Ant Colony Optimisation for the Travelling Salesman Problem

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Contents

1 Introduction

Before defining the Travelling Salesman Problem and proving properties about it, a number of graph theoretic concepts must first be defined. Therefore, what follows is a sub-section that introduces a number of graph theoretic concepts which are required for the Travelling Salesman Problem.

1.1 Some Graph Theory

Graph theory is the study of the structure called a graph. A graph can be defined formally as shown in definition 1.1.1 below.

Definition 1.1.1 (Graph). *A graph G is a pair (V,E) were V is any non empty finite set called the set of vertices of G, and $E \subseteq \{\{u,v\} : u,v \in V$ and $u \neq v\}$ is called the set of edges of G [1]. A graph G defined by the pair (V,E) is denoted by G(V,E) or G.*

A graph defined using definition 1.1.1 is called an undirected graph. There is also the concept of a directed graph were $E \subseteq \{(u,v) : u,v \in V$ and $u \neq v\}$ [1]. However, in this thesis it can be assumed that any graph that will be considered is undirected unless otherwise stated. It can also be assumed that there are no edges between same vertices unless otherwise stated. The discussion will now proceed by introducing more graph theoretic terminology, along with examples for these terminologies.

When 2 vertices are joined by an edge, they are said to be adjacent. This is defined formally in definition 1.1.2 below.

Definition 1.1.2. *Given a graph G(V,E), $\forall$ u,v $\in$ V, u and v are said to be adjacent if $\{u,v\} \in E$. [2]*

Any graph $G(V,E)$ can also be represented pictorially by drawing the vertices of G using circles, and by representing the edges of G using lines between adjacent vertices.As a result in this thesis, a graph is sometimes given formally using sets, or as a pictorial representation assuming that one can be converted to another. Example 1.1.3 below depicts how a graph can be represented pictorially.

Example 1.1.3. Consider the graph $G(V,E)$ such that $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\}$.
Then G can be represented pictorially as :
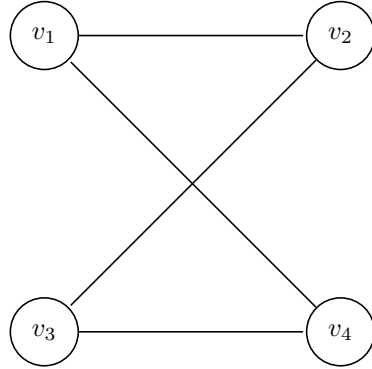
There are many other examples of graphs, one of them being the complete graph on n vertices.

Definition 1.1.4 (Complete Graph). *A graph G(V,E) is said to be complete if $\forall\ v,w \in V\ v \neq w$, v is adjacent to w. The complete graph on n vertices is denoted by K_n.* [3]

Given any graph $G(V, E)$, one can also define concepts that lie within G, one of them being a path.

Definition 1.1.5 (Path). *Given a graph G(V,E), a path in G joining any 2 vertices $u,v \in V$, is a sequence of vertices $u = u_1,\ u_2,\ ...,u_n = v$ in which no vertex is repeated and, $\forall\ u_i,u_{i+1}\ i \in [n\text{-}1],\ \{u_i, u_{i+1}\} \in E$ [4]. A path on n vertices is denoted by P_n.*

Definition 1.1.5 can now be used to define cycles and connectivity in a graph.

Definition 1.1.6 (Connected Graph). *A graph G(V,E) is said to be connected if $\forall\ u,v \in V\ u \neq v$, u and v are joined by a path.* [5]

Definition 1.1.7 (Cycle). *Given a graph G(V,E), a cycle in G is a path P_n n ≥ 4, such that, the starting vertex and the end vertex are equal.* [6]

By definitions 1.1.5, 1.1.7 above, it is clear that a cycle is a special instance of a path, with the only difference being that in a cycle, the first vertex and the last vertex are equal. Another thing worth mentioning is that according to definitions 1.1.5 and 1.1.7, cycles and paths are sequences of vertices and not actual graphs. However, this is not the case because they can be represented easily as graphs. For example, given the path/cycle $u_1, u_2, ..., u_n$ a new graph $G(V, E)$ can be created such that, $V(G) = \{u_1, u_2, ..., u_n\}$ and $E(G) = \{\{u_i, u_{i+1}\} : i \in [n-1]\}$. For example, consider the cycle v1 v2 v3 v4 v1, then the graph $G(V, E)\ V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\}$ depicted in example 1.1.3, is the graph representing this cycle. Since this construction can be done, cycles can be both treated as graphs and sequences. This will later be useful when defining Hamiltonian cycles. For better understanding of definitions 1.1.4, 1.1.5, 1.1.6, 1.1.7 example 1.1.8 is constructed.

Example 1.1.8. Consider the graph $G(V,E)$ below:
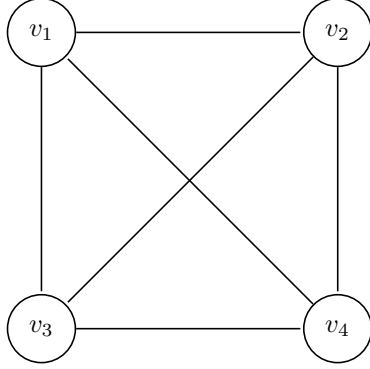
Since every vertex in G is adjacent to every other vertex, G must be complete. Therefore, it must also be connected because, there is a path P_2 between any 2 distinct vertices of G. This graph is denoted by K_4 since it is the complete graph on 4 vertices.

Some examples of paths in G are:

1. v_1 v_2 v_3 v_4
2. v_1 v_4
3. v_4 v_3 v_1

Some examples of cycles in G are:

1. v_1 v_2 v_3 v_4 v_1
2. v_1 v_4 v_2 v_3 v_1
3. v_4 v_3 v_1 v_4

Another important terminology is that of subgraphs.

Definition 1.1.9 (Subgraph)**.** *Given a graph $G(V,E)$ and a graph $H(V',E')$, H is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. [7]*

After defining some important concepts, the next step is to extend definition 1.1.1 to define another class of graphs called weighted graphs. It must be noted that all definitions presented so far apply also to weighted graphs.

Definition 1.1.10 (Weight Function)**.** *Given a graph $G(V,E)$ a weight function is a function $f : E \mapsto \Re$ [7]. The real numbers assigned to each edge are called weights.*

Definition 1.1.11. *A weighted graph is a graph $G(V,E)$ with a weight function f [7]. This is denoted by the triple $G(V,E,f)$.*

According to Bondy and Murty [8], weighted graphs occur regularly in applied graph theory. For example, a railway network can be represented by a weighted graph were, the vertices are the set of towns in the railway network, and there are edges between 2 vertices in the graph if there is a direct route from one town to another, without visiting other towns in the process. In addition

to that, the shortest path between 2 towns in the network may be required. It is clear that in order to try and solve such problems, the total weight of a subgraph must first be defined.

Definition 1.1.12 (Weight of a Subgraph). *Given a weighted graph G(V,E,f), the total weight of any subgraph H(V′,E′,f) of G is:*

$$\sum_{e \in E'} f(e)$$

. [8]

It is important to note that by definition 1.1.9, any weighted graph G is a subgraph of itself, therefore, it's weight can be calculated. This is highlighted in Example 1.1.13 below.

Example 1.1.13. Consider the weighted graph *G(V,E,f)* such that, *G(V,E)* is the graph in example 1.1.8 with weight function *f* such that,
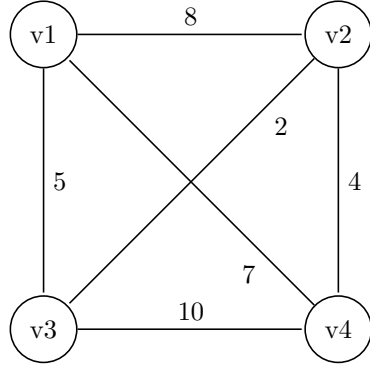$f(\{v_1, v_2\}) = 8$
$f(\{v_1, v_3\}) = 5$
$f(\{v_2, v_3\}) = 2$
$f(\{v_3, v_4\}) = 10$
$f(\{v_2, v_4\}) = 4$
$f(\{v_4, v_1\}) = 7$
Then according to definition 1.1.12, the weight of *G* is 36.

According to Guichard [9], trees are another useful class of graphs.

Definition 1.1.14 (Tree). *A tree is a connected graph with no cycles.* [9]

Having defined much of the basic graph theoretic concepts, it is now time to define harder concepts that use previous definitions. It is important to note that these concepts can be applied to both weighted and unweighted graphs.

Definition 1.1.15 (Spanning Subgraph). *H(V′,E′) is a spanning subgraph of G(V,E) if H is a subgraph of G and V′ = V.* [10]

There are many spanning subgraphs, however the ones that are relevant to this thesis are spanning trees and spanning cycles, the latter mostly known as Hamiltonian cycles.

Definition 1.1.16 (Spanning Tree). *A graph H is a spanning tree of G if H is a tree and H is a spanning subgraph of G.* [10]

Definition 1.1.17 (Hamiltonian Cycle). *Given a graph G, c is a Hamiltonian cycle in G if c is a cycle and c is a spanning subgraph of G. Also, a graph that contains a Hamiltonian cycle is called a Hamiltonian graph.* [11]

It is worth mentioning that definition 1.1.17 holds because, cycles can be represented as graphs by the construction discussed earlier. What follows now is an example that illustrates better definitions 1.1.15, 1.1.16 and 1.1.17.

2 The Travelling Salesman Problem

2.1 Heuristics

2.2 The Ant Colony Algorithm

3 Experimental Data

4 Conclusion

References

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