Physics 410 - Homework 1

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1 Question 1

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, c = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

1.1

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \tag{1}$$

1.2

$$5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 13 \end{bmatrix} \tag{2}$$

1.3

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 10 \tag{3}$$

1.4

$$\begin{aligned} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} &= \begin{bmatrix} (2-6)- \\ (1-9)+ \\ (2-6) \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 9 \\ -4 \end{bmatrix} \end{aligned} \tag{4}$$

2 Question 2

$$b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, c = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\vec{b} \cdot \vec{c} &= |b||c| \cos\theta \\
\frac{\vec{b} \cdot \vec{c}}{|b||c|} &= \cos\theta \\
\cos^{-1} \frac{\vec{b} \cdot \vec{c}}{|b||c|} &= \theta
\end{aligned} \tag{5}$$

$$\begin{aligned}
\vec{b} \cdot \vec{c} &= 12 \\
|b| &= \sqrt{21} \\
|c| &= \sqrt{21} \\
\cos^{-1} \frac{12}{\sqrt{21}\sqrt{21}} &= \theta \\
\cos^{-1} \frac{4}{7} &= \theta \\
55.15^\circ &= \theta
\end{aligned} \tag{6}$$

3 Question 3

$$r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \, s = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

3.1

$$\begin{aligned}
p_x &= (0 * 0) - (0 * 1) = 0 \\
p_y &= (0 * 1) - (1 * 0) = 0 \\
p_z &= (1 * 1) - (0 * 1) = 1
\end{aligned} \tag{7}$$

3.2

$$\begin{aligned}
\vec{p} \cdot \vec{r} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= 0
\end{aligned} \tag{8}$$

3.3

$$\begin{aligned}
\vec{p} \cdot \vec{s} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= 0
\end{aligned} \tag{9}$$

3.4

$\vec{p}$ points $+\hat{z}$ which is consistent with the right-hand rule.

3.5

$$\begin{aligned} |\vec{p}| &= \vec{p} \cdot \vec{p} \\ &= \sqrt{0+0+1} \\ &= 1 \end{aligned} \tag{10}$$

$$\begin{aligned} |\vec{r}| &= \vec{r} \cdot \vec{r} \\ &= \sqrt{1+0+0} \\ &= 1 \end{aligned} \tag{11}$$

$$\begin{aligned} |\vec{s}| &= \vec{s} \cdot \vec{s} \\ &= \sqrt{1+1+0} \\ &= \sqrt{2} \end{aligned} \tag{12}$$

$$\begin{aligned} \cos^{-1} \frac{\vec{r} \cdot \vec{s}}{|r||s|} &= \theta \\ \cos^{-1} \frac{1}{\sqrt{2}} &= \theta \\ \frac{\pi}{4} &= \theta \end{aligned} \tag{13}$$

$$\begin{aligned} |r| \cdot |s| \sin\left(\frac{\pi}{4}\right) &= |\vec{p}| \\ \sqrt{2}\frac{\sqrt{2}}{2} &= 1 \\ 1 &= 1 \end{aligned} \tag{14}$$

4 Question 4

4.1

$$\vec{r} \cdot \vec{r} = \sum_{n=1}^{3} r_n s_n = r_1 r_1 + r_2 r_2 + r_3 r_3 \tag{15}$$

Pythagoras's theorem states:

$$a^2 + b^2 = c^2 \tag{16}$$

where c is magnitude.. when we generalize this to r we can see

$$\begin{aligned} r^2 &= {r_1}^2 + {r_2}^2 + {r_3}^2 \\ |r| &= \sqrt{{r_1}^2 + {r_2}^2 + {r_3}^2} \end{aligned} \tag{17}$$

4.2

Substituting $r + s$ into equation above...

$$\begin{aligned}
|r+s| &= \sqrt{(r_1+s_1)^2+(r_2+s_2)^2+(r_3+s_3)^2} \\
&= \sqrt{(r_1^2+2r_1s_1+s_1^2)+(r_2^2+2r_2s_2+s_2^2)+(r_3^2+2r_3s_3+s_3^2)} \\
&= \sqrt{(r_1^2+r_2^2+r_3^2)+(2r_1s_1+2r_2s_2+2r_3s_3)+(s_1^2+s_3^2+s_2^2)} \\
|r+s| &= \sqrt{|r|+2(\vec{r}\cdot\vec{s})+|s|} \\
\frac{|r+s|^2-|r|-|s|}{2} &= (\vec{r}\cdot\vec{s})
\end{aligned} \tag{18}$$

Pythagoras's theorem above tells us that $|r|$ is axis independent. Since we can see in the equation above that only magnitudes of vectors $|r + s|$, $|r|$, and $|s|$, we can conclude that $r \cdot s$ is an axis independent quantity. (This also makes reasonable sense since the dot product is a scalar quantity to begin with and therefore has no direction i.e. no dependence on a axis / coordinate system)

5 Question 5

$$F(v) = \frac{P}{v}$$

5.1 velocity

$$\begin{aligned}
m\frac{dv}{dt} &= \frac{P}{v} \\
m\frac{vdv}{Pdt} &= 1 \\
\frac{m}{P}vdv &= dt \\
\frac{m}{P}\int_{v_0}^{v} v'dv' &= \int_0^t dt' \\
\frac{m}{P}(\frac{v^2}{2}-\frac{{v_0}^2}{2}) &= t \\
v^2 &= (\frac{2Pt}{m}+{v_0}^2) \\
v &= \sqrt{(\frac{2Pt}{m}+{v_0}^2)}
\end{aligned} \tag{19}$$

5.2 position

$$\begin{aligned}\frac{dx}{dt} &= \sqrt{(\frac{2Pt}{m} + {v_0}^2)} \\ \frac{dx}{dt} &= \sqrt{{v_0}^2}\sqrt{(\frac{2Pt}{m{v_0}^2} + 1)} \\ \frac{1}{v_0}\int_{x_0}^{x} dx &= \int_0^t \sqrt{(\frac{2Pt'}{m{v_0}^2} + 1)}dt' \\ \text{Let } u = \frac{2Pt'}{m{v_0}^2} + 1 \\ \frac{1}{v_0}(x - x_0) &= \frac{m{v_0}^2 t^{\frac{3}{2}}}{3P} \\ x &= \frac{m{v_0}^3 t^{\frac{3}{2}}}{3P} + x_0\end{aligned} \tag{20}$$

5.3 kinetic energy

$$\begin{aligned}KE &= \frac{mv^2}{2} \\ &= \frac{m(\sqrt{(\frac{2Pt}{m} + {v_0}^2)})^2}{2} \\ &= \frac{(2Pt + m{v_0}^2)}{2} \\ &= Pt + \frac{(m{v_0}^2)}{2}\end{aligned} \tag{21}$$

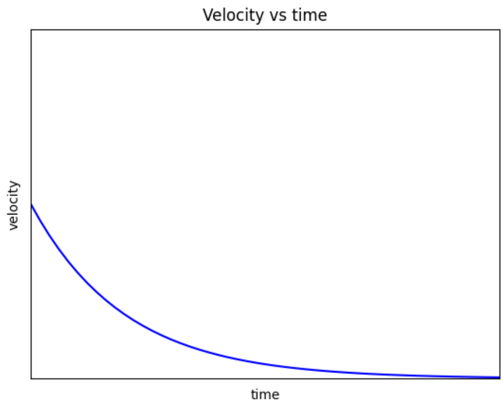
$\frac{(m{v_0}^2)}{2}$ is a constant, therefore the kinetic energy has a linear relationship with time t.

6 Question 6

$$\begin{aligned}v_{y_0} &= 2v_{\text{ter}} \\ &= 2\frac{mg}{b}\end{aligned}$$

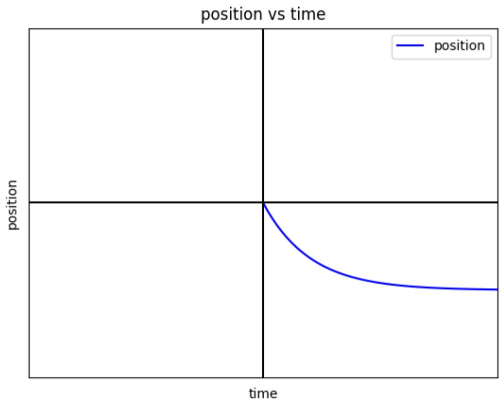
6.1 velocity

$$
\begin{aligned}
m\frac{dv_y}{dt} &= mg - bv_y \\
\frac{dv_y}{dt} &= g - \tau v_y \\
\frac{dv_y}{g - \tau v_y} &= dt \\
\int_{v_0}^{v} \frac{dv_y}{g - \tau v_y} &= \int_0^t dt \\
\int_{v_0}^{v} \frac{dv_y}{g - \tau v_y} &= t \\
\text{Let } u = g - \tau v_y & \\
-\frac{1}{\tau}\int_{v_0}^{v} \frac{du}{u} &= t \\
-\frac{1}{\tau}(\ln|v| - \ln|v_0|) &= t \\
\ln|\frac{v}{v_0}| &= -\tau t \\
v &= e^{-\tau t} v_0 \\
&= e^{-\tau t} 2\frac{mg}{b}
\end{aligned}
\tag{22}
$$



6.2 position

$$\begin{aligned}
\frac{dx}{dt} &= e^{-\tau t} 2\frac{mg}{b} \\
\int_{x_0}^{x} dx &= 2\frac{mg}{b}\int_0^t e^{-\tau t} dt \\
x - x_0 &= -2\frac{mg}{b\tau}(e^{-\tau t} - 1) \\
x &= -2\frac{g}{\tau^2}(e^{-\tau t} - 1) + x_0
\end{aligned} \tag{23}$$



7 Question 7

7.1

$$v_{ter} = g\tau,\ \tau = \frac{b}{m}$$

$$\begin{aligned}
v_y(t) &= v_{ter}(1 - e^{-\frac{t}{\tau}}) \\
&= g\tau(1 - [1 - \frac{t}{\tau} + \frac{t^2}{2\tau} - \frac{t^3}{6\tau}]) \\
&= g\tau(1 - [1 - \frac{t}{\tau} + \frac{t^2}{2\tau} - \frac{t^3}{6\tau}])
\end{aligned} \tag{24}$$

With small t, we can assume t^2 and t^3 to be negligible

$$\begin{aligned}
&= g\tau(+\frac{t}{\tau}) \\
&= gt
\end{aligned}$$

7.2

$$v_{y_0} = 0$$

$$
\begin{aligned}
y(t) &= v_{ter}t + (v_{y_0} - v_{ter})\tau(1 - e^{-\frac{t}{\tau}}) \\
&= v_{ter}t - v_{ter}\tau(1 - e^{-\frac{t}{\tau}}) \\
&= g\tau t - g\tau\tau(1 - [1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2}]) \\
&= g\tau t - g\tau(t - \frac{t^2}{2\tau}) \\
&= gt - g(t - \frac{t^2}{2}) \\
&= gt - (gt - \frac{gt^2}{2}) \\
&= \frac{gt^2}{2}
\end{aligned} \tag{25}
$$

8 Question 8

8.1

$$
\begin{aligned}
m\frac{dv}{dt} &= -cv^{\frac{3}{2}} \\
-\frac{mdv}{cv^{\frac{3}{2}}dt} &= 1 \\
-\frac{m}{c}\int_{v_0}^{v}\frac{dv'}{v'^{\frac{3}{2}}} &= \int_0^t dt' \\
-\frac{m}{c}(-\frac{2}{v^{\frac{1}{2}}} + \frac{2}{{v_0}^{\frac{1}{2}}}) &= t \\
(-\frac{2}{v^{\frac{1}{2}}} + \frac{2}{{v_0}^{\frac{1}{2}}}) &= -\frac{ct}{m} \\
\frac{2}{v^{\frac{1}{2}}} &= \frac{ct}{m} + \frac{2}{{v_0}^{\frac{1}{2}}} \\
\frac{2}{\frac{ct}{m} + \frac{2}{{v_0}^{\frac{1}{2}}}} &= v^{\frac{1}{2}} \\
\pm\sqrt{\frac{2}{\frac{ct}{m} + \frac{2}{{v_0}^{\frac{1}{2}}}}} &= v
\end{aligned} \tag{26}
$$

8.2

$$
\begin{aligned}
\sqrt{\frac{2}{\frac{ct}{m} + \frac{2}{{v_0}^{\frac{1}{2}}}}} &= 0 \\
\frac{2}{\frac{ct}{m} + \frac{2}{{v_0}^{\frac{1}{2}}}} &= 0 \\
2 &= 0
\end{aligned} \tag{27}
$$

The above equation concludes the mass m will not come to rest.

9 Question 9

9.1

9.1.1 velocity

$$
\begin{aligned}
F &= -mg - bv \\
m\frac{dv}{dt} &= -mg - bv \\
\frac{m\frac{dv}{dt}}{-mg - bv} &= 1 \\
m\int_{v_0}^{v} \frac{dv'}{-mg - bv'} &= \int_0^t dt' \\
\text{let } u = -mg - bv',\ du = -bdv' \\
-\frac{m}{b}\int_{u_0}^{u} \frac{1}{u} du &= t \\
-\frac{m}{b}\ln(u)|_{u_0}^{u} &= t \\
\ln(-mg - bv) - \ln(-mg - bv_0) &= -\frac{bt}{m} \\
\ln|\frac{-mg - bv}{-mg - bv_0}| &= -\frac{bt}{m} \\
\frac{-mg - bv}{-mg - bv_0} &= e^{-\frac{bt}{m}} \\
v &= \frac{e^{-\frac{bt}{m}}(-mg - bv_0) + mg}{-b}
\end{aligned} \tag{28}
$$

9.1.2 position

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{e^{-\frac{bt}{m}}(-mg - bv_0) + mg}{-b} \\
\int_{y_0}^{y} dy' &= \int_0^t \frac{e^{-\frac{bt'}{m}}(-mg - bv_0) + mg}{-b} dt' \\
y - y_0 &= \int_0^t \frac{e^{-\frac{bt'}{m}}(-mg - bv_0)}{-b} dt' + \int_0^t \frac{mg}{-b} dt' \\
y - y_0 &= \frac{(-mg - bv_0)}{-b}\int_0^t e^{-\frac{bt'}{m}} dt' + \frac{mgt}{-b} \\
y &= y_0 + \frac{(-mg) - bv_0}{-b}\frac{m}{-b}(e^{\frac{-bt}{m}} - 1) + \frac{mgt}{-b}
\end{aligned} \tag{29}
$$

9.2

9.2.1 time

$$
\begin{aligned}
v &= \frac{e^{-\frac{bt}{m}}(-mg - bv_0) + mg}{-b} \\
\frac{-bv - mg}{-mg - bv_0} &= e^{-\frac{bt}{m}} \\
\ln|\frac{-bv - mg}{-mg - bv_0}| &= -\frac{bt}{m} \\
t &= \frac{m}{-b}\ln|\frac{-mg}{-mg - bv_0}|
\end{aligned} \tag{30}
$$

9.2.2 position

$$
\begin{aligned}
y_{max} &= y_0 + \frac{(-mg) - bv_0}{-b}\frac{m}{-b}(e^{\frac{-b\frac{m}{-b}\ln|\frac{-mg}{-mg-bv_0}|}{m}} - 1) + \frac{mg\frac{m}{-b}\ln|\frac{-mg}{-mg-bv_0}|}{-b} \\
y_{max} &= y_0 + \frac{(-mg) - bv_0}{-b}\frac{m}{-b}(\frac{-mg}{-mg - bv_0} - 1) + \frac{mg\frac{m}{-b}\ln|\frac{-mg}{-mg-bv_0}|}{-b} \\
y_{max} &= y_0 + \frac{(-mg) - bv_0}{-b}\frac{m}{-b}(\frac{-mg}{-mg - bv_0} - 1) + \frac{m^2}{b^2}g\ln|\frac{-mg}{-mg - bv_0}|
\end{aligned} \tag{31}
$$

9.3

$$
\begin{aligned}
\ln|\frac{-mg}{-mg - bv_0}| = -\ln|\frac{-mg - bv_0}{-mg}| = -\ln|1 + \frac{bv_0}{mg}| \\
-\ln|1 + \frac{bv_0}{mg}| = -(\frac{bv_0}{mg} - \frac{1}{2}(\frac{bv_0}{mg})^2)
\end{aligned}
$$

$$
\begin{aligned}
y_{max} &= y_0 + \frac{(-mg) - bv_0}{-b}\frac{m}{-b}(\frac{-mg}{-mg - bv_0} - 1) + \frac{m^2}{b^2}g - (\frac{bv_0}{mg} - \frac{1}{2}(\frac{bv_0}{mg})^2) \\
y_{max} &= y_0 + \frac{m}{-b}(\frac{mg}{b} - \frac{mg - bv_0}{b}) + \frac{m^2}{b^2}g - (\frac{bv_0}{mg} - \frac{1}{2}(\frac{bv_0}{mg})^2) \\
y_{max} &= y_0 + \frac{m}{-b}(\frac{mg}{b} - \frac{mg - bv_0}{b}) + \frac{-mv_0}{b} + \frac{{v_0}^2}{2g} \\
y_{max} &= y_0 + \frac{m}{-b}(\frac{mg - mg - bv_0}{b}) + \frac{-mv_0}{b} + \frac{{v_0}^2}{2g} \\
y_{max} &= y_0 + \frac{m}{-b}(-v_0) + \frac{-mv_0}{b} + \frac{{v_0}^2}{2g} \\
y_{max} &= y_0 + \frac{{v_0}^2}{2g}
\end{aligned} \tag{32}
$$

10 Question 10

10.1

In polar coordinates:

$$\vec{r} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y} \tag{33}$$

$$|\vec{r}| = \sqrt{\cos^2(\phi) + \sin^2(\phi)} = 1 \tag{34}$$

$$\begin{aligned} \hat{r} &= \frac{\vec{r}}{|\vec{r}|} \\ &= \frac{\cos(\phi)\hat{x} + \sin(\phi)\hat{y}}{1} \\ &= \cos(\phi)\hat{x} + \sin(\phi)\hat{y} \end{aligned} \tag{35}$$

In polar coordinates we know: $\hat{\phi}$ is orthogonal to $\hat{r}$, which geometrically means that $\hat{\phi}$ is rotated $90°$ from $\hat{r}$. We will use this fact to deduce an expression for $\hat{\phi}$

$$\hat{\phi} = -\cos(\phi)\hat{x} + \sin(\phi)\hat{y} \tag{36}$$

You could also have defined $\hat{\phi}$ with a negative sin instead of cos.

10.2

11 Question 11

11.1

$$v(t) = c \tag{37}$$

$$\dot{v}(t) = \frac{d}{dt}v(t) = 0 \tag{38}$$

$$v(t) \cdot \dot{v}(t) = v(t) \cdot 0 = 0 \tag{39}$$

The dot product of orthogonal vectors is 0, therefore $v(t)$ and $\dot{v}(t)$ are orthogonal.

11.2

$$\begin{aligned} \vec{v} \cdot \frac{d\vec{v}}{dt} &= 0 \\ \frac{dv^2}{dt} &= 0 \\ \text{From this result we know that } \vec{v} \cdot \vec{v} &= c \\ |v| \cdot |v| \cos\theta &= c \\ \text{Since this is the cosine between the same vector } \theta = 0,\ \cos\theta &= 1 \\ |v| \cdot |v| &= c \\ |v|^2 &= c \\ |v| &= \sqrt{c} \end{aligned} \tag{40}$$

From this result we can conclude that $|v|$ is a constant.