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A keynote talk to ICMETE2020
"Zaitsev Neighborhood for Cellular Automata"

A Generalized Neighborhood for Cellular Automata

Dmitry Zaitsev

http://member.acm.org/~daze

Basic references

- Zaitsev D.A. A generalized neighborhood for cellular automata, Theoretical Computer Science, 666 (2017), 21-35.
- Zaitsev D.A. hm, hn, hmn - Generators of canvas for Petri net models of hypertorus (hypercube) grid with Moore's, von-Neumann's, and generalized neighborhoods, respectively, 2015, https://github.com/dazeorgacm/hmn
- Zaitsev D.A. htgen - Generator of hypertorus Petri net models, 2015, https://github.com/dazeorgacm/htgen
- Zaitsev D.A. ts - Torus Simulator: simulator of traffic within multidimensional torus interconnect, 2020, https://github.com/dazeorgacm/ts
- Zaitsev, D.A., Tymchenko, S.I., Shtefan, N.Z. Switching vs Routing within Multidimensional Torus Interconnect, PIC&ST2020, October 6-9, 2020, Kharkiv, Ukraine, http://picst.org

Neighborhood in hypercube (hypertorus) for microelectronics and telecommunications

- Connection of nodes in square (multidimensional) lattice
- Network-on-chip, 2D, 3D
- Interconnect for supercomputers and clusters, 3D, 5D, 6D

Torus interconnect

- Three-dimensional torus network: IBM Blue Gene/L and Blue Gene/P, and the Cray XT3
- Five-dimensional torus network: IBM Blue Gene/Q
- Six-dimensional torus network: Fujitsu K computer, PRIMEHPC FX10 – three-dimensional torus 3D mesh interconnect Tofu
- Fugaku, ~0.5 exaflops – TOFU interconnect D

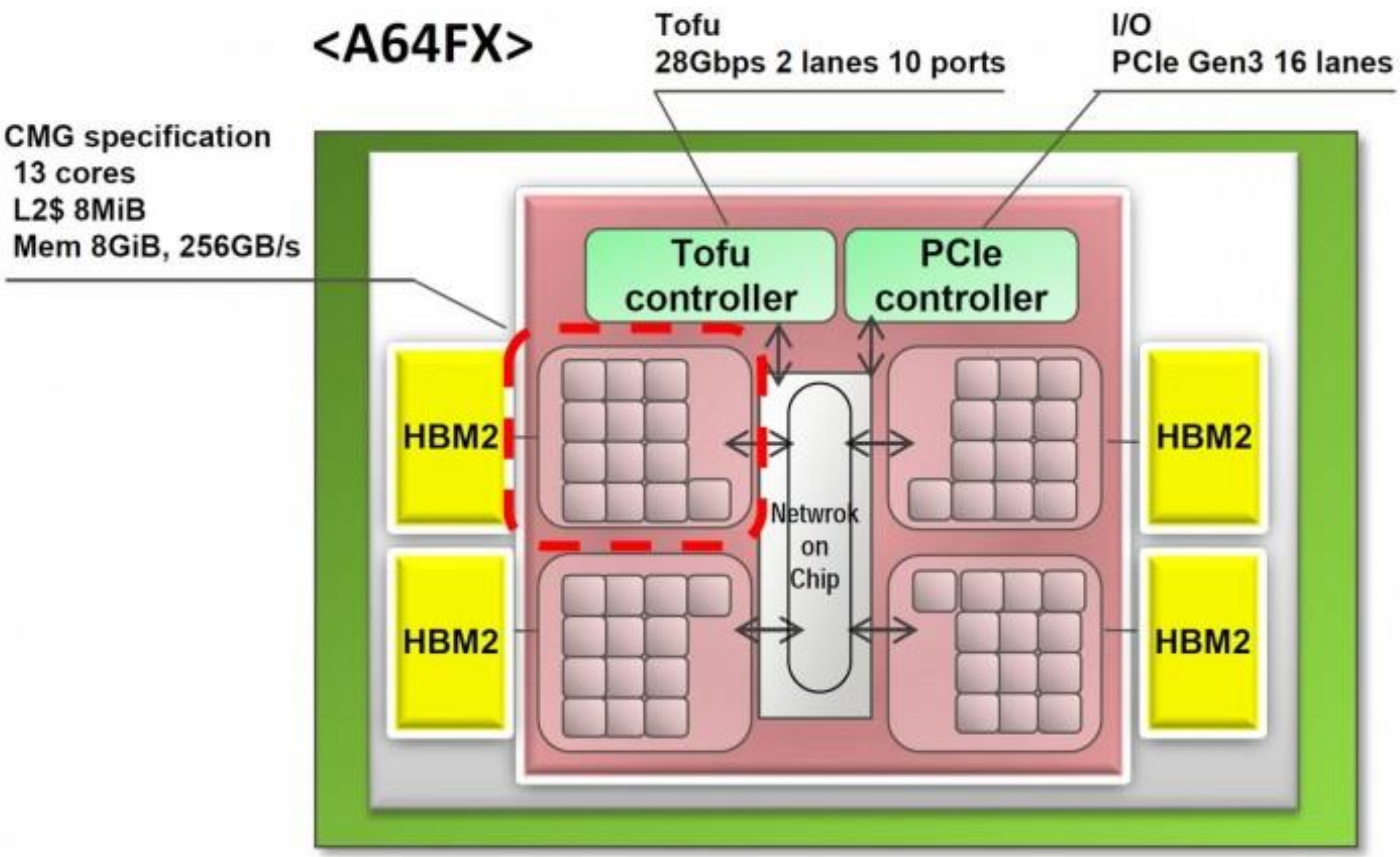
Fugaku (Fujitsu, RIKEN)

A64FX

A64FX® Microarchitecture Manual

http://github.com/fujitsu/A64FX

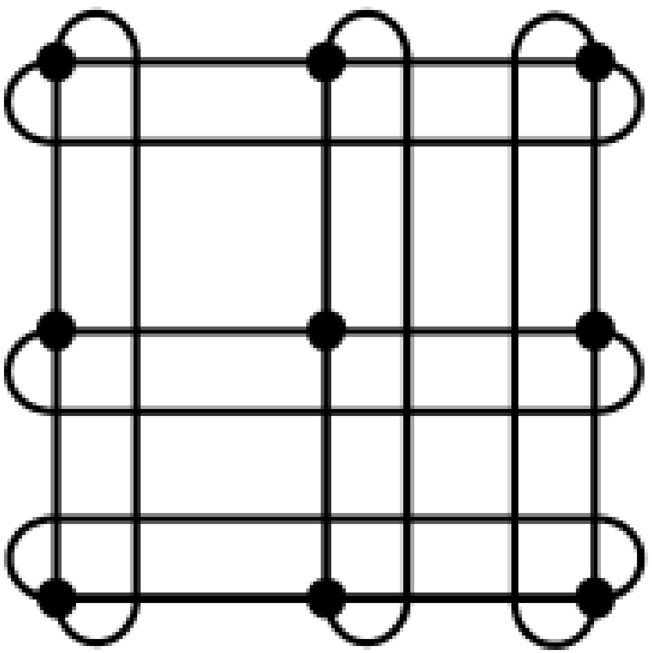
A64FX block diagram



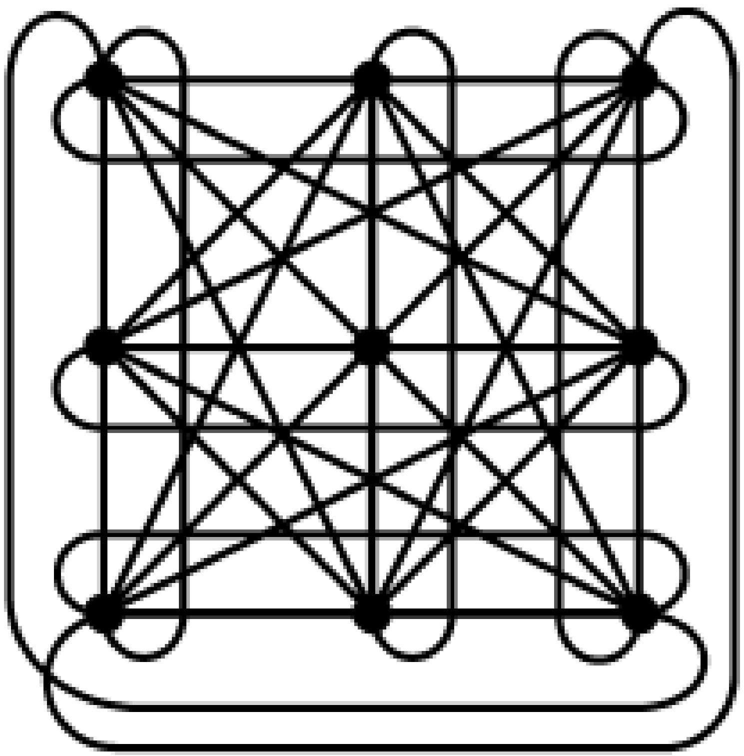
Neighborhood of torus interconnect

- Von Neumann neighborhood
- Mixed mesh and von Neumann interconnect
- Moore neighborhood is too dense in multidimensional space
- Generalized neighborhood is flexible, density is adjusted using a parameter
- Cross-By-Pass-Torus can by implemented as a generalized neighborhood with radius > 1

Neighborhoods in 2D torus



Von Neumann

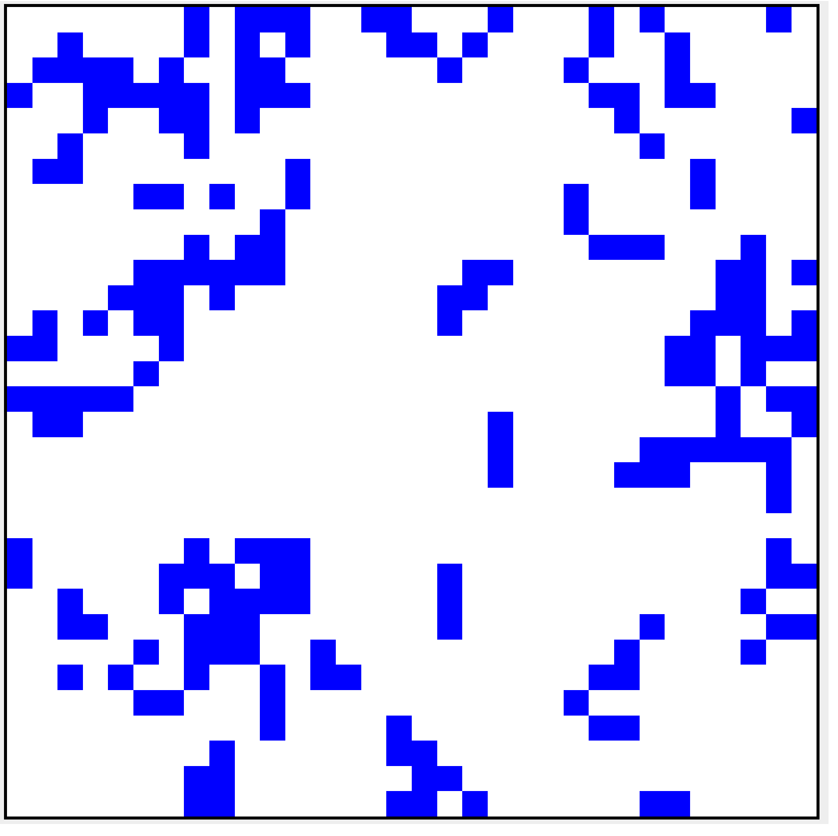


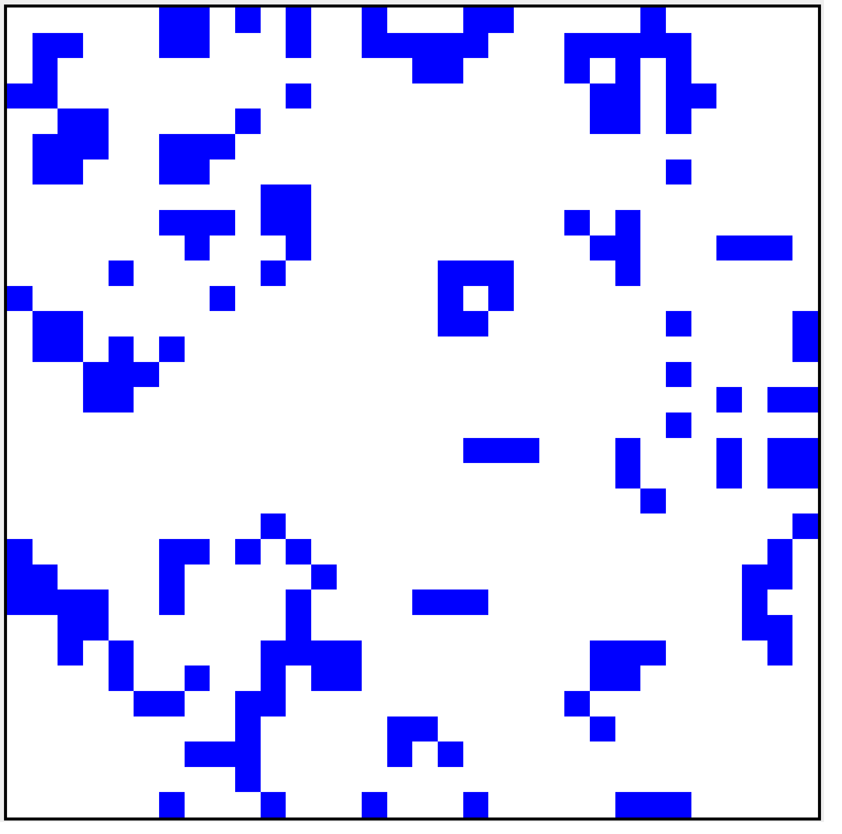
Moore

Cellular Automata

- Lattice of cells in d-dimensional space
- Cell state in the next tact depends on the states of its neighbors in the current tact
- 1D, elementary ({0,1}), rule 110 – Turing-complete
- 2D – Game of Life
- 3D – Simulation of Flows of Water
- Synchronous, Asynchronous, Totalistic, Stochastic etc

2D CA simulation example





FiatLux simulator, Nazim Fatès

http://fiatlux.loria.fr

Applications of Cellular Automata

- Network Routing
- Cracks in metal constructions
- Spreading of viruses
- Spreading of insects (mosquito)
- Modelling climate change
- Cloud dynamics simulation
- Artificial brain dynamics

Conventional neighborhoods for cellular automata

- Von Neumann – Manhattan distance of 1
- Moore – Chebyshev distance of 1
- Using radius r
- Diamond shaped

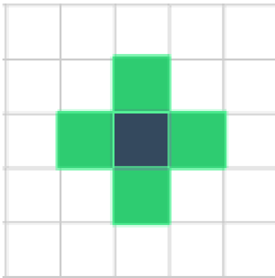
Chebyshev distance – maximal distance on coordinates: $L^\infty(\vec{i'}, \vec{i}) = \max_j \left(|\vec{i'} - \vec{i}|\right)$

Manhattan distance – sum of distances on coordinates (taxicab): $L^1(\vec{i'}, \vec{i}) = \sum_j \left(|\vec{i'} - \vec{i}|\right)$

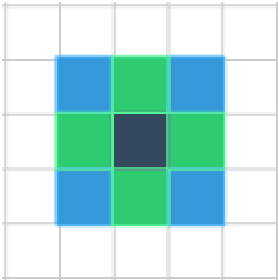
Generalized k-neighborhood in d-dimensional lattice

- Sharp k-neighborhood – exactly k coordinates change, their differences belong to {-1,1} (Chebyshev distance of 1 restricted by Manhattan distance of k)
- k-neighborhood – no more than k coordinates change, their differences belong to {-1,1}
- Parameter k: $1 \leq k \leq d$
- 1-neighborhood = von Neumann
- d-neighborhood = Moore

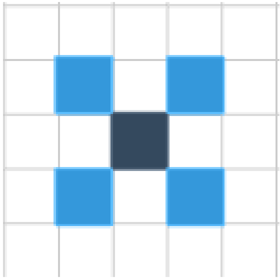
2D case



1-neighborhood,
sharp 1-neighborhood,
von Neumann neighborhood
4 neighbors



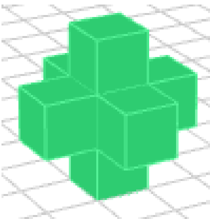
2-neighborhood,
Moore neighborhood
8 neighbors



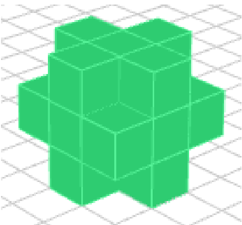
sharp 2-neighborhood
4 neighbors

Neighborhoods on slides are drawn in Voxelbuilder

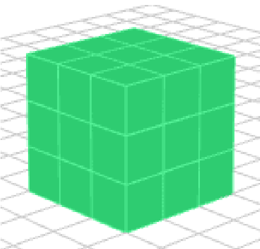
3D case



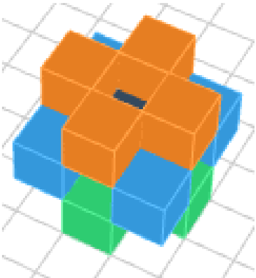
1-neighborhood,
sharp 1-neighborhood,
von Neumann neighborhood
6 neighbors



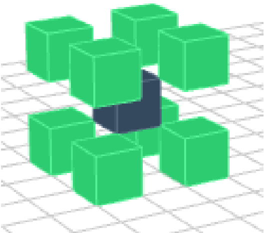
2-neihgborhood
18 neighbors



3-neighborhood,
Moore neighborhood
26 neighbors



sharp 2-neighborhood,
12 neighbors



sharp 3-neighborhood
8 neighbors

4D case

1-neiborhood von Neumann	2-neiborhood	3-neiborhood	4-neiborhood Moore
sharp 1	sharp 1 + sharp 2	sharp 1 + sharp 2 + sharp 3 = 2-neighborhood + sharp 3	sharp 1 + sharp 2 + sharp 3 + sharp 4 = 3-neighborhood + sharp 4
Sharp neighborhoods			
Sharp 1	Sharp 2	Sharp 3	Sharp 4
(-1,0,0,0), (1,0,0,0), (0,-1,0,0), (0,1,0,0), (0,0,-1,0), (0,0,1,0), (0,0,0,-1), (0,0,0,1) 8 neighbors	(-1,-1,0,0), (-1,1,0,0), (1,-1,0,0), (1,1,0,0), (-1,0,-1,0), (-1,0,1,0), (1,0,-1,0), (1,0,1,0), (-1,0,0,-1), (-1,0,0,1), (1,0,0,-1), (1,0,0,1), (0,-1,-1,0), (0,-1,1,0), (0,1,-1,0), (0,1,1,0), (0,-1,0,-1), (0,-1,0,1), (0,1,0,-1), (0,1,0,1), (0,0,-1,-1), (0,0,-1,1), (0,0,1,-1), (0,0,1,1) 24 neighbors	(-1,-1,-1,0),(-1,-1,1,0), (-1,1,-1,0), (-1,1,1,0), (1,-1,-1,0), (1,-1,1,0), (1,1,-1,0), (1,1,1,0), (-1,-1,0,-1),(-1,-1,0,1), (-1,1,0,-1), (-1,1,0,1), (1,-1,0,-1), (1,-1,0,1), (1,1,0,-1), (1,1,0,1), (-1,0,-1,-1),(-1,0,-1,1), (-1,0,1,-1), (-1,0,1,1), (1,0,-1,-1), (1,0,-1,1), (1,0,1,-1), (1,0,1,1), (0,-1,-1,-1),(0,-1,-1,1), (0,-1,1,-1), (0,-1,1,1), (0,1,-1,-1), (0,1,-1,1), (0,1,1,-1), (0,1,1,1) 24 neighbors	(-1,-1,-1,-1), (-1,-1,-1,1), (-1,-1,1,-1), (-1,-1,1,1), (-1,1,-1,-1), (-1,1,-1,1), (-1,1,1,-1), (-1,1,1,1), (1,-1,-1,-1), (1,-1,-1,1), (1,-1,1,-1), (1,-1,1,1), (1,1,-1,-1), (1,1,-1,1), (1,1,1,-1), (1,1,1,1) 16 neighbors

Tasks solved

The number of neighbors in k-neighborhood for d-dimensional lattice counted for:

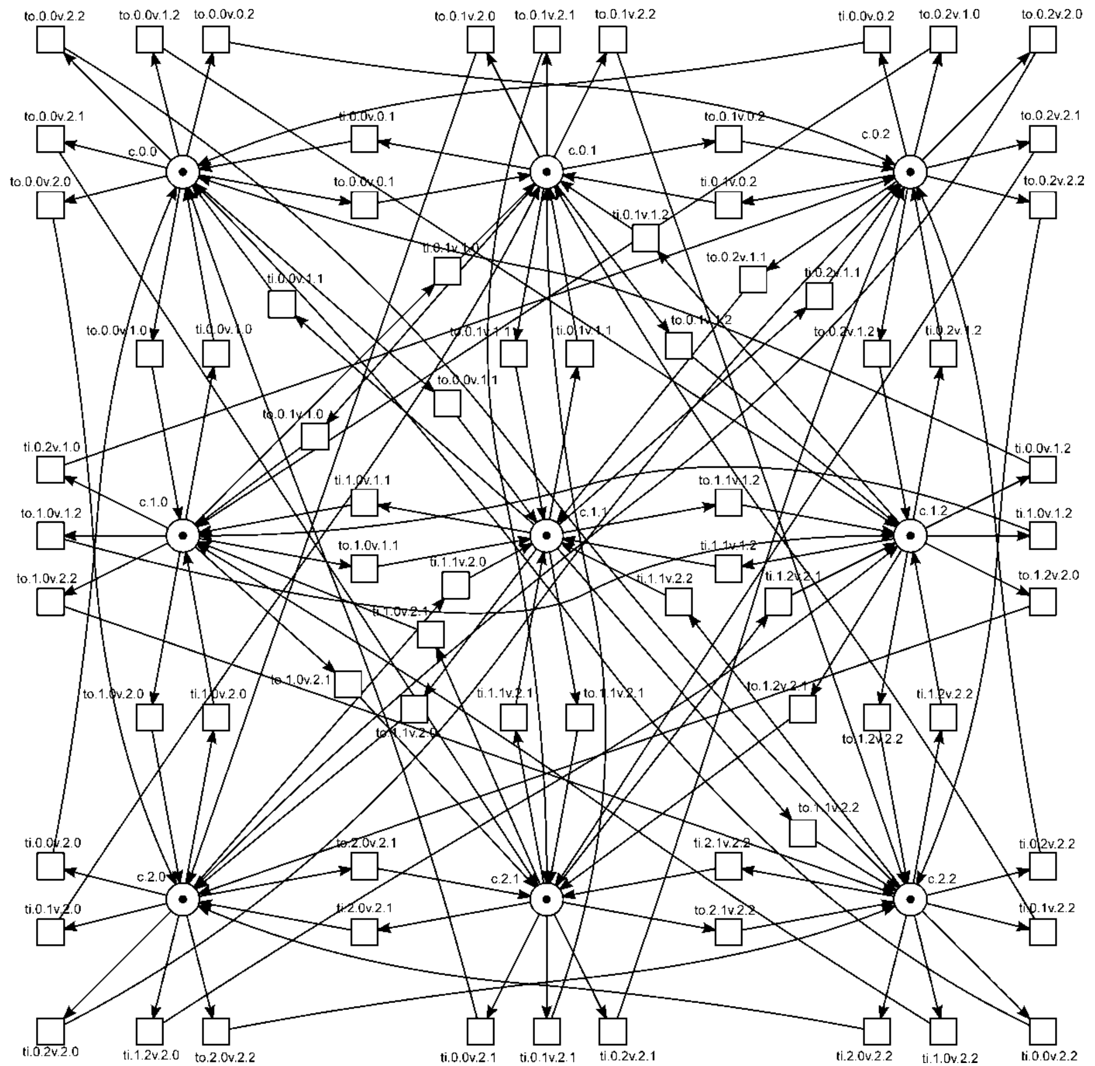
- Infinite lattice
- Finite hypercubes and hypertoruses
- k-neighborhood of radius r

Connection with oeis.org

- The On-Line Encyclopedia of Integer Sequences
- OEIS A005843 – von Neumann
- OEIS A024023 – Moore
- OEIS A013609 – sharp k-neigborhood
- New: OEIS A265014 – k-neigborhood
- New: OEIS A266213 – sharp, radius r, diamond shaped
- OEIS A008288 – Delannoy numbers – partial sums of a new OEIS A266213

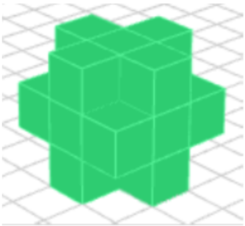
Generator of canvas for finite grids

- https://github.com/dazeorgacm/hmn

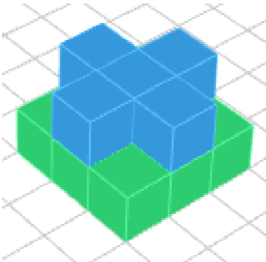


k-neighborhood in finite d-cubes

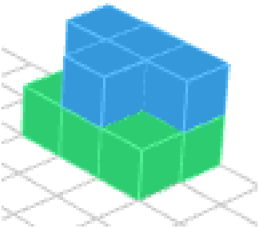
An example of 2-neighborhood in a 3-cube



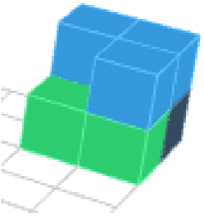
inside 3-cube
18 neighbors



on 2-cube bound,
(a square)
13 neighbors



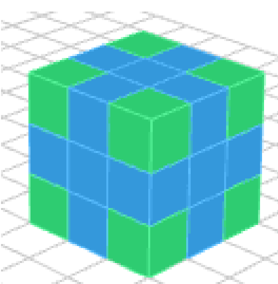
on 1-cube bound,
(a segment)
9 neighbors



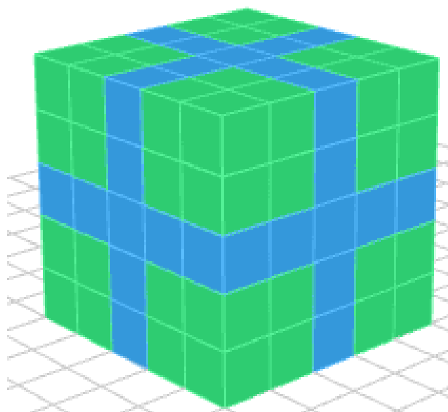
on 0-cube bound,
(a point)
6 neighbors

Neighborhoods of radius r

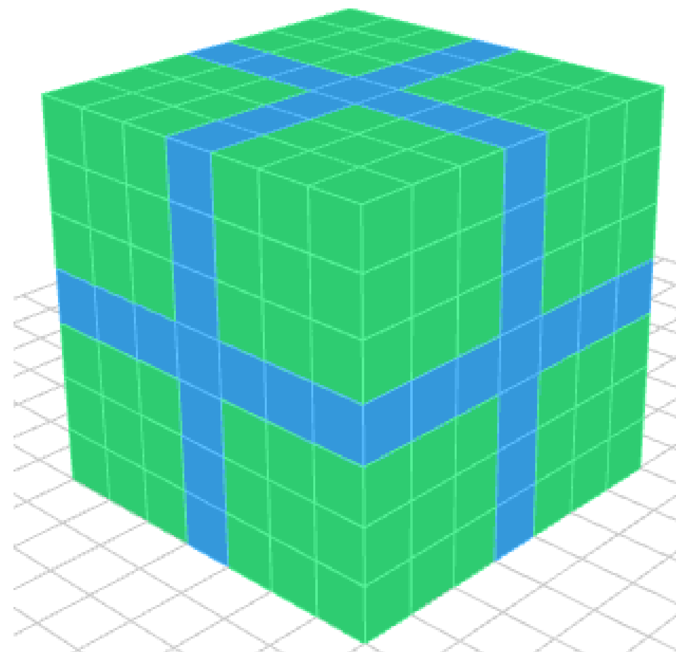
Moore's – for k=d,
an example for d=3



r=1
26 neighbors



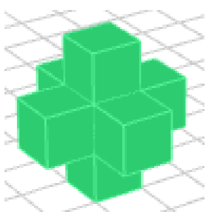
r=2
124 neighbors



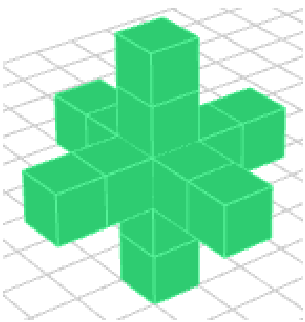
r=3
342 neighbors

Neighborhoods of radius r

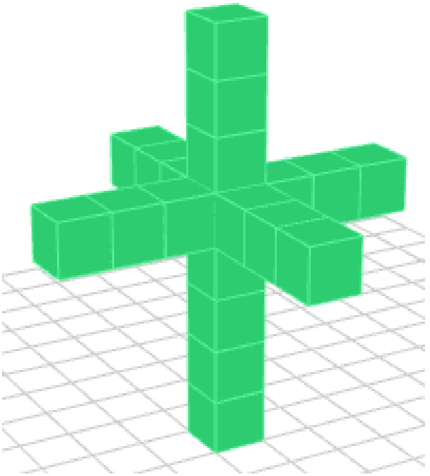
Narrow von Neumann's – for k=1,
an example for d=3



r=1
6 neighbors



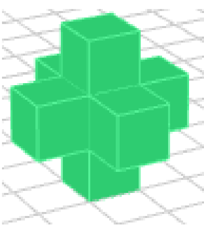
r=2
12 neighbors



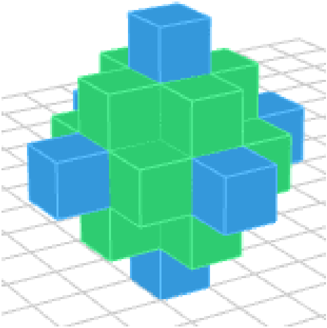
r=3
18 neighbors

Neighborhoods of radius r

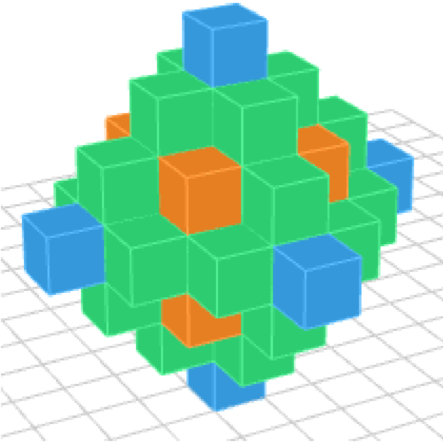
Diamond-shaped von Neumann's – Manhattan distance r, an example for d=3



r=1
6 neighbors



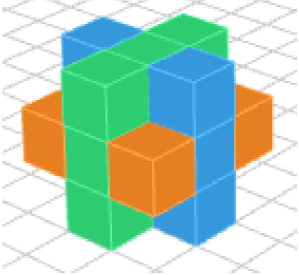
r=2
24 neighbors



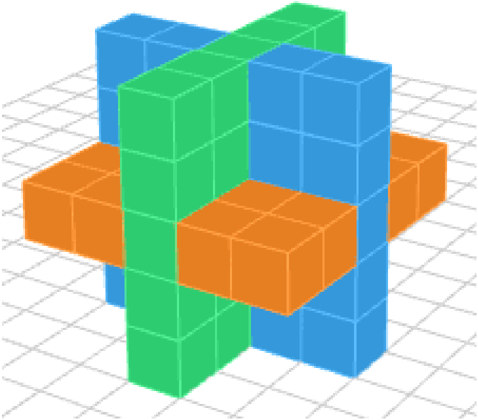
r=3
56 neighbors

k-neighborhoods of radius r

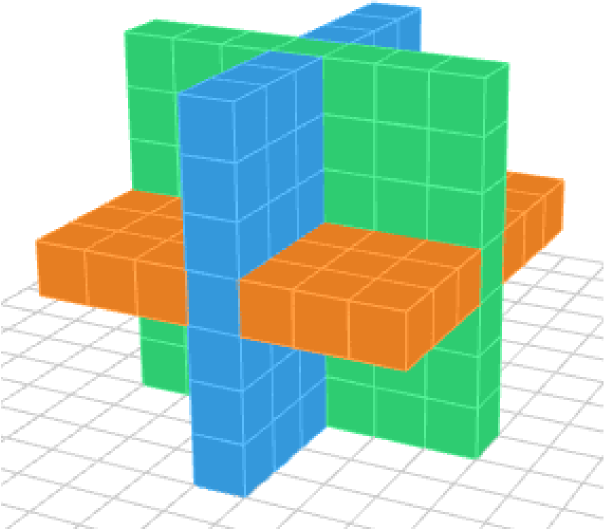
An example of 2-neighborhood in 3-dimensional lattice



r=1
18 neighbors



r=2
60 neighbors

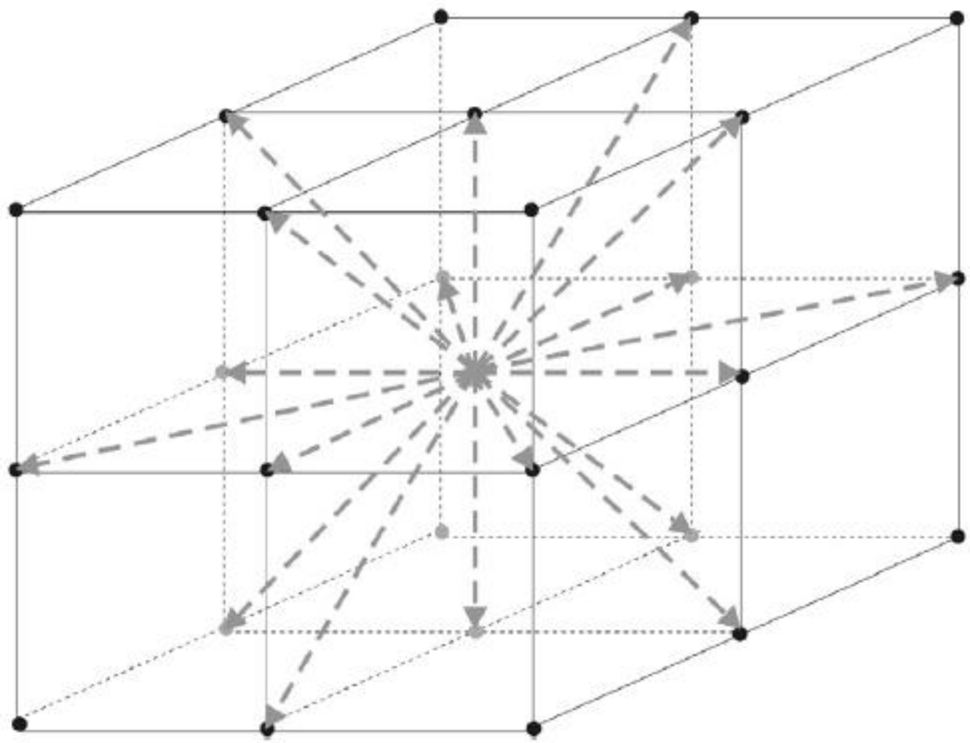


r=3
126 neighbors

List of derived formulae

- $K(d,k)$ – number of neighbors in a k-neighborhood
- $M(d,k,m)$ – number of neighbors in a k-neighborhood for a cell situated at a (d - m)-cube bound
- $T(d,k,n)$ – the number of connections for a k-neighborhood in a d-dimensional lattice of size n
- $R_{diamond}(d, r)$ – number of neighbors in a diamond-shaped neighborhood of radius r
- $R(d, k, r)$ – number of neighbors in a k-neighborhood of radius r

2-neighborhood in 3-dimensional case



$(0,0,-1),\ (0,0,1),\ (0,-1,0),\ (0,1,0),\ (-1,0,0),\ (1,0,0),$

$(0,-1,-1),\ (0,-1,1),\ (0,1,-1),\ (0,1,1),\ (-1,0,-1),\ (-1,0,1),$

$(1,0,-1),\ (1,0,1),\ (-1,-1,0),\ (-1,1,0),\ (1,-1,0),\ (1,1,0)$

Number of neighbors in *k*-neighborhood

$$\hat{K}(d,k) = |S(d,k)| = 2^k C_d^k$$

Known OEIS A013609

$$K(d,k) = |G(d,k)| = \sum_{j=1}^{k} \hat{K}(d,j) = \sum_{j=1}^{k} 2^j C_d^j$$

$$K(d,k) = K(d,k-1) + K(d-1,k) + K(d-1,k-1) - 2K(d-1,k-2),$$

$$K(d,k) = 2K(d-1,k-1) + K(d-1,k) + 2.$$

A new OEIS sequence OEIS A265014

Number of Neighbors

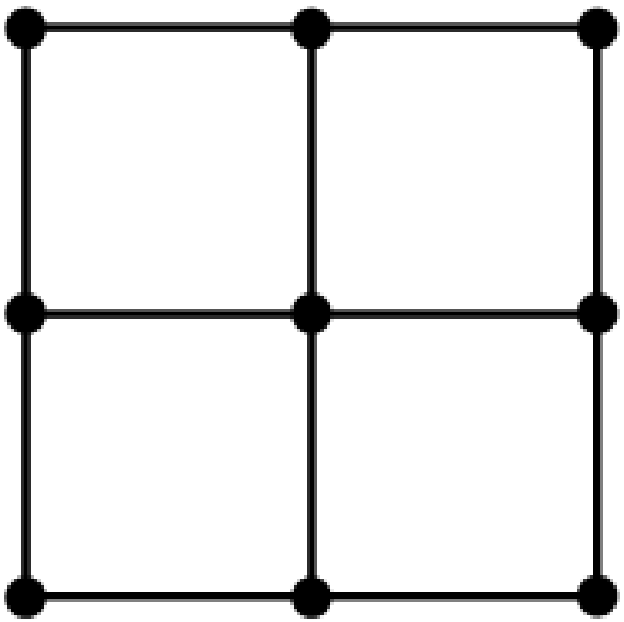
Table 1
The number of $(d-k)$-cube bounds in a d-hypercube – the number of neighbors in a sharp k-neighborhood.

d \ k	1	2	3	4	5	6	7	8	...
1	2								
2	4	4							
3	6	12	8						
4	8	24	32	16					
5	10	40	80	80	32				
6	12	60	160	240	192	64			
7	14	84	280	560	672	448	128		
8	16	112	448	1120	1792	1792	1024	256	
...	...	...	...	...	...	...	...	...	...

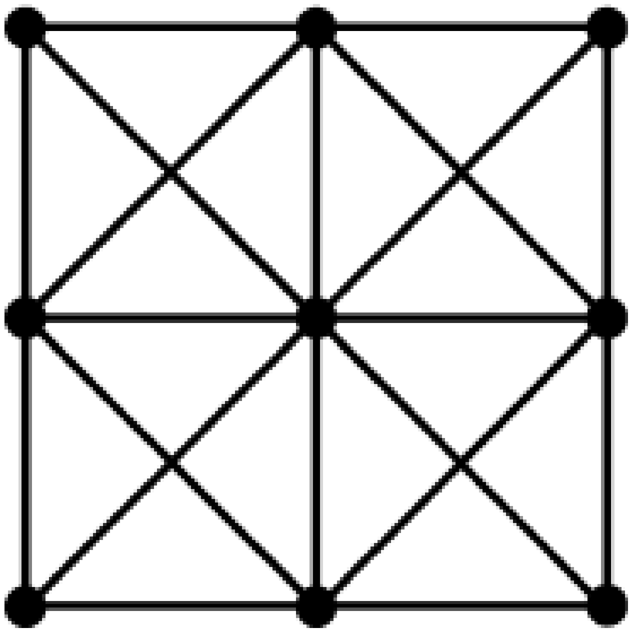
Table 2
The number of neighbors in a generalized k-neighborhood, from von Neumann's $(k=1)$ to Moore's $(k=d)$.

d \ k	1	2	3	4	5	6	7	8	...
1	2								
2	4	8							
3	6	18	26						
4	8	32	64	80					
5	10	50	130	210	242				
6	12	72	232	472	664	728			
7	14	98	378	938	1610	2058	2186		
8	16	128	576	1696	3488	5280	6304	6560	
...	...	...	...	...	...	...	...	...	...

Finite hypercube

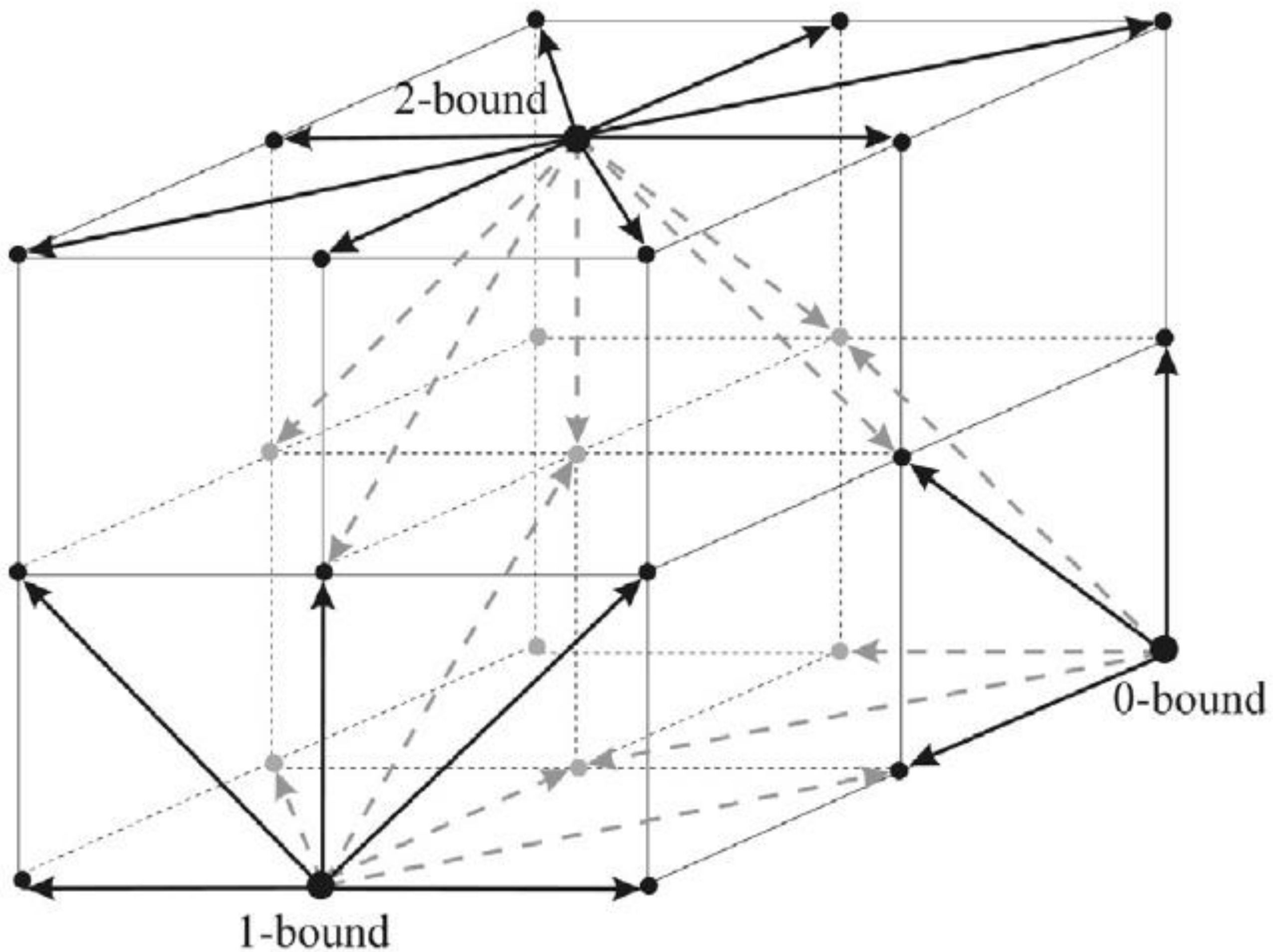


Von Neumann



Moore

2-neighborhood in 3-dimensional case



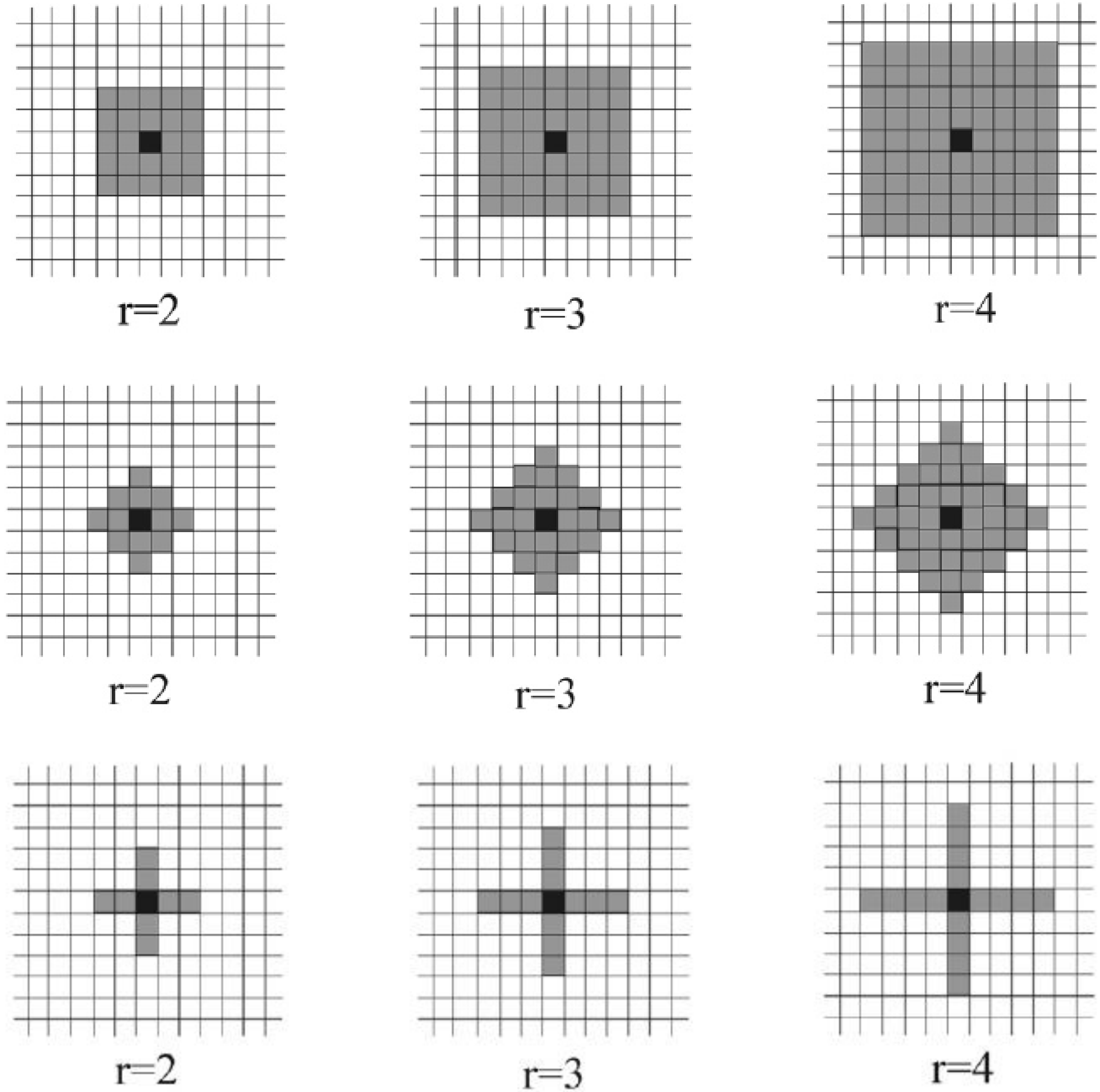
Number of neighbors and connections

$$\hat{M}(d,k,m) = \sum_{\substack{k_1+k_2=k \\ k_1 \le m \\ k_2 \le d-m}} C_m^{k_1} C_{d-m}^{k_2} 2^{k_2}$$

$$M(d,k,m) = \sum_{j=1}^{k} \hat{M}(d,j,m) = \sum_{j=1}^{k} \sum_{j_1=\max(0,m+j-d)}^{\min(j,m)} C_m^{j_1} C_{d-m}^{j-j_1} 2^{j-j_1}$$

$$T(d,k,n) = \frac{\sum_{m=0}^{d} C_d^m 2^m (n-2)^{d-m} M(d,k,m)}{2}$$

Radius



Decomposition of Delannoy numbers

$$\hat{R}_{diamond}(d,r) = \sum_{k=1}^{\min(d,r)} C_{r-1}^{k-1} C_d^k 2^k$$

A new OEIS sequence OEIS A266213

$$R_{diamond}(d,r) = \sum_{k=1}^{\min(d,r)} C_r^k C_d^k 2^k$$

$$D(d,r) = R_{diamond}(d,r) + 1$$

Known OEIS A008288

Number of neighbors in a generalized k-neighborhood of radius r

$$R(d, k, r) = \sum_{j=1}^{k} C_d^j (2r)^j$$

$$R(3, 2, 2) = C_3^1 4^1 + C_3^2 4^2 = 3 \cdot 4 + 3 \cdot 16 = 60$$

$(-2, -2, 0)$, $(-2, 2, 0)$, $(2, -2, 0)$, $(2, 2, 0)$, $(-2, 0, -2)$, $(-2, 0, 2)$,
$(2, 0, -2)$, $(2, 0, 2)$, $(0, -2, -2)$, $(0, -2, 2)$, $(0, 2, -2)$, $(0, 2, 2)$,
$(-2, -1, 0)$, $(-2, 1, 0)$, $(2, -1, 0)$, $(2, 1, 0)$, $(-2, 0, -1)$, $(-2, 0, 1)$,
$(2, 0, -1)$, $(2, 0, 1)$, $(0, -2, -1)$, $(0, -2, 1)$, $(0, 2, -1)$, $(0, 2, 1)$,
$(-1, -2, 0)$, $(-1, 2, 0)$, $(1, -2, 0)$, $(1, 2, 0)$, $(-1, 0, -2)$, $(-1, 0, 2)$,
$(1, 0, -2)$, $(1, 0, 2)$, $(0, -1, -2)$, $(0, -1, 2)$, $(0, 1, -2)$, $(0, 1, 2)$.

Conclusions

- k-neighborhood introduced in d-dimensional lattice
- its extreme cases give von Neumann’s (sparsest) and Moore’s (densest) neighborhoods
- number of neighbors counted for infinite lattice and finite hypercubes and hypertoruses
- number of neighbors counted for k-neighborhoods of radius r
- two new sequences A265014 and A266213 accepted by OEIS; partials sums of A266213 produce Delannoy numbers

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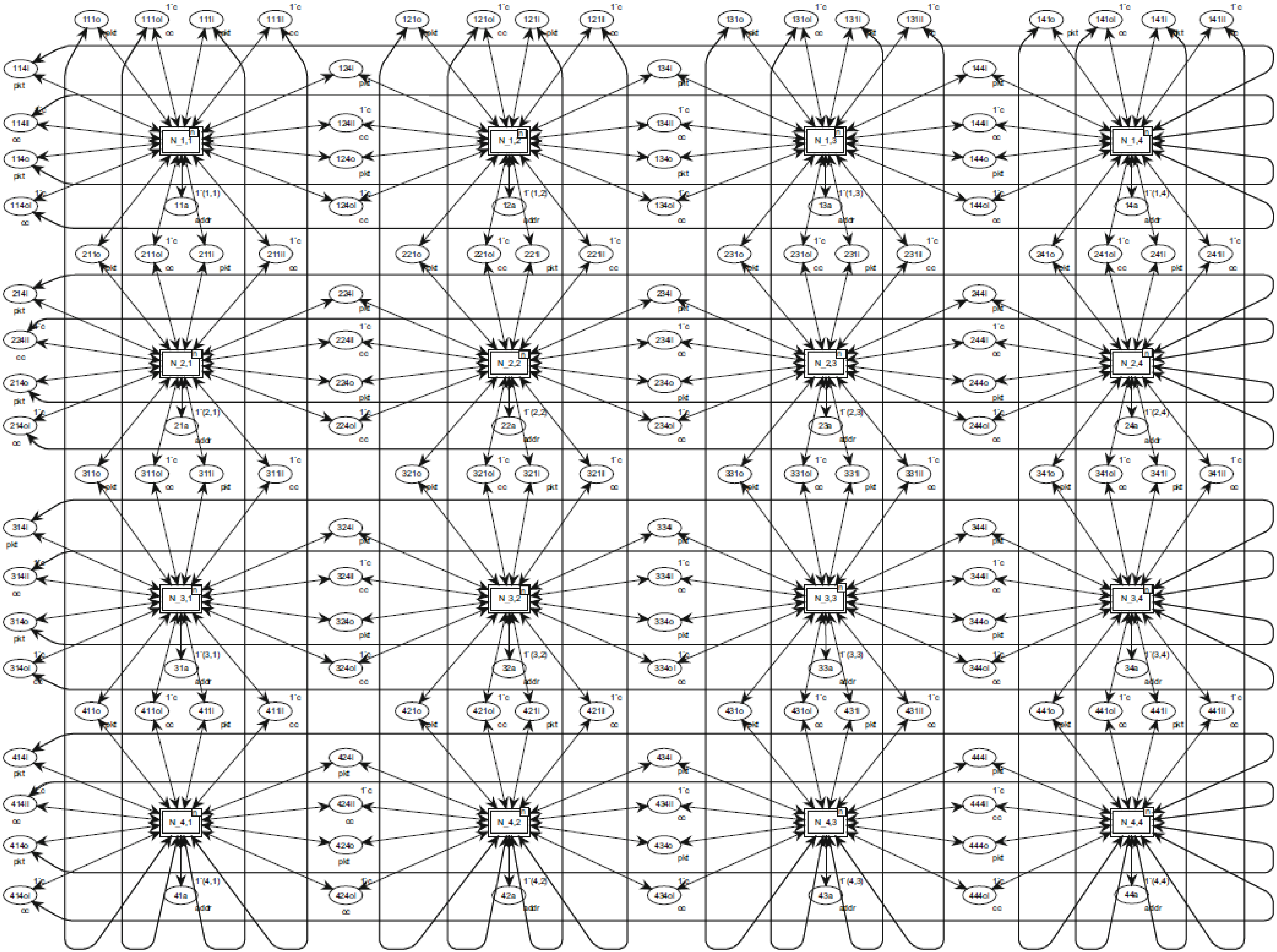
Future research directions

- Modify and adjust ts
- Register new OEIS sequences
- Routing via CA
- Combine hypertorus, hypercube with meshes
- Use 2 parameters: (k,k')-neighborhood
- Use reverse definition of k'-neigborhood decreasing from Moore neighborhood
- Specify sequences using hypergeometric

http://member.acm.org/~daze


N_1,1
N_1,2
N_1,3
N_1,4
N_2,1
N_2,2
N_2,3
N_2,4
N_3,1
N_3,2
N_3,3
N_3,4
N_4,1
N_4,2
N_4,3
N_4,4