ADACS Homework 3

By: *Dillon Allen*

Scenario

The equations governing a rigid body is

$$\frac{d}{dt}H = 0 \Longrightarrow \frac{d}{dt}([I])\omega = 0$$

$$\frac{d}{dt}([I]\omega) = [I]\dot{\omega} + \omega \times ([I]\omega)$$

$$\dot{\omega} = -[I]^{-1}(\omega \times ([I]\omega)$$

Problem 1

Expand the equations and write in the form

$$\frac{d}{dt}\omega = f(\omega, t)$$

Solution:

$$[I]\omega = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{pmatrix}$$

$$\omega \times [I]\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{pmatrix} = \begin{pmatrix} \omega_y\omega_z(I_z - I_y) \\ \omega_x\omega_z(I_x - I_z) \\ \omega_x\omega_y(I_y - I_x) \end{pmatrix}$$

$$[I]^{-1} = \begin{pmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{pmatrix}$$

$$\frac{d}{dt}\omega = -[I]^{-1}(\omega \times [I]\omega)$$

$$\frac{d}{dt}\omega = \begin{pmatrix} -I_x^{-1} & 0 & 0 \\ 0 & -I_y^{-1} & 0 \\ 0 & 0 & -I_z^{-1} \end{pmatrix}\begin{pmatrix} \omega_y\omega_z(I_z - I_y) \\ \omega_x\omega_z(I_x - I_z) \\ \omega_x\omega_y(I_y - I_x) \end{pmatrix} = \begin{pmatrix} \omega_y\omega_z\left(\frac{I_y - I_z}{I_x}\right) \\ \omega_x\omega_z\left(\frac{I_z - I_x}{I_y}\right) \\ \omega_x\omega_y\left(\frac{I_x - I_y}{I_z}\right) \end{pmatrix}$$

Thus, the three differential equations are

$$\dot{\omega}_x = \omega_y \omega_z \left(\frac{I_y - I_z}{I_x}\right)$$

$$\dot{\omega}_y = \omega_x \omega_z \left(\frac{I_z - I_x}{I_y}\right)$$

$$\dot{\omega}_z = \omega_x \omega_y \left(\frac{I_x - I_y}{I_z}\right)$$

Problem 2

Write the expression for the magnitude of the total momentum (h) and the rotational kinetic energy (T)

Solution:

The equation for the total momentum is

$$\mathbf{h} = [I]\omega$$

$$\mathbf{h} = \begin{pmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{pmatrix}$$

$$\|h\| = \sqrt{I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2}$$

The equation for the rotational Kinetic Energy is

$$T_{\text{rot}} = \frac{1}{2} \omega^T [I] \omega$$

$$T_{\text{rot}} = \frac{1}{2} \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{pmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{pmatrix} = \frac{1}{2} \left(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 \right)$$

Problem 3 (Function at bottom)

Run the problem for the following conditions

- [I] = diag([100,200,300])
- $\omega_x = 0,\ \omega_y = 1 \frac{\text{deg}}{\text{sec}},\ \omega_z = 30\ \text{rpm}$
- time span of 20 seconds

And find the following:

- Plot the angular rates as a function of time
- Plot the total momentum and kinetic energy vs time
- Show the total momentum and kinetic energy are constant
- Show that the resulting motion is sinusoidal with period of about 2 seconds.

Solution:

```
% Initial Conditions
Imat1 = diag([100, 200, 300]);

w_initial1_x = 0;
w_initial1_y = 1 * (pi/180); % deg/s -> rad/s
w_initial1_z = 30 * (2*pi/60);  % rpm -> rad/s

w_initial1 = [w_initial1_x; w_initial1_y; w_initial1_z];

tspan1 = 0:0.1:20; % seconds

[w1, momenta_matrix1, totMomentum1, Trot1] = ...
                            AngularRateODESolver(Imat1, w_initial1, tspan1)
```

```
w1 = 201×3
         0    0.0175    3.1416
   -0.0054    0.0166    3.1416
   -0.0103    0.0141    3.1416
   -0.0141    0.0103    3.1416
   -0.0166    0.0054    3.1416
   -0.0175   -0.0000    3.1416
   -0.0166   -0.0054    3.1416
   -0.0141   -0.0103    3.1416
   -0.0103   -0.0141    3.1416
   -0.0054   -0.0166    3.1416
      ⋮

momenta_matrix1 = 1×201 cell
```

...

	1	2	3	4	5	6	7	8
1	[0;3.490...	[-0.5394...	[-1.0259...	[-1.4122...	[-1.6601...	[-1.7453...	[-1.6602...	[-1.4118...

```
totMomentum1 = 1×201 cell
```

...

	1	2	3	4	5	6	7	8
1	942.4843	942.4843	942.4843	942.4843	942.4843	942.4843	942.4843	942.4843

```
Trot1 = 1×201 cell
```

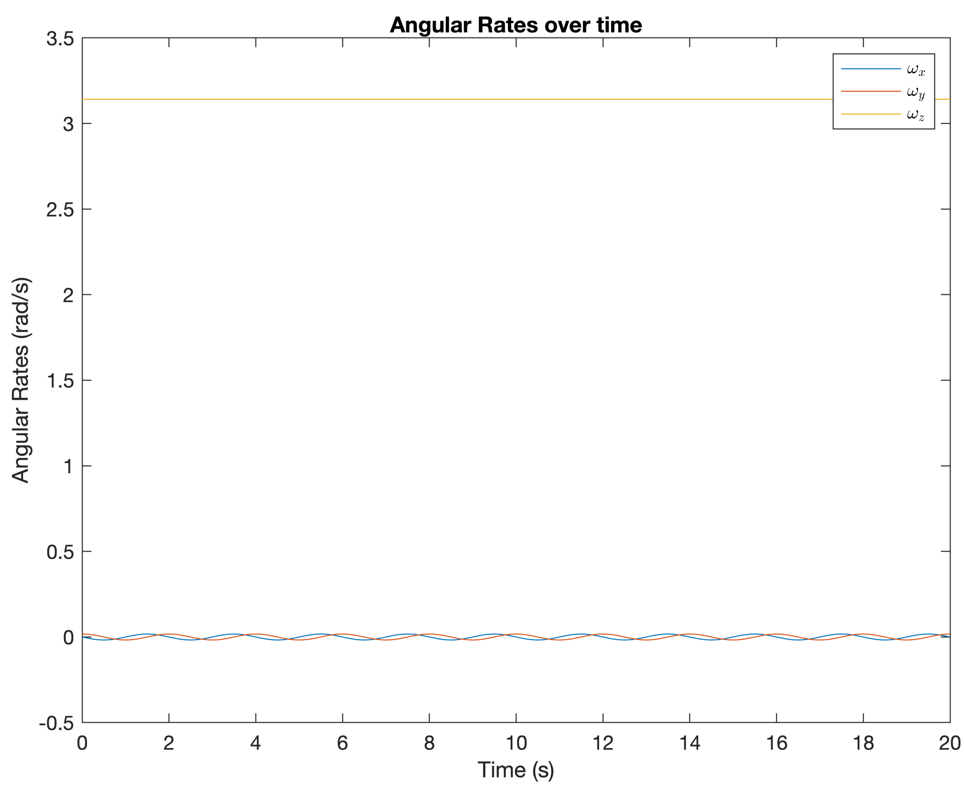
...

	1	2	3	4	5	6	7	8
1	1.4805e+03	1.4805e+03	1.4805e+03	1.4805e+03	1.4805e+03	1.4805e+03	1.4805e+03	1.4805e+03

```
% Peak Finding for Period
[~,locs] = findpeaks(w1(:,1),tspan1);
period = max(diff(locs))
```

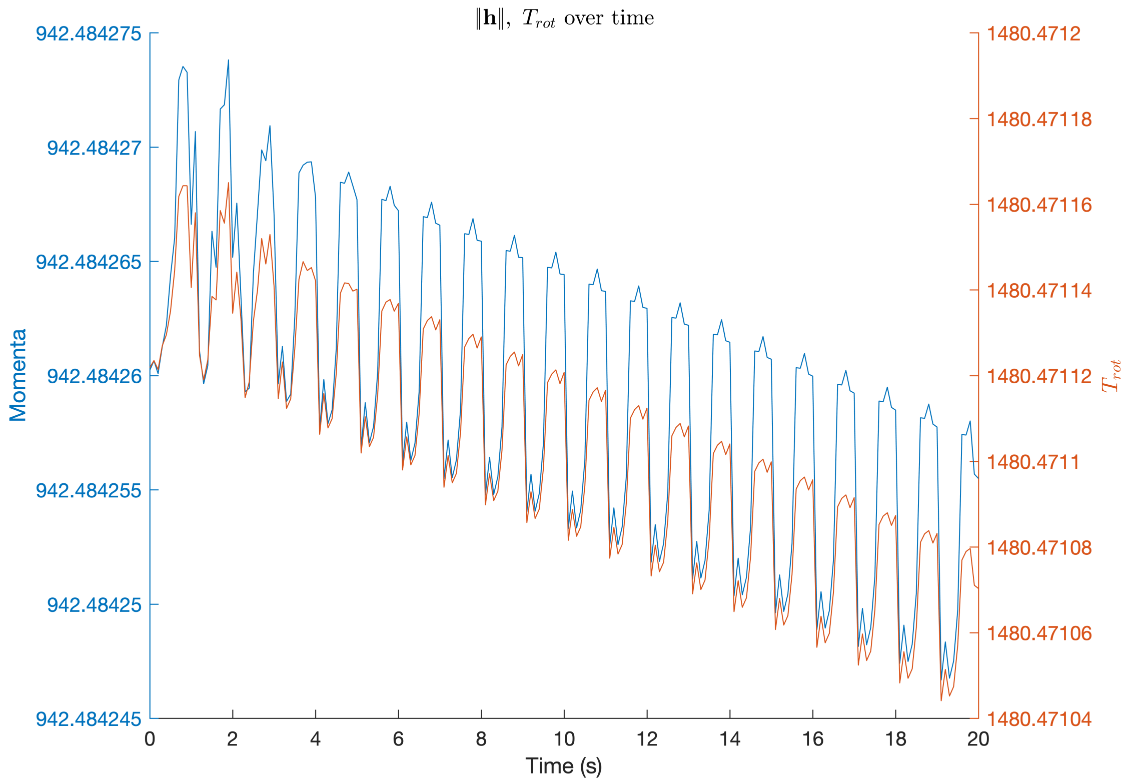
```
period = 2
```

```
% Angular Rate Plot
figure(1)
plot(tspan1, w1(:,1));
hold on
plot(tspan1, w1(:,2));
plot(tspan1, w1(:,3));
hold off;
title("Angular Rates over time");
xlabel("Time (s)");
ylabel("Angular Rates (rad/s)");
legend("$\omega_x$", "$\omega_y$", "$\omega_z$", 'Interpreter',"latex")
```




```
% Total Momentum and Kinetic Energy Plots

figure(2)
hold on
yyaxis left
plot(tspan1,cell2mat(totMomentum1))
ylabel("Momenta")
xlabel("Time (s)");
yyaxis right
plot(tspan1, cell2mat(Trot1));
ylabel("$T_{rot}$", "Interpreter","latex")
hold off;
title("$|\!| \textbf{h} |\!|,\ T_{rot} $ over time","Interpreter","latex")
```



Problem 4

With the same body, re-run the simulation with new parameters:

- [I] = diag([100, 200, 300]);
- $\omega_x = 1\,\frac{\text{deg}}{\text{sec}},\ \omega_y = 30\ \text{rpm},\ \omega_z = 0$
- time span: 20 seconds

Solution:

```
% Initial Conditions

% Moment of Inertia Matrix
Imat2 = diag([100,200,300]);

% Initial Angular Rates
w_initial2_x = 1 * (pi/180); % deg/s -> rad/s
w_initial2_y = 30 * (2*pi/60); % rpm -> rad/s
w_initial2_z = 0;
w_initial2 = [w_initial2_x; w_initial2_y; w_initial2_z];

tspan2 = 0:0.1:20;
```

```
[w2, momenta_matrix2, totMomentum2, Trot2] = ...
                            AngularRateODESolver(Imat2, w_initial2, tspan2)
```

```
w2 = 201×3
    0.0175    3.1416         0
    0.0177    3.1416   -0.0018
    0.0186    3.1416   -0.0037
    0.0201    3.1416   -0.0058
    0.0223    3.1416   -0.0080
    0.0251    3.1415   -0.0104
    0.0289    3.1415   -0.0133
    0.0335    3.1415   -0.0165
    0.0393    3.1414   -0.0203
    0.0464    3.1413   -0.0248
     ⋮

momenta_matrix2 = 1×201 cell
```

	1	2	3	4	5	6	7	8
1	[1.7453;...	[1.7741;...	[1.8614;...	[2.0101;...	[2.2252;...	[2.5137;...	[2.8851;...	[3.3515;...

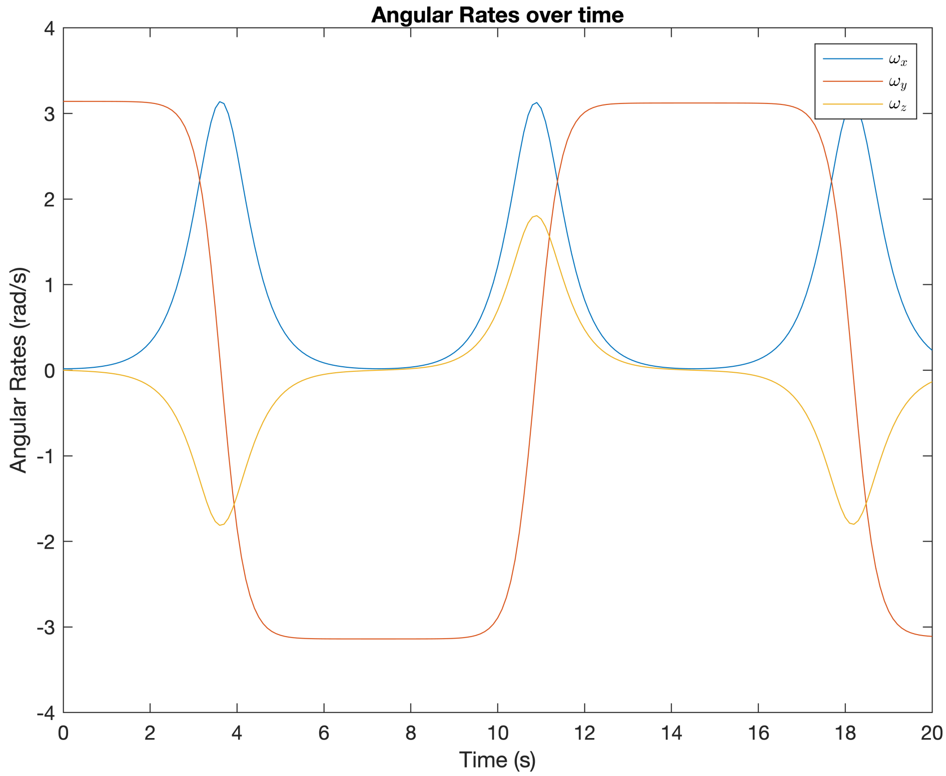
```
totMomentum2 = 1×201 cell
```

	1	2	3	4	5	6	7	8
1	628.3210	628.3210	628.3210	628.3210	628.3210	628.3210	628.3209	628.3209

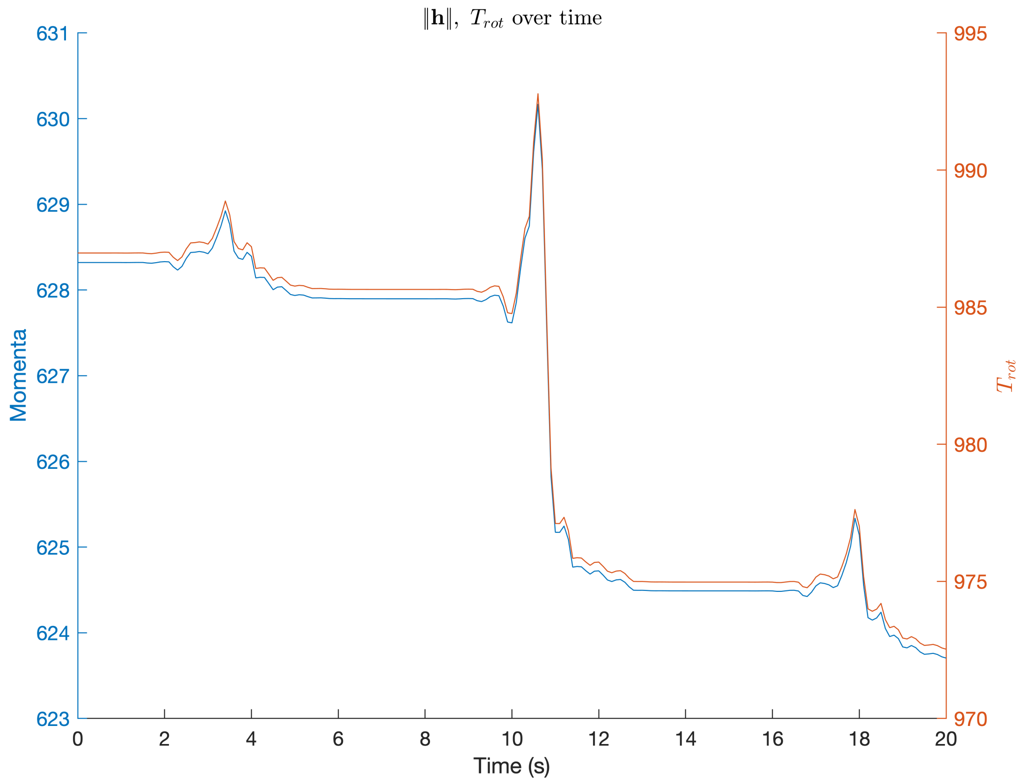
```
Trot2 = 1×201 cell
```

	1	2	3	4	5	6	7	8
1	986.9757	986.9757	986.9757	986.9757	986.9757	986.9757	986.9755	986.9756

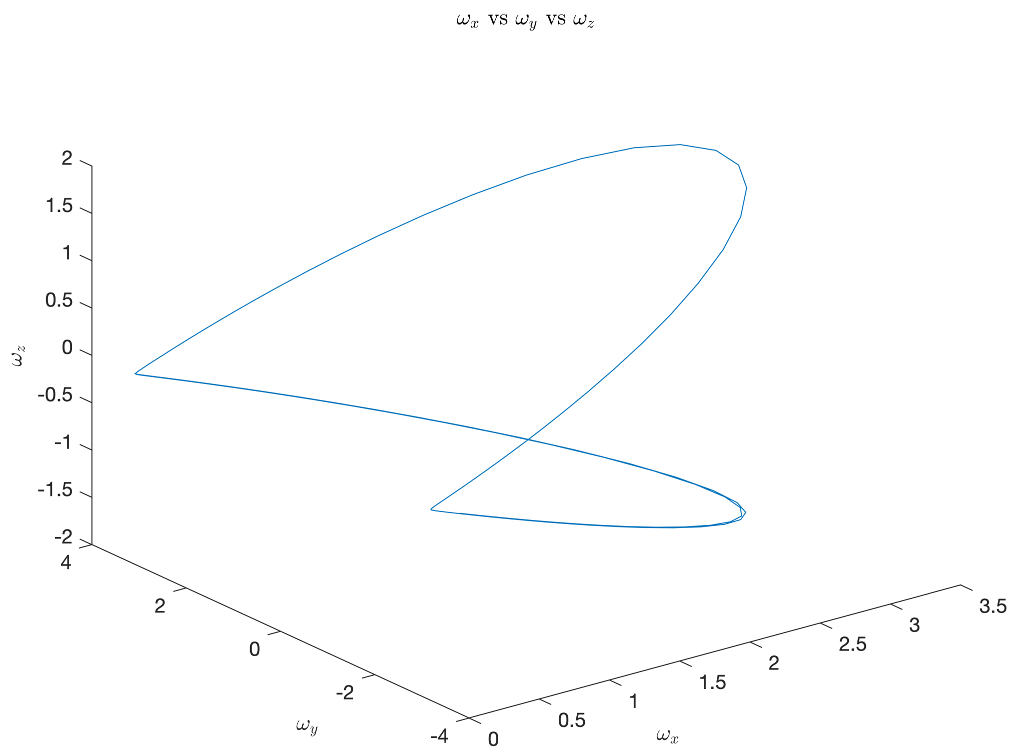
```
% Angular Rates Plot
figure(3)
plot(tspan2, w2(:,1));
hold on
plot(tspan2, w2(:,2));
plot(tspan2, w2(:,3));
hold off;
title("Angular Rates over time");
xlabel("Time (s)");
ylabel("Angular Rates (rad/s)");
legend("$\omega_x$", "$\omega_y$", "$\omega_z$", 'Interpreter',"latex")
```




```
% Total Momentum and Kinetic Energy plot
figure(4)
hold on
yyaxis left
plot(tspan2,cell2mat(totMomentum2))
ylabel("Momenta")
xlabel("Time (s)");
yyaxis right
plot(tspan2, cell2mat(Trot2));
ylabel("$T_{rot}$", "Interpreter","latex")
hold off;
title("$|\!| \textbf{h} |\!|,\ T_{rot} $ over time","Interpreter","latex")
```




```
figure(5)
% As shown in class, I wanted to recreate the plot of w_x vs w_y vs w_z
plot3(w2(:,1), w2(:,2), w2(:,3));
title("$\omega_x$ vs $\omega_y$ vs $\omega_z$", "Interpreter","latex");
xlabel("$\omega_x$", "Interpreter", "latex");
ylabel("$\omega_y$", "Interpreter","latex");
zlabel("$\omega_z$", "Interpreter","latex");
```



Function

Write a matlab function to solve this ODE, given initial conditions and a timespan

Output the following:

- Angular Rates (ω's)
- momenta along each axis
- Total momentum (magnitude) and kinetic energy

Solution:

```
function [w, momenta_matrix, totMomentum, Trot] = ...
                               AngularRateODESolver(Imat, w_init, tspan)
    % Inertia Values
    Ix = Imat(1,1);
    Iy = Imat(2,2);
    Iz = Imat(3,3);

    % differential equation here
    func = @(t,w) [((Iy - Iz)/Ix)*w(2)*w(3);
                   ((Iz - Ix)/Iy)*w(1)*w(3);
                   ((Ix - Iy)/Iz)*w(1)*w(2)];

    % ode45 solver
    opts = odeset('AbsTol',1E-12);
    [t,w] = ode45(func, tspan, w_init, opts);
```

```
    for i = 1:length(tspan)
        momenta_matrix{:,i} = [Ix * w(i,1); Iy * w(i,2); Iz*w(i,3)];
        totMomentum{i} = norm(momenta_matrix{:,i},2); % Taking Euclidean 2-norm
        Trot{i} = 0.5*dot([w(i,1); w(i,2);w(i,3)], Imat*[w(i,1);w(i,2);w(i,3)]);
    end
end
```