Derivation of Hamilton's Equations

Define the Hamiltonian from the Lagrangian.

$$\mathcal{H} = \sum_j p_j \dot{q}_j - \mathcal{L}$$

Take the partial of $\mathcal{H}$ with respect to q_i. This equation is like Newton's 2nd law.

$$\frac{\partial \mathcal{H}}{\partial q_i} = \sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i} - \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{\partial \mathcal{H}}{\partial q_i} = \sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i} - \sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{\partial \mathcal{H}}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} = -\dot{p}_i$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

Take the partial of $\mathcal{H}$ with respect to p_i. This usually reproduces the definition of the canonical momentum.

$$\frac{\partial \mathcal{H}}{\partial p_i} = \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i} + \dot{q}_i - \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i}$$

$$\frac{\partial \mathcal{H}}{\partial p_i} = \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i} + \dot{q}_i - \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i}$$

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

And now for the time derivative.

$$\frac{d\mathcal{H}}{dt} = \sum_j \dot{p}_j \frac{\partial \mathcal{H}}{\partial p_j} + \sum_j \dot{q}_j \frac{\partial \mathcal{H}}{\partial q_j} + \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t}$$

$$\frac{d\mathcal{H}}{dt} = \frac{\partial H}{\partial t} = \sum_j p_j \ddot{q}_j - \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j - \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

And it's very straight forward to relate the time derivative of an observable O to the Hamiltonian.

$$\frac{dO}{dt} = \{O, H\} + \frac{\partial O}{\partial t}$$

$$\{O, H\} = \sum_j \left[\frac{\partial O}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial O}{\partial p_j} \frac{\partial H}{\partial q_j} \right]$$