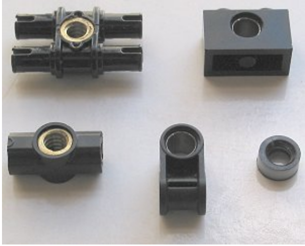
Structure and dynamics of complex networks

February 25, 2020



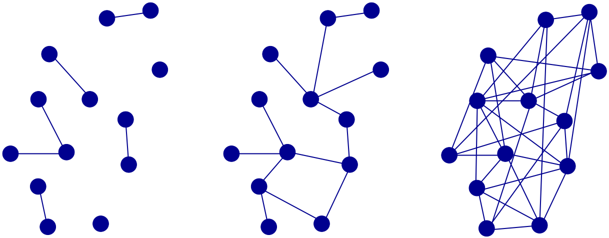
COMPONENTS

Components

Undirected case

Component

A component in an undirected network corresponds to a maximal set of nodes in which a path exists between any pair of nodes.



$n_c = 7$ $n_c = 1$ $n_c = 1$

Giant component

- Most networks we encounter contain a **giant** component.

→ What is a "giant component"?

Giant component

Giant component

A network (graph) with system size $N \to \infty$ contains a giant component if the relative size of this component remains finite, (larger than zero):

$$\lim_{N\to\infty} \frac{S_1}{N} > 0$$

How to generalize the concept of components for the directed case?

Components

Directed networks

Strongly connected component

A strongly connected component is a maximal set of nodes in which a directed path exists between any pair of nodes.

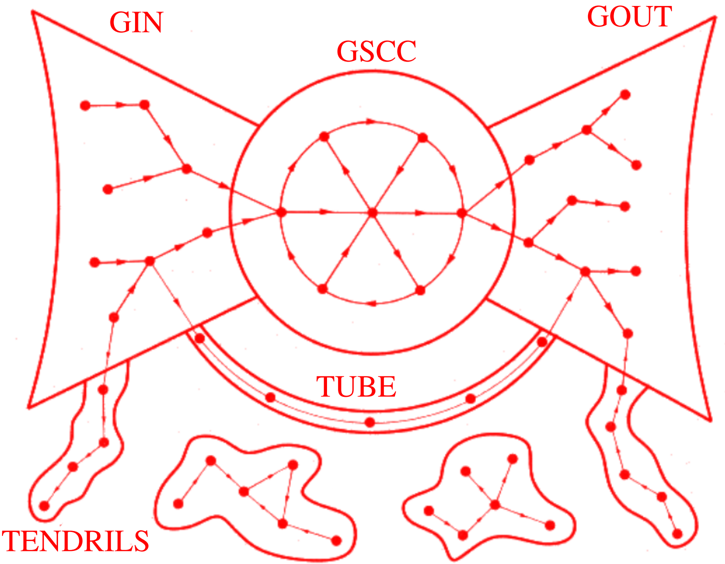
Weakly connected component

A weakly connected component is a maximal set of nodes in which an undirected path exists between any pair of nodes.

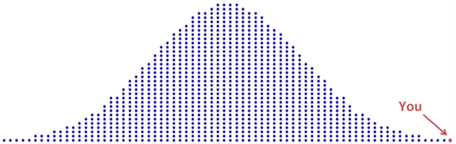
Components of the Internet

How does the structure of a large directed network like the Internet looks like on the large scale from the point of view of components?

Components of the Internet



Advanced network characteristics



DEGREE DISTRIBUTION

Degree distribution

Degree distribution

Degree distribution $\longrightarrow$ **probability distribution of the node degrees.**

Degree distribution

Degree distribution $\longrightarrow$ **probability distribution of the node degrees.**

Degree distribution

- The degree distribution of a network, $p(k)$ is equal to the probability that a randomly chosen node has a degree k.

Degree distribution
Finite networks

Degree distribution

Finite networks

Degree distribution

- For a finite network with N nodes,

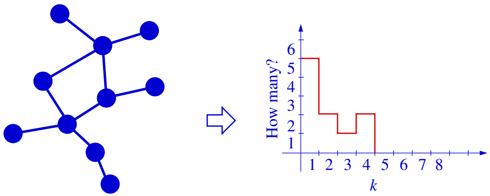
$$p(k) = \frac{N_k}{N},$$

where N_k denotes the number of nodes with degree k.

- Thus, $p(k)$ is simply the fraction of nodes with degree k.

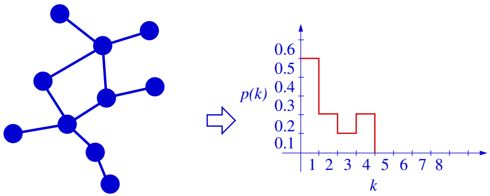
Calculating $p(k)$

- The calculation of the degree distribution is very similar to the construction of a histogram.
- We count for each degree value k how many nodes have that degree,
- and we divide it by the total number of nodes.



Calculating $p(k)$

- The calculation of the degree distribution is very similar to the construction of a histogram.
- We count for each degree value k how many nodes have that degree,
- and we divide it by the total number of nodes.



$p(k)$ in the Erdős-Rényi model

- What is the degree distribution of an Erdős-Rényi graph where N nodes are linked independently with probability p?

$p(k)$ in the Erdős-Rényi model

- What is the degree distribution of an Erdős-Rényi graph where N nodes are linked independently with probability p?
  - The number of edges between i and j:

$$\begin{aligned}\mathcal{P}(e_{ij} = 1) &= p \\ \mathcal{P}(e_{ij} = 0) &= 1 - p\end{aligned}$$

(This is the Bernoulli distribution.)

$p(k)$ in the Erdős-Rényi model

- What is the degree distribution of an Erdős-Rényi graph where N nodes are linked independently with probability p?
  - The number of edges between i and j:

$$\begin{aligned}\mathcal{P}(e_{ij} = 1) &= p \\ \mathcal{P}(e_{ij} = 0) &= 1 - p\end{aligned}$$

  (This is the Bernoulli distribution.)

  - A given node can choose from $N - 1$ possible neighbors, which are attached independently,

$\rightarrow$ $p(k)$ follows a binomial distribution:

$$p(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

$p(k)$ in the Erdős-Rényi model

- We usually neglect the -1:

$$p(k) = \binom{N}{k} p^k (1-p)^{N-k},$$

- and approximate in the large N limit the binomial distribution by the Poisson distribution:

$$\begin{aligned} p(k) &= \binom{N}{k} p^k (1-p)^{N-k} \\ &\downarrow \\ p(k) &\simeq \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}, \qquad \langle k \rangle = Np \end{aligned}$$

$p(k)$ in the Erdős-Rényi model

- We usually neglect the -1:

$$p(k) = \binom{N}{k} p^k (1-p)^{N-k},$$

- and approximate in the large N limit the binomial distribution by the Poisson distribution:

$$\begin{aligned} p(k) &= \binom{N}{k} p^k (1-p)^{N-k} \\ &\downarrow \\ p(k) &\simeq \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}, \qquad \langle k \rangle = Np \end{aligned}$$

$p(k)$ in the Erdős-Rényi model

From binomial to Poisson distribution

$$p(k) = \frac{N(N-1)\cdots(N-k+1)}{k!}\frac{\langle k\rangle^k}{N^k}\left(1-\frac{\langle k\rangle}{N}\right)^N\left(1-\frac{\langle k\rangle}{N}\right)^{-k}$$
$$= \frac{\langle k\rangle^k}{k!}\underbrace{\left(1-\frac{\langle k\rangle}{N}\right)^N}_{\simeq e^{-\langle k\rangle}}\underbrace{\frac{N(N-1)\cdots(N-k+1)}{N^k}}_{\simeq 1}\underbrace{\left(1-\frac{\langle k\rangle}{N}\right)^{-k}}_{\simeq 1}$$

The last two factors converge to 1:

$$\lim_{N\to\infty}\frac{N(N-1)\cdots(N-k+1)}{N^k} = 1,$$
$$\lim_{N\to\infty}\left(1-\frac{\langle k\rangle}{N}\right)^{-k} = 1,$$

whereas

$$\lim_{N\to\infty}\left(1-\frac{\langle k\rangle}{N}\right)^{N} = e^{-\langle k\rangle}$$

Thus,

$$p(k) \simeq \frac{\langle k\rangle^k}{k!}e^{-\langle k\rangle}.$$

$p(k)$ in the Erdős-Rényi model

From binomial to Poisson distribution

$$\begin{aligned} p(k) &= \frac{N(N-1)\cdots(N-k+1)}{k!}\frac{\langle k\rangle^k}{N^k}\left(1-\frac{\langle k\rangle}{N}\right)^N\left(1-\frac{\langle k\rangle}{N}\right)^{-k} \\ &= \frac{\langle k\rangle^k}{k!}\underbrace{\left(1-\frac{\langle k\rangle}{N}\right)^N}_{\simeq e^{-\langle k\rangle}}\underbrace{\frac{N(N-1)\cdots(N-k+1)}{N^k}}_{\simeq 1}\underbrace{\left(1-\frac{\langle k\rangle}{N}\right)^{-k}}_{\simeq 1} \end{aligned}$$

The last two factors converge to 1:

$$\lim_{N\to\infty}\frac{N(N-1)\cdots(N-k+1)}{N^k} = 1,$$

$$\lim_{N\to\infty}\left(1-\frac{\langle k\rangle}{N}\right)^{-k} = 1,$$

whereas

$$\lim_{N\to\infty}\left(1-\frac{\langle k\rangle}{N}\right)^{N} = e^{-\langle k\rangle}$$

Thus,

$$p(k) \simeq \frac{\langle k\rangle^k}{k!}e^{-\langle k\rangle}.$$

$p(k)$ in the Erdős-Rényi model

From binomial to Poisson distribution

$$p(k) = \frac{N(N-1)\cdots(N-k+1)}{k!}\frac{\langle k\rangle^k}{N^k}\left(1-\frac{\langle k\rangle}{N}\right)^N\left(1-\frac{\langle k\rangle}{N}\right)^{-k}$$

$$= \frac{\langle k\rangle^k}{k!}\underbrace{\left(1-\frac{\langle k\rangle}{N}\right)^N}_{\simeq e^{-\langle k\rangle}}\underbrace{\frac{N(N-1)\cdots(N-k+1)}{N^k}}_{\simeq 1}\underbrace{\left(1-\frac{\langle k\rangle}{N}\right)^{-k}}_{\simeq 1}$$

The last two factors converge to 1:

$$\lim_{N\to\infty}\frac{N(N-1)\cdots(N-k+1)}{N^k} = 1,$$

$$\lim_{N\to\infty}\left(1-\frac{\langle k\rangle}{N}\right)^{-k} = 1,$$

whereas

$$\lim_{N\to\infty}\left(1-\frac{\langle k\rangle}{N}\right)^N = e^{-\langle k\rangle}$$

Thus,

$$p(k) \simeq \frac{\langle k\rangle^k}{k!}e^{-\langle k\rangle}.$$

$p(k)$ in the Erdős-Rényi model

From binomial to Poisson distribution

$$\begin{aligned}
p(k) &= \frac{N(N-1)\cdots(N-k+1)}{k!}\frac{\langle k\rangle^k}{N^k}\left(1-\frac{\langle k\rangle}{N}\right)^N\left(1-\frac{\langle k\rangle}{N}\right)^{-k} \\
&= \frac{\langle k\rangle^k}{k!}\underbrace{\left(1-\frac{\langle k\rangle}{N}\right)^N}_{\simeq e^{-\langle k\rangle}}\underbrace{\frac{N(N-1)\cdots(N-k+1)}{N^k}}_{\simeq 1}\underbrace{\left(1-\frac{\langle k\rangle}{N}\right)^{-k}}_{\simeq 1}
\end{aligned}$$

The last two factors converge to 1:

$$\lim_{N\to\infty}\frac{N(N-1)\cdots(N-k+1)}{N^k} = 1,$$

$$\lim_{N\to\infty}\left(1-\frac{\langle k\rangle}{N}\right)^{-k} = 1,$$

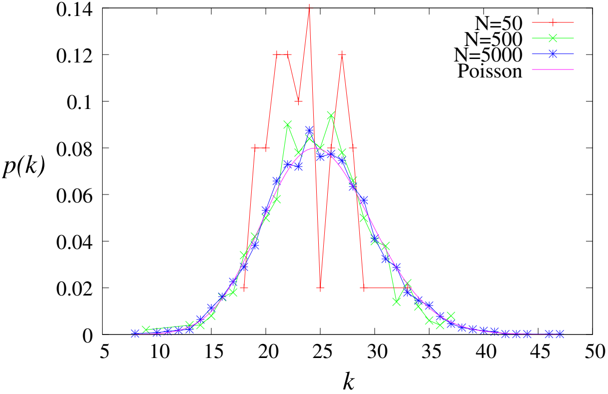
whereas

$$\lim_{N\to\infty}\left(1-\frac{\langle k\rangle}{N}\right)^N = e^{-\langle k\rangle}$$

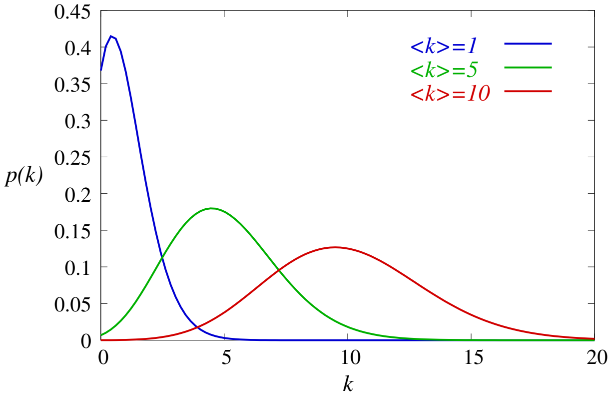
Thus,

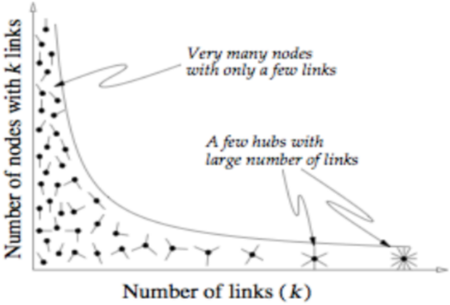
$$p(k) \simeq \frac{\langle k\rangle^k}{k!}e^{-\langle k\rangle}.$$

$p(k)$ in the Erdős-Rényi model



$p(k)$ in the Erdős-Rényi model





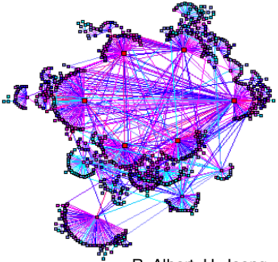
SCALE-FREE NETWORKS

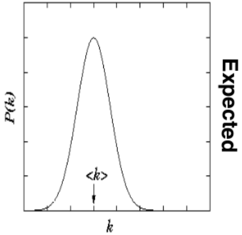
Degree distribution of real networks

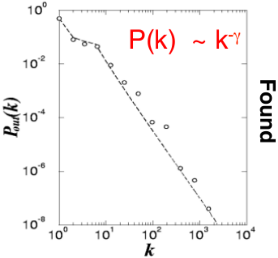
Nodes: **WWW documents**
Links: **URL links**

Over 3 billion documents

ROBOT: collects all URL's found in a document and follows them recursively







R. Albert, H. Jeong, A-L Barabasi, ***Nature***, 401 130 (1999).

(from the slides of A.-L. Barabási)

Plotting power-law $p(k)$

- Usually we plot a power-laws on log-log scale. Why?

Plotting power-law $p(k)$

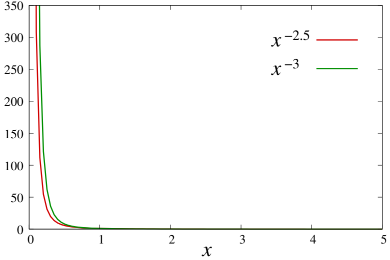
- Usually we plot a power-laws on log-log scale. Why?
  - Power-law: $p(k) \simeq ck^{-\gamma}$,

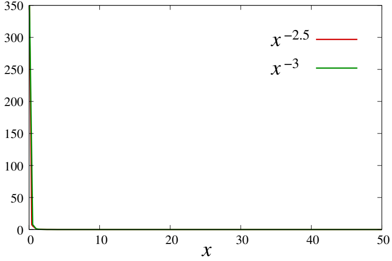
  $\rightarrow$ $\ln p(k) \simeq \ln c - \gamma \cdot \ln k$.
  - In a log-log plot

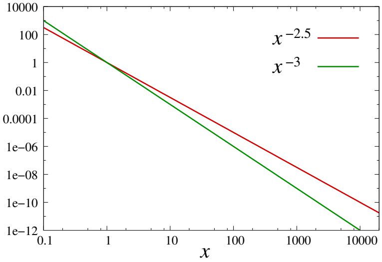
$$
\begin{aligned}
x &\rightarrow \ln k, \\
y = f(x) &\rightarrow \ln p(k), \\
\rightarrow f(x) &= \ln c - \gamma \cdot x
\end{aligned}
$$

  - Thus, on log-log scale a power-law looks like a linear function, with a slope equal to γ.

Plotting power-laws
Illustration

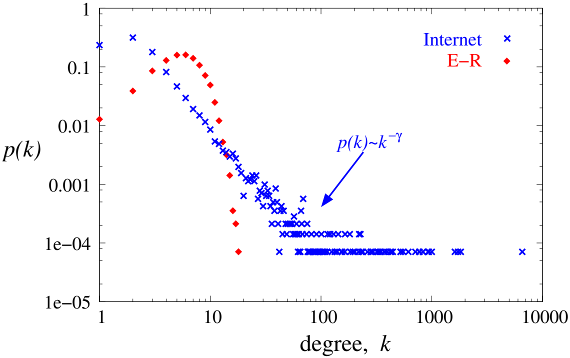






Power-law vs Poisson distribution

The $p(k)$ of the Internet at the level of AS, compared to the $p(k)$ of an Erdős–Rényi graph with the same number of nodes and links:



Scale-free networks

Scale-free networks

- A network is called **scale-free** if the tail of its degree distribution **decays as a power-law**,
$$p(k) \sim k^{-\gamma}.$$
- The exponent γ is often referred to as the node degree exponent or node degree decay exponent.

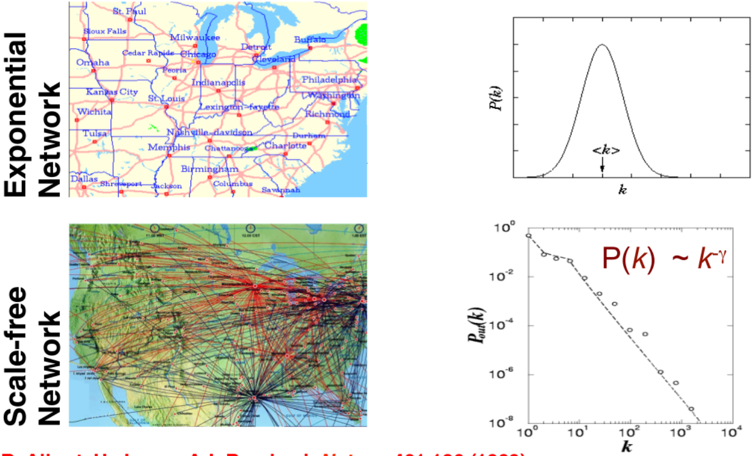
Degree distribution of real networks

What do you expect?

R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

(from the slides of A.-L. Barabási)

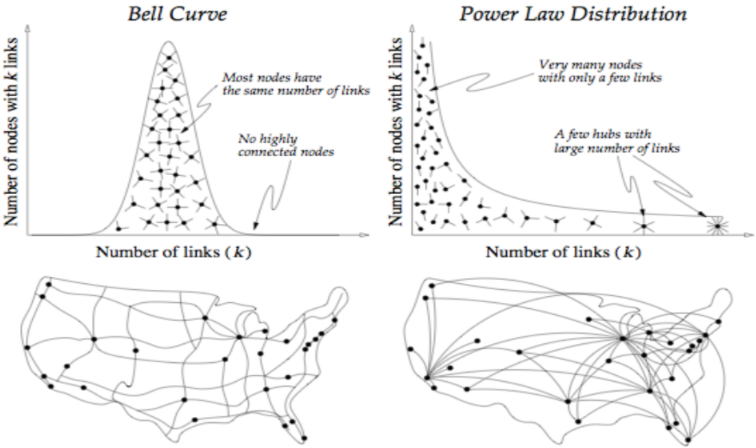
Degree distribution of real networks



R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

(from the slides of A.-L. Barabási)

Fundamental difference between scale-free and Poisson-distribution!

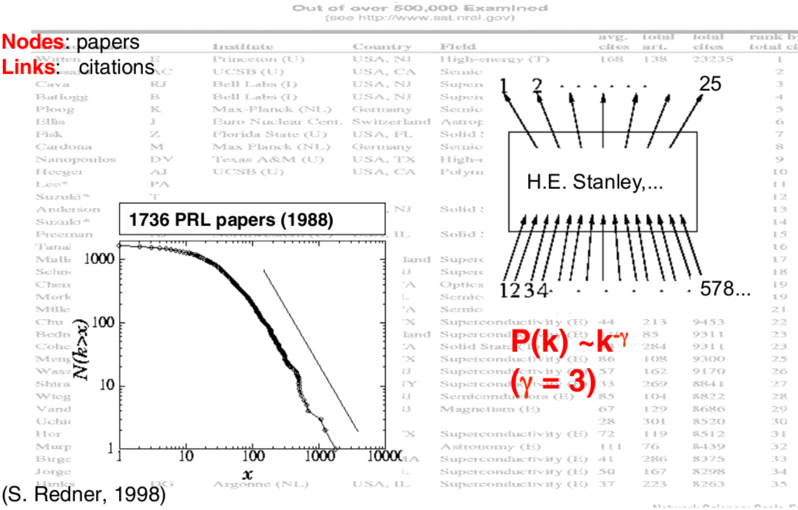


(from the slides of A.-L. Barabási)

Fundamental difference between scale-free and Poisson-distribution!

- The two distributions are strikingly different: **HUBS!**
  - $\langle k \rangle << k$.
  - Thus, no “typical” degree.
  - The degree distribution is very **inhomogeneous** and skewed.

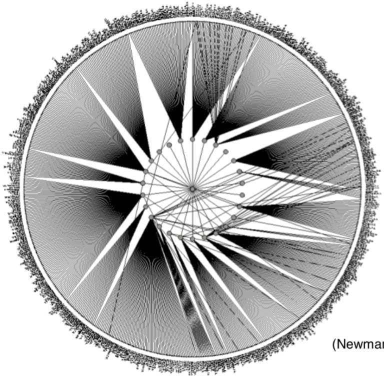
Scale-free $p(k)$ everywhere

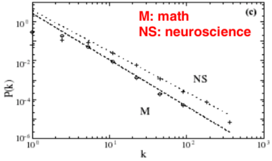


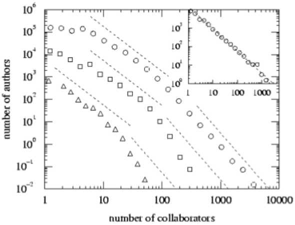
(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

Nodes: scientist (authors)
Links: joint publication







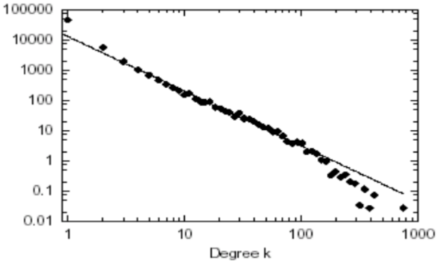
(Newman, 2000, Barabasi et al 2001)

(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

Nodes: online user
Links: email contact

Kiel University log files
112 days, N=59,912 nodes



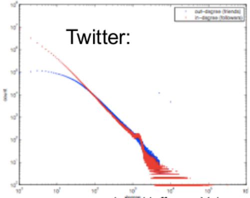
(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

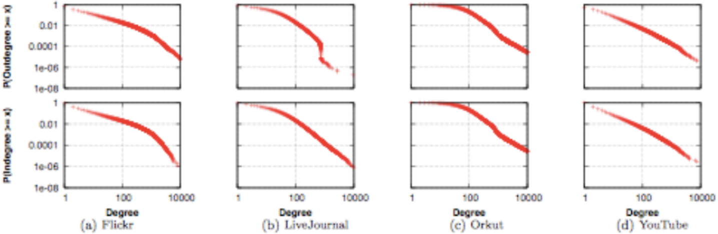
Nodes: online user
Links: email contact

All distribtions show a fat-tail behavior:
there are orders of magnitude spread in the degrees

Alan Mislove, Measurement and Analysis of Online Social Networks

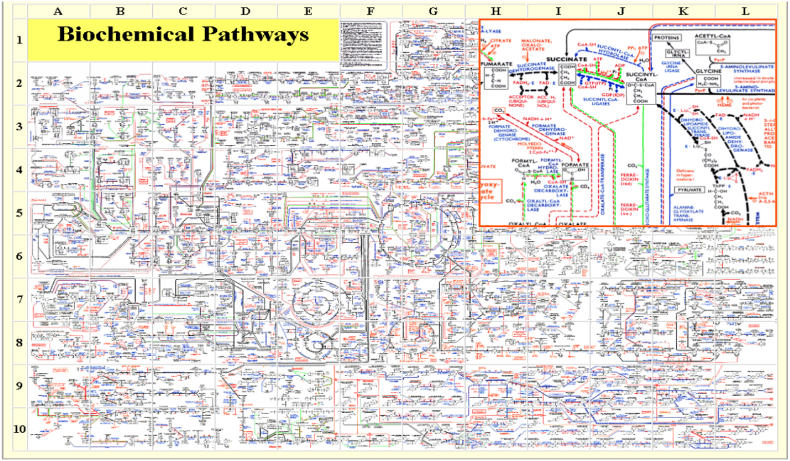


Jake Hoffman, Yahoo,



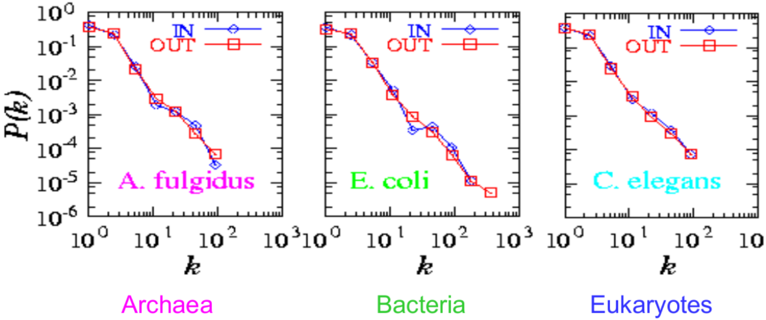
(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere



(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere



Organisms from all three domains of life are scale-free!

$$P_{in}(k) \approx k^{-2.2}$$

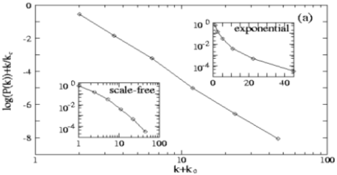
$$P_{out}(k) \approx k^{-2.2}$$

H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.L. Barabasi, *Nature*, 407 651 (2000)

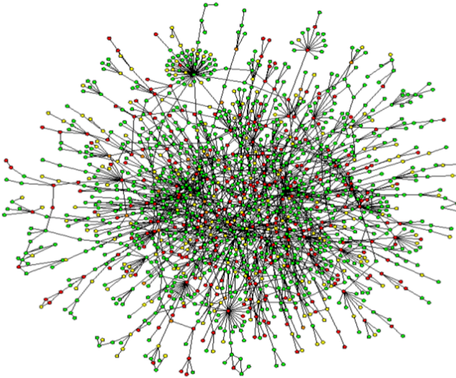
(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

Nodes: proteins
Links: physical interactions-binding



$$P(k) \sim (k + k_0)^{-\gamma} \exp\left(-\frac{k + k_0}{k_\tau}\right)$$

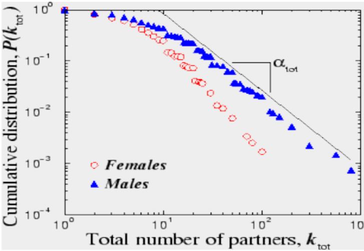


(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

(from the slides of A.-L. Barabási)

Nodes: people (Females; Males)
Links: sexual relationships



4781 Swedes; 18-74;
59% response rate.

Liljeros et al. Nature 2001

Why “scale-free”?

Why are networks with a power-law degree distribution called “**scale-free**”?

Scaling

- What is **scaling**?

Scaling

- What is **scaling**?

  Suppose we are interested in two quantities, A and B. The quantity B is scaling

  - linearly with A if $B \propto A$,
  - quadratically with A if $B \propto A^2$,
  - etc.

Scaling

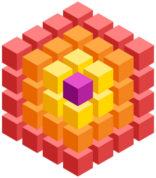
- What is **scaling**?

  Suppose we are interested in two quantities, A and B.
  The quantity B is scaling

  - linearly with A if $B \propto A$,
  - quadratically with A if $B \propto A^2$,
  - etc.

- A trivial example:
  How does the volume of a square scale with the length of its edges?

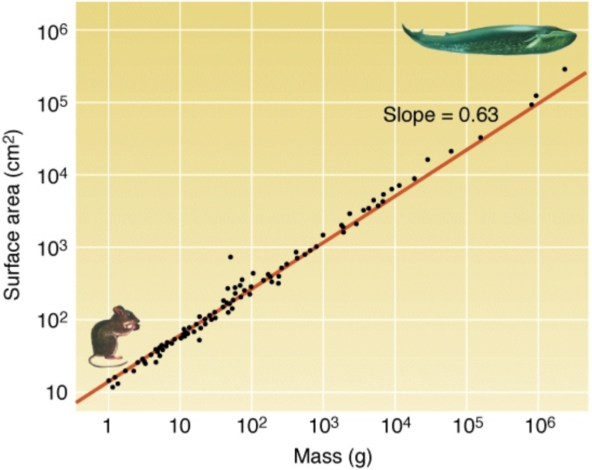
$$V = l^3$$



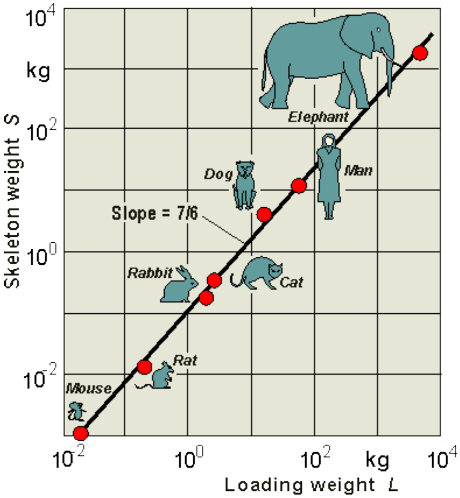
Scaling in nature

A prominent example of scaling laws in nature is provided by allometric scaling.

Allometric scaling
Body surface



Allometric scaling
Body parts



Allometric scaling

Body parts

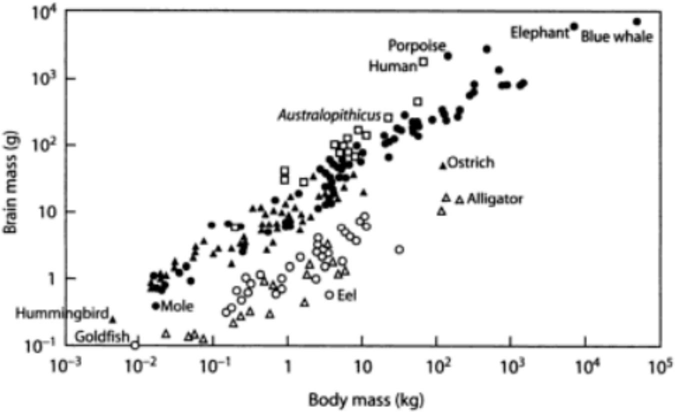
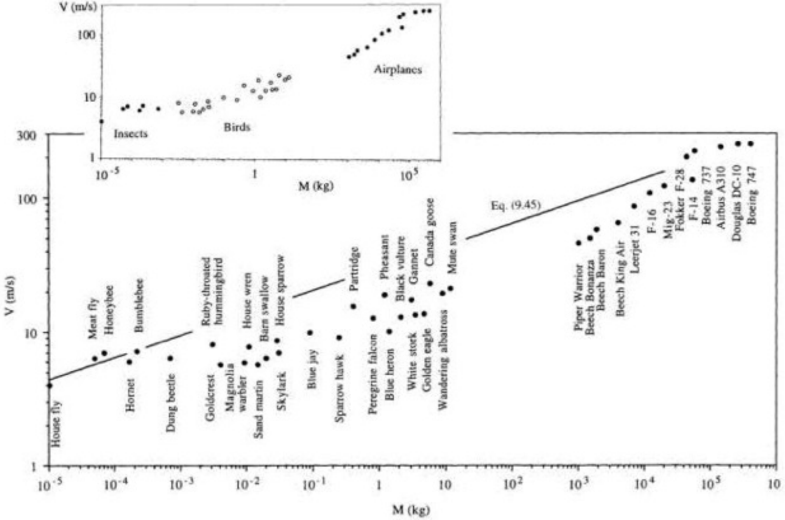


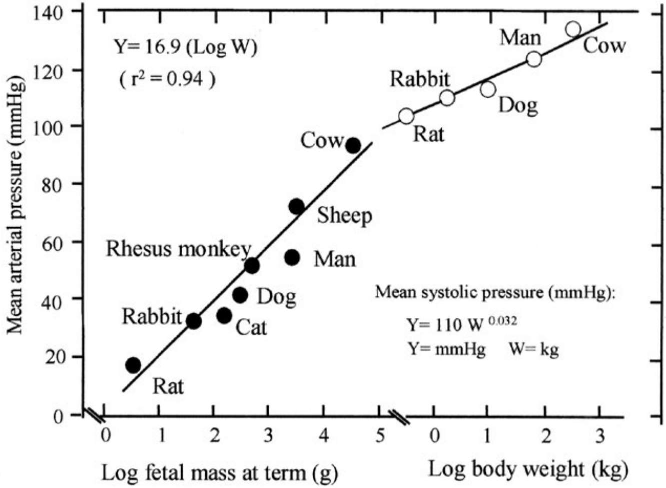
Figure 15. Brain size of 200 species of vertebrates plotted against body size on a log-log graph. Primates are open squares; other mammals are solid dots; birds are solid triangles; bony fishes are open circles; and reptiles are open triangles. (After H. J. Jerison, *The Evolution of the Brain and Intelligence*, 1973)

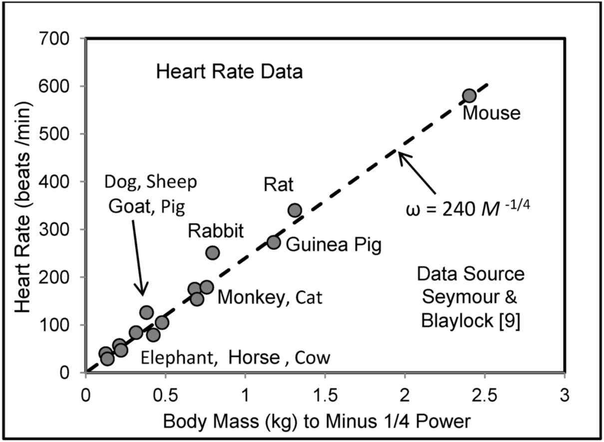
Allometric scaling

Velocity



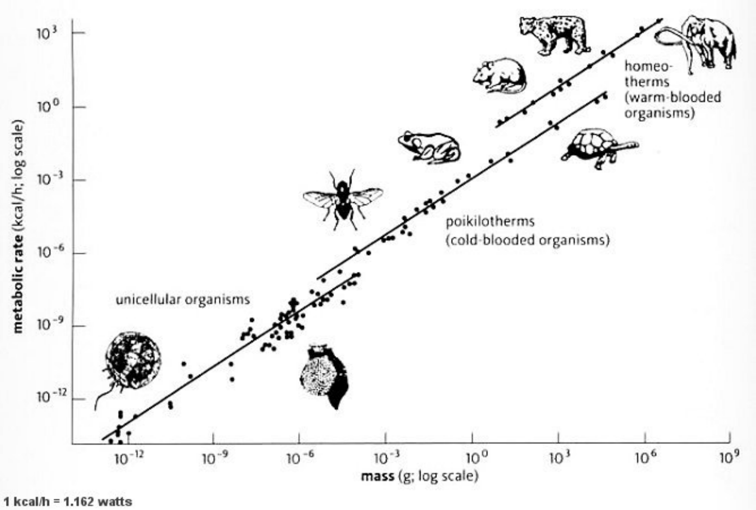
Allometric scaling
Vascular system





Heart Rate Data
Heart Rate (beats /min)
700
600
500
400
300
200
100
0
0
0.5
1
1.5
2
2.5
3
Body Mass (kg) to Minus 1/4 Power
Mouse
Rat
Guinea Pig
Rabbit
Dog, Sheep
Goat, Pig
Monkey, Cat
Elephant, Horse , Cow
ω = 240 M -1/4
Data Source
Seymour &
Blaylock [9]

Allometric scaling
Metabolic rate



Scaling function

Scaling function

Definition: a function $F(x)$ is **scaling** if

$$F(a \cdot x) = g(a) \cdot F(x),$$

thus, changing the argument is equivalent to multiplying $F(x)$ with a constant.

Power-laws are scaling

Assume $F(x) = b \cdot x^{\gamma}$.

$$\rightarrow F(a \cdot x) = b \cdot (a \cdot x)^{\gamma} = b \cdot a^{\gamma} \cdot x^{\gamma} = a^{\gamma} \cdot b \cdot x^{\gamma} = a^{\gamma} \cdot F(x).$$

(In addition, it can be proven that all scaling functions are power-laws).

Scaling function

Scaling function

Definition: a function $F(x)$ is **scaling** if

$$F(a \cdot x) = g(a) \cdot F(x),$$

thus, changing the argument is equivalent to multiplying $F(x)$ with a constant.

Power-laws are scaling

Assume $F(x) = b \cdot x^{\gamma}$.

$$\rightarrow F(a \cdot x) = b \cdot (a \cdot x)^{\gamma} = b \cdot a^{\gamma} \cdot x^{\gamma} = a^{\gamma} \cdot b \cdot x^{\gamma} = a^{\gamma} \cdot F(x).$$

(In addition, it can be proven that all scaling functions are power-laws).

Scaling function

Scaling function

Definition: a function $F(x)$ is **scaling** if

$$F(a \cdot x) = g(a) \cdot F(x),$$

thus, changing the argument is equivalent to multiplying $F(x)$ with a constant.

Power-laws are scaling

Assume $F(x) = b \cdot x^{\gamma}$.

$$\to F(a \cdot x) = b \cdot (a \cdot x)^{\gamma} = b \cdot a^{\gamma} \cdot x^{\gamma} = a^{\gamma} \cdot b \cdot x^{\gamma} = a^{\gamma} \cdot F(x).$$

(In addition, it can be proven that all scaling functions are power-laws).

Scaling function

Scaling function

Definition: a function $F(x)$ is **scaling** if

$$F(a \cdot x) = g(a) \cdot F(x),$$

thus, changing the argument is equivalent to multiplying $F(x)$ with a constant.

Power-laws are scaling

Assume $F(x) = b \cdot x^{\gamma}$.

$$\rightarrow F(a \cdot x) = b \cdot (a \cdot x)^{\gamma} = b \cdot a^{\gamma} \cdot x^{\gamma} = a^{\gamma} \cdot b \cdot x^{\gamma} = a^{\gamma} \cdot F(x).$$

(In addition, it can be proven that all scaling functions are power-laws).

Scaling function

Scaling function

Definition: a function $F(x)$ is **scaling** if

$$F(a \cdot x) = g(a) \cdot F(x),$$

thus, changing the argument is equivalent to multiplying $F(x)$ with a constant.

Power-laws are scaling

Assume $F(x) = b \cdot x^{\gamma}$.

$$\rightarrow F(a \cdot x) = b \cdot (a \cdot x)^{\gamma} = b \cdot a^{\gamma} \cdot x^{\gamma} = a^{\gamma} \cdot b \cdot x^{\gamma} = a^{\gamma} \cdot F(x).$$

(In addition, it can be proven that all scaling functions are power-laws).

Scaling distributions

What about the scaling of probability distributions?

Scaling distribution

A probability distribution is **scaling** if the $\rho(x)$ density function behaves as a power-law

$$\rho(x) \sim x^{-\alpha},$$

(at least on a reasonably wide interval).

Scaling distributions

What about the scaling of probability distributions?

Scaling distribution

A probability distribution is **scaling** if the $\rho(x)$ density function behaves as a power-law

$$\rho(x) \sim x^{-\alpha},$$

(at least on a reasonably wide interval).

Vilfredo Pareto:

- The 80-20 rule: Approximately 80% of the land (money, wealth, etc.) is owned by less than the 20% of the population.

Vilfredo Pareto:

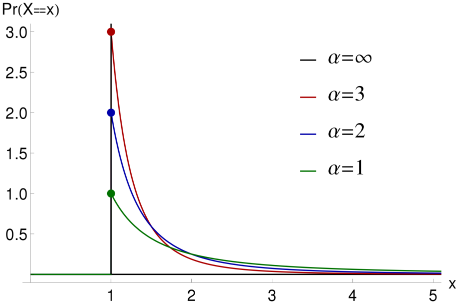
- The 80-20 rule: Approximately 80% of the land (money, wealth, etc.) is owned by less than the 20% of the population.
- The distribution of wealth:

$$\rho(x) = \begin{cases} \frac{\alpha x_{\min}}{x^{\alpha+1}} & x > x_{\min} \\ 0 & x < x_{\min} \end{cases}$$

Vilfredo Pareto:

- The 80-20 rule: Approximately 80% of the land (money, wealth, etc.) is owned by less than the 20% of the population.
- The distribution of wealth:

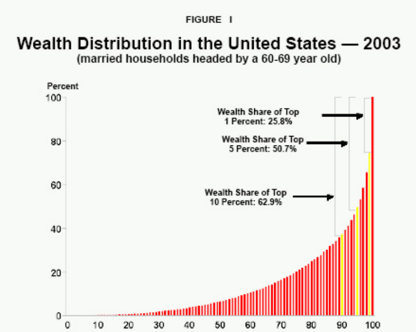
$$\rho(x) = \begin{cases} \frac{\alpha x_{\min}}{x^{\alpha+1}} & x > x_{\min} \\ 0 & x < x_{\min} \end{cases}$$



Vilfredo Pareto:

- The 80-20 rule: Approximately 80% of the land (money, wealth, etc.) is owned by less than the 20% of the population.
- The distribution of wealth:

$$\rho(x) = \begin{cases} \frac{\alpha x_{\min}}{x^{\alpha+1}} & x > x_{\min} \\ 0 & x < x_{\min} \end{cases}$$

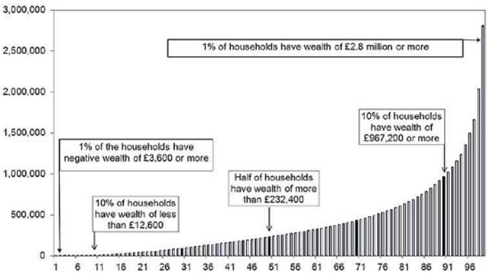


Vilfredo Pareto:

- The 80-20 rule: Approximately 80% of the land (money, wealth, etc.) is owned by less than the 20% of the population.
- The distribution of wealth:

$$\rho(x) = \begin{cases} \frac{\alpha x_{\min}}{x^{\alpha+1}} & x > x_{\min} \\ 0 & x < x_{\min} \end{cases}$$

Figure 1: Distribution of total wealth between households, 2008-10, GB



- Jean-Baptiste Estoup (1868–1950),
- George Kingsley Zipf (1902–1950):

G. K. Zipf

- Jean-Baptiste Estoup (1868–1950),
- George Kingsley Zipf (1902–1950):

> The frequency of words is inversely proportional to their rank in the frequency table.

G. K. Zipf

- Jean-Baptiste Estoup (1868–1950),
- George Kingsley Zipf (1902–1950):

The frequency of words is inversely proportional to their rank in the frequency table.

- the frequency of the 2^{nd} most frequent word is $1/2$ of the frequency of the most frequent word,

G. K. Zipf

Scaling distributions
Zipf's law

- Jean-Baptiste Estoup (1868–1950),
- George Kingsley Zipf (1902–1950):

The frequency of words is inversely proportional to their rank in the frequency table.

- the frequency of the 2$^{\text{nd}}$ most frequent word is $1/2$ of the frequency of the most frequent word,
- the frequency of the 3$^{\text{rd}}$ most frequent word is $1/3$ of the frequency of the most frequent word,
- etc.

G. K. Zipf

- Jean-Baptiste Estoup (1868–1950),
- George Kingsley Zipf (1902–1950):

The frequency of words is inversely proportional to their rank in the frequency table.

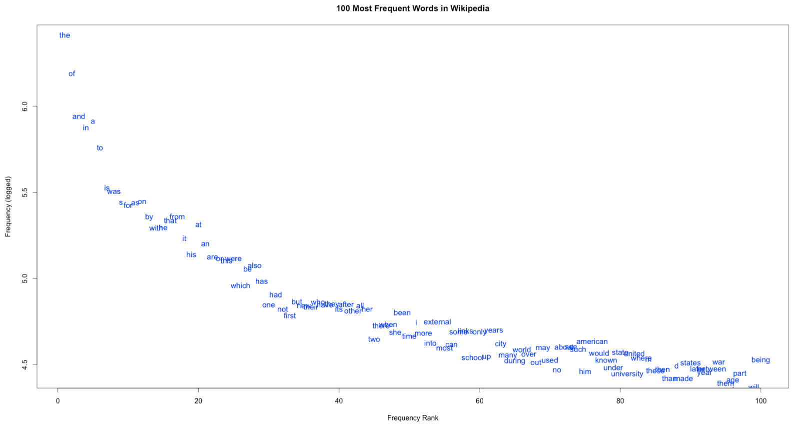
- the frequency of the 2$^{\text{nd}}$ most frequent word is $1/2$ of the frequency of the most frequent word,
- the frequency of the 3$^{\text{rd}}$ most frequent word is $1/3$ of the frequency of the most frequent word,
- etc.

G. K. Zipf

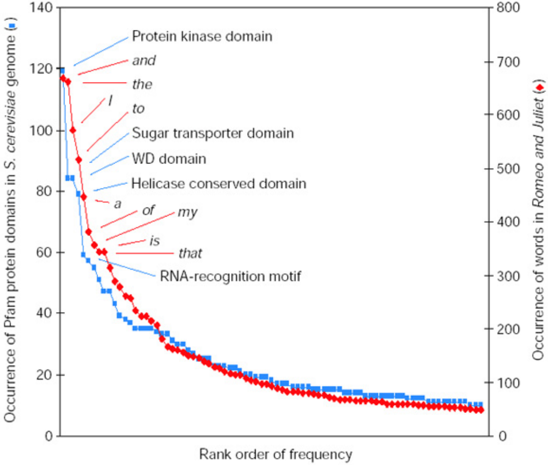
$$p(q) = \frac{C}{q^{\gamma}}, \qquad C = \sum_{k=1}^{N} \frac{1}{q^{\gamma}}, \qquad (\gamma = 1)$$

Scaling distributions

Zipf's law



Scaling distributions
Zipf's law



Why “scale-free”?

Why are networks with a power-law degree distribution called **“scale-free”**?

Why “scale-free”?

Why are networks with a power-law degree distribution called **“scale-free”**?

- power-law $p(k)$ → scaling distribution.
- no “typical degree” → no typical scale for the degrees.

Scale-free networks
Measured γ exponents

Scale-free degree distribution: $p(k) \sim k^{-\gamma}$.

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference
WWW	$325,729$	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, Barabási 1999
WWW	4×10^7	7		2.38	2.1				Kumar *et al.* 1999
WWW	2×10^8	7.5	$4,000$	2.72	2.1	16	8.85	7.61	Broder *et al.* 2000
WWW, site	$260,000$				1.94				Huberman, Adamic 2000
Internet, domain*	$3,015$ - $4,389$	3.42 - 3.76	$30-40$	2.1 - 2.2	2.1 - 2.2	4	6.3	5.2	Faloutsos 1999
Internet, router*	$3,888$	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos 1999
Internet, router*	$150,000$	2.66	60	2.4	2.4	11	12.8	7.47	Govindan 2000
Movie actors*	$212,250$	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási, Albert 1999
Coauthors, SPIRES*	$56,627$	173	$1,100$	1.2	1.2	4	2.12	1.95	Newman 2001b,c
Coauthors, neuro.*	$209,293$	11.54	400	2.1	2.1	6	5.01	3.86	Barabási *et al.* 2001
Coauthors, math*	$70,975$	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási *et al.* 2001
Sexual contacts*	2810			3.4	3.4				Liljeros *et al.* 2001
Metabolic, E. coli	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong *et al.* 2000
Protein, S. cerev.*	1870	2.39		2.4	2.4				Mason *et al.* 2000
Ythan estuary*	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya, Solé 2000
Silwood park*	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya, Solé 2000
Citation	$783,339$	8.57			3				Redner 1998
Phone-call	53×10^6	3.16		2.1	2.1				Aiello *et al.* 2000
Words, cooccurence*	$460,902$	70.13		2.7	2.7				Cancho, Solé 2001
Words, synonyms*	$22,311$	13.48		2.8	2.8				Yook *et al.* 2001

Scale-free $p(k)$ and Zeta function

How to turn $p(k) \sim k^{-\gamma}$ into a normalized distribution?

Scale-free $p(k)$ and Zeta function

How to turn $p(k) \sim k^{-\gamma}$ into a normalized distribution?

$$p(k) = Ck^{-\gamma} \quad \rightarrow \quad C = \frac{1}{\sum_k k^{-\gamma}}$$

Scale-free $p(k)$ and Zeta function

How to turn $p(k) \sim k^{-\gamma}$ into a normalized distribution?

$$p(k) = Ck^{-\gamma} \quad \rightarrow \quad C = \frac{1}{\sum_k k^{-\gamma}}$$

Assuming $k > 0$ (no isolated nodes), C^{-1} is actually the Riemann Zeta function!

Scale-free $p(k)$ and Zeta function

How to turn $p(k) \sim k^{-\gamma}$ into a normalized distribution?

$$p(k) = Ck^{-\gamma} \quad \rightarrow \quad C = \frac{1}{\sum_k k^{-\gamma}}$$

Assuming $k > 0$ (no isolated nodes), C^{-1} is actually the Riemann Zeta function!
The Rieman Zeta function:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$$

Scale-free $p(k)$ and Zeta function

How to turn $p(k) \sim k^{-\gamma}$ into a normalized distribution?

$$p(k) = Ck^{-\gamma} \quad \rightarrow \quad C = \frac{1}{\sum_k k^{-\gamma}}$$

Assuming $k > 0$ (no isolated nodes), C^{-1} is actually the Riemann Zeta function!
The Rieman Zeta function:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$$

$$\rightarrow \quad p(k) = \frac{k^{-\gamma}}{\zeta(\gamma)}.$$