Lecture 23

10.2 Calculus with parametric curve

Recall: A parametric equation is a pair of functions

$$x = f(t), \quad y = g(t)$$

Tangents

Suppose f and g are differentiable functions. Assume moreover that y is a differentiable function of x. Then by the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Hence,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

Rem 1) $\frac{dy}{dt}$ (resp. $\frac{dx}{dt}$) is the velocity in the vertical (resp. horizontal) direction.
The slope of the tangent is the ratio of these

2) Have horizotal tangents when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$
Have vertical tangents when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

3) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad \left(\neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} \right)$$

Ex 1: C is defined by $x = t^2$, $y = t^3 - 3t$
a) Show C has 2 tangents at $(3,0)$ and find their equations
b) Find horizontal and vertical tangents
c) Determine when C is concave upward or downward
d) Sketch C

a) $y = t(t^2-3)$ so $y = 0 \Leftrightarrow t = 0, \pm\sqrt{3}$
$(3,0)$ on C correspond to $t = \sqrt{3}$ and $t = -\sqrt{3}$ (i.e. C intersects itself)

$$\frac{dy}{dx} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

For $t = \pm\sqrt{3}$ we get $\frac{dy}{dx} = \pm \frac{6}{2\sqrt{3}} = \pm\sqrt{3}$
Hence, the tangents at $(3,0)$ are

$$y = \sqrt{3}(x-3) \text{ and } y = -\sqrt{3}(x-3)$$

b) Horizontal tangents:

$$\frac{dy}{dt} = 0 \Leftrightarrow t^2 = 1 \Leftrightarrow t = \pm 1 \qquad \left(\frac{dx}{dt}(\pm 1) = \pm 2\right)$$

The corresponding points on C are $(1,-2)$ and $(1,2)$
Vertical tangents:

$$\frac{dx}{dt} = 0 \Leftrightarrow 2t = 0 \Leftrightarrow t = 0 \qquad \left(\frac{dy}{dt}(0) = -3\right)$$

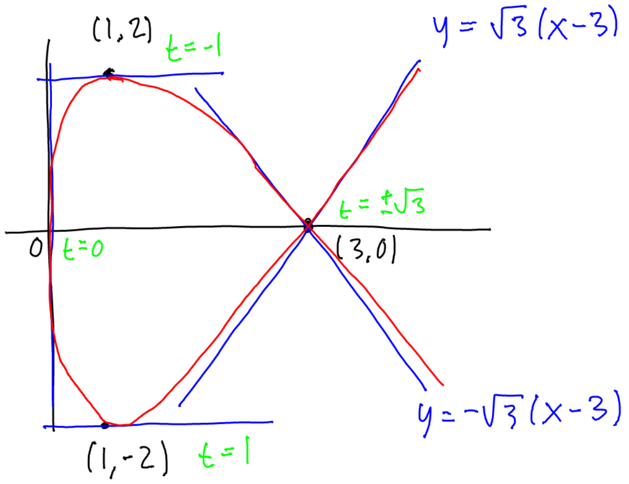
The corresponding point on C is $(0,0)$

c) $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t} = \dfrac{3(t^2+1)}{4t^3}$

concave upwards $\Leftrightarrow \dfrac{d^2y}{dx^2} > 0 \Leftrightarrow t > 0$

concave downward $\Leftrightarrow t < 0$

d)



Ex 2 a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$ at $\theta = \frac{\pi}{3}$

b) Find horizontal and vertical tangents

a) $\dfrac{dy}{dx} = \dfrac{r\sin\theta}{r(1-\cos\theta)} = \dfrac{\sin\theta}{1-\cos\theta}$

When $\theta = \frac{\pi}{3}$

$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$, $y = r\left(1-\frac{1}{2}\right) = \frac{r}{2}$

$\dfrac{dy}{dx} = \dfrac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \sqrt{3}$ slope

Equation for tangent at $\theta = \frac{\pi}{3}$

$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right)$

b) Horizontal tangent

$\dfrac{dy}{dt} = 0 \Leftrightarrow \theta = n\pi$, n integer

$\dfrac{dx}{dt} \neq 0$ also $\Rightarrow \theta = (2n-1)\pi$

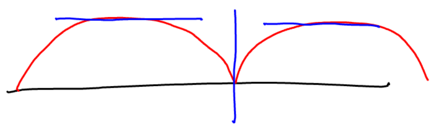
Corresponding points:

$((2n-1)\pi r, 2r)$

$\dfrac{dx}{d\theta} = 0 \Leftrightarrow \theta = 2\pi n$

l'Hospital

$\lim\limits_{\theta \to 2n\pi^+} \dfrac{dy}{dx} = \lim\limits_{\theta \to 2n\pi^+} \dfrac{\cos\theta}{\sin\theta} = \infty$



This is a vertical tangent

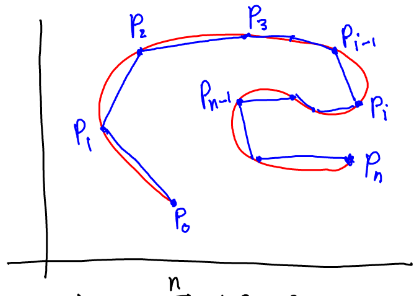
Arc Length

Want: the length of a curve given by parametric equations on interval $[\alpha, \beta]$

Idea: Approximate by line segments

Divide $[\alpha,\beta]$ into n subintervals of equal width Δt

- denote the endpoints of these subintervals by $t_0, t_1, \ldots, t_n$



$P_i := (f(t_i), g(t_i))$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2}$$

$$\boxed{\begin{array}{l}\text{Mean value thm} \Rightarrow \text{there exist } t_{i-1} \le t_i^*, t_i^{**} \le t_i \text{ s.t.} \\ f(t_i) - f(t_{i-1}) = f'(t_i^*)\Delta t \quad , \quad g(t_i) - g(t_{i-1}) = g'(t_i^{**})\Delta t\end{array}}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{f'(t_i^*)^2 + g'(t_i^{**})^2}\ \Delta t$$

(can be shown that the limit is the same as if $t_i^* = t_i^{**}$)

$$= \int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

Thm If the curve C is transversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex 4 (unit circle) $x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^{2\pi} 1\, dt = 2\pi \qquad \text{(as it should be)}$$

Warning: we could also have used the representation

$$x = \sin 2t \quad , \quad y = \cos 2t \qquad 0 \le t \le 2\pi$$

Then we would get

$$\frac{dx}{dt} = 2\cos 2t \quad , \quad \frac{dy}{dt} = -2\sin 2t$$

so $\displaystyle\int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\, dt = \int_0^{2\pi} 2\, dt = 4\pi$

This is because we go around the circle twice as t runs from 0 to 2π

Ex 5: Find the length of one arch of the cycloid $x = r(\theta - \sin\theta),\ y = r(1-\cos\theta),\ 0 \le t \le 2\pi$

The arch is transversed exactly once on $0 \le \theta \le 2\pi$

$$\frac{dx}{d\theta} = r(1-\cos\theta) \quad , \quad \frac{dy}{d\theta} = r\sin\theta$$

$$L = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{1 + \cos^2\theta - 2\cos\theta + \sin^2\theta}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta \qquad 1 - \cos\theta = 2\sin\left(\frac{\theta}{2}\right) \ge 0 \text{ for } 0 \le \theta \le 2\pi$$

$$= r\int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right)\, d\theta$$

$= 2r\left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}$

$= 8r$