where $\hat{n}_{\rm p}$ and $\hat{n}_{\rm n}$ are the particle number operators for protons and neutrons respectively. The sums in these operators run over all proton (labels π) or neutron (labels ν) single-particle states. The eigenvalue equations for the number operators are

$$\hat{n}_{\rm p}|\pi_1\,\pi_2\ldots\pi_Z\,\nu_1\,\nu_2\ldots\nu_N\rangle = Z|\pi_1\,\pi_2\ldots\pi_Z\,\nu_1\,\nu_2\ldots\nu_N\rangle\ , \tag{4.17}$$

$$\hat{n}_{\rm n}|\pi_1\,\pi_2\ldots\pi_Z\,\nu_1\,\nu_2\ldots\nu_N\rangle = N|\pi_1\,\pi_2\ldots\pi_Z\,\nu_1\,\nu_2\ldots\nu_N\rangle\ . \tag{4.18}$$

4.2 Operators and Their Matrix Elements

In this section we discuss the occupation number representation of one- and two-body operators. In addition, the matrix elements of spherical tensors are discussed in considerable detail.

4.2.1 Occupation Number Representation of One-Body Operators

Let t be a *one-body operator*, e.g. the kinetic energy of a particle. A one-body operator is an operator which acts on the coordinates, including spin, of only one particle at a time. If this is particle number i, the one-body operator is $t(i)$. In coordinate representation it is $t(\boldsymbol{x}_i)$, where $\boldsymbol{x}_i$ comprises both the space and spin variables. The total effect of an operator T acting on a nucleus is obtained by summing the actions on individual nucleons, i.e. $T = \sum_{i=1}^{A} t(\boldsymbol{x}_i)$. A familiar example of a one-body operator is the kinetic-energy operator which depends only on the space variables $\boldsymbol{r}_i$.

It can be proved that a general one-body operator can be expressed in occupation number representation as

$$\boxed{\begin{aligned} T &= \sum_{i=1}^{A} t(\boldsymbol{x}_i) = \sum_{\alpha\beta} t_{\alpha\beta} c_\alpha^\dagger c_\beta\ , \\ t_{\alpha\beta} &\equiv \langle\alpha|T|\beta\rangle = \int \phi_\alpha^\dagger(\boldsymbol{x}) t(\boldsymbol{x}) \phi_\beta(\boldsymbol{x}) {\rm d}^3\boldsymbol{r}\ . \end{aligned}} \tag{4.19}$$

Example 4.1
The angular momentum operators J_z and $J_\pm$ can be expressed in occupation number representation as

$$J_z = \hbar \sum_\alpha m_\alpha c_\alpha^\dagger c_\alpha\ , \quad J_\pm = \hbar \sum_\alpha m_\alpha^\mp c_\alpha^\dagger c_{\alpha\mp1}\ , \tag{4.20}$$

where

$$m_\alpha^\pm \equiv \sqrt{(j_a \pm m_\alpha + 1)(j_a \mp m_\alpha)}\ , \quad c_{\alpha\pm1} \equiv c_{a,m_\alpha\pm1}\ . \tag{4.21}$$

This representation of the angular momentum operators can be conveniently used in the commutation relations (2.12) to test whether a set of operators

one-body transition density. The transition densities characterize the many-nucleon properties of the initial (i) and final (f) nuclear states. They do not contain any information about the transition operator beyond its one-body character.

4.2.3 Occupation Number Representation of Two-Body Operators

A *two-body operator* $v(i,j)$, e.g. the potential energy of interaction, acts simultaneously on the observables of two particles i and j. One can sum all the pairwise actions of this operator to produce the total action V. In coordinate representation its expression is

$$V = \sum_{i<j} v(\boldsymbol{x}_i, \boldsymbol{x}_j) = \tfrac{1}{2} \sum_{i \neq j} v(\boldsymbol{x}_i, \boldsymbol{x}_j) \ , \tag{4.26}$$

where we allow also a spin dependence for the two-body operator. It can be shown that in occupation number representation

$$\boxed{\begin{aligned} V &= \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} v_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma \ , \\ v_{\alpha\beta\gamma\delta} &= \int \phi_\alpha^\dagger(\boldsymbol{x}_1) \phi_\beta^\dagger(\boldsymbol{x}_2) v(\boldsymbol{x}_1, \boldsymbol{x}_2) \phi_\gamma(\boldsymbol{x}_1) \phi_\delta(\boldsymbol{x}_2) \mathrm{d}^3\boldsymbol{r}_1 \mathrm{d}^3\boldsymbol{r}_2 \ . \end{aligned}} \tag{4.27}$$

In addition to spatial coordinates and spin variables the two-body operator $v(i,j)$ can contain isospin dependence. In that case the single-particle wave functions carry also isospin labels. An isospin-dependent two-nucleon interaction is an important practical example.

It is convenient to write the two-nucleon interaction V by using an *antisymmetrized two-nucleon interaction matrix element* $\bar{v}_{\alpha\beta\gamma\delta}$:

$$\boxed{V = \tfrac{1}{4} \sum_{\alpha\beta\gamma\delta} \bar{v}_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma \ , \quad \bar{v}_{\alpha\beta\gamma\delta} \equiv v_{\alpha\beta\gamma\delta} - v_{\alpha\beta\delta\gamma} \ ,} \tag{4.28}$$

with the symmetry properties[3]

$$\bar{v}_{\alpha\beta\gamma\delta} = -\bar{v}_{\beta\alpha\gamma\delta} = -\bar{v}_{\alpha\beta\delta\gamma} = \bar{v}_{\beta\alpha\delta\gamma} = \bar{v}^*_{\gamma\delta\alpha\beta} \ . \tag{4.29}$$

These symmetries are helpful in actual calculations. It is worth noting that the antisymmetrized matrix element (4.29) is the same as the *normalized* and *antisymmetrized* two-body matrix element defined by some authors, e.g. [17]:

$$\bar{v}_{\alpha\beta\gamma\delta} = {}_{\text{n.as.}}\langle \alpha\beta | V | \gamma\delta \rangle_{\text{n.as.}} \ . \tag{4.30}$$

[3] See also (13.127).