Physics 410 - Homework 5

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Problem 1

$$d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \tag{1}$$

$$\begin{aligned}
d\vec{s} &= rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \\
|d\vec{s}| &= \sqrt{r^2 d\theta^2 + r^2\sin^2 d\phi^2} \\
&= r\sqrt{d\theta^2 + \sin^2 d\phi^2} \\
&= rd\theta\sqrt{1 + \sin^2 \frac{d\phi}{d\theta}^2} \\
&= rd\theta\sqrt{1 + \sin^2 \phi'(\theta)^2} \\
L = \int_{\theta_1}^{\theta_2} |ds| &= R\int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2 \phi'(\theta)^2} d\theta
\end{aligned} \tag{2}$$

Problem 2

$$\begin{aligned}
&\int_0^p (y'^2 + yy' + y^2)dx \\
&ds = \sqrt{dx^2 + dy^2} \\
&y'(x) = 0;\ y''(x) = 0
\end{aligned} \tag{3}$$

$$y(x) = c_1 e^x + c_2 e^{-x} \tag{4}$$

Point: $P(0,0)$

$$\begin{aligned}
&y(0) = c_1 e^0 + c_2 e^{-0} = 0 \\
&c_1 = -c_2
\end{aligned} \tag{5}$$

Point: $P(1,1)$

$$\begin{aligned} y(1) &= c_1e^1 - c_2e^{-1} = 1 \\ &= c_1(e - \frac{1}{e}) = 1 \\ c_1 &= (e - \frac{1}{e})\frac{(e^2-1)}{e} \\ c_1 &= \frac{e}{e^2-1} \end{aligned} \tag{6}$$

$$y(x) = \frac{e^{x+1} - e^{-x+1}}{e^2-1} \tag{7}$$

Problem 3

Setup

$$y = y(x)$$
$$\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}dx$$

Partials

$$\frac{\partial f}{\partial y} = 0 \tag{8}$$

$$\begin{aligned} \frac{\partial f}{\partial y'} &= (1+y^{\partial})^{\frac{1}{2}} \\ &= \frac{y'\sqrt{x}}{\sqrt{1+y'^2}} \end{aligned} \tag{9}$$

Euler-Lagrange

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} &= 0 \\ -\frac{d}{dx}\frac{\partial f}{\partial y'} &= 0 \\ -\frac{d}{dx}\frac{y'\sqrt{x}}{\sqrt{1+y'^2}} &= 0 \end{aligned} \tag{10}$$

Therefore $\frac{\partial f}{\partial y'}$ must not be constant.

Solve for y'

$$\begin{aligned}
\frac{y'\sqrt{x}}{\sqrt{1+y'^2}} &= C \\
y'\sqrt{x} &= C\sqrt{1+y'^2} \\
y'^2 x &= C^2(1+y'^2) \\
y'^2 x - C^2 y'^2 &= C^2 \\
y'^2(x-C^2) &= C^2 \\
y' &= \sqrt{\frac{C^2}{(x-C^2)}} \\
y' &= \frac{C}{\sqrt{(x-C^2)}}
\end{aligned} \tag{11}$$

Solve for y

$$\begin{aligned}
\int y' &= \int \frac{C}{\sqrt{(x-C^2)}} dx \\
y(x) &= C\int \frac{1}{\sqrt{(x-C^2)}} dx \\
u = x - c^2 \ ; \ du = dx \\
y(x) &= C\int \frac{1}{\sqrt{u}} du \\
y(x) &= C_1(2\sqrt{u}) + C_2 \\
y(x) &= C_1 2(\sqrt{x - {C_1}^2}) + C_2
\end{aligned} \tag{12}$$

Problem 4

Setup

$$E_o = E_f$$

Solve for v_f

$$\begin{aligned}
\frac{1}{2}m{v_o}^2 &= \frac{1}{2}m{v_f}^2 - mgy \\
v_f &= \sqrt{{v_o}^2 + 2gy}
\end{aligned} \tag{13}$$

Equation for t

$$\begin{aligned}
ds &= \sqrt{dx^2 + dy^2} \\
&= dy\sqrt{\frac{dx}{dy}^2 + 1} \\
&= dy\sqrt{x'(y)^2 + 1} \\
t &= \int_1^2 \frac{\sqrt{x'(y)^2 + 1}}{\sqrt{{v_o}^2 + 2gy}} dy
\end{aligned} \tag{14}$$

Euler-Lagrange

$$\begin{gathered}
f(x, x', y) = \frac{\sqrt{x'(y)^2 + 1}}{\sqrt{{v_o}^2 + 2gy}} dy \\
\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'} \\
\tfrac{\partial f}{\partial x} = 0 \ ; \ \tfrac{d}{dy}\tfrac{\partial f}{\partial x'} = 0
\end{gathered} \tag{15}$$

Solve for x'

$$\begin{aligned}
\frac{\partial f}{\partial x'} &= \frac{\partial f}{\partial x'}\frac{\sqrt{x'^2 + 1}}{\sqrt{{v_o}^2 + 2gy}} \\
&= \frac{x'}{\sqrt{(x'^2 + 1)({v_o}^2 + 2gy)}}
\end{aligned} \tag{16}$$

$$\begin{aligned}
\frac{d}{dy}\frac{x'}{\sqrt{(x'^2 + 1)({v_o}^2 + 2gy)}} &= 0 \\
\frac{x'}{\sqrt{(x'^2 + 1)({v_o}^2 + 2gy)}} &= C
\end{aligned} \tag{17}$$

$$
\begin{aligned}
\frac{x'^2}{(x'^2+1)(v_o{}^2+2gy)} &= C^2 \\
x'^2 &= C^2(x'^2+1)(v_o{}^2+2gy) \\
x'^2 &= C^2(x'^2 v_o{}^2 + 2gyx'^2 + v_0{}^2 + 2gy) \\
x'^2 &= (C^2x'^2 v_o{}^2 + C^2 2gyx'^2 + C^2 v_0{}^2 + 2gy) \\
x'^2 - C^2x'^2v_o{}^2 - C^2 2gyx'^2 &= (C^2v_0{}^2 + 2gy) \\
x'^2(1 - C^2v_o{}^2 - C^2 2gy) &= (C^2v_0{}^2 + 2gy) \\
x'^2 &= \frac{(C^2v_0{}^2+2gy)}{(1-C^2v_o{}^2 - C^2 2gy)} \\
x' &= \sqrt{\frac{(C^2v_0{}^2+2gy)}{(1-C^2v_o{}^2 - C^2 2gy)}}
\end{aligned}
\tag{18}
$$

Solve for x

$$
\begin{aligned}
y &= a(1-\cos\theta) - \frac{v_o{}^2}{2g} \\
\int x' &= \int \sqrt{\frac{(C^2v_0{}^2+2gy)}{(1-C^2v_o{}^2 - C^2 2gy)}} \\
x &= \int \sqrt{\frac{(C^2v_0{}^2+2ga(1-\cos\theta) - \frac{v_o{}^2}{2g})}{(1-C^2v_o{}^2 - C^2 2ga(1-\cos\theta) - \frac{v_o{}^2}{2g})}}
\end{aligned}
\tag{19}
$$

This integral is particularly nasty. I am not really sure what to do, I plugged it into a calculator but there is no dy to indicate what this is being integrated over. I really have no clue what to do and I just added an x + c just because.

$$
x = \left[\sqrt{\frac{(C^2v_0{}^2+2ga(1-\cos\theta) - \frac{v_o{}^2}{2g})}{(1-C^2v_o{}^2 - C^2 2ga(1-\cos\theta) - \frac{v_o{}^2}{2g})}}\right] x + c \tag{20}
$$

Problem 5

Do Taylor's problem 6.27
Hint: In the end, look at ratios like $\frac{y'}{x'}$. How is that related to $\frac{dy}{dx}$?

Euler-Lagrange

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{d}{du}\frac{\partial f}{\partial x'}\\ \frac{\partial f}{\partial y} &= \frac{d}{du}\frac{\partial f}{\partial y'}\\ \frac{\partial f}{\partial z} &= \frac{d}{du}\frac{\partial f}{\partial z'}\end{aligned}$$

$$S = \int_{u_1}^{u_2} f[x(u), y(u), z(u), x'(u), y'(u), z'(u), u]du \tag{21}$$

$$dS = \sqrt{dx^2 + dy^2 + dz^2} \tag{22}$$

$$f(x(u), y(u), z(u), x'(u), y'(u), z'(u), u) = \sqrt{x'^2 + y'^2 + z'^2} \tag{23}$$

Partials

$$\begin{aligned}0 &= \frac{d}{du}\frac{\partial f}{\partial x'}\\ 0 &= \frac{d}{du}\frac{\partial f}{\partial y'}\\ 0 &= \frac{d}{du}\frac{\partial f}{\partial z'}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x} f(x(u), y(u), z(u), x'(u), y'(u), z'(u), u) &= \frac{\partial}{\partial x}\sqrt{x'^2 + y'^2 + z'^2}\\ &= \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}}\end{aligned} \tag{24}$$

Same for y' and z'

$$\begin{aligned}0 &= \frac{d}{du}\frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}}\\ 0 &= \frac{d}{du}\frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}}\\ 0 &= \frac{d}{du}\frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}}\end{aligned}$$

Looking at the ratios of the partials

$$\frac{x'}{y'} = \frac{dx}{dy} = \frac{c_1}{c_2} \tag{25}$$

$$\frac{x'}{z'} = \frac{dx}{dz} = \frac{c_1}{c_3} \tag{26}$$

$$\frac{y'}{z'} = \frac{dy}{dz} = \frac{c_2}{c_3} \tag{27}$$

Therefore we can conclude:

$$\frac{dx}{c_1} = \frac{dy}{c_2} = \frac{dz}{c_3} \tag{28}$$

Which is the equation for a straight line in three dimensions.

Problem 6

Do Taylor's problem 7.4
Note: The plane being described here is like a sheet of plywood resting at an angle, and the x coordinate is the "sideways" direction on the surface of the plywood.

Setup

$$S = \int_{t_1}^{t_2} \mathcal{L} dt$$
$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\dot{q}_i}$$
$$\mathcal{L} = T - U$$

Lagrangian

$$T = \frac{mv^2}{2} = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} \tag{29}$$

$$U = -mgy\sin(\alpha) \tag{30}$$

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{m(\dot{x}^2 + \dot{y}^2)}{2} + mgy\sin(\alpha) \end{aligned} \tag{31}$$

For x

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\ 0 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\end{aligned} \tag{32}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{\partial \mathcal{L}}{\partial \dot{x}}\frac{m(\dot{x}^2+\dot{y}^2)}{2} + mgy\sin(\alpha) \\ &= \frac{m\dot{x}^2}{2}\end{aligned} \tag{33}$$

$$\frac{d}{dt}\frac{m\dot{x}^2}{2} = m\ddot{x} \tag{34}$$

$$\boxed{m\ddot{x} = 0}$$

For y

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \tag{35}$$

$$\frac{\partial \mathcal{L}}{\partial y} = mg\sin(\alpha) \tag{36}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{m\dot{y}^2}{2} \tag{37}$$

$$\frac{d}{dt}\frac{m\dot{y}^2}{2} = m\ddot{y} \tag{38}$$

$$m\ddot{y} = mg\sin(\alpha) \tag{39}$$

$$\boxed{\ddot{y} = g\sin(\alpha)}$$

Acceleration in y, no acceleration in x. This is what we expected.

Problem 7

Do Taylor's problem 7.10
Hint: Begin by sketching the coordinate axes and the cone, reading the description carefully to make sure you understand how the cone is arranged and how its proportions are described. To clarify the instructions in the last sentence: your job is to find expressions for each of x, y and z in terms of ρ and ϕ, and also to find expressions for each of ρ and ϕ in terms of x, y and z.

$$\tan\alpha = \frac{\rho}{z}$$
$$z = \frac{\rho}{\tan\alpha}$$

$$\rho = \sqrt{x^2 + y^2}$$
$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$

Problem 8

Do Taylor's problem 7.15

Setup

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}(m_1 + m_2)(\dot{x})^2 + m_2 g x\end{aligned}$$

$$T = \frac{1}{2}m_1(\dot{x})^2 + \frac{1}{2}m_2(\dot{x})^2$$

$$\frac{1}{2}(m_1 + m_2)(\dot{x})^2$$

$$U = -m_2 g x$$

Partial

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}$$
$$m_2 g = (m_1 + m_2)\ddot{x} \tag{40}$$
$$\boxed{\ddot{x} = \frac{m_2 g}{(m_1 + m_2)}}$$

$$\frac{\partial\mathcal{L}}{\partial x} = m_2 g \tag{41}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = (m_1 + m_2)\dot{x} \tag{42}$$

$$\frac{d}{dt}(m_1 + m_2)\dot{x} = (m_1 + m_2)\ddot{x} \tag{43}$$

Problem 9

Do Taylor's problem 7.15

Be sure to answer all the questions it asks, even though they are not called out as parts a, b, c, etc.

Setup

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}m(\dot{r} + (r\dot{\phi}))^2\end{aligned}$$

$$T = \frac{1}{2}m(\dot{r} + (r\dot{\phi}))^2$$

$$U = 0$$

Partials

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 \tag{44}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \tag{45}$$

$$\frac{d}{dt}m\dot{r} = m\ddot{r} \tag{46}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\ mr\dot{\phi}^2 &= m\ddot{r} \\ r\dot{\phi}^2 &= \ddot{r}\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{d^2r}{dt^2} &= r\omega^2 \\ &= Ae^{\omega t} + Be^{-\omega t}\end{aligned} \tag{48}$$

At origin at rest

There is no energy

If $r_o > 0$

$$r(0) = A + B \tag{49}$$

Show $r(t)$ grows exponentially

$$\begin{aligned}r(t) &= Ae^{\omega t} + B\frac{1}{\omega t} \\ \text{As } t &\to \infty \\ r(t) &= Ae^{\omega t}\end{aligned} \tag{50}$$

As t grows, the B coefficient will get smaller and smaller, the equation essentially becomes
$$r(t) = Ae^{\omega t}$$

Explain your results in terms of $m\omega^2 r$

$$m\ddot{r} = m\omega^2 r \tag{51}$$

This was explained in equation 46.