ADCS Final

Dillon Allen

Question 1

1.1 For 2-1-2 Euler Transform with angles θ_1, θ_2, θ_3, what angles, if any, are undefined?

Ans: $\theta_2 = 0, \pi$ makes the second rotation an identity matrix and makes identifying the other angles impossible.

1.2 Spacecraft is halfway between Earth and Mars. What sources of disturbance do we need to consider?

Since we are not nearby any planets and outside of any sources of magnetic fields, our disturbance force will be SRP.

1.3 Use at least 2 criteria to show that [A] is not a valid DCM

$$[A] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

We will use two criteria, det(A) = +1 and we will check to see if the only real eigenvalue is +1.

```
A = [-1 0 0; 0 sqrt(3)/2 1/2; 0 -1/2 sqrt(3)/2];

% check one, does detA = +/- 1?
detA = det(A);
if(detA == 1)
    fprintf("detA = +1, valid DCM")
else
    fprintf("Not a valid determinant value, %f. Not a DCM", detA);
end
```

```
Not a valid determinant value, -1.000000. Not a DCM
```

```
% Check via inspection
evalsA = eig(A)
```

```
evalsA = 3×1 complex
   0.8660 + 0.5000i
   0.8660 - 0.5000i
  -1.0000 + 0.0000i
```

```
for i = 1:3
    if isreal(evalsA(i))
        if evalsA(i) == 1
            frpintf("valid e-val")
        else
            fprintf("not valid e-val: %f", evalsA(i))
        end
    end
```

```
end
```

```
not valid e-val: -1.000000
```

Problem 1.4 State any two conditions under which this vector equation is valid

$$\dot{\omega} + \omega \times [I]\omega = 0$$

1. $\omega = 0$
2. ω is constant and $I_x = I_y = I_z$

Problem 1.5

We would place the torque rods along each axis. Although the magnetic field may pass through an axis, if the spacecraft is rotating itll trade off which wheel is being de-saturated.

Question 2

$$[I] = \begin{pmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Problem 2.1 What are the principal moments of inertia?

```
I = [6 -2 0; -2 6 0; 0 0 7]
```

```
I = 3×3
     6    -2     0
    -2     6     0
     0     0     7
```

```
[V,E] = eig(I)
```

```
V = 3×3
   -0.7071         0   -0.7071
   -0.7071         0    0.7071
         0    1.0000         0
E = 3×3
     4     0     0
     0     7     0
     0     0     8
```

```
Ip = V' * I * V
```

```
Ip = 3×3
    4.0000         0         0
         0    7.0000         0
         0         0    8.0000
```

```
% Confirming that we get the initial MOI matrix when working backwards
I_old = inv(V')*Ip*inv(V)
```

```
I_old = 3×3
     6    -2     0
    -2     6     0
```

```
     0     0     7
```

Problem 2.2 Is the spacecraft stable in simple spin about the Z axis?

Yes, $I_z > I_x$ and $I_z > I_y$.

Problem 2.3 How does the answer change if the spacecraft is quasi-rigid?

The spin is still stable along the Z-axis, as $I_z > I_x$.

Problem 2.4 Is it stable about the other two axes if the spacecraft is quasi-rigid?

No, it will nutate until it spins across its major axis.

Problem 3

Given a 3-2-1 Inertial to Body Euler Angles and their Rates

$$\psi = 10^{\circ} \ \ \theta = -15^{\circ} \ \ \phi = 20^{\circ}$$

$$\dot{\psi} = 1^{\circ}/s \ \ \dot{\theta} = 1^{\circ}/s \ \ \dot{\phi} = 0^{\circ}/s$$

Problem 3.1 Find the Angular Velocity in the Body Frame

We will use the following equation from the notes for a 3-2-1 body rates (Final Review, Slide 22)

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + [A_\theta] \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + [A_\theta][A_\psi] \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

```
psi       = deg2rad(10);
theta     = deg2rad(-15);
phi       = deg2rad(20);
psi_dot   = deg2rad(1);
theta_dot = deg2rad(1);
phi_dot   = 0;

A_psi = [cos(psi) sin(psi) 0; -sin(psi) cos(psi) 0; 0 0 1];
A_theta = [cos(theta) 0 -sin(theta); 0 1 0; -sin(theta) 0 cos(theta)];
A_phi = [1 0 0; 0 cos(phi) sin(phi); 0 -sin(phi) cos(phi)];

omega_br = [phi_dot; 0; 0] + A_theta * [0; theta_dot; 0] + ...
            A_theta * A_psi * [0; 0; psi_dot];
fprintf("body rates are {%f, %f, %f} (rad/s)", ...
        omega_br(1), omega_br(2), omega_br(3));
```

```
body rates are {0.004517, 0.017453, 0.016859} (rad/s)
```

Problem 3.2 Find the DCM from Body to Inertial Frame

```
DCM_bi = A_phi * A_theta * A_psi
```

```
DCM_bi = 3×3
    0.9513    0.1677    0.2588
   -0.0760    0.9408    0.3304
    0.2989   -0.2946    0.9077
```

Problem 3.3 Find the Angular Velocity in the Inertial Frame

```
omega_i = DCM_bi * omega_br;
fprintf("Inertial Angular Velocities: {%f, %f, %f} (rad/s)", ...
        omega_i(1), omega_i(2), omega_i(3));
```

```
Inertial Angular Velocities: {0.011588, 0.021646, 0.011511} (rad/s)
```

Problem 4

Problem 4.1 Derive the 1-2-3 DCM for angles ϕ, θ, ψ respectively.

$$A_{123} = A_{\phi\theta\psi} = A_\psi A_\theta A_\phi$$

$$A_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & sin(\phi) \\ 0 & -sin(\phi) & cos(\phi) \end{pmatrix}$$

$$A_\theta = \begin{pmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{pmatrix}$$

$$A_\psi = \begin{pmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_{\phi\theta\psi} = \begin{pmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & sin(\phi) \\ 0 & -sin(\phi) & cos(\phi) \end{pmatrix}$$

$$A_{\phi\theta\psi} = \begin{pmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta) & sin(\theta)sin(\phi) & -cos(\phi)sin(\theta) \\ 0 & cos(\phi) & sin(\phi) \\ sin(\theta) & -cos(\theta)sin(\phi) & cos(\theta)cos(\phi) \end{pmatrix}$$

$$A_{\phi\theta\psi} = \begin{pmatrix} cos(\theta)cos(\psi) & cos(\psi)sin(\theta)sin(\phi) + cos(\phi)sin(\psi) & -cos(\phi)cos(\psi)sin(\theta) + sin(\phi)sin(\psi) \\ -cos(\theta)sin(\psi) & cos(\phi)cos(\psi) - sin(\theta)sin(\phi)sin(\psi) & cos(\psi)sin(\phi) + cos(\phi)sin(\theta)sin(\psi) \\ sin(\theta) & -cos(\theta)sin(\phi) & cos(\theta)cos(\phi) \end{pmatrix}$$

Problem 4.2 Show that gimbal lock occurs at $\theta = \dfrac{3\pi}{2}$

$$A_{\phi,\frac{3\pi}{2},\psi} = \begin{pmatrix} 0 & -cos(\psi)sin(\phi) + cos(\phi)sin(\psi) & cos(\phi)cos(\psi) + sin(\phi)sin(\psi) \\ 0 & cos(\phi)cos(\psi) + sin(\phi)sin(\psi) & cos(\psi)sin(\phi) - cos(\phi)sin(\psi) \\ -1 & 0 & 0 \end{pmatrix}$$

From here, we are unable to isolate any of the euler angles, so they have become undefined.

Problem 5

Problem 5.1 Obtain the I2B DCM via the TRIAD Algorithm

```
ui = [0.8273 0.5541 -0.0920]
```

```
ui = 1×3
    0.8273    0.5541   -0.0920
```

```
vi = [-0.8285 0.5522 -0.0955]
```

```
vi = 1×3
   -0.8285    0.5522   -0.0955
```

```
ui_bar = ui / norm(ui)
```

```
ui_bar = 1×3
    0.8273    0.5541   -0.0920
```

```
vi_bar = cross(ui_bar, vi) / norm(cross(ui_bar, vi))
```

```
vi_bar = 1×3
   -0.0023    0.1671    0.9859
```

```
wi_bar = cross(ui_bar, vi_bar) / norm(cross(ui_bar, vi_bar))
```

```
wi_bar = 1×3
    0.5617   -0.8155    0.1395
```

```
ub = [-0.1517 -0.9669 0.2050]
```

```
ub = 1×3
   -0.1517   -0.9669    0.2050
```

```
vb = [-0.8393 0.4494 -0.3044]
```

```
vb = 1×3
   -0.8393    0.4494   -0.3044
```

```
ub_bar = ub / norm(ub)
```

```
ub_bar = 1×3
   -0.1517   -0.9669    0.2050
```

```
vb_bar = cross(ub_bar, vb) / norm(cross(ub_bar, vb))
```

```
vb_bar = 1×3
    0.2177   -0.2350   -0.9473
```

```
wb_bar = cross(ub_bar, vb_bar) / norm(cross(ub_bar, vb_bar))
```

```
wb_bar = 1×3
    0.9641   -0.0991    0.2462
```

```

A_i2b = [ub_bar.' vb_bar.' wb_bar.']*[ui_bar.' vi_bar.' wi_bar.']'
```

```
A_i2b = 3×3
    0.4156   -0.8339    0.3631
   -0.8551   -0.4943   -0.1566
    0.3100   -0.2455   -0.9185
```

Problem 5.2 Using the DCM, Derive the i2b quaternion

```
q4 = 0.5 * sqrt(1 + A_i2b(1,1) + A_i2b(2,2) + A_i2b(3,3))
```

```
q4 = 0.0264
```

```
q1 = 0.25 * (A_i2b(2,3) - A_i2b(3,2)) / q4
```

```
q1 = 0.8409
```

```
q2 = 0.25 * (A_i2b(3,1) - A_i2b(1,3)) / q4
```

```
q2 = -0.5022
```

```
q3 = 0.25 * (A_i2b(1,2) - A_i2b(2,1)) / q4
```

```
q3 = 0.2001
```

```
q_i2b = [q1; q2; q3; q4]
```

```
q_i2b = 4×1
    0.8409
   -0.5022
    0.2001
    0.0264
```

Problem 5.3 Using the DCM, derive the 3-2-1 Euler Angles

```
theta = asin(-A_i2b(1,3));
phi = atan2(A_i2b(2,3),A_i2b(3,3));
psi = atan2(A_i2b(1,2),A_i2b(1,1));
fprintf("phi = %f (deg), theta = %f (deg), psi = %f (deg)",...
        rad2deg(phi), rad2deg(theta), rad2deg(psi));
```

```
phi = -170.326727 (deg), theta = -21.292912 (deg), psi = -63.512360 (deg)
```

Problem 6

Problem 6.1 Derive the PID Gains from

$$I_y\ddot{\theta} = -K_p\theta - K_I\int\theta - K_d\dot{\theta}$$

while comparing to the closed loop characteristic equation

$$\left(s^2 + 2\omega_n\zeta s + \omega_n^2\right)\left(s + \frac{1}{T}\right) = 0$$

First, we shall expand the characteristic equation, and then take the laplace transform of the PID equation to equate coefficients

$$s^3 + \frac{1}{T}s^2 + 2\omega_n\zeta s^2 + \frac{2\omega_n\zeta}{T}s + \omega_n^2 s + \frac{\omega_n^2}{T}$$

Now, taking the Laplace Transform of the PID law, we have

$$I_y s^2\Theta(s) + K_p\Theta(s) + \frac{K_I}{s}\Theta(s) + K_d s\Theta(s) = 0$$

$$I_y\Theta(s)\left(s^3 + \frac{K_p}{I_y}s + \frac{K_I}{I_y} + \frac{K_d}{I_y}s^2\right) = 0$$

Equating, we get

$$K_{iy} = I_y\left(\frac{\omega_n^2}{T}\right),\ K_{py} = I_y\left(\frac{2\omega_n\zeta}{T} + \omega_n^2\right),\ K_{dy} = I_y\left(2\omega_n\zeta + \frac{1}{T}\right)$$

Problem 6.2 Derive the PID gains for a different PID law

This time, the equation for the PID law is

$$I_y\ddot{\theta} = K_{py}\left(\theta + \frac{1}{T_i}\int\theta d\tau + T_d\dot{\theta}\right)$$

We will now follow the same procedure to determine $K_{py},\ T_i,\ T_d$.

We have

$$I_y\Theta(s)\left(s^3 - \frac{K_{py}T_d}{I_y}s^2 - \frac{K_{py}}{I_y}s - \frac{K_{py}}{I_yT_i}\right) = 0$$

equating coefficients, we get

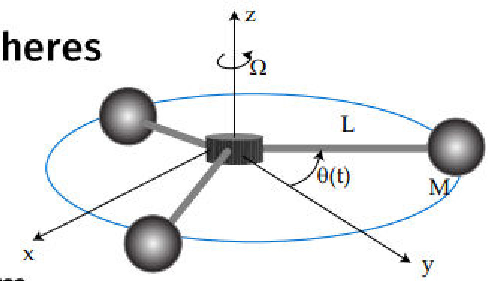
$$K_{py} = -I_y\left(\frac{2\omega_n\zeta}{T} + \omega_n^2\right)$$

$$-\frac{K_{py}T_d}{I_y} = \frac{1}{T} + 2\omega_n\zeta \Rightarrow T_d = \frac{\left(\frac{1}{T} + 2\omega_n\zeta\right)}{\frac{2\omega_n\zeta}{T} + \omega_n^2} \Rightarrow T_d = \frac{1 + 2\omega_n\zeta T}{\omega_n^2 T + 2\omega_n\zeta}$$

$$-\frac{K_{py}}{I_y T_i} = \frac{\omega_n^2}{T} \Rightarrow T_i = -\frac{K_{py}T}{I_y\omega_n^2} \Rightarrow T_i = \frac{2\zeta}{\omega_n} + T$$

Problem 7

- **A design for a space station consists of three spheres of mass M, distributed symmetrically in the X-Y plane around a central hub. Assume that the hub and connecting rods are relatively massless and that the spheres can be treated as point masses. M = 10,000 kg, L = 20 m, Ω=5 rpm.**



Problem 7.1 What is the Inertia about the Z-Axis?

We will use the point mass formulation for MoI

$$I_z = \sum m_i(x_i^2 + y_i^2)$$

WLOG, we can assume M1 is at $\theta = 0$, M2 is at $\theta = \frac{2\pi}{3}$, and M3 is at $\theta = \frac{4\pi}{3}$. Through trigonometry,

$x_i^2 + y_i^2 = L^2$, due to the pythagorean identity with sines and cosines.

```
M       = 100000;       % kg
L       = 20;           % m
Omega   = 5 * pi/30;    % rad/s
Iz      = 3 * M* L^2;   % kg-m^2
fprintf("The MoI about Z is %f kg-m^2", Iz);
```

```
The MoI about Z is 120000000.000000 kg-m^2
```

Problem 7.2 Show that the Inertia about X and Y axes are $1.5ML^2$, irrespective of θ

Since this is in the $X - Y$ plane, we know $z_i = 0$. Thus, for I_x, we have the formula

$$I_x = \sum_i m_i y_i^2$$

and for I_y, we have

$$I_y = \sum_i m_i x_i^2$$

Now, assume we perturb our initial angles by θ'. That means $y_1 = sin\left(\frac{2\pi}{3} + \theta'\right)$, $y_2 = sin\left(\frac{4\pi}{3} + \theta'\right)$,

$y_3 = sin(2\pi + \theta')$

$$I_x = \sum_i m_i y_i^2 = ML^2\left(sin\left(\frac{2\pi}{3} + \theta'\right)^2 + sin\left(\frac{4\pi}{3} + \theta'\right)^2 + sin(2\pi + \theta')^2\right)$$

$$\Big)^2 = \left(sin\left(\frac{2\pi}{3}\right)cos(\theta') + cos\left(\frac{2\pi}{3}\right)sin(\theta')\right)^2 = \left(\frac{\sqrt{3}}{2}cos(\theta') - \frac{1}{2}sin(\theta')\right)^2 = \frac{3}{4}cos^2(\theta') + \frac{1}{4}sin^2(\theta') - \frac{\sqrt{3}}{4}sin$$

Following the same logic with sine addition formulas, we also get

$$sin\left(\frac{4\pi}{3} + \theta'\right)^2 = \frac{3}{4}cos^2(\theta') + \frac{1}{4}sin^2(\theta') + \frac{\sqrt{3}}{4}sin(\theta')cos(\theta')$$

$$sin(2\pi + \theta')^2 = sin^2(\theta')$$

Adding together, we get

$$I_x = \sum_i m_i y_i^2 = ML^2\left(2 \times \frac{3}{4}cos^2(\theta') + 2 \times \frac{1}{4}sin^2(\theta') + sin^2(\theta')\right) = \frac{3}{2}ML^2(cos^2(\theta') + sin^2(\theta')) = \frac{3}{2}ML^2$$

Similarly, we will do the same for I_y,

$$I_y = \sum_i m_i x_i^2 = ML^2\left(cos\left(\frac{2\pi}{3} + \theta'\right)^2 + cos\left(\frac{4\pi}{3} + \theta'\right)^2 + cos(2\pi + \theta')^2\right)$$

$$I_y = ML^2\left(cos^2(\theta') + \frac{1}{2}cos^2(\theta') + \frac{3}{2}sin^2(\theta')\right) = \frac{3}{2}ML^2$$

Problem 7.3 Show that the spin about the Z-axis is stable

```
Ix = 1.5 * M * L^2;
Iy = 1.5 * M * L^2;
fprintf("X and Y Axis MOI: %f kg-m^2", Ix);
```

```
X and Y Axis MOI: 60000000.000000 kg-m^2
```

```
if (Iz > Ix && Iz > Iy)
    fprintf("Iz > Ix and Iz > Iy, thus spin is stable about Z");
else
    fprintf("Iz < Ix OR Iz < Iy, so spin is not stable about Z");
end
```

```
Iz > Ix and Iz > Iy, thus spin is stable about Z
```

Problem 7.4 Find the Nutation Period

The nutation frequency is defined as

$$\lambda = \Omega\left(\frac{I_z - I_x}{I_x}\right)$$

so the Nutation period would be

$$T = \frac{1}{\lambda} = \frac{I_x}{\Omega(I_z - I_x)}$$

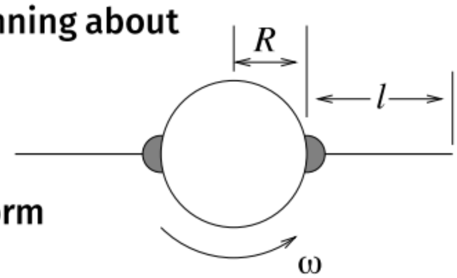
```
nutationFreq   = Omega * (Iz - Ix) / Ix; % 1/s
nutationPeriod = 1 / nutationFreq;       %s
fprintf("The nutation period is %f (sec)", nutationPeriod);
```

```
The nutation period is 1.909859 (sec)
```

Problem 8

- **The spherical vehicle shown (not to scale) is spinning about its axis of symmetry in space under zero torque. The spin is to be slowed by releasing the two identical masses, which will slide frictionlessly off the massless rods. Each mass weighs 5 kg, and the central body (which is uniform in density) has characteristics:**

  $m_{hub} = 50\ kg \quad R = 1\ m \quad l = 4\ m \quad \omega = 10\ rpm$



Problem 8.1 Calculate the Inertia of the central body

From the notes, the inertia for the sphere rotating about its axis of symmetry is $I = \frac{2}{5} M R^2$.

```
mHub   = 50;          % kg
rHub   = 1;           % m
l_rod  = 4;           % m
omega  = 10 * pi/30;  % rad/s

Ihub = 2 * mHub * rHub^2 / 5;
fprintf("The inertia of the hub is %f kg-m^2", Ihub);
```

```
The inertia of the hub is 20.000000 kg-m^2
```

Problem 8.2 Calculate the spacecraft inertia at the time shown in the picture

Each point mass m is a distance r away, and rotating about the same axis. We will sum the individual inertias to get the total spacecraft inertia

```
Mpm = 5;                % kg
r   = rHub;             % m
Ipm = Mpm * r^2;        % kg-m^2
Isc = Ihub + 2 * Ipm;   % kg-m^2
```

```
fprintf("The inertia of the spacecraft is %f (kg-m^2)", Isc);
```

```
The inertia of the spacecraft is 30.000000 (kg-m^2)
```

Problem 8.3 What is the spin rate when the masses reach the tip?

As the masses reach the tip, the MOI is increased so the spin rate is decreased, due to conservation of angular momentum.

First we must find the inertia when the point masses are at the edge of the spacecraft

```
r = rHub + l_rod;                % m
Ipm_end = Mpm * r^2;             % kg-m^2
Isc_end = Ihub + 2 * Ipm_end;    % kg-m^2
fprintf("The inertia of the s/c with the point masses at the end is %f (kg-m^2)", Isc_e
```

```
The inertia of the s/c with the point masses at the end is 270.000000 (kg-m^2)
```

Now, by conservation of angular momentum we have

$$I_{sc,end}\omega_f = I_{sc}\omega_0$$

```
omega_f = Isc * omega / Isc_end; % rad/s
fprintf("The angular rate of the s/c once the masses are extended is %f (rad/s) ", omeg
```

```
The angular rate of the s/c once the masses are extended is 0.116355 (rad/s)
```

Problem 8.4 At the edge, the masses fly off the wire. What is their speed at this time?

Linear speed is calculated as

$$v = r\omega$$

```
v = r * omega_f;
fprintf("The speed of the masses once they fly off is %f (m/s)", v);
```

```
The speed of the masses once they fly off is 0.581776 (m/s)
```

Problem 9

If only 2 thrusters are used at any time, find the combinations for torques

along +X, -X, +Y, -Y, +Z, -Z

```
% Write all the distance vectors from cm to thruster point
rA1 = [-.3, -.25, .45];   FA1 = [0, 1, 0];
rA2 = [-.3, -.25, -.45]; FA2 = [0, 1, 0];
rA3 = [.3, -.25, -.45];   FA3 = [0, 1, 0];
rA4 = [.3, -.25, .45];    FA4 = [0, 1, 0];
rB1 = rA1;               FB1 = [1, 0, 0];
rB2 = rA2;               FB2 = [1, 0, 0];
rB3 = [-.3, .25, .45];   FB3 = [1, 0, 0];
rB4 = [-.3, .25, -.45]; FB4 = [1, 0, 0];
TA1 = cross(rA1,FA1);
TA2 = cross(rA2, FA2);
TA3 = cross(rA3, FA3);
```

```
TA4 = cross(rA4, FA4);
TB1 = cross(rB1, FB1);
TB2 = cross(rB2, FB2);
TB3 = cross(rB3, FB3);
TB4 = cross(rB4, FB4);
```

+X

A2 and A3

-X

A1 and A4

+Y

B1 and B3

-Y

B2 and B4

+Z

A3, A4; B1, B2;

-Z

A1, A2; B3, B4

Problem 10

Problem 10.1 A plant has 3 roots at -2, -3, and -5. It has no zeros. Write the transfer function.

```
plantTF = zpk([], [-2, -3, -5],1)
```

```
plantTF =

          1
  -----------------
  (s+2) (s+3) (s+5)

Continuous-time zero/pole/gain model.
```

Problem 10.2 Is the system stable?

Yes, the system is stable. The denominator has no change in signs, and all roots are in the LH side of the complex plane.

Problem 10.3 Consider the general 3rd order system

$$\frac{1}{s^3 + as^2 + bs + c}$$

using Routh-Hurwitz Criterion, show that the system is stable iff

$$a > 0,\ c > 0,\ b > \frac{c}{a}$$

From the RH Criterion, we know that all signs need to stay the same. Since the sign on s^3 is positive ($a_3 = 1$), we need $c > 0$. Calculating $a_2 - \frac{a_3 a_0}{a_1}$ from the notes, we get

$$a - \frac{1 \times c}{b} > 0 \Rightarrow a - \frac{c}{b} > 0 \Rightarrow b > \frac{c}{a}$$

our last condition for b to be positive is for $a > 0$. Thus, we have shown all the conditions for stability, given that $a_3 = 1$.

Problem 10.4 Close the loop with $C(s) = 1000$. What is the CLTF?

$$G_{CL} = \frac{CP}{1 + CP}$$

where

$$C = 1000,\ P = \frac{1}{(s+2)(s+3)(s+5)}$$

Putting this together, our CLTF is

$$G_{CL} = \frac{1000}{(s+2)(s+3)(s+5) + 1000}$$

Problem 10.5 What is the characteristic equation for the CLTF?

The characteristic equation is the denominator of G_{CL}

$$(s+2)(s+3)(s+5) + 1000$$

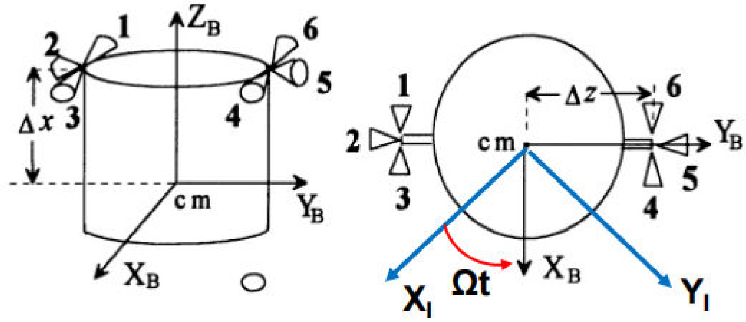
$$s^3 + 10s^2 + 31s + 30 + 1000 = s^3 + 10s^2 + 31s + 1030$$

Problem 10.6 Using the RH Criterion, show that $C(s) = 1000$ destabilizes the system

From 10.3, we know that $b > \frac{c}{a}$, with $a = 10,\ b = 31,\ c = 1030$. Since $b \not> \frac{1030}{10}$, the system fails the RH Criterion and is not stable.

Problem 11

- A spacecraft is spinning about the Z_B axis
- It is controlled by thrusters (force = F).
- The spin inertia of the spacecraft is I_s, and the transverse inertias are I_t.
- The spacecraft is spinning at Ω rad/s.



Problem 11.1 What is the torque when Thrusters 1 and 6 are fired?

$$F_1 = (F_x, 0, 0),\ F_6 = (F_x, 0, 0)$$

$$r_{cm,1} = (0, -\Delta z, \Delta x),\ r_{cm,6} = (0, \Delta z, \Delta x)$$

$$T_1 = r_{cm,1} \times F_1 = (0, F_x\Delta x, F_x\Delta z)$$

$$T_6 = r_{cm,6} \times F_6 = (0, F_x\Delta x, -F_x\Delta z)$$

$$T_1 + T_6 = (0, 2F_x\Delta x, 0)$$

Problem 11.2 Write the DCM at time t, with Ωt being the angle from the inertial to body frame.

Since it is only rotating about z, our DCM is just a z rotation

$$A_{\Omega t} = \begin{pmatrix} cos(\Omega t) & sin(\Omega t) & 0 \\ -sin(\Omega t) & cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 11.3 Write the linearized Equations of Motion for when the Thrusters are fired

Using the vector equation

$$T_{ext} = I\dot{\omega} + \omega^{\times} I\omega$$

$$\begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} = \begin{pmatrix} I_t & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_s \end{pmatrix} \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} + \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} I_t\omega_x \\ I_t\omega_y \\ I_s\omega_z \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2F\Delta x \\ 0 \end{pmatrix} = \begin{pmatrix} I_t\dot{\omega}_x - I_t\omega_z\omega_y + I_s\omega_y\omega_z \\ I_t\dot{\omega}_y + I_t\omega_z\omega_x - I_s\omega_x\omega_z \\ I_s\dot{\omega}_z \end{pmatrix}$$

Now, we will linearize by setting $\omega_z \rightarrow \omega_z + \Omega$, and all products of angular velocities go to zero.

$$\begin{pmatrix} 0 \\ 2F\Delta x \\ 0 \end{pmatrix} = \begin{pmatrix} I_t\dot{\omega}_x - I_t(\omega_z + \Omega)\omega_y + I_s\omega_y(\omega_z + \Omega) \\ I_t\dot{\omega}_y + I_t(\omega_z + \Omega)\omega_x - I_s\omega_x(\omega_z + \Omega) \\ I_s\dot{\omega}_z \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2F\Delta x \\ 0 \end{pmatrix} = \begin{pmatrix} I_t\dot{\omega}_x - I_t\Omega\omega_y + I_s\omega_y\Omega \\ I_t\dot{\omega}_y + I_t\Omega\omega_x - I_s\omega_x\Omega \\ I_s\dot{\omega}_z \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \frac{1}{I_t}2F\Delta x \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\omega}_x + \left(\frac{I_s - I_t}{I_t}\Omega\right)\omega_y \\ \dot{\omega}_y - \left(\frac{I_s - I_t}{I_t}\Omega\right)\omega_x \\ I_s\dot{\omega}_z \end{pmatrix}$$

So our equations of motion are:

$$\dot{\omega}_x + \left(\frac{I_s - I_t}{I_t}\Omega\right)\omega_y = 0$$

$$\dot{\omega}_y - \left(\frac{I_s - I_t}{I_t}\Omega\right)\omega_x = \frac{2}{I_t}F\Delta x$$

$$\dot{\omega}_z = 0$$