Phys273 - Homework 6

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October 8, 2022

1 Problem 1

1.1 Question a

$$v_g = \frac{d\omega}{dk} \tag{1}$$

$$\omega_{max} = \frac{d\omega}{dk} = 0 \tag{2}$$

From the two equations above, we can conclude that when ω is at ω_{max} for a traveling wave, v_g will be equal to 0.

1.2 Question b

For $k > k_m$ the group velocity will be positive. At ω_{max} the group velocity will be 0. For $k < k_m$ the group velocity will be decreasing as it approaches 0. Since there is not other maxima or minima, the graph of ω vs k will be decreasing from the point of the maxima, and therefore $\frac{d\omega}{dk}$ will be increasing from 0, giving it a positive value. Since group velocity is $\frac{d\omega}{dk}$, it will be positive for $k > k_m$.

2 Problem 2

2.1 Question a

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} - v^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} - L^2 v^2 \frac{\partial^4 \psi(x,t)}{\partial x^4} = 0$$

Assume solution of $\psi(x,t) = A \exp[i(kx \pm w(k)t)]$

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = -Aw(k)^2 \exp[i(kx \pm w(k)t)] \tag{3}$$

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = -Ak^2 \exp[i(kx \pm w(k)t)] \tag{4}$$

$$\frac{\partial^4 \psi(x,t)}{\partial x^4} = Ak^4 \exp[i(kx \pm w(k)t)] \tag{5}$$

Substitute values into equation above

$$\begin{aligned}
-Aw(k)^2\exp[i(kx \pm w(k)t)] - (-v^2Ak^2\exp[i(kx \pm w(k)t)]) - L^2v^2Ak^4\exp[i(kx \pm w(k)t)] = 0 \\
-w(k)^2 + v^2k^2 - L^2v^2k^4 = 0 \\
-w(k)^2 = -v^2k^2 + L^2v^2k^4 \\
w(k) = \sqrt{v^2k^2 - L^2v^2k^4}
\end{aligned} \tag{6}$$

2.2 Question b

$$\omega_{max} = \frac{d\omega}{dk}$$

2.2.1 Show $k = \frac{1}{L\sqrt{2}}$

$$\begin{aligned}
k \text{ at } \omega_{max} &= \frac{d\sqrt{v^2k^2 - L^2v^2k^4}}{dk} \\
&= \frac{d}{dk}vk\frac{d}{dk}\sqrt{1 - L^2k^2}
\end{aligned} \tag{7}$$

Product rule, set = 0

$$\begin{aligned}
k &= \frac{d}{dk}vk\frac{d}{dk}\sqrt{1 - L^2k^2} \\
&= (v\sqrt{1 - L^2k^2}) + (\frac{-2vL^2k^2}{2\sqrt{1 - L^2k^2}}) \\
\text{Multiply by } \sqrt{1 - L^2k^2} \\
&= v(1 - L^2k^2) - vL^2k^2 \\
&= (v - vL^2k^2) - vL^2k^2 = 0 \\
&= v = 2vL^2k^2 \\
k^2 &= \frac{1}{2L^2} \\
k &= \frac{1}{L\sqrt{2}}
\end{aligned} \tag{8}$$

Since L is a length value - k will always be positive

2.2.2 Show $\omega_{max} = \frac{1}{2L}$

$$\begin{aligned}
\omega(\frac{1}{L\sqrt{2}}) &= \sqrt{v^2(\frac{1}{L\sqrt{2}})^2 - L^2v^2(\frac{1}{L\sqrt{2}})^4} \\
&= \sqrt{v^2(\frac{1}{L^22}) - L^2v^2(\frac{1}{L^44})} \\
&= \sqrt{(\frac{v^2}{2L^2}) - (\frac{v^2}{4L^2})} \\
&= \sqrt{(\frac{2v^2}{4L^2}) - (\frac{v^2}{4L^2})} \\
&= \sqrt{(\frac{v^2}{4L^2})} \\
\omega_{max} &= \frac{v}{2L}
\end{aligned} \tag{9}$$