EEG inverse problem: TP1 - Gibbs sampler

SISEA - Problèmes Inverses

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1 Prerequisites

We aim to reconstruct the unknown source vector $\boldsymbol{s}$ given its observation :

$$\boldsymbol{x} = \mathbf{A}\boldsymbol{s} + \boldsymbol{n}, \tag{1}$$

where $\mathbf{A}$ is the known $N \times D$ mixing matrix ($N \ll D$), $\boldsymbol{x}$ the observation column vector ($N \times 1$), and $\boldsymbol{n}$ the white Gaussian noise of unknown variances σ_n^2.
To impose sparsity in the solution, it is further supposed that $\boldsymbol{s}$ follows a Bernoulli-Gaussian (BG) model, an independent, identically distributed (iid) process defined in two stages. Firstly, the sparse nature is governed by the Bernoulli law:

$$P(\boldsymbol{q}) = \lambda^L (1-\lambda)^{D-L} \tag{2}$$

with $\boldsymbol{q} = [q_1, \ldots, q_D]^{\mathrm{t}}$ a binary sequence and $L = \sum_{d=1}^{D} q_d$, the number of non-zero entries of $\boldsymbol{q}$. Secondly, amplitudes $\boldsymbol{s} = [s_1, \ldots, s_D]^{\mathrm{t}}$ are assumed iid zero-mean Gaussian conditionally to $\boldsymbol{q}$:

$$\boldsymbol{s} \mid \boldsymbol{q} \ \sim \ \mathcal{N}(\mathbf{0}, \sigma_s^2 \ \mathrm{diag}(\boldsymbol{q})), \tag{3}$$

where $\mathrm{diag}(\boldsymbol{q})$ denotes a diagonal matrix whose diagonal is $\boldsymbol{q}$.
The problem becomes the estimation of $\Theta = \{\boldsymbol{s}, \boldsymbol{q}, \lambda, \sigma_n^2, \sigma_s^2\}$ given $\boldsymbol{x}$, for which the joint posterior distribution writes :

$$P(\Theta \,|\, \boldsymbol{x}) \propto \ g(\boldsymbol{x} - \mathbf{A}\boldsymbol{s}; \sigma_n^2 \mathbf{I}_N)\, P(\boldsymbol{q}; \lambda)\, g(\boldsymbol{s}; \sigma_s^2 \ \mathrm{diag}(\boldsymbol{q}))\, P(\sigma_n^2) P(\sigma_s^2) P(\lambda) \tag{4}$$

where $g(\cdot; \mathbf{R})$ denotes the centered Gaussian density of covariance $\mathbf{R}$. A conjugate prior law is adopted for λ:

$$\lambda \sim Be(?, ?)$$

where Be represents the **Beta** distribution. For simplicity, fixed values are assumed for $\sigma_n^2 =?$ and $\sigma_s^2 =?$ The hyper-parameters (noted by ?) are to be adjusted according to the problem context.

2 Gibbs Sampler

According to the Monte Carlo principle, a posterior mean estimator of the unknown random variables can be approximated by:

$$\widehat{\Theta} = \frac{1}{I-J} \sum_{k=J+1}^{I} \Theta^{(k)}, \tag{5}$$

where the sum extends over the last $I-J$ samples. In the MCMC framework, the samples are generated iteratively, so that asymptotically converges in distribution to the joint posterior probability in eq. (4).

	Metropolis-Hastings algorithm
1	current configuration $\Theta^{(k)}$
2	draw Θ' with a proposition law $Q(\Theta' \mid \Theta^{(k)})$
3	accept $\Theta^{(k+1)} = \Theta'$ with probability $\min\left\{1, \dfrac{P(\Theta')Q(\Theta^{(k)} \mid \Theta')}{P(\Theta^{(k)})Q(\Theta' \mid \Theta^{(k)})}\right\}$
4	repeat from 1

	Gibbs algorithm
1	current configuration $\Theta^{(k)}$
2	choose (cyclically or randomly) i
3	draw $\Theta_i^{(k+1)} \sim \Theta_i \mid \Theta^{(k)} \setminus \Theta_i^{(k)}, \boldsymbol{z}$
4	repeat from 1

Exercise 2.1 *write the following conditional probabilities from the joint probability in Eq (4):*

1. $q_d \mid \Theta \setminus \{q_d, s_d\}, \boldsymbol{x}$,
2. $s_d \mid \Theta \setminus \{s_d\}, \boldsymbol{x}$,
3. $\lambda \mid \Theta \setminus \lambda, \boldsymbol{x}$,

N.B. the Beta distribution writes $Beta(x; \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

Exercise 2.2 *From the Gibbs sampling scheme, derive the pseudocode in Tab. 1*

Table 1: Pseudocode for the Gibbs sampler

`% Initialization`
$\boldsymbol{q} \leftarrow \boldsymbol{0}$; $\boldsymbol{s} \leftarrow \boldsymbol{0}$
Sample λ using the prior law and choose appropriate values for σ_n^2, σ_s^2
repeat
  `% --------------------- Step 1: Sample` (q_i, s_i) `---------------------`
  $\boldsymbol{e} \leftarrow \boldsymbol{x} - \mathbf{A}\boldsymbol{s}$
  for $i = 1$ to D **do**
    $\boldsymbol{e}_i \leftarrow \boldsymbol{e} + \mathbf{A}_i s_i$ `%` $\mathbf{A}_i$ `is the` i`-th column of` $\mathbf{A}$
    $\sigma_i^2 \leftarrow \sigma_n^2 \sigma_s^2 / (\sigma_n^2 + \sigma_s^2 \|\mathbf{A}_i\|^2)$
    $\mu_i \leftarrow (\sigma_i^2/\sigma_n^2)\mathbf{A}_i^{\mathrm{t}} \boldsymbol{e}_i$
    $\nu_i \leftarrow \lambda(\sigma_i/\sigma_s)\exp\left(\mu_i^2/(2\sigma_i^2)\right)$
    $\lambda_i \leftarrow \nu_i/(\nu_i + 1 - \lambda)$
    Sample $q_i \sim Bi(\lambda_i)$
    Sample $s_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ if $q_i = 1$, $s_i = 0$ otherwise
    $\boldsymbol{e} \leftarrow \boldsymbol{e}_i - \mathbf{A}_i s_i$ `% Update` $\boldsymbol{e}$
  end for
  `% --------------------- Step 2: Sample` λ `-----------------------`
  Sample $\lambda \sim \mathrm{Be}(? + L, ? + D - L)$, with $L = \sum_d q_d$
until Convergence

Exercise 2.3 *Implement the Gibbs sampler according to the pseudocode by completing the Matlab function Gibbs_sampler.m.*

Exercise 2.4 *Apply the Gibbs sampler to solve the EEG inverse problem given the example of simulated EEG data:*

- *Integrate the function Gibbs_sampler.m in the script TP_inverse_problems.m and visualize the estimated solution.*
- *Compare the estimated active dipoles with the original sources.*
- *Analyze the influence of the SNR (ranging from 0.1 to 10) on the obtained solution.*