Networks theoretical practice problems no.2.

1. Suppose we generate a Watts-Strogatz random graph made of $N = 1000$ nodes, where every node is connected to its first and second neighbours along the ring (this means that $q = 2$), and we set the random rewiring probability to $\beta = 10^{-4}$. What do you expect, is this going to result in a small world network?

   Our very simple estimate for the critical β_c where we expect the network to turn into a small-world network can be obtained from $\beta_c \cdot N \cdot q = 1$, or in other words $\beta_c = \frac{1}{N \cdot q}$. In the present case this gives $\beta_c = \frac{1}{1000 \cdot 2} = 5 \cdot 10^{-4}$. Since the proposed β is smaller, our expectation is that the resulting network is not going to be a small-world network since the expected number of randomly rewired links is very low.

2. Suppose we modify the B-A model in the following way: each node is given a uniform fitness value $a \in [0, m]$, and the probability for an already existing node i to gain a new link is proportional to $k_i - a$:

$$\mathcal{P}_i \sim k_i - a.$$

→ Derive the decay exponent for $p(k)$ in the mean-field approximation for the **large k regime**.

Hint: since we are interested only in the tail of $p(k)$ where $k >> 1$, in order to be able to solve the differential equation in a similar way to the original B-A model, neglect a beside k_i after writing down the differential equation.

Similarly to the original B-A model we assume that for large t

$$N \simeq t,$$
$$M \simeq mt.$$

Since we have an additional uniform fitness parameter $a \in [0, m]$, the probability for choosing i is

$$\mathcal{P}_i \sim k_i - a.$$

By using the normalisation condition that $\sum \mathcal{P}_i = 1$ we can write

$$\mathcal{P}_i = \frac{k_i - a}{\sum_j (k_j - a)} = \frac{k_i - a}{2M - Na},$$

where we have used that $\sum_i k_i = 2M$. *By substituting* $N \simeq t, M \simeq mt$ *we obtain*

$$\mathcal{P}_i = \frac{k_i + a}{2mt - at}.$$

Based on that, differential equation for k_i *takes the form of*

$$\frac{\partial k_i}{\partial t} = m\mathcal{P}_i = m\frac{k_i - a}{2mt - at} = \frac{k_i - a}{t(2 - a/m)}.$$

Since we are interested in the large degree regime where $k >> 1$, *we can neglect* a *besides* k_i *in the nominator, thus,*

$$\frac{\partial k_i}{\partial t} = \frac{k_i - a}{t(2 - a/m)} \simeq \frac{k_i}{t(2 - a/m)}.$$

Solving the diff. eq.:

$$\frac{\partial k_i}{k_i} = \frac{\partial t}{t(2 - a/m)} \quad \to \quad \int \frac{dk_i}{k_i} = \int \frac{dt}{t(2 - a/m)}$$
$$\to \quad \ln k_i = \frac{1}{2 - a/m} \ln t + \text{const.} \quad \to \quad k_i(t) = ct^{\frac{1}{2-a/m}}$$

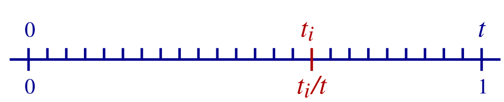
The constant c *is determined by the boundary condition that at* $t = t_i$, *(the appearance of node* i*) we have* $k_i = m$*:*

$$k_i(t = t_i) = m = ct_i^{\frac{1}{2-a/m}} \quad \to \quad c = mt_i^{-\frac{1}{2-a/m}}$$
$$\to \quad k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2-a/m}}.$$

The cumulative degree distribution:

$$P(k) \equiv \mathcal{P}(k_i < k) = \mathcal{P}\left(m(t/t_i)^{\frac{1}{2-a/m}} < k\right) = \mathcal{P}\left(t/t_i < (k/m)^{2-a/m}\right) =$$
$$\mathcal{P}\left(t_i/t > (m/k)^{2-a/m}\right).$$

The lengths of the time steps:



Thus,

$$\begin{aligned} P(k) &= 1 - \left(\frac{m}{k}\right)^{2-a/m} \\ \to p(k) &= 2m^2 k^{-3+a/m} \end{aligned}$$

This means that now we can control the exponent $\gamma = 3 - a/m$ with the help of the parameter a. When $a = 0$ we recover the original B-A model with $\gamma = 3$, and when $a = m$, we obtain $\gamma = 2$.

3. Is preferential attachment really necessary in the B-A model for achieving a scale-free degree distribution? Let us find out, by modifying the model in such a way that the new nodes are choosing simply uniformly at random from the already existing nodes when connecting into the network. Derive the $p(k)$ in this model following the same steps we went through in the slides for the original B-A model.

   If the new links are connected randomly with uniform probability:

$$\mathcal{P}_i = \frac{1}{N(t)} \simeq \frac{1}{t},$$
$$\frac{\partial k_i}{\partial t} = m\mathcal{P}_i = \frac{m}{t},$$

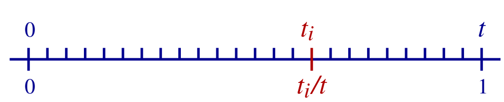
   leading to

$$k_i(t) = m\ln(t/t_i) + m,$$

   The cumulative degree distribution:

$$\begin{aligned} P(k) \equiv &\mathcal{P}(k_i < k) = \mathcal{P}(m\ln(t/t_i) + m < k) = \\ &\mathcal{P}(\ln(t/t_i) < k/m - 1) = \mathcal{P}(t/t_i < e^{k/m-1}) = \\ &\mathcal{P}(t_i/t > e^{1-k/m}), \end{aligned}$$

   The lengths of the time steps:



   Thus,

$$P(k) = 1 - e^{1-k/m}$$
$$p(k) = \frac{e^{1-k/m}}{m}$$

According to this result, if we have uniform attachment instead of preferential attachment, the degree distribution develops into an exponential distribution, and not into a scaling power-law distribution. Therefore, the preferential attachment is indeed a necessary ingredient of the B-A model.