Conformal-Dual

vladimir.gurvich

April 2022

1 Introduction

2 Preliminaries (from the "Dually conformal hypergraphs" project)

Corollary 2.1. *Let $\mathcal{H} = (V, E)$ be a hypergraph with dimension d and maximum degree Δ having no isolated vertices, given by an edge-vertex incidence matrix and a doubly-linked representation of its incident pairs, and let $S \subseteq V$. Then, there exists an algorithm running in time*

$$\mathcal{O}\left(d|E| \cdot \min\left\{\Delta^{|S|}, \left(\frac{|E|}{|S|}\right)^{|S|}\right\}\right)$$

that determines if S is a subtransversal of $\mathcal{H}$. In particular, if $|S| = \mathcal{O}(1)$, the complexity is $\mathcal{O}(d|E|\Delta^{|S|})$.

Theorem 2.2 (Folklore)**.** *For every Sperner hypergraph $\mathcal{H}$, the following properties are equivalent.*

1. *$\mathcal{H}$ is conformal.*
2. *$\mathcal{H}$ is the clique hypergraph of some graph.*
3. *$\mathcal{H}$ is the clique hypergraph of its co-occurrence graph.*

3 CDC graphs

The complexity of the Transversal Conformality problem, or equivalently, of the problem of recognizing dually conformal hypergraphs, is open. In this section, we focus on an interesting special case of the problem, namely when $\mathcal{H}$ is the clique hypergraph of some graph.

Definition 3.1. We say that a graph G is *clique dually conformal (CDC)* if its clique hypergraph is dually conformal.

Lemma 3.2. *For every graph G, the following two conditions are equivalent.*

1. *G is CDC.*
2. *The maximal cliques of G^c are exactly the minimal clique transversals of G.*

Example 3.3. *[To be refined.] Let G be the 5-vertex path, with vertices $1, \ldots, 5$ in order. Then $\{2, 4\}$ is a subtransversal (since there is a non-covering selection), and similarly for $\{2, 3\}$, and $\{3, 4\}$. But $\{2, 3, 4\}$ is not a subtransversal. Hence, P_5 is not CDC.*

3.1 Dually conformal pairs of Sperner hypergraphs and their supporting graphs

Given a graph G, we denote by $\mathcal{C}(G)$ its clique hypergraph, that is, the hypergraph formed by the maximal cliques of G. To every graph G we also associate another graph called the *clique-dual graph* of G, denoted by G^c and defined as the graph $G(\mathcal{H}^d)$ where $\mathcal{H} = \mathcal{C}(G)$. In words, G^c is the co-occurrence graph of the hypergraph of minimal clique transversals of G. Note that $V(G^c) = V(G)$ and two distinct vertices in $V(G)$ are adjacent in G^c if and only if they belong to a common minimal clique transversal of G. Two graphs G_1 and G_2 with the same vertex set are said to be *clique-dual to each other* if $G_2 = G_1^c$ and $G_1 = G_2^c$.

A *dually conformal pair of Sperner hypergraphs* is a pair $(\mathcal{H}_1, \mathcal{H}_2)$ of Sperner hypergraphs such that $\mathcal{H}_2 = \mathcal{H}_1^d$ and both $\mathcal{H}_1$ and $\mathcal{H}_2$ are conformal. To each dually conformal pair $(\mathcal{H}_1, \mathcal{H}_2)$ of Sperner hypergraphs we can naturally associate *a pair of supporting graphs* (G_1, G_2) such that $G_i = G(\mathcal{H}_i)$ for $i = 1, 2$.

Observation 3.4. *Let $(\mathcal{H}_1, \mathcal{H}_2)$ be a dually conformal pair of Sperner hypergraphs and let (G_1, G_2) be the corresponding pair of supporting graphs. Then G_1 and G_2 are clique-dual to each other.*

Proof. For $i \in \{1, 2\}$, since $\mathcal{H}_i$ is Sperner and conformal, we have by Theorem 2.2 that $\mathcal{H}_i$ is the clique hypergraph of its co-occurrence graph G_i. Thus, by the definition of the clique-dual graph, we infer that $G_1^c = G(\mathcal{H}_1^d) = G(\mathcal{H}_2) = G_2$. Similarly, $G_2^c = G_1$. □

Corollary 3.5. *If a graph G is clique dually conformal, then $(G^c)^c = G$.*

Proof. Let G be clique dually conformal, let $G_1 = G$, let $G_2 = G^c$, let $\mathcal{H}_1$ be the clique hypergraph of G_1, and let $\mathcal{H}_2$ be the clique hypergraph of G_2. Note that G_2 is the co-occurrence graph of $\mathcal{H}_1^d$ and for $i \in \{1, 2\}$, the graph G_i is the co-occurrence graph of $\mathcal{H}_i$. By Observation 3.4, it thus suffices to prove that $\mathcal{H}_1$ and $\mathcal{H}_2$ form a dually conformal pair of Sperner hypergraphs. By construction, $\mathcal{H}_1$ and $\mathcal{H}_2$ are both Sperner and conformal. Furthermore, by assumption on $G = G_1$, the hypergraph $\mathcal{H}_1^d$ is conformal. Since $\mathcal{H}_1^d$ is Sperner and conformal, we have by Theorem 2.2 that $\mathcal{H}_1^d$ is the clique hypergraph of its co-occurrence graph G_2. Thus, $\mathcal{H}_2 = \mathcal{H}_1^d$ and the hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ indeed form a dually conformal pair of Sperner hypergraphs. □

Question 1. *Is it true that a graph G is clique dually conformal if and only if $(G^c)^c = G$?*

NO: there is a counterexample on 9 vertices.

An *edge clique cover* of a graph G is a set of cliques of G covering all edges of G.

Lemma 3.6. *Two graphs with the same vertex set are clique-dual to each other if and only if for each of the two graphs, the family of its minimal clique transversals forms an edge clique cover of the other graph.*

Proof. Let G_1 and G_2 be two graphs with the same vertex set. Assume that G_1 and G_2 are clique-dual to each other. By symmetry, it suffices to prove that the minimal clique transversals of G_1 form an edge clique cover of G_2. We have $G_2 = G_1^c$, that is, G_2 is the co-occurrence graph of the hypergraph of minimal clique transversals of G_1. Thus, every minimal clique transversal of G_1 is a clique in G_2. Furthermore, for every edge uv of G_2 there exists a minimal clique transversal T of G_1 such that $\{u, v\} \subseteq T$. It follows that the minimal clique transversals of G_1 form an edge clique cover of G_2.

Assume now that for each of the two graphs, the family of its minimal clique transversals forms an edge clique cover of the other graph. By symmetry, it suffices to prove that $G_2 = G_1^c$. We have $V(G_1^c) = V(G_1) = V(G_2)$. Consider two distinct vertices u and v in $V(G_2) = V(G_1^c)$.

If $uv \in E(G_2)$, then there exists a minimal clique transversal T of G_1 such that $\{u, v\} \subseteq T$ and consequently $uv \in E(G_1^c)$. Thus, $E(G_2) \subseteq E(G_1^c)$. Similarly, if $uv \in E(G_1^c)$, then there exists a minimal clique transversal T of G_1 such that $\{u, v\} \subseteq T$. The fact that the family of minimal clique transversals of G_1 forms an edge clique cover of G_2 implies that T is a clique in G_2. Since $\{u, v\} \subseteq T$, we obtain that u and v are adjacent in G_2. We thus have $E(G_1^c) \subseteq E(G_2)$ and consequently $E(G_2) = E(G_1^c)$, that is, $G_2 = G_1^c$. □

3.2 General properties (?)

Closed under substitution (in particular, they generalize cographs).

P_5, C_5, the net, and the 3-sun are not CDC.

The class is not hereditary. The graph obtained from the net by adding a new vertex adjacent to the triangle, is CDC.

3.3 Split CDC graphs

In this section, we characterize CDC split graphs. A graph $G = (V, E)$ is said to be *split* if it has a *split partition*, that is, a pair (K, I) such that K is a clique, I is an independent set, $K \cap I = \emptyset$, and $K \cup I = V$.

We will use a characterization of minimal clique transversals of split graphs from [2]. Given a graph G and a set of vertices $X \subseteq V(G)$, we denote by $N_G(X)$ the set of all vertices in $V(G) \setminus X$ that have a neighbor in X. Moreover, given a vertex $v \in X$, an *X-private neighbor* of v is any vertex $w \in N_G(X)$ such that $N_G(w) \cap X = \{v\}$.

Proposition 3.7 (Milanič and Uno [2])**.** *Let G be a split graph with a split partition (K, I) such that I is a maximal independent set and let $X \subseteq V(G)$. Let $K' = K \cap X$ and $I' = I \cap X$. Then X is a minimal clique transversal of G if and only if the following conditions hold:*

(i) $K' \neq \emptyset$ if K is a maximal clique.

(ii) $I' = I \setminus N_G(K')$.

(iii) Every vertex in K' has a K'-private neighbor in I.

We first describe the structure of the clique-dual graph of a split graph G.

Lemma 3.8. *Let G be a split graph with a split partition (K, I) such that I is a maximal independent set, and let u and v be two distinct vertices of G. Then the following holds:*

(i) If $u, v \in K$, then $uv \in E(G^c)$ if and only if the sets $N_G(u) \cap I$ and $N_G(v) \cap I$ are incomparable with respect to inclusion.

(ii) If $u \in K$ and $v \in I$, then $uv \in E(G^c)$ if and only if $uv \notin E(G)$.

(iii) If $u, v \in I$ and K is a maximal clique in G, then $uv \in E(G^c)$ if and only if the sets $K \setminus N_G(u)$ and $K \setminus N_G(v)$ have a non-empty intersection.

(iv) If $u, v \in I$ and K is not a maximal clique in G, then $uv \in E(G^c)$.

Proof. Recall that u and v are adjacent in G^c if and only if they belong to a common minimal clique transversal of G. We use Proposition 3.7 to prove the four properties in order.

For claim (i), set $K' = \{u, v\}$ and $I' = I \setminus N_G(K')$. By Proposition 3.7, the set $K' \cup I'$ is a minimal clique transversal of G if and only if every vertex in K' has a K'-private neighbor in

I. This is equivalent to the condition that the sets $N_G(u) \cap I$ and $N_G(v) \cap I$ are incomparable with respect to inclusion.

Consider now claim (ii). Assume first that $uv \in E(G^c)$. Then there exists a minimal clique transversal $K' \cup I'$ of G such that $u \in K' \subseteq K$ and $v \in I' \subseteq I$. By (ii) of Proposition 3.7, we have $I' = I \setminus N_G(K')$. Hence, u and v are non-adjacent in G. Conversely, assume that $uv \notin E(G)$. Let $K' = \{u\}$ and $I' = I \setminus N_G(u)$. Then $v \in I'$. Since I is a maximal independent set in G, vertex u must have a neighbor in I, and thus properties (i)–(iii) from Proposition 3.7 hold for the sets K' and I'. It follows that $K' \cup I'$ is a minimal clique transversal of G containing u and v, and hence $uv \in E(G^c)$.

Next we show claim (iii). Assume first that $uv \in E(G^c)$. Then there exists a minimal clique transversal $K' \cup I'$ of G such that $K' \subseteq K$ and $\{u, v\} \subseteq I' \subseteq I$. By (i) from Proposition 3.7, the set K' is nonempty. Since we also have $I' = I \setminus N_G(K')$, every vertex in K' is adjacent to neither u nor v. This implies that in G, vertices u and v have a common non-neighbor in K. Conversely, assume that the sets $K \setminus N_G(u)$ and $K \setminus N_G(v)$ have a non-empty intersection. Let w be an arbitrary vertex in this intersection. Let $K' = \{w\}$ and $I' = I \setminus N_G(K')$. Then $K' \neq \emptyset$ and $\{u, v\} \subseteq I'$. Furthermore, since I is a maximal independent set in G, vertex w must have a neighbor in I, and thus properties (i)–(iii) from Proposition 3.7 hold for the sets K' and I'. Hence $K' \cup I'$ is a minimal clique transversal of G containing u and v, which implies that $uv \in E(G^c)$.

Finally, we prove claim (iv). Note that we have $I = K' \cup I'$ where $K' = \emptyset$ and $I' = I \setminus N_G(K')$. Since K is not a maximal clique, conditions (i)–(iii) from Proposition 3.7 are all satisfied, and hence I is a minimal clique transversal of G. Consequently, since $\{u, v\} \subseteq I$, we infer that $uv \in E(G^c)$. $\square$

We are now ready to characterize CDC split graphs. In order to state the characterization, we need to introduce some further notation and definitions.

Definition 3.9. Let $\mathcal{H} = (V, E)$ be a hypergraph. We say that $\mathcal{H}$ has the *Sperner-private property* (or *SP property* for short) if for every inclusion-maximal subfamily $F \subseteq E$ of hyperedges such that the hypergraph (V, F) is Sperner, there exists a collection of vertices $(v_f : f \in F)$ such that for all $f \in F$, the vertex $v_f \in V$ is an *F-private element of f*, that is, $\{e \in F : v_f \in e\} = \{f\}$.

Definition 3.10. Let G be a split graph with a split partition (K, I). We say that G:

- has the *Sperner-private (SP) property* if the hypergraph $(I, \{N_G(v) \cap I : v \in K\})$ has the SP property,
- is 2*-well-dominated* if all inclusion-minimal subsets $S \subseteq I$ such that $K \subseteq N_G(S)$ are of size two.

Theorem 3.11. *Let G be a split graph with a split partition (K, I) such that I is a maximal independent set. Then G is clique dually conformal if and only if the following two conditions hold.*

1. *G has the SP property.*
2. *If K is a maximal clique in G, then G is* 2*-well-dominated.*

Proof. Recall that by definition a graph G is clique dually conformal if its clique hypergraph is dually conformal. By the definitions of dual conformality and of the clique-dual graph G^c, this is equivalent to the condition that every maximal clique of the clique-dual graph G^c is a minimal clique transversal of G.

Assume first that every maximal clique of the clique-dual graph G^c is a minimal clique transversal of G. We first show that G has the SP property, or equivalently, that the hypergraph $\mathcal{H} = (I, \{N_G(v) \cap I : v \in K\})$ has the SP property. Let F be an inclusion-maximal family of hyperedges of $\mathcal{H}$ such that the hypergraph (I, F) is Sperner. For each $f \in F$, there exists a vertex u_f of G such that $u_f \in K$ and $f = N_G(u_f) \cap I$. Let $K_F = \{u_f : f \in F\}$. We claim that K_F is a clique in the clique-dual graph G^c. Consider an arbitrary pair of distinct vertices u and u' in K_F. Since the hypergraph (I, F) is Sperner, the sets $N_G(u) \cap I$ and $N_G(u') \cap I$ are incomparable with respect to inclusion. By claim (i) of Lemma 3.8, the vertices u and u' are adjacent in G^c. Hence, K_F is a clique in G^c. Let C be a maximal clique in G^c such that $K_F \subseteq C$. By assumption, C is a minimal clique transversal of G. Thus, writing $C = K' \cup I'$ where $K' \subseteq K$ and $I' \subseteq I$, properties (i)–(iii) from Proposition 3.7 hold for the sets K' and I'. In particular, since $K_F \subseteq K'$, property (iii) implies that for every hyperedge $f \in F$, the corresponding vertex $u_f \in K_F$ has, in the graph G, a K'-private neighbor v_f in I. By construction of the hypergraph $\mathcal{H}$, we conclude that $(v_f : f \in F)$ is a collection of vertices of $\mathcal{H}$ such that for each hyperedge $f \in F$, the vertex v_f is an F-private element of f. Thus, $\mathcal{H}$ has the SP property.

Next, we show that if K is a maximal clique in G, then G is 2-well-dominated. Assume that K is a maximal clique in G and consider an arbitrary inclusion-minimal subset $S \subseteq I$ such that $K \subseteq N_G(S)$. Since K is a maximal clique in G, the set S is of size at least two. Suppose for a contradiction that $|S| \geq 3$. We claim that S is a clique in G^c. Consider two distinct vertices $u, v \in S$. By the minimality of S, we have $K \nsubseteq N_G(u) \cup N_G(v)$, and thus by claim (iii) of Lemma 3.8, u and v are adjacent in G^c. It follows that S is a clique in G^c, as claimed. Let $C = K' \cup I'$ be a maximal clique in G^c such that $S \subseteq C$, $K' \subseteq K$, and $I' \subseteq I$. Since $K \subseteq N_G(S)$, every vertex in K is adjacent in G to a vertex in S, which by claim (ii) of Lemma 3.8 implies that every vertex in K is non-adjacent in G^c to a vertex in S. Thus, $K' = \emptyset$. Recall the assumption that every maximal clique of the clique-dual graph G^c is a minimal clique transversal of G. In particular, C is a minimal clique transversal of G. However, since K is a maximal clique of G, this contradicts the fact that $C \cap K = K' = \emptyset$. This shows that G is 2-well-dominated.

Let us now prove that the stated conditions are also sufficient for the CDC property. Assume thus that G has the SP property and, furthermore, that if K is a maximal clique in G, then G is 2-well-dominated. We need to show that every maximal clique of the clique-dual graph G^c is a minimal clique transversal of G. Let $C = K' \cup I'$ be an arbitrary maximal clique of G^c with $K' \subseteq K$ and $I' \subseteq I$. To complete the proof of our claim, we verify that properties (i)–(iii) from Proposition 3.7 hold for the sets K' and I'.

We first establish property (i). Suppose for a contradiction that K is a maximal clique in G but $K' = \emptyset$. Since $C = I'$ is a maximal clique in G^c, every vertex in K is non-adjacent in G^c with a vertex in I'. By claim (ii) of Lemma 3.8, this implies that $K \subseteq N_G(I')$. Thus, there exists an inclusion-minimal set $S \subseteq I'$ such that $K \subseteq N_G(S)$. Since K is a maximal clique in G, our assumption on G implies that G is 2-well-dominated. This means that $S = \{x, y\}$ for two distinct vertices $x, y \in I'$. However, by claim (i) of Lemma 3.8 the fact that $K \subseteq N_G(\{x, y\})$ implies that x and y are non-adjacent in G^c, contradicting the fact that I' is a clique in G^c. Thus, property (i) of Proposition 3.7 holds.

Next we establish property (ii) of Proposition 3.7. Claim (ii) of Lemma 3.8 implies that no vertex in K' is adjacent in G with a vertex in I', that is, $I' \subseteq I \setminus N_G(K')$. Suppose that the inclusion is strict. Then there exists a vertex $u \in I \setminus (I' \cup N_G(K'))$. We consider two cases depending on whether K' is empty or not. Suppose first that $K' = \emptyset$. By property (i) of Proposition 3.7, we have that K is not a maximal clique. Thus I is a clique in G^c by claim (iv) of Lemma 3.8. Since $K' = \emptyset$, we have $I' \subseteq I$, and the maximality of I' implies that $I' = I$. However, this contradicts the fact that $u \in I \setminus I'$. It remains to analyze the case when $K' \neq \emptyset$.

Note that $u \notin C$ and therefore, by the maximality of C, there exists a vertex $v \in C$ that is not adjacent to u in G^c. The choice of u implies that u is not adjacent in G to any vertex in K'. By claim (ii) of Lemma 3.8, this means that u is adjacent in G^c to every vertex in K'. In particular, the vertex v cannot belong to K' and must therefore belong to I'. Since u and v are two vertices in I that are non-adjacent in G^c, we obtain from claim (iv) of Lemma 3.8 that K is a maximal clique in G and, furthermore, by claim (iii) of Lemma 3.8, that $K \subseteq N_G(\{u, v\})$. By the assumption of this case, we have $K' \neq \emptyset$, thus there exists a vertex $w \in K'$. Since u is not adjacent in G to w, we must have $vw \in E(G)$. Consequently, by claim (ii) of Lemma 3.8, we have $vw \notin E(G^c)$, contradicting the fact that C is a clique in G^c. This shows that property (ii) of Proposition 3.7 holds.

Finally, we show that property (iii) of Proposition 3.7 holds, that is, that every vertex in K' has a K'-private neighbor in I. By claim (i) of Lemma 3.8, for every two distinct vertices u and v in K', the sets $N_G(u) \cap I$ and $N_G(v) \cap I$ are incomparable with respect to inclusion. Thus, by the SP property of G, there exists a collection of vertices $(v_x : x \in K')$ such that for all $x \in K'$, the vertex $v_x \in I$ is a K'-private neighbor of x. Thus, property (iii) of Proposition 3.7 holds.

Thus, we conclude that C is indeed a minimal clique transversal of G. □

Theorem 3.12. *Let $\mathcal{H} = (V, E)$ be a hypergraph. There exists an algorithm running in time $\mathcal{O}(|V||E|^2)$ that determines if $\mathcal{H}$ has the SP property.*

Proof. We prove the theorem by showing that the condition that $\mathcal{H}$ does not have the SP property is equivalent to the following condition: there exists a hyperedge $e \in E$ such that e is a subset of the union of hyperedges of $\mathcal{H}$ that are incomparable with e (with respect to inclusion). Let us first argue that this is enough. To verify this condition, we iterate over all hyperedges $e \in E$, and compute the union of the incomparable hyperedges. For each of the $\mathcal{O}(|E|)$ hyperedges, the above computation can be done in time $\mathcal{O}(|V||E|)$.

To see that this reformulation is equivalent with the lack of SP property, note that by definition we must have a Sperner subfamily $F \subseteq E$ and a hyperedge $f \in F$ such that f does not have an F-private element. This implies that f is a subset of the hyperedges in $F \setminus \{f\}$. Note also that all these hyperedges are incomparable with f since F is Sperner. To complete our proof, we need to show that if there exists a hyperedge $e \in E$ such that e is a subset of the union of hyperedges of $\mathcal{H}$ that are incomparable with e, then we can construct a Sperner subfamily $F \subseteq E$ containing e such that e is a subset of the union of the hyperedges in $F \setminus \{e\}$. To see this, consider all hyperedges in E that are incomparable with e, and choose a minimal subfamily that contains e as a subset. Such a minimal subfamily together with e must be Sperner. □

Corollary 3.13. *There exists an algorithm running in time $\mathcal{O}(|V|^8)$ that determines if a given graph $G = (V, E)$ is a CDC split graph.*

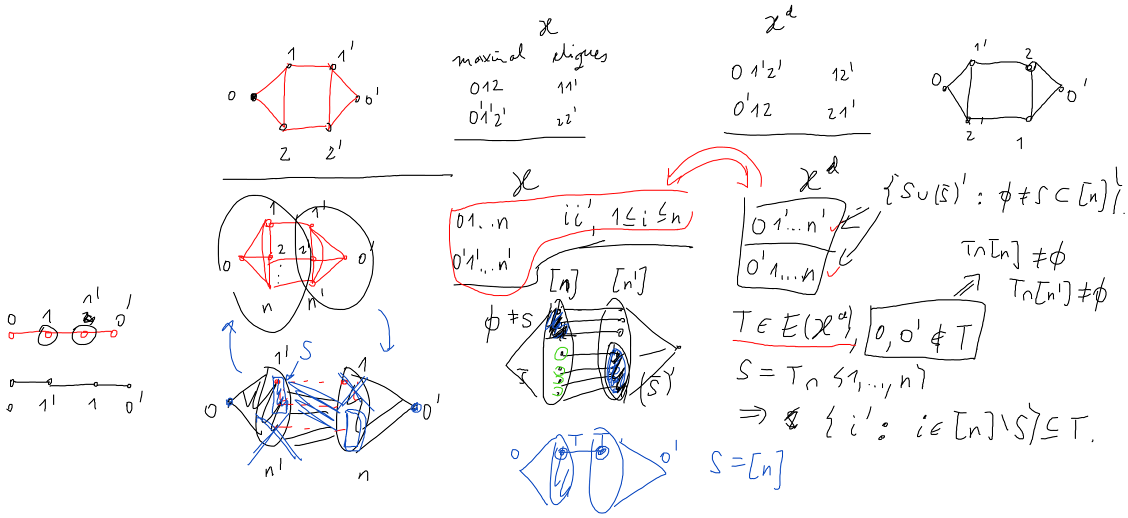
Proof. Given a graph $G = (V, E)$, we can test in time $\mathcal{O}(|V| + |E|)$ if G is split and if this is the case, compute a split partition (K, I) of G [1]. If K contains a vertex with no neighbors in I, we remove it from K and add it to I. This can also be done in linear time since the algorithm from [1] first computes the vertex degree, and K contains a vertex with no neighbors in I if and only if K contains a vertex with degree $|K| - 1$.

We may thus assume that (K, I) is a split partition of G such that I is a maximal independent set. We now apply Theorem 3.11 and test whether G has the SP property and whether it is 2-well-dominated when K is a maximal clique. To test the SP property, we first compute the hypergraph $\mathcal{H} = (I, \{N_G(v) \cap I : v \in K\})$. This can be done in time $\mathcal{O}(|K||I|) = \mathcal{O}(|V|^2)$. We have $|V(\mathcal{H})| = |I| = \mathcal{O}(|V|)$ and $|E(\mathcal{H})| \leq |K| = \mathcal{O}(|V|)$. By Theorem 3.12, we can determine in time $\mathcal{O}(|V(\mathcal{H})||E(\mathcal{H})|^2) = \mathcal{O}(|V|^3)$ if $\mathcal{H}$ has the SP property. If $\mathcal{H}$ has the SP property, then we conclude that G is not a CDC graph. If $\mathcal{H}$ does have the SP property and K is not a maximal clique in G (which we can test in linear time), then we conclude that G is a CDC graph. If $\mathcal{H}$ does have the SP property and K is a maximal clique in G, then we still need to test if G is 2-well-dominated. Note that since K is a maximal clique, every set $S \subseteq I$ such that $K \subseteq N_G(S)$ has size at least two. It thus suffices to verify that the hypergraph $\mathcal{H}$ does not contain any subtransversal of size three. For each of the $\mathcal{O}(|V|^3)$ subsets $S \subseteq I$ of size three, we apply Corollary 2.1 to verify in time $\mathcal{O}(|V(\mathcal{H})||E(\mathcal{H})|^4) = \mathcal{O}(|V|^5)$ if S is a subtransversal of $\mathcal{H}$. If no such set is a subtransversal of $\mathcal{H}$, then G is 2-well-dominated, and we conclude that G is a CDC graph. Otherwise, we conclude that G is not a CDC graph. The total time complexity of the algorithm is $\mathcal{O}(|V|^8)$. $\square$

3.4 Infinite families of CDC graphs

The 6-vertex examples generalize to infinite families.

One of them is as follows:



We can generalize the "fork" (the graph that looks like letter "E") to an infinite family by we taking a subdivided star with all branches of length two, except one, which is of length one. That is, for an integer $n \geq 2$ we consider the graph G with vertex set $\{v_0, v_1, \ldots, v_n\} \cup \{v'_0, v'_1, \ldots, v'_n\}$ and edge set $\{v_0v'_0\} \cup \{v_0v_i : 1 \leq i \leq n\} \cup \{v_iv'_i : 1 \leq i \leq n\}$. The dual of the clique hypergraph $\mathcal{C}(G)$ has the following hyperedges: the set $\{v'_0, v_1, \ldots, v_n\}$ and all sets of the form $\{v_0\} \cup \{v_j : j \in S\} \cup \{v'_j : j \in \{1, \ldots, n\} \setminus S\}$ for all subsets S of $\{1, \ldots, n\}$. This hypergraph is conformal, since it consists of precisely the maximal cliques of the graph with vertex set $V(G)$ in which vertex v_0 has a unique non-neighbor v'_0, vertex v'_0 is simplicial, with neighborhood $\{v_1, \ldots, v_n\}$, and the subgraph induced by $V(G) \setminus \{v_0, v'_0\}$ is a complete graph minus a perfect matching $\{v_jv'_j : 1 \leq j \leq n\}$.

Another family is as follows.

Fix an integer $n \geq 1$ and consider the graph G with vertex set $\{v_1, \ldots, v_{2n}\}$, in which two distinct vertices v_i and v_j are adjacent if and only if $|i - j| < n$. The maximal cliques of G are precisely the sets $C_j = \{v_i : j \leq i \leq j + n - 1\}$ where $j \in \{1, \ldots, n+1\}$. Note that each maximal clique of G has size n. Let S be a minimal transversal of the maximal cliques of G. Then S contains a vertex from C_1, that is, a vertex v_i with $1 \leq i \leq n$. Let v_i be the vertex in $S \cap \{v_1, \ldots, v_n\}$ with the largest index. Similarly, S contains a vertex from C_{n+1}, that is, a vertex v_j with $n+1 \leq j \leq 2n$. Let v_j be the vertex in $S \cap \{v_{n+1}, \ldots, v_{2n}\}$ with the smallest index. The minimality of S implies that $S \cap \{v_1, \ldots, v_n\} = \{v_i\}$ since if $v \in (S \cap \{v_1, \ldots, v_n\}) \setminus \{v_i\}$, then the set $S \setminus \{v\}$ is also a clique transversal of G. Similarly, $S \cap \{v_{n+1}, \ldots, v_{2n}\} = \{v_j\}$. Thus, every minimal transversal of the maximal cliques of G contains exactly one vertex from $\{v_1, \ldots, v_n\}$ and exactly one vertex from $\{v_{n+1}, \ldots, v_{2n}\}$. It follows that the co-occurrence graph is bipartite and hence, conformality holds. A more precise description of the clique transversal hypergraph of G (and thus of its co-occurrence graph) can also be easily obtained: a set $S = \{v_i, v_j\}$ with $v_i \in \{v_1, \ldots, v_n\}$ and $v_j \in \{v_{n+1}, \ldots, v_{2n}\}$ is a minimal clique transversal of G if and only if $|j - i| \leq n$.

References

[1] P. L. Hammer and B. Simeone. The splittance of a graph. *Combinatorica*, 1(3):275–284, 1981.

[2] M. Milanič and Y. Uno. Upper clique transversals of graphs. 2021. Manuscript.