Physics 410 - Homework 4

Dillon Walton

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1 Problem 1

$$\text{max displacement} = 0.25m$$
$$\text{max speed} = 2.0\tfrac{m}{s}$$

$$\begin{aligned} F &= -kx \\ m\ddot{x} &= -kx \\ m\ddot{x} + kx &= 0 \end{aligned} \tag{1}$$

Assume solution of $x = A\cos{(wt + \phi)}$

$$\begin{aligned} x &= A\cos{(\omega t + \phi)} \\ \dot{x} &= -A\omega\sin{(\omega t + \phi)} \\ \ddot{x} &= -A\omega^2\cos{(\omega t + \phi)} \end{aligned} \tag{2}$$

$$\begin{aligned} m\ddot{x} + kx &= 0 \\ m - A\omega^2\cos{(\omega t + \phi)} + kA\cos{(\omega t + \phi)} &= 0 \\ -m\omega^2 + k &= 0 \\ \omega &= \sqrt{\frac{k}{m}} \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{x} &= -A\omega\sin{(\omega t + \phi)} \\ 2 &= 0.25\omega \\ \omega &= 8 \end{aligned} \tag{4}$$

$$\begin{aligned} T &= \frac{1}{f} \\ &= \frac{2\pi}{\omega} \\ &= \frac{1}{4}\pi \end{aligned} \tag{5}$$

2 Problem 2

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ \dot{x}(t) &= -A\omega\sin(\omega t + \phi) \\ \ddot{x}(t) &= -A\omega^2\cos(\omega t + \phi) \end{aligned} \tag{6}$$

$$\begin{aligned} A\cos(\phi) &= x_0 \\ \cos(\phi) &= \frac{x_0}{A} \\ \cos^2(\phi) &= \frac{{x_0}^2}{A^2} \end{aligned} \tag{7}$$

$$\begin{aligned} -A\omega\sin(\phi) &= v_0 \\ -A\omega\sqrt{1-\cos^2(\phi)} &= v_0 \\ -A\omega\sqrt{1-\frac{{x_0}^2}{A^2}} &= v_0 \\ -A\omega\sqrt{\frac{A^2-{x_0}^2}{A^2}} &= v_0 \\ A^2\omega^2(\frac{A^2-{x_0}^2}{A^2}) &= {v_0}^2 \\ \omega^2(A^2-{x_0}^2) &= {v_0}^2 \\ A &= \sqrt{\frac{{v_0}^2}{\omega^2}+{x_0}^2} \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{A\cos(\phi)}{-A\omega\sin(\phi)} &= \frac{x_0}{v_0} \\ \frac{\cot(\phi)}{-\omega} &= \frac{x_0}{v_0} \\ \cot(\phi) &= \frac{-x_0\omega}{v_0} \\ \phi &= \cot^{-1}(\frac{-x_0\omega}{v_0}) \end{aligned} \tag{9}$$

3 Problem 3

$$U(r) = U_0(\frac{r}{R} + \lambda^2\frac{R}{r}) \tag{10}$$

$$
\begin{aligned}
r_0 &= \dot{U}(r) = 0 \\
&= U_0 \frac{d}{dr}[\frac{r}{R} + \lambda^2 \frac{R}{r}] = 0 \\
&= [\frac{1}{R} - \lambda^2 \frac{R}{r^2}] = 0 \\
\frac{1}{R} &= \lambda^2 \frac{R}{r^2} \\
1 &= \lambda^2 \frac{R^2}{r^2} \\
r^2 &= \lambda^2 R^2 \\
r_0 &= \lambda R
\end{aligned} \tag{11}
$$

$$
U(r_0) = U_0(\frac{r_0}{R} + \lambda^2 \frac{R}{r_0}) \tag{12}
$$

$$
\begin{aligned}
U(r_0 + x) &= U_0(\frac{r_0 + x}{R} + \lambda^2 \frac{R}{r_0 + x}) \\
&= U_0(\frac{r_0 + x}{R} + \lambda^2 \frac{R}{r_0}(1 + \frac{x}{r_0})^{-1}) \\
&= U_0(\frac{r_0 + x}{R} + \lambda^2 \frac{R}{r_0}(1 - \frac{x}{r_0} + \frac{x^2}{{r_0}^2})) \\
&= U_0(\frac{r_0}{R} + \frac{x}{R} + \frac{\lambda^2 R}{r_0} - \frac{\lambda^2 R x}{{r_0}^2} + \frac{\lambda^2 R x^2}{{r_0}^3}) \\
r_0 = \lambda R & \\
&= U_0(\lambda + \frac{x}{R} + \lambda - \frac{x}{R} + \frac{x^2}{\lambda R^2}) \\
&= 2U_0\lambda + \frac{1}{2}\frac{2U_0 x^2}{\lambda R^2}
\end{aligned} \tag{13}
$$

$$
\omega = \sqrt{\frac{k}{m}} \tag{14}
$$

$$
= \sqrt{\frac{2U_0}{m\lambda R^2}} \tag{15}
$$

4 Problem 4

$$
U(r) = A[(e^{(\frac{r_0 - r}{s})} - 1)^2 - 1]
$$
$$
A, r_0, s > 0
$$

4.1

$$
\begin{aligned}
\frac{dU}{dr} &= A\frac{d}{dr}(e^{\frac{r_0-r}{s}}-1)^2 \\
&= A\frac{d}{dr}(e^{\frac{r_0-r}{s}}-1)(e^{\frac{r_0-r}{s}}-1) \\
&= A\frac{d}{dr}(e^{2\frac{r_0-r}{s}}-2e^{\frac{r_0-r}{s}}+1) \\
&= A\frac{d}{dr}(e^{2\frac{r_0-r}{s}})-\frac{d}{dr}2e^{\frac{r_0-r}{s}} \\
&= A(-\frac{2}{s}e^{\frac{r_0-r}{s}}+\frac{2}{s}e^{\frac{r_0-r}{s}}) \\
r = r_0 & \\
&= A(-\frac{2}{s}e^{\frac{0}{s}}+\frac{2}{s}e^{\frac{0}{s}}) \\
&= A(-\frac{2}{s}+\frac{2}{s}) \\
\dot{U}(r) &= 0
\end{aligned}
\tag{16}
$$

4.2

$$
\begin{aligned}
U(r) &= A[(e^{(\frac{r_0-r}{s})}-1)^2-1] \\
&= A[(e^{(\frac{x}{s})}-1)^2-1] \\
&= A[(1+\frac{-x}{s}-1)^2-1] \\
&= A([\frac{x}{s}]^2-1) \\
&= -A+\frac{Ax^2}{s^2} \\
&= -A+\frac{1}{2}\frac{2Ax^2}{s^2}
\end{aligned}
\tag{17}
$$

Since $A, r_0, s > 0$, we know k is positive, and therefore this is a local minimum.

5 Problem 5

5.1

$$
\begin{aligned}
\omega_0 &= 2\pi f_0 \\
\frac{\omega_0}{2\pi} &= f_0 \\
T_0 &= \frac{2\pi}{\omega_0} \\
4 &= \frac{2\pi}{\omega_0} \\
\omega_0 &= \frac{\pi}{2}
\end{aligned}
\tag{18}
$$

5.2

$$\omega_1 = \frac{2\pi}{4.01} \tag{19}$$

$$\begin{aligned}
\frac{2\pi}{4.01} &= \sqrt{(\frac{\pi}{2})^2 - \beta^2} \\
(\frac{2\pi}{4.01})^2 &= (\frac{\pi}{2})^2 - \beta^2 \\
\beta^2 &= (\frac{\pi}{2})^2 - (\frac{2\pi}{4.01})^2 \\
\beta^2 &= 2.467 - 2.455 \\
\beta^2 &= 0.012 \\
\beta &= 0.111\frac{rad}{s}
\end{aligned} \tag{20}$$

5.3

$$x(t) = Ae^{-\beta t}\cos{(\omega_1 t + \delta)} \tag{21}$$

$$\begin{aligned}
x(t) &= 1.00e^{-0.111*20.05} \\
&= 1e^{-2.225} \\
&= 0.108cm
\end{aligned} \tag{22}$$