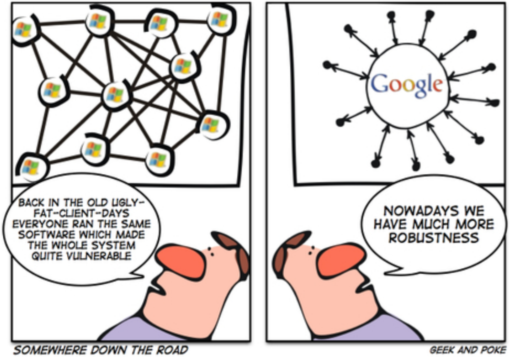
Structure and dynamics of complex networks

April 7, 2020

ROBUSTNESS AND PERCOLATION



WHAT IS ROBUSTNESS?

Robustness of complex networks

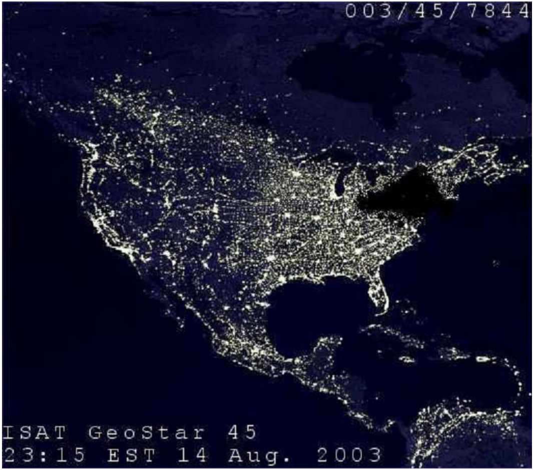
Some complex networks seem to be robust in the following general sense:

- Cell: gene mutations...
- Internet: router breakdowns...
- etc.

Although a part of the nodes (links) are down, the system as a whole is functioning normally.

Robustness of complex networks

However, if too many nodes or links are removed, the whole system breaks down:



Robustness of complex networks

- How to describe in quantitative terms the breakdown of a network under node or link removal?
- How does the network structure affect the robustness?

Robustness of complex networks

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Thus, as long as the network contains a **giant component**, it is functioning, since the existence of the giant component ensures that most of the nodes can reach each other.

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→ Based on this we can give a simple formulation of what we mean by the robustness of a complex network.

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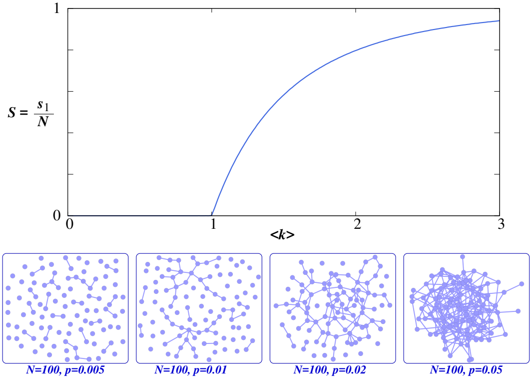
- As long as the network contains a **giant component**, it is considered to be still functioning, however, when the giant component is destroyed, the system is down.
- Thus, the **critical f_c at which the giant component is destroyed** provides a simple indicator of the robustness:
  - If f_c is small, the network is fragile,
  - If f_c is large, the network is robust.

Robustness of complex networks

Since the giant component is a really important concept in this chapter, before actually examining the robustness of complex networks, it is a good idea to deepen our knowledge about the emergence or disappearance of the giant component in networks.



GIANT COMPONENT IN THE ERDŐS-RÉNYI GRAPH

Giant component in the E-R graph

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  - if the linking probability p is $p = 0$, all nodes are isolated, thus, we have no giant component.
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$\rightarrow$ Let's find out!

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  Although for any given E-R graph size N we have a simple linear relation between p and $\langle k \rangle$ in the form of $\langle k \rangle = Np$, when comparing E-R graphs of different size, it is more natural to take $\langle k \rangle$ as the other parameter beside N instead of p.

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- Similarly, when measuring the sizes of the components, it is a good idea to switch for any component α from the number of nodes in the component s_α to the relative component size, given by

$$S_\alpha = \frac{s_\alpha}{N}.$$

  Again, when comparing E-R graphs of different sizes this is a natural thing to do, since the relative size is always between 0 and 1.

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Relative size of the **largest component** as a function of $\langle k \rangle$:

- Please take a look at the accompanying notebook where we examine this by simulations for finite size.

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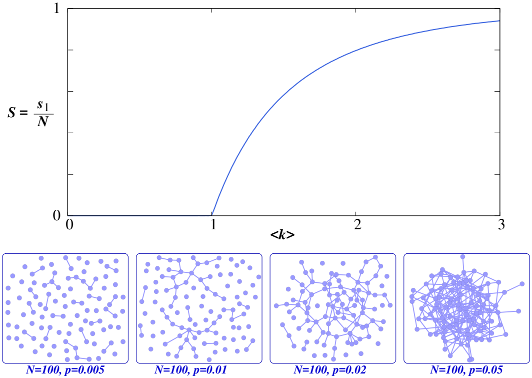
Relative size of the **largest component** as a function of $\langle k \rangle$:

- Please take a look at the accompanying notebook where we examine this by simulations for finite size.
- In case of an infinitely large E-R graph we would obtain the following curve (where the graphs below are just illustrations):



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  - However, the probability that i is NOT in the giant component is u by definition, thus

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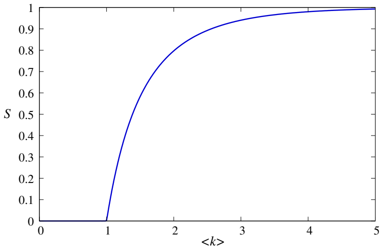
This equation can be solved numerically for any $\langle k \rangle$, giving the S in the figure.
Please take a look at the jupyter notebook, where we calculate S.



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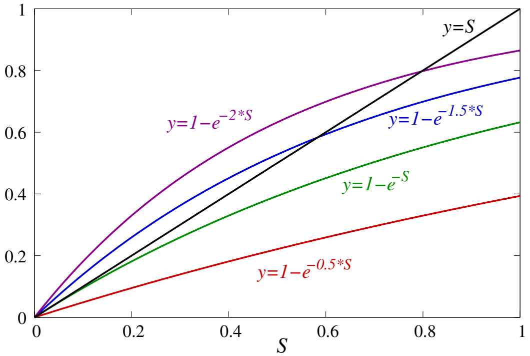

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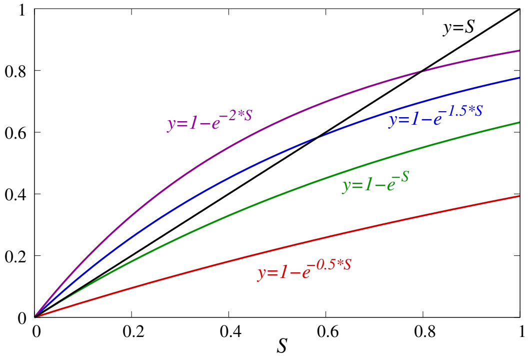

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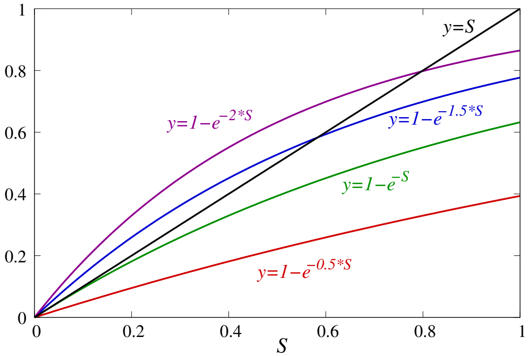

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- **Thus, the transition point is at $k = 1$, and for $k \geq 1$ we have a giant component in the E-R graph.**

PERCOLATION TRANSITION

Phase transitions

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- Phase transitions have several **universal features**, and we can discuss them in a more or less uniform framework.

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  - in case of the E-R graph this is the average degree, $\langle k \rangle$.

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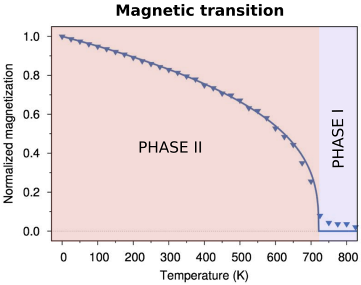
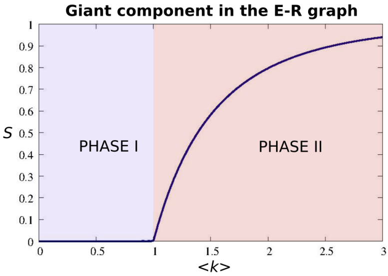
  - in case of the traffic jam this can be the size of the jammed region,
  - in case of the E-R graph this can be the (relative) size of the largest connected component.

Phase transitions

When plotting the **order parameter** as a function of the **control parameter**, phase transitions look quite similar:





Percolation transition

Beside the ferromagnetic transition, there is another phase transition in statistical physics that is even more closely related to the giant component in the E-R graph:

THE PERCOLATION TRANSITION!!

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- Percolation is the passing of a liquid through some porous material or filter.

→ Thus, when making coffee, you are actually percolating water through ground coffee beans, and this device for making coffee (often found in Hungarian households) is called a percolator:

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- However, this works only if the small pores are connected enough so that together they make up a large connected cavity...
- So before actually setting up the machinery for oil mining in this way, it is good to know whether this is the case or not. Actually it turns out that by examining the structure of a smaller rock sample we can make a pretty good guess about that.
- The theoretical models for the large scale structures made up by small cavities in porous materials are called **percolation models**.

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- We introduce a probability p for the lattice points to be occupied by a cavity, and the **cavities are placed absolutely at random**.

  Thus, at $p = 0$ we have no cavities at all, whereas at $p = 1$ all lattice points are occupied by a small cavity.

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- We introduce a probability p for the lattice points to be occupied by a cavity, and the **cavities are placed absolutely at random**.

  Thus, at $p = 0$ we have no cavities at all, whereas at $p = 1$ all lattice points are occupied by a small cavity.
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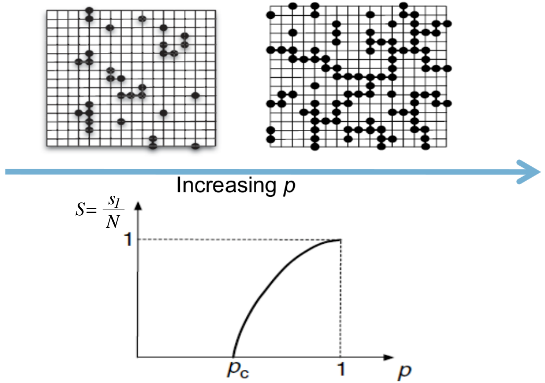
→ We have **two phases** (percolating cluster and small clusters), and the transition from one to the other is called

THE PERCOLATION TRANSITION!

Percolation transition

Percolation on lattices

- a node is occupied with probability p,
- at p_c a spanning cluster emerges.
- control parameter is p, the order parameter is the relative size of the largest connected cluster.



Percolation transition

Analogy between the percolation transition and the emergence of the giant component in the E-R graph:

	Percolation		**E-R graph**
phase I.:	no giant cluster	$\leftrightarrow$	no giant component
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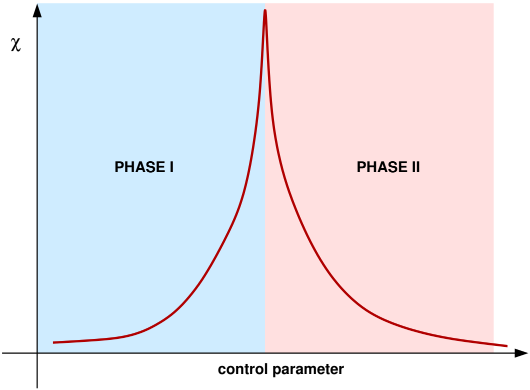
Although no one ever tried to percolate a liquid through a network, still, based on this analogy, the emergence of the giant component in the E-R graph at $\langle k\rangle = 1$ is often referred to as

THE PERCOLATION TRANSITION OF THE E-R GRAPH !



SUSCEPTIBILITY

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- insertion of just a single extra car can induce a large traffic jam,
- insertion of just a single extra link can induce the emergence of the giant component in the random graph.

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- Similarly to the control parameter and the order parameter, the actual definition or formula of the susceptibility is different for each system.
- Nevertheless, the behaviour of the susceptibility is the same across all systems:

  When plotted as a function of the control parameter, it has a **strong, sharp peak at the transition point**.

  For macroscopic (or more precisely, infinitely large) systems χ is actually divergent at the transition point.

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→ **A simple idea is to define χ as the expected change in the component size for the initially separated components that are merged because of the extra added link.**

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  - the probability for choosing the 2nd node from a component of size s can be written as $s \cdot p(s)$ (since choosing the node from a given component is proportional to the size of the component).
  - However, when choosing the 2nd node from a component of size s, the component size change from the point of view of the first node is s, thus, if we take into account all possibilities we obtain
  $$\chi = \sum s \cdot s \cdot p(s) = \sum s^2 p(s) = \langle s^2 \rangle,$$
  which means that χ is equal to the second moment of the component size distribution!

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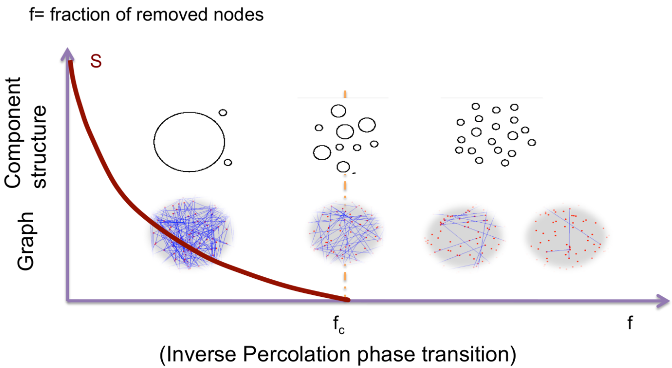
- Please take a look at the accompanying jupyter notebook, where we calculate and plot χ for the E-R graph.



RESILIENCE AGAINST RANDOM NODE REMOVAL

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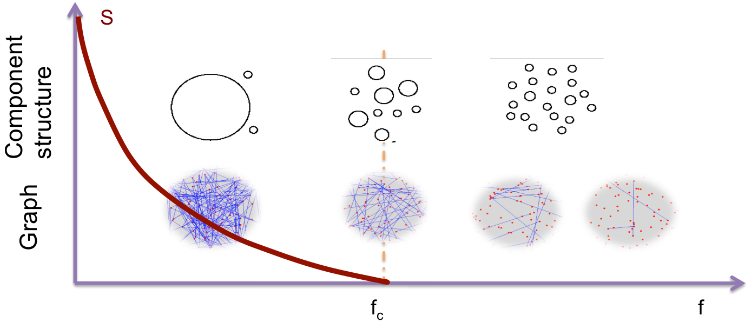
- **Please take a look at the accompanying jupyter notebook where we simulate the random node removal process on both a real network and an E-R graph.**

Random node removal: “inverse percolation"

Illustration:

f= fraction of removed nodes



(from the slides of A.-L. Barabási)

Condition for the existence of a giant component

- In the following we are going to derive a simple, general condition for the existence of a giant component in networks.
- What can we use that for?
- If we see that it is not fulfilled any more, then based on that we know that the giant component has been destroyed!

Condition for the existence of a giant component

- Assume a network with a giant component.
- The average degree in a giant component: $\langle k \rangle_{\mathrm{G}} \geq 2$

$\leftrightarrow$

- Let $\mathcal{P}(k_i \mid i \leftrightarrow j)$ be the **conditional probability** for node i to have degree k_i given it is connected to some random other node j.

- Suppose we have a **random network** with a **GC**. Assuming that a node connected to at least one other node is actually linked to the GC we obtain

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- Accordingly, an E-R graph has a GC if

$$p(k) \simeq \frac{\langle k\rangle^k e^{-\langle k\rangle}}{k!}, \rightarrow \left\langle k^2\right\rangle = \langle k\rangle\left(1+\langle k\rangle\right),$$

$$\rightarrow \frac{\left\langle k^2\right\rangle}{\langle k\rangle} = 1+\langle k\rangle \geq 2$$

$$\rightarrow \langle k\rangle \geq 1.$$

(The same result was derived in different way in the 2nd part of the lecture.)

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  - The smallest possible value of f for which the condition is not fulfilled defines the critical f, where we reach the transition point.

Network breakdown

Assuming the random removal of f fraction of nodes from a network characterized by a given $p(k)$ (and corresponding $\langle k \rangle$, $\langle k^2 \rangle$):

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- Simple estimate for $\langle k'\rangle$ after the removal if $f \ll 1$:
  - f fraction of removed nodes induces f fraction of removed links,

  $\rightarrow$ in other words, $1-f$ fraction of the links remain, giving $\langle k'\rangle = (1-f)\langle k\rangle$.

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- Therefore, the degree distribution after the removal is

$$p'(k') = \sum_{k=k'}^{\infty} p(k) \binom{k}{k'} f^{k-k'} (1-f)^{k'}.$$

The critical f

- The modified $\langle k' \rangle$ after the removal:

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The sum is done over the triangle, thus, it can be replaced as $\sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} = \sum_{k=0}^{\infty} \sum_{k'=0}^{k}$

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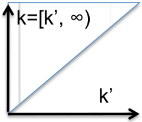

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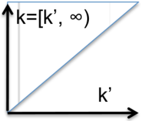

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$$\begin{aligned} \langle k^2 \rangle &= \langle k \rangle (1 + \langle k \rangle), \\ \rightarrow f_c &= 1 - \frac{1}{\langle k \rangle}, \\ \rightarrow \langle k' \rangle_{f_c} &= (1 - f_c) \langle k \rangle = 1. \end{aligned}$$

Thus, in order to destroy an E-R graph we have to keep on removing the links until reaching $\langle k' \rangle = 1$.

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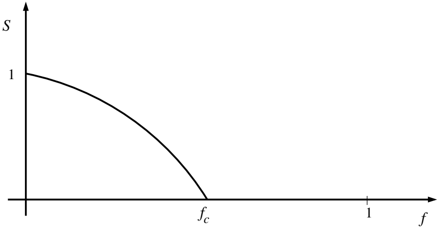

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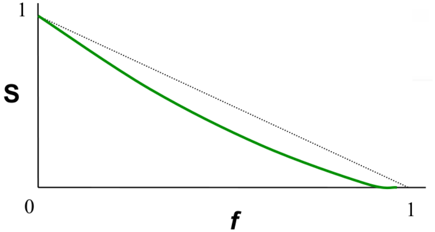
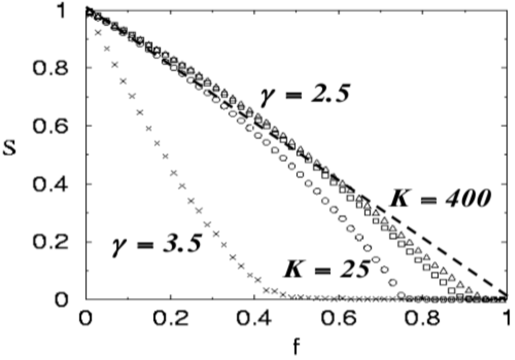


Extreme robustness of scale-free networks

Real world examples:

- Internet, router level map: $N = 228,263$, $\gamma = 2.1 \pm 0.1$
  $\rightarrow f_c = 0.962$
- Internet, AS level map: $N = 11,164$, $\gamma = 2.1 \pm 0.1$
  $\rightarrow f_c = 0.996$

Figure from the paper by Barabási and co-workers:



Extreme robustness of scale-free networks

Why do we see scale-free networks everywhere?

→ A very plausible answer is their extreme robustness!

The price of robustness

This extreme robustness has a price...

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This extreme robustness has a price...

Scale-free networks are vulnerable to targeted attack!

Attack tolerance

How to “attack” a scale-free network?

→ Removing nodes in the order of their degree.

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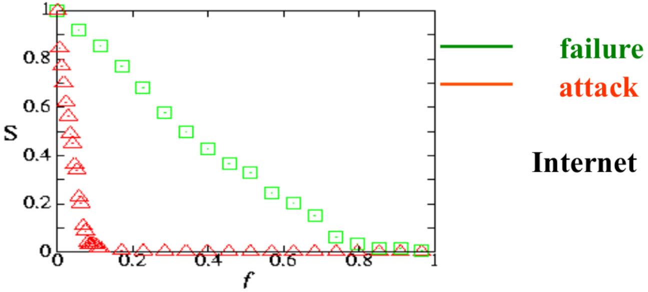

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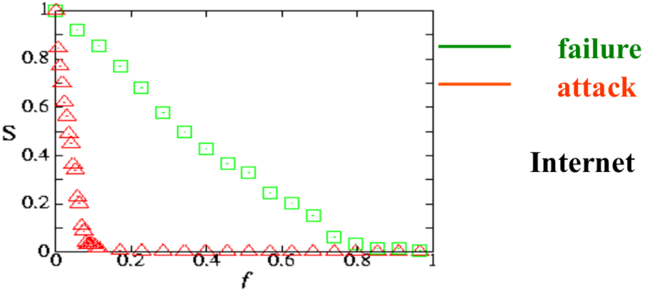
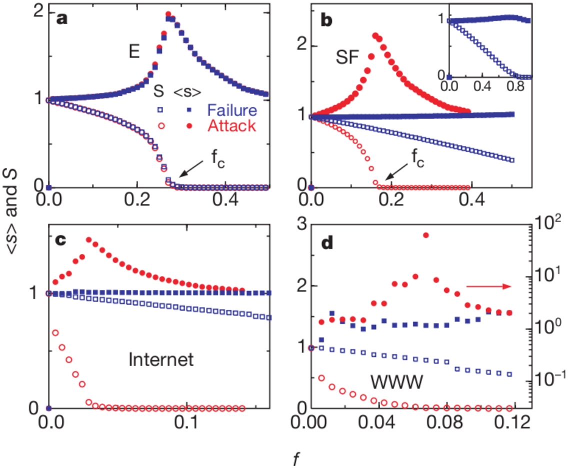


Please take a look at the accompanying jupyter notebook where we simulate also attack tolerance on networks.

Error and attack tolerance

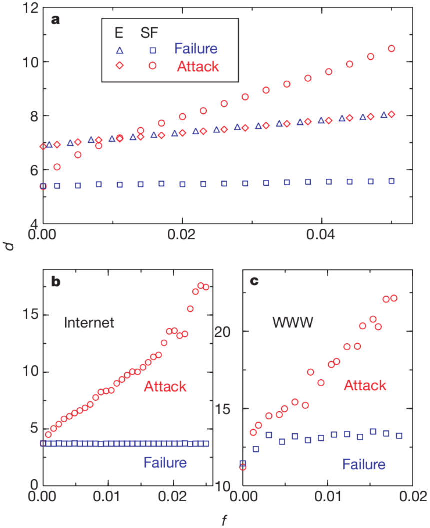
Figure from the paper by Albert et. al., showing the behaviour of S and χ:



R. Albert, H. Jeong and A.-L. Barabási, *Nature* **406**, 378 (2000)

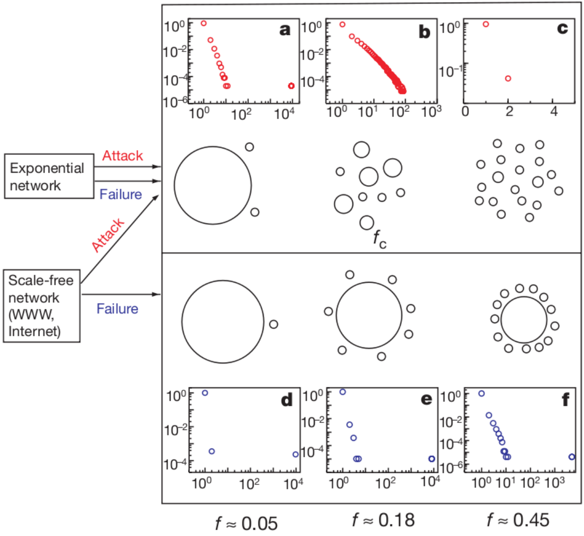
Error and attack tolerance

Figure from the paper by Albert et. al., showing the behaviour of the average distance:



Error and attack tolerance

Summary figure from the paper by Albert et. al.:



Summary

Network robustness short summary

- Scale-free networks are extremely robust against random node removal. This property is connected to the divergence of the second moment of the degree distribution.
- However, in the mean time scale-free networks are also fragile against targeted attack, since taking out the HUBS is producing a lot of damage.
- We examined these properties with the help of the giant component, within the framework of percolation transitions.

Further reading on network robustness (not compulsory):

Network science book by A.-L. Barabási, chapter 8:
`http://networksciencebook.com/chapter/8`