Project 1: Finite Abelian Groups

Lemma. *Suppose that* $m = \sum r_i \alpha_i$ *for some* $r_i \in \{0, 1, \cdots, p^{k_i} - 1\}$ *with* $gcd(r_i, p) = 1$ *for all* i*. To ease notation, assume that* $r_1 \neq 0$ *and* $k_1 = max\{k_i | r_i \neq 0\}$*. Then* $M \cong \langle m \rangle \times M_2 \times M_3 \times \cdots \times M_n$*.*

Proof. Note that it is sufficient to prove that $M_1 \cong \langle m \rangle$, or equivalently, since both are cyclic groups, $p^{k_1} = |M_1| = |m|$.
The identity in G will be denoted as 0, the same symbol used to denote the additive identity in $\mathbb{Z}$. Note that the identity in $\langle m \rangle$ is $(0, \cdots, 0, \cdots, 0)$.

$$p^{k_1} m = p^{k_1}(r_1 x_1, \cdots, r_i x_i, \cdots, r_n x_n) = (r_1(p^{k_1} x_1), \cdots, p^{k_1} r_i x_i, \cdots, p^{k_1} r_n x_n)$$

$p^{k_1} x_1 = 0$ because x_1 is a generator for M_1, so $|x_1| = p^{k_1}$.
And for all other $1 < i \leq n$, since $k_1 = max\{k_i | r_i \neq 0\}$, $k_1 = k_i + \beta_i$, where $\beta_i \geq 0$. Then, $p^{k_1} r_i x_i = r_i p^{\beta_i}(p^{k_i} x_i) = r_i p^{\beta_i}(0) = 0$. Thus,

$$p^{k_1} m = (0, \cdots, 0, \cdots, 0)$$

Now let $y \in \mathbb{Z}$ such that $0 < y < p^{k_1}$. Then,

$$ym = y(r_1 x_1, \cdots, r_i x_i, \cdots, r_n x_n) = (y r_1 x_1, \cdots, y r_i x_i, \cdots, y r_n x_n)$$

Now, y has at most $k_1 - 1$ factors of p in its unique prime factorization. Also, since $gcd(r_1, p) = 1$, r_1 has no factors of p. Then, $y r_1$ has at most $k_1 - 1$ factors of p. That is, p^{k_1} does not divide $y r_1$. Thus, with the assumption that $r_1 \neq 0$, $y r_1 x_1 \neq 0$, so

$$ym \neq (0, \cdots, 0, \cdots, 0)$$

Therefore, $|m| = p^{k_1}$, and thus, $M \cong \langle m \rangle \times M_2 \times M_3 \times \cdots \times M_n$. $\square$