E(M_2)) = M_1 \oplus M_2$$

such that $\{a, b, e_{a,b}\}$ is a circuit, for every $(a, b) \in E(M_1) \times E(M_2)$. That is, $M_1 \boxtimes M_2$ is obtained from $M_1 \oplus M_2$ by placing $e_{a,b}$ on the line between a and b, for every pair (a, b) in $E(M_1) \times E(M_2)$.

Theorem 1.1. *Let M be a binary matroid with no prism-minor.*
  (i) *If M is internally 4-connected, then M has rank at most 5, and is isomorphic to a minor of $\mathrm{AG}(3,2) \boxtimes U_{1,1}$.*
  (ii) *If M is 3-connected but not internally 4-connected, and M has an internally 4-connected minor with at least 6 elements that is not isomorphic to $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$, then M is isomorphic to one of the sporadic matroids **S1**, **S2**, **S3**, **S4**, or **S5**, defined in Table 3 of Section 2.2.*

Theorem 1.1 has a peculiar consequence (Corollary 1.2). Apart from a finite number of exceptions, every 3-connected binary matroid with no prism-minor can be constructed using 3-sums starting from copies of only two matroids: $M(K_4)$ and F_7.

Recall that a *parallel extension* of the matroid M is a matroid M' with an element $e \in E(M')$ such that $M'\backslash e = M$, and e is in a parallel pair of M'. A *cycle* of a binary matroid is a (possibly empty) disjoint union of circuits. If M_1 and M_2 are binary matroids such that (i) $|E(M_1)|, |E(M_2)| \geq 7$, (ii) $E(M_1) \cap E(M_2) = T$, where T is a triangle of both M_1 and M_2, (iii) T does not contain a cocircuit in either M_1 or M_2, then the *3-sum* of M_1 and M_2 (denoted $M_1 \oplus_3 M_2$) is defined. It is a binary matroid on the set $(E(M_1) \cup E(M_2)) - T$, and the cycles of $M_1 \oplus_3 M_2$ are exactly the sets of the form $(Z_1 - Z_2) \cup (Z_2 - Z_1)$, where Z_i is a cycle of M_i for $i = 1, 2$, and $Z_1 \cap T = Z_2 \cap T$.

Corollary 1.2.   (i) *If M is a 3-connected binary matroid with no prism-minor, then either M is an internally 4-connected minor of $\mathrm{AG}(3,2) \boxtimes U_{1,1}$, or M is one of **S1**, **S2**, **S3**, **S4**, or **S5**, or M can be constructed from copies of $M(K_4)$ and F_7 using parallel extensions and 3-sums.*
  (ii) *If M is a 3-connected binary matroid with no $M(K_5\backslash e)$-minor, then either M is an internally 4-connected minor of $(\mathrm{AG}(3,2) \boxtimes U_{1,1})^*$, or M is one of **S1***, **S2***, **S3***, **S4***, or **S5***, or M can be constructed from copies of $M(K_4)$, F_7, and $M^*(K_{3,3})$ using parallel extensions and 3-sums.*

Proof. Suppose that M is a counterexample to (i), chosen so that $|E(M)|$ is as small as possible. Theorem 1.1(i) means that M is not internally 4-connected. Therefore $M = M_1 \oplus_3 M_2$ for some matroids M_1 and M_2 (see [15, (2.9)]). Both M_1 and M_2 are minors of M [15, (4.1)], so neither M_1

nor M_2 has a prism-minor. Moreover, both M_1 and M_2 have fewer elements than M, and $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ are both 3-connected [15, (4.3)], so $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ satisfy Corollary 1.2(i).

Assume that $\mathrm{si}(M_1)$ is one of **S1**, **S2**, **S3**, **S4**, or **S5**. In Section 2.2 we certify that each of these 5 matroids contains an internally 4-connected minor N such that $|E(N)| \geq 6$ and N is not isomorphic to $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$. Thus $\mathrm{si}(M_1)$, and hence M, contains an internally 4-connected minor other than $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$. This is a contradiction to Theorem 1.1(ii), as M is not one of **S1**, **S2**, **S3**, **S4**, or **S5**. Therefore $\mathrm{si}(M_1)$ is not one of the 5 sporadic matroids.

Assume that $\mathrm{si}(M_1)$ is internally 4-connected. As M_1 is a term in a 3-sum, it contains a triangle T, and T does not contain a cocircuit of M_1. This means that $r(M_1) \geq 3$. Since $\mathrm{si}(M_1)$ is 3-connected, it follows that $\mathrm{si}(M_1)$ has at least 6 elements. Now M contains $\mathrm{si}(M_1)$ as a minor, and M is not one of the sporadic matroids. Theorem 1.1(ii) implies that $\mathrm{si}(M_1)$ is isomorphic to $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$. But M_1 contains a triangle, so $\mathrm{si}(M_1)$ is not isomorphic to F_7^* or $M(K_{3,3})$. Hence $\mathrm{si}(M_1)$ is isomorphic to $M(K_4)$ or F_7.

On the other hand, if $\mathrm{si}(M_1)$ is not internally 4-connected, then by the inductive hypothesis, M_1 can be constructed from copies of $M(K_4)$ and F_7 using parallel extensions and 3-sums. Thus, in either case, M_1 (and M_2, by an identical argument), can be constructed from copies of $M(K_4)$ and F_7 using parallel extensions and 3-sums. Therefore the same statement holds for M. This contradiction completes the proof of (i). The proof of (ii) is almost identical: if M is a minimal counterexample, then M can be expressed as $M_1 \oplus_3 M_2$, and if $\mathrm{si}(M_1)$ is internally 4-connected, then it must be isomorphic to $M(K_4)$, F_7, F_7^*, or $M^*(K_{3,3})$. But $\mathrm{si}(M_1) \not\cong F_7^*$, as $\mathrm{si}(M_1)$ must contain a triangle. $\square$

Our main theoretical tool in the proof of Theorem 1.1 is a chain theorem for internally 4-connected binary matroids [2]. This theorem tells us that if a counterexample to Theorem 1.1 exists, then it has at most three more elements than one of the known internally 4-connected prism-free binary matroids, and therefore has rank at most 8. Thus the proof of the theorem is reduced to a (large) finite case analysis to demonstrate that none of the internally 4-connected prism-free binary matroids of rank up to 5 can be extended and/or coextended by up to three elements to form a new internally 4-connected prism-free binary matroid. This case analysis is performed by computer, but to increase confidence in the correctness of the result, we conducted this analysis in two totally independent ways.

The first search relies upon the MACEK software package by Petr Hliněný [4]. Macek has the facility to extend or coextend matroids while avoiding specified minors and using this it is relatively easy to verify that there are no new internally 4-connected prism-free binary matroids within three extension/coextension steps of any of the known ones. The second technique

involves the construction from first principles of a database of *all* simple prism-free binary matroids of rank up to 8. Having constructed such a database, verifying that Theorem 1.1 holds merely requires checking that none of the rank 6, 7 or 8 prism-free binary matroids are internally 4-connected. Both these computer searches are described in more detail in Section 4.

2. Listing matroids

In this section we explicitly list the 42 internally 4-connected prism-free binary matroids, and the five sporadic matroids that appear in the statement of Theorem 1.1.

2.1. **The internally 4-connected prism-free binary matroids.** Let $\mathcal{M}$ be the set of internally 4-connected minors of $\mathrm{AG}(3,2) \boxtimes U_{1,1}$ which, by Theorem 1.1, is exactly the set of internally 4-connected binary matroids with no prism-minors.

There are 42 matroids in $\mathcal{M}$. It is easy to see that the only internally 4-connected binary matroids on at most 5 elements are $U_{0,0}$, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$, so from this point we content ourselves with listing the 36 matroids in $\mathcal{M}$ that have at least 6 elements.

The only rank-3 members of $\mathcal{M}$ are $M(K_4)$ and F_7, which (for the sake of consistency), we will denote with **M1** and **M2**, respectively.

Let e be an element of $\mathrm{AG}(3,2)$. It is an easy exercise to show that contracting e from $\mathrm{AG}(3,2) \boxtimes U_{1,1}$ and then simplifying produces a rank-4 binary matroid with 15 elements, which is therefore isomorphic to $\mathrm{PG}(3,2)$. Hence $\mathcal{M}$ contains every rank-4 internally 4-connected binary matroid. Every such matroid can be obtained by deleting columns from the matrix representing $\mathrm{PG}(3,2)$ which is shown at the head of Table 1. Each row of the table corresponds to an internally 4-connected binary matroid with rank 4. The empty entries in that row correspond to columns which should be deleted to obtain a representation. For example, **M9** is represented by the matrix

$$\begin{bmatrix} 1 & 0 & & 1 & 1 & 1 & 0 & 0 & & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & & 1 & 0 & 0 & 1 & 1 & & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & & 0 & 1 & 0 & 1 & 0 & & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & & 0 & 0 & 1 & 0 & 1 & & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

which has been obtained by deleting three particular columns from the matrix representing $\mathrm{PG}(3,2)$.

We use a similar format to describe the rank-5 members of $\mathcal{M}$. Let A be the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	
	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	
	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	
	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	
M3					•	•	•				•	•	•		•	F_7^*
M4					•	•	•	•			•	•	•	•	•	$M^*(K_{3,3})$
M5					•	•	•	•	•	•	•	•	•	•		$M(K_5)$
M6					•	•	•	•	•		•	•	•	•	•	
M7		•			•	•	•	•	•		•	•	•	•	•	
M8					•	•	•	•	•	•	•	•	•	•	•	
M9	•	•			•	•	•	•	•		•	•	•	•	•	
M10	•				•	•	•	•	•	•	•	•	•	•	•	
M11	•	•			•	•	•	•	•	•	•	•	•	•	•	
M12	•	•	•		•	•	•	•	•	•	•	•	•	•	•	
M13	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	PG(3,2)

TABLE 1. Internally 4-connected rank-4 restrictions of PG(3,2).

Then $[I_4|A]$ is a GF(2)-representation of AG(3,2). It follows that AG(3,2) $\boxtimes$ $U_{1,1}$ is represented over GF(2) by the matrix

$$\left[\begin{array}{c|c|c|c|c} \mathbf{0} & I_4 & A & I_4 & A \\ \hline 1 & \mathbf{0}^T & \mathbf{0}^T & \mathbf{1}^T & \mathbf{1}^T \end{array}\right],$$

where $\mathbf{0}$ (respectively $\mathbf{1}$) is the 4×1 vector of all zeros (respectively ones).

A representation of any rank-5 member of $\mathcal{M}$ can be obtained by deleting columns from this matrix and each row of Table 2 corresponds to a rank-5 member of $\mathcal{M}$. We note here that **M14** is $M(K_{3,3})$, and **M16** is the regular matroid R_{10}.

2.2. **Sporadic matroids.** In this section we describe the 5 sporadic matroids that appear in the statement of Theorem 1.1. For the sake of brevity we represent an n-element binary matroid M with a string of numbers $m_1, \ldots, m_n$ between 1 and $2^{r(M)} - 1$. A representation of M over GF(2) can be obtained by taking the binary representations of $m_1, \ldots, m_n$, each of which has $r(M)$ bits, and taking these binary representations to be the columns of a matrix A. We use the convention that the least significant bit of the binary representation will be in the bottom row of A, and the most significant will be in the top row. Thus the sequence

$$1, 2, 3, 4, 6, 8, 9, 12,$$

	c1	c2	c3	c4	c5	c6	c7	c8	c9	c10	c11	c12	c13	c14	c15	c16	c17
	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1
	0	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1
	0	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1
	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0
	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
M14		•	•	•	•	•	•			•	•	•					
M15	•	•	•	•	•	•	•			•	•					•	
M16		•	•	•	•	•	•			•	•	•				•	
M17	•	•	•	•	•	•	•	•		•	•		•				
M18	•	•	•	•	•	•	•			•	•	•				•	
M19		•	•	•	•	•	•	•		•	•	•	•				
M20	•	•	•	•	•	•	•	•		•	•		•	•			
M21	•	•	•	•	•	•	•	•		•	•		•				•
M22	•	•	•	•	•	•	•	•		•	•	•	•				
M23		•	•	•	•	•	•	•	•	•	•	•	•				
M24	•	•	•	•	•	•	•	•		•	•		•			•	•
M25	•	•	•	•	•	•	•	•	•	•	•	•	•				
M26	•	•	•	•	•	•	•	•		•	•	•	•	•			
M27		•	•	•	•	•	•	•	•	•	•	•	•	•			
M28	•	•	•	•	•	•	•	•	•	•	•	•	•	•			
M29	•	•	•	•	•	•	•	•		•	•	•	•	•	•		
M30		•	•	•	•	•	•	•	•	•	•	•	•	•	•		
M31	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•		
M32	•	•	•	•	•	•	•	•		•	•	•	•	•	•	•	
M33		•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
M34	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
M35		•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
M36	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•

TABLE 2. Internally 4-connected rank-5 restrictions of $\text{AG}(3,2)\boxtimes U_{1,1}$.

corresponds to the following matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We also use the numbers $m_1, \ldots, m_n$ to represent the corresponding elements in $E(M)$. With this notational convention, the 5 sporadic matroids are presented in Table 3.

By deleting or contracting at most two elements from any of these sporadic matroids, we can obtain one of the internally 4-connected matroids listed in Section 2.1, as we now certify.

- Contracting 16 from **S1** produces **M5**.

Name	Size	Rank	Elements
S1	11	5	$1, 4, 5, 8, 9, 14, 15, 16, 22, 27, 29$
S2	11	5	$1, 3, 4, 8, 9, 14, 15, 16, 22, 27, 29$
S3	12	5	$1, 3, 4, 5, 8, 9, 14, 15, 16, 22, 27, 29$
S4	12	6	$1, 2, 4, 8, 15, 16, 32, 42, 44, 49, 56, 63$
S5	13	5	$1, 2, 3, 4, 5, 8, 9, 14, 15, 16, 22, 27, 29$

TABLE 3. The sporadic 3-connected prism-free binary matroids.

- Contracting 16 from **S2** produces **M6**.
- Deleting 3 or 5 from **S3** produces **S1** or **S2** respectively.
- Contracting 1 from **S4** produces **S1**.
- Deleting 29 from **S5** produces **M20**.

3. APPLYING A CHAIN THEOREM

We reduce the proof of Theorem 1.1 to a finite case check by using a chain theorem for internally 4-connected binary matroids, but before stating this theorem, we define the families of graphs appearing in it.

For $n \geq 3$, the *planar quartic ladder* on $2n$ vertices consists of two disjoint cycles

$$\{u_0u_1, u_1u_2, \ldots, u_{n-2}u_{n-1}, u_{n-1}u_0\} \cup \{v_0v_1, v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0\}$$

and two perfect matchings

$$\{u_0v_0, u_1v_1, \ldots, u_{n-1}v_{n-1}\} \cup \{u_0v_{n-1}, u_1v_0, \ldots, u_{n-1}v_{n-2}\}.$$

Each planar quartic ladder contains all smaller planar quartic ladders as minors, and the smallest planar quartic ladder is the *octahedron* on 6 vertices. For $n \geq 3$, the *Möbius quartic ladder* on $2n - 1$ vertices consists of a Hamilton cycle

$$\{v_0v_1, v_1v_2, \ldots, v_{2n-3}v_{2n-2}, v_{2n-2}v_0\}$$

and the set of edges

$$\{v_iv_{i+n-1}, v_iv_{i+n} \mid 0 \leq i \leq n - 1\},$$

where subscripts are read modulo $2n - 1$. Each quartic Möbius ladder contains all smaller quartic Möbius ladders as minors, and the smallest quartic Möbius ladder is the complete graph K_5 on 5 vertices. Figure 2 shows one member of each of these families.

Finally, the *terrahawk* is obtained from the cube by adding a new vertex, and making it adjacent to the four vertices in a face of the cube. Figure 3 shows diagrams of the cube, the octahedron, and the terrahawk. With these definitions in hand, we can state the chain theorem:

Theorem 3.1 (Chun, Mayhew, Oxley [2])**.** *Let M be an internally 4-connected binary matroid such that $|E(M)| \geq 7$. Then M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$ unless M or its dual*

is the cycle matroid of a planar quartic ladder or Möbius quartic ladder, or a terrahawk.
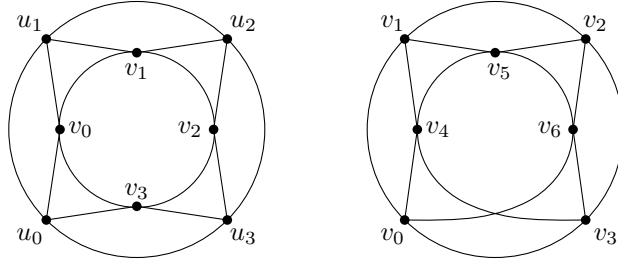

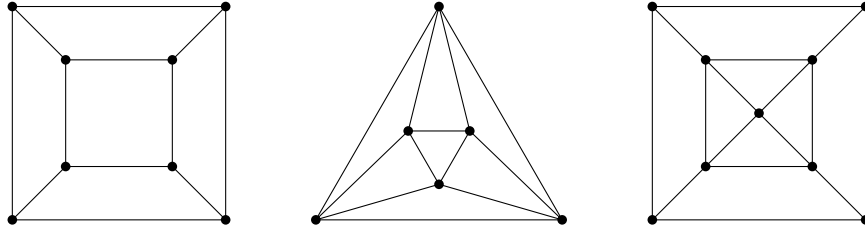FIGURE 2. A planar quartic ladder, and a Möbius quartic ladder.

FIGURE 3. The cube, the octahedron, and the terrahawk.

The next lemma shows that the exceptional cases in Theorem 3.1 all have prism minors except for $M(K_5)$.

Lemma 3.2. *The cycle and bond matroids of (i) the terrahawk, (ii) the planar quartic ladders, and (iii) the Möbius quartic ladders with at least* 7 *vertices, all have the triangular prism as a minor. Moreover, the bond matroid of the Möbius quartic ladder with* 5 *vertices has the triangular prism as a minor.*

Proof. The cycle matroid of the terrahawk, which is self-dual, clearly has a cube minor, which itself clearly has a triangular prism minor. The planar quartic ladders all have an octahedron minor, while their duals all have a cube minor, and both the octahedron and the cube have a triangular prism minor. A Möbius quartic ladder on at least seven vertices contains the Möbius quartic ladder with seven vertices, which in turn contains a triangular prism minor (see Figure 4). Since the smallest Möbius quartic ladder is isomorphic to K_5, the bond matroid of a Möbius quartic ladder contains $M^*(K_5)$, and hence $M^*(K_5 \backslash e)$. □

Assume that M is a minimal counterexample to Theorem 1.1(i). Then M is internally 4-connected, and $|E(M)| \geq 7$, as otherwise M is certainly a minor of $AG(3,2) \boxtimes U_{1,1}$. Note that $M(K_5)$ is a minor of $AG(3,2) \boxtimes U_{1,1}$,
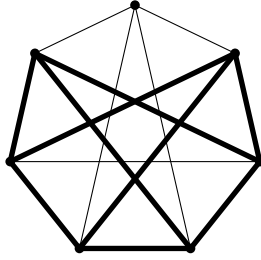
FIGURE 4. The Möbius quartic ladder on seven vertices.

so $M \not\cong M(K_5)$. This fact, and Lemma 3.2, implies M is not the cycle or bond matroid of a quartic ladder, or of the terrahawk. Thus Theorem 3.1 says that M contains an internally 4-connected minor N satisfying $1 \leq |E(M)| - |E(N)| \leq 3$. The minimality of M implies that N is one of the internally 4-connected matroids listed in Section 2.1. Therefore, if a counterexample exists, we can find it by extending and coextending by at most three elements, starting with the known internally 4-connected matroids. Next we show that the extensions and/or coextensions required to find M starting from N can be chosen to maintain 3-connectivity at each step.

Let $n \geq 3$ be an integer. Recall that the *wheel* graph $\mathcal{W}_n$ is obtained from a cycle with n vertices by adding a new vertex and joining it to each of the n vertices in the cycle.

Lemma 3.3. *Let M be an internally 4-connected binary matroid such that $|E(M)| \geq 7$ and M does not have a prism-minor. If M is not isomorphic to $M(K_5)$, $M(K_{3,3})$ or $M^*(K_{3,3})$, then there is a sequence $M_0, \ldots, M_t$ of 3-connected matroids such that:*

    (i) M_0 is internally 4-connected,
    (ii) $M_t = M$,
    (iii) $1 \leq t \leq 3$, and
    (iv) M_{i+1} is a single-element extension or coextension of M_i, for every $i \in \{0, \ldots, t-1\}$.

Proof. By Theorem 3.1 and Lemma 3.2, M contains an internally 4-connected minor N such that $1 \leq |E(M)| - |E(N)| \leq 3$. We let M_0 be equal to N. If N is not a wheel, then the result follows immediately from Seymour's Splitter Theorem (see [9, Theorem 12.1.2]). Therefore we must assume that N is isomorphic to $M(\mathcal{W}_n)$ for some n. If $n \geq 4$, then $M(\mathcal{W}_n)$ is not internally 4-connected, so $N \cong M(\mathcal{W}_3) \cong M(K_4)$. If M does not have any larger wheel as a minor, then we can again apply the Splitter Theorem and deduce that the result holds. Therefore we assume that M contains $\mathcal{W}_4$ as a minor. As $|E(\mathcal{W}_4)| = 8$, and $|E(M)| \leq |E(K_4)| + 3 = 9$, and $M(\mathcal{W}_4)$ is not internally 4-connected, we deduce that M is a single-element extension or coextension of $M(\mathcal{W}_4)$.

Let us assume that M is an extension of $M(\mathcal{W}_4)$. Recall that $M(\mathcal{W}_4)$ is represented over $\mathrm{GF}(2)$ by the following matrix.

$$\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The uniqueness of representations over $\mathrm{GF}(2)$ (see [9, Proposition 6.6.5]) means that a representation of M is obtained by adding a column to this matrix. If this new column contains a zero in the first row, then $\{a, e, h\}$ is a triad of M, and $\{a, b, e\}$ is a triangle. This leads to a violation of internal 4-connectivity. Therefore the new column contains a one in the first row. Repeating this argument shows that the new column must contain ones everywhere. Now it is easy to see that $M \cong M^*(K_{3,3})$, contradicting the hypotheses of the lemma. A dual argument shows that if M is a coextension of $M(\mathcal{W}_4)$, then $M \cong M(K_{3,3})$, so we are done. $\square$

4. Computer searches

We use two independent computer searches to verify the following lemmas, and thereby prove Theorem 1.1. Section 4.1 outlines a technique using MACEK to extend the known internally 4-connected prism-free binary matroids, while Section 4.2 describes an exhaustive search for all prism-free binary matroids of rank up to 8. The results from both searches were in total agreement.

Lemma 4.1. *Let M be an internally 4-connected binary matroid with no prism-minor. Then M is one of the 42 matroids described in Section 2.1.*

Lemma 4.2. *Let M be a binary matroid with no prism-minor. If M is 3-connected but not internally 4-connected, and M contains an internally 4-connected minor with at least 6 elements that is not isomorphic to $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$, then M is one of the 5 sporadic matroids* **S1**, **S2**, **S3**, **S4**, *or* **S5**.

4.1. **Using MACEK.** It is straightforward to use MACEK to find all 3-connected binary matroids M satisfying the following properties:

(i) M does not have a prism-minor;
(ii) there is a sequence $M_0, \ldots, M_t$ of 3-connected matroids where
    (a) M_0 has at least 6 elements, and is one of the internally 4-connected matroids listed in Section 2.1,
    (b) $M_t = M$,
    (c) $1 \leq t \leq 3$,
    (d) M_{i+1} is a single-element extension or coextension of M_i, for all $i \in \{0, \ldots, t-1\}$;

(iii) M does not contain a minor N where $|E(N)| = |E(M_0)| + 1$, and N is isomorphic to one of the internally 4-connected matroids listed in Section 2.1.

For example, if M_0 is **M4**, then we would use the command

```
./macek -pGF2 '@ext-forbid Prism M5 M6 M15 M16;!extend bbb' M4
```

to find M, since **M5**, **M6**, **M15**, and **M16** are the matroids in Section 2.1 that are a single element larger than **M4**.

Assume that M is a counterexample to Lemma 4.1, chosen so that $|E(M)|$ is as small as possible. Then M certainly has an $M(K_4)$-minor [9, Corollary 12.2.13], but M is not isomorphic to $M(K_4)$. Therefore $|E(M)| \geq 7$. Since $M(K_5)$, $M(K_{3,3})$, and $M^*(K_{3,3})$ all appear in Section 2.1, M is isomorphic to none of these matroids. We consider the sequence of matroids $M_0, \ldots, M_t$ supplied by Lemma 3.3. As there are no internally 4-connected binary matroids with four or five elements, it is certainly the case that $|E(M_0)| \geq 6$. Moreover, M_0 is listed in Section 2.1, by the minimality of M. We assume that M and M_0 have been chosen so that t is as small as possible. This means that M cannot contain a minor N such that $|E(N)| = |E(M_0)| + 1$ and N is listed in Section 2.1, or else we would have chosen M_0 to be N instead. Thus condition (iii) in the list above holds. Therefore M will be found in the MACEK search we described at the beginning of this section. But the MACEK search uncovers no internally 4-connected matroids. We conclude that Lemma 4.1 holds.

Now assume that M is a minimal counterexample to Lemma 4.2. Then M contains an internally 4-connected minor M_0 such that $|E(M_0)| \geq 6$ and M_0 is not isomorphic to $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$. Lemma 4.1 implies that M_0 is one of the matroids listed in Section 2.1. By Seymour's Splitter Theorem, there is a sequence of 3-connected matroids $M_0, \ldots, M_t$ such that $M_t = M$, and each matroid in the sequence is a single-element extension or coextension of the previous matroid. We assume that M and M_0 have been chosen so that $|E(M)| - |E(M_0)| = t$ is as small as possible.

Suppose that M contains as a minor an internally 4-connected matroid with exactly one more element than M_0. By the minimality of t, this internally 4-connected matroid must be $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$. This implies M_0 has at most 8 elements, and is therefore isomorphic to $M(K_4)$, F_7, or F_7^*. This contradiction shows that M and M_0 obey condition (iii), described above.

Certainly $t \geq 1$. If M_{t-1} is internally 4-connected, then $t = 1$. Assume that M_{t-1} is not internally 4-connected. By the minimality of t, M_{t-1} is not a counterexample to Lemma 4.2, so it must be one of the sporadic matroids. Every such matroid has an internally 4-connected minor that is at most two elements smaller. Moreover, none of these internally 4-connected matroids is isomorphic to $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$. It follows from this that $M_0 = M_{t-3}$ or $M_0 = M_{t-2}$, and therefore $t \leq 3$. Therefore M will be uncovered by the MACEK procedure we described earlier. However,

when we apply the MACEK search procedure to the internally 4-connected matroids in Section 2.1 other than $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$, we produce no 3-connected matroids other than the sporadic matroids. We conclude that Lemma 4.2 is true.

4.2. **An exhaustive search process.** In this section we describe the exhaustive search process that was used to compute the list of simple binary matroids with no $M^*(K_5 \backslash e)$-minor of rank up to 8. This is accomplished by mapping the problem into a graph-theoretic context and then using an orderly algorithm designed for graphs to perform the computations.

First, note that a simple binary matroid of rank at most r can be identified with a set of points in the projective space $\mathrm{PG}(r-1, 2)$. Given the matroid M, we can take the columns of an arbitrary representing matrix, which are non-zero as M is simple, to be vectors in $\mathrm{GF}(2)^r$ (padding them with zeros if M has rank less than r). Conversely any set of points in $\mathrm{PG}(r-1, 2)$ determines a matroid of rank at most r simply by taking the unique non-zero vector representing each point to be the columns of a matrix. The importance of this identification is that the unique representability of binary matroids ensures that the natural concepts of equivalence in each context coincide. In particular, two simple binary matroids are isomorphic if and only if the two corresponding subsets of $\mathrm{PG}(r-1, 2)$ are equivalent under the automorphism group of the projective space, which is the projective general linear group $\mathrm{PGL}(r, 2)$.

Lemma 4.3. *Let X and Y denote sets of points in the projective space* $\mathrm{PG}(r-1, 2)$. *Then the matroids determined by X and Y are isomorphic if and only if there is an element of $\mathrm{PGL}(r, 2)$ mapping X to Y.* □

To express this in a graph-theoretic context, we need to work with a graph whose automorphism group is $\mathrm{PGL}(r, 2)$. So let Γ_r denote the point-hyperplane incidence graph of $\mathrm{PG}(r-1, 2)$; this is a bipartite graph with $2(2^r - 1)$ vertices of which $2^r - 1$ are "point-type" vertices and $2^r - 1$ are "hyperplane-type" vertices. The automorphism group of this graph is $\mathrm{PGL}(r, 2) \times 2$ where the extra factor of 2 arises from an automorphism that exchanges points with hyperplanes. Let $\mathcal{P}_r$ denote the "point-type" vertices of Γ_r. Thus any simple binary matroid M of rank at most r can be identified with a subset $\mathcal{P}_r(M)$ of $\mathcal{P}_r$, and the automorphism group of Γ_r fixing $\mathcal{P}_r(M)$ is the automorphism group of M.

Brendan McKay's graph isomorphism and canonical labelling program `nauty` can find the automorphism group and canonical labelling of a graph with a given set of vertices distinguished (i.e., a coloured graph). Therefore two simple binary matroids M and N of rank at most r are isomorphic if and only if $|M| = |N|$ and the canonically labelled isomorph of Γ_r with $\mathcal{P}_r(M)$ distinguished is identical to the canonically labelled isomorph of Γ_r with $\mathcal{P}_r(N)$ distinguished.

4.2.1. *An orderly algorithm.* Suppose first that our task is to compute *all* the simple binary matroids up to some fixed rank r — a simple modification to this basic algorithm will permit the computation of simple binary matroids excluding any particular matroid or set of matroids.

As in the previous section, let Γ_r be the bipartite point-hyperplane graph of $PG(r-1, 2)$, let $\mathcal{P}_r$ denote the points of $PG(r-1, 2)$ and let $G = PGL(r, 2)$ be the subgroup of $\text{Aut}(\Gamma_r)$ that fixes $\mathcal{P}_r$. Then our aim is the following:

Compute one representative of each G-orbit on subsets of $\mathcal{P}_r$.

Let $\mathcal{L}_k$ be a set containing one representative from each G-orbit on k-subsets of the points. Then the algorithm below shows how to compute $\mathcal{L}_{k+1}$ from $\mathcal{L}_k$ with no explicit isomorphism tests between pairs of $(k+1)$-subsets.

ALGORITHM 1

For each k-subset $X \in \mathcal{L}_k$
- Compute the group G_X fixing X setwise
- For each orbit representative x of G_X on $\mathcal{P}_r \backslash X$
  - ◇ Let $Y = X \cup \{x\}$
  - ◇ Compute the group G_Y and the corresponding canonically labelled graph
  - ◇ Add Y to $\mathcal{L}_{k+1}$ if and only if x is in the same G_Y-orbit as the lowest canonically labelled vertex of Y.

Algorithm 1: An orderly progression from $\mathcal{L}_k$ to $\mathcal{L}_{k+1}$

Theorem 4.4. *With the notation above, if $\mathcal{L}_k$ contains exactly one representative of each G-orbit on k-subsets of $\mathcal{P}_r$, then the set $\mathcal{L}_{k+1}$ produced by Algorithm 1 contains exactly one representative of each G-orbit on $(k+1)$-subsets of $\mathcal{P}_r$.*

Proof. Suppose that Y is a $(k + 1)$-subset of $\mathcal{P}_r$. We need to show that $\mathcal{L}_{k+1}$ contains exactly one isomorph of Y. Consider the canonically labelled version of Γ_r with Y distinguished, let y be the element of Y with the lowest canonical label and set $X = Y \backslash \{y\}$. Then by the inductive hypothesis, $\mathcal{L}_k$ contains some isomorph of X. When this isomorph is processed by Algorithm 1, all of its single-element extensions will be considered including an isomorph of Y which will then be accepted. Hence an isomorph of Y is accepted at least once. An isomorph of Y can be accepted only as an extension of an isomorph of X, and hence different k-subsets cannot yield isomorphic $(k + 1)$-subsets. Finally note that if both $X \cup x_1$ and $X \cup x_2$ are accepted as extensions of X, then they cannot be isomorphic. If they were isomorphic then some automorphism would map x_1 to x_2 (as they are both in the orbit of the lowest canonically labelled vertex), and hence some automorphism fixing X would map x_1 to x_2, contradicting the fact that only

one representative of each orbit of G_X is considered for addition to X as it is processed. □

4.2.2. *Modification of this algorithm.* This algorithm is easily modified to produce only those matroids that exclude a given minor, say M. (Henceforth we will identify a matroid with the corresponding k-subset of $\mathcal{P}_r$ and mix the graph or matroid terminology interchangeably, even in a single sentence!)

If M has k elements, then the algorithm is run normally until the set $\mathcal{L}_k$ has been produced. One of these k-sets is equivalent to M and we then set $\mathcal{L}'_k = \mathcal{L}_k\backslash M$ which, by definition, contains every k-element simple binary matroid of rank up to r with no minor isomorphic to M. If Algorithm 1 is then applied to $\mathcal{L}'_k$ then the resulting list will certainly contain every $(k+1)$-element matroid without an M-minor, but probably will introduce some new matroids that *do* contain an M-minor. Therefore, a two-step process is used to produce $\mathcal{L}'_{k+1}$ from $\mathcal{L}'_k$:

(1) Use Algorithm 1 to compute $\mathcal{L}'_k \to \mathcal{M}_{k+1}$
(2) Form $\mathcal{L}'_{k+1}$ by removing any matroids with an M-minor from $\mathcal{M}_{k+1}$.

The second stage of this process would be prohibitively expensive to perform if each matroid had to be directly tested for the presence of a minor isomorphic to M. However we can exploit the fact that the lists $\mathcal{L}'_\ell$ for $\ell \leq k$ jointly contain all smaller simple binary matroids with no minor isomorphic to M. As each candidate matroid is created, we test that (the simplifications) of all of its single-element deletions and contractions are contained in these lists, thereby certifying that the matroid is M-free.

Once the list of all simple prism-free binary matroids of rank up to 8 has been found, a straightforward calculation verifies that there are no internally 4-connected prism-free binary matroids of rank 6, 7 or 8 and hence Lemma 4.1 holds. A slightly more elaborate computation determining the minor order on the prism-free binary matroids confirms that Lemma 4.2 is also true.

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