Physics 410 - Homework 3

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1 Problem 1

1.1

$$\begin{aligned} l &= \vec{r} \times \vec{p} \\ &= rmv \sin\phi \\ \phi = 90^\circ \\ &= rmv \\ \omega = \frac{v}{r},\ v = \omega r \\ &= r^2 m\omega \end{aligned} \tag{1}$$

1.2

$$\begin{aligned} \frac{d}{dt}(\frac{dA}{dt}) = 0 = \text{Kepler's Second Law} \\ \frac{dA}{dt} &= \frac{1}{2}\omega r^2 \\ \frac{dA}{dt} &= \frac{1}{2}\frac{v}{r}r^2 \\ \frac{dA}{dt} &= \frac{1}{2}vr \\ \frac{d}{dt}(\frac{dA}{dt}) &= (\frac{1}{2}vr)\frac{d}{dt} \\ &= \frac{1}{2}(v\dot{r} + r\dot{v}) \\ &= \frac{1}{2}(0+0) \\ \frac{d}{dt}(\frac{dA}{dt}) &= 0 \end{aligned} \tag{2}$$

2 Problem 2

2.1

Prove $\dot{l} = 0$

Since the force is always being applied parallel to r, the momentum never changes

$$
\begin{aligned}
\dot{l} &= \Gamma^{ext} \\
\Gamma^{ext} &= \vec{F} \times \vec{r} \\
\Gamma^{ext} = 0 &= \dot{l}
\end{aligned} \tag{3}
$$

This is true since the force is being applied parallel to the radius at all points. Therefore their cross product is 0 and there is no change in angular momentum.

2.2

$$I = mr^2$$

$$l_{final} = I_{final}\omega_{final}$$

$$l_{initial} = I_{initial}\omega_{initial}$$

Since angular momentum, l, is constant in the system, we can set the $l_{initial} = l_{final}$, and solve for ω_{final}

$$
\begin{aligned}
l_{initial} &= l_{final} \\
mr^2_{initial}\omega_{initial} &= mr^2_{final}\omega_{final} \\
r^2_{initial}\omega_{initial} &= r^2_{final}\omega_{final} \\
\omega_{final} &= \frac{r^2_{initial}\omega_{initial}}{r^2_{final}}
\end{aligned} \tag{4}
$$

2.3

$$KE = \frac{1}{2}I\omega^2$$

We can calculate the kinetic energy before and after pulling the string to calculate the work done.

$$KE_{initial} = \frac{1}{2}mr^2_{initial}\omega^2_{initial}$$

$$KE_{final} = \frac{1}{2}mr^2_{final}\omega^2_{final}$$

$$
\begin{aligned}
KE_{initial} + W &= KE_{final} \\
\frac{1}{2}mr^2_{initial}\omega^2_{initial} + W &= \frac{1}{2}mr^2_{final}\omega^2_{final} \\
\frac{1}{2}mr^2_{initial}\omega^2_{initial} + W &= \frac{1}{2}mr^2_{final}(\frac{r^2_{initial}\omega_{initial}}{r^2_{final}})^2 \\
W &= \frac{1}{2}mr^2_{final}(\frac{r^4_{initial}\omega^2_{initial}}{r^4_{final}}) - \frac{1}{2}mr^2_{initial}\omega^2_{initial} \\
&= \frac{1}{2}mr^2_{initial}\omega^2_{initial}(\frac{r^2_{initial}}{r^2_{final}} - 1)
\end{aligned} \tag{5}
$$

Since the work is non-zero and $(\frac{r_{initial}^2}{r_{final}^2} - 1)$ will be a positive, we know that work is positive and therefore $KE_{final} > KE_{initial}$

3 Problem 3

$$W = \int_0^P F \cdot dr = \int_0^P (F_x dx + F_y dy)$$
$$F = (x^2, 2xy)$$

3.1

$$\int_0^P (F_x dx + F_y dy) = \int_0^Q (F_x dx) + \int_0^P (F_y dy)$$

Since x and y both go from 0 to 1, we can replace the bounds of integration

$$\begin{aligned}
&= \int_0^1 (x^2 dx) + \int_0^1 (2xydy) \\
&= \frac{x^3}{3}|_0^1 + \int_0^1 (21ydy) \\
&= \frac{1}{3} + 2\int_0^1 (\frac{y^2}{2}dy) \\
&= \frac{1}{3} + 2[\frac{y^2}{2}|_0^1] \\
&= \frac{1}{3} + 1 \\
&= \frac{4}{3}
\end{aligned} \tag{6}$$

3.2

$$\begin{aligned}
\int_0^P (F_x dx + F_y dy) &= \int_0^Q (F_x dx) + \int_0^P (F_y dy) \\
&= \int_0^Q (x^2 dx) + \int_0^P (2xydy) \\
&= \int_0^Q (x^2 dx) + \int_0^P (2x^3 2xdx) \\
&= \int_0^1 (x^2 dx) + \int_0^1 (2x^3 2xdx) \\
&= \frac{1}{3} + \frac{4}{5} \\
&= \frac{17}{15}
\end{aligned} \tag{7}$$

3.3

$$\begin{gathered} 0 \leq t \leq 1 \\ F = (t^6, 2t^5) \end{gathered}$$

$$\begin{aligned} \int_0^Q (F_x dx) + \int_0^P (F_y dy) &= \int_0^1 t^6 3t^2 dt + \int_0^1 2t^5 2t dt \\ &= 3\int_0^1 t^8 dt + 4\int_0^1 t^6 dt \\ &= (3 * \frac{1}{9}) + (4 * \frac{1}{7}) \\ &= \frac{1}{3} + \frac{4}{7} \\ &= \frac{19}{21} \end{aligned} \tag{8}$$

4 Problem 4

$$E = U + T$$

$$U = mgR\cos\phi \tag{9}$$

$$E_0 = mgR \tag{10}$$

$$E_f = mgR\cos\phi + \frac{mv^2}{2} \tag{11}$$

Using conservation of energy we know the $E_0 = E_f$

$$\begin{aligned} E_0 &= E_f \\ mgR &= mgR\cos\phi + \frac{mv^2}{2} \\ gR - gR\cos\phi &= \frac{v^2}{2} \\ 2gR(1-\cos\phi) &= v^2 \\ v &= \sqrt{2gR(1-\cos\phi)} \end{aligned} \tag{12}$$

The puck will leave the sphere when the normal force is 0.

$$\begin{aligned} m\dot{v} &= -F_N + F_g \\ m\frac{v^2}{r} &= -F_N + mg\cos\phi \\ F_N &= -m\frac{v^2}{r} + mg\cos\phi \end{aligned} \tag{13}$$

Solving for when $F_N = 0$ and plugging in our value for the velocity in terms of ϕ

$$
\begin{aligned}
F_N &= -m\frac{v^2}{R} + mg\cos\phi \\
0 &= -m\frac{(\sqrt{2gR(1-\cos\phi)})^2}{R} + mg\cos\phi \\
m\frac{(\sqrt{2gR(1-\cos\phi)})^2}{R} &= mg\cos\phi \\
2(1-\cos\phi) &= \cos\phi \\
2 - 2\cos\phi &= \cos\phi \\
2 &= 3\cos\phi \\
\frac{2}{3} &= \cos\phi \\
\phi &= 48.18^\circ
\end{aligned} \tag{14}
$$

However, it should leave the surface at a max 45° as it does not make physical sense for the puck to gain momentum in the $-\hat{x}$ which would occur when the puck basically loops around the sphere when $\phi = 48^\circ$. For this reason, I calculated the sin as well and got an answer of $\phi = 41.81^\circ$, which might be more reasonable.

5 Problem 5

5.1 $f(x, y, z) = ax^2 + bxy + cy^2$

$$
\begin{aligned}
&\frac{\partial}{\partial x}(ax^2 + bxy + cy^2) \\
&\quad = \frac{\partial}{\partial x}ax^2 + \frac{\partial}{\partial x}bxy + 0 \\
&\quad = 2ax + by
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial y}(ax^2 + bxy + cy^2) \\
&\quad = ax^2 + \frac{\partial}{\partial y}(bxy) + \frac{\partial}{\partial y}(cy^2) \\
&\quad = 0 + bx + 2cy \\
&\quad = bx + 2cy
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial z}(ax^2 + bxy + cy^2) \\
&\quad = 0
\end{aligned} \tag{17}
$$

5.2 $g(x, y, z) = \sin(axyz^2)$

$$
\begin{aligned}
&\frac{\partial}{\partial x}(\sin(axyz^2)) \\
&\quad = \frac{\partial}{\partial x}(\sin(axyz^2)) * \frac{\partial}{\partial x}(axyz^2) \\
&\quad = \cos(axyz^2)(ayz^2)
\end{aligned} \tag{18}
$$

$$\begin{aligned}
&\frac{\partial}{\partial y}(\sin(axyz^2)) \\
&\qquad = \frac{\partial}{\partial y}(\sin(axyz^2)) * \frac{\partial}{\partial y}(axyz^2) \\
&\qquad = \cos(axyz^2)(axz^2)
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\frac{\partial}{\partial z}(\sin(axyz^2)) \\
&\qquad = \frac{\partial}{\partial z}(\sin(axyz^2)) * \frac{\partial}{\partial z}(axyz^2) \\
&\qquad = \cos(axyz^2)(2axyz)
\end{aligned} \tag{20}$$

5.3 $h(x,y,z) = ae^{\frac{xy}{z^2}}$

$$\begin{aligned}
&\frac{\partial}{\partial x}(ae^{\frac{xy}{z^2}}) \\
&\qquad = a\frac{\partial}{\partial x}(e^{\frac{xy}{z^2}}) * \frac{\partial}{\partial x}(\frac{xy}{z^2}) \\
&\qquad = a(e^{\frac{xy}{z^2}})(\frac{y}{z^2})
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\frac{\partial}{\partial y}(ae^{\frac{xy}{z^2}}) \\
&\qquad = a\frac{\partial}{\partial y}(e^{\frac{xy}{z^2}}) * \frac{\partial}{\partial y}(\frac{xy}{z^2}) \\
&\qquad = a(e^{\frac{xy}{z^2}})(\frac{x}{z^2})
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{\partial}{\partial z}(ae^{\frac{xy}{z^2}}) \\
&\qquad = a\frac{\partial}{\partial z}(e^{\frac{xy}{z^2}}) * \frac{\partial}{\partial z}(\frac{xy}{z^2}) \\
&\qquad = a(e^{\frac{xy}{z^2}})(-\frac{2xy}{z^3})
\end{aligned} \tag{23}$$

6 Problem 6

$$U(r) = k(x^2 + y^2 + z^2)$$
$$\vec{F} \cdot dr = W = -U$$

$$\begin{aligned}
&\vec{F} \cdot dr = -U \\
&\quad \vec{F} \cdot dr = -[k(x^2 + y^2 + z^2)] \\
&\quad \vec{F} = -\frac{d}{dr}k(x^2 + y^2 + z^2) \\
&\quad = -k\frac{d}{dr}r^2 \\
&\quad = -k2r \\
&\quad = -2k(\sqrt{x^2 + y^2 + z^2})
\end{aligned} \tag{24}$$

7 Problem 7

$$\nabla \times F = 0$$

7.1 $F = k(x, 2y, 3z)$

$$\begin{aligned}\nabla \times F &= ([\frac{\partial}{\partial y}k3z] - [\frac{\partial}{\partial z}k2y])\hat{i} + ([\frac{\partial}{\partial z}kx] - [\frac{\partial}{\partial x}k3z])\hat{j} + ([\frac{\partial}{\partial x}k2y] - [\frac{\partial}{\partial y}kx])\hat{k} \\ &= 0\hat{i} + 0\hat{j} + 0\hat{k} \\ &= 0\end{aligned} \tag{25}$$

F is conservative

$$\begin{aligned}U &= -\int \vec{F} \cdot dr \\ &= -[\int F_x dx + \int F_y dy + \int F_z dz] \\ &= -[k\int xdx + 2k\int ydy + 3k\int zdz] \\ &= -k[(\frac{x^2}{2} + c_1) + 2(\frac{y^2}{2} + c_2) + 3(\frac{z^2}{2} + c_3)]\end{aligned} \tag{26}$$

Now we can verify $F = -\nabla U$

$$\begin{aligned}F &= -\nabla U \\ &= [\frac{\partial}{\partial x}(-\frac{kx^2}{2} - kc_1) + \frac{\partial}{\partial y}(-\frac{2ky^2}{2} - 2kc_2) + \frac{\partial}{\partial z}(-\frac{3kz^2}{2} - 3kc_3)] \\ &= [kx + k2y + k3z] \\ &= k(x, 2y, 3z)\end{aligned} \tag{27}$$

7.2 $F = k(y, x, 0)$

$$\begin{aligned}\nabla \times F &= ([\frac{\partial}{\partial y}0] - [\frac{\partial}{\partial z}kx])\hat{i} + ([\frac{\partial}{\partial z}ky] - [\frac{\partial}{\partial x}0])\hat{j} + ([\frac{\partial}{\partial x}kx] - [\frac{\partial}{\partial y}ky])\hat{k} \\ &= 0\hat{i} + 0\hat{j} + k[1-1]\hat{k} \\ &= 0\end{aligned} \tag{28}$$

F is conservative

$$\begin{aligned}U &= -\int \vec{F} \cdot dr \\ &= -[\int F_x dx + \int F_y dy + \int F_z dz] \\ &= -[ky\int dx + kx\int dy + \int 0dz] \\ &= -[ky(x + c_1) + kx(y + c_2)]\end{aligned} \tag{29}$$

Now we can verify $F = -\nabla U$

$$
\begin{aligned}
F &= -\nabla U \\
&= [\frac{\partial}{\partial x}(kyx + kyc_1) + \frac{\partial}{\partial y}(kxy + kxc_2) + \frac{\partial}{\partial z}(0)] \\
&= -[ky + kx + 0] \\
&= k(y, x, 0)
\end{aligned} \tag{30}
$$

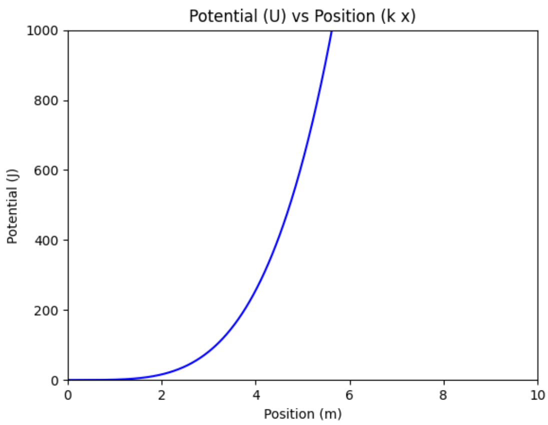
7.3 $F = k(-y, x, 0)$

$$
\begin{aligned}
\nabla \times F &= ([\frac{\partial}{\partial y}0] - [\frac{\partial}{\partial z}x])\hat{i} + ([\frac{\partial}{\partial z}y] - [\frac{\partial}{\partial x}0])\hat{j} + ([\frac{\partial}{\partial x}x] - [-\frac{\partial}{\partial y}y])\hat{k} \\
&= 0\hat{i} + 0\hat{j} + [1 + 1]\hat{k} \\
&= 2\hat{k}
\end{aligned} \tag{31}
$$

F is not conservative

8 Problem 8

8.1



8.2

$$
\begin{aligned}
t &= \int_{x_0}^{x} \frac{x'}{\dot{x}(x')} \\
&= \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}}
\end{aligned}
$$

$$U = kx^4$$

$$x_{max} = A$$

At x_{max}, $KE = 0$ therefore $E = U(A^4)$

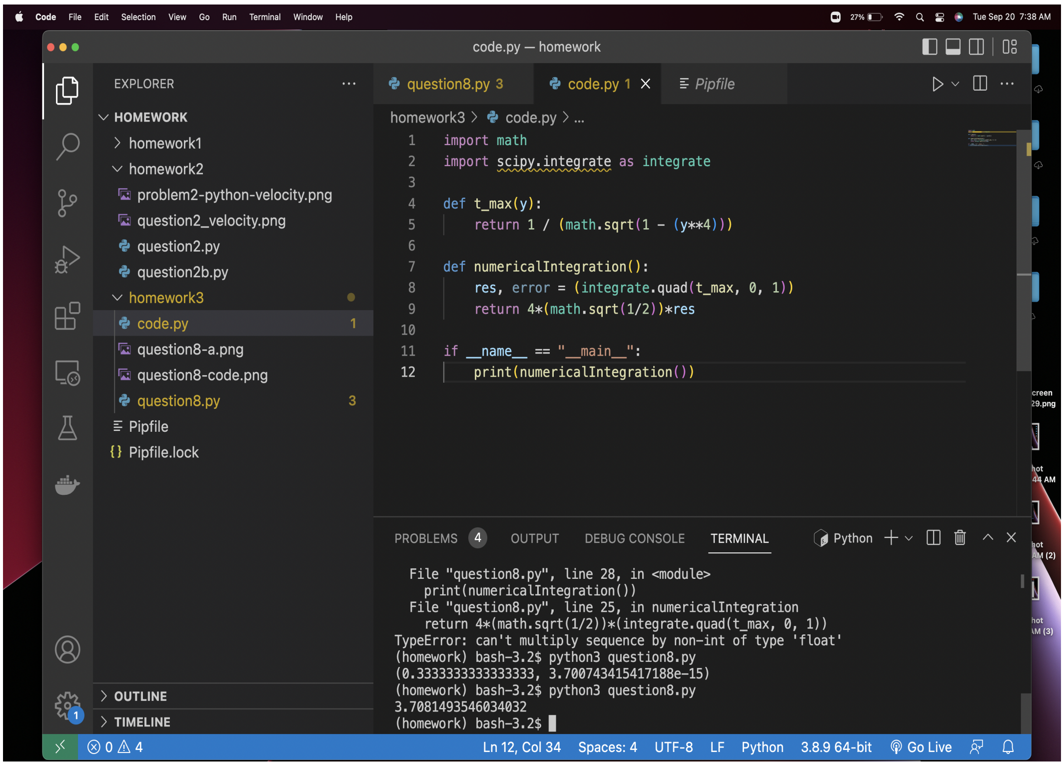
$$\begin{aligned} t_{max} &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx'}{\sqrt{E - U(x')}} \\ &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx'}{\sqrt{kA^4 - kx'^4}} \\ &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx'}{\sqrt{k(A^4 - x'^4)}} \end{aligned} \tag{32}$$

8.3

$$\begin{aligned} t_{max} &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx'}{\sqrt{k(A^4 - x'^4)}} \\ &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx'}{\sqrt{kA^4}\sqrt{(1 - \frac{kx'^4}{kA^4})}} \\ &= \sqrt{\frac{m}{2}} \frac{1}{\sqrt{kA^4}} \int_0^A \frac{dx'}{\sqrt{(1 - \frac{x'^4}{A^4})}} \\ &= \sqrt{\frac{m}{2}} \frac{1}{A^2\sqrt{k}} \int_0^A \frac{dx'}{\sqrt{(1 - \frac{x'^4}{A^4})}} \\ y = \frac{x}{A},\ dy = \frac{1}{A}dx & \\ &= \sqrt{\frac{m}{2}} \frac{1}{A^2\sqrt{k}} \int_0^A \frac{1}{\sqrt{(1 - y^4)}dyA} \\ &= \sqrt{\frac{m}{2k}} \frac{1}{A} \int_0^A \frac{1}{\sqrt{(1 - y^4)}dy} \\ \text{Multiply by 4 to get full period} & \\ &= 4\sqrt{\frac{m}{2k}} \frac{1}{A} \int_0^A \frac{1}{\sqrt{(1 - y^4)}dy} \end{aligned} \tag{33}$$

8.4

$$\begin{aligned} t_{max} &= 4\sqrt{\frac{1}{2}} \int_0^1 \frac{1}{\sqrt{(1 - y^4)}dy} \\ &= 3.708s \end{aligned} \tag{34}$$



9 Problem 9

$$U(\phi) = mgl(1 - \cos\phi)$$
$$4t = \tau$$
$$T = \frac{1}{2}m(l\dot{\phi})^2$$

$$\begin{aligned} E(\phi) &= U(\phi) + T(\phi) \\ &= mgl(1 - \cos\phi) + \frac{1}{2}m(l\dot{\phi})^2 \end{aligned} \tag{35}$$

Using this equation for energy we can solve for $\dot{\phi}$ in terms of ϕ

$$\begin{aligned} E(\phi) &= U(\phi) + T(\phi) \\ E(\phi) &= mgl(1 - \cos\phi) + \frac{1}{2}m(l\dot{\phi})^2 \\ \frac{E(\phi)}{ml^2} - \frac{mgl(1 - \cos\phi)}{l^2} &= \dot{\phi}^2 \\ \sqrt{\frac{E(\phi)}{ml^2} - \frac{g(1 - \cos\phi)}{l}} &= \dot{\phi} \end{aligned} \tag{36}$$

Since this is a conservative system, we know there is no change in energy and therefore $\delta E = 0$

$$\begin{aligned} U(\phi) &= mg(1-\cos\phi) \\ \frac{U}{mg} &= (1-\cos\phi) \\ \phi &= \cos^{-1}(1-\frac{U}{mg}) \end{aligned} \tag{37}$$

$$\begin{aligned} U(\phi) &= KE \\ mgl(1-\cos\phi) &= \frac{1}{2}m(l\dot{\phi})^2 \\ 2g(1-\cos\phi) &= \dot{\phi}^2 \\ \dot{\phi} &= \sqrt{2g(1-\cos\phi)} \end{aligned} \tag{38}$$

$$t = \int_0^{\Phi} \frac{d\phi}{\sqrt{2g(1-\cos\phi)}} \tag{39}$$

10 Problem 10

10.1

$$\begin{aligned} \tau &= \tau_0 = 2\pi\sqrt{\frac{l}{g}} \\ \tau &=_0 \frac{2}{\pi}\int_0^1 \frac{du}{\sqrt{1-u^2}} \\ &= \tau_0\frac{2}{\pi} * (\frac{\pi}{2}) \\ \tau &= \tau_0 \end{aligned} \tag{40}$$

$$\begin{aligned} \tau &= \tau_0\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-u^2}}(1+\frac{1}{2}A^2u^2)du \\ &= \tau_0[1+\frac{1}{4}\sin^2(\frac{\Phi}{2})] \end{aligned} \tag{41}$$