Phys273 - Final

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1 Problem 1

1.1 Part a

Define $f(\omega)$

$$\begin{aligned} f(\omega) &= \frac{(\gamma\omega)^2}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]} \\ &= \frac{1}{\frac{(\omega_0^2-\omega^2)^2}{(\gamma\omega)^2}+1} \end{aligned} \tag{1}$$

At FWHM $f(\omega)=\frac{1}{2}$, therefore

$$\begin{aligned} \frac{1}{\frac{(\omega_0^2-\omega^2)^2}{(\gamma\omega)^2}+1} &= \frac{1}{2} \\ \frac{(\omega_0^2-\omega^2)^2}{(\gamma\omega)^2}+1 &= 2 \\ \frac{(\omega_0^2-\omega^2)^2}{(\gamma\omega)^2} &= 1 \\ (\omega_0^2-\omega^2)^2 &= (\gamma\omega)^2 \\ \omega_0^2-\omega^2 &= \pm\gamma\omega \end{aligned} \tag{2}$$

Define system of equations:

$$\omega_0^2-\omega_1^2 = \gamma\omega_1 \text{ and } \omega_0^2-\omega_2^2 = -\gamma\omega_2 \tag{3}$$

Using elimination to solve for γ

$$\begin{aligned} \gamma &= \omega_2-\omega_1 \\ &= 0.25\omega_0 \end{aligned} \tag{4}$$

We know solution for damped, driven oscillators at steady state

$$x_p = A_p\cos(\omega t+\phi) \tag{5}$$

$$
\begin{aligned}
A_p &= \frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}} \\
\text{Where } F &= \frac{F_d}{m} \text{ and } F_d = F_0\cos\omega t
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\tan\phi &= -\frac{\gamma\omega}{\omega_0^2-\omega^2} \\
\gamma &= \frac{\omega_0}{4} \\
&= \frac{\omega_0\omega}{4(\omega^2-\omega_0^2)}
\end{aligned} \tag{7}
$$

1.2 Part b

For damped oscillator

$$
\begin{aligned}
F_{\text{net}} &= F_{\text{spring}} + F_{\text{damping}} \\
&= -kx + -b\dot{x} \\
m\ddot{x} &= -kx + -b\dot{x} \\
\ddot{x} &= -\omega_0^2 x + -\gamma\dot{x} \\
\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0
\end{aligned} \tag{8}
$$

Assume solution of $x = ce^{\alpha t}$

$$x = ce^{\alpha t} \tag{9}$$

$$\dot{x} = c\alpha e^{\alpha t} \tag{10}$$

$$\ddot{x} = c\alpha^2 e^{\alpha t} \tag{11}$$

Sub into equation

$$
\begin{aligned}
c\alpha^2 e^{\alpha t} + \gamma c\alpha e^{\alpha t} + \omega_0^2 ce^{\alpha t} &= 0 \\
\alpha^2 + \gamma\alpha + \omega_0^2 &= 0 \\
\text{Since } c = 0 \text{ is trivial}
\end{aligned} \tag{12}
$$

Underdamped oscillator $\gamma < 2\omega_0$

$$
\begin{aligned}
c\alpha^2 e^{\alpha t} + \gamma c\alpha e^{\alpha t} + \omega_0^2 ce^{\alpha t} &= 0 \\
\alpha^2 + \gamma\alpha + \omega_0^2 &= 0 \\
\text{Since } c = 0 \text{ is trivial} \\
\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}
\end{aligned} \tag{13}
$$

Define ω_u

$$\omega_u = \omega_0\sqrt{1 - (\frac{\gamma}{2\omega_0})^2} \tag{14}$$

α has two solutions and is complex

$$\alpha_1 = \frac{-\gamma}{2} + i\omega_u$$
$$\alpha_2 = \frac{-\gamma}{2} - i\omega_u \tag{15}$$

X has 2 independent solutions

$$X = x_1 + x_2 \tag{16}$$
$$x_1 = c_1 e^{\alpha_1 t}$$
$$x_2 = c_2 e^{\alpha_2 t}$$

$$\begin{aligned} X &= c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} \\ &= c_1 e^{\frac{-\gamma}{2} + i\omega_u t} + c_2 e^{\frac{-\gamma}{2} - i\omega_u t} \end{aligned} \tag{17}$$

X must be real

$$x_1 = \bar{x_2}$$
$$c_1 = \bar{c_2}$$

$$\begin{aligned} X &= e^{-\frac{\gamma}{2}t}[c_0 e^{i(\omega_u t + \phi)} + c_0 e^{-i(\omega_u t + \phi)}] \\ &= 2c_0 e^{-\frac{\gamma}{2}t} \cos(\omega_u t + \phi) \\ &= A e^{-\frac{\gamma}{2}t} \cos(\omega_u t + \phi) \end{aligned} \tag{18}$$

In terms of ω_0

$$\begin{aligned} \gamma &= \frac{\omega_0}{4} \\ \omega_u &= \omega_0 \sqrt{1 - (\frac{\gamma}{2\omega_0})^2} \\ X &= A e^{-\frac{\omega_0}{8}t} \cos(\omega_0 \sqrt{1 - (\frac{\omega_0}{8\omega_0})^2} t + \phi) \\ &= A e^{-\frac{\omega_0}{8}t} \cos(\omega_0 \sqrt{1 - \frac{1}{64}} t + \phi) \\ &= A e^{-\frac{\omega_0}{8}t} \cos(\omega_0 \frac{\sqrt{63}}{8} t + \phi) \\ a &= \frac{1}{8} \\ b &= \frac{\sqrt{63}}{8} \end{aligned} \tag{19}$$

1.3 Part c

$$x(t) = A e^{-a\omega_0 t} \cos(b\omega_0 t + \phi)$$
$$\dot{x}(t) = -A[a\omega_0 e^{-a\omega_0 t} \cos(b\omega_0 t + \phi) - e^{-a\omega_0 t} b\omega_0 \sin(b\omega_0 t + \phi)]$$

When: $x(t=0)=0$

$$\begin{aligned} x(t=0) &= Ae^0\cos(0+\phi) \\ &= A\cos(\phi) \\ A=0 &\text{ is trivial} \\ 0 &= \cos(\phi) \\ \phi &= \frac{n\pi}{2}, n=1,2,3... \end{aligned} \tag{20}$$

When: $\dot{x}(t=0)=v_0$

$$\begin{aligned} \dot{x}(t=0) &= -A[a\omega_0 e^{-a\omega_0 t}\cos(b\omega_0 t+\phi) - e^{-a\omega_0 t}b\omega_0\sin(b\omega_0 t+\phi)] \\ &= -A[a\omega_0 e^0\cos(0+\phi) - e^0 b\omega_0\sin(0+\phi)] \\ &= -A[a\omega_0 e^0\cos(\frac{n\pi}{2}) - e^0 b\omega_0\sin(\frac{n\pi}{2})] \\ v_0 &= Ab\omega_0\sin(\frac{n\pi}{2}) \\ \frac{v_0}{b\omega_0\sin(\frac{n\pi}{2})} &= A \\ \pm\frac{v_0}{b\omega_0} &= A \end{aligned} \tag{21}$$

Subbing in solution for a and b

$$\phi = \frac{n\pi}{2}, n=1,2,3... \tag{22}$$

$$A = \pm\frac{v_0}{\frac{\sqrt{63}}{8}\omega_0} \tag{23}$$

2 Problem 2

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = Re\left[A_f e^{i\omega_f t}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} + A_m e^{i\omega_m t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + A_s e^{i\omega_s t}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}\right] \tag{24}$$

$$= A_f\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}\cos(\sqrt{2+\sqrt{2}}\omega_0 t+\phi_f) + A_m\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\cos(\sqrt{2}\omega_0 t+\phi_m) + A_s\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}\cos(\sqrt{2-\sqrt{2}}\omega_0 t+\phi_s) \tag{25}$$

Expanded form

$$\begin{aligned} x_1(t) &= A_f\cos(\sqrt{2+\sqrt{2}}\omega_0 t+\phi_f) + A_m\cos(\sqrt{2}\omega_0 t+\phi_m) + A_s\cos(\sqrt{2-\sqrt{2}}\omega_0 t+\phi_s) \\ x_2(t) &= -\sqrt{2}A_f\cos(\sqrt{2+\sqrt{2}}\omega_0 t+\phi_f) + \sqrt{2}A_s\cos(\sqrt{2-\sqrt{2}}\omega_0 t+\phi_s) \\ x_3(t) &= A_f\cos(\sqrt{2+\sqrt{2}}\omega_0 t+\phi_f) - A_m\cos(\sqrt{2}\omega_0 t+\phi_m) + A_s\cos(\sqrt{2-\sqrt{2}}\omega_0 t+\phi_s) \end{aligned} \tag{26}$$

Derivatives

$$\begin{aligned}
\dot{x}_1(t) &= -\sqrt{2+\sqrt{2}}\omega_0 A_f \sin(\sqrt{2+\sqrt{2}}\omega_0 t + \phi_f) - \sqrt{2}\omega_0 A_m \sin(\sqrt{2}\omega_0 t + \phi_m) - \sqrt{2-\sqrt{2}}\omega_0 A_s \sin(\sqrt{2-\sqrt{2}}\omega_0 t + \phi_s) \\
\dot{x}_2(t) &= \sqrt{2}\sqrt{2+\sqrt{2}}\omega_0 A_f \sin(\sqrt{2+\sqrt{2}}\omega_0 t + \phi_f) - \sqrt{2}\sqrt{2-\sqrt{2}}\omega_0 A_s \sin(\sqrt{2-\sqrt{2}}\omega_0 t + \phi_s) \\
\dot{x}_3(t) &= -\sqrt{2+\sqrt{2}}\omega_0 A_f \sin(\sqrt{2+\sqrt{2}}\omega_0 t + \phi_f) + \sqrt{2}\omega_0 A_m \sin(\sqrt{2}\omega_0 t + \phi_m) - \sqrt{2-\sqrt{2}}\omega_0 A_s \sin(\sqrt{2-\sqrt{2}}\omega_0 t + \phi_s)
\end{aligned} \tag{27}$$

Initial conditions

$$x_1 = x_3 = 0, x_2 = A$$

$$\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$$

Define

$$\begin{aligned}
a_f &= A_f \cos\phi_f \\
b_f &= -A_f \sin\phi_f \\
c_m &= A_m \cos\phi_m \\
d_m &= -A_m \sin\phi_m \\
g_s &= A_s \cos\phi_s \\
h_s &= -A_s \sin\phi_s
\end{aligned}$$

$$\begin{aligned}
x_1(t) &= a_f \cos(\sqrt{2+\sqrt{2}}\omega_0 t) + b_f \sin(\sqrt{2+\sqrt{2}}\omega_0 t) + c_m \cos(\sqrt{2}\omega_0 t) + d_m \sin(\sqrt{2}\omega_0 t) + g_s \cos(\sqrt{2-\sqrt{2}}\omega_0 t) + h_s \sin(\sqrt{2-\sqrt{2}}\omega_0 t) \\
x_2(t) &= -\sqrt{2}a_f \cos(\sqrt{2+\sqrt{2}}\omega_0 t) - \sqrt{2}b_f \sin(\sqrt{2+\sqrt{2}}\omega_0 t) + \sqrt{2}g_s \cos(\sqrt{2-\sqrt{2}}\omega_0 t) + \sqrt{2}h_s \sin(\sqrt{2-\sqrt{2}}\omega_0 t) \\
x_3(t) &= a_f \cos(\sqrt{2+\sqrt{2}}\omega_0 t) + b_f \sin(\sqrt{2+\sqrt{2}}\omega_0 t) - c_m \cos(\sqrt{2}\omega_0 t) - d_m \sin(\sqrt{2}\omega_0 t) + g_s \cos(\sqrt{2-\sqrt{2}}\omega_0 t) + h_s \sin(\sqrt{2-\sqrt{2}}\omega_0 t)
\end{aligned} \tag{28}$$

Apply: $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$ - All cos terms will go to sin and 0 out, all sin terms will go to cos and become 1 at t = 0. First $\dot{x}_1$

$$\begin{aligned}
\dot{x}_1(t=0) &= \sqrt{2+\sqrt{2}}b_f + \sqrt{2}d_m + \sqrt{2-\sqrt{2}}h_s = 0 \\
\dot{x}_2(t=0) &= -\sqrt{2}\sqrt{2+\sqrt{2}}b_f + \sqrt{2}\sqrt{2-\sqrt{2}}h_s = 0 \\
\dot{x}_3(t=0) &= \sqrt{2+\sqrt{2}}b_f - \sqrt{2}d_m + \sqrt{2-\sqrt{2}}h_s = 0
\end{aligned} \tag{29}$$

$\dot{x}_2$ tells us

$$\begin{aligned}
0 &= -\sqrt{2}\sqrt{2+\sqrt{2}}b_f + \sqrt{2}\sqrt{2-\sqrt{2}}h_s \\
b_f &= h_s
\end{aligned} \tag{30}$$

Subtract $\dot{x}_1 - \dot{x}_3$

$$\begin{aligned} \dot{x}_1(t=0) &= \sqrt{2+\sqrt{2}}b_f + \sqrt{2}d_m + \sqrt{2-\sqrt{2}}h_s = 0 \\ -\dot{x}_3(t=0) &= \sqrt{2+\sqrt{2}}b_f - \sqrt{2}d_m + \sqrt{2-\sqrt{2}}h_s = 0 \\ 0 &= 0 + 2\sqrt{2}d_m + 0 \\ 0 &= d_m \end{aligned} \tag{31}$$

Add $x_1 + x_3$

$$\begin{aligned} x_1(t=0) &= a_f + c_m + g_s = 0 \\ +x_3(t=0) &= a_f - c_m + g_s = 0 \\ 0 &= a_f + g_s \\ -a_f &= g_s \end{aligned} \tag{32}$$

Subtract $x_1 - x_3$

$$\begin{aligned} x_1(t=0) &= a_f + c_m + g_s = 0 \\ -x_3(t=0) &= a_f - c_m + g_s = 0 \\ 0 &= 2c_m \\ 0 &= c_m \end{aligned} \tag{33}$$

x_2 **tells us**

$$\begin{aligned} x_2(t=0) &= -\sqrt{2}a_f + \sqrt{2}g_s = A \\ A &= -a_f + g_s \\ \text{sub in values from equation 33} \\ A &= g_s + g_s \\ \frac{A}{2} &= g_s \\ -\frac{A}{2} &= a_f \end{aligned} \tag{34}$$

Final values

$$\begin{aligned} a_f &= -\frac{A}{2} \\ b_f &= 0 \\ c_m &= 0 \\ d_m &= 0 \\ g_s &= \frac{A}{2} \\ h_s &= 0 \end{aligned}$$

x(t) final

$$
\begin{aligned}
x_1(t) &= -\frac{A}{2}\cos\left(\sqrt{2+\sqrt{2}}\omega_0 t\right) + \frac{A}{2}\cos\left(\sqrt{2-\sqrt{2}}\omega_0 t\right) \\
&= -\frac{A}{2}\cos(\omega_f t) + \frac{A}{2}\cos(\omega_s t) \\
&= \frac{A}{2}(-\cos(\omega_f t) + \cos(\omega_s t)) \\
x_2(t) &= \frac{A\sqrt{2}}{2}\cos(\omega_f t) + \frac{A\sqrt{2}}{2}\cos(\omega_s t) \\
&= \frac{A\sqrt{2}}{2}(\cos(\omega_f t) + \cos(\omega_s t)) \\
x_3(t) &= \frac{A}{2}(\cos(\omega_f t) - \cos(\omega_s t))
\end{aligned} \tag{35}
$$

3 Problem 3

$$\psi(x,t) = A\sin\left(\omega\sqrt{\frac{\mu}{T}}x\right)\cos(\omega t + \phi)$$

3.1 Part a

Justification for choice of sin: From our boundary condition x = 0, we know x = 0 is a node and therefore $\psi(x=0,t) = 0$ for all t. This will require $\psi(x,t) \propto \sin kx \cos(\omega t + \phi)$ in order to satisfy the boundary condition. If we chose a function in the form $\psi(x,t) \propto \cos kx \cos(\omega t + \phi)$, our function would not be 0 at $x = 0$, and would not satisfy the boundary condition

Justification for spatial dependence involving $\omega\sqrt{\frac{\mu}{T}}x$

$$k = \frac{2\pi}{\lambda} \tag{36}$$

$$v = \frac{\omega}{k} = \sqrt{\frac{T_s}{\mu}} \tag{37}$$

$$\omega = 2\pi\nu_f \tag{38}$$

ν_f is frequency. Written as ν_f to distinguish it from velocity v

$$
\begin{aligned}
v &= \frac{\omega}{k} \\
k &= \frac{\omega}{v} \\
&= \frac{\omega}{\sqrt{\frac{T_s}{\mu}}} \\
&= \omega\sqrt{\frac{\mu}{T_s}}
\end{aligned} \tag{39}
$$

Since we know $\psi(x,t) \propto \sin kx \cos(\omega t + \phi)$, we can sub our value of k from equation 38 to get $\psi(x,t) = A\sin(\omega\sqrt{\frac{\mu}{T_s}}x)\cos(\omega t + \phi)$

3.2 Part b

Use Newton's Second to get equation for net force

$$F_{spring} = -K_{\mathrm{s\ p}}\psi(x,t)$$

$$F_{tension} = -T\sin\theta$$

$$\text{Using small angle approximation: } \sin\theta \approx \tan\theta \approx \theta \approx \text{slope} \approx \frac{\partial\psi(x,t)}{\partial x}$$

$$F_{tension} = -T\frac{\partial\psi(x,t)}{\partial x}$$

$$\begin{aligned}
m\frac{\partial^2\psi(x,t)}{\partial t^2} &= F_{spring} + F_{tension}\\
m\frac{\partial^2\psi(x,t)}{\partial t^2} &= -K_{\mathrm{s\ p}}\psi(x,t) - T\frac{\partial\psi(x,t)}{\partial x}\\
m\frac{\partial^2\psi(x,t)}{\partial t^2} + K_{\mathrm{s\ p}}\psi(x,t) &= -T\frac{\partial\psi(x,t)}{\partial x}\\
\text{For } x &= L\\
m\frac{\partial^2\psi(x,t)}{\partial t^2} + K_{\mathrm{s\ p}}\psi(x,t) &= -T\frac{\partial\psi(x,t)}{\partial x}\\
\text{For } x &= L\\
m\frac{\partial^2\psi(L,t)}{\partial t^2} + K_{\mathrm{s\ p}}\psi(L,t) &= -T\frac{\partial\psi(L,t)}{\partial x}
\end{aligned} \tag{40}$$

3.3 Part c

Boundary condition for a free end

$$\frac{\partial\psi(x,t)}{\partial x} = 0$$

$$\begin{aligned}
\psi(x,t) &= A\sin(\omega\sqrt{\frac{\mu}{T_s}}x)\cos(\omega t+\phi)\\
\frac{\partial\psi(x,t)}{\partial x} &= A\omega\sqrt{\frac{\mu}{T_s}}\cos(\omega\sqrt{\frac{\mu}{T_s}}x)\cos(\omega t+\phi)\\
\frac{\partial^2\psi(x,t)}{\partial t^2} &= -A\omega^2\sin(\omega\sqrt{\frac{\mu}{T_s}}x)\cos(\omega t+\phi)
\end{aligned}$$

$$\begin{aligned}
m\frac{\partial^2\psi(L,t)}{\partial t^2} + K_{\mathrm{s\ p}}\psi(L,t) &= -T\frac{\partial\psi(L,t)}{\partial x}\\
m - A\omega^2\sin(\omega\sqrt{\frac{\mu}{T_s}}x)\cos(\omega t+\phi) + K_{\mathrm{s\ p}}A\sin(\omega\sqrt{\frac{\mu}{T_s}}x)\cos(\omega t+\phi) &= -T\frac{\partial\psi(L,t)}{\partial x}
\end{aligned} \tag{41}$$

Input $\omega_0 = \omega = \sqrt{\frac{K_{sp}}{m}}$ and $x = L$

$$
\begin{aligned}
m - A\sqrt{\frac{K_{sp}}{m}}^2 \sin(\sqrt{\frac{K_{sp}}{m}}\sqrt{\frac{\mu}{T_s}}L)\cos(\sqrt{\frac{K_{sp}}{m}}t+\phi) + K_{\mathrm{s\ p}}A\sin(\sqrt{\frac{K_{sp}}{m}}\sqrt{\frac{\mu}{T_s}}L)\cos(\sqrt{\frac{K_{sp}}{m}}t+\phi) &= -T\frac{\partial\psi(L,t)}{\partial x} \\
m - A\frac{K_{sp}}{m}\sin(\sqrt{\frac{K_{sp}\mu}{mT_s}}L)\cos(\sqrt{\frac{K_{sp}}{m}}t+\phi) + K_{\mathrm{s\ p}}A\sin(\sqrt{\frac{K_{sp}\mu}{mT_s}}L)\cos(\sqrt{\frac{K_{sp}}{m}}t+\phi) &= -T\frac{\partial\psi(L,t)}{\partial x} \\
-AK_{sp}\sin(\sqrt{\frac{K_{sp}\mu}{mT_s}}L)\cos(\sqrt{\frac{K_{sp}}{m}}t+\phi) + K_{\mathrm{s\ p}}A\sin(\sqrt{\frac{K_{sp}\mu}{mT_s}}L)\cos(\sqrt{\frac{K_{sp}}{m}}t+\phi) &= -T\frac{\partial\psi(L,t)}{\partial x} \\
0 &= -T\frac{\partial\psi(L,t)}{\partial x} \\
\text{T is not 0 therefore} \\
0 &= \frac{\partial\psi(L,t)}{\partial x}
\end{aligned} \tag{42}
$$

When ω is oscillating at ω_0 at $x = L$, the spring and the string have the same frequency and amplitude and therefore act as if they were one single component

3.4 Part d

Apply boundary condition where ω is oscillating at ω_0 at $x = L$

$$
\begin{aligned}
0 &= \frac{\partial\psi(L,t)}{\partial x} \\
&= A\omega\sqrt{\frac{\mu}{T_s}}\cos(\omega\sqrt{\frac{\mu}{T_s}}L)\cos(\omega t+\phi) \\
\text{A} = 0 \text{ is trivial therefore:} \\
0 &= \cos(\omega\sqrt{\frac{\mu}{T_s}}L) \\
(n+\frac{1}{2})\pi &= (\omega\sqrt{\frac{\mu}{T_s}}L) \\
\frac{(n+\frac{1}{2})\pi}{\omega\sqrt{\frac{\mu}{T_s}}} &= L \\
n &= 0, 1, 2...
\end{aligned} \tag{43}
$$

4 Problem 4

$$\begin{aligned}
\vec{E} &= E_0 Re(\frac{\hat{x}+i\hat{y}}{\sqrt{2}}e^{i(kz-\omega t)}) \\
&= E_0 Re([\frac{\hat{x}e^{i(kz-\omega t)}}{\sqrt{2}} + \frac{i\hat{y}e^{i(kz-\omega t)}}{\sqrt{2}}]) \\
&= E_0 Re([\frac{\hat{x}[\cos{(kz-\omega t)} + i\sin{(kz-\omega t)}]}{\sqrt{2}} + \frac{i\hat{y}[\cos{(kz-\omega t)} + i\sin{(kz-\omega t)}]}{\sqrt{2}}]) \\
&= \frac{E_0}{\sqrt{2}} Re([[\hat{x}\cos{(kz-\omega t)} + i\hat{x}\sin{(kz-\omega t)}] + [i\hat{y}\cos{(kz-\omega t)} + ii\hat{y}\sin{(kz-\omega t)}]]) \\
&= \frac{E_0}{\sqrt{2}}(\hat{x}\cos{(kz-\omega t)} - \hat{y}\sin{(kz-\omega t)})
\end{aligned}$$

4.1 Part a

x component

$$E_x = \frac{E_0}{\sqrt{2}}(\cos{(kz-\omega t)}) \tag{44}$$

y component

$$E_y = -\frac{E_0}{\sqrt{2}}\sin{(kz-\omega t)} \tag{45}$$

Circular polarization: Plug $z = 0$, sin is odd so $-sin(\theta) = sin(-\theta)$

$$\begin{aligned}
E &= \frac{E_0}{\sqrt{2}}(\cos{(0-\omega t)})\hat{x} + \frac{E_0}{\sqrt{2}}\sin{(0+\omega t)}\hat{y} \\
\frac{E}{\sqrt{2}} &= E_0(\cos{(\omega t)})\hat{x} + E_0\sin{(\omega t)}\hat{y}
\end{aligned} \tag{46}$$

Phase difference of $\frac{\pi}{2}$ between x and y components, therefore there is circular polarization

4.2 Part b

$$\begin{aligned}
\vec{B} &= \frac{E_0}{c} Re(\frac{\hat{x}+i\hat{y}}{\sqrt{2}}e^{i(kz-\omega t)}) \\
&= \frac{E_0}{c\sqrt{2}}(\hat{x}\cos{(kz-\omega t)} - \hat{y}\sin{(kz-\omega t)})
\end{aligned}$$

Define

$$k \times E = \omega B$$

Maxwell's first tells us E is perpendicular to k therefore we can define

$$\begin{aligned}
kE &= \omega B \\
E &= \frac{\omega}{k} B \\
c &= \frac{\omega}{k} \\
E &= cB \\
\frac{E}{c} &= B \\
\frac{E_0(\cos(kz-\omega t))\hat{x} - E_0 \sin(kz-\omega t)\hat{y}}{\sqrt{2}c} &= B \\
B &= \frac{E_0}{\sqrt{2}c}(\cos(kz-\omega t))\hat{x} - \sin(kz-\omega t)\hat{y})
\end{aligned} \tag{47}$$

4.3 Part c

$$S = \frac{1}{\mu_0}(E \times B)$$

For traveling wave

$$= \frac{1}{\mu_0} E \frac{E}{c} \hat{k}$$

Substitute: $\frac{1}{\mu_0} = c^2 \epsilon_0$

$$= c\epsilon E^2 \hat{k}$$
$$= c\epsilon E^2 \hat{z}$$

$$\begin{aligned}
E^2 &= (\frac{E_0}{\sqrt{2}}(\hat{x}\cos(kz-\omega t) - \hat{y}\sin(kz-\omega t)))^2 \\
&= \frac{E_0^2}{2}(\cos^2(kz-\omega t) + \sin^2(kz-\omega t)) \\
&= \frac{E_0^2}{2}
\end{aligned} \tag{48}$$

Final Answer

$$S = c\epsilon \frac{E_0^2}{2} \hat{z} \tag{49}$$

Subbing in answer from below for energy density

$$S = c(\frac{E_0^2}{2})^2 \epsilon_0 \hat{z} \tag{50}$$

Since c is constant, E_0 is constant, and ϵ_0 is constant, we can see that the Poynting vector is independent of space and time.

4.4 Part d

$$\begin{aligned}
\epsilon &= \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 \\
\text{substitute } B = \frac{E}{c} \\
&= \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}(\frac{E}{c})^2 \\
&= \frac{E^2}{2}(\epsilon_0 + \frac{1}{c\mu_0}^2) \\
\text{substitute } \mu_0 = \frac{1}{c^2\epsilon_0} \\
&= \frac{E^2}{2}\epsilon_0 + \epsilon_0 \\
\epsilon &= E^2\epsilon_0
\end{aligned}$$

Final Answer

$$\begin{aligned}
\epsilon &= E^2\epsilon_0 \\
\epsilon &= \frac{E_0^2}{2}\epsilon_0
\end{aligned} \tag{51}$$

4.5 Part e

$$\begin{aligned}
\vec{E} &= \frac{E_0(\hat{x}\cos(\theta) + \hat{y}sin(\theta))}{\sqrt{2}}\cos((kz - \omega t) + \delta) \\
\delta &= \frac{\pi}{2} for circularly polarized wave
\end{aligned}$$

$$\frac{E_0}{\sqrt{2}}(\hat{x}\cos(kz - \omega t) - \hat{y}\sin(kz - \omega t)) \times \hat{x}\cos(\theta) + \hat{y}\sin\theta$$

Is going to retturn to you:

$$\vec{E} = \frac{E_0(\hat{x}\cos(\theta) + \hat{y}sin(\theta))}{\sqrt{2}}\cos((kz - \omega t) + \frac{\pi}{2})$$

Thank you for all your help this semester and I hope this year goes well for you. Best regards - Dillon Walton