Lecture 19

11.8 Power series

Def: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

where x is a variable and c_i's are constants called the coefficients

May converge for some values of x and diverge for others. The sum is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

whose domain is the set of all x for which the series converges
"polynomial with infinitely many terms.

Ex $\sum_{n=0}^{\infty} x^n$ geometric series
converge for $-1 < x < 1$ and diverge when $|x| \geq 1$

More generally: series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

is called a "power series in $(x-a)$", or a "power series centered at a" or a "power series about a"

convention: $(x-a)^0 = 1$ even if $x = a$

Note: always converge to c_0 for $x = a$

Ex 1: For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Ratio test: (assume $x \neq 0$)

$$\lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

Hence, series diverges for $x \neq 0$. Only converges for $x = 0$.

Ex 3: Find the domain of the Bessel function of order 0 (famous function with applications in physics and chemenstry)

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

Ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} ((n+1)!)^2} \cdot \frac{2^{2n} (n!)^2}{(-1)^n x^{2n}} \right|$$

$$= \frac{x^{2n+2}}{2^{2n+2} (n+1)^2 (n!)^2} \cdot \frac{2^{2n} (n!)^2}{x^{2n}}$$

$$= \frac{x^2}{4(n+1)^2} \to 0 < 1 \qquad \text{for all } x$$

$\Rightarrow$ series converges for all x. Hence, domain of J_0 is $(-\infty, \infty)$

It turns out that the kinds of domains for x we have seen are the only possible ones

Thm For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities

(i) The series converges only for $x=a$

(ii) The series converges for all x

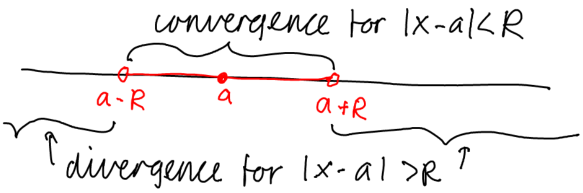
(iii) There is a positive number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$

Notation:

- R is called radius of convergence. In case (i) $R=0$ and in (ii) $R=\infty$
- interval of convergence : interval of all x where the series converge. In case (i) it is $\{a\}$ and in (ii) it is $(-\infty,\infty)$

In case (iii) there are 4 possibilities for the interval of convergence

$(a-R,\ a+R)\ ,\quad (a-R,a+R]\ ,\quad [a-R,\ a+R)\ ,\quad [a-R,\ a+R]$



o may or may not be included

Rem: in general the ratio test (or sometimes the root test) should be used to find R, but it fails at the endpoints so there we need other tests

Ex 4: Find radius of convergence and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \frac{\sqrt{n+1}}{(-3)^n x^n} \right| = \left| 3x \frac{\sqrt{n+1}}{\sqrt{n+2}} \right| = 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} \longrightarrow 3|x| \text{ as } n\to\infty$$

$\Rightarrow$ converges if $3|x|<1 \Leftrightarrow |x|<\frac{1}{3}$ interval $(-\frac{1}{3},\frac{1}{3})$

diverges if $3|x|>1 \Leftrightarrow |x|>\frac{1}{3}$

$\Rightarrow R=\frac{1}{3}$

Endpoints:

$x=-\frac{1}{3}$: $\quad \sum_{n=0}^{\infty} \frac{(-3)^n(-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ divergent since $\sim$ p-series with $p=\frac{1}{2}$

$x=\frac{1}{3}$: $\quad \sum_{n=0}^{\infty} \frac{(-3)^n(\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by the alternating series test

So interval of convergence is $(-\frac{1}{3},\frac{1}{3}]$

Class exercise:

Ex 5: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \frac{3^{n+1}}{n(x+2)^n} \right| = \left| \frac{1}{3} \frac{n+1}{n} (x+2) \right| = \left(1+\frac{1}{n}\right) \frac{|x+2|}{3} \longrightarrow \frac{|x+2|}{3} \text{ as } n\to\infty$$

$\Rightarrow$ convergent if $|x+2|<3$ and divergent if $|x+2|>3$

$\Rightarrow R=3$ $\qquad \Updownarrow$

$\qquad -5<x<1$

Endpoints

$x = -5$: $\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$ diverge

$x = 1$: $\sum_{n=0}^{\infty} \frac{n}{3} = \frac{1}{3} \sum_{n=0}^{\infty} n$ diverge

Hence, interval of convergence is $(-5, 1)$

11.9: Representations of functions as power series

Why you should love power series:

1) They are simple and have nice properties e.g. wrt differentiating and integrating and also work nicely for computer calculations
2) We can approximate other functions by power series so we can take advantage of these nice properties also when dealing with more complicated functions

Basic example:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

We can regard this as expressing the function $\frac{1}{1-x}$ as a power series

Can use this to get more examples

<u>Ex 2</u>: Find a power series representation for $\frac{1}{x+2}$

$$\frac{1}{2+x} = \frac{1}{2(1-(-\frac{x}{2}))} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \quad \text{when } \frac{|x|}{2} < 1 \Leftrightarrow |x| < 2$$

interval of convergence $(-2, 2)$

<u>Ex 3</u>: Find a power series representation of $\frac{x^3}{x+2}$

$$\frac{x^3}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$$

Class exercise:

<u>Ex 1</u>: Express $\frac{1}{1+x^2}$ as the sum of a power series and find interval of convergence

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

convergence $|-x^2| < 1 \Leftrightarrow x^2 < 1 \Leftrightarrow |x| < 1$

interval of convergence is $(-1, 1)$

Differentiation and integration of power series

<u>Thm</u> If the power series $\sum c_n (x-a)^n$ has radius of convergence R, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable (and therefore continues) on $(a-R, a+R)$ and

(i) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

(ii) $\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$

The radii of convergence of the power series in (i) and (ii) are both R

Rem: This gives us ways of integrating a function even if it doesn't have a nice antiderivative

In fact we can approximate any function as we will see next lecture