MATH 111, Assignment 3 (due 09/14/21)

MATH 111, Assignment 3 Solutions

Name: Dillon Allen, Julio Hernandez **Date:** 09/11/2021
Group Name: Group 2

1. (50 points) Negate the following statements. Give the statement *and* its negation in both symbolic form, then also give negation in the form of a sentence. Finally, write the converse of each statement when possible.
   (i) The numbers x and y are both odd.
   Solution: The statement can be written in $P \wedge Q$ in symbolic form. Thus, the statement reads $(x \text{ is odd}) \wedge (y \text{ is odd})$. The negation of the symbolic form is

$$\neg(x \text{ is odd}) \wedge (y \text{ is odd}) = (x \text{ is even}) \vee (y \text{ is even})$$

   The negation in word form can be described as, The number x is even or the number y is even.
   (ii) Let ABC be an equilateral triangle or a right triangle.
   Solution: P: ABC is an equilateral triangle
   Q: ABC is a right triangle.
   This statement is of the form $P \vee Q$, so the negation of this is

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

   The negation in sentence form is that ABC is not an equilateral triangle and ABC is not a right triangle.
   (iii) $\sqrt{2}$ is a real number that is not rational.
   Solution: P: $\sqrt{2} \in \mathbb{R}$
   Q: $\sqrt{2} \notin \mathbb{Q}$
   Symbolically, this is an and statement that is of the form $P \wedge Q$. The negation of this is $\neg(P \wedge Q) = \neg P \vee \neg Q$. In a sentence, this reads as $\sqrt{2}$ is not a real number or $\sqrt{2}$ is a rational number.
   (iv) If x is a real number then $x^2 + 1 > 0$.
   Solution: P: $x \in \mathbb{R}$
   Q: $x^2 + 1 > 0$
   Symbolically this is a $P \implies Q$ statement. The negation of this is is $\neg(P \implies Q) = P \wedge \neg Q$. In a symbolic form the negation this reads as $(x \in \mathbb{R}) \wedge (x^2 + 1 \leq 0)$. In a sentence the negation reads as x is a real number and one added to the square of x is less than or equal to 0. The converse of this statement would be that if $x^2 + 1 > 0$ then x is a real number.
   (v) The number x is positive, but the number y is not positive.

Solution:The original statement in symbolic form is $(x > 0) \wedge (y < 0)$. The negation of this statement is

$$\begin{aligned}\neg((x > 0) \wedge (y < 0)) &= \neg(x > 0) \vee \neg(y < 0) \\ &= (x \leq 0) \vee (y \geq 0)\end{aligned}$$

In sentence form, this statement says that the number x is less than or equal to zero or the number y is greater than or equal to zero.

(vi) If x is a rational number and $x \neq 0$, then $\tan(x)$ is not a rational number.
Solution: The original statement in symbolic form would be

$$(x \in \mathbb{Q}) \wedge (x \neq 0) \implies (\tan(x) \notin \mathbb{Q})$$

. The negation of the following statement would be

$$\begin{gathered}\neg((P \wedge Q) \implies R) \\ (P \wedge Q) \wedge \neg R\end{gathered}$$

Symbolically, the negation says

$$((x \in \mathbb{Q}) \wedge (x \neq 0)) \wedge (\tan(x) \in \mathbb{Q})$$

In a sentence, this negation says that x is a rational number and $x \neq 0$ and $\tan(x)$ is a rational number.
For the converse, the statement would be $R \implies (P \wedge Q)$. In sentence form, this would say if $\tan(x)$ is not a rational number then x is a rational number and $x \neq 0$.

(vii) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.
Solution: Symbolically, this statement reads

$$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R},\ (x \neq y) \implies (x^2 + y^2 > 0)$$

This is a $P \implies Q$ statement, with the negation being $\neg(P \implies Q) = P \wedge \neg Q$. Therefore, the negation of this statement is

$$\begin{aligned}\neg(\forall x \in \mathbb{R}\ \forall y \in \mathbb{R},\ (x \neq y) \implies (x^2 + y^2 > 0)) &= \exists x \in \mathbb{R}\ \exists y \in \mathbb{R},\ \neg((x \neq y) \implies (x^2 + y^2 > 0)) \\ &= \exists x \in \mathbb{R}\ \exists y \in \mathbb{R},\ (x \neq y) \wedge \neg(x^2 + y^2 > 0) \\ &= \exists x \in \mathbb{R}\ \exists y \in \mathbb{R},\ (x \neq y) \wedge (x^2 + y^2 \leq 0)\end{aligned}$$

The negation, in sentence form, says that there exists some real numbers x and y such that if $x \neq y$ then $x^2 + y^2 \leq 0$.
The converse of this statement says that $x^2 + y^2 > 0$ implies that x and y are real numbers where $x \neq y$.

(viii) There exists a rational number whose square is 2.
Solution Symbolically this statement read as

$$\exists x \in Q,\ x^2 = 2.$$

This is a P statement with the negation being $\neg P$

$$\neg(\exists x \in Q, x^2 = 2)$$
$$(\forall x \in Q, x^2 \neq 2)$$

This negation in as a sentence means says that for all $x \in \mathbb{Q}$ there is no Q that's $x^2 \neq 2$

(ix) If the set B is contained in the set A, then B without A is nonempty.
Solution:
P: The set B is contained in set A.
Q: B without A is nonempty.
$P \implies Q$ statement. Therefore, the negation of this would be

$$\neg(P \implies Q) = P \wedge \neg Q$$

In a sentence, this negation reads as: The set B is contained in set A and B without A is empty.
The converse of this sentence is if set B without A is nonempty, then the set B is contained in the set A.

(x) For every prime number p, either p is odd or p is 2.
Solution: Symbolically, the statement can be written as

$$\text{for all primes } p,\ (p \text{ odd}) \vee (p = 2)$$

The statement is of the form $\forall x, P \vee Q$ so the general negation is $\exists x, \neg P \wedge \neg Q$. So, the negation of this statement is

$$\text{There exists some prime } p,\ \neg(p \text{ odd}) \wedge \neg(p = 2)$$

$$\text{There exists some prime } p, \text{ such that },\ (p \text{ even}) \wedge (p \neq 2)$$

In sentence form, there exists a prime p such that p is even and p is not equal to 2.

2. (30 points) Write each of the following as English sentences. Say whether they are true or false. Prove your claim (this may need examples and/or counterexamples)

(i) $(\forall x \in \mathbb{N})(x \geq 1)$.
Solution: For all natural numbers x, such that x is greater than or equal to 1
We know this to be true as natural numbers are defined as $\mathbb{N} = \{1, 2, 3, \dots\}$. Therefore if we were to take the number 2 as an example, we get $2 \geq 1$. Which we know satisfies our condition. The lowest number we could use would be 1 and if we plug that in we get $1 \geq 1$. By this we know that all natural numbers will satisfy our condition.

(ii) $(\exists! x \in \mathbb{R})(x \geq 0 \wedge x \leq 0)$.
Solution: There is exactly one number such that x is greater than or equal to 0 and also less than or equal to 0. This is a true statement for only the number 0.

Proof. We will prove this by taking 3 cases, for $x > 0$, $x = 0$, and $x < 0$.
Case 1: Taking $x > 0$, let $x = 5$. $5 \geq 0$, which is true, but $5 \leq 0$ is false. Therefore, the logical statement $P \wedge Q$ is false.
Case 2: Let $x = 0$. $0 \geq 0$ is true and $0 \leq 0$ is also true. So, logically, $T \wedge T = T$. This is a true statement.

Case 3: Taking $x < 0$, let $x = -2$. $-2 \geq 0$ is false, whereas $-2 \leq 0$ is true. $F \wedge T = F$, therefore this case is false.
Therefore, the only case that satisfies this claim is when $x = 0$. $\square$

(iii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0$.
Solution:For all real numbers x, there is some number y such that their sum is zero. This is a true statement. For every value x, we have the equation $x = -y$. So no matter what value of x we have, there will exist a y value that is the additive inverse of x.

(iv) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 0$.
Solution: For exists some real number x such that for all real numbers y, their sum is zero. This statement is false, because you can not have a value for x that is the additive inverse for all numbers y.

(v) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x \geq y$.
Solution: There is a real number x that is bigger than all real numbers y. This is a true statement, which has to do with the infinities of the real numbers. I can always find some number bigger than the previous ones suggested. Whenever you can guess a number x, I can always guess 10^x. This argument can continue building until we exhaust all numbers, but will come up with new real numbers in the process.

3. (20 points) Translate each of the following sentences into symbolic logic.
 (i) Even numbers are divisible by 2.
 Solution: Symbolically, this can be rewritten into a $P \implies Q$ conditional with:
 P: n is an even number.
 Q: n is divisible by 2.
 Therefore, we have $P \implies Q$ rewriting the statement as "if n is an even number then it is divisible by 2".
 (ii) If f is a polynomial, f' is constant whenever the degree of f is less than 2.
 Solution: Let $p(x)$ represent all polynomial functions and deg(f) represent the degree of some function f. Then
 $$(\forall f \in p(x)),\ (\deg(f) < 2) \implies (f' = \text{constant})$$
 (iii) If x is prime, then $\sqrt{x}$ is not a rational number.
 Solution: $x \in \{\text{primes}\} \implies \sqrt{x} \notin \mathbb{Q}$
 (iv) Every set is contained in itself.
 Solution: P: A is a set.
 Q: A is a subset of itself.
 $P \implies Q$ says that if A is a set then A is a subset of itself.
 (v) There exist two integers whose product is negative, but whose sum is positive.
 Solution:
 This is a $P \wedge Q$ statement.
 $(\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}),\ (x * y < 0) \wedge (x + y > 0)$

Extra Credit: Write each of the following as English sentences. Say whether they are true or false. Prove your claim (this may need examples and/or counterexamples)

(i) $(\forall x \in \mathbb{N})(x \text{ is prime} \wedge x \neq 2 \implies x \text{ is odd})$.

(ii) $(\exists ! x \in \mathbb{R})(\log_e x = 1)$.
(iii) $\forall x \in \mathbb{R}, x + 2 > x$.
(iv) $\exists x \in \mathbb{N}, 2x + 3 \geq 6x + 7$.