Structure and dynamics of complex networks

March 26, 2020

THE BARABÁSI-ALBERT MODEL

Random graphs vs real networks

	E-R	W-S
- sparseness, $\langle k \rangle$?	**OK**	**OK**
-the small-world effect, $\langle \ell \rangle$?	**OK**	**OK**
-large local clustering coeff., $\langle C \rangle$?	**NO!**	**OK**
-scale-free $p(k)$?	**NO!**	**NO!**

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How to generate a **scale-free** random graph in a simple way?

The Barabási-Albert model

Growing networks...

- Erdős-Rényi model: fixed N, (static)
- Watts-Strogatz model: fixed N, (static)
- Real networks?

The Barabási-Albert model

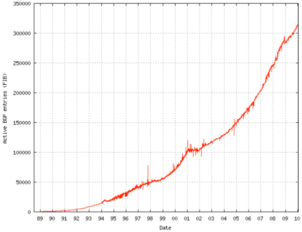
Growing networks...

Actor network

Number of movies in IMDB

Herr II, Bruce W., Ke, Weimao, Hardy, Elisha, and Börner, Katy. (2007) Movies and Actors: Mapping the Internet Movie Database. In Conference Proceedings of 11th Annual Information Visualization International Conference (IV 2007), Zurich, Switzerland, July 4-6, pp. 465-469.

Internet



Growth of the Internet routing table

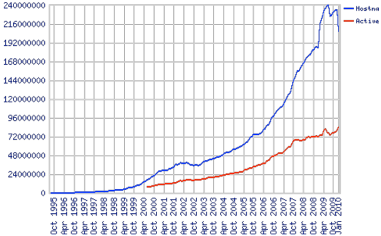
http://www.trainsignaltraining.com/ccna-ipv6

(from the slides of A.-L. Barabási)

The Barabási-Albert model

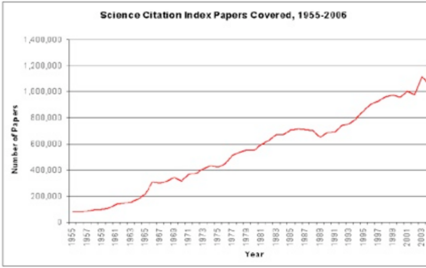
Growing networks...

WWW



http://website101.com/define-ecommerce-web-terms-definitions/

Scientific Publications



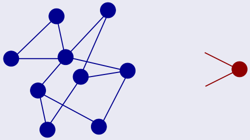
http://www.kk.org/thetechnium/archives/2008/10/the_expansion_o.php

(from the slides of A.-L. Barabási)

The Barabási-Albert model

The Barabási-Albert model

- Growing model: one new node in one step with m new links.



- The other end of the new links is attached according to the preferential attachment rule, i.e., we choose from the 'old' nodes with probabilities proportional to their degree:

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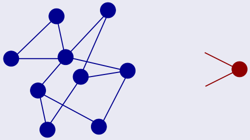

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The Barabási-Albert model
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- However, here we are going to discuss a more simple (approximate) analytic derivation of the $p(k)$.

Main stages of the forthcoming calculation:

1) Calculation of the $k_i(t)$, corresponding to the degree of node i at time step t, given that node i appeared as the new node at a given time step t_i.
2) Based on the formula obtained for $k_i(t)$, the derivation of $p(k)$.

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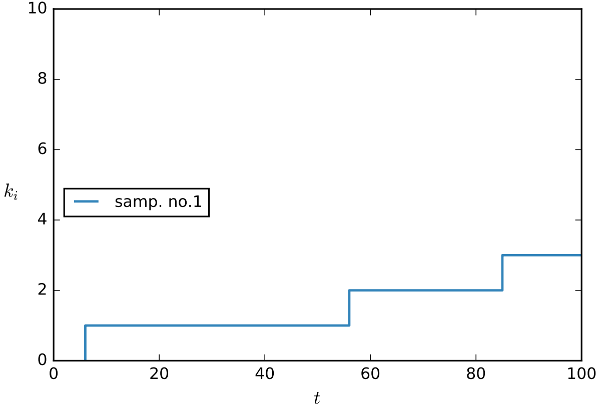
Let us first take a look at the degree as a function of time in simulation results!

The degree of a particular node as a function of t in simulations:



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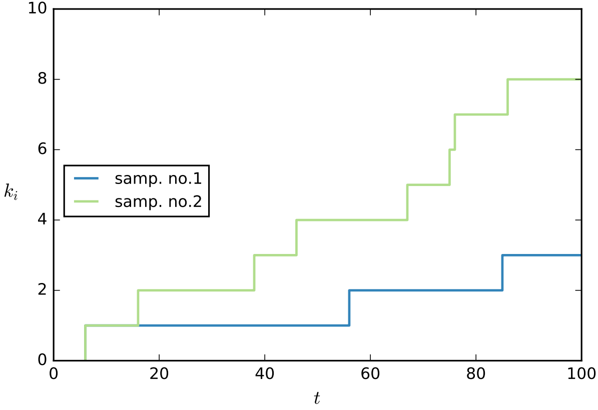

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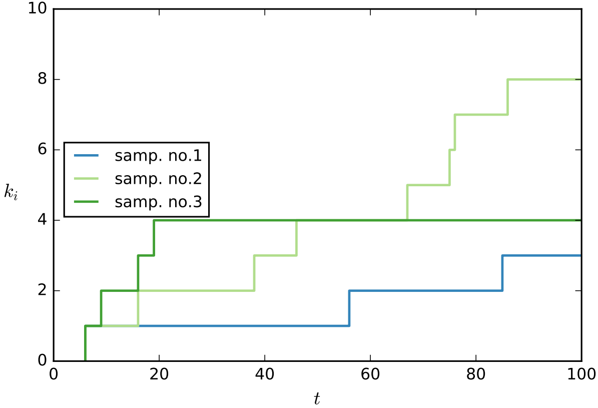

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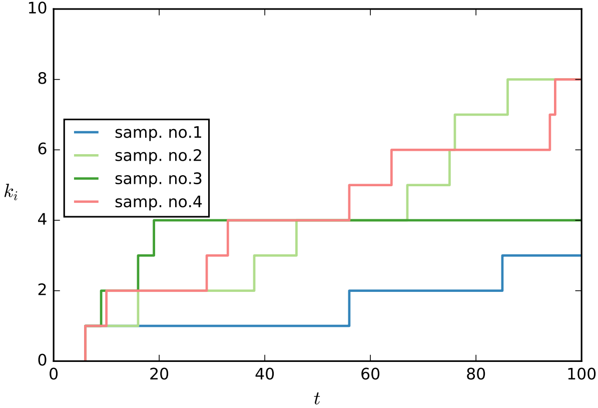

The Barabási-Albert model
Time dependent degree

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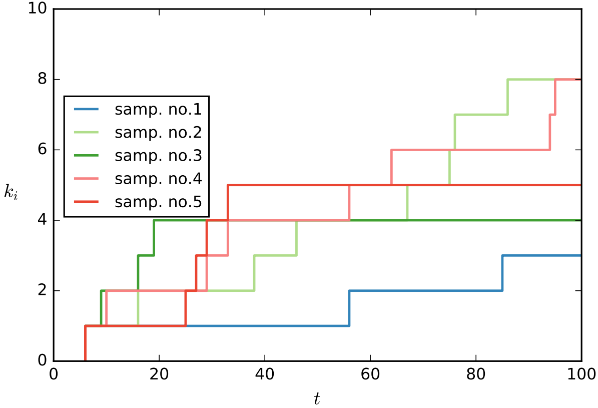

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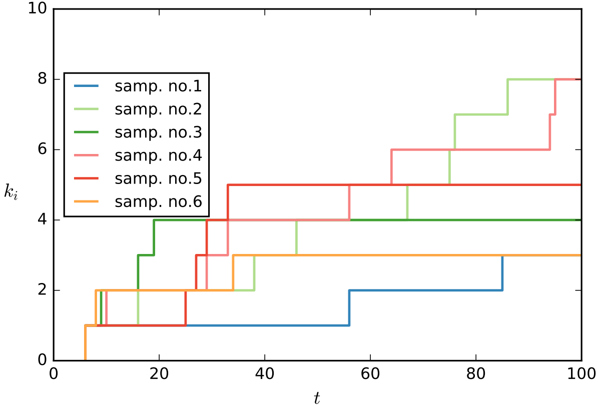

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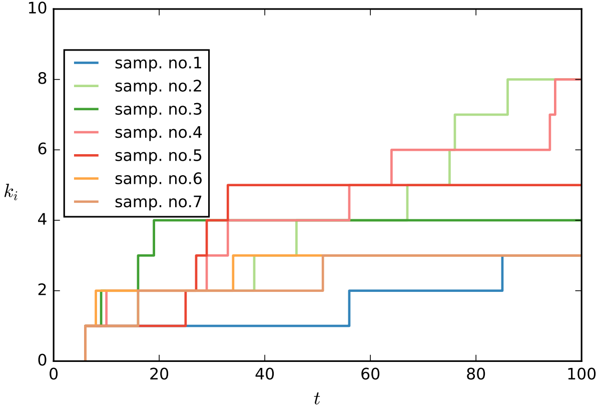

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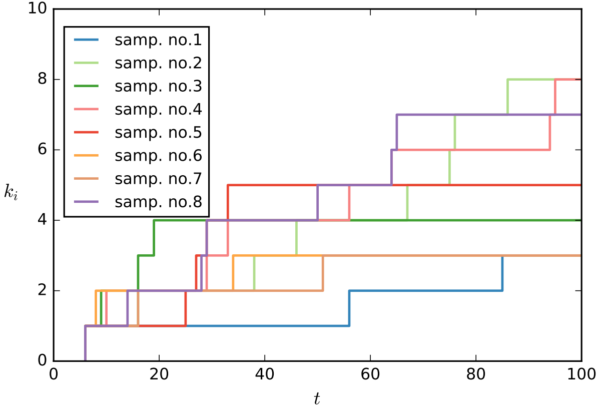

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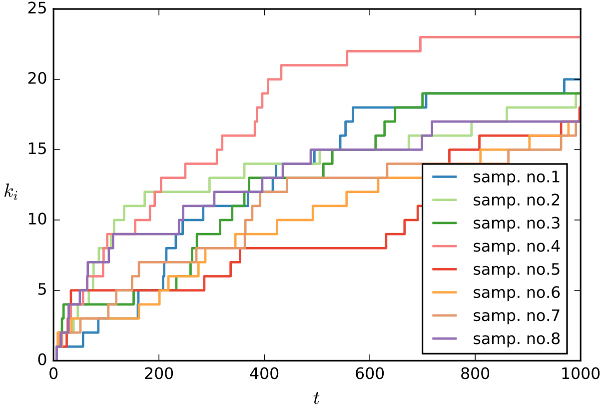

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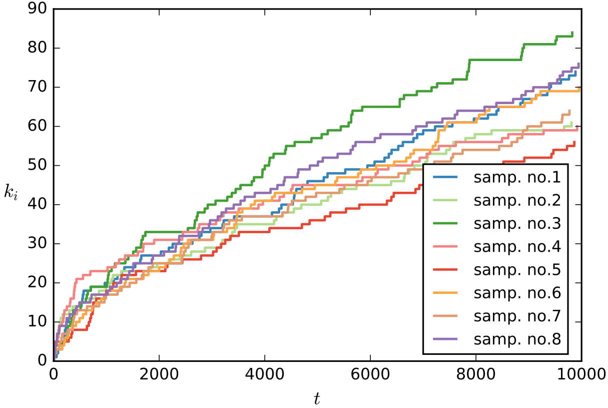

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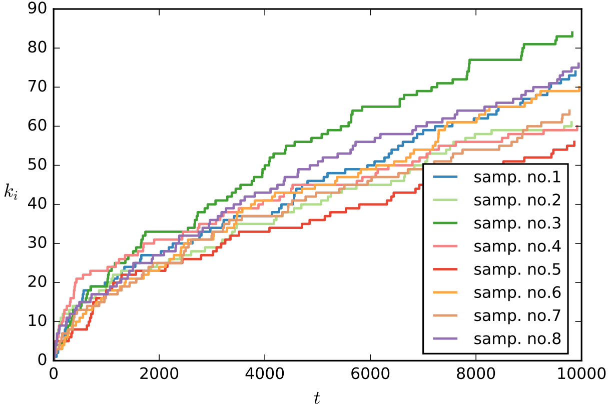

We can see a clear trend. Next, we are going to derive an analytic formula for the average $k_i(t)$!

The Barabási-Albert model
Analytic $p(k)$

- For large t the number of nodes and links is

$$N \simeq t,$$
$$M \simeq mt.$$

- The probability of choosing i:

$$\mathcal{P}_i = \frac{k_i}{\sum_j k_j}.$$

→ The approximate change of k_i in a time step:

$$\frac{\Delta k_i}{\Delta t} \simeq m\mathcal{P}_i \quad \to \quad \frac{\partial k_i}{\partial t} = m\frac{k_i}{\sum_j k_j}.$$

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$$\rightarrow \quad \ln k_i = \frac{1}{2} \ln t + \text{const.} \quad \rightarrow \quad k_i(t) = ct^{\frac{1}{2}}$$

- How to determine the constant c?

At $t = t_i$, (the appearance of node i): $k_i = m$, leading to

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}.$$

- Thus, the degree of any node is increasing as $k(t) \sim t^{\beta}$, where $\beta = 1/2$ is the so-called dynamical exponent.

The Barabási-Albert model
Dynamical exponent

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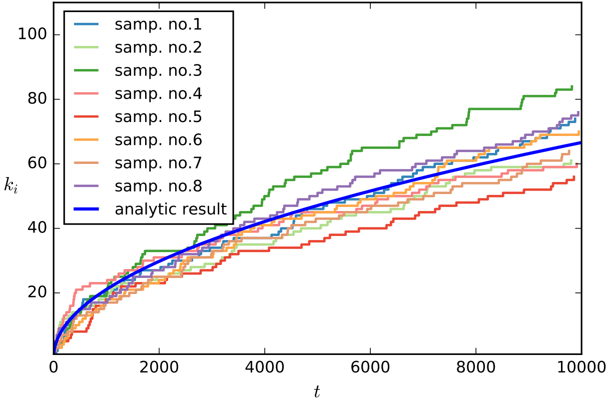

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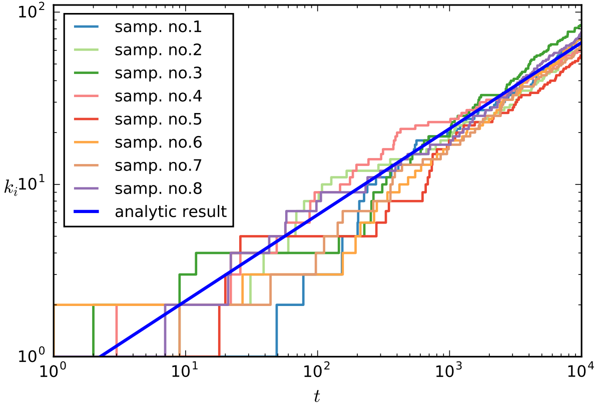

- How to calculate the degree distribution from $k_i(t)$?

$\rightarrow$ The easiest way is by considering the cumulative distribution first.

The Barabási-Albert model

Analytic $p(k)$

- The cumulative degree distribution:

$$P(k) \equiv \mathcal{P}(k_i < k) = \mathcal{P}(m(t/t_i)^{1/2} < k) = \mathcal{P}(t/t_i < (k/m)^2) = \mathcal{P}(t_i/t > (m/k)^2).$$

- The lengths of the time steps:

$$P(k) = 1 - \left(\frac{m}{k}\right)^2$$

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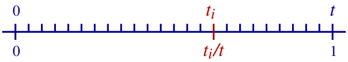

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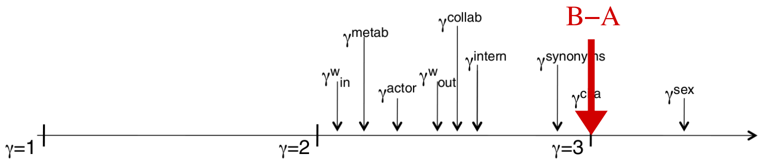
- The degree distribution is **SCALE-FREE** with $\gamma = 3$!

The Barabási-Albert model



<k²> diverges | <k²> finite

<k> diverges | <k> finite

Ultra small world behavior | Small world

Regime full of anomalies… | The scale-free behavior is relevant | Behaves like a random network

(from the slides of A.-L. Barabási)

Average distance in scale-free networks

$$\langle l \rangle \sim \begin{cases} \text{const.} & \gamma \le 2 \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

$\gamma \le 2$ and $2 < \gamma < 3$: Ultra Small World

$\gamma > 3$: Small World

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  - The degree of node i is given by $k_i(t) = m(\frac{t}{t_i})^{1/2}$
  - The probability of a connection to j introduced at $t = t_j$:

$$\mathcal{P}(i-j) = m\frac{k_i}{2mt} = \frac{k_i}{2t} = \frac{m\left(\frac{t_j}{t_i}\right)^{\frac{1}{2}}}{2t_j} = \frac{m}{2}(t_it_j)^{-\frac{1}{2}}.$$

- As the next step, let us write down the expected number of links between the neighbours of a node l, (introduced at t_l), at the end of the node generation process $t = N$:

$$\begin{aligned}
n_l &= \frac{1}{2}\sum_{t_i=1}^{N}\sum_{t_j=1}^{N}\mathcal{P}(l-i)\mathcal{P}(l-j)\mathcal{P}(i-j) \\
&\simeq \frac{1}{2}\int_1^N dt_i \int_1^N dt_j \mathcal{P}(l-i)\mathcal{P}(l-j)\mathcal{P}(i-j) \\
&= \frac{m^3}{16}\int_1^N dt_i \int_1^N dt_j (t_l t_i)^{-\frac{1}{2}}(t_l t_j)^{-\frac{1}{2}}(t_i t_j)^{-\frac{1}{2}} \\
&= \frac{m^3}{16 t_l}\int_1^N dt_i \frac{1}{t_i}\int_1^N dt_j \frac{1}{t_j} = \frac{m^3}{16 t_l}(\ln N)^2 .
\end{aligned}$$

- The number of "edge places" between the neighbors of l at the end $(t = N)$:

$$\frac{k_l(k_l - 1)}{2} \simeq \frac{k_l^2}{2} = \frac{1}{2}\left[m\left(\frac{t}{t_l}\right)^{\frac{1}{2}}\right]^2 = \frac{m^2 N}{2t_l}$$

- Thus, the clustering coefficient of l:

$$C_l = \frac{m^3}{16t_l}(\ln N)^2 \frac{2t_l}{m^2 N} = \frac{m(\ln N)^2}{8N}.$$

- Since we choose a general l, and C_l does not depend on l, our result for C in the B-A model:

$$C = \frac{m(\ln N)^2}{8N}.$$

$\langle C \rangle = \frac{m(\ln N)^2}{8N}$ → **Is this good or bad?**

→ Let's compare it to the Erdős-Rényi model:

- Erdős-Rényi: $\langle C \rangle \simeq \frac{\langle k \rangle}{N} \sim \frac{1}{N}$
- Barabási-Albert: $\langle C \rangle = \frac{m(\ln N)^2}{8N} \sim \frac{(\ln N)^2}{N}$

Slower decay with N, however, no decay at all in real networks:

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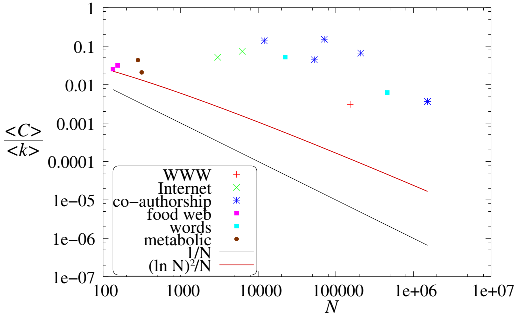

The Barabási-Albert model

Summary

- **Growth** and **preferential attachment**,
- Dynamical exponent: $k(t) \sim t^{\beta}$, $\beta = \frac{1}{2}$,
- **SCALE-FREE**: $p(k) \sim k^{-3}$, $\gamma = \mathbf{3}$,
- Small-world: $\langle l \rangle \sim \frac{\ln N}{\ln \ln N}$,
- $C \sim \frac{(\ln N)^2}{N}$.

Variations of the B-A model

Remaining challenges, problems?

Variations of the B-A model

Remaining challenges, problems:

- Original B-A: $\gamma = 3$.

  $\updownarrow$

  In real networks $2 \leq \gamma \leq 3$.

- Original B-A: oldest nodes have highest degree.

  $\updownarrow$

  Not always in real systems, e.g., Google.

- Low clustering coefficient.

- A simple solution to the problem of **tunable** γ **exponent** is given by one of the practice exercise, involving a **uniform additional fitness** beside the degree.

B-A model and fitness

To enable **newcomers overtake old nodes** in the B-A model we need a further **multiplicative fitness**:

- We introduce a fitness parameter η for the nodes, drawn from some distribution $\rho(\eta)$.
- The probability for node i to acquire new links:

$$\mathcal{P}_i \sim \eta_i k_i \quad \rightarrow \quad \mathcal{P}_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

→ The time evolution of the degree now depends also on the fitness:

$$k_i(t) \sim t^{\beta(\eta_i)}, \quad \beta(\eta) = c\eta$$

- This also means that now a later coming node with higher fitness (and higher β exponent) can actually overtake older but less fit nodes.

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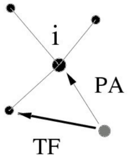

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Network models

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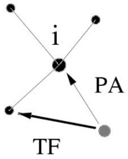
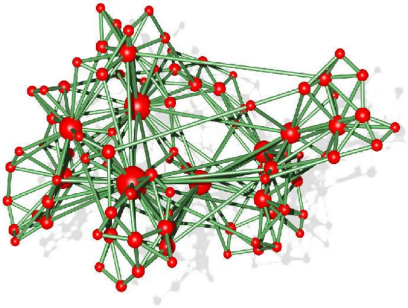
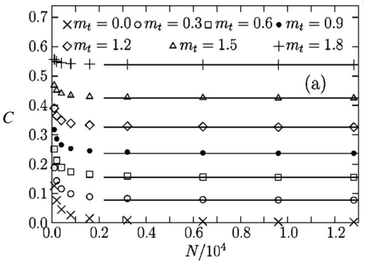


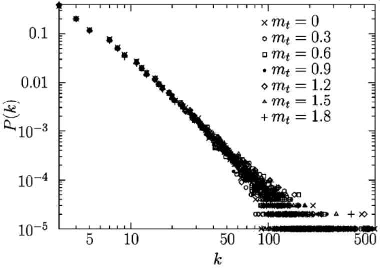


HK graph with p = 0.9

Scale-free model with high clustering

- Holme-Kim model:





$\rightarrow$ tunable clustering coefficient!

(The larger p we set, the larger is $\langle C \rangle$ in the resulting network).

P. Holme and B. J. Kim, *Phys. Rev. E* **65**, 026107 (2002)

Further reading on the Barabási-Albert model (not compulsory):

- Network science book by A.-L. Barabási, chapter 5: `http://networksciencebook.com/chapter/5`
- Wikipedia: `https://en.wikipedia.org/wiki/Barabasi-Albert_model`

THE CONFIGURATION MODEL

The configuration model
Motivation

- What if we do not care about the **mechanism** leading to scale-free $p(k)$ and are interested only in the **actual form** of the degree distribution?

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- What if we do not care about the **mechanism** leading to scale-free $p(k)$ and are interested only in the **actual form** of the degree distribution?
- How to generate a random network with arbitrary $p(k)$?

The configuration model

The configuration model

- The goal is to generate a network with N nodes and a given $p(k)$ corresponding to a random sample from all possible realisations.
- We first draw N times from $p(k)$ to obtain the degree sequence.
- Then the “half links” are joined to connect the nodes with each other.

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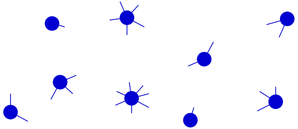

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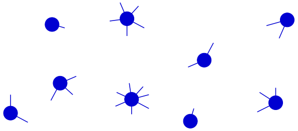

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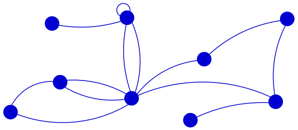

The configuration model

- If you want a simple graph (without self-connections and multi-edges), connecting the nodes is the hard part...
- How would you start?

The configuration model

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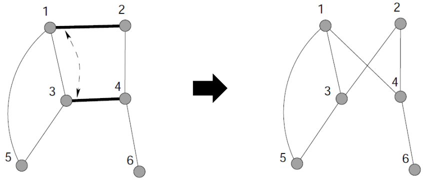
- The aim is to get rid of any extra correlation coming from the node connection process, but also preserve the degree of the nodes.
- How can we do that?

Randomisation

Link randomisation

An intuitive idea is link randomisation:

- in every step, choose 2 links at random,
- and swap one end of the links.
- (Of course, we have to check that we do not introduce self connections or multi edges before actually carrying out the swap).



Randomization
Node-randomization

Another possibility is node randomisation:

- in every step, choose 2 nodes at random,
- and swap one link on the nodes.
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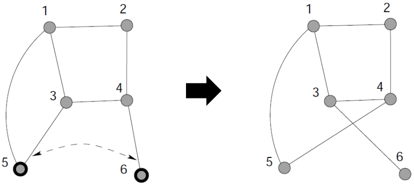

Randomisation

How long do we have to keep on swapping the links?

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→ A general rule of the thumb is that the number of swaps should be high enough so that on average every link was swapped a few times, i.e., a the number of swaps should be something like 3 or 5 times the number of links.

Randomisation and the configuration model

Now that we can randomise networks in a way that the node degrees remain the same, comparing any given real network to its configuration model counterpart has become very easy:

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We just **randomise a copy** of the original network (again, the number of carried out rewirings should be a few times the number of links), and the **resulting graph can be viewed as a sample from the configuration model**, where $p(k)$ **is exactly the same** as in the original network!

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$\rightarrow$ The configuration model is also a very important random 'baseline' to which we can compare a given real network.

Randomisation

- Any other application of randomisation apart from the configuration model?

Measuring significance by randomization

- Real networks are not completely random nor completely regular...
- To what extent is a given feature (e.g., $\langle \ell \rangle$, C, k_{nn}, etc.) is due to some intrinsic structure, and to what extent is it random?

$\rightarrow$ Let's randomise the network!