Physics 401 - Homework 5

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Problem 1

$$|\psi\rangle = |+\rangle_x \qquad |\phi\rangle = |-\rangle_x$$

Part A

$$|\psi\rangle \otimes |\phi\rangle$$

Express in z basis...

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \tag{1}$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \tag{2}$$

Now take tensor product between two states...

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= (\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle) * (\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle) \\ &= \frac{1}{2}(|{+}{+}\rangle - |{+}{-}\rangle - |{-}{+}\rangle + |{-}{-}\rangle) \end{aligned} \tag{3}$$

Part B

$$a_1 = \begin{array}{c} 1 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 \end{array} \tag{4}$$

$$a_2 = \begin{array}{c} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \end{array} \tag{5}$$

$$a_3 = \begin{array}{c} 0 \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \end{array} \tag{6}$$

$$a_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{7}$$

Part C

$$\begin{aligned} P_{+_1,+_2} &= \frac{1}{2}|((\langle+|_1 \langle+|_2) * (|+\rangle_1 |+\rangle_2 - |+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 + |-\rangle_1 |-\rangle_2)|^2 \\ &= \frac{1}{2}|((\langle+|_1 |+\rangle_1 \langle+|_2 |+\rangle_2 - \langle+|_1 |+\rangle_1 \langle+|_2 |-\rangle_2 - \langle+|_1 |-\rangle_1 \langle+|_2 |+\rangle_2 + \langle+|_1 |-\rangle_1 \langle+|_2 |-\rangle_2)|^2 \\ &= \frac{1}{2}|1 * 1|^2 \\ &= \frac{1}{2} \end{aligned} \tag{8}$$

Problem 2

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2)$$

Part A

Normalization condition...

$$\begin{aligned} \langle\psi|\psi\rangle &= 1 \\ &= \frac{1}{\sqrt{2}}((\langle+|_1 \langle-|_2 - \langle-|_1 \langle+|_2) * \frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2) \\ &= \frac{1}{2}((\langle+|+\rangle_1 \langle-|-\rangle_2 - \langle+|-\rangle_1 \langle-|+\rangle_2 - \langle-|+\rangle_1 \langle+|-\rangle_2 + \langle-|-\rangle_1 \langle+|+\rangle_2) \\ &= \frac{1}{2}(1+1) \\ &= 1 \end{aligned} \tag{9}$$

Normalization condition proven

Part B

$$S_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \; S_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; S_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

If $|\psi\rangle$ is an eigenstate of S_{1z} then...

$$S_{1z} |\psi\rangle = \lambda |\psi\rangle$$

Lets see...

$$\begin{aligned} S_{1z}\left|\psi\right\rangle &= \frac{1}{\sqrt{2}}(S_{1z}\left|+\right\rangle_1\left|-\right\rangle_2 - S_{1z}\left|-\right\rangle_1\left|+\right\rangle_2) \\ &= \frac{1}{\sqrt{2}}(\frac{\hbar}{2}\left|+\right\rangle_1\left|-\right\rangle_2 - \frac{-\hbar}{2}\left|-\right\rangle_1\left|+\right\rangle_2) \end{aligned} \tag{10}$$

$$\lambda\left|\psi\right\rangle \neq \frac{1}{\sqrt{2}}\frac{\hbar}{2}(\left|+\right\rangle_1\left|-\right\rangle_2 + \left|-\right\rangle_1\left|+\right\rangle_2)$$

Therefore the ψ is not an eigenstate of S_{1z}

Part C

Operators can only act on their individual particles, so basically this should be the sum of the $S_{1,z}$ operator with $\left|\psi\right\rangle$ and the $S_{2,z}$ operator with $\left|\psi\right\rangle$. Then we will add them together to see if we get an eigenvalue equation. We know $S_{1,z}\left|\psi\right\rangle$ from part b, so we will calculate $S_{2,z}\left|\psi\right\rangle$. We are looking for something along the lines of $(S_{1,z}+S_{2,z})\left|\psi\right\rangle = \lambda\left|\psi\right\rangle$

$$\begin{aligned} S_{2,z}\left|\psi\right\rangle &= \frac{1}{\sqrt{2}}(\left|+\right\rangle_1 S_{2,z}\left|-\right\rangle_2 - \left|-\right\rangle_1 S_{2,z}\left|+\right\rangle_2) \\ &= \frac{1}{\sqrt{2}}(\left|+\right\rangle_1\frac{-\hbar}{2}\left|-\right\rangle_2 - \left|-\right\rangle_1\frac{\hbar}{2}\left|+\right\rangle_2) \\ &= \frac{1}{\sqrt{2}}\frac{\hbar}{2}(-\left|+\right\rangle_1\left|-\right\rangle_2 - \left|-\right\rangle_1\left|+\right\rangle_2) \end{aligned} \tag{11}$$

Therefore the ψ is not an eigenstate of $S_{2,z}$

What about $S_{1,z}+S_{2,z}$??

$$\begin{aligned} (S_{1,z}+S_{2,z})\left|\psi\right\rangle &= \frac{1}{\sqrt{2}}\frac{\hbar}{2}(\left|+\right\rangle_1\left|-\right\rangle_2 + \left|-\right\rangle_1\left|+\right\rangle_2) + \frac{1}{\sqrt{2}}\frac{\hbar}{2}(-\left|+\right\rangle_1\left|-\right\rangle_2 - \left|-\right\rangle_1\left|+\right\rangle_2) \\ &= -\frac{1}{\sqrt{2}}\frac{\hbar}{2}(-\left|+\right\rangle_1\left|-\right\rangle_2 - \left|-\right\rangle_1\left|+\right\rangle_2) + \frac{1}{\sqrt{2}}\frac{\hbar}{2}(-\left|+\right\rangle_1\left|-\right\rangle_2 - \left|-\right\rangle_1\left|+\right\rangle_2) \\ &= -(-\frac{1}{\sqrt{2}}\frac{\hbar}{2} + \frac{1}{\sqrt{2}}\frac{\hbar}{2})(\left|+\right\rangle_1\left|-\right\rangle_2 + \left|-\right\rangle_1\left|+\right\rangle_2) \\ &= (0)(\left|+\right\rangle_1\left|-\right\rangle_2 + \left|-\right\rangle_1\left|+\right\rangle_2) \end{aligned} \tag{12}$$

We can conclude that $\left|\psi\right\rangle$ is an eigenstate of $(S_{1,z}+S_{2,z})$ with $\lambda = 0$

Part D

$$\left|\psi\right\rangle_{general} = \frac{1}{\sqrt{2}}(\left|+\right\rangle_{1n}\left|-\right\rangle_{2n} - \left|-\right\rangle_{1n}\left|+\right\rangle_{2n})$$

We know this state is entangled and perfectly anti-correlated. We will prove this below, and then we will take our $|\psi\rangle$ in the $\hat{z}$ and measure it in $\hat{n}$ to see if they have the same probability.

$$
\begin{aligned}
P_{+1n,-2n} &= \frac{1}{2}|((\langle +|_{1n} \langle -|_{2n}) * (|\psi\rangle_{general})|^2 \\
&= \frac{1}{2}|((\langle +|_{1n} \langle -|_{2n}) * (|+\rangle_{1n} |-\rangle_{2n} - |-\rangle_{1n} |+\rangle_{2n})|^2 \\
&= \frac{1}{2}|((\langle +|_{1n} |+\rangle_{1n} \langle -|_{2n} |-\rangle_{2n} - \langle +|_{1n} |-\rangle_{1n} \langle -|_{2n} |+\rangle_{2n})|^2 \\
&= \frac{1}{2}|1 * 1|^2 \\
&= \frac{1}{2}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
P_{-1n,+2n} &= \frac{1}{2}|((\langle -|_{1n} \langle +|_{2n}) * (|\psi\rangle_{general})|^2 \\
&= \frac{1}{2}|((\langle -|_{1n} \langle +|_{2n}) * (|+\rangle_{1n} |-\rangle_{2n} - |-\rangle_{1n} |+\rangle_{2n})|^2 \\
&= \frac{1}{2}|((\langle -|_{1n} |+\rangle_{1n} \langle +|_{2n} |-\rangle_{2n} - \langle -|_{1n} |-\rangle_{1n} \langle +|_{2n} |+\rangle_{2n})|^2 \\
&= \frac{1}{2}| - (1 * 1)|^2 \\
&= \frac{1}{2}
\end{aligned}
\tag{14}
$$

From these two calculations we know that for $|\psi\rangle_{general}$, the results are perfectly anti-correlated. Now we will see how $|\psi\rangle$ looks in the $\hat{n}$...

$$
\begin{aligned}
P_{+1n,-2n} &= \frac{1}{2}|((\langle +|_{1n} \langle -|_{2n}) * (|\psi\rangle)|^2 \\
&= \frac{1}{2}|\big(\cos\frac{\theta}{2} \langle +|_1 + e^{-i\phi} \sin\frac{\theta}{2} \langle -|_1 \big)\big(\sin\frac{\theta}{2} \langle +|_2 - e^{-i\phi} \cos\frac{\theta}{2} \langle -|_2 \big) * (|+\rangle_1 |-\rangle \\
&= \frac{1}{2}|((\cos\frac{\theta}{2} \langle +|_1 |+\rangle_1 + e^{-i\phi} \sin\frac{\theta}{2} \langle -|_1 |+\rangle_1 \big) \\
&\big(\sin\frac{\theta}{2} \langle +|_2 |-\rangle_2 - e^{-i\phi} \cos\frac{\theta}{2} \langle -|_2 |-\rangle_2 \big) \\
&-(\cos\frac{\theta}{2} \langle +|_1 |-\rangle_1 + e^{-i\phi} \sin\frac{\theta}{2} \langle -|_1 |-\rangle_1 \big) \\
&\big(\sin\frac{\theta}{2} \langle +|_2 |+\rangle_2 - e^{-i\phi} \cos\frac{\theta}{2} \langle -|_2 |+\rangle_2 \big))|^2 \\
&= \frac{1}{2}|((\cos\frac{\theta}{2}\big)\big(- e^{-i\phi} \cos\frac{\theta}{2}\big) - \big(e^{-i\phi} \sin\frac{\theta}{2}\big)\big(\sin\frac{\theta}{2}\big))|^2 \\
&= \frac{1}{2}|((- e^{-i\phi} \cos^2\frac{\theta}{2}\big) - \big(e^{-i\phi} \sin^2\frac{\theta}{2}\big))|^2 \\
&= \frac{1}{2}|((- e^{-i\phi} \cos^2\frac{\theta}{2}\big) - \big(e^{-i\phi} \sin^2\frac{\theta}{2}\big))| * |((- e^{i\phi} \cos^2\frac{\theta}{2}\big) - \big(e^{i\phi} \sin^2\frac{\theta}{2}\big))| \\
&= \frac{1}{2}\big(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\big) \\
&= \frac{1}{2}
\end{aligned}
\tag{15}
$$

Since we know these particles are anti-correlated, we know that this result above indicates the opposite must be true as well (i.e. $P_{-1n,+2n} = \frac{1}{2}$), therefore the $|\psi\rangle$ and $|\psi\rangle_{general}$ are physically equivalent.

Part E

$$|\alpha\rangle = \frac{1}{2}(|+\rangle_1 |+\rangle_2 - |+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2 - |-\rangle_1 |-\rangle_2)$$

This is not an entangled state because it can be decomposed to the two states:

$$|\alpha_1\rangle = \frac{1}{2}(|+\rangle + |-\rangle) \tag{16}$$

$$|\alpha_2\rangle = \frac{1}{2}(|+\rangle - |-\rangle) \tag{17}$$

Any entangled state cannot be decomposed and therefore the α could not be fully described without reference to the second particle. In this case, we can fully describe the state of particle 1 without needing to know or care about the state of particle 2.

Problem 3

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_{1,z} |-\rangle_{2,z} - |-\rangle_{1,z} |+\rangle_{2,z})$$
$$\theta = 120^\circ \qquad \varphi = 0^\circ$$

Part A

For this problem we will basically just take what we did in problem 2D and go from there. I think the way we solve this is calculate the probability of $|\psi\rangle$ in $\hat{n}$ given the properties and see what we are left with.

$$\begin{aligned}
P_{+1n,-2n} &= \frac{1}{2}|((\langle +|_{1n} \langle -|_{2n}) * (|\psi\rangle)|^2 \\
&= \frac{1}{2}|\big(\frac{1}{2}\langle +|_1 + \frac{\sqrt{3}}{2}\langle -|_1\big)\big(\frac{\sqrt{3}}{2}\langle +|_2 - \frac{1}{2}\langle -|_2\big) * (|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2)|^2 \\
&= \frac{1}{2}|((\frac{1}{2})(-\frac{1}{2}) - ((\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2}))|^2 \\
&= \frac{1}{2}| - (\frac{1}{4}) - (\frac{3}{4})|^2 \\
&= \frac{1}{2}
\end{aligned} \tag{18}$$

$$
\begin{aligned}
P_{-1n,+2n} &= \frac{1}{2}|(\langle -|_{1n}\langle +|_{2n}) * (|\psi\rangle)|^2 \\
&= \frac{1}{2}|(\frac{\sqrt{3}}{2}\langle +|_1 - \frac{1}{2}\langle -|_1)(\frac{1}{2}\langle +|_2 + \frac{\sqrt{3}}{2}\langle -|_2) * (|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2)|^2 \\
&= \frac{1}{2}|(\frac{\sqrt{3}}{2}\langle +|_1 - \frac{1}{2}\langle -|_1)(\frac{1}{2}\langle +|_2 + \frac{\sqrt{3}}{2}\langle -|_2) * (\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - (-\frac{1}{2}\frac{1}{2}))|^2 \\
&= \frac{1}{2}|\frac{3}{4}\frac{1}{4}|^2 \\
&= \frac{1}{2}
\end{aligned} \tag{19}
$$

The probabilities are $\frac{1}{2}$ for each case, 1 for the total probability. Every case they will have equal and opposite measurement outcomes because the calculation above proves they are perfectly anti-correlated.

Part B

I used Patrick Banner's guest lecture on entanglement in 371 last year to come to this conclusion
We do not need math for this. Her outcomes will be the same regardless, so she will have $\frac{1}{2}$ for $+\frac{\hbar}{2}$ and $\frac{1}{2}$ for $-\frac{\hbar}{2}$. The reason for this is that the measurement of Bob does not "change" Alice outcomes. Entanglement just says that if you know one, the other is correlated, and we can calculate that correlation based on the state. The outcomes are still random, and the probabilities do not change. Even though we do not measure Bob's particle, because of entanglement, we know what it is.

Problem 4

$$
|\beta\rangle = a\,|+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}}|+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}}|-\rangle_1 |+\rangle_2
$$

Part A

Normalization condition...

$$
\begin{aligned}
\langle\beta|\beta\rangle &= 1 \\
&= (a\langle +|_1\langle +|_2 + \frac{i}{\sqrt{6}}\langle +|_1\langle -|_2 + \frac{2}{\sqrt{6}}\langle -|_1\langle +|_2) * (a\,|+\rangle_1|+\rangle_2 - \frac{i}{\sqrt{6}}|+\rangle_1|-\rangle_2 + \frac{2}{\sqrt{6}}|-\rangle_1|+\rangle_2) \\
&= (a^2 + \frac{1}{6} + \frac{4}{6}) \\
1 &= (a^2 + \frac{5}{6}) \\
a^2 &= \frac{1}{6} \\
a &= \frac{1}{\sqrt{6}}
\end{aligned} \tag{20}
$$

Verification...

$$
\begin{aligned}
|\frac{1}{\sqrt{6}}|^2 + |\frac{i}{\sqrt{6}}|^2 + |\frac{2}{\sqrt{6}}| = \\
\frac{1}{6} + \frac{1}{6} + \frac{4}{6} = 1
\end{aligned} \tag{21}
$$

Part B

You can think about this $|\psi\rangle$ as:

$$
|\psi\rangle = \frac{1}{\sqrt{6}} \begin{matrix} 1 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 \end{matrix} + \frac{i}{\sqrt{6}} \begin{matrix} 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \end{matrix} + \frac{2}{\sqrt{6}} \begin{matrix} 0 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \end{matrix} + 0 \begin{matrix} 0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 1 \end{matrix}
$$

To get the probability, knowing that this is normalized, we will just do what we do to calculate the probability for all the cases when particle 1 is up:

$$
P_{+1} = P_{++} + P_{+-}
$$

$$
\begin{aligned}
P_{++} &= |(\langle+|_1 \langle+|_2) * |\psi\rangle|^2 \\
&= |(\langle+|_1 \langle+|_2) * (\frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 |^2 \\
&= |\frac{1}{\sqrt{6}}|^2 \\
&= \frac{1}{6}
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
P_{+-} &= |(\langle+|_1 \langle-|_2) * |\psi\rangle|^2 \\
&= |(\langle+|_1 \langle-|_2) * (\frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 |^2 \\
&= |\frac{i}{\sqrt{6}}|^2 \\
&= \frac{1}{6}
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
P_{+1} &= \frac{1}{6} + \frac{1}{6} \\
&= \frac{1}{3}
\end{aligned} \tag{24}
$$

Part C

For P_{++}

$$
\begin{aligned}
P_{++} &= |(\langle+|_1 \langle+|_2) * |\psi\rangle|^2 \\
&= |(\langle+|_1 \langle+|_2) * (\frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 |^2 \\
&= |\frac{1}{\sqrt{6}}|^2 \\
&= \frac{1}{6}
\end{aligned} \tag{25}
$$

For P_{+-} and P_{-+}

In this case we will add the probabilities for each case together to get the total probability

$$\begin{aligned} P_{+-} &= |(\langle +|_1 \langle -|_2) * |\psi\rangle|^2 \\ &= |(\langle +|_1 \langle -|_2) * (\frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 |^2 \\ &= |\frac{i}{\sqrt{6}}|^2 \\ &= \frac{1}{6} \end{aligned} \tag{26}$$

$$\begin{aligned} P_{-+} &= |(\langle -|_1 \langle +|_2) * |\psi\rangle|^2 \\ &= |(\langle +|_1 \langle -|_2) * (\frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 |^2 \\ &= |\frac{2}{\sqrt{6}}|^2 \\ &= \frac{4}{6} \end{aligned} \tag{27}$$

$$P_{+-} + P_{-+} = \frac{5}{6} \tag{28}$$

Part D

$$\langle\beta| (S_{1,z} + S_{2,z}) |\beta\rangle = \langle\beta| \left(S_{1,z} |\beta\rangle + S_{2,z} |\beta\rangle \right) \tag{29}$$

First: $S_{1,z} |\beta\rangle$

$$\begin{aligned} S_{1,z} |\beta\rangle &= \frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 \\ &= \frac{1}{\sqrt{6}} S_{1,z} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} S_{1,z} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} S_{1,z} |-\rangle_1 |+\rangle_2 \\ &= \left(\frac{1}{\sqrt{6}} \frac{\hbar}{2} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} \frac{\hbar}{2} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} \frac{-\hbar}{2} |-\rangle_1 |+\rangle_2 \right) \\ &= \frac{\hbar}{2} \left(\frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 - \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 \right) \end{aligned}$$

Second: $S_{2,z} |\beta\rangle$

$$\begin{aligned} S_{2,z} |\beta\rangle &= \frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 \\ &= \frac{1}{\sqrt{6}} |+\rangle_1 S_{2,z} |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 S_{2,z} |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 S_{2,z} |+\rangle_2 \\ &= \frac{1}{\sqrt{6}} |+\rangle_1 \frac{\hbar}{2} |+\rangle_2 - \frac{i}{\sqrt{6}} |+\rangle_1 \frac{-\hbar}{2} |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 \frac{\hbar}{2} |+\rangle_2 \\ &= \frac{\hbar}{2} \left(\frac{1}{\sqrt{6}} |+\rangle_1 |+\rangle_2 + \frac{i}{\sqrt{6}} |+\rangle_1 |-\rangle_2 + \frac{2}{\sqrt{6}} |-\rangle_1 |+\rangle_2 \right) \end{aligned}$$

Calculate the expectation value:

$$\langle\beta|\left(S_{1,z}|\beta\rangle + S_{2,z}|\beta\rangle\right) = \langle\beta|S_{1,z}|\beta\rangle + \langle\beta|S_{2,z}|\beta\rangle \tag{30}$$

$$\begin{aligned}\langle\beta|S_{1,z}|\beta\rangle &= \left(\frac{1}{\sqrt{6}}\langle+|_1\langle+|_2 + \frac{i}{\sqrt{6}}\langle+|_1\langle-|_2 + \frac{2}{\sqrt{6}}\langle-|_1\langle+|_2\right) * \frac{\hbar}{2}\left(\frac{1}{\sqrt{6}}|+\rangle_1|+\rangle_2 - \frac{i}{\sqrt{6}}|+\rangle_1|-\rangle_2 - \frac{2}{\sqrt{6}}|-\rangle_1|+\rangle_2\right)\\ &= \frac{\hbar}{2}\left(\frac{1}{6} + \frac{1}{6} - \frac{4}{6}\right)\\ &= \frac{\hbar}{2}\left(-\frac{1}{3}\right)\end{aligned} \tag{31}$$

$$\begin{aligned}\langle\beta|S_{2,z}|\beta\rangle &= \left(\frac{1}{\sqrt{6}}\langle+|_1\langle+|_2 + \frac{i}{\sqrt{6}}\langle+|_1\langle-|_2 + \frac{2}{\sqrt{6}}\langle-|_1\langle+|_2\right) * \frac{\hbar}{2}\left(\frac{1}{\sqrt{6}}|+\rangle_1|+\rangle_2 + \frac{i}{\sqrt{6}}|+\rangle_1|-\rangle_2 + \frac{2}{\sqrt{6}}|-\rangle_1|+\rangle_2\right)\\ &= \frac{\hbar}{2}\left(\frac{1}{6} - \frac{1}{6} + \frac{4}{6}\right)\\ &= \frac{\hbar}{2}\left(+\frac{1}{3}\right)\end{aligned} \tag{32}$$

$$\langle\beta|\left(S_{1,z} + S_{2,z}\right)|\beta\rangle = 0 \tag{33}$$

Problem 5

Part A

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Operating on $|1\rangle$

$$\begin{aligned}\sigma_x|1\rangle &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\end{aligned} \tag{34}$$

Operating on $|0\rangle$

$$\begin{aligned}\sigma_x|1\rangle &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{aligned} \tag{35}$$

Part B

$$
\begin{aligned}
H &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \\
&= \frac{1}{\sqrt{2}}\left(\sigma_1 + \sigma_3\right)
\end{aligned} \tag{36}
$$

Part C

$$
\begin{aligned}
H \cdot \sigma_x \cdot H &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \sigma_z
\end{aligned} \tag{37}
$$

Part D

$$
\begin{aligned}
H \cdot \sigma_z \cdot H &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \sigma_x
\end{aligned} \tag{38}
$$

Eigenvalue Equations

$$
\begin{aligned}
S_{1,z}\left|+\right\rangle_1 &= +\frac{\hbar}{2}\left|+\right\rangle_1 \\
S_{1,z}\left|-\right\rangle_1 &= -\frac{\hbar}{2}\left|-\right\rangle_1
\end{aligned} \tag{39}
$$