Physics 401 - Homework 4

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Problem 1

Part A

$$S_n \doteq \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \tag{1}$$

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{n} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$\begin{aligned}
S_n &= S \cdot \hat{n} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta \\
&= \frac{\hbar}{2}\left(\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i(\sin\theta\sin\phi) \\ i(\sin\theta\sin\phi) & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}\right) \\
&= \frac{\hbar}{2}\left(\begin{pmatrix} \cos\theta & (\sin\theta\cos\phi - i(\sin\theta\sin\phi)) \\ (\sin\theta\cos\phi + i(\sin\theta\sin\phi)) & -\cos\theta \end{pmatrix}\right) \\
&= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}
\end{aligned}$$

Part B

$$\begin{aligned}
|\begin{smallmatrix}\frac{\hbar}{2}\cos\theta-\lambda & \frac{\hbar}{2}\sin\theta e^{-i\phi}\\ \frac{\hbar}{2}\sin\theta e^{i\phi} & -\frac{\hbar}{2}\cos\theta-\lambda\end{smallmatrix}| &= 0\\
-(\cos^2\theta\frac{\hbar^2}{4}-\lambda^2)-(\sin^2\theta\frac{\hbar^2}{4}) &= 0\\
(-\cos^2\theta\frac{\hbar^2}{4}+\lambda^2)-(\sin^2\theta\frac{\hbar^2}{4}) &= 0\\
\lambda^2-\cos^2\theta\frac{\hbar^2}{4}-\sin^2\theta\frac{\hbar^2}{4} &= 0\\
\lambda^2-\frac{\hbar^2}{4}(\cos^2\theta+\sin^2\theta) &= 0\\
\lambda^2 &= \frac{\hbar^2}{4}\\
\lambda &= \frac{\hbar}{2}
\end{aligned} \tag{2}$$

$$|+\rangle_n = a\,|+\rangle + b\,|-\rangle \tag{3}$$

$$\begin{aligned}
\begin{pmatrix}\frac{\hbar}{2}\cos\theta-\frac{\hbar}{2} & \frac{\hbar}{2}\sin\theta e^{-i\phi}\\ \frac{\hbar}{2}\sin\theta e^{i\phi} & -\frac{\hbar}{2}\cos\theta-\frac{\hbar}{2}\end{pmatrix}[\begin{smallmatrix}a\\b\end{smallmatrix}] &= 0\\
\frac{\hbar}{2}\begin{pmatrix}\cos\theta-1 & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta-1\end{pmatrix}[\begin{smallmatrix}a\\b\end{smallmatrix}] &= 0\\
\begin{pmatrix}\cos\theta-1 & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta-1\end{pmatrix}[\begin{smallmatrix}a\\b\end{smallmatrix}] &= 0\\
a(\cos\theta-1)+b\sin\theta e^{-i\phi} &= 0\\
a(\cos\theta-1) &= -b\sin\theta e^{-i\phi}\\
a &= -b\frac{\sin\theta e^{-i\phi}}{(\cos\theta-1)}\\
a &= -b\frac{\sin\theta\cos\phi-i\sin\theta\sin\phi)}{(\cos\theta-1)}
\end{aligned} \tag{4}$$

***** Insert trig identities *****

I honestly could not do all the trig

$$\begin{aligned}
|+\rangle_n &= \cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}e^{i\phi}\,|-\rangle\\
|-\rangle_n &= \sin\frac{\theta}{2}\,|+\rangle - \cos\frac{\theta}{2}e^{i\phi}\,|-\rangle
\end{aligned} \tag{5}$$

Part C

$$|+\rangle_y = \frac{1}{\sqrt{2}}[|+\rangle + i\,|-\rangle] \tag{6}$$

$$
\begin{aligned}
P_{+y} &= \left|{}_y\langle +|+\rangle_n\right|^2 \\
&= \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{1\pi}{3} \\ \sin\frac{1\pi}{3}e^{i\frac{3\pi}{2}}\end{pmatrix}\right|^2 \\
&= \left|\begin{pmatrix}\frac{1}{2\sqrt{2}} & -\frac{i\frac{\sqrt{3}}{2}-i}{\sqrt{2}}\end{pmatrix}\right|^2 \\
&= \left|\begin{pmatrix}\frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}}\end{pmatrix}\right|^2 \\
&= \left|\frac{1-\sqrt{3}}{2\sqrt{2}}\right|^2 \\
&= 0.07 \\
&= 7\%
\end{aligned} \tag{7}
$$

Part D

$$
|+\rangle_x = \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle] \tag{8}
$$

$$
\begin{aligned}
P_{+x} &= \left|{}_x\langle +|+\rangle_n\right|^2 \\
&= \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{3\pi}{8} \\ \sin\frac{3\pi}{8}e^{i\frac{2\pi}{3}}\end{pmatrix}\right|^2 \\
&= \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0.999 \\ 0.0205e^{i\frac{2\pi}{3}}\end{pmatrix}\right|^2 \\
&= \left|\frac{0.999}{\sqrt{2}} + \frac{0.0205e^{i\frac{2\pi}{3}}}{\sqrt{2}}\right|^2 \\
&= \left|0.699 + 0.125i\right|^2 \\
&= (0.699 + 0.125i)(0.699 - 0.125i) \\
&= 0.687 \\
&= 68.7\%
\end{aligned} \tag{9}
$$

Problem 2

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$$

$$S_y \doteq \frac{\hbar}{2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}$$

$$S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

$$
\begin{aligned}
[S_x, S_y] &= S_x S_y - S_y S_x \\
&= \left(\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} * \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right) - \left(\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} * \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) \\
&= (\frac{\hbar^2}{4})\left(\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\right) \\
&= (\frac{\hbar^2}{4})\begin{pmatrix}2i & 0\\ 0 & -2i\end{pmatrix} \\
&= (\frac{\hbar^2}{2})i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
&= i\hbar S_z
\end{aligned} \tag{10}
$$

Problem 3

Part A

$$|\psi\rangle = |-\rangle_y \tag{11}$$

$$|-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\ 1\end{pmatrix} \tag{12}$$

$$
\begin{aligned}
< S_z > &=_y \langle -| \, S_z \, |-\rangle_y \\
&= (-i \quad 1)\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}i\\ 1\end{pmatrix} \\
&= (-i \quad 1)\begin{pmatrix}i\\ -1\end{pmatrix} \\
&= (1 - 1) \\
&= 0
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\triangle S_z &= \sqrt{< {S_z}^2 > - < S_z >^2} \\
&= \sqrt{< {S_z}^2 >} \\
&= \frac{1}{2}\sqrt{{}_y\langle - | \, S_z^2 \, | - \rangle_y} \\
&= \frac{1}{2}(-i \quad 1)\frac{\hbar^2}{4}(\begin{matrix}1 & 0 \\ 0 & -1\end{matrix})(\begin{matrix}1 & 0 \\ 0 & -1\end{matrix})(\begin{matrix}i \\ 1\end{matrix}) \\
&= \frac{1}{2}(-i \quad 1)\frac{\hbar^2}{4}(\begin{matrix}1 & 0 \\ 0 & 1\end{matrix})(\begin{matrix}i \\ 1\end{matrix}) \\
&= \frac{1}{2}(-i \quad 1)\frac{\hbar^2}{4}(\begin{matrix}i \\ 1\end{matrix}) \\
&= \frac{1}{2}\frac{\hbar^2}{4}(-i \quad 1)(\begin{matrix}i \\ 1\end{matrix}) \\
&= \sqrt{\frac{\hbar^2}{4}} \\
&= \pm\frac{\hbar}{2}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
< S_x > &=_y \langle - | \, S_x \, | - \rangle_y \\
&= \frac{\hbar}{4}(-i \quad 1)(\begin{matrix}0 & 1 \\ 1 & 0\end{matrix})(\begin{matrix}i \\ 1\end{matrix}) \\
&= \frac{\hbar}{4}(-i \quad 1)(\begin{matrix}1 \\ i\end{matrix}) \\
&= \frac{\hbar}{4}(-i + i) \\
&= 0
\end{aligned} \tag{15}
$$

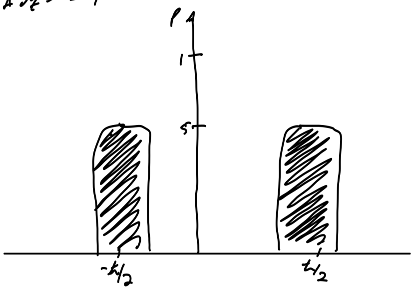
$$
\begin{aligned}
\triangle S_x &= \sqrt{< {S_x}^2 > - < S_x >^2} \\
&= \sqrt{< {S_x}^2 >} \\
&= \sqrt{\frac{1}{2}{}_y \langle - | \, S_x^2 \, | - \rangle_y} \\
&= \frac{\hbar^2}{4}\frac{1}{2}(-i \quad 1)(\begin{matrix}0 & 1\\ 1 & 0\end{matrix})(\begin{matrix}0 & 1\\ 1 & 0\end{matrix})(\begin{matrix}i\\ 1\end{matrix}) \\
&= \frac{\hbar^2}{4}\frac{1}{2}(-i \quad 1)(\begin{matrix}1 & 0\\ 0 & 1\end{matrix})(\begin{matrix}i\\ 1\end{matrix}) \\
&= \frac{\hbar^2}{4}\frac{1}{2}(-i \quad 1)(\begin{matrix}i\\ 1\end{matrix}) \\
&= \frac{\hbar^2}{4}\frac{1}{2}2 \\
&= \sqrt{\frac{\hbar^2}{4}} \\
&= \pm\frac{\hbar}{2}
\end{aligned} \tag{16}
$$

3. A

$|\psi_{in}\rangle = |-\rangle_y$

$\langle S_z \rangle = 0$

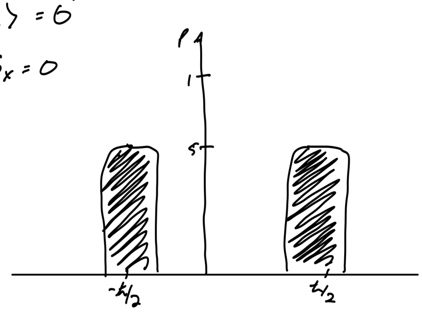
$\Delta S_z = \pm\hbar/2$



$|\psi_{in}\rangle = |-\rangle_y$

$\langle S_x \rangle = 0$

$\Delta S_x = 0$



Y es these both make physical sense since you are changing basis. You will then have the familiar 50% distribution we are used to seeing.

Part B

$$|\psi\rangle = \frac{1}{\sqrt{5}}(|+\rangle + 2i\,|-\rangle) \tag{17}$$

$$\begin{aligned}
< S_z > &= \langle\psi| S_z |\psi\rangle \\
&= (\tfrac{1}{\sqrt{5}} \quad -\tfrac{2i}{\sqrt{5}})\frac{\hbar}{2}(\begin{matrix}1 & 0\\ 0 & -1\end{matrix})(\begin{matrix}\frac{1}{\sqrt{5}}\\ \frac{2i}{\sqrt{5}}\end{matrix}) \\
&= (\tfrac{1}{\sqrt{5}} \quad -\tfrac{2i}{\sqrt{5}})\frac{\hbar}{2}(\begin{matrix}\frac{1}{\sqrt{5}}\\ -\frac{2i}{\sqrt{5}}\end{matrix}) \\
&= \frac{\hbar}{2}(\tfrac{1}{5} \quad -2i)(\begin{matrix}\frac{1}{\sqrt{5}}\\ -2i\end{matrix}) \\
&= \frac{\hbar}{2}(\frac{1}{5} + (-\frac{4}{5})) \\
&= -\frac{3\hbar}{10}
\end{aligned} \tag{18}$$

$$\begin{aligned}
\triangle S_z &= \sqrt{\frac{25\hbar^2}{100} - \frac{9\hbar^2}{100}} \\
&= \sqrt{\frac{16\hbar^2}{100}} \\
&= \frac{2\hbar}{5}
\end{aligned} \tag{19}$$

$$\begin{aligned}
< S_z >^2 &= (\frac{3\hbar}{10}) \\
&= \frac{9\hbar^2}{100}
\end{aligned} \tag{20}$$

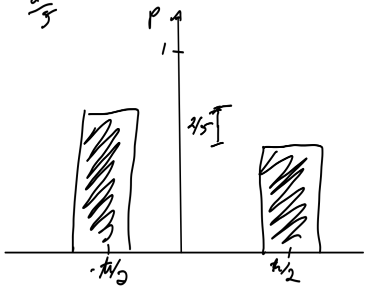
$$\begin{aligned}
< {S_z}^2 > &= (\tfrac{1}{\sqrt{5}} \quad -\tfrac{2i}{\sqrt{5}})\frac{\hbar^2}{4}(\begin{matrix}1 & 0\\ 0 & -1\end{matrix})(\begin{matrix}1 & 0\\ 0 & -1\end{matrix})(\begin{matrix}\frac{1}{\sqrt{5}}\\ \frac{2i}{\sqrt{5}}\end{matrix}) \\
&= (\tfrac{1}{\sqrt{5}} \quad -\tfrac{2i}{\sqrt{5}})\frac{\hbar^2}{4}(\begin{matrix}1 & 0\\ 0 & 1\end{matrix})(\begin{matrix}\frac{1}{\sqrt{5}}\\ \frac{2i}{\sqrt{5}}\end{matrix}) \\
&= \frac{\hbar^2}{4}(\tfrac{1}{\sqrt{5}} \quad -\tfrac{2i}{\sqrt{5}})(\begin{matrix}\frac{1}{\sqrt{5}}\\ \frac{2i}{\sqrt{5}}\end{matrix}) \\
&= \frac{\hbar^2}{4}(\tfrac{1}{\sqrt{5}} \quad -\tfrac{2i}{\sqrt{5}})(\begin{matrix}\frac{1}{\sqrt{5}}\\ \frac{2i}{\sqrt{5}}\end{matrix}) \\
&= \frac{\hbar}{2}
\end{aligned} \tag{21}$$

3B.

$|\psi_w\rangle = \frac{1}{\sqrt{5}}\left(|+\rangle + 2i\,|-\rangle\right)$

$\langle S_x \rangle = \frac{-3\hbar}{10}$

$\Delta S_x = \frac{2\hbar}{5}$



I was certainly confused on what was going on in this final one, partly because I am a little exhausted and partly because the numbers were not easy to play with in my head. I believe it physically makes sense, I am not sure I can explain why. I do know that there was no change of basis or previous measurements that would indicate these values to be unreasonable.

Problem 4

$$|\psi\rangle = \frac{1}{\sqrt{5}}(|+\rangle + 2i\,|-\rangle) \tag{22}$$

Part A

$$|+\rangle_n = \cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}e^{i\phi}\,|-\rangle$$

Since we know the values of our coefficients for the desired state we can just solve.

$$
\begin{aligned}
\frac{1}{\sqrt{5}} &= \cos\frac{\theta}{2} \\
\frac{\theta}{2} &= \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \\
\theta &= 2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \\
&= 126.87
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
e^{i\phi}\sin\left(\frac{\theta}{2}\right) &= \frac{2}{\sqrt{5}} \\
e^{i\phi}\sin\left(\frac{126.87}{2}\right) &= \frac{2}{\sqrt{5}} \\
e^{i\phi}(0.894) &= \frac{2}{\sqrt{5}} \\
e^{i\phi} &= 0 \\
\phi &= 0
\end{aligned} \tag{24}
$$

Part B

T his is addressed in McIntyre far more clearly than I am capable of, however this concept allowed me to truly be comfortable with the irreversible aspects of QM. Mathematically, projecting a vector onto another basis will certainly change it's value, even if it is 0 in one basis. It can have a value in another basis and even more importantly, it can have a value in the same basis if it leaves and then returns later. Using the fifth postulate, we know:

$$
|\psi'\rangle = \frac{P_n |\psi\rangle}{\sqrt{\langle\psi|\psi\rangle\, P_n\psi}} \tag{25}
$$

S ince we know what we've just measured, we can rewrite our equation above to find the output state:

$$
\begin{aligned}
|\psi'\rangle &= \frac{P_n |\psi\rangle}{\sqrt{\langle\psi|\psi\rangle\, P_n\psi}} \\
&= \frac{P_+ |\psi\rangle}{\sqrt{\langle\psi|\psi\rangle\, P_+\psi}} \\
&= |+\rangle
\end{aligned} \tag{26}
$$

U sing the information we are giving in the problem, I would believe this answer makes sense. The reason is the the numerator will give you only the "up" part of ψ while the denominator will normalize it for you to give you a perfect 100% in the spin-up state.