Phys273 - Homework 4

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1 Problem 1

Function: $y(x) = Ax(L-x)$
Interval: $0 \leq x \leq L$
Period: L

1.1 Trigonometric

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos{(\frac{2\pi nx}{L})} + b_n \sin{(\frac{2\pi nx}{L})} \tag{1}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx \tag{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2\pi nx}{L}dx \tag{3}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi nx}{L}dx \tag{4}$$

1.1.1 Calculation for a_0

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L Ax(L-x)dx \\ &= \frac{A}{L}\int_0^L xL - x^2 dx \\ &= \frac{A}{L}\Big[\int_0^L xLdx - \int_0^L x^2 dx\Big] \\ &= \frac{A}{L}\Big[L\frac{L^2}{2} - \frac{L^3}{3}\Big] \\ &= \frac{AL^2}{6} \end{aligned} \tag{5}$$

1.1.2 Calculation for a_n

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L Ax(L-x)\cos{(\frac{2\pi nx}{L})}dx \\ &= \frac{AL^2(-\pi L^3 ncos(2\pi n) - \pi n + \sin{(2\pi n)})}{2\pi^3 n^3} \end{aligned} \tag{6}$$

$\cos 2\pi n = 1$ for all integer n = 0, 1, 2 ...
$\sin 2\pi n = 0$ for all integer n = 0, 1, 2 ...

$$a_n = \frac{AL^2(-\pi L^3 n - \pi n)}{2\pi^3 n^3} \tag{7}$$

1.1.3 Calculation for b_n

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L Ax(L-x)\sin{(\frac{2\pi nx}{L})}dx \\ &= -\frac{AL^2\sin{(2\pi n)}}{2\pi^2 n^2} \end{aligned} \tag{8}$$

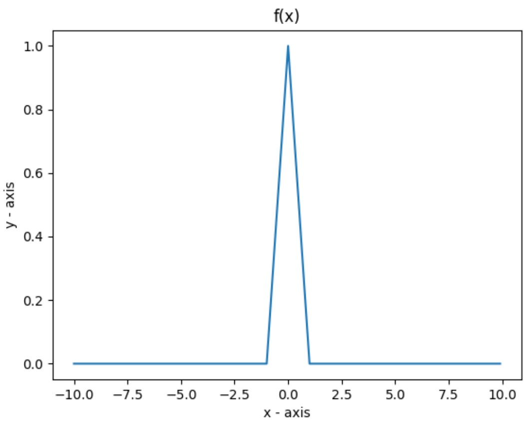
$\sin 2\pi n = 0$ for all integer n = 0, 1, 2 ...

$$b_n = 0 \tag{9}$$

1.1.4 Final answer for f(x)

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{n} a_n \cos{(\frac{2\pi nx}{L})} \\ &= \frac{AL^2}{6} + \sum_{n=1}^{n} \frac{AL^2(-\pi L^3 n - \pi n)}{2\pi^3 n^3}\cos{(\frac{2\pi nx}{L})} \end{aligned} \tag{10}$$

2 Problem 2



$$f(x) = \begin{cases} 1-|x| & |x| < 1 \\ 0 & \text{Otherwise} \end{cases} \tag{11}$$

2.1 Proof f(x) is even

if f(x) is an even function: f(x) = f(-x)

$$f(x) = 1 - |x| \tag{12}$$

$$\begin{aligned} f(-x) &= 1 - |-x| \\ &= 1 - |x| \\ &= f(x) \end{aligned} \tag{13}$$

Therefore C(k) will be even and real
$f_o(x) = 0$ therefore $C_o(k) = 0$
($f_o(x)$ is odd part of f(x), $C_o(k)$ is odd part of C(k))

2.2 Calculation of C(k)

f(x) = $1 - |x|$ on interval $-1 < x < 1$

$$\begin{aligned} C(k) &= \frac{1}{2\pi}\int_{-1}^{1} f(x)\exp\left[-ikx\right]dx \\ &= \frac{1}{2\pi}\int_{-1}^{1}\Big(f_e(x) + f_o(x)\Big)\Big(\cos(kx) + -i\sin(kx)\Big)dx \\ &= \frac{1}{2\pi}\int_{-1}^{1}\Big(f_e(x)\cos(kx) + f_e(x) - i\sin(kx) + f_o(x)\cos(kx) + f_o(x) - i\sin(kx)\Big)dx \end{aligned} \tag{14}$$

Simplify with $f_o(x) = 0$ and distribute integral...

$$C(k) = \frac{1}{2\pi}\Big(\int_{-1}^{1} f_e(x)\cos(kx)dx\Big) + \frac{1}{2\pi}\Big(\int_{-1}^{1} f_e(x) - i\sin(kx)dx\Big) \tag{15}$$

Simplify with $\int_{-1}^{1} f_e(x) - i\sin(kx)dx$ is an odd function and will be 0 over the interval

$$C(k) = \frac{1}{2\pi}\int_{-1}^{1} f_e(x)\cos(kx)dx \tag{16}$$

$$f_e(x) = f(x)$$

$$\begin{aligned} C(k) &= \frac{1}{2\pi}\int_{-1}^{1} f(x)\cos(kx)dx \\ &= \frac{1}{2\pi}\int_{-1}^{1}(1 - |x|)\cos(kx)dx \\ &= \frac{1}{2\pi}\Big(\int_{-1}^{0}(1 + x)\cos(kx)dx\Big) + \frac{1}{2\pi}\Big(\int_{0}^{1}(1 - x)\cos(kx)dx\Big) \end{aligned} \tag{17}$$

2.2.1 C(k) over interval $-1 < x < 0$

$$\frac{1}{2\pi}\int_{-1}^{0}(1+x)\cos(kx)dx \tag{18}$$

$$\begin{gathered} u = (1+x) \\ du = dx \\ dv = \cos kx dx \\ v = \tfrac{1}{k}\sin kx \end{gathered}$$

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-1}^{0}(1+x)\cos(kx)dx \\ &= \Big[(1+x)\frac{1}{k}\sin kx\Big]_{-1}^{0} - \int_{-1}^{0}\frac{1}{k}\sin kx dx \\ &= 0 - \frac{1}{k}\Big[-\frac{1}{k}\cos kx\Big]_{-1}^{0} \\ &= 0 + \frac{1}{k^2}\Big[\cos(0) - \cos(-k)\Big] \\ &= 0 + \frac{1}{k^2}\Big[1 - \cos(k)\Big] \\ &= \frac{1-\cos(k)}{2\pi k^2} \end{aligned} \tag{19}$$

($\frac{1}{2\pi}$ added at the end for convenience)

2.2.2 C(k) final

Because C(k) is even on the interval $-1 \leq x \leq 1$, the final integral will just be 2 * the integral calculated in step 2.2.1. This is property of even functions over a domain of equal and opposite values. Therefore:

$$C(k) = \frac{1-cos(k)}{\pi k^2} \tag{20}$$