Physics 412 - Equations

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Vector fields

Intuition: Imagine fluid flow
The magnitude of the field (i.e. the strength of the field at a point) can be visualized as the density of the field lines at that point

Gradient

$$\nabla f = < f_x, f_y, f_z > \tag{1}$$

Will yield you a vector field of change in each direction. Gradient vector field

∇f points in direction of steepest ascent
$-\nabla f$ points in direction of steepest drop

$$\vec{F} = \nabla f \tag{2}$$

f is the potential function for $\vec{F}$, If this f exists such that $\vec{F} = \nabla f$, then $\vec{F}$ is a conservative vector field. and the function f is the potential function for $\vec{F}$

If you know $\vec{F}$, you can then work backward to find its potential function

$$f(x,y) = \int F_x dx + \int F_y dy \tag{3}$$

Divergence

$$\vec{\nabla} \cdot \vec{F} = scalar \tag{4}$$

Will describe how much fluid is flowing into or out of that region: if negative it is a sink, if positive it is a source

$Fluid\ in > Fluid\ out = +$ indicates some sort of source / spontaneous generation
Vice-versa and in any direction

Curl

$$\vec{\nabla} \times \vec{F} \tag{5}$$

Will describe how much fluid is flowing into or out of that region: if negative it is a sink, if positive it is a source

Contour plots

$$f(x, y) = k \tag{6}$$

You can set that k equal to any constant and you will see how things play out with the function. Think elevation map

Level surfaces

$$f(x, y, z) = k \tag{7}$$

Surfaces

$$z = f(x, y) \tag{8}$$

Remember that functions of 2 variables are just surfaces in 3d

Parameterization

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \tag{9}$$

You define a t to hit every single point on the curve. Every point in your 2 d function.

the curve you define will now have a direction of motion

t	x	y
t	x(t)	y(t)
2	x(2)	y(2)
n	x(n)	y(n)

Curves and line integrals

Curve is **smooth** if $r'(t)$ is continuous and $r'(t) \neq 0$ for all t

Line integral

$$\int_C f(x, y)ds = \int_a^b f(h(t), g(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt \tag{10}$$

$$\int_C f(x, y)ds = \int_{-C} f(x, y)ds \tag{11}$$

$$\int_C f(x, y, z)ds \tag{12}$$

ds indicates we are moving along the curve C
ds is a line integral of f with respect to arc length

As long as you touch each point on C from a to b one time and one time only, the value of your line integral will be independent of the parameterization you choose

Arclength

$$L = \int_a^b ds \tag{13}$$

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \tag{14}$$

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = ||\vec{r}'(t)|| \tag{15}$$

Line integrals of vector functions

$$\vec{F}(x, y, z) = \vec{r}(t) \tag{16}$$

$\vec{r}(t)$ is the parameterization of $\vec{F}(x, y, z)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \tag{17}$$

Unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} \tag{18}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_C \vec{F} \cdot \vec{T}(t) ds \\ &= \int_a^b \vec{F} \cdot \frac{\vec{r}'(t)}{||\vec{r}'(t)||} ||\vec{r}'(t)|| dt \\ &= \int_a^b \vec{F}(\vec{r}'(t)) \cdot \vec{r}'(t) dt \end{aligned} \tag{19}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-C} \vec{F} \cdot d\vec{r} \tag{20}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (Pdx + Qdy + Rdz) \tag{21}$$

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \tag{22}$$

Fun facts with line integrals

- $\int_C \nabla f \cdot d\vec{r}$ is independent of path
- if $\vec{F}$ is conservative $\int_C \vec{F} \cdot d\vec{r}$ is independent of path
- if $\int_C \vec{F} \cdot d\vec{r}$ is independent of path, $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C
- if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C, $\int_C \vec{F} \cdot d\vec{r}$ is independent of path
- Region D is open if it does not contain any boundary points
- Region D is connected is we can connect any two points in the region with a path that lies completely in D (like linear combination)
- Region D is simply connected if it is connected and contains no holes

FTC line integrals

Coordinate systems

Just like in linear algebra / quantum, you just use whichever coordinate system (i.e. - basis) makes it easiest to solve the problem at hand. Each of them has advantages, pros and cons that you should easily be able to recognize off the bat.

Approximations and their valid uses

Chapter 2

Coulomb Law

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{rr^2}\hat{rr} \tag{23}$$

$$rr = r - r'$$

Q is a test charge
rr is the distance from r (location of Q) to r' (location of q)
Remember to think about whether the force is attractive or repulsive, and confirm this in the sign you use for Q and q

$$\vec{F} = Q\vec{E} \tag{24}$$

Discrete charge distributions

$$\vec{E}(\vec{r}) \doteq \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{rr_i^2}\hat{rr}_i$$

Continuous charge distributions

$$\vec{E}(\vec{r}) \doteq \frac{1}{4\pi\epsilon_0}\int \frac{1}{rr_i^2}\hat{rr}_i dq$$

$$\textit{Distributed on line} = \lambda(\text{charge per unit length}) \rightarrow dq = \lambda dl' \tag{25}$$
$$\textit{Distributed on surface} = \sigma(\text{charge per unit area}) \rightarrow dq = \sigma da' \tag{26}$$
$$\textit{Distributed through volume} = \rho(\text{charge per unit volume}) \rightarrow dq = \rho d\tau' \tag{27}$$

$\vec{E}(\vec{r})$ can be thought of as the force per unit charge if you placed a test charge at point P
Just replace dq with the distribution if the charge is continuous
Always consider the symmetries at play as it will simplify your calculations
Coulombs is really the $\vec{E}(\vec{r})$ in 3d

Asymptotic Analysis

Flux through surface

- pg 69 in Griffiths

$$\begin{aligned} \Phi_E &= \int_S \vec{E} \cdot d\vec{a} \\ &= \frac{q}{\epsilon_0} \end{aligned} \tag{28}$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc} \tag{29}$$

F lux is a measure of the $\vec{E}$ field lines passing through a closed surface at any given point. It can only be impacted by charges on the inside, since any charge outside the closed surface will have field lines that pass in one side and out the other.

T he surface can be any shape. As long as it encloses the charge you will find eqn above to be true.

T he number of field lines is proportional to the strength of the charge - think back to magnitude of a vector field

F lux is the basis of Gauss law, since you have an $\vec{E}$ in the equation, you can see that when there is symmetry you can calculate what the electric field is from the flux.

N umber of field lines is constant from a source no matter your distance from source / sink.

Potential

$$V(r) = -\int_{\infty}^{r} E(r') \cdot dl' \tag{30}$$

Gauss Law

Using Divergence theorem

D ivergence theorem relates the surface integral of a vector field over a close surface (flux) to a volume integral of the divergence of the vector field in the volume enclosed

$$\oint_S E \cdot d\vec{a} = \int_\nu (\nabla \cdot E) d\tau$$

$$Q_{enc} = \int_\nu \rho d\tau$$

$$\int_\nu (\nabla \cdot E) d\tau = \int_\nu (\frac{\rho}{\epsilon_0}) d\tau \tag{31}$$

Differential form

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \tag{32}$$

Y ou should easily be able to go back and fourth between integral and differential form of Gauss law. This is good practice.

Faraday Law

Faraday Law

Green's theorem

Closed loop C creates region D

Curve C has positive orientation if we take it to be traced out counter-clockwise - ie region D is alwasy left of curve C as you trace

Stoke's theorem