

MONADIC NON-DEFINABILITY AND GAIN-GRAPHIC MATROIDS

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ABSTRACT. We present an analogue of a Myhill-Nerode characterisation which will allow us to prove that classes of hypergraphs cannot be defined by sentences in the counting monadic second-order logic of hypergraphs. We apply this to classes of gain-graphic matroids, and show that if the group Γ is not uniformly locally finite, then the class of Γ -gain-graphic matroids is not monadically definable. (A group is uniformly locally finite if and only if there is a maximum size amongst subgroups generated by at most k elements, for every k .) In addition, we define the conviviality graph of a group, and show that if the group Γ has an infinite conviviality graph, then the class of Γ -gain-graphic matroids is not monadically definable. This will be useful in future constructions.

1. INTRODUCTION

The monadic second-order logic of matroids was introduced and developed by Hliněný [6, 7]. We use CMSO_1 to refer to the *counting monadic second-order logic* of matroids (and more generally, hypergraphs). This language has predicates that let us say when a subset of the domain has cardinality congruent to p modulo q , for any appropriate pair p and q . The fragment of CMSO_1 that does not use these predicates is denoted by MSO_1 . (This language has been at various times denoted by MSOL , MS_M , and MS_0 .) The following conjectures appear in [5].

Conjecture 1.1. *Let Γ be a finite group. The class of Γ -gain-graphic matroids is MSO_1 -definable.*

Conjecture 1.2. *Let Γ be an infinite group. The class of Γ -gain-graphic matroids is not MSO_1 -definable.*

The first and third authors have shown that Conjecture 1.1 is true. However, this positive result requires a structural theorem giving us control over the representations of frame matroids by biased graphs. This will allow us to construct a monadic transduction taking frame matroids as input and producing their biased-graphic representations as output. The proof of this structural theorem will be lengthy and is still work in progress, so for now we claim this definability result without proof.

In contrast to this positive result, Conjecture 1.2 is false. If, for example, Γ is the direct product of an infinite number of copies of \mathbb{Z}_2 , then the class of Γ -gain-graphic matroids is MSO_1 -definable. In fact, the third author and

Ben-Shahar have built a hierarchy of infinite groups which are constructed by alternately using direct products and a special semi-direct product. Each of the infinite groups in this hierarchy gives rise to an MSO_1 -definable class of gain-graphic matroids. This proof again requires the structural representation theorem, so we do not provide a proof here.

The purpose of this article is to instead present some negative results: proving that for certain infinite groups Γ , the class of Γ -gain-graphic matroids is not CMSO_1 -definable. We do this by using a variant on the Myhill-Nerode characterisation of regular languages [12, 13]. Say that \mathcal{L} is a language of finite strings. The Myhill-Nerode characterisation constructs an equivalence relation on finite strings: two strings w_1 and w_2 are equivalent (relative to \mathcal{L}) if there exists no string z such that exactly one of w_1z and w_2z is in \mathcal{L} . (Juxtaposition of strings indicates concatenation.) Now \mathcal{L} is regular if and only if there are finitely many equivalence classes in this relation. Lemma 3.7 provides an analogue of this characterisation for hypergraphs. Our analogue of concatenation is a coloured sum (Definition 3.3). The result of such a sum is a hypergraph. The summing operation allows us to define an equivalence relation (Definition 3.5) on hypergraphs in exactly the same way that Myhill-Nerode defines an equivalence relation on strings. Monadic definability of a class implies that the index of this relation is finite (Lemma 3.7). With this result in hand, we can prove the non-definability of some properties of hypergraphs. We do so by exhibiting representatives to demonstrate that there is an infinite number of equivalent classes.

We note some other similar results in the literature. First, Lemma 3.7 is a generalisation of Lemmas 1.3 and 1.4 in [11], since those lemmas are specific to particular types of matroid summing operation. Lemma 3.7 is independent from [2, Corollary 3], since the notion of a hypergraph sum in that work bounds the number of hyperedges that intersect both sides of the sum. Our notion of a sum does not require any such bound. Our lemma is also independent of the tool created by Kotek and Makowsky [10, Theorem 3.5], as the binary operation of matroid sum (defined in terms of independent subsets) is not *smooth* (using their language).

Let Γ be a (multiplicative) group. Then Γ has *finite exponent* if there is some positive integer p such that h^p is the identity for every $h \in \Gamma$. If every finite subset of Γ generates a finite subgroup then Γ is *locally finite*. Assume there is a function g_Γ taking positive integers to positive integers such that any subgroup of Γ generated by at most k elements has size at most $g_\Gamma(k)$. In this case Γ is *uniformly locally finite*. It is immediate that a uniformly locally finite group is locally finite. Moreover, any subgroup generated by a single element has order at most $g_\Gamma(1)$, and hence we see that Γ has finite exponent. So uniform local-finiteness is a sufficient condition for Γ to be locally finite with finite exponent. The restricted version of the famous Burnside problem says that it is also necessary. This result is known to be true thanks to the work of Zel'manov [16, 17]. To reiterate: a group is

uniformly locally finite if and only if it has finite exponent and is locally finite.

Our main theorem is as follows:

Theorem 5.7. *Let Γ be an infinite group that is not uniformly locally finite. The class of Γ -gain-graphic matroids is not CMSO_1 -definable.*

It is reasonable to ask if the other direction of Theorem 5.7 holds: is it true for any Γ that if the class of Γ -gain-graphic matroids is not CMSO_1 -definable, then Γ is not uniformly locally finite? This is more naturally asked in the contrapositive: if Γ is uniformly locally finite, then is the class of Γ -gain-graphic matroids CMSO_1 -definable? A counting argument shows that this cannot be the case. There are uncountably many infinite groups that are uniformly locally finite. It is not hard to show that in addition, there are uncountably many distinct classes of Γ -gain-graphic matroids, where Γ is uniformly locally finite. Since there are only countably many CMSO_1 -sentences, it follows that the converse of Theorem 5.7 cannot hold.

In fact, it is possible to go further, and explicitly construct an infinite group Γ such that Γ is uniformly locally finite and the class of Γ -gain-graphic matroids is not CMSO_1 -definable. We do not provide the description of such a group here, but we develop a tool that will aid in this construction. Let F be a finite subgroup of an infinite group Γ . The *F-conviviality graph* of Γ carries information about how copies of F are embedded in Γ : specifically, how these copies of F relate to other finite subgroups of Γ . As a second application of the Myhill-Nerode lemma, we prove our second main theorem.

Theorem 6.4. *Let Γ be a group. If Γ has a finite subgroup F such that the F -conviviality graph of Γ is infinite, then the class of Γ -gain-graphic matroids is not CMSO_1 -definable.*

The structure of the paper is as follows: In Section 2 we cover some fundamental notions of hypergraphs, monadic logic, matroids, and biased graphs. We use ultrafilters and ultraproducts to simplify the proof of Theorem 5.7, and these concepts are explained in Section 2.4. Section 3 gives the context and proof for an analogue of the Myhill-Nerode characterisation for monadically-defined classes of hypergraphs. Section 4 is a purely matroidal section, establishing properties of the ‘gluing’ operation that we use. In Section 5 we use our assembled tools to prove Theorem 5.7. Section 6 introduces the notion of an F -conviviality graph and proves Theorem 6.4.

2. PRELIMINARIES

We write \mathbb{N} for the set of positive integers. If n is in \mathbb{N} , we write $[n]$ for $\{1, 2, \dots, n\}$. We write 2^U for the power set of the set U . If I is a set and i is an element then we write $I + i$ for $I \cup \{i\}$. We regard each function as a set of ordered pairs. So if $\sigma: X \rightarrow Y$ is a function and x is an element not in X then $\sigma + (x, y)$ is the function with domain $X + x$ which takes each element of x to its image under σ and which takes x to y . Graphs may have

loops and parallel edges. If G is a graph and X is a set of edges, then $G[X]$ is the subgraph with X as its set of edges. The vertices of $G[X]$ are exactly the vertices of G that are incident with at least one edge in X . We will very frequently blur the distinction between sets of edges and subgraphs. For example, a cycle may be a set of edges or it may be a subgraph, according to which is more convenient for us.

2.1. Hypergraphs. A *hypergraph* consists of a finite set E and a collection \mathcal{I} of subsets of E . We refer to E as the *ground set* of the hypergraph, and call the members of \mathcal{I} the *hyperedges*.

The foundations of matroid theory can be found in Oxley [14]. A *matroid* is a hypergraph where the collection of hyperedges is non-empty and is closed downwards under subset containment, and furthermore, whenever I and J are hyperedges satisfying $|I| < |J|$, then there is an element $e \in J - I$ such that $I + e$ is a hyperedge. The hyperedges of a matroid are called *independent sets*. The *dependent* subsets are the subsets of the ground set that are not independent. A dependent subset that does not properly contain a dependent subset is a *circuit*. A (matroidal) *loop* is an element e such that $\{e\}$ is a circuit. A *coloop* is an element that is in no circuit. A matroid is *simple* if every subset of size at most two is independent.

Let $M = (E, \mathcal{I})$ be a matroid. If X is a subset of E then the *rank* of X , written $r(X)$, is the maximum cardinality of an independent subset of X . Thus $r(X) = |X|$ if and only if X is independent. We write $M|X$ for the matroid

$$(X, \{I \in \mathcal{I} : I \subseteq X\}).$$

and we refer to this as the *restriction* of M to X . A *flat* is a subset $F \subseteq E$ such that $r(F + x) > r(F)$ for every $x \in E - F$. An intersection of flats is also a flat. The *closure* of X , written $\text{cl}(X)$, is the intersection of all flats that contain X . Assume X and Y are disjoint sets. Then $r(X) + r(Y) \geq r(X \cup Y)$ by submodularity of the rank function [14, Lemma 1.3.1]. If $r(X) + r(Y) = r(X \cup Y)$ then the pair of sets is *skew*. This is the case if and only if there is no circuit contained in $X \cup Y$ that contains elements of both X and Y [14, Proposition 4.2.1]. A *separation* of the matroid is a partition of E into a skew pair of non-empty sets.

Let E be a finite set of vectors from a vector space V . Declare a subset of E to be a hyperedge if and only if it is linearly independent. The resulting hypergraph is a *representable matroid*.

2.2. Monadic second-order logic. In this section we construct *counting monadic second-order logic* for hypergraphs, which we denote by $\text{CMSO}_1^{\text{hyp}}$. In the context of this article, monadic second-order logic always applies to hypergraphs, so we omit the superscript and write CMSO_1 . Formulas will be constructed using variables from the set $\{Z_1, Z_2, \dots\}$. The *atomic formulas* are as follows.

- $Z_i \subseteq Z_j$ is an atomic formula for $i, j \in \mathbb{N}$. We declare

$$\text{Var}(Z_i \subseteq Z_j) = \text{Free}(Z_i \subseteq Z_j) = \{Z_i, Z_j\}.$$

- $\text{hyp}(Z_i)$ is an atomic formula for $i \in \mathbb{N}$. In this case

$$\text{Var}(\text{hyp}(Z_i)) = \text{Free}(\text{hyp}(Z_i)) = \{Z_i\}.$$

- $|Z_i| \equiv p \pmod q$ is an atomic formula for $i \in \mathbb{N}$, where p and q are integers satisfying $q > 1$ and $0 \leq p < q$. We define

$$\text{Var}(|Z_i| \equiv p \pmod q) = \text{Free}(|Z_i| \equiv p \pmod q) = \{Z_s\}.$$

Any *formula* in CMSO_1 is built using the following rules.

- Every atomic formula is a formula.
- If ψ is a formula then $\neg\psi$ is a formula and $\text{Var}(\neg\psi) = \text{Var}(\psi)$ while $\text{Free}(\neg\psi) = \text{Free}(\psi)$.
- If ψ is a formula and Z_s is in $\text{Free}(\psi)$ then $\exists Z_s \psi$ is a formula and $\text{Var}(\exists Z_s \psi) = \text{Var}(\psi)$ while $\text{Free}(\exists Z_s \psi) = \text{Free}(\psi) - \{Z_s\}$.
- Assume that ψ_1 and ψ_2 are formulas such that

$$(\text{Var}(\psi_i) - \text{Free}(\psi_i)) \cap \text{Free}(\psi_{3-i}) = \emptyset$$

for $i = 1, 2$. Then $\psi_1 \wedge \psi_2$ is a formula. We declare

$$\text{Var}(\psi_1 \wedge \psi_2) = \text{Var}(\psi_1) \cup \text{Var}(\psi_2) \quad \text{and}$$

$$\text{Free}(\psi_1 \wedge \psi_2) = \text{Free}(\psi_1) \cup \text{Free}(\psi_2).$$

If φ is a CMSO_1 -formula, then any variable in $\text{Free}(\varphi)$ is a *free variable* of φ . Any variable in $\text{Var}(\varphi) - \text{Free}(\varphi)$ is a *bound variable* of φ . We use $\text{Bound}(\varphi)$ to denote the set of bound variables in φ . If $\text{Var}(\varphi) = \text{Bound}(\varphi)$ then φ is a CMSO_1 -sentence.

The collection of CMSO_1 -formulas that we construct without using any atomic formula of the form $|\cdot| \equiv p \pmod q$ is *monadic second-order logic* for hypergraphs, which we denote by MSO_1 . Let δ be a positive integer. We say that a CMSO_1 -formula is δ -confined if it can be constructed without using a predicate of the form $|\cdot| \equiv p \pmod q$ where $q > \delta$. Note that a formula is 1-confined if and only if it is an MSO_1 -formula.

We have now discussed the syntax of monadic second-order logic for hypergraphs. Let us move to the semantics. Let φ be a CMSO_1 -formula and let $M = (E, \mathcal{I})$ be a hypergraph. An *interpretation* of φ in M is a function $\theta: \text{Free}(\varphi) \rightarrow 2^E$. We define what it means for φ to be *satisfied* by (M, θ) . If φ is $Z_i \subseteq Z_j$ then φ is satisfied if $\theta(Z_i) \subseteq \theta(Z_j)$. If φ is $\text{hyp}(Z_i)$ then φ is satisfied if $\theta(Z_i)$ is in \mathcal{I} . Next, if φ is $|Z_i| \equiv p \pmod q$ then φ is satisfied if $|\theta(Z_i)|$ is congruent to p modulo q .

Now we assume that φ is not atomic. We define satisfaction recursively. If φ is $\neg\psi$ then φ is satisfied if ψ is *not* satisfied by (M, θ) . If φ is $\exists Z_s \psi$, then φ is satisfied if there exists a subset $X \subseteq E$ such that ψ is satisfied by $(M, \theta + (Z_s, X))$. Finally, if φ is $\psi_1 \wedge \psi_2$, then we let θ_i be the restriction of θ to $\text{Free}(\psi_i)$ for $i = 1, 2$. Now φ is satisfied if ψ_i is satisfied by (M, θ_i) for all $i \in \{1, 2\}$. If φ is a CMSO_1 -sentence, then φ has no free variables. In

this case, an interpretation is the empty function, and we may speak of φ being satisfied by M , rather than (M, θ) .

Let \mathcal{M} be a class of hypergraphs. Assume there is a CMSO₁-sentence φ such that φ is satisfied by a hypergraph if and only if that hypergraph belongs to \mathcal{M} . In this case we say that \mathcal{M} is CMSO₁-*definable*. If φ is a MSO₁-sentence then \mathcal{M} is MSO₁-*definable*.

2.3. Gain-graphic matroids. Let G be a graph with edge-set E and vertex-set V . A *bicycle* of G is a subset $X \subseteq E$ such that X is minimal with respect to $G[X]$ being connected and containing at least two cycles. Every bicycle is a *handcuff* or a *theta subgraph*. The first of these consists of two cycles with at most one vertex in common, along with a unique minimal path joining the two cycles. (Note that this path may consist of a single vertex that is in both cycles — in this case the handcuff is *tight* and otherwise it is a *loose*.) A theta subgraph consists of two distinct vertices and three pairwise internally-disjoint paths that join them. A *linear class* is a set \mathcal{B} of cycles such that no theta subgraph in G contains exactly two cycles in \mathcal{B} . In this case, we say that (G, \mathcal{B}) is a *biased graph*. Any cycle that belongs to \mathcal{B} is said to be *balanced*.

When $\Omega = (G, \mathcal{B})$ is a biased graph we can define $F(\Omega)$, the *frame matroid* of Ω . The ground set of $F(\Omega)$ is the edge-set of G . The circuits are the edge-sets of balanced cycles and the edge-sets of bicycles that contain no balanced cycle. Note that a loop edge in the graph G will only be a loop in the matroid $F(\Omega)$ if that edge comprises a balanced cycle.

The rank of $F(\Omega)$ is obtained by subtracting the number of connected components in Ω that contain no unbalanced cycles from the number of vertices in Ω . A *line* of a matroid is a rank-2 flat. The intersection of two distinct lines has rank at most one. A line is *long* if it contains at least four rank-1 flats.

Proposition 2.1. *Let $\Omega = (G, \mathcal{B})$ be a biased graph. Assume that the element e is contained in two distinct long lines of $F(\Omega)$. Then e is a loop edge of G .*

Proof. We can let $\{a, b, c, e\}$ and $\{x, y, z, e\}$ be sets such that any 3-element subset of either is a circuit. We can also assume that no rank-2 flat contains both sets. Assume that $\{a, b, c\}$ is the edge-set of a balanced cycle in Ω . Then a and b form a path of two edges, and c must form a circuit with these edges. This is only possible if $\{a, b, c\}$ is also the set of edges in a balanced cycle, which means c and e are parallel edges. Therefore $\{a, b, c, e\}$ is the edge-set of a theta subgraph, but the cycle comprising c and e is not balanced, since $\{a, c, e\}$ is a circuit. Therefore this theta subgraph contains exactly two balanced cycles, and we have a contradiction. Exactly the same argument shows that no three edges from $\{a, b, c, e\}$ or $\{x, y, z, e\}$ form a balanced cycle. This means that $\{a, b, c\}$ is a bicycle that contains no balanced cycle. Subsequently $G[\{a, b, c\}]$ contains exactly two vertices. Let

these vertices be p and q . Then e is not incident with any vertex not in $\{p, q\}$. The same argument shows that $G[\{x, y, z\}]$ has exactly two vertices, s and t , and e is not incident with any vertex not in $\{s, t\}$. If $\{s, t\} = \{p, q\}$ then there is a rank-2 flat that contains $\{a, b, c, x, y, z, e\}$, contrary to assumption. Therefore $\{s, t\}$ and $\{p, q\}$ have at most one vertex in common, which means that e is incident with at most one vertex. Thus e is a loop, as we claimed. \square

Let G be a graph and let Γ be a (multiplicative) group. A Γ -*gaining* of G is a function σ that takes as input any triple (e, u, v) such that either e is an edge joining the distinct vertices u and v , or e is a loop incident with u and $u = v$. The codomain of σ is Γ . We require that if $u \neq v$ then $\sigma(e, u, v) = \sigma(e, v, u)^{-1}$. Now we say that (G, σ) is a Γ -*gain-graph*. If $\Omega = (G, \mathcal{B})$ is a biased graph, and there exists a Γ -gaining σ such that $\mathcal{B} = \mathcal{B}(\sigma)$, then Ω is Γ -*gainable*. If e is a loop edge of G incident with the vertex u and $\sigma(e, u, u)$ is not the identity, then we say e is an *unbalanced loop*.

Let W be a walk of G and let $v_0, e_0, v_1, e_1, \dots, e_{n-1}, v_n$ be the sequence of vertices and edges in W . Then $\sigma(W)$ is

$$\sigma(e_0, v_0, v_1) \sigma(e_1, v_1, v_2) \cdots \sigma(e_{n-1}, v_{n-1}, v_n).$$

Let $\mathcal{B}(\sigma)$ be the collection of cycles C in G such that $\sigma(C)$ is the identity of Γ . (If this condition holds, then it will hold no matter which starting point and orientation of the cycle is chosen.) Then $\mathcal{B}(\sigma)$ is a linear class of cycles [15, Proposition 5.1]. We write $F(G, \sigma)$ to denote the frame matroid $F(G, \mathcal{B}(\sigma))$. Any such matroid is said to be Γ -*gain-graphic* matroid.

If σ is a Γ -gaining and ρ is a function from $V(G)$ to Γ , then σ^ρ is the Γ -gaining that takes (e, u, v) to

$$\rho(u)^{-1} \sigma(e, u, v) \rho(v)$$

when $u \neq v$. (And which takes any tuple (e, u, u) to $\sigma(e, u, u)$.) It is easy to see that $\mathcal{B}(\sigma^\rho) = \mathcal{B}(\sigma)$, and therefore $F(G, \sigma^\rho) = F(G, \sigma)$. We say that σ^ρ is obtained from σ by *switching*. Let T be a maximal forest of G . By performing an appropriately chosen switching, we can obtain a Γ -gaining that takes (e, u, v) to the identity of Γ whenever e is an edge in T [15, Lemma 5.3].

2.4. Ultrapowers. Here we will give a brief description of the theory of ultraproducts. All this material is standard; see for instance [8, Section 8.5].

Definition 2.2. An *ultrafilter* on \mathbb{N} is a set $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ such that

- (i) For all $S \in \mathcal{U}$, for all $S \subseteq T \subseteq \mathbb{N}$, $T \in \mathcal{U}$.
- (ii) For all $S_1, S_2 \in \mathcal{U}$, $S_1 \cap S_2 \in \mathcal{U}$.
- (iii) For all $S \subseteq \mathbb{N}$, exactly one of S and $\mathbb{N} \setminus S$ is in \mathcal{U} . In particular, $\emptyset \notin \mathcal{U}$.

Remark 2.3. For any $n \in \mathbb{N}$, $\{S : n \in S \subseteq \mathbb{N}\}$ is an ultrafilter on \mathbb{N} . We say an ultrafilter \mathcal{U} on \mathbb{N} is *non-principal* if it is not of the above form. By the axiom of choice, such an ultrafilter must exist.

Remark 2.4. An ultrafilter \mathcal{U} on \mathbb{N} is non-principal iff it contains no finite set, or equivalently iff it contains every cofinite set. This follows from properties (ii) and (iii) of the definition.

Definition 2.5. Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a sequence of groups, and let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} . Then there is a relation $\sim_{\mathcal{U}}$ on $\prod_i \Gamma_i$ defined by

$$(a_i) \sim_{\mathcal{U}} (b_i) \Leftrightarrow \{i : a_i = b_i\} \in \mathcal{U}$$

It follows from the definition of an ultrafilter that $\sim_{\mathcal{U}}$ is a $\prod_i \Gamma_i$ -invariant equivalence relation. Now the ultraproduct is defined as $\prod_i \Gamma_i / \mathcal{U} = \prod_i \Gamma_i / \sim_{\mathcal{U}}$. It is easy to see that $\prod_i \Gamma_i / \mathcal{U}$ is still a group. We will only be interested in the case where all the G_i are equal. In this case, the ultraproduct is referred to as an *ultrapower*, and the ultrapower of Γ is written as $\Gamma^{\mathcal{U}}$.

Definition 2.6. Let

$$\{s_i(x_1, x'_1, \dots, x_l, x'_l) : i \in S\} \quad \text{and} \quad \{t_i(x_1, x'_1, \dots, x_l, x'_l) : i \in T\}$$

be two sets of strings in $\{x_1, x'_1, \dots, x_l, x'_l\}^*$. Given a group Γ , and elements $g_1, \dots, g_l \in \Gamma$, there is a natural evaluation map taking an s_i or t_i and returning an element of Γ . We will write this $s_i(g_1, g_1^{-1}, \dots, g_l, g_l^{-1})$. Given some l , a group Γ and two such sets of strings $\{s_i\}$, $\{t_i\}$, let us say Γ *solves the pair* $(\{s_i\}, \{t_i\})$ if there are elements g_1, \dots, g_l such that for all $i \in S$,

$$s_i(g_1, g_1^{-1}, \dots, g_l, g_l^{-1}) = \text{Id}$$

and for all $i \in T$,

$$t_i(g_1, g_1^{-1}, \dots, g_l, g_l^{-1}) \neq \text{Id}$$

The next proposition is a special case of Łoś's Theorem. We include a proof to make this section more self-contained.

Proposition 2.7. Fix some l , and let $\{s_i : i \in S\}$, $\{t_i : i \in T\}$ be two finite lists of strings from $\{x_1, x'_1, \dots, x_l, x'_l\}^*$. Let Γ be a group and \mathcal{U} be a non-principal ultrafilter on \mathbb{N} . Then the following are equivalent:

- (i) Γ solves the pair $(\{s_i : i \in S\}, \{t_i : i \in T\})$.
- (ii) $\Gamma^{\mathcal{U}}$ solves the pair $(\{s_i : i \in S\}, \{t_i : i \in T\})$.

Proof. We first do the forward direction. Let α be the map $\Gamma \rightarrow \Gamma^{\mathcal{U}}$ given by

$$g \mapsto (g, g, g, \dots) / \sim_{\mathcal{U}}$$

Note that α is injective, as by definition $\emptyset \notin \mathcal{U}$, so for any $g_1 \neq g_2$, we have $(g_1, g_1, g_1, \dots) \not\sim_{\mathcal{U}} (g_2, g_2, g_2, \dots)$. Thus, α is a group embedding. So given $g_1, \dots, g_l \in \Gamma$ satisfying condition (i), $\alpha(g_1), \dots, \alpha(g_l)$ satisfies condition (ii).

Now we do the backward direction. Let $(g_k^{(1)})_{k \in \mathbb{N}}, \dots, (g_k^{(l)})_{k \in \mathbb{N}} \in \Gamma^{\mathcal{U}}$ satisfy condition (ii). Thus, for each $1 \leq i \leq n$,

$$S_i = \{k : s_i(g_k^{(1)}, (g_k^{(1)})^{-1}, \dots, g_k^{(l)}, (g_k^{(l)})^{-1}) = \text{Id}\} \in \mathcal{U}$$

and for each $1 \leq j \leq m$,

$$T_j = \{k : t_j(g_k^{(1)}, (g_k^{(1)})^{-1}, \dots, g_k^{(l)}, (g_k^{(l)})^{-1}) \neq \text{Id}\} \in \mathcal{U}$$

By property (ii) in the definition of ultrafilters, $\bigcap S_i \cap \bigcap T_j \in \mathcal{U}$, and is therefore nonempty. Let k be an element in this set. It then follows that $g_k^{(1)} \dots g_k^{(l)}$ satisfy condition (i). \square

Corollary 2.8. *For any group Γ and non-principal ultrafilter \mathcal{U} on \mathbb{N} , a biased graph Ω is Γ -gainable if and only if it is $\Gamma^{\mathcal{U}}$ -gainable.*

Proof. It is straightforward to see that Ω is Γ -gainable if and only if there exist elements of Γ satisfying a certain list of equations and inequations. By the above proposition, such elements exist in Γ if and only if they exist in $\Gamma^{\mathcal{U}}$. \square

Corollary 2.9. *For any group Γ and non-principal ultrafilter \mathcal{U} on \mathbb{N} , the class of Γ -gain-graphic matroids is equal to the class of $\Gamma^{\mathcal{U}}$ -gain-graphic matroids.*

Corollary 2.10. *For any group Γ and non-principal ultrafilter \mathcal{U} on \mathbb{N} , the following statements are equivalent.*

- (i) Γ is uniformly locally finite.
- (ii) $\Gamma^{\mathcal{U}}$ is uniformly locally finite
- (iii) $\Gamma^{\mathcal{U}}$ is locally finite.

Proof. First, (i) \Rightarrow (ii). Assume for a contradiction that $\Gamma^{\mathcal{U}}$ is not uniformly locally finite, but Γ is. There exists a K such that $\Gamma^{\mathcal{U}}$ contains arbitrarily large subgroups generated by at most K elements. Let $g_{\Gamma}(K)$ be the maximum size of a subgroup of Γ generated by at most K elements. Let h_1, \dots, h_K be elements of $\Gamma^{\mathcal{U}}$ that generate a subgroup of more than $g_{\Gamma}(K)$ elements. Then there is a finite list of inequalities certifying that $|\langle \{h_1, \dots, h_K\} \rangle| > g_{\Gamma}(K)$. We apply Proposition 2.7 to this list of inequalities to deduce that there are elements $g_1, \dots, g_K \in \Gamma$ also satisfying those inequalities. But then $|\langle \{g_1, \dots, g_K\} \rangle| > g_{\Gamma}(K)$, yielding a contradiction.

The fact that (ii) implies (iii) is obvious from definitions, as discussed in the introduction. To prove (iii) \Rightarrow (i), we assume Γ is not uniformly locally finite, and prove that $\Gamma^{\mathcal{U}}$ is not locally finite. Our assumption means that there is some positive integer K such that there is a sequence of tuples $(g_{i,j})_{1 \leq i \leq K, j \in \mathbb{N}}$ in Γ generating subgroups of strictly increasing size. Consider

$$\{(g_{i,1}, g_{i,2}, \dots) / \sim_{\mathcal{U}} : 1 \leq i \leq K\} \subseteq \Gamma^{\mathcal{U}}$$

Call these elements h_1, \dots, h_K . Suppose they generate a finite subgroup of $\Gamma^{\mathcal{U}}$, and let the size of this subgroup be R . This finiteness is witnessed by

a finite sequence of equalities $\{s_j(h_1, h_1^{-1}, \dots, h_K, h_K^{-1}) = \text{Id} : j \in J\}$ (e.g. let these equalities imply all the equalities in the Cayley table of this group, thus forcing that any set satisfying those equalities is a quotient of this finite group). For each $j \in J$, let $S_j = \{i \in \mathbb{N} : s_j(g_{1,i}, g_{1,i}^{-1}, \dots, g_{K,i}, g_{K,i}^{-1}) = \text{Id}\}$. Then by the definition of $\sim_{\mathcal{U}}$, each S_j is in \mathcal{U} , and by property (ii) of ultrafilters, $S = \bigcap_j S_j$ is also in \mathcal{U} , and therefore nonempty. As in Remark 2.4, since the ultrafilter is non-principal, S must be infinite. Fix an index i in S larger than R . Then $\{g_{1,i}, \dots, g_{K,i}\}$ satisfy all the s_j , so they must generate a subgroup of size at most R . But by definition of the sequence $g_{i,j}$, $\{g_{1,i}, \dots, g_{K,i}\}$ generates a subgroup of size at least i for each i . Contradiction. The proposition follows. \square

3. A MYHILL-NERODE ANALOGUE

In this section we develop an analogue of the Myhill-Nerode characterisation of regular languages [12, 13] (see also [9, Section 3.4]). The Myhill-Nerode characterisation relies on an equivalence relation on strings, which is defined via the operation of concatenation. In [11] we developed an idea that was inspired by Myhill-Nerode, but which used an equivalence relation on matroids defined via the (matroidal) operation of amalgamation. In this section we generalise this technique to hypergraphs. In order to establish an analogue of concatenation of strings, we develop the idea of coloured systems and coloured complements. These can be glued together to form a hypergraph using an operation that we call a coloured sum. This gives us the hypergraph analogue of concatenation that we use to construct a Myhill-Nerode-style equivalence.

Definition 3.1 (*C*-coloured system). Let C be a finite set. A *C*-coloured system is a pair (U, c) , where c is a function from 2^U to C .

Definition 3.2 (*C*-coloured complement). Let C be a finite set. A *C*-coloured complement is a finite set V along with a function $d: 2^V \times C \rightarrow \{0, 1\}$.

Definition 3.3 (*C*-coloured sum). Let C be a finite set and let (U, c) be a *C*-coloured system. Let (V, d) be a *C*-coloured complement, where U and V are disjoint. The *C*-coloured sum is

$$(U \cup V, \{X \cup Y : X \subseteq U, Y \subseteq V, d(Y, c(X)) = 1\}).$$

We denote this hypergraph by $(U, c) \boxplus (V, d)$.

Remark 3.4. The models that we have in mind for coloured sums are versions of matroid sums. Standard matroid operations, such as 1-, 2-, and 3-sums can all be expressed as *C*-coloured sums, as can amalgams over finite sets.

To illustrate, we let M and N be matroids on the ground sets $U \cup \{p\}$ and $V \cup \{p\}$ respectively, where U , V , and $\{p\}$ are pairwise disjoint, and p is not a loop or coloop in either matroid. We define the function $c: 2^U \rightarrow [3]$ as

follows. If $X \subseteq U$ is dependent in M , then set $c(X) = 1$. If X is independent and p is contained in the closure of X , then set $c(X) = 2$. Otherwise, X is independent and p is not in the closure of X . In this case, set $c(X)$ to be 3. Now (U, c) is a $[3]$ -coloured system. Let $d: 2^V \times [3] \rightarrow \{0, 1\}$ be the function such that (Y, i) is taken to 1 if Y is an independent subset of N and either $i = 3$, or $i = 2$ and Y does not span p . All other pairs are taken to 0. It is not difficult to check that the $[3]$ -coloured sum $(U, c) \boxplus (V, d)$ is the hypergraph of independent sets of the 2-sum of M and N along the basepoint p .

Definition 3.5. Let \mathcal{M} be a family of hypergraphs and let C be a finite set. We define the relation $\sim_{\mathcal{M}, C}$ on C -coloured systems. Let (U_1, c_1) and (U_2, c_2) be two such systems. Then

$$(U_1, c_1) \sim_{\mathcal{M}, C} (U_2, c_2)$$

holds if, for every C -coloured complement (V, d) such that $V \cap U_1 = V \cap U_2 = \emptyset$, we have

$$(U_1, c_1) \boxplus (V, d) \in \mathcal{M} \leftrightarrow (U_2, c_2) \boxplus (V, d) \in \mathcal{M}.$$

It is clear that $\sim_{\mathcal{M}, C}$ is an equivalence relation.

Definition 3.6. Let s, t , and δ be positive integers and let φ be a δ -confined CMSO₁-formula. We define the integer $\Lambda_\varphi(s, t, \delta)$. If φ is $|Z_i| \equiv p \pmod q$, then we set $\Lambda_\varphi(s, t, \delta)$ to be $(\delta!)^s$. If φ is $\text{hyp}(Z_i)$, then $\Lambda_\varphi(s, t, \delta)$ is t^s , and if φ is $Z_i \subseteq Z_j$, then $\Lambda_\varphi(s, t, \delta)$ is 2^{s^2} .

We have defined $\Lambda_\varphi(s, t, \delta)$ when φ is atomic. Now assume φ is $\neg\psi$. In this case we set $\Lambda_\varphi(s, t, \delta)$ to be $\Lambda_\psi(s, t, \delta)$. If φ is $\psi_1 \wedge \psi_2$, then we set $\Lambda_\varphi(s, t, \delta)$ to be the product $\Lambda_{\psi_1}(s, t, \delta)\Lambda_{\psi_2}(s, t, \delta)$. Finally, we assume that φ is $\exists Z_i \psi$. In this case we set $\Lambda_\varphi(s, t, \delta)$ to be

$$2^{\Lambda_\psi(s, t, \delta)}.$$

Lemma 3.7. Let t, δ , and s be positive integers and let φ be a δ -confined CMSO₁-sentence with s variables. Let \mathcal{M} be the class of hypergraphs that satisfy φ . If C is a set with cardinality t , then $\sim_{\mathcal{M}, C}$ has at most $\Lambda_\varphi(s, t, \delta)$ equivalence classes.

We illustrate Lemma 3.7 and build intuition by using it to strengthen Theorem 1.1 in [11].

Theorem 3.8. The class of representable matroids is not CMSO₁-definable.

We note that this does indeed strengthen [11, Theorem 1.1], since that result applies only to MSO₁-definability.

Proof of Theorem 3.8. Assume for a contradiction that φ is a CMSO₁-sentence that is satisfied exactly by the hypergraphs that are representable matroids. Let s be the number of variables in φ and let δ be the smallest positive integer such that φ is δ -confined. Let \mathcal{M} be the class of representable matroids.

For any prime number p , let (U_p, \mathcal{I}_p) be a matroid isomorphic to the finite projective plane $\text{PG}(2, p)$. Set c_p to be the function which takes every dependent subset of U_p to 1 and every independent subset to 2. Then $M_p = (U_p, c_p)$ is a $[2]$ -coloured system. Because there are infinitely many prime numbers [3, Proposition 20], Lemma 3.7 tells us that there are distinct primes, p_1 and p_2 , such that $M_{p_1} \sim_{\mathcal{M}, [2]} M_{p_2}$ holds.

Let N be a matroid (V, \mathcal{I}) that is isomorphic to $\text{PG}(2, p_1)$. Let $d: 2^V \times [2] \rightarrow \{0, 1\}$ be the function taking (Y, i) to 1 when Y is independent in N and $i = 2$. Other pairs are taken to 0. Then $M_{p_i} \boxplus (V, d)$ is isomorphic to the matroidal direct sum $\text{PG}(2, p_i) \oplus N$. Therefore $M_{p_1} \boxplus (V, d)$ is representable over $\text{GF}(p_1)$ [14, Proposition 4.2.11]. On the other hand, both $\text{PG}(2, p_1)$ and $\text{PG}(2, p_2)$ are isomorphic to minors of $M_{p_2} \boxplus (V, d)$ [14, 4.2.19]. It follows from [14, Proposition 3.2.4] and [1, Proposition 7.3] that if $M_{p_2} \boxplus (V, d)$ is representable over a field, then that field must simultaneously have subfields isomorphic to $\text{GF}(p_1)$ and $\text{GF}(p_2)$, an impossibility. To summarise, $M_{p_1} \boxplus (V, d)$ is representable and $M_{p_2} \boxplus (V, d)$ is not. Therefore $M_{p_1} \not\sim_{\mathcal{M}, [2]} M_{p_2}$, which is a contradiction. \square

The proof of Lemma 3.7 involves several technical definitions. However, the basic idea is not too complicated. A *cleft* (Definition 3.9) is a certificate that two C -coloured set systems are not equivalent under $\sim_{\mathcal{M}, C}$. Definition 3.10 introduces the idea of a piece of information carried by each C -coloured system. The exact form of this piece of information will depend on the structure of φ , but the important point is that there are at most $\Lambda_\varphi(s, t, \delta)$ values that this information can take (Proposition 3.14). Furthermore, if two C -coloured systems carry the same piece of information, then there can be no cleft that divides them (Corollary 3.13). Lemma 3.7 follows from these steps.

Definition 3.9 (Cleft). Let C be a finite set and for $i = 1, 2$, let $M_i = (U_i, c_i)$ be a C -coloured system. Let φ be a CMSO_1 -formula and for $i = 1, 2$, let σ_i be a function from $\text{Free}(\varphi)$ to 2^{U_i} . A φ -cleft for (M_1, σ_1) and (M_2, σ_2) consists of a C -coloured complement (V, d) and a function $\tau: \text{Free}(\varphi) \rightarrow 2^V$ such that $V \cap U_1 = V \cap U_2 = \emptyset$ and the following holds: for $i = 1, 2$, if we define N_i to be

$$M_i \boxplus (V, d)$$

and let θ_i be the function taking each $Z_i \in \text{Free}(\varphi)$ to $\sigma_i(Z_i) \cup \tau(Z_i)$, then φ is satisfied by exactly one of (N_1, θ_1) and (N_2, θ_2) .

So when φ is a sentence that defines the class \mathcal{M} , there is a φ -cleft exactly when M_1 and M_2 are not equivalent under $\sim_{\mathcal{M}, C}$.

Definition 3.10. Let $S \subseteq \{Z_1, Z_2, \dots\}$ be a finite set of variables, let C be a finite set, and let δ be a positive integer. We are going to define a function $R_{S, C, \delta}$ which takes as input any triple (M, φ, σ) , where:

- $M = (U, c)$ is a C -coloured system,

- φ is a δ -confined CMSO₁-formula such that $\text{Var}(\varphi) \subseteq S$, and
- σ is a function from $S - \text{Bound}(\varphi)$ to 2^U .

We define the output of $R_{S,C,\delta}$ recursively. Let T stand for $S - \text{Bound}(\varphi)$, so that σ is a function from T to 2^U . If φ is an atomic formula, then $\text{Bound}(\varphi) = \emptyset$ and $T = S$. In this case, $R_{S,C,\delta}(M, \varphi, \sigma)$ is defined as follows.

- (i) If φ is $|Z_i| \equiv p \pmod q$ for some $Z_i \in S$, then $R_{S,C,\delta}(M, \varphi, \sigma)$ is the function $r_1: S \rightarrow \{0, 1, \dots, \delta! - 1\}$, where $r_1(Z_i)$ is the residue of $|\sigma(Z_i)|$ modulo $\delta!$ for each $Z_i \in S$.
- (ii) If φ is $\text{hyp}(Z_i)$ for some $Z_i \in S$, then $R_{S,C,\delta}(M, \varphi, \sigma)$ is the function $r_2: S \rightarrow C$, where $r_2(Z_i)$ is $c(\sigma(Z_i))$ for each $Z_i \in S$.
- (iii) If φ is $Z_i \subseteq Z_j$ for some $Z_i, Z_j \subseteq S$, then $R_{S,C,\delta}(M, \varphi, \sigma)$ is the function $r_3: S \times S \rightarrow \{0, 1\}$, where $r_3(Z_i, Z_j) = 1$ if and only if $\sigma(Z_i) \subseteq \sigma(Z_j)$.

We have now defined $R_{S,C,\delta}(M, \varphi, \sigma)$ in the case that φ is atomic. Assume now that φ is not atomic. If φ is $\neg\psi$ then $\text{Bound}(\varphi) = \text{Bound}(\psi)$ and we declare $R_{S,C,\delta}(M, \varphi, \sigma)$ to be equal to $R_{S,C,\delta}(M, \psi, \sigma)$. Next assume that $\varphi = \psi_1 \wedge \psi_2$. Note that $\text{Bound}(\varphi) = \text{Bound}(\psi_1) \cup \text{Bound}(\psi_2)$ and

$$\text{Bound}(\psi_i) \cap \text{Free}(\psi_{3-i}) = \emptyset$$

for $i = 1, 2$. Let σ_i be the function

$$\sigma \cup \{(Z_s, \emptyset) : Z_s \in \text{Bound}(\psi_{3-i}) - \text{Bound}(\psi_i)\}.$$

Thus σ_i is a function from $S - \text{Bound}(\psi_i)$ to subsets of U . We set $R_{S,C,\delta}(M, \varphi, \sigma)$ to be the ordered pair

$$(R_{S,C,\delta}(M, \psi_1, \sigma_1), R_{S,C,\delta}(M, \psi_2, \sigma_2)).$$

Finally we assume that $\varphi = \exists Z_i \psi$. In this case

$$\text{Bound}(\psi) = \text{Bound}(\varphi) + Z_i.$$

We declare $R_{S,C,\delta}(M, \varphi, \sigma)$ to be the set

$$\{R_{S,C,\delta}(M, \psi, \sigma + (Z_i, X)) : X \subseteq U\}.$$

That is, we consider every extension of σ by an ordered pair of the form (Z_i, X) . We then let $R_{S,C,\delta}(M, \varphi, \sigma)$ be the set of all outputs produced by $R_{S,C,\delta}$ operating on these extensions.

We observe that $R_{S,C,\delta}(M, \varphi, \sigma)$ is either a function with S or $S \times S$ as its domain, or it is a ordered pair, or it is a set. Thus two outputs $R_{S,C,\delta}(M_1, \varphi, \sigma_1)$ and $R_{S,C,\delta}(M_2, \varphi, \sigma_2)$ are equal if they are equal as functions, pairs, or sets.

Definition 3.11 (Sympathetic). Let $S \subseteq \{Z_1, Z_2, \dots\}$ be a finite set of variables, let C be a finite set, and let δ be a positive integer. Let $M = (U, c)$ be a C -coloured system. Let φ be a δ -confined CMSO₁-formula satisfying $\text{Var}(\varphi) \subseteq S$ and let σ be a function from $S - \text{Bound}(\varphi)$ to 2^U . Let $\Pi = (V, d)$ be a C -coloured complement, where we assume that U and V are disjoint.

Let τ be a function from $\text{Free}(\varphi)$ to 2^V . We are going to define what it means for $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) to be *sympathetic*.

If φ is atomic then $R_{S,C,\delta}(M, \varphi, \sigma)$ is a function. First assume that φ is $|Z_i| \equiv p \pmod q$, so that $R_{S,C,\delta}(M, \varphi, \sigma)$ is the function r_1 from S to $\{0, 1, \dots, \delta! - 1\}$. In this case we say that $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic if

$$r_1(Z_i) + |\tau(Z_i)| \text{ is congruent to } p \text{ modulo } q.$$

Next assume that φ is $\text{hyp}(Z_i)$, so $R_{S,C,\delta}(M, \varphi, \sigma)$ is the function $r_2: S \rightarrow C$. We declare that $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic if

$$d(\tau(Z_i), r_2(Z_i)) = 1.$$

In the next case we assume φ is $Z_i \subseteq Z_j$, so that $R_{S,C,\delta}(M, \varphi, \sigma)$ is the function $r_3: S \times S \rightarrow \{0, 1\}$. Then $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic if $r_3(Z_i, Z_j) = 1$ and $\tau(Z_i) \subseteq \tau(Z_j)$.

We will now assume that φ is not atomic. Assume that φ is $\neg\psi$. Then $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic if and only if $R_{S,C,\delta}(M, \psi, \sigma)$ and (Π, τ) are *not* sympathetic.

Next assume that φ is $\psi_1 \wedge \psi_2$. We define σ_i exactly as in Definition 3.10, so that it is a function from $S - \text{Bound}(\psi_i)$ to 2^U . For $i = 1, 2$, let τ_i be the restriction of τ to $\text{Free}(\psi_i)$. Now $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic if and only if $R_{S,C,\delta}(M, \psi_i, \sigma_i)$ and (Π, τ_i) are sympathetic, for $i = 1, 2$.

Finally, assume that φ is $\exists Z_i \psi$. Then $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic if and only if there exist subsets $X \subseteq U$ and $Y \subseteq V$ such that $R_{S,C,\delta}(M, \psi, \sigma + (Z_i, X))$ and $(\Pi, \tau + (Z_i, Y))$ are sympathetic.

Proposition 3.12. *Let C be a finite set and let δ be a positive integer. Let $M = (U, c)$ be a C -coloured system. Let φ be a δ -confined CMSO₁-formula. Set S to be $\text{Var}(\varphi)$ and let σ be a function from $\text{Free}(\varphi)$ to 2^U . Let $\Pi = (V, d)$ be a C -coloured complement where we assume U and V are disjoint, and let τ be a function from $\text{Free}(\varphi)$ to 2^V . Set E to be $U \cup V$ and define $\theta: \text{Free}(\varphi) \rightarrow 2^E$ so that $\theta(Z_s) = \sigma(Z_s) \cup \tau(Z_s)$ for every free variable Z_s . Let N be the hypergraph $M \boxplus \Pi$. Then φ is satisfied by (N, θ) if and only if $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic.*

Proof. The proof is by induction on the number of steps required to construct φ . First assume that φ is atomic. Then $R_{S,C,\delta}(M, \varphi, \sigma)$ is one of the functions r_1, r_2 , or r_3 . Assume that φ is $|Z_i| \equiv p \pmod q$. Then φ is satisfied by (N, θ) if and only if

$$|\theta(Z_i)| = |\sigma(Z_i)| + |\tau(Z_i)|$$

is congruent to p modulo q . Note that $|\sigma(Z_i)|$ can be expressed as $r_1(Z_i) + \kappa\delta!$ for some integer κ , by the definition of the r_1 function. Since φ is δ -confined it follows that $q \leq \delta$ and therefore q divides $\delta!$. It follows that $|\sigma(Z_i)| + |\tau(Z_i)|$ is congruent to p modulo q if and only if $r_1(Z_i) + |\tau(Z_i)|$ is congruent to p modulo q . This is true if and only if $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic. Therefore we are done in the case that φ is $|Z_i| \equiv p \pmod q$.

Assume that φ is the formula $\text{hyp}(Z_i)$. Then φ is satisfied by (N, θ) if and only if $\theta(Z_i) = \sigma(Z_i) \cup \tau(Z_i)$ is a hyperedge of $M \boxplus \Pi$. This set of hyperedges is

$$\{X \cup Y : X \subseteq U, Y \subseteq V, d(Y, c(X)) = 1\},$$

so φ is satisfied by (N, θ) if and only if

$$1 = d(\tau(Z_i), c(\sigma(Z_i))) = d(\tau(Z_i), r_2(Z_i)).$$

This is exactly what it means for $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) to be sympathetic.

Next we assume that φ is $Z_i \subseteq Z_j$. Then φ is satisfied by (N, θ) if and only if $\sigma(Z_i) \subseteq \sigma(Z_j)$ and $\tau(Z_i) \subseteq \tau(Z_j)$. This is true if and only if $r_3(Z_i, Z_j) = 1$ and $\tau(Z_i) \subseteq \tau(Z_j)$, which in turn is true if and only if $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic.

We are now done with the case that φ is atomic. Therefore we consider the case that φ is $\neg\psi$. Hence φ is satisfied by (N, θ) if and only if ψ is not satisfied by (N, θ) . By induction, this is the case if and only if $R_{S,C,\delta}(M, \psi, \sigma)$ and (Π, τ) are not sympathetic, which is precisely the condition for $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) to be sympathetic.

The next case is when φ is $\psi_1 \wedge \psi_2$. For $i = 1, 2$, we let τ_i (respectively θ_i) be the restriction of τ (respectively θ) to $\text{Free}(\psi_i)$. Define σ_i to be

$$\sigma \cup \{(Z_s, \emptyset) : Z_s \in \text{Bound}(\psi_{3-i}) - \text{Bound}(\psi_i)\}.$$

Now θ_i takes each $Z_s \in \text{Free}(\psi)$ to $\sigma_i(Z_s) \cup \tau_i(Z_s)$. We see that φ is satisfied by (N, θ) if and only if ψ_i is satisfied by (N, θ_i) for $i = 1, 2$. By induction, this is true if and only if $R_{S,C,\delta}(M, \psi_i, \sigma_i)$ and (Π, τ_i) are sympathetic for $i = 1, 2$. This is true if and only if $R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic.

Finally, we assume that φ is $\exists Z_s \psi$. We argue as follows.

- φ is satisfied by (N, θ)
- \Leftrightarrow there exists $D \subseteq E$ such that ψ is satisfied by $(N, \theta + (Z_s, D))$
- \Leftrightarrow there exists $D \subseteq E$ such that ψ is satisfied by (N, θ') where we
 - define $\sigma' = \sigma + (Z_s, D \cap U)$ and $\tau' = \tau + (Z_s, D \cap V)$
 - and θ' maps each $Z_t \in \text{Free}(\psi)$ to $\sigma'(Z_t) \cup \tau'(Z_t)$
- \Leftrightarrow there exist $X \subseteq U$ and $Y \subseteq V$ such that ψ is satisfied by (N, θ')
 - where we define $\sigma' = \sigma + (Z_s, X)$ and $\tau' = \tau + (Z_s, Y)$
 - and θ' maps each $Z_t \in \text{Free}(\psi)$ to $\sigma'(Z_t) \cup \tau'(Z_t)$
- \Leftrightarrow there exist $X \subseteq U$ and $Y \subseteq V$ such that
 - $R_{S,C,\delta}(M, \psi, \sigma + (Z_s, X))$ and $(\Pi, \tau + (Z_s, Y))$
 - are sympathetic
- $\Leftrightarrow R_{S,C,\delta}(M, \varphi, \sigma)$ and (Π, τ) are sympathetic

Now the proof of Proposition 3.12 is complete. \square

Corollary 3.13. *Let C be a finite set and let δ be a positive integer. For $i = 1, 2$ let $M_i = (U_i, c_i)$ be a C -coloured system. Let φ be a δ -confined CMSO₁-formula and for $i = 1, 2$ let σ_i be a function from $\text{Free}(\varphi)$ to 2^{U_i} . Let S be $\text{Var}(\varphi)$. If there is a φ -cleft for (M_1, σ_1) and (M_2, σ_2) , then*

$$R_{S,C,\delta}(M_1, \varphi, \sigma_1) \neq R_{S,C,\delta}(M_2, \varphi, \sigma_2).$$

Proof. Assume for a contradiction that

$$R_{S,C,\delta}(M_1, \varphi, \sigma_1) = R_{S,C,\delta}(M_2, \varphi, \sigma_2)$$

and yet we have a φ -cleft for (M_1, σ_1) and (M_2, σ_2) . Let this cleft consist of the C -coloured complement $\Pi = (V, d)$ and the function $\tau: \text{Free}(\varphi) \rightarrow 2^V$. For $i = 1, 2$, let $\theta_i: \text{Free}(\varphi) \rightarrow 2^{U \cup V}$ be the function taking each $Z_s \in \text{Free}(\varphi)$ to $\sigma_i(Z_s) \cup \tau(Z_s)$. Let N_i be $M_i \boxplus (V, d)$. Definition 3.9 means that φ is satisfied by exactly one of (N_1, θ_1) and (N_2, θ_2) .

Proposition 3.12 says that φ is satisfied by (N_1, θ_1) if and only if $R_{S,C,\delta}(M_1, \varphi, \sigma_1)$ and (Π, τ) are sympathetic. As

$$R_{S,C,\delta}(M_1, \varphi, \sigma_1) = R_{S,C,\delta}(M_2, \varphi, \sigma_2)$$

holds, this is the case if and only if $R_{S,C,\delta}(M_2, \varphi, \sigma_2)$ and (Π, τ) are sympathetic, which holds if and only if φ is satisfied by (N_2, θ_2) . Now we have contradicted our earlier conclusion. \square

Recall that the integer $\Lambda_\varphi(s, t, \delta)$ was described in Definition 3.6.

Proposition 3.14. *Let $S \subseteq \{Z_1, Z_2, \dots\}$ be a finite set of variables. Let C be a finite set, and let δ be a positive integer. Let φ be a δ -confined CMSO₁-formula such that $\text{Var}(\varphi) \subseteq S$. As M ranges over all C -coloured systems (U, c) , and σ ranges over all functions from $S - \text{Bound}(\varphi)$ to 2^U , the number of values taken by $R_{S,C,\delta}(M, \varphi, \sigma)$ is no greater than $\Lambda_\varphi(|S|, |C|, \delta)$.*

Proof. The proof is by induction on the number of steps needed to construct φ . Assume φ is $|Z_i| \equiv p \pmod q$. Then $R_{S,C,\delta}(M, \varphi, \sigma)$ is a function r_1 from S to $\{0, 1, \dots, \delta! - 1\}$. The number of such functions is at most $(\delta!)^{|S|} = \Lambda_\varphi(|S|, |C|, \delta)$, so the result holds. Similarly, if φ is $\text{hyp}(Z_i)$, then $R_{S,C,\delta}(M, \varphi, \sigma)$ is a function r_2 from S to C , and there are at most $|C|^{|S|} = \Lambda_\varphi(|S|, |C|, \delta)$ such functions. If φ is $Z_i \subseteq Z_j$, then $R_{S,C,\delta}(M, \varphi, \sigma)$ is the function r_3 from $S \times S$ to $\{0, 1\}$, and there are at most $2^{|S|^2}$ such functions. So the result holds in the case that φ is atomic.

If φ is $\neg\psi$, then $\text{Bound}(\varphi) = \text{Bound}(\psi)$ and $R_{S,C,\delta}(M, \varphi, \sigma) = R_{S,C,\delta}(M, \psi, \sigma)$ for every choice of M and σ . Furthermore, $\Lambda_\varphi(|S|, |C|, \delta) = \Lambda_\psi(|S|, |C|, \delta)$. Thus the result holds by induction. Therefore we will assume that φ is $\psi_1 \wedge \psi_2$. We recall that $\text{Bound}(\varphi) = \text{Bound}(\psi_1) \cup \text{Bound}(\psi_2)$ and that no variable is bound in exactly one of ψ_1 and ψ_2 . Recall also that when σ is a function from $S - \text{Bound}(\varphi)$ to 2^U , then

$$\sigma_i = \sigma \cup \{(Z_s, \emptyset) : Z_s \in \text{Bound}(\psi_{3-i}) - \text{Bound}(\psi_i)\}$$

for $i = 1, 2$. Now $R_{S,C,\delta}(M, \varphi, \sigma)$ is the ordered pair

$$(R_{S,C,\delta}(M, \psi_1, \sigma_1), R_{S,C,\delta}(M, \psi_2, \sigma_2)).$$

So the number of values taken by $R_{S,C,\delta}(M, \varphi, \sigma)$ is at most the product of the numbers of values taken by $R_{S,C,\delta}(M, \psi_1, \sigma_1)$ and $R_{S,C,\delta}(M, \psi_2, \sigma_2)$. By induction, this is at most $\Lambda_{\psi_1}(|S|, |C|, \delta) \Lambda_{\psi_2}(|S|, |C|, \delta)$, which is equal to $\Lambda_{\varphi}(|S|, |C|, \delta)$.

Now we must assume that φ is $\exists Z_s \psi$. Then $R_{S,C,\delta}(M, \varphi, \sigma)$ is a set of outputs of the form $R_{S,C,\delta}(M, \psi, \sigma + (Z_s, X))$. By induction the number of such subsets is at most

$$2^{\Lambda_{\psi}(|S|, |C|, \delta)} = \Lambda_{\varphi}(|S|, |C|, \delta)$$

so the proof is complete. \square

Proof of Lemma 3.7. Let s , t , and δ , be positive integers, and let φ be a δ -confined CMSO₁-sentence with s variables. Let S be the set of variables in φ . Let \mathcal{M} be the class of hypergraphs that satisfy φ and let C be a set with cardinality t . We claim that the number of equivalence classes of $\sim_{\mathcal{M},C}$ is no greater than $\Lambda_{\varphi}(s, t, \delta)$. Let M_1, M_2, \dots be representatives of these equivalence classes, so that each M_i is a C -coloured system. Since $M_i \sim_{\mathcal{M},C} M_j$ when $i \neq j$, we can find a C -coloured complement (V, d) such that exactly one of $M_i \boxplus (V, d)$ and $M_j \boxplus (V, d)$ is in \mathcal{M} . Because $\text{Free}(\varphi)$ is empty, this complement is a φ -cleft for M_i and M_j . Corollary 3.13 now implies that $R_{S,C,\delta}(M_i, \varphi, \emptyset) \neq R_{S,C,\delta}(M_j, \varphi, \emptyset)$ when $i \neq j$. Therefore the number of equivalence classes under $\sim_{\mathcal{M},C}$ is no greater than the number of values taken by $R_{S,C,\delta}(M, \varphi, \emptyset)$ as M ranges over all C -coloured systems. This is at most $\Lambda_{\varphi}(s, t, \delta)$, by Proposition 3.14, so we are done. \square

4. AMALGAMS

Let M_1 and M_2 be matroids with ground sets E_1 and E_2 , rank functions r_1 and r_2 , and closure operators cl_1 and cl_2 . Let ℓ be $E_1 \cap E_2$. We assume that $M_1|_{\ell} = M_2|_{\ell}$ and we denote this shared restriction by N . If M is a matroid on the ground set $E_1 \cup E_2$ and $M|_{E_i} = M_i$ for $i = 1, 2$, then we say M is an *amalgam* of M_1 and M_2 .

A matroid is *modular* if $r(F) + r(F') = r(F \cap F') + r(F \cup F')$ whenever F and F' are flats. Assume that N is a modular matroid. Then [14, Theorem 11.4.10] tells us that we can obtain an amalgam of M_1 and M_2 by setting the rank of any subset $X \subseteq E_1 \cup E_2$ to be

$$(1) \quad \min\{r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_N(Y \cap \ell) : X \subseteq Y \subseteq E_1 \cup E_2\}$$

The resulting matroid is the *proper amalgam* of M_1 and M_2 . We refer to the set ℓ as the *amalgam base* and we denote the proper amalgam by $\text{Amal}_{\ell}(M_1, M_2)$, or $\text{Amal}(M_1, M_2)$ if the amalgam base is clear from the context. We can easily check that every rank-2 matroid is modular. The following result is slightly more powerful than Proposition 4.1 in [11] because it allows the matroids to be non-simple.

Proposition 4.1. *For $i = 1, 2$, let M_i be a matroid with ground set E_i , rank function r_i , and closure operator cl_i . Let $\ell = E_1 \cap E_2$, where $M_1|_\ell = M_2|_\ell$ and $r_1(\ell) = 2$. Let X be a subset of $E_1 \cup E_2$. If $X \cap E_1$ is dependent in M_1 or if $X \cap E_2$ is dependent in M_2 , then X is dependent in $\text{Amal}(M_1, M_2)$. If $X \cap E_1$ is independent in M_1 and $X \cap E_2$ is independent in M_2 , then X is dependent in $\text{Amal}(M_1, M_2)$ if and only if*

- (i) $\ell \subseteq \text{cl}_1(X \cap E_1)$ and $X - E_1$ is not skew with ℓ in M_2 ,
- (ii) $\ell \subseteq \text{cl}_2(X \cap E_2)$ and $X - E_2$ is not skew with ℓ in M_1 , or
- (iii) $\text{cl}_1(X - E_2) \cap \text{cl}_2(X - E_1)$ contains a non-loop element.

Proof. We use N to denote $M_1|_\ell = M_2|_\ell$. Assume $X \cap E_1$ is dependent in M_1 . Then $X \cap E_1$ is dependent in $\text{Amal}(M_1, M_2)$ since $\text{Amal}(M_1, M_2)|_{E_1} = M_1$. Thus X is dependent in $\text{Amal}(M_1, M_2)$. By symmetry we conclude that if $X \cap E_1$ is dependent in M_1 or if $X \cap E_2$ is dependent in M_2 , then X is dependent in $\text{Amal}(M_1, M_2)$. Henceforth we assume $X \cap E_1$ is independent in M_1 and $X \cap E_2$ is independent in M_2 .

Assume statement (i) holds. Then

$$r_2((X - E_1) \cup \ell) < r_2(X - E_1) + r_2(\ell) = r_2(X - E_1) + 2.$$

Let Y be $X \cup \ell$, so that

$$r_1(Y \cap E_1) = r_1((X \cap E_1) \cup \ell) = r_1(X \cap E_1)$$

because ℓ is in the closure of $X \cap E_1$. Now

$$\begin{aligned} |X| &= |X \cap E_1| + |X - E_1| \\ &= r_1(X \cap E_1) + r_2(X - E_1) \\ &> r_1(X \cap E_1) + r_2((X - E_1) \cup \ell) - 2 \\ &= r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_N(Y \cap \ell), \end{aligned}$$

so (1) implies the rank of X in $\text{Amal}(M_1, M_2)$ is less than $|X|$ and therefore X is dependent in $\text{Amal}(M_1, M_2)$. By symmetric arguments, we see that if (i) or (ii) holds, then X is dependent in $\text{Amal}(M_1, M_2)$.

Next we assume that (iii) holds. Let y be a non-loop element in

$$\text{cl}_1(X - E_2) \cap \text{cl}_2(X - E_1)$$

so that y is necessarily in ℓ . As y is in $\text{cl}_1(X - E_2)$ there is a circuit of M_1 contained in $(X - E_2) \cup y$ that contains y . Since $X \cap E_1$ contains no such circuit, it follows that y is not in X .

Assume $X \cap \ell$ is non-empty and let x be an arbitrary element of this set. Assume that $\{x, y\}$ is dependent in M_1 . Since $X \cap E_1$ is independent in M_1 it follows that x is not a loop. Therefore $\{x, y\}$ is a circuit. There is a circuit contained in $(X - E_1) \cup y$ that contains y . Performing circuit elimination on this circuit and $\{x, y\}$ produces a circuit of M_1 contained in $X \cap E_1$. This is a contradiction, so $\{x, y\}$ is independent in M_1 .

Assume $X \cap \ell$ contains distinct elements, x and x' . As $X \cap E_1$ is independent in M_1 we see that $\{x, x'\}$ is independent in M_1 . The previous paragraph

shows that $\{x, y\}$ and $\{x', y\}$ are independent in M_1 , so $r(N) = 2$ implies that $\{x, x', y\}$ is a circuit. By performing circuit elimination on $\{x, x', y\}$ and a circuit contained in $(X - E_2) \cup y$ that contains y , we obtain a circuit of M_1 contained in $X \cap E_1$. This contradiction means that $|X \cap \ell| \in \{0, 1\}$.

Let Y be $X \cup y$. Because y is in $\text{cl}_1(X - E_2)$ it follows that $r_1(X \cap E_1) = r_1(Y \cap E_1)$. Similarly, $r_2(X \cap E_2) = r_2(Y \cap E_2)$. If $X \cap \ell = \emptyset$ then $0 = |X \cap \ell| = r_N(Y \cap \ell) - 1$. Now assume that $X \cap \ell = \{x\}$. Since $\{x, y\}$ is independent in N , we have $r_N(Y \cap \ell) = 2$, so $|X \cap \ell| = r_N(Y \cap \ell) - 1$ holds in either case. Now we see that

$$\begin{aligned} |X| &= |X \cap E_1| + |X \cap E_2| - |X \cap \ell| \\ &= r_1(X \cap E_1) + r_2(X \cap E_2) - |X \cap \ell| \\ &= r_1(Y \cap E_1) + r_2(Y \cap E_2) - (r_N(Y \cap \ell) - 1) \\ &> r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_N(Y \cap \ell). \end{aligned}$$

Again we see that X is dependent in $\text{Amal}(M_1, M_2)$, and this completes the proof of the ‘if’ direction.

For the ‘only if’ direction, we assume that X is dependent in $\text{Amal}(M_1, M_2)$. As $X \cap E_1$ is independent in M_1 and $X \cap E_2$ is independent in M_2 , it follows that X is contained in neither E_1 nor E_2 . There is some set Y such that $X \subseteq Y \subseteq E_1 \cup E_2$ and $|X| > r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_N(Y \cap \ell)$. Assume that amongst all such sets, Y has been chosen so that it is as small as possible. If y is an element in $Y - (X \cup E_2)$, then we could replace Y with $Y - y$. Therefore no such element exists. By symmetry it follows that $Y - X \subseteq \ell$.

If Y contains a loop element y , then y is in $Y - X$, since $X \cap E_1$ and $X \cap E_2$ are independent in M_1 and M_2 respectively. But in this case we could replace Y with $Y - y$, so Y contains no loops.

If $Y = X$, then $Y \cap E_1$ is independent in M_1 and $Y \cap E_2$ is independent in M_2 , so $|X| > r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_N(Y \cap \ell) = |Y| = |X|$. This contradiction means that $Y - X$ is non-empty.

Claim 4.1.1. If y is in $Y - X$, then

$$y \in \text{cl}_1((Y - y) \cap E_1) \cap \text{cl}_2((Y - y) \cap E_2) \text{ but } y \notin \text{cl}_N((Y - y) \cap \ell).$$

Proof. The minimality of Y means that

$$\begin{aligned} r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_N(Y \cap \ell) \\ < r_1((Y - y) \cap E_1) + r_2((Y - y) \cap E_2) - r_N((Y - y) \cap \ell). \end{aligned}$$

and the result follows. \square

Claim 4.1.2. $|Y \cap \ell| \in \{1, 2\}$ and if $|Y \cap \ell| = 2$, then $Y \cap \ell$ is independent in N .

Proof. Assume y and y' are distinct elements of $Y \cap \ell$. If $\{y, y'\}$ is dependent then we can assume without loss of generality that y is not in X , or else $X \cap \ell$ is dependent in N . Since Y contains no loops, it follows that $\{y, y'\}$

is a circuit of N . This means that y is in $\text{cl}_N((Y - y) \cap \ell)$, and we have contradiction to Claim 4.1.1. Therefore $\{y, y'\}$ is independent.

We know that $Y - X \subseteq \ell$ is non-empty so $Y \cap \ell$ contains at least one element. Let y be such an element. Assume $Y \cap \ell$ contains three distinct elements and let y_1 and y_2 be distinct elements in $(Y \cap \ell) - y$. Then $\{y_1, y_2\}$ is independent by the previous paragraph, so $\{y_1, y_2\}$ spans N . This means that y is in $\text{cl}_N((Y - y) \cap \ell)$, a contradiction. We conclude that $|Y \cap \ell| < 3$. \square

Assume $X \cap \ell$ contains distinct elements x and x' . Then $\{x, x'\}$ is independent by Claim 4.1.2. In this case $\{x, x'\}$ spans N . We can let y be an element in $Y - X$, and now y is in $\text{cl}_N((Y - y) \cap \ell)$, contradicting Claim 4.1.1. Hence $|X \cap \ell|$ is in $\{0, 1\}$.

Claim 4.1.3. $X - E_1$ is not skew with ℓ in M_2 and $X - E_2$ is not skew with ℓ in M_1 .

Proof. Let y be an arbitrary element of $Y - X$. Claim 4.1.1 implies there is a circuit C of M_2 contained in $Y \cap E_2$ that contains y . If C contains an element of $X - E_1$, then it certifies that $X - E_1$ and ℓ are not skew in M_2 . So assume that C is contained in $Y \cap \ell$. But now y is contained in $\text{cl}_N((Y - y) \cap \ell)$, and we have a contradiction. The claim follows by symmetry. \square

If ℓ is contained in $\text{cl}_1(X \cap E_1)$ or $\text{cl}_2(X \cap E_2)$, then Claim 4.1.3 implies that statement (i) or (ii) holds. In this case we have nothing left to prove, so we assume that neither $X \cap E_1$ nor $X \cap E_2$ spans ℓ .

Assume that $Y \cap \ell$ contains a single element y . Then $(Y - y) \cap E_1 = X - E_2$ and $(Y - y) \cap E_2 = X - E_1$. Claim 4.1.1 implies that statement (iii) holds. Therefore we assume that $|Y \cap \ell| \neq 1$, so Claim 4.1.2 implies that $Y \cap \ell = \{y, y'\}$ for distinct elements y and y' .

Assume that $X \cap \ell$ is non-empty, and therefore contains a single element x . Without loss of generality, we assume that $x = y'$. Now $X \cap E_1 = (Y - y) \cap E_1$, so $\text{cl}_1(X \cap E_1)$ contains both x and y by Claim 4.1.1. As $\{x, y\}$ is independent in N by Claim 4.1.2, it follows that $\text{cl}_1(X \cap E_1)$ contains ℓ , contradicting our earlier assumption. Therefore $X \cap \ell$ is empty and $Y \cap E_1 = (X \cap E_1) \cup \{y, y'\}$.

We reason as follows.

$$\begin{aligned}
r_1(X \cap E_1) &< r_1((X \cap E_1) \cup \ell) \quad (\text{since } \ell \not\subseteq \text{cl}_1(X \cap E_1)) \\
&= r_1((X \cap E_1) \cup \text{cl}_N(\{y, y'\})) \\
&\leq r_1(\text{cl}_1((X \cap E_1) \cup \{y, y'\})) \\
&= r_1((X \cap E_1) \cup \{y, y'\}) \\
&\leq r_1((X \cap E_1) \cup \ell) \\
&< r_1(X \cap E_1) + 2 \quad (\text{since } X \cap E_1 \text{ and } \ell \text{ are not skew})
\end{aligned}$$

We see that $r_1(X \cap E_1) + 1 = r_1((X \cap E_1) \cup \{y, y'\}) = r_1(Y \cap E_1)$. Symmetrically, $r_2(Y \cap E_2) = r_2(X \cap E_2) + 1$. Therefore

$$\begin{aligned} |X| &> r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_N(Y \cap \ell) \\ &= r_1(X_1 \cap E_1) + r_2(X \cap E_2) = |X_1 \cap E_1| + |X \cap E_2| = |X|. \end{aligned}$$

This final contradiction completes the proof. \square

Definition 4.2. Let G_1 and G_2 be graphs such that $V(G_1) \cap V(G_2) = \{u, v\}$. We use $G_1 \oplus_{uv} G_2$ to denote the graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2)$. Let $e \in E(G_1) \cup E(G_2)$ be an edge and let $w \in V(G_1) \cup V(G_2)$ be a vertex. Then e and w are incident in $G_1 \oplus_{uv} G_2$ if and only if they are incident in G_1 or G_2 .

We note that $\{u, v\}$ is a vertex cut-set of the graph $G_1 \oplus_{uv} G_2$.

Definition 4.3 (Gain-graph amalgam). Let Γ be a group, and for $i = 1, 2$ let $\Omega_i = (G_i, \sigma_i)$ be a Γ -gain-graph such that $V(G_1) \cap V(G_2) = \{u, v\}$. Set ℓ to be $E(G_1) \cap E(G_2)$. We assume the following conditions hold:

- (i) $G_1[\ell] = G_2[\ell]$,
- (ii) $\sigma_1(e, x, y) = \sigma_2(e, x, y)$ whenever (e, x, y) is in the domains of both σ_1 and σ_2 ,

Under these circumstances, the *gain-graph amalgam* $\Omega_1 \oplus_{uv} \Omega_2$ is defined. Let σ be the union of σ_1 and σ_2 . Then $\Omega_1 \oplus_{uv} \Omega_2$ is the Γ -gain-graph $(G_1 \oplus_{uv} G_2, \sigma)$. We say that $\{u, v\}$ is the *base* of the amalgam.

Lemma 4.4. Let Γ be a group and for $i = 1, 2$, let $\Omega_i = (G_i, \sigma_i)$ be a Γ -gain-graph. Assume that $V(G_1) \cap V(G_2) = \{u, v\}$ and let $\ell = E(G_1) \cap E(G_2)$. We assume the following conditions hold.

- (i) $G_1[\ell] = G_2[\ell]$,
- (ii) $\sigma_1(e, x, y) = \sigma_2(e, x, y)$ whenever (e, x, y) is in the domains of both σ_1 and σ_2 ,
- (iii) ℓ contains unbalanced loop edges incident with u and v , and
- (iv) whenever P_i is a path of G_i from u to v for each $i = 1, 2$, and $\sigma_1(P_1) = \sigma_2(P_2)$, then there is a non-loop edge $e \in \ell$ such that $\sigma_1(e, u, v) = \sigma_2(e, u, v) = \sigma_1(P_1) = \sigma_2(P_2)$.

Then $F(\Omega_1 \oplus_{uv} \Omega_2) = \text{Amal}_\ell(F(\Omega_1), F(\Omega_2))$.

Proof. Let G be $G_1 \oplus_{uv} G_2$ and let σ be the union of σ_1 and σ_2 . We will prove the lemma by showing that a circuit of $F(G, \sigma)$ is dependent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$ and a circuit in $\text{Amal}(F(\Omega_1), F(\Omega_2))$ is dependent in $F(G, \sigma)$. For $i = 1, 2$, let E_i stand for $E(G_i)$.

We start by assuming that C is a circuit of $F(G, \sigma)$. This means that C is either a balanced cycle in $(G, \mathcal{B}(\sigma))$, or it is a bicycle containing no balanced cycles. If C is contained in G_i for some i , then C is dependent in $F(\Omega_i)$. In this case C is dependent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$, by Proposition 4.1. So we will assume that both $C - E_1$ and $C - E_2$ are non-empty.

First assume that C is a theta subgraph consisting of three paths between vertices x and y . Any cut-set in this subgraph consisting of two vertices is contained in one of the three paths from x to y . Therefore u and v are both on the same path of C from x to y . From this it follows that we can assume without loss of generality that $C - E_1$ is a path joining u to v . Therefore $C \cap E_1$ contains an unbalanced cycle made of two of the paths in C joining x to y . Let q be an unbalanced loop incident with either u or v . Now $(C \cap E_1) \cup q$ is connected and contains two unbalanced cycles, including q . This shows that q is in $\text{cl}_1(C \cap E_1)$. Since we have a guarantee that ℓ contains unbalanced loops incident with u and v , it now follows that $\text{cl}_1(C \cap E_1)$ contains ℓ . The union of $C - E_1$ with loops incident with u and v forms a circuit contained in $\ell \cup (C - E_1)$. This circuit contains elements from both ℓ and $C - E_1$. Therefore these sets are not skew in $F(\Omega_2)$. Now Proposition 4.1 tells us that C is dependent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$.

Next assume that C is a handcuff. Then C contains two edge-disjoint cycles, both unbalanced. Let these cycles be C_1 and C_2 and let P be the path of C that joins a vertex of C_1 to a vertex of C_2 . Note that P may comprise a single vertex. Assume that both $C - E_1$ and $C - E_2$ contain cycles of C . Then P contains a vertex $w \in \{u, v\}$. Let $q \in \ell$ be an unbalanced loop that is incident with w . Then $C - E_1$ contains an unbalanced cycle and a path joining this cycle to w , so the union of q and $C - E_1$ contains a handcuff with two unbalanced cycles. This shows that q is in $\text{cl}_2(C - E_1)$. The same argument shows that q is also in $\text{cl}_1(C - E_2)$. Proposition 4.1 now shows that C is dependent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$. Therefore we assume without loss of generality that $C - E_1$ does not contain a cycle.

Now $G[C - E_1]$ is a forest with at least one edge, and therefore it contains at least two degree-one vertices. But $G[C]$ contains no such vertex, so any degree-one vertex in $G[C - E_1]$ is incident with edges in both $C - E_1$ and $C \cap E_1$. There are at most two vertices (u and v) incident with edges in both these sets. This shows that $G[C - E_1]$ contains exactly two degree-one vertices, and these vertices are u and v . Thus $C - E_1$ is a subpath of $G[C]$ and its end vertices are u and v . No internal vertex of this path has degree three in $G[C]$, or else it would be a vertex incident with edges in both $C - E_1$ and $C \cap E_1$, and the only such vertices are u and v . In particular, the vertex in both C_1 and P is not an internal vertex of $C - E_1$. The same applies to the vertex in both C_2 and P . Now, up to symmetry, there are two possibilities: the path $C - E_1$ is contained in C_1 , or is contained in P .

First consider the case that $C - E_1$ is contained in C_1 . Then $G[C \cap E_1]$ contains the unbalanced cycle C_2 as well as the vertices u and v . Since ℓ contains unbalanced loops incident with u and v , it follows that $\text{cl}_1(C \cap E_1)$ contains ℓ . Also, $C - E_1$ is a path joining u and v , and therefore we can find a circuit of $F(\Omega_2)$ contained in $(C - E_1) \cup \ell$ that contains elements from both $C - E_1$ and ℓ . Again we see that C is dependent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$.

Next we assume that $C - E_1$ is a subpath of P . One component of $C \cap E_1$ contains an unbalanced cycle and u . The other contains an unbalanced cycle

and v . As before, we can argue that $\text{cl}_1(C \cap E_1)$ contains ℓ , and that ℓ and $C - E_1$ are not skew, so once again C is dependent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$.

The remaining case is that C is a balanced cycle of (G, σ) . In this case C contains both u and v . For $i = 1, 2$, let P_i be the path of G_i from u to v that is contained in C . Since C is balanced, we see that $\sigma(C)$ is the identity, and therefore $\sigma_1(P_1) = \sigma_2(P_2)$. The hypotheses mean that there is an edge e in ℓ joining u to v such that $\sigma(e, u, v) = \sigma_1(P_1) = \sigma_2(P_2)$. This means that the union of P_i with e is a balanced cycle of Ω_i . So e is in $\text{cl}_i(C - E_{3-i})$. Proposition 4.1 now tells us that C is dependent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$.

We have concluded the argument that a circuit of $F(G, \sigma)$ is dependent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$. Now we will assume that C is a circuit of $\text{Amal}(F(\Omega_1), F(\Omega_2))$. If either $C - E_1$ or $C - E_2$ is empty, then C is a dependent subset of $F(\Omega_1)$ or $F(\Omega_2)$. In this case C is dependent in $F(G, \sigma)$ and we are done. So we will assume that $C - E_1$ and $C - E_2$ are both non-empty. Therefore $C \cap E_i$ is an independent subset of $F(\Omega_i)$ for $i = 1, 2$. Now we can apply Proposition 4.1 and deduce that statement (i), (ii), or (iii) from that result holds.

Symmetrical arguments will deal with both (i) and (ii), so we assume the former holds. Then ℓ is contained in $\text{cl}_1(C \cap E_1)$ and $C - E_1$ is not skew with ℓ in $F(\Omega_2)$. Let w be an arbitrary vertex in $\{u, v\}$ and let q be an unbalanced loop incident with w . If q is in $C \cap E_1$, then there is a connected component of $C \cap E_1$ that contains an unbalanced cycle (namely q) and w . If q is not in $C \cap E_1$, then $(C \cap E_1) \cup q$ contains a circuit that contains q , and this circuit must be a handcuff. In either case $C \cap E_1$ contains a connected component that contains both an unbalanced cycle and w . We choose such a component and call it H_w .

Now let C' be a circuit of $F(\Omega_2)$ that is contained in $\ell \cup (C - E_1)$ and which contains edges from both ℓ and $C - E_1$. Since $G[C']$ is connected, it follows that $G[C' - E_1]$ contains at least one of u and v . Assume that $G[C' - E_1]$ contains a cycle and the vertex u . Then the union of $C' - E_1$ and H_u contains a connected component that contains two distinct cycles. Now it follows that C contains a circuit of $F(G, \sigma)$. The same argument applies if $C' - E_1$ contains v instead of u . Therefore we must assume that $G[C' - E_1]$ is a forest. Then $C' - E_1$ contains at least two degree-one vertices. Since C' contains no such vertex, it follows that any degree-one vertex of $C' - E_1$ must share a common vertex with an edge in ℓ . Therefore $C' - E_1$ has exactly two degree-one vertices, and in fact it is a path of G_2 between u and v . We consider the union of this path with H_u and H_v . Note that the union is a connected subgraph of C . If H_u and H_v are distinct, then this connected subgraph contains two distinct cycles, and therefore C is dependent in $F(G, \sigma)$. So we assume H_u and H_v are the same connected component. This component contains a path of G_1 from u to v . The union of this path with $C' - E_1$ is a cycle and this cycle is distinct from the cycle of $C \cap E_1$ contained in $H_u = H_v$. We have once again found a connected component of $G[C]$ that contains two distinct cycles, so C is dependent

in $F(G, \sigma)$. We have now concluded the argument when case (i) holds in Proposition 4.1.

We can now assume that case (iii) holds. This means we can choose an edge $e \in \ell$ and for each $i = 1, 2$, we can let C_i be a circuit of $F(\Omega_i)$ such that $e \in C_i \subseteq (C - E_{3-i}) \cup e$. Assume that e is a loop edge incident with $w \in \{u, v\}$. Then each C_i is a handcuff, and $C - E_{3-i}$ contains a connected component that contains a cycle and the vertex w . Now $G[C]$ contains a connected component with two distinct cycles and we are done. So we assume e is an edge joining u and v . Then $C - E_{3-i}$ contains a path from u to v . This means that there is a cycle of C containing edges from both $C - E_1$ and $C - E_2$. If either $C_1 - e$ or $C_2 - e$ contains a cycle, then C contains a component that contains two cycles, so we assume that both $C_1 - e$ and $C_2 - e$ are forests. Since C_1 and C_2 have no degree-one vertices, it now follows that both these subgraphs are balanced cycles. For $i = 1, 2$, let P_i be the path $C_i - e$, directed from u to v . Because C_1 and C_2 are balanced, we conclude that $\sigma_1(P_1) = \sigma(e, u, v) = \sigma_2(P_2)$. This means that the union of P_1 and P_2 is a balanced cycle of (G, σ) . Therefore C contains a circuit of $F(G, \sigma)$ and this completes the proof. \square

Lemma 4.5. *For $i = 1, 2$, let $\Omega_i = (G_i, \mathcal{B}_i)$ be a biased graph without balanced loops and assume that $V(G_1) \cap V(G_2) = \{u, v\}$. Assume also that in Ω_1 and Ω_2 , every vertex is incident with an unbalanced loop, and that each vertex w has two distinct neighbours w_1 and w_2 such that for each $i = 1, 2$, there is an unbalanced 2-edge cycle containing w and w_i . Let $\Omega = (G, \mathcal{B})$ be a biased graph such that G has no isolated vertices and $\text{Amal}(F(\Omega_1), F(\Omega_2))$ is equal to $F(\Omega)$. Then G is isomorphic to $G_1 \oplus_{uv} G_2$.*

Proof. Let L be a set containing exactly one unbalanced loop incident with each vertex in $G_1 \oplus_{uv} G_2$. Then $L \cap E_i$ is a basis of $F(\Omega_i)$ for each $i = 1, 2$. It is easy to verify, using Proposition 4.1, that L is independent in $\text{Amal}(F(\Omega_1), F(\Omega_2))$. Let e be an arbitrary element of $\text{Amal}(F(\Omega_1), F(\Omega_2))$ that is not in L . We can assume without loss of generality that e is an edge of Ω_1 . Either e is an unbalanced loop that has a common vertex with a loop in L , or it is non-loop edge of Ω_1 . In these cases, e is in a 2- or 3-element circuit of $F(\Omega_1)$ that is contained in $L \cup e$. This circuit is also a circuit of $\text{Amal}(F(\Omega_1), F(\Omega_2))$. It now follows that L spans $\text{Amal}(F(\Omega_1), F(\Omega_2))$, and is therefore a basis of this matroid. Let x and y be an arbitrary pair of distinct vertices contained in an unbalanced 2-edge cycle of either Ω_1 or Ω_2 . The edges from the 2-edge cycle along with unbalanced loops incident with x and y form a long line of either $F(\Omega_1)$ or $F(\Omega_2)$, and it is not difficult to see that this line is also a line in $\text{Amal}(F(\Omega_1), F(\Omega_2))$. Now the hypotheses imply that each element of L is in two distinct long lines of $\text{Amal}(F(\Omega_1), F(\Omega_2))$. Now Proposition 2.1 implies that the elements of L are all unbalanced loops in Ω . Since L is independent in $F(\Omega)$, we have no more than one element of L incident with any given vertex of G , for any two loops incident with the same vertex form a handcuff. If there exists some

vertex of G not incident with a loop in L , then this vertex is incident with an edge e (since G has no isolated vertices), and now e is not spanned by L . This is a contradiction, so every vertex of G is incident with exactly one loop in L . This induces a bijection between the vertices of $G_1 \oplus_{uv} G_2$ and G . Any edge of $G_1 \oplus_{uv} G_2$ that is not in L is incident with at most two elements of L , and this shows that the bijection between the vertices of $G_1 \oplus_{uv} G_2$ and G gives us an isomorphism between the two graphs. \square

5. UNIFORMLY LOCALLY FINITE GROUPS

In this section, we will prove that if Γ is not a uniformly locally finite group, the class of Γ -gain-graphic matroids is not CMSO_1 -definable.

Definition 5.1. Let Γ be a group. A subset $\{a_1, \dots, a_n\}$ is a *generating set* if it is closed under inverses. Then, for any element $g \in \langle a_1, \dots, a_n \rangle$ there is a string of the $\{a_i\}$ which evaluates to g . Let $f_{\{a_i\}}(g)$ be the minimum length of such a string (letting $f_{\{a_i\}}(\text{Id}) = 0$). Note that for $g_1, g_2 \in \langle a_1, \dots, a_n \rangle$, $f_{\{a_i\}}(g_1 g_2) \leq f_{\{a_i\}}(g_1) + f_{\{a_i\}}(g_2)$. When there is only one generating set in the context, we will drop the subscript and write $f(x) = f_{\{a_i\}}(x)$.

Definition 5.2. For $n, N \in \mathbb{N}$, the graph $H_{n,N}^*$ is defined in Figure 1.

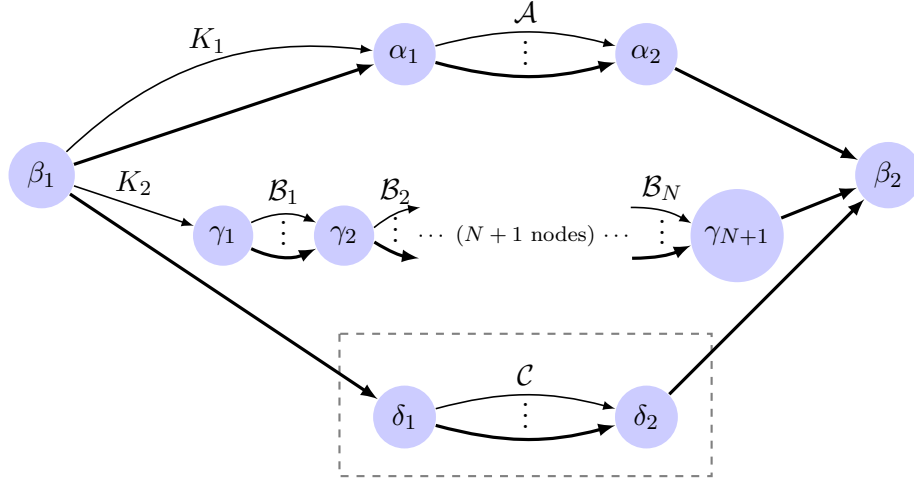


FIGURE 1. The graph $H_{n,N}^*$.

For ease of notation, we will refer to vertices by lower-case Greek letters, and edges by upper-case Roman letters. Let $G_{\ell,N}^*$ be the subgraph enclosed by the dotted line, and let ℓ_N^* be its edge-set. Let \mathcal{T} be the collection of bolded edges. Note that each of $\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_N$ and \mathcal{C} are collections of edges:

$$\begin{aligned} \mathcal{A} &= \{A_{\text{Id}}, A_1, \dots, A_n, A_s\} \\ \mathcal{B}_i &= \{B_{i,\text{Id}}, B_{i,1}, \dots, B_{i,n}\} \text{ for all } 1 \leq i \leq N \\ \mathcal{C} &= \{C_{\text{Id}}, C_1, \dots, C_n, C_s\} \end{aligned}$$

\mathcal{T} has a single element in each of these: the elements $A_{\text{Id}}, B_{i,\text{Id}} : 1 \leq i \leq N$, and C_{Id} . Therefore, we have added a single bold line in each collection in the diagram.

Next, we define $H_{n,N}$ as the union of $H_{n,N}^*$ and the following new edges:

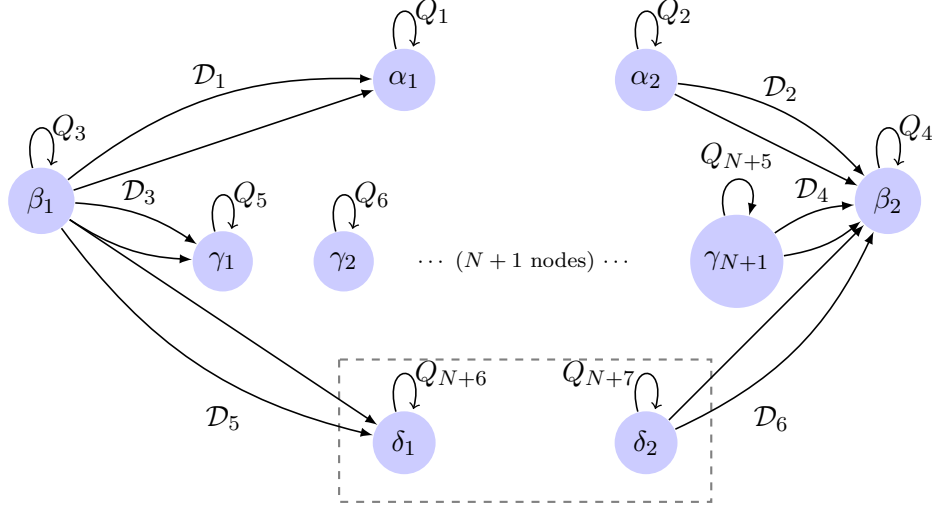


FIGURE 2. The graph $H_{n,N} \setminus H_{n,N}^*$.

Each $\mathcal{D}_i : 1 \leq i \leq 6$ is a collection of two edges: $\mathcal{D}_i = \{D_{i,1}, D_{i,2}\}$. Each Q_i is a single loop edge. Let $G_{\ell,N} = G_{\ell,N}^* \cup \{Q_{N+6}, Q_{N+7}\}$, and let ℓ_N be its edge-set. We will use these new edges at only one point in the argument.

Suppose we have some group Γ , a generating set $\{a_1, \dots, a_n\}$, some $s, \mathfrak{M} \in \langle a_1, \dots, a_n \rangle$. Then, $H_{n,N}^*$ has a Γ -gaining $\sigma^* = \sigma^*(\Gamma, \{a_1, \dots, a_n\}, s, \mathfrak{M})$, defined as follows:

$$\begin{aligned} \sigma^*(T) &= \text{Id} \quad \forall T \in \mathcal{T} \\ \sigma^*(A_j) &= a_j \quad \forall 1 \leq j \leq n \\ \sigma^*(B_{i,j}) &= a_j \quad \forall 1 \leq j \leq n, 1 \leq i \leq N \\ \sigma^*(C_j) &= a_j \quad \forall 1 \leq j \leq n \end{aligned}$$

And $\sigma^*(K_1) = \sigma^*(K_2) = \mathfrak{M}$, $\sigma^*(A_s) = \sigma^*(C_s) = s$. Note all these edges are oriented as in the diagrams, so that, for example, $\sigma^*(A_j) = a_j$ is shorthand for $\sigma^*(A_j, \alpha_1, \alpha_2) = a_j$. Secondly, given elements

$$d_{i,j} : 1 \leq i \leq 6, 1 \leq j \leq 2, \quad \text{and} \quad q_i : 1 \leq i \leq N+7$$

of Γ , we can define $\sigma = \sigma(\Gamma, \{a_1, \dots, a_n\}, s, \mathfrak{M}, \{d_{i,j}\}, \{q_i\})$, a Γ -gaining of $H_{n,N}$ extending σ^* , by

$$\begin{aligned} \sigma(D_{i,j}) &= d_{i,j} \quad \forall 1 \leq i \leq 6, 1 \leq j \leq 2 \\ \sigma(Q_i) &= q_i \quad \forall 1 \leq i \leq N+7 \end{aligned}$$

These edges are also oriented as in the diagrams. We will always assume

$$\begin{aligned}
 & f_{\{a_i\}}(s) = N, \\
 (\dagger) \quad & f_{\{a_i\}}(\mathfrak{M}) \geq 2N + 1, \text{ and} \\
 & \text{the } \{d_{i,j}\}, \text{ and } \{q_i\} \text{ are chosen so that} \\
 & \text{no cycle containing a } D_{i,j} \text{ or } Q_i \text{ is balanced}
 \end{aligned}$$

Subject to this assumption, the balanced cycles of σ depend only on the choice of $\{a_1, \dots, a_n\}$ and s .

Let us also define $\Omega(\Gamma, \{a_1, \dots, a_n\}, s, \mathfrak{M}, \{d_{i,j}\}, \{q_i\})$ as the induced biased graph on $H_{n,N}$.

Remark 5.3. Let Γ be an infinite group, and let S be a finite collection of strings over the characters $\Gamma \cup \{x\}$, where each string contains exactly one copy of x . Given an element $g \in \Gamma$ and $s \in S$, there is an evaluation $s(g) \in \Gamma$, where we replace x with g . Since Γ is infinite, there will always be an element $g \in \Gamma$ such that for all $s \in S$, $s(g) \neq \text{Id}$. Thus, we can always find elements satisfying condition (\dagger) in Definition 5.2.

Since we will use it several times, we note a specialisation of [13, Lemma 5.3]:

Fact 5.4. *Let Γ be a group, G a finite graph, and σ a Γ -gaining of G . Let G_2 be a subgraph such that in $(G_2, \sigma|_{G_2})$ every cycle is balanced. Then, there is a Γ -gaining σ_2 of G , with the same balanced and unbalanced cycles as σ , such that for all $e \in G_2$, $\sigma_2(e) = \text{Id}$.*

Lemma 5.5. *Let \mathcal{M} be a CMSO_1 -definable class of matroids. Let Γ be a group, and let $\{(\Xi_j, \sigma_j) : j \in J\}$ be a collection of Γ -gained finite graphs, over an index set J , all containing a fixed copy of a fixed graph G_ℓ , with edge-set ℓ . Thus, without loss of generality, we may take $G_\ell = \Xi_i \cap \Xi_j$ for any $i \neq j$. Assume also that G_ℓ has exactly two vertices, and in every (Ξ_j, σ_j) each vertex has an unbalanced loop edge. Then, there is a finite partition $J = J_1 \cup \dots \cup J_n$ such that for all l , for all $j_1, j_2 \in J_l$, and for all k ,*

$$\text{Amal}_\ell(F(\Xi_{j_1}, \sigma_{j_1}), F(\Xi_k, \sigma_k)) \in \mathcal{M} \leftrightarrow \text{Amal}_\ell(F(\Xi_{j_2}, \sigma_{j_2}), F(\Xi_k, \sigma_k)) \in \mathcal{M}.$$

That is, whether the amalgam is in the class depends only on where in the partition the amalgam components reside.

Proof. Let cl_j be the matroid closure operator of $F(\Xi_j, \sigma_j)$, and let E_j be the edge-set of Ξ_j for each j . We finitely colour each $\mathcal{P}(E_j)$ as follows: Given $X \subseteq E_j$,

$$t_j(X) = (\text{cl}_j(X) \cap \ell, \text{cl}_j(X \setminus \ell) \cap \ell,$$

$$\text{Is } X \text{ dependent?}, \text{Is } X \setminus \ell \text{ skew with } \ell?)$$

Where the last two terms are boolean values. Let T be the set of colours (which does not depend on j). The pairs (E_j, t_j) are then T -coloured systems.

Note that for any j, k , and any set $X \cup Y \subseteq \text{Amal}_\ell(F(\Xi_j, \sigma_j), F(\Xi_k, \sigma_k))$, where $X \subseteq E_j$ and $Y \subseteq E_k \setminus \ell$, whether $X \cup Y$ is dependent depends only on $t_j(X)$ and Y , by Proposition 4.1. Thus, for each j , let

$$d_j : \mathcal{P}(E_j \setminus \ell) \times T \rightarrow \{0, 1\}$$

be the function which, given a subset Y and colour $c \in T$, returns whether $X \cup Y$ would be independent for any $X \subseteq E_k$ such that $t_k(X) = c$. Given this definition, for any j, k , we have

$$\text{Amal}_\ell(F(\Xi_j, \sigma_j), F(\Xi_k, \sigma_k)) \cong (E_j, t_j) \boxplus (E_k, d_k).$$

By Lemma 3.7, there is a finite partition $J = J_1 \cup \dots \cup J_n$ such that for all $l \in [n]$, all $j_1, j_2, k \in J$,

$$(E_{j_1}, t_{j_1}) \boxplus (E_k, d_k) \in \mathcal{M} \leftrightarrow (E_{j_2}, t_{j_2}) \boxplus (E_k, d_k) \in \mathcal{M}.$$

By the above isomorphism, this concludes the proof. \square

The next theorem shows that an infinite group Γ must be locally finite in order for the class of Γ -gain-graphic matroids to be CMSO_1 -definable.

Theorem 5.6. *Let Γ be a group, and let $\{a_1, \dots, a_n\} \subseteq \Gamma$ be a generating set which generates an infinite subgroup. Then the class of Γ -gain-graphic matroids is not CMSO_1 -definable.*

Proof. We may restrict to the case where the $\{a_i\}$ contain two elements $\alpha_1 \neq \alpha_2$ such that $\alpha_1^{-1} \neq \alpha_2$, since we may freely add new pairs $\{g, g^{-1}\} \subseteq \langle a_1 \dots a_n \rangle$ to the generating set. For each $1 \leq N < \omega$, fix some

$$s_N, \mathfrak{M}_N, d_{i,j} : 1 \leq i \leq 6, 1 \leq j \leq 2 \quad \text{and} \quad q_i : 1 \leq i \leq 10$$

all in Γ satisfying condition (\dagger) in Definition 5.2. Let σ_N be $\sigma(\Gamma, \{a_1, \dots, a_n\}, s_N, \mathfrak{M}_N, \{d_{i,j}\}, \{q_i\})$, the Γ -gaining of $H_{n,N}$. Then, for ease of notation, for each $N \geq 1$, let $H_N = (H_{n,N}, \sigma_N)$, and let Ω_N be the associated biased graph. Note that any H_N contains $G_{\ell,N}$, and the $\{G_{\ell,N}\}$ are all isomorphic as Γ -gained graphs. So, we could arrange that the H_n are all sub-graphs of some larger graph \mathbb{G} , and that the $G_{\ell,N}$ are actually equal, and for any N, N' distinct, $H_N \cap H_{N'} = G_{\ell,N}$. Thus, given any two $H_N, H_{N'}$, we can construct both the matroid amalgam $\text{Amal}(F(H_N), F(H_{N'}))$ over the base $F(\ell_N) = F(\ell_{N'})$, and the graph amalgam $H_{n,N} \oplus_{\delta_1, \delta_2} H_{n,N'}$ over the base $\{\delta_1, \delta_2\} = \{\delta'_1, \delta'_2\}$. Throughout the proof, when we amalgamate it will always be over these bases. By Lemma 5.5, if the class of Γ -gain-graphic matroids is CMSO_1 -definable, there is a finite partition of ω such that for any N, N' , whether $\text{Amal}(F(H_N), F(H_{N'}))$ is a Γ -gain-graphic matroid depends only on which classes in the partition N and N' belong to. Thus, for a contradiction it suffices to show that the amalgam $\text{Amal}(F(H_N), F(H_{N'}))$ is a Γ -gain matroid if and only if $N = N'$ for all $N, N' \geq 1$. There are two directions to this.

First, suppose $\text{Amal}(F(H_N), F(H_{N'}))$ is a Γ -gain-graphic matroid, built from some Γ -gained graph (X, τ_0) , with associated biased graph Ω . Without loss of generality, we may assume X has no isolated vertices. Note

that X has the same edge-set as $Y = H_{n,N} \oplus H_{n,N'}$. We claim that the triple $\Omega_N, \Omega'_N, \Omega$ satisfies the conditions of Lemma 4.5. Certainly no loop edges in H_N or $H_{N'}$ are balanced. The desired unbalanced loops are supplied by $Q_1, \dots, Q_{N+7}, Q'_1, \dots, Q'_{N'+7}$. Next, for the desired unbalanced cycles, note that by assumption each $\mathcal{A}, \mathcal{A}', \mathcal{B}_i, \mathcal{B}'_i, \mathcal{C}, \mathcal{C}'$ contains an unbalanced cycle, since we assumed $\{a_i\}$ contains two distinct elements which are not reciprocals of each other. Note also by property (\dagger) in Definition 5.2, each \mathcal{D}_i and \mathcal{D}'_i contains an unbalanced cycle. Finally, by assumption $\text{Amal}(F(H_N), F(H_{N'})) = F(X, \tau_0)$ and X contains no isolated vertices. Thus, applying the lemma, we conclude $X \cong Y$. So, by our assumption, there is a Γ -gaining τ_0 of Y such that $F(Y, \tau_0) = \text{Amal}(F(H_N), F(H_{N'}))$. Let us distinguish the edges in $F(H_N)$ from those in $F(H_{N'})$ by adding a tick to those in $F(H_{N'})$, and doing the same for collections of edges. Thus, $K_1 \in F(H_N)$, $K'_1 \in F(H_{N'})$, and e.g. $C_1 = C'_1$, since these edges lie in the amalgamation base. By Fact 5.4, without loss of generality we may assume $\tau_0(\mathcal{T} \cup \mathcal{T}') = \{\text{Id}\}$. Then, let $\tau = \tau_0|_{H_{n,N}}$, $\tau' = \tau_0|_{H_{n,N'}}$ be the two natural restrictions of τ_0 . By switching, we can assume that $\tau(\mathcal{C}_{\text{Id}}) = \text{Id}$. Let h_s stand for the element $\tau(\mathcal{C}_s)$, and let h_j stand for the element $\tau(\mathcal{C}_j)$ for all j . For each $j \in \{1, \dots, n\}$, by considering the balanced cycle $C_j^{-1} \delta_1 \beta_1 \alpha_1 A_j \alpha_2 \beta_2 \delta_2$ (where we omit an edge if it is in \mathcal{T}), we must have $\tau(A_j) = \tau(C_j) = h_j$. Similarly, we must have $\tau(A_s) = h_s$. Next, observe that since we have the balanced cycle

$$\alpha_1 \alpha_2 \beta_2 \gamma_{N+1} \gamma_N \cdots \gamma_1 K_2^{-1} \beta_1 K_1,$$

we must have $\tau(K_1) = \tau(K_2)$. For each $1 \leq i \leq N$ and each $1 \leq j \leq n$, we consider the balanced cycle

$$\alpha_1 A_i \alpha_2 \beta_2 \gamma_{N+1} \gamma_N \cdots \gamma_{i+1} B_{i,j}^{-1} \gamma_i \gamma_{i-1} \cdots \gamma_1 K_2^{-1} \beta_1 K_1 \alpha_1,$$

and conclude $\tau(B_{i,j}) = \tau(A_j) = h_j$. Now, let $\eta : \{1, \dots, N\} \rightarrow \{h_1, \dots, h_n\} \cup \{\text{Id}\}$ be a string of length at most N in the h_i . There is a path P_η from γ_1 to γ_{N+1} such that for each k , the edge P_η takes between γ_k and γ_{k+1} is mapped to $\eta(k)$ by τ . Let C_η be the cycle obtained from P_η by appending $\gamma_{N+1} \beta_2 \alpha_2 A_s^{-1} \alpha_1 K_1^{-1} \beta_1 K_2 \gamma_1$. Thus, $\tau(C_\eta) = \tau(P_\eta) h_s^{-1}$. Since $f_{\{a_i\}}(s) = N$, there exists η a string of length N such that $\sigma_N(C_\eta) = \text{Id}$, but not one of smaller length. So there exists η of length N such that $\tau(C_\eta) = \text{Id}$ but not one of smaller length. In other words, $f_{\{h_i\}}(h_s) = N$. Symmetrically, if we let $\tau'(C_j) = h'_j$ for all j , and $\tau'(C_s) = h'_s$, then $f_{\{h'_i\}}(h'_s) = N'$. But the edges in \mathcal{C} are all in the amalgamation base, so τ and τ' agree on \mathcal{C} , so

$$N = f_{\{h_i\}}(h_s) = f_{\{h'_i\}}(h'_s) = N'.$$

Now we have shown the ‘only if’ direction.

We prove the other direction. Fix some $N \geq 1$. We must show that $\text{Amal}(F(H_N), F(H_N))$ is a Γ -gain-graphic matroid. As above, let us distinguish the two copies of $H_{n,N}$ by adding a tick to the names of the components of the second copy. Thus, $K'_1 \in H_{n,N'}$, $K_1 \notin H_{n,N}$. Let σ'_N be the gaining

σ_N , applied to $H_{n,N'}$. Note that σ_N and σ'_N agree on the amalgamation base.

Let Y be $H_{n,N} \oplus H_{n,N'}$. Our goal is to use Lemma 4.4 to construct a Γ -gaining of Y such that the corresponding frame matroid is equal to $\text{Amal}(F(H_N), F(H_{N'}))$. We observe that we cannot simply use $(H_{n,N} \oplus_{\delta_1, \delta_2} H_{n,N'}, \sigma_N \cup \sigma'_N)$, as for instance the cycle

$$\delta_1 \beta_1 K_1 \alpha_1 \alpha_2 \beta_2 \delta_2 \beta'_2 \alpha'_2 \alpha'_1 (K'_1)^{-1} \beta'_1 \delta_1$$

(where, as before, we omit an edge if it is in $\mathcal{T} \cup \mathcal{T}'$) would be balanced, whereas that cycle is not balanced in the matroid amalgam. Thus, we need our gaining to have different values on K_1 and K'_1 .

Choose some $\mathfrak{M}' \in \Gamma$ such that $f_{\{a_i\}}(\mathfrak{M}') = 4N + 2$. Let us define a gaining $\tau^{*'} of $H_{n,N'}$ which operates as follows:$

$$(A, u, v) \mapsto \begin{cases} \sigma'_N(A, u, v) : A \in H_{n,N'}, A \notin \{K'_1, K'_2\} \\ \mathfrak{M}' : (A, u, v) \in \{(K'_1, \beta'_1, \alpha'_1), (K'_2, \beta'_1, \gamma'_1)\} \\ (\mathfrak{M}')^{-1} : (A, u, v) \in \{(K'_1, \alpha'_1, \beta'_1), (K'_2, \gamma'_1, \beta'_1)\} \end{cases}$$

Then, let us extend this to τ' a Γ -gaining of $H_{n,N'}$ by first letting $\tau(Q'_i) = \sigma_N(Q_i)$ for $i = N+6, N+7$, and then choosing the remaining values of $\tau(Q'_i)$ such that no Q'_i is a balanced loop, and then choosing values of $\tau(D'_{i,j})$ such that no cycle containing a $D'_{i,j}$ is balanced. We can do this since the group is infinite. Note τ' agrees with σ_N on $\ell_N = \ell_{N'}$.

We claim $F(H_{n,N'}, \tau') = F(H_{n,N'}, \sigma'_N)$. Note both sides agree that any cycle containing a $D'_{i,j}$ or a Q'_i is unbalanced. So the problem reduces to showing that

$$F(H_{n,N'}, \tau^{*'}) = F(H_{n,N'}, \sigma^{*'}_N).$$

Note that $\tau^{*'} disagrees with $\sigma^{*'}_N$ only on K'_1 and K'_2 . Further, since K'_1 and K'_2 are incident with a common vertex, and$

$$\tau^{*'}((K'_1)^{-1} K'_2) = \sigma^{*'}_N((K'_1)^{-1} K'_2) = \text{Id},$$

the only cycles they could possibly disagree on are those containing exactly one of K'_1 and K'_2 . But for any such cycle, both $\tau^{*'} and $\sigma^{*'}_N$ agree that it is unbalanced. So as desired$

$$F(H_{n,N'}, \tau') = F(H_{n,N'}, \sigma'_N).$$

Now that we have this identity, we want to show $F(H_{n,N} \oplus_{\ell_N} H_{n,N'}, \sigma_N \cup \tau')$ is the matroid $\text{Amal}(F(H_N), F(H_{N'})) = \text{Amal}(F(H_N), F(H_{n,N'}, \tau'))$, by Lemma 4.4. We must prove that this data satisfies its conditions. Note (i), (ii) are immediate from the definition of τ' , and (iii) is satisfied by Q_{N+6} and Q_{N+7} , so there is only one condition of Lemma 4.4 remaining: to show that for any paths $P \subseteq H_{n,N}$, $P' \subseteq H_{n,N'}$, both starting at δ_1 and ending at δ_2 , and $\sigma_N(P) = \tau'(P')$, there is some edge E from δ_1 to δ_2 such that $\sigma_N(E) = \sigma_N(P) = \tau'(P')$. Fix such P, P' . Their union is a balanced cycle, so they must lie in $H_{n,N}^*$ and $H_{n,N'}^*$, respectively. Then, in particular,

$f_{\{a_i\}}(P) = f_{\{a_i\}}(P')$. By inspection $f_{\{a_i\}}(P) \leq 3N + 1$. If P' passes through K'_1 or K'_2 , $f_{\{a_i\}}(P') \geq 3N + 2$. Therefore, P' passes through neither K'_1 nor K'_2 . Hence, $\tau'(P') \in \{\text{Id}, a_1, \dots, a_n, s\}$, and so there is some edge $E \in \mathcal{A}$ such that $\sigma_N(E) = \sigma_N(P) = \tau'(P')$. The conditions of the Lemma are satisfied, so $\text{Amal}(F(H_N), F(H_{N'})) = F(H_{n,N} \oplus_{\ell_N} H_{n,N'}, \sigma_N \cup \tau')$ is a Γ -gain graphic matroid. \square

In the next result, we show how Theorem 5.7 follows from the locally finite case (Theorem 5.6) by using ultraproducts.

Theorem 5.7. *Let Γ be an infinite group that is not uniformly locally finite. The class of Γ -gain-graphic matroids is not CMSO_1 -definable.*

Proof. Assume Γ is not uniformly locally finite. Let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} , and let $\Gamma^{\mathcal{U}}$ be the ultrapower of Γ . Corollary 2.10 says that $\Gamma^{\mathcal{U}}$ is not locally finite. Therefore Theorem 5.6 implies that the class of $\Gamma^{\mathcal{U}}$ -gain-graphic matroids is not CMSO_1 -definable. Corollary 2.9 now tells us that the class of Γ -gain-graphic matroids is not CMSO_1 -definable. \square

6. THE CONVIVIALITY GRAPH

Recall that a monomorphism is an injective homomorphism.

Definition 6.1 (Conviviality graph). We fix the finite group F . Let H be an infinite group which has a copy of F as a subgroup. Let \mathcal{H} be the set of all pairs (Γ, ψ) , where

- (i) Γ is a finite group,
- (ii) there exists a monomorphism from Γ into H , and
- (iii) ψ is a monomorphism from F into Γ .

Define the equivalence relation \sim on \mathcal{H} such that $(\Gamma_1, \psi_1) \sim (\Gamma_2, \psi_2)$ if and only if there exists an isomorphism $\theta: \Gamma_1 \rightarrow \Gamma_2$ such that $\psi_2 = \theta \circ \psi_1$. Say that (Γ_1, ψ_1) and (Γ_2, ψ_2) are representatives of two equivalence classes. If there are monomorphisms $\theta_1: \Gamma_1 \rightarrow H$ and $\theta_2: \Gamma_2 \rightarrow H$ such that $\theta_1 \circ \psi_1 = \theta_2 \circ \psi_2$, then we say that (Γ_1, ψ_1) and (Γ_2, ψ_2) are F -convivial in H . It is easy to see that the choice of representatives does not change whether the pair is F -convivial, so we can think of conviviality as being a relation on equivalence classes. Note that every equivalence class is convivial with itself, since if $(\Gamma_1, \psi_1) \sim (\Gamma_2, \psi_2)$, then there is an isomorphism $\theta: \Gamma_1 \rightarrow \Gamma_2$ witnessing this, and then for any θ_2 , a monomorphism of Γ_2 into H , $\theta_2 \circ \theta$ is a monomorphism of Γ_1 into H , and $\theta_2 \circ \theta \circ \psi_1 = \theta_2 \circ \psi_2$, by the definition of θ . The *elementary F -conviviality graph* of Γ has the set of equivalence classes \mathcal{H}/\sim as its vertex-set, where (Γ_1, ψ_1) and (Γ_2, ψ_2) are adjacent if and only if they are F -convivial.

We now define the equivalence relation \approx on the vertices of the elementary conviviality graph so that two vertices are equivalent if they have exactly the same neighbours. Note that this requires that the vertices are adjacent since every vertex is self-adjacent. Now the *F -conviviality graph* of H has

the equivalence classes of \approx as its vertices. Two equivalence classes are adjacent in the conviviality graph if and only if representative vertices from those classes are adjacent.

Definition 6.2. Given finite groups $\Gamma_1 \leq \Gamma_2$, the graph $\Lambda_{\Gamma_1, \Gamma_2}^*$ is defined by Figure 3.

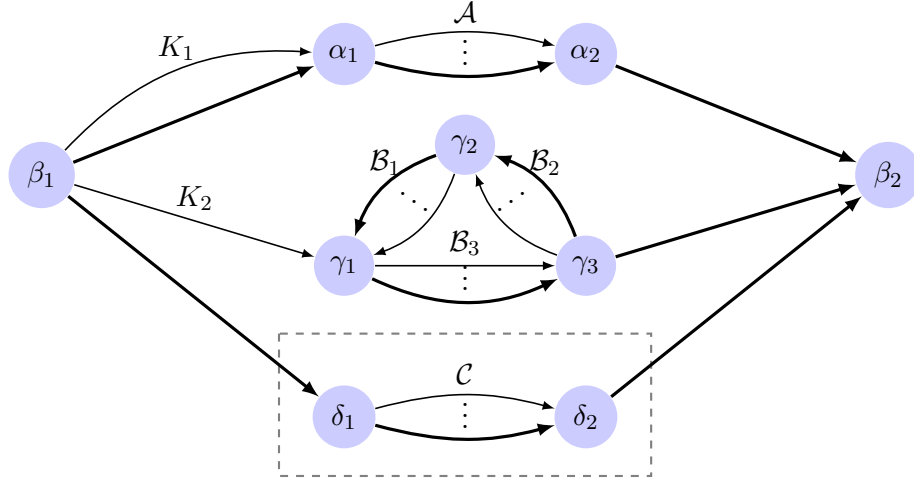


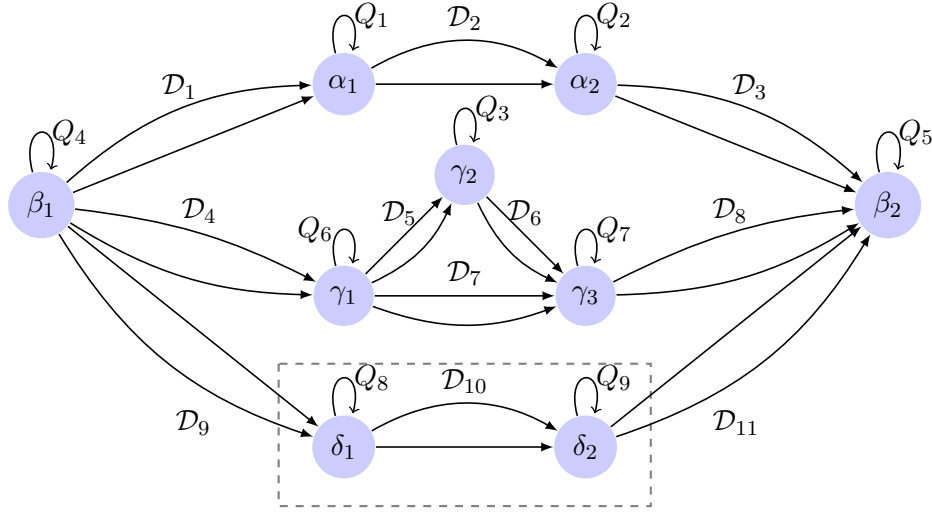
FIGURE 3. The graph $\Lambda_{\Gamma_1, \Gamma_2}^*$

For ease of notation, we will refer to vertices by lower-case Greek letters, and edges by upper-case Roman letters. Let $\ell_{\Gamma_1}^*$ be the edge set of the subgraph enclosed by the dotted line (we will see its structure depends only on Γ_1). Let \mathcal{T} be the collection of bolded edges. Note that each of $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and \mathcal{C} are collections of edges:

$$\begin{aligned}\mathcal{A} &= \{A_g : g \in \Gamma_1\} \\ \mathcal{B}_i &= \{B_{i,g} : g \in \Gamma_2\} : i = 1, 2, 3 \\ \mathcal{C} &= \{C_g : g \in \Gamma_1\}\end{aligned}$$

\mathcal{T} has a single element in each of these: the elements $A_{\text{Id}}, B_{1,\text{Id}}, B_{2,\text{Id}}, B_{3,\text{Id}}$, and C_{Id} . Therefore, we have added a single bold line in each collection in the diagram.

Next, we define $\Lambda_{\Gamma_1, \Gamma_2}$ as the union of $\Lambda_{\Gamma_1, \Gamma_2}^*$ and the following new edges:

FIGURE 4. The graph $\Lambda_{\Gamma_1, \Gamma_2} \setminus \Lambda_{\Gamma_1, \Gamma_2}^*$

Each $\mathcal{D}_i : 1 \leq i \leq 11$ is a collection of two edges: $\mathcal{D}_i = \{D_{i,1}, D_{i,2}\}$. Each Q_i is a single loop edge. Let $\ell_{\Gamma_1} = \ell_{\Gamma_1}^* \cup \{Q_8, Q_9\} \cup \mathcal{D}_{10}$. We will use these new edges at only one point in the argument.

Suppose we have an infinite group Γ_3 containing Γ_2 , and \mathfrak{M} some element in Γ_3 . Then, $\Lambda_{\Gamma_1, \Gamma_2}^*$ has a Γ_2 -gaining $\sigma^* = \sigma^*(\Gamma_1, \Gamma_2, \mathfrak{M})$, defined as follows:

$$\begin{aligned} \sigma^*(T) &= \text{Id} \quad \forall T \in \mathcal{T} \\ \sigma^*(A_g) &= g \quad \forall g \in \Gamma_1 \\ \sigma^*(B_g) &= g \quad \forall g \in \Gamma_2 \\ \sigma^*(C_g) &= g \quad \forall g \in \Gamma_1 \end{aligned}$$

And $\sigma^*(K_1) = \sigma^*(K_2) = \mathfrak{M}$. Note all these edges are oriented as in the diagrams. Secondly, given elements $d_{i,j} : 1 \leq i \leq 11, 1 \leq j \leq 2$, and $q_i : 1 \leq i \leq 9$ of Γ , we can define $\sigma = \sigma(\Gamma_1, \Gamma_2, \mathfrak{M}, \{d_{i,j}\}, \{q_i\})$, a Γ_3 -gaining of $H_{n,N}$ extending σ^* , by

$$\begin{aligned} \sigma(D_{i,j}) &= d_{i,j} \quad \forall 1 \leq i \leq 11, 1 \leq j \leq 2 \\ \sigma(Q_i) &= q_i \quad \forall 1 \leq i \leq 9 \end{aligned}$$

All these edges are also oriented as in the diagrams. We will always assume

- (†) $\mathfrak{M} \notin \Gamma_2$, and
the $\{d_{i,j}\}$, and $\{q_i\}$ are chosen so that
no cycle containing a $D_{i,j}$ or Q_i is balanced

Subject to this assumption, the balanced cycles of σ depend only on the isomorphism type of the pair (Γ_1, Γ_2) .

Note that by Remark 5.3, we can always find such $\{d_{i,j}\}$ and $\{q_i\}$.

Proposition 6.3. *Let Γ be an infinite group, and let Γ_0 be a finite subgroup. Fix two representatives (H_1, ϕ_1) and (H_2, ϕ_2) of vertices of the elementary Γ_0 -conviviality graph. Fix, for each $k = 1, 2$, some data $(\mathfrak{M}^{(k)}, \{d_{i,j}^{(k)}\}, \{q_i^{(k)}\})$ satisfying condition (\dagger) in Definition 6.2 with respect to the pair (Γ_0, H_k) , and let (Λ_k, σ_k) be the gained graph*

$$(\Lambda_{\phi_k(\Gamma_0), H_k}, \sigma(\phi_k(\Gamma_0), H_k, \mathfrak{M}^{(k)}, \{d_{i,j}^{(k)}\}, \{q_i^{(k)}\})).$$

We assume $\ell_{\phi_1(\Gamma_0)} = \ell_{\phi_2(\Gamma_0)}$. Then (H_1, ϕ_1) and (H_2, ϕ_2) are Γ_0 -convivial in Γ if and only if the matroid amalgam of $F(\Lambda_1, \sigma_1)$ and $F(\Lambda_2, \sigma_2)$ over $\ell_{\phi_1(\Gamma_0)}$ is Γ -gainable.

Proof. Fix (H_1, ϕ_1) , (H_2, ϕ_2) , $\mathfrak{M}^{(k)}$, $\{d_{j,k}^{(k)}\}$ and $\{q_j^{(k)}\}$, and (Λ_1, σ_1) , (Λ_2, σ_2) as above. For each $k = 1, 2$, let

$$(\Lambda_k^*, \sigma_k^*) = (\Lambda_{\phi_k(\Gamma_0), H_k}^*, \sigma^*(\phi_k(F), H_k, \mathfrak{M}^{(k)})).$$

As in Theorem 5.6, let us distinguish the edges in Λ_1 and Λ_2 by adding a tick to those edges in Λ_2 . We have assumed that $\ell_{\sigma_1(\Gamma_0)}$ and $\ell_{\sigma_2(\Gamma_0)}$ are equal. In addition, we assume $V(\Lambda_1) \cap V(\Lambda_2) = \{\delta_1, \delta_2\}$. Therefore, for example, $Q_8 = Q'_8$ and $\delta_1 = \delta'_1$. Whenever we gain-graph amalgamate in this proof, it will be over the base $\{\delta_1, \delta_2\}$. Whenever we matroid amalgamate, it will be over the base $\ell_{\phi_1(\Gamma_0)}$, which is the set of edges adjacent only to δ_1 and δ_2 .

The proof has two directions. First, suppose the matroid amalgam is Γ -gain-graphic. Let this be witnessed by a graph X with Γ -gaining τ_0 , without loss of generality with no isolated vertices. As in Theorem 5.6, we wish to apply Lemma 4.5. We note that (Λ_1, σ_1) and (Λ_2, σ_2) have no balanced loop edges. The desired unbalanced loops are supplied by $Q_1, \dots, Q_{10}, Q'_1, \dots, Q'_{10}$. The desired unbalanced cycles are supplied by the $\mathcal{D}_i, \mathcal{D}'_i$. By definition, X has no isolated vertices and

$$F(X, \tau_0) \cong \text{Amal}_{\ell_{\phi_1(\Gamma_0)}}(F(\Lambda_1, \sigma_1), F(\Lambda_2, \sigma_2)).$$

Thus Lemma 4.5 implies $X \cong \Lambda_1 \oplus \Lambda_2$.

Define τ to be the restriction $\tau_0|_{\Lambda_1}$, and let τ' be $\tau_0|_{\Lambda_2}$. By Fact 5.4, we may assume that $\tau_0(\mathcal{T} \cup \mathcal{T}') = \{\text{Id}\}$. For any fixed element $g \in \phi_1(\Gamma_0)$, by considering the balanced cycle $\delta_1 C_g \delta_2 \beta_2 \alpha_2 A_g \alpha_1 \beta_1 \delta_1$ (where we omit an edge if it is in \mathcal{T}), we deduce that $\tau(C_g) = \tau(A_g)$. Similarly, the balanced cycle $\alpha_1 A_{\text{Id}} \alpha_2 \beta_2 \gamma_3 B_{3,\text{Id}} \gamma_1 K_2 \beta_1 K_1$ implies that $\tau(K_1) = \tau(K_2)$, and also the balanced cycle $\alpha_1 A_g \alpha_2 \beta_2 \gamma_3 B_{3,g} \gamma_1 K_2 \beta_1 K_1$ implies that $\tau(B_{3,g}) = \tau(A_g)$. Finally, by considering the list of balanced cycles $\gamma_1 B_{3,g} \gamma_3 B_{2,h} \gamma_2 B_{1,g+h}$, and noting that $\tau(B_{i,\text{Id}}) = \text{Id}$ for all i , we deduce that for all $g \in \phi_1(\Gamma_0)$, and all i, j , $\tau(B_{i,g}) = \tau(B_{j,g})$, and also for all $g, h \in H_1$, $\tau(B_{1,g})\tau(B_{1,h}) = \tau(B_{1,gh})$. Similarly for all $g \in \phi_2(\Gamma_0)$, $\tau'(A'_g) = \tau'(B'_{3,g}) = \tau'(C'_g)$; for all $g \in H_2$, and all i, j , $\tau'(B'_{i,g}) = \tau'(B'_{j,g})$; for all $g, h \in H_2$, $\tau'(B_{i,g})\tau'(B_{i,h}) = \tau'(B_{i,gh})$. For $i = 1, 2$, let $\chi_i : H_i \rightarrow \Gamma$

be the function defined by

$$g \mapsto \begin{cases} \tau(B_g) : i = 1 \\ \tau'(B'_g) : i = 2 \end{cases}$$

By the above results, for each i , χ_i is an embedding of H_i into Γ , and since for all $g \in \Gamma_0$ we have $C_{\phi_1(g)} = C'_{\phi_2(g)}$, it follows that $\chi_1 \circ \phi_1 = \chi_2 \circ \phi_2$. Thus (H_1, ϕ_1) and (H_2, ϕ_2) are Γ_0 -convivial in Γ .

Now we prove the other direction. Let (H_1, ϕ_1) and (H_2, ϕ_2) be Γ_0 -convivial in Γ , witnessed by embeddings $\chi_i : H_i \rightarrow \Gamma$ such that $\chi_1 \circ \phi_1 = \chi_2 \circ \phi_2$. For ease of notation, let us identify H_i with its image under χ_i for each i , so now each H_i is a specific subgroup of Γ . Note that since $\chi_1 \circ \phi_1 = \chi_2 \circ \phi_2$, this is consistent with our assumption $\ell_{\phi_1(\Gamma_0)} = \ell_{\phi_2(\Gamma_0)}$. We will define a Γ -gaining τ_0 on $\Lambda_1 \oplus \Lambda_2$ such that $F(\Lambda_1 \oplus \Lambda_2, \tau_0) \cong \text{Amal}(F(\Lambda_1, \sigma_1), F(\Lambda_2, \sigma_2))$.

As in Theorem 5.6, we need to change the gaining of a few edges to make sure there are no ‘unintentional’ balanced cycles.

Fix some $\mathfrak{M}'_1 \in \Gamma \setminus \langle H_1, H_2 \rangle$, $\mathfrak{M}'_2 \in \Gamma \setminus \langle H_1, H_2, \mathfrak{M}'_1 \rangle$. Let

$$\Delta_+ = \{(\gamma_1, \gamma_2), (\gamma_2, \gamma_3), (\gamma_3, \gamma_1)\}.$$

Then define a partial gaining τ^* which maps (E, u, v) to the following elements:

$$\begin{cases} \text{Id} : E \in \mathcal{T} \\ g : E \in \{B_{1,g}, B_{2,g}, B_{3,g}\} \text{ for some } g \in H_1, (u, v) \in \Delta_+ \\ \phi_1(g) : (E, u, v) \in \{(A_{\phi_1(g)}, \alpha_1, \alpha_2), (C_{\phi_1(g)}, \delta_1, \delta_2)\} \text{ for some } g \in \Gamma_0 \\ \mathfrak{M}'_1 : (E, u, v) \in \{(K_1, \beta_1, \alpha_1), (K_2, \beta_1, \gamma_1)\} \end{cases}$$

We extend it to a full gaining using the rule $\tau^*(E, u, v) = \tau^*(E, v, u)^{-1}$. Let $\tau^{*'}$ be defined symmetrically, with $\tau^{*'}(K'_1) = \tau^{*'}(K'_2) = \mathfrak{M}'_2$. Note they agree on $\ell_{\phi_1(\Gamma_0)}$, the amalgamation base. Thus, we can define $\tau_0^* = \tau^* \cup \tau^{*'}$. Next, we define τ_0 by extending τ_0^* such that any cycle containing any $D_{i,j}, D'_{i,j}, Q_i$ or Q'_i is unbalanced. We can do this since Γ is infinite. Now we must show $F(\Lambda_1 \oplus \Lambda_2, \tau_0)$ is as desired.

Claim 6.3.1. $F(\Lambda_1, \tau_0|_{\Lambda_1}) \cong F(\Lambda_1, \sigma_1)$, and similarly for Λ_2

Proof. By symmetry, it suffices to prove the result for Λ_1 . Note that the LHS and RHS both agree that any cycle containing any $D_{i,j}$ or Q_i is unbalanced, so it suffices to check $F(\Lambda_1^*, \tau_0|_{\Lambda_1^*}) \cong F(\Lambda_1^*, \sigma_1^*)$. But note that $\tau_0|_{\Lambda_1^*} = \sigma^*(\phi_1(\Gamma_0), H_1, \mathfrak{M}'_1)$. Since $\mathfrak{M}'_1 \notin H_1$, the balanced cycles induced by this gaining depend only on the isomorphism type of $(\phi_1(\Gamma_0), H_1)$ (as noted in Definition 6.2). Thus, since $(\phi_1(\Gamma_0), H_1) \cong (\phi_1(\Gamma_0), H_1)$, and $\sigma_1^* = \sigma^*(\phi_1(\Gamma_0), H_1, \mathfrak{M}^{(1)})$ we have the desired isomorphism. \square

This implies

$$\text{Amal}(F(\Lambda_1, \sigma_1), F(\Lambda_2, \sigma_2)) \cong \text{Amal}(F(\Lambda_1, \tau_0|_{\Lambda_1}), F(\Lambda_2, \tau_0|_{\Lambda_2}))$$

Now, we aim to apply Lemma 4.4, with $\Omega_i = (\Lambda_1, \tau_0|_{\Lambda_i})$ for $i = 1, 2$. The conclusion of that lemma would imply

$$\text{Amal}(F(\Lambda_1, \tau_0|_{\Lambda_1}), F(\Lambda_2, \tau_0|_{\Lambda_2})) \cong F(\Lambda_1 \oplus \Lambda_2, \tau_0),$$

proving the proposition. So it suffices to check that for any two paths $P_1 \subseteq \Lambda_1$, $P_2 \subseteq \Lambda_2$, each going from δ_1 to δ_2 , if $P_1 \cup P_2$ is balanced, then there is an edge E going from δ_1 to δ_2 such that $\tau_0(P_1) = \tau_0(P_2) = \tau_0(E)$. Fix some such P_1, P_2 . Note that if either contains any $D_{i,j}, D'_{i,j}, Q_i$ or Q'_i , then $P_1 \cup P_2$ is automatically unbalanced, which would yield a contradiction. Since $\mathfrak{M}'_2 \notin \langle H_1, H_2, \mathfrak{M}'_1 \rangle$, by the same reasoning we know $K'_1, K'_2 \notin P_2$. Similarly $K_1, K_2 \notin P_1$. By inspecting Definition 6.2, we deduce that $\tau_0(P_1) \in \phi_1(\Gamma_0)$, since it can only consist of an element of \mathcal{C} , or an element of \mathcal{A} and some elements of \mathcal{T} . Let $\tau_0(P_1) = \phi_1(g)$. Then $\tau_0(P_1) = \tau_0(P_2) = \tau_0(C_{\phi_1(g)})$, as desired. Thus, we have satisfied the conditions of the lemma, so $\text{Amal}(F(\Lambda_1, \tau_0|_{\Lambda_1}), F(\Lambda_2, \tau_0|_{\Lambda_2})) \cong F(\Lambda_1 \oplus \Lambda_2, \tau_0)$, which concludes the proof. \square

Theorem 6.4. *Let Γ be a group. If Γ has a finite subgroup Γ_0 such that the Γ_0 -conviviality graph of Γ is infinite, then the class of Γ -gain-graphic matroids is not CMSO_1 -definable.*

Proof. Note that if Γ is finite, then for any $\Gamma_0 \leq \Gamma$, the Γ_0 -conviviality graph of Γ must be finite, so the theorem is vacuously true. Therefore, suppose Γ is infinite. Fix the pairs $(H_1, \phi_1), (H_2, \phi_2)$, representatives of vertices in the elementary Γ_0 -conviviality graph of Γ . Also fix, for each $k = 1, 2$, some data

$$(\mathfrak{M}^{(k)}, \{d_{i,j}^{(k)}\}, \{q_i^{(k)}\})$$

satisfying condition (\dagger) in Definition 6.2 with respect to the pair (Γ_0, H_k) , and let

$$(\Lambda_k, \sigma_k) = (\Lambda_{\phi_k(\Gamma_0), H_k}, \sigma(\phi_k(\Gamma_0), H_k, \mathfrak{M}^{(k)}, \{d_{i,j}^{(k)}\}, \{q_i^{(k)}\})).$$

Then, by Proposition 6.3, the amalgam $\text{Amal}(F(\Lambda_1, \sigma_1), F(\Lambda_2, \sigma_2))$ is Γ -gain-graphic if and only if $(H_1, \phi_1), (H_2, \phi_2)$ are Γ_0 -convivial. Note by Lemma 5.5 if the class of Γ -gain-graphic matroids is CMSO_1 -definable, there is a finite partition of $\{(H, \phi) \text{ in the elementary } \Gamma_0\text{-conviviality graph of } \Gamma\}$ such that whether $\text{Amal}(F(\Lambda_1, \sigma_1), F(\Lambda_2, \sigma_2))$ is Γ -gain-graphic depends only on which classes (H_1, ϕ_1) and (H_2, ϕ_2) are in. But then, by the Proposition, whether (H_1, ϕ_1) and (H_2, ϕ_2) are Γ_0 -convivial would depend only on which classes they are in, which would imply the Γ_0 -conviviality graph is finite. \square

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