Structure and dynamics of complex networks

March 3, 2020

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$$C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\gamma} dk} = \frac{1}{\frac{1}{1-\gamma} \left[k^{1-\gamma}\right]_{k_{\min}}^{\infty}} = \frac{\gamma - 1}{k_{\min}^{1-\gamma}}.$$

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- Thus, the continuous $p(k)$ is given by

$$p(k) = (\gamma - 1)\frac{k^{-\gamma}}{k_{\min}^{1-\gamma}}$$

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→ Let's calculate it with the help of our newly introduced continuous formalism!

Variance and the 2nd moment

- The 2nd moment of $p(k)$ can be written as

$$\langle k^2 \rangle \equiv \int_{k=k_{\min}}^{\infty} k^2 p(k) dk = \frac{\gamma - 1}{k_{\min}^{1-\gamma}} \int_{k=k_{\min}}^{\infty} k^{2-\gamma} dk =$$

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$$3 - \gamma > 0 \quad \rightarrow \quad \langle k^m \rangle = \infty$$

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→ For $\gamma < 3$, the $\langle k^2 \rangle$ is divergent!

Scale-free networks

Measured γ exponents

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference
WWW	$325,729$	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, Barabási 1999
WWW	4×10^7	7		2.38	2.1				Kumar *et al.* 1999
WWW	2×10^8	7.5	$4,000$	2.72	2.1	16	8.85	7.61	Broder *et al.* 2000
WWW, site	$260,000$				1.94				Huberman, Adamic 2000
Internet, domain*	$3,015$ - $4,389$	3.42 - 3.76	$30 - 40$	2.1 - 2.2	2.1 - 2.2	4	6.3	5.2	Faloutsos 1999
Internet, router*	$3,888$	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos 1999
Internet, router*	$150,000$	2.66	60	2.4	2.4	11	12.8	7.47	Govindan 2000
Movie actors*	$212,250$	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási, Albert 1999
Coauthors, SPIRES*	$56,627$	173	$1,100$	1.2	1.2	4	2.12	1.95	Newman 2001b,c
Coauthors, neuro.*	$209,293$	11.54	400	2.1	2.1	6	5.01	3.86	Barabási *et al.* 2001
Coauthors, math*	$70,975$	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási *et al.* 2001
Sexual contacts*	2810			3.4	3.4				Liljeros *et al.* 2001
Metabolic, E. coli	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong *et al.* 2000
Protein, S. cerev.*	1870	2.39		2.4	2.4				Mason *et al.* 2000
Ythan estuary*	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya, Solé 2000
Silwood park*	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya, Solé 2000
Citation	$783,339$	8.57			3				Redner 1998
Phone-call	53×10^6	3.16		2.1	2.1				Aiello *et al.* 2000
Words, cooccurence*	$460,902$	70.13		2.7	2.7				Cancho, Solé 2001
Words, synonyms*	$22,311$	13.48		2.8	2.8				Yook *et al.* 2001

Most measured γ are smaller than 3.

$\rightarrow$ $\langle k^2 \rangle$ **diverges in the $N \rightarrow \infty$ limit!**

Divergence of the variance and σ

- The variance of the degree:

$$\mathrm{Var}(k) \equiv \left\langle (k - \langle k \rangle)^2 \right\rangle = \left\langle k^2 \right\rangle - \langle k \rangle^2 .$$

→ The variance is diverging as well!

- The standard deviation of the degree:

$$\sigma(k) \equiv \sqrt{\mathrm{Var}(k)} = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}.$$

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Divergence of the variance

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WWW: $\langle k \rangle = 7 \pm \infty$
Internet: $\langle k \rangle = 3.5 \pm \infty$
Coauthorship: $\langle k \rangle = 11.5 \pm \infty$
etc.

The $\langle k \rangle$ is not meaningful due to the large fluctuations!

Consequences of the scale-free $p(k)$

Summary of the consequences of the scale-free $p(k)$:

- we plot $p(k)$ on **log-log scale**
- **HUBS!**
- **divergent $\langle k^2 \rangle$!** (for $\gamma < 3$)

$\rightarrow$ no “typical” degree,
($\rightarrow$ anomalous percolation),
($\rightarrow$ anomalous spreading)

Average distance in scale-free networks

- Do scale-free networks have the small-world property?

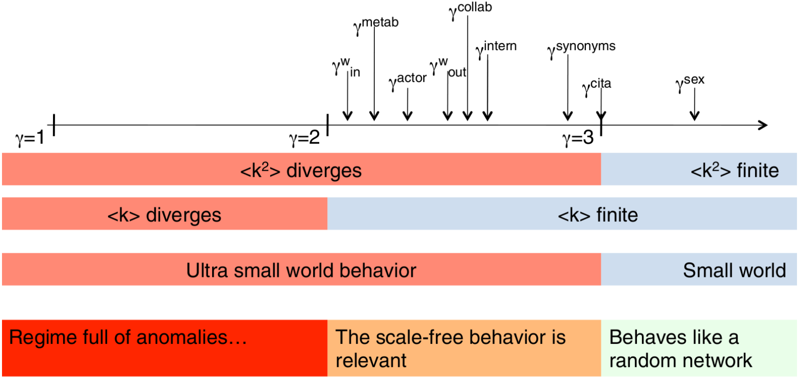
Average distance in scale-free networks

$$\langle l \rangle \sim \begin{cases} \text{const.} & \gamma \le 2 \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

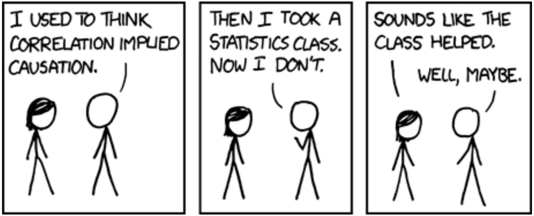
The first two cases ($\gamma \le 2$ and $2 < \gamma < 3$): Ultra Small World

The case $\gamma > 3$: Small World

Summary of the behavior of scale-free networks



(from the slides of A.-L. Barabási)

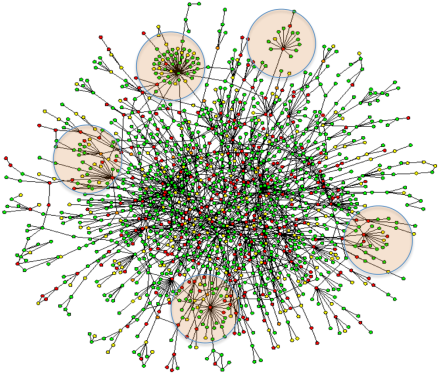


DEGREE CORRELATIONS

Disassortative “mixing” in PPI networks

- Hubs tend to link to small degree nodes in PPI networks...

→ What is the probability for having a link between nodes of degree k_i and k_j in a random graph?



→ If $k_i = 50$, $k_j = 13$, $L = 1746$, we have
$p_{50,13} = 0.15 \quad \leftrightarrow \quad p_{2,1} = 0.0004$

Yet, we see many links between degree 2 and 1 nodes, and no links between the hubs...

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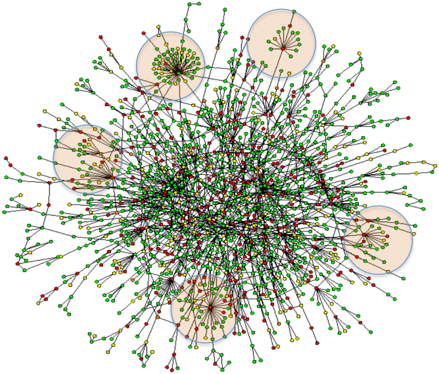

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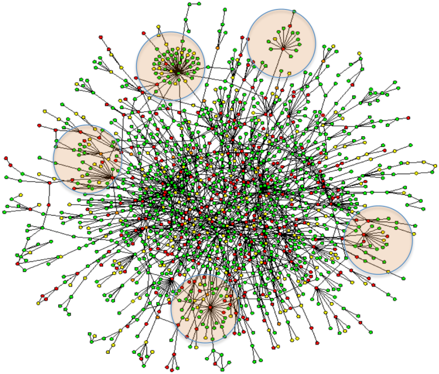
$$P(\text{link } i - j) = \frac{k_i k_j}{2L}$$



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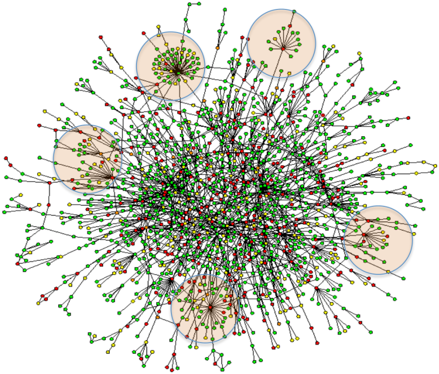

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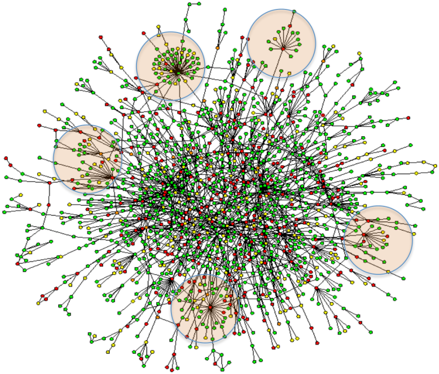

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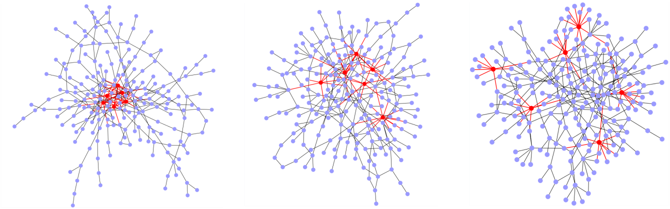
Assortative and disassortative networks

Assortativity and disassortativity

- **Assortative network**: small degree nodes tend to connect to other small degree nodes, hubs tend to link to each other.
- Neutral network: nodes connect to each other at random.
- **Disassortative network**: hubs avoid linking to each other, instead they connect to small degree nodes.

Illustration:



assortative neutral dissasortative

How to describe assortativity?

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Full statistical description:

- Def.: let $P(k' \mid k)$ denote the **conditional probability** for finding a **node with degree** k' at one end of a link, **given** the node at the other end has **degree** k.
- In principle, $P(k' \mid k)$ encodes all info about whether the network is assortative or disassortative.
- How to measure this in practice?
  By definition:

$$P(k' \mid k) = \frac{P(\text{link between } k' \text{ and } k)}{P(\text{link on } k)}.$$

→ Def.: let $E_{k',k}$ count the number of links between nodes of degree k' and k, and

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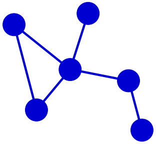
- $E_{k',k}$: number of links between nodes of degree k' and k, links between nodes with the same degree count twice!



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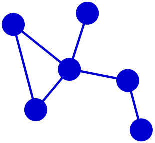

$\rightarrow E_{k',k}$:

$k=$	1	2	3	4
1	0	1	0	1
2	1	2	0	3
3	0	0	0	0
4	1	3	0	0

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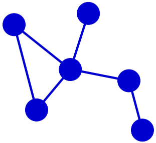

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$\rightarrow$ If we are going to measure $E_{k',k}$, we might as well "forget" $P(k' \mid k)$, and examine what does assortativity mean in turns of $E_{k',k}$.

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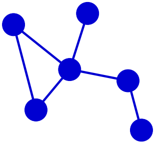

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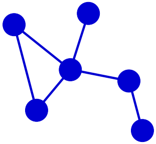

$\rightarrow e_{k',k}$:

$k =$	1	2	3	4
1	0	$\frac{1}{12}$	0	$\frac{1}{12}$
2	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{4}$
3	0	0	0	0
4	$\frac{1}{12}$	$\frac{1}{4}$	0	0

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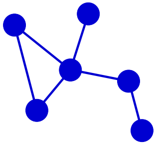

$$\rightarrow e_{k',k}: \quad \begin{array}{c|cccc} k= & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \frac{1}{12} & 0 & \frac{1}{12} \\ 2 & \frac{1}{12} & \frac{1}{6} & 0 & \frac{1}{4} \\ 3 & 0 & 0 & 0 & 0 \\ 4 & \frac{1}{12} & \frac{1}{4} & 0 & 0 \end{array}$$

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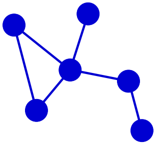

$\rightarrow e_{k',k}$:

$k =$	1	2	3	4
1	0	$\frac{1}{12}$	0	$\frac{1}{12}$
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4	$\frac{1}{12}$	$\frac{1}{4}$	0	0

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Advanced network characteristics

Full statistical description

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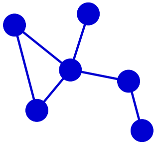

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What is q_k?

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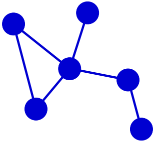

$\rightarrow e_{k',k}$:

$k =$	1	2	3	4
1	0	$\frac{1}{12}$	0	$\frac{1}{12}$
2	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{4}$
3	0	0	0	0
4	$\frac{1}{12}$	$\frac{1}{4}$	0	0
q_k	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$

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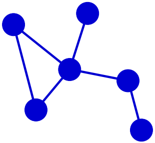

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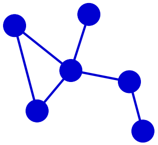

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$\rightarrow$ In a neutral network with no degree correlations: $e_{k'k} = q_{k'} \cdot q_k$.

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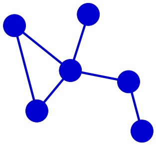

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Thus, the **deviations from this value are the signatures of degree correlations**.