Is the Maxwell-Boltzmann distribution applicable to any arbitrary potential in non-relativistic classical mechanics? If we believe in equal *a priori* probabilities in the micro-canonical ensemble, then for reasonably well-behaved potentials that don't depend on momentum (and with no magnetic fields because a vector potential messes up the derivation), I think we should believe in the Maxwell-Boltzmann distribution.

In 3D, the distribution of velocities is proportional to

$$v^2 e^{-\beta m v^2/2} \tag{1}$$

In 2D, we lose a factor of v because we integrate over a circle instead of a sphere

$$v e^{-\beta m v^2/2} \tag{2}$$

And in 1D,

$$e^{-\beta m v^2/2} \tag{3}$$

Let prove that the 1D simple harmonic oscillator obeys the Maxwell-Boltzmann distribution. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{4}$$

For simplicity, let's work in units where $m = \omega = 1$. Then the energy is

$$E = \frac{1}{2}(v^2 + x^2) \tag{5}$$

At energy E, and setting t=0 to coincide with zero speed, the velocity is given by

$$v(t) = \sqrt{2E}\sin(t) \tag{6}$$

The velocity distribution of that oscillator is given by $P(v)dv = |2dt/T|$, where $T = 2\pi$ is the period, but I will drop constants whenever I want. Using $dv = \sqrt{2E}\cos(t)$,

$$P(v)dv \propto \frac{dv}{\sqrt{2E}\cos(t)} = \frac{dv}{\sqrt{2E - (\sqrt{2E}\sin(t))^2}} = \frac{dv}{\sqrt{2E - v^2}} \tag{7}$$

if $v^2 < 2E$. Otherwise, $P(v)dv = 0$.

We can now take a collection of oscillators connected to a heat bath. The probability that an oscillator has energy E is proportional to $e^{-\beta E}$. We can then multiply $P(v)dv$ by $e^{-\beta E}$ and integrate over energies. The density of states is constant. The limits of integration are $v^2/2$ to infinity because oscillators with energy less than $v^2/2$ are never at velocity v.

$$\int_{v^2/2}^{\infty} \frac{e^{-\beta E}}{\sqrt{2E - v^2}}\, \mathrm{d}E = \sqrt{\frac{\pi}{2\beta}} e^{-\beta v^2/2} \tag{8}$$

And we've recovered the 1D Maxwell-Boltzmann distribution!