Physics 401 - Homework 2

Dillon Walton

September 2022

1 Problem 1

1.1

$$N(4cm) = \frac{3}{16} = 18.75\% \tag{1}$$

1.2

$$< L >= 5.25cm \tag{2}$$

1.3

$$N(< L >) = \frac{0}{16} = 0\% \tag{3}$$

1.4

$$< L^2 >= 31.75cm \tag{4}$$

1.5

$$\begin{aligned} \sigma &= \sqrt{< L^2 > - < L >^2} \\ &= \sqrt{31.75 - 27.56} \\ &= 2.04cm \end{aligned} \tag{5}$$

1.6

$$\begin{aligned} P(< L > \pm\sigma) &= \frac{11}{16} \\ &= 68.75\% \end{aligned} \tag{6}$$

Given what we know about probabilities, this makes the most sense as $\approx \frac{2}{3}$ of the values should be included within σ.

2 Problem 2

2.1

I did not include work for proving they were orthogonal since generally, it was just taking the difference of the two ket coefficients, and it did not require work.

2.1.1 ψ_1

A normalized ket orthogonal to $|\psi_1\rangle$:

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle \\ &= \frac{1}{5} + \frac{4}{5} \\ &= 1 \end{aligned} \tag{7}$$

2.1.2 ψ_2

A normalized ket orthogonal to $|\psi_2\rangle$:

$$\begin{aligned} |\phi_2\rangle &= \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle \\ &= (\frac{1}{\sqrt{3}} + i\frac{\sqrt{2}}{\sqrt{3}})(\frac{1}{\sqrt{3}} - i\frac{\sqrt{2}}{\sqrt{3}}) \\ &= (\frac{1}{3} + \frac{2}{3}) \\ &= 1 \end{aligned} \tag{8}$$

2.1.3 ψ_3

A normalized ket orthogonal to $|\psi_3\rangle$:

$$\begin{aligned} |\phi_3\rangle &= \frac{e^{\frac{-i\pi}{4}}}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \\ &= |\frac{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{\sqrt{2}}|^2 + |\frac{1}{\sqrt{2}}|^2 \\ &= (\frac{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{\sqrt{2}})(\frac{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{\sqrt{2}}) + \frac{1}{2} \\ &= \frac{\frac{2}{4} + \frac{2}{4}}{2} + \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1 \end{aligned} \tag{9}$$

2.2

2.2.1 $\langle\psi_2|\psi_3\rangle$

$$\begin{aligned}
\langle\psi_2|\psi_3\rangle &= [\frac{\sqrt{2}}{\sqrt{3}}\langle+| + \frac{i}{\sqrt{3}}\langle-|][\frac{1}{\sqrt{2}}|+\rangle - \frac{e^{\frac{i\pi}{4}}}{\sqrt{3}}|-\rangle] \\
&= \frac{\sqrt{2}}{\sqrt{2}\sqrt{3}} + \frac{i(-1-i)}{2\sqrt{3}} \\
&= \frac{1}{\sqrt{3}} + \frac{1-i}{2\sqrt{3}} \\
&= \frac{3-i}{2\sqrt{3}}
\end{aligned} \tag{10}$$

2.2.2 $\langle\psi_3|\psi_2\rangle$

$$\begin{aligned}
\langle\psi_3|\psi_2\rangle &= [\frac{1}{\sqrt{2}}\langle+| - \frac{e^{\frac{-i\pi}{4}}}{\sqrt{2}}\langle-|][\frac{\sqrt{2}}{\sqrt{3}}|+\rangle - \frac{i}{\sqrt{3}}|-\rangle] \\
&= [\frac{\sqrt{2}}{\sqrt{2}\sqrt{3}} + \frac{i\frac{\sqrt{2}}{2}}{\sqrt{2}\sqrt{3}} + \frac{\frac{\sqrt{2}}{2}}{\sqrt{2}\sqrt{3}}] \\
&= \frac{2}{2\sqrt{3} + \frac{1}{2\sqrt{3}}} + \frac{i}{2\sqrt{3}} \\
&= \frac{3+i}{2\sqrt{3}}
\end{aligned} \tag{11}$$

2.3

They are complex conjugates.

3 Problem 3

3.1

3.1.1 $|\psi_1\rangle$

$$\begin{aligned}
\psi_1 &= \frac{3}{5}|+\rangle - \frac{4}{5}|-\rangle \\
&= \frac{9}{25} + \frac{16}{25} \\
&= 1
\end{aligned} \tag{12}$$

3.1.2 $|\psi_2\rangle$

$$\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{5}}\,|+\rangle + \frac{2i}{\sqrt{5}}\,|-\rangle \\
&= (\frac{1}{\sqrt{5}} + \frac{2i}{\sqrt{5}})(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}) \\
&= \frac{1}{5} + \frac{4}{5} \\
&= 1
\end{aligned} \tag{13}$$

3.1.3 $|\psi_3\rangle$

$$\begin{aligned}
|\psi_3\rangle &= \frac{1}{\sqrt{5}}\,|+\rangle - \frac{2e^{\frac{i\pi}{4}}}{\sqrt{5}}\,|-\rangle \\
&= \frac{1}{5} + |\frac{(\frac{2\sqrt{2}}{2} + i2\frac{\sqrt{2}}{2})(\frac{2\sqrt{2}}{2} - 2i\frac{\sqrt{2}}{2})}{5}| \\
&= \frac{1}{5} + \frac{(\sqrt{2} + i\sqrt{2})(\sqrt{2} - i\sqrt{2})}{5} \\
&= \frac{1}{5} + \frac{2+2}{5} \\
&= 1
\end{aligned} \tag{14}$$

3.2

3.2.1 $|\psi_1\rangle$

$$\langle +|\psi_1\rangle = \frac{9}{25} \tag{15}$$

This is really straight forward, so excuse lack of supporting work

3.2.2 $|\psi_2\rangle$

$$\langle +|\psi_2\rangle = \frac{1}{5} \tag{16}$$

This is really straight forward, so excuse lack of supporting work

3.2.3 $|\psi_3\rangle$

$$\langle +|\psi_2\rangle = \frac{1}{5} \tag{17}$$

This is really straight forward, so excuse lack of supporting work

3.3

$$\begin{aligned}
P(\psi_1)_{+x} &= |(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|)(\frac{3}{5}|+\rangle - \frac{4}{5}|-\rangle)|^2 \\
&= |\frac{3}{5\sqrt{2}} + \frac{-4}{5\sqrt{2}}|^2 \\
&= |\frac{-1}{\sqrt{2}5}|^2 \\
&= \frac{1}{50}
\end{aligned} \tag{18}$$

3.4

$$\begin{aligned}
P(\psi_3)_{+y} &= |(\frac{1}{\sqrt{2}}\langle +| + i\langle -|)(\frac{1}{\sqrt{5}}|+\rangle - \frac{2e^{\frac{i\pi}{4}}}{\sqrt{5}}|-\rangle)|^2 \\
&= |\frac{1}{\sqrt{10} + \frac{-2i(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})}{\sqrt{5}}}|^2 \\
&= |\frac{1}{\sqrt{10}} + \frac{-i(\sqrt{2}+i\sqrt{2})}{\sqrt{5}}|^2 \\
&= |\frac{1}{\sqrt{10}} - \frac{\sqrt{2}}{\sqrt{5}} + \frac{i\sqrt{2}}{\sqrt{5}}|^2 \\
&= |\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} + \frac{2i}{\sqrt{10}}|^2 \\
&= |\frac{-1}{\sqrt{10}} + \frac{2i}{\sqrt{10}}|^2 \\
&= (\frac{-1}{\sqrt{10}} + \frac{2i}{\sqrt{10}})(\frac{-1}{\sqrt{10}} - \frac{2i}{\sqrt{10}}) \\
&= \frac{1}{10} + \frac{4}{10} \\
&= \frac{1}{2}
\end{aligned} \tag{19}$$

4 Problem 4

4.1

4.1.1 S_z

Possible measurements are $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

4.1.2

$$\begin{aligned}
P_+ &= \langle + | \psi \rangle \\
&= |((\langle + |_z + \langle - |_z)(\frac{2}{\sqrt{13}} |+\rangle + i\frac{3}{\sqrt{13}} |-\rangle)|^2 \\
&= |\frac{2}{\sqrt{26}} |+\rangle + i\frac{3}{\sqrt{26}} |-\rangle |^2 \\
&= (\frac{2}{\sqrt{26}} + i\frac{3}{\sqrt{26}})(\frac{2}{\sqrt{26}} - i\frac{3}{\sqrt{26}}) \\
&= (\frac{4}{26} + \frac{9}{26}) \\
&= \frac{1}{2}
\end{aligned} \tag{20}$$

$$\begin{aligned}
P_- &= \langle - | \psi \rangle \\
&= |((\langle + |_z + \langle - |_z)(\frac{2}{\sqrt{13}} |+\rangle + i\frac{3}{\sqrt{13}} |-\rangle)|^2 \\
&= |\frac{2}{\sqrt{26}} |+\rangle + i\frac{3}{\sqrt{26}} |-\rangle |^2 \\
&= (\frac{2}{\sqrt{26}} + i\frac{3}{\sqrt{26}})(\frac{2}{\sqrt{26}} - i\frac{3}{\sqrt{26}}) \\
&= (\frac{4}{26} + \frac{9}{26}) \\
&= \frac{1}{2}
\end{aligned} \tag{21}$$

4.2

4.2.1 S_x

Possible measurements are $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. We have the measurement already on the x axis, we simply square the coefficients of the kets.

4.2.2

$$P_+ = \frac{4}{13} \tag{22}$$

$$P_- = \frac{9}{13} \tag{23}$$

This is pretty straight forward, since we did not need to change our basis, we could just square the coefficient (or take the complex conjugate if needed) to get the probability of the spin up and spin down outcomes.

5 Problem 5

$$\begin{aligned}
P(\psi) &= e^{i\beta} |\psi\rangle \\
&= e^{i\beta} a |+\rangle + e^{i\beta} b |-\rangle \\
&= |(a\cos\beta + ai\sin\beta)|^2 + |(b\cos\beta + bi\sin\beta)|^2 \\
&= (a\cos\beta + ai\sin\beta)(a\cos\beta - ai\sin\beta) + (b\cos\beta + bi\sin\beta)(b\cos\beta - bi\sin\beta) \\
&= (a^2\cos^2\beta + a^2\sin^2\beta) + (b^2\cos^2\beta + b^2\sin^2\beta) \\
&= (a^2 + b^2)(\cos^2\beta + \sin^2\beta) \\
&= a^2 + b^2 = P(\psi)
\end{aligned} \tag{24}$$

6 Problem 6

$$|\psi_0\rangle = \frac{2}{\sqrt{5}} |+\rangle + \frac{1}{\sqrt{5}} |-\rangle$$

6.1 P_+

$$\begin{aligned}
P_+ &= \langle +|\psi_0\rangle \\
&= \frac{4}{5} \\
&= 80\%
\end{aligned} \tag{25}$$

6.2 P_-

$$\begin{aligned}
P_- &= \langle -|\psi_0\rangle \\
&= \frac{1}{5} \\
&= 20\%
\end{aligned} \tag{26}$$

6.3 Probability total

$$\begin{aligned}
P_- &= \langle \psi_0|\psi_0\rangle \\
&= |\frac{2}{\sqrt{5}}|^2 + |\frac{1}{\sqrt{5}}|^2 \\
&= \frac{4}{5} + \frac{1}{5} \\
&= 1
\end{aligned} \tag{27}$$

6.4

This answer is not unique, there is several examples in the questions above which have the same probability distribution. For example: equation 13 and 14.