Scientific Research

Solving Linear Systems via Composition of their Clans

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Abstract: The decomposition technique for linear system solution is proposed. This technique may be combined with any known method of linear system solution. An essential acceleration of computations is obtained. Methods of linear system solution depend on the set of numbers used. For integer and especially for natural numbers all the known methods are hard enough. In this case the decomposition technique described allows the exponential acceleration of computations.

Keywords: linear system, decomposition, clans, acceleration of computation

1. Introduction

The task of linear system solution is a classical task of linear algebra [10]. There are a variety of known methods for linear system solution. It should be noted that the choice of the method depends essentially on the set of numbers used. More precise it depends on the algebraic structure of the variables and coefficients. The solution of system in fields and bodies, for example, in rational or real numbers is the most studied. These algebraic structures allow the everywhere defined operation of division. It is convenient to develop the simple and powerful methods. Precise and approximate methods are distinguished. The most popular is Gauss method consisting in consequent elimination of variables and obtaining the triangular form of the matrix.

Ring algebraic structures such as integer numbers require special methods, as the division is not the everywhere-defined operation in a ring. There is known the universal method using unimodular transformations of matrix to obtain Smith normal form [10]. Result matrix has diagonal form and allow the simple representation of solutions. Unfortunately this method is exponential. Up to present time were suggested a few more complex but polynomial methods [3,7,8]. The most interesting is founded on consequent solution of system in the classes' fields of deductions modulo of prime numbers to construct a target general integer solution of the system [3].

The development of such areas of computer science as Petri net theory [6,13], logic programming [5], artificial intellect [1] require to solve integer systems over the set of nonnegative integer numbers. Notice that nonnegative integer numbers form algebraic structure of monoid. In monoid even the operation of subtraction is not everywhere defined. All the known methods of integer system

solution in nonnegative integer numbers are exponential [2,4,9,12]. It makes significant difficulties in the application of these methods to real-life systems analysis.

For example, if the complexity of method is about 2^q, where q is the dimension of the system, then to solve the system with dimension 100 it is required about 10^{30} operations. A computer with the performance 10^9 operations per second will have executed this job in 10^{21} seconds or in about 10^{13} years.

Therefore, the task of the effective methods development for linear system solution, especially in nonnegative numbers, is enough significant. The goal of present paper is to introduce the decomposition technique of linear system solution. Application of this technique allows an essential acceleration of computations. In the case the exponential complexity of the source method of system solution the acceleration is exponential too.

2. Basic Concepts

Let us consider a homogeneous linear system

$$A \cdot \overline{x} = 0 \qquad (1)$$

where A is the matrix of coefficients with dimension of $m \times n$ and $\overline{x}$ is the vector-column of variables with dimension of n. So we have the system of m equations with n unknowns. We do not define now more precisely the sets of variables and coefficients. We suppose only that there is a method to solve a system (1) and to represent the general solution in the form

$$\overline{x} = \overline{y} \cdot G \qquad (2)$$

where matrix G is the matrix of basis solutions and $\overline{y}$

is the vector-row of independent variables. Each row of matrix G is a basis solution. To generate a particular solution of system the values of components of vector $\bar{y}$ may be chosen arbitrarily with respect to the set of values of variables $\bar{x}$. It should be noted that system (1) has always at least the trivial zero solution. For the brevity we shall name further homogeneous system an inconsistent in the case it has only the trivial solution.

Let us consider a nonhomogeneous system

$$A \cdot \bar{x} = \bar{b} \tag{3}$$

where $\bar{b}$ is the vector-column of free elements with dimension of m. We suppose also that there is a method to solve the system (3) and to represent the general solution in the form

$$\bar{x} = \bar{x}' + \bar{y} \cdot G \tag{4}$$

where $\bar{y} \cdot G$ is the general solution of corresponding homogenous system (1) and $\bar{x}'$ is the minimal particular solution of the nonhomogeneous system (3).

According to classical algebra [10], results represented above are true for arbitrary fields and rings. Moreover, according to [4], they are true also for monoid solutions with a ring structure of matrix A and vector $\bar{b}$. In this case the set of minimal solutions $\bar{x}'$ of nonhomogeneous system should be used.

3. Decomposition of System

Let us decompose the system (1) of homogeneous equations

$$A \cdot \bar{x} = 0$$

Above system may be considered as the predicate

$$L(\bar{x}) = L_1(\bar{x}) \wedge L_2(\bar{x}) \wedge ... \wedge L_m(\bar{x}) \tag{5}$$

where $L_i(\bar{x})$ are the equations of the system:

$$L_i(\bar{x}) = (\bar{a}^i \cdot \bar{x} = 0)$$

and $\bar{a}^i$ is the i-th row of the matrix A. We imply that $\bar{a}^i$ is a nonzero vector so at least one component of $\bar{a}^i$ is nonzero. Moreover, we shall denote as X the set of variables of system.

Let us consider the set of equations $\Im = \{L_i\}$ of system L. We shall introduce relations over the set $\Im$.

Definition 3.1. *Near relation:* two equations $L_i, L_j \in \Im$ are *near* and denoted as $L_i \, \mathbf{o} \, L_j$ if and only if $\exists x_k \in X$: $a_{i,k}, a_{j,k} \neq 0$, $sign(a_{i,k}) = sign(a_{j,k})$.

Lemma 3.1. Near relation is reflexive and symmetric.
Proof.

a) Reflexivity: $L_i \, \mathbf{o} \, L_i$. As $\bar{a}^i$ is a nonzero vector so there exists $x_k \in X$: $a_{i,k} \neq 0$. Then it is trivial that

$sign(a_{i,k}) = sign(a_{i,k})$.

b) Symmetry: $L_i \, \mathbf{o} \, L_j \Rightarrow L_j \, \mathbf{o} \, L_i$. It is symmetric according to symmetry of equality sign: $sign(a_{i,k}) = sign(a_{j,k}) \Rightarrow sign(a_{j,k}) = sign(a_{i,k})$.

For each equation chosen we may construct the set of near equations. It is useful to consider the transitive closure of near relation. We introduce the special notation for this transitive closure.

Definition 3.2. *Clan relation:* two equations $L_i, L_j \in \Im$ *belong to the same clan* and are denoted as $L_i \, \mathbf{O} \, L_j$ if and only if there exists a sequence (may be empty) of equations $L_{l_1}, L_{l_2}, ..., L_{l_k}$ such that: $L_i \, \mathbf{o} \, L_{l_1} \, \mathbf{o} ... \mathbf{o} \, L_{l_k} \, \mathbf{o} \, L_j$.

Theorem 3.1. Clan relation is the equivalence relation.

Proof. We have to prove that relation above is reflexive, symmetric and transitive. We shall implement this in a constructive way.

a) Reflexivity: $L_i \, \mathbf{O} \, L_i$. As Definition 3.2 allows the empty sequence and near relation is reflexive according to Lemma 3.1, so clan relation is reflexive.

b) Symmetry: $L_i \, \mathbf{O} \, L_j \Rightarrow L_j \, \mathbf{O} \, L_i$. As $L_i \, \mathbf{O} \, L_j$, so, according to Definition 3.2, we may construct the sequence $L_{l_1}, L_{l_2}, ..., L_{l_k}$ such as $L_i \, \mathbf{o} \, L_{l_1} \, \mathbf{o} ... \mathbf{o} \, L_{l_k} \, \mathbf{o} \, L_j$. Since near relation is symmetry according to Lemma 3.1, we may construct the reversed sequence $L_{l_k}, L_{l_{k-1}}, ..., L_{l_1}$ such as $L_j \, \mathbf{o} \, L_{l_k} \, \mathbf{o} ... \mathbf{o} \, L_{l_1} \, \mathbf{o} \, L_i$, then $L_j \, \mathbf{O} \, L_i$.

c) Transitivity: $L_i \, \mathbf{O} \, L_j, L_j \, \mathbf{O} \, L_l \Rightarrow L_i \, \mathbf{O} \, L_l$. As $L_i \, \mathbf{O} \, L_j$, $L_j \, \mathbf{O} \, L_l$, so, according to Definition 3.2, we may construct two sequences $s = L_{i_1}, L_{i_2}, ..., L_{i_k}$ and $V = L_{l_1}, L_{l_2}, ..., L_{l_r}$ such that $L_i \, \mathbf{o} \, L_{i_1} \, \mathbf{o} ... \mathbf{o} \, L_{i_k} \, \mathbf{o} \, L_j$ and $L_j \, \mathbf{o} \, L_{l_1} \, \mathbf{o} ... \mathbf{o} \, L_{l_r} \, \mathbf{o} \, L_l$. Let us consider the concatenation $s L_j V$. This sequence connects the elements L_i and L_l with the chain of elements satisfying the near relation. Thus $L_i \, \mathbf{O} \, L_l$.

Corollary. Clan relation defines the partition of the set $\Im$: $\Im = \bigcup_j C^j$, $C^i \bigcap C^j = \varnothing$, $i \neq j$.

Definition 3.3. *Clan:* element of the partition $\{\Im, \mathbf{O}\}$ will be named a *clan* and denoted C^j. The variables $X^j = X(C^j) = \{x_i \, | \, x_i \in X, \exists L_k \in C^j : a_{k,i} \neq 0\}$ will be named *variables of clan* C^j.

Definition 3.4. *Internal variable:* variable $x_i \in X(C^j)$ is an *internal variable of the clan* C^j if and only if for

all the other clans C^l, $l \neq j$ it is true $x_i \notin X^l$. The set of internal variables of clan C^j we shall denote as $\overset{\jmath}{X}{}^j$.

Definition 3.5. *Contact variable:* variable $x_i \in X$ is a *contact variable* if and only if clans C^j and C^l exist such as $x_i \in X^j$, $x_i \in X^l$. The set of all the contact variables we shall denote as X^0. We shall also denote the set of contact variables of clan C^j as $\overset{\backprime}{X}{}^j$ so $X^j = \overset{\jmath}{X}{}^j \mathbf{U} \overset{\backprime}{X}{}^j$ and $\overset{\jmath}{X}{}^j \mathbf{I} \overset{\backprime}{X}{}^j = \varnothing$.

Lemma 3.3. An arbitrary contact variable $x_i \in X^0$ cannot appear in different clans with the same sign.

Proof. We shall assume the contrary. Let variable $x_i^* \in X$ appears in different clans $C^{j_1}, C^{j_2}, ..., C^{j_k}$ with the same sign, where $k > 1$. Thus, according to Definition 3.3, equations $L_{l_1} \in C^{j_1}$, $L_{l_2} \in C^{j_2}$, ... , $L_{l_k} \in C^{j_k}$ exist such as $a_{l_1,i} \neq 0$, $a_{l_2,i} \neq 0$, ..., $a_{l_k,i} \neq 0$ and $sign(a_{l_1,i}) = sign(a_{l_2,i}) = ... = sign(a_{i_k,i})$. Then, according to Definitions 3.2 and 3.3, equations $L_{l_1}, L_{l_2}, ..., L_{l_k}$ belong to the same clan. So we have obtained the contradiction.

Lemma 3.4. An arbitrary contact variable $x_i \in X^0$ belongs to two clans exactly.

Proof. We shall implement the proof from the contrary. Let us assume that a contact variable $x_i \in X$ belongs to q different clans and $q \neq 2$. We shall consider separately two cases:

a) $q < 2$. Than, according to Definition 3.5, variable x_i is not a contact variable.

b) $q > 2$ We have contradiction with Lemma 3.3 so there are only two signs: plus and minus.

Corollary. An arbitrary contact variable $x_i \in X^0$ is contained in different clans with opposite signs.

Definition 3.6. Clan C^j will be named an *input clan of the contact variable* $x_i \in X^0$ and denoted as $I(x_i)$ if and only if it contains this variable with the sign plus.

Definition 3.7. Clan C^j will be named an *output clan of the contact variable* $x_i \in X^0$ and denoted as $O(x_i)$ if and only if it contains this variable with the sign minus.

It should be noted that analogous classification into input and output might be introduced for the contact variables of the clan.

Thus we have obtained on the one hand the partition of equations into clans and on the other hand the partition of variables into internal and contact. Let us make a new enumeration of equations and variables. The enumeration of equations we start from the equations of the first clan and so on up to the last clan. The enumeration of variables we start from the contact variables and than continue with internal variables of clans in ascending order. Then we order sets X and $\mathfrak{I}$ according to the new enumeration.

As a result we obtain a block form of matrix A:

$$A = \begin{Vmatrix} A^{0,1} & \overset{\backprime}{A}{}^1 & 0 & 0 & 0 \\ A^{0,2} & 0 & \overset{\backprime}{A}{}^2 & 0 & 0 \\ \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{M} \\ A^{0,k} & 0 & 0 & 0 & \overset{\backprime}{A}{}^k \end{Vmatrix}$$

With respect to clans and subsets of variables it may be more convenient to represent the matrix in Table 1.

Let us consider more detailed the structure of matrixes for the contact variables. According to Definition 3.5, $\overset{\backprime}{X}{}^j$ denotes the contact variables of the clan C^j. Then we have $X^0 = \mathbf{U}_j \overset{\backprime}{X}{}^j$ but this is not a partition of the set X^0 as each contact variable $x_i \in X^0$, according to Lemma 3.4, belongs to two clans exactly. Further we shall use matrixes $\overset{\backprime}{A}{}^j$ also. Notice that unlike of $A^{0,j}$ that contains values for all contact places X^0 the matrix $\overset{\backprime}{A}{}^j$ contains values only for contact variables $\overset{\backprime}{X}{}^j$ of the clan C^j. In other words the matrix $\overset{\backprime}{A}{}^j$ contains only nonzero columns of the matrix $A^{0,j}$.

As a result of Lemma 3.4 application to matrix A we conclude that each contact variable $x_i \in X^0$ appears in

Table 1. **The block structure of the system matrix**

Clan/Variables	X^0	$\overset{\backprime}{X}{}^1$	$\overset{\backprime}{X}{}^2$	$\ldots$	$\overset{\backprime}{X}{}^k$
C^1	$A^{0,1}$	$\overset{\backprime}{A}{}^1$	0	$\ldots$	0
C^2	$A^{0,2}$	0	$\overset{\backprime}{A}{}^2$	$\ldots$	0
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
C^k	$A^{0,k}$	0	0	$\ldots$	$\overset{\backprime}{A}{}^k$

two of matrixes $A^{0,j}$, $j = \overline{1,k}$ exactly and besides, it appears in one matrix with positive coefficients and in another matrix with negative coefficients.

4. Solution of Decomposed System

At first we shall solve the system separately for each clan. If we consider only variables of the clan we have the system

$$A^j \cdot \overline{x}^j = 0 \qquad (6)$$

where

$$A^j = \left\| \overset{(}{A}{}^j \quad \overset{)}{A}{}^j \right\|, \quad \overline{x}^j = \left\| \overset{\mathbf{(}}{\overline{x}}{}^j \atop \overset{\mathbf{)}}{\overline{x}}{}^j \right\|.$$

System (6) will be denoted also as $L^{C^j}(\overline{x})$. Notice that values of variables $X \setminus X^j$ may be chosen arbitrarily. More detailed

$$L^{C^j}(\overline{x}) = \underset{L_l \in C^j}{\textstyle\&} L_l(\overline{x})$$

Theorem 4.1. If system (1) has a nontrivial solution then each system (6) has a nontrivial solution.

Proof. Let us consider the representation (5) of the system (1). As, according to Theorem 3.1, clan relation defines a partition of the set of equations, so representation (5) may be written in the form

$$L(\overline{x}) = L^{C_1}(\overline{x}) \wedge L^{C_2}(\overline{x}) \wedge \ldots \wedge L^{C_k}(\overline{x}) \qquad (7)$$

Thus, an arbitrary solution of (1) is the solution of each system (6).

Corollary. If at least one of systems (6) is inconsistent then the entire system (1) is inconsistent.

Let us the general solution of the system (6), according to (2), has the form

$$\overline{x}^j = \overline{y}^j \cdot G^j$$

As each $x_i \in \overset{(}{X}{}^j$ is contained exactly in one system (6), so for all the internal variables of clans we have

$$\overset{\mathbf{)}}{\overline{x}}{}^j = \overline{y}^j \cdot \overset{\mathbf{)}}{G}{}^j$$

Each contact variable, according to Lemma 3.4, belongs to two systems exactly. Thus, its values have to be equal

$$\overline{x}_i^j = \overline{x}_i^l \text{ or } \overline{y}^j \cdot G_i^j = \overline{y}^l \cdot G_i^l ,$$

where G_i^j denotes the column of matrix G^j corresponding to variable x_i.

Therefore, we obtain the system

$$\begin{cases} \overline{x}^j = \overline{y}^j \cdot G^j, \quad j = \overline{1,k}, \\ \overline{y}^j \cdot G_i^j = \overline{y}^l \cdot G_i^l, \quad x_i \in X^0, \quad C^j = O(x_i), \quad C^l = I(x_i). \end{cases} \qquad (8)$$

Theorem 4.2. System (8) is equivalent to the system (1).

Proof. As Expression (7) is the equivalent representation of the source system (1) and there is a partition of the set of variables X into contact and internal, so above reasoning proves the theorem.

Let us consider more detailed the equations of system (8) for the contact variables. Equation

$$\overline{y}^j \cdot G_i^j = \overline{y}^l \cdot G_i^l$$

may be represented in a block form as

$$\left\| \overline{y}^j \quad \overline{y}^l \right\| \cdot \left\| \begin{matrix} G_i^j \\ -G_i^l \end{matrix} \right\| = 0$$

We enumerate all the variables $\overline{y}^j$ in such a manner to obtain the joint vector

$$\overline{y} = \left\| \overline{y}^1 \quad \overline{y}^2 \quad \ldots \quad \overline{y}^k \right\|$$

and assemble G_i^j, G_i^l into the joint matrix F. Thus, we obtain the system

$$\overline{y} \cdot F = 0 \quad \text{or} \quad F^T \cdot \overline{y}^T = 0 \qquad .(9)$$

System obtained has the form (1) so the general solution of system has the form (2):

$$\overline{y} = \overline{z} \cdot R \qquad (10)$$

Let us compose the joint matrix G of systems' (6) solutions for all the clans in such a manner that

$$\overline{x} = \overline{y} \cdot G \qquad (11)$$

This matrix has a block structure as follows

$$G = \left\| \begin{matrix} S^1 & \overset{\mathbf{)}}{G}{}^1 & 0 & 0 & 0 \\ S^2 & 0 & \overset{\mathbf{)}}{G}{}^2 & 0 & 0 \\ \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{M} \\ S^k & 0 & 0 & 0 & \overset{\mathbf{)}}{G}{}^k \end{matrix} \right\|$$

The structure of the first column of above block representation that corresponds to contact variables is complex enough. For each contact variable $x_i \in X^0$ we create the column so that either block S^j or S^l contain nonzero elements according to $\overset{\mathbf{)}}{G}{}^j$ or $\overset{\mathbf{)}}{G}{}^l$ where $C^j = O(x_i)$, $C^l = I(x_i)$. Really, as a contact variable belong to two clans, so its value may be calculated either according to general solution of input clan or according to general solution of output clan.

Let us substitute (10) into (11):

$$\overline{x} = \overline{z} \cdot R \cdot G$$

Therefore

$$\overline{x} = \overline{z} \cdot H, \quad H = R \cdot G, \qquad (12)$$

Theorem 4.3. Expressions (12) represent the general

solution of homogeneous system (1).

Proof. As only equivalent transformations were used, so above reasoning proves the theorem.

Let us implement analogous transformations for a nonhomogeneous system (3). The general solution for each clan, according to (4), has the form

$$\overline{x}^j = \overline{x}'^j + \overline{y}^j \cdot G^j$$

Equations for contact variables may be represented as follows

$$\overline{x}_i'^j + \overline{y}^j \cdot G_i^j = \overline{x}_i'^l + \overline{y}^l \cdot G_i^l$$

and further

$$\overline{y}^j \cdot G_i^j - \overline{y}^l \cdot G_i^l = \overline{b}_i', \ \ \overline{b}_i' = \overline{x}_i'^l - \overline{x}_i'^j$$

or in the joint matrix form

$$\overline{y} \cdot F = \overline{b}' \ \ \text{or} \ \ F^T \cdot \overline{y}^T = \overline{b}'^T .$$

The general solution of above system, according to (4), may be represented as

$$\overline{y} = \overline{y}' + \overline{z} \cdot R .$$

Using the joint matrix G we may write

$$\overline{x} = \overline{x}' + \overline{y} \cdot G$$

or

$$\overline{x} = \overline{x}' + (\overline{y}' + \overline{z} \cdot R) \cdot G = \overline{x}' + \overline{y}' \cdot G + \overline{z} \cdot R \cdot G$$

and finally

$$\overline{x} = \overline{y}'' + \overline{z} \cdot H , \ \ \overline{y}'' = \overline{x}' + \overline{y}' \cdot G , \ \ H = R \cdot G . \quad (13)$$

Theorem 4.4. Expressions (13) represent the general solution of nonhomogeneous system (2).

Proof. As only equivalent transformations were used, so above reasoning proves the theorem.

5. Description of Algorithm

Solution of linear homogeneous system of Equations (1) with decomposition technique consists in:

Stage 1. Decompose system (1) into set of clans: $\{C^j\}$.

Stage 2. Solve the system (6): $\overline{x}^j = \overline{y}^j \cdot G^j$ for each clan C^j . If at least one of systems (6) is inconsistent, then the entire system (1) is inconsistent. Stop.

Stage 3. Construct and solve system (9) for contact variables: $\overline{y} = \overline{z} \cdot R$. If this system is inconsistent, then the entire system (1) is inconsistent. Stop.

Stage 4. Compose matrix G and calculate matrix H of basis solutions: $H = R \cdot G$. Stop.

It should be noted that Stages 2, 3 use a known method of linear system solution. The choice of this method is defined by the sets of numbers used. For example, this is Gauss method for rational numbers, unimodular transformations for integer numbers and Toudic method for nonnegative integer solutions. Moreover, analogous technique, according to Theorem 4.4, may be implemented for nonhomogeneous system.

It has to describe more precisely now the decomposition algorithm used at Stage 1. Let us q is a maximum dimension of matrix A : $q = \max(m, n)$. Clan relation may be calculated in a standard way, as transitive closure of near relation but this way is hard enough from the computational point of view.

We propose the following algorithm of decomposition (Figure 1) with a total complexity about q^3 :

```
j := 1;
while  ℑ ≠ ∅
do
      C^j := ∅;
      curC := L, (L ∈ ℑ);
      ℑ := ℑ \ L;
      do
      newC := ∅;
      for  L'  in  curC
      for  L''  in  ℑ
      if  L' ∘ L''  then do  ℑ := ℑ \ L'';   newC := newC ∪ L'';  od;
      C^j := C^j ∪ curC;
      curC := newC;
      od until  newC = ∅
      j := j + 1;
  od;
```

Figure 1. Algorithm of system decomposition

It creates clans C^j out of the source set of equations $\Im$ of the system. To create the transitive closure the algorithm compares each equation of the set $curC$ with each equation of $\Im$. Equations included into current clan are extracted from the set $\Im$. The usage of two auxiliary subsets $curC$ and $newC$ allow the once comparison of each pair of equations. As the calculation of near relation is linear, so the total complexity of the algorithm is about q^3.

Let us $M(q)$ is the time complexity of solution for linear system of size q. We shall estimate the total complexity of linear system solution with decomposition technique. Let us the source system (1) is decomposed in k clans and p is the maximal number of either contact or clans' internal variables. Thus $q \le (k+1) \cdot p$. The following expression estimates the complexity of system solution with decomposition technique:

$$V(q) \le ((k+1) \cdot p)^3 + k \cdot M(p) + M(p) + ((k+1) \cdot p)^3.$$

Each of four addends of this expression estimates the complexity of the corresponding stage. In more simplified form it is represented as

$$V(q) \approx 2 \cdot (k+1)^3 \cdot p^3 + (k+1) \cdot M(p) \approx k^3 \cdot p^3 + k \cdot M(p).$$

Let us estimate the acceleration of computations under the decomposition technique usage. The target expression is

$$Acc(q) = \frac{M(q)}{k^3 \cdot p^3 + k \cdot M(p)}$$

It is enough clear that even for polynomial methods of system solution with a degree higher than 3 we obtain acceleration greater than one. Now we estimate the acceleration for exponential methods with complexity $M(q) = 2^q$ such, for example, as Toudic method [9,12].

$$AccE(q) = \frac{2^q}{k^3 \cdot p^3 + k \cdot 2^p} \approx \frac{2^q}{2^p} = 2^{q-p}.$$

The acceleration obtained is exponential. It is good enough result.

6. Example

Let us solve the homogeneous system (1) of 9 equations with 10 variables. The matrix of the system is as follows

$$A = \begin{Vmatrix} 1 & 0 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

Stage 1. *Decomposition.* Above system is decomposed into two clans

$$C^1 = \{L_1, L_2, L_5, L_6\}, \quad C^2 = \{L_3, L_4, L_7, L_8, L_9\}.$$

It has the following subsets of clans' variables

$$X^1 = \{x_3, x_6, x_8, x_{10}, x_1, x_2, x_7\},$$

$$X^2 = \{x_3, x_6, x_8, x_{10}, x_4, x_5, x_9\},$$

contact variables

$$X^0 = \overset{\backprime}{X}{}^1 = \overset{\backprime}{X}{}^2 = \{x_3, x_6, x_8, x_{10}\}$$

and internal variables

$$\overset{\backprime}{X}{}^1 = \{x_1, x_2, x_7\}, \quad \overset{\backprime}{X}{}^2 = \{x_4, x_5, x_9\}.$$

According to new enumeration of variables

$$nx = \begin{pmatrix} 3 & 6 & 8 & 10 & 1 & 2 & 7 & 4 & 5 & 9 \end{pmatrix}$$

and new enumeration of equations

$$nL = \begin{pmatrix} 1 & 2 & 5 & 6 & 3 & 4 & 7 & 8 & 9 \end{pmatrix}$$

the matrix A has the form

$$A = \begin{Vmatrix} 2 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{Vmatrix}$$

Stage 2. *Solution of systems for clans.* Application of Toudic method gives us the following matrixes of basis solutions for clans:

$$G^1 = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix},$$

$$G^2 = \begin{Vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix}$$

with respect to vectors of free variables $\bar{y}^1 = (y_1^1, y_2^1, y_3^1)$ and $\bar{y}^2 = (y_1^2, y_2^2)$ correspondingly.

Stage 3. *Solution of system for contact variables.* The system for contact variables has the form:

$$\begin{cases} y_1^1 - y_1^2 = 0, \\ y_1^1 - y_1^2 = 0, \\ y_2^1 - y_1^2 = 0, \\ y_2^1 - y_1^2 = 0. \end{cases}$$

It should be noted that equations correspond to contact variables x_3, x_6, x_8, x_{10} and first and second equations coincide as well as third and fourth. The general solution of this system is

$$R = \begin{Vmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

Stage 4. *Composition of target solution.* As it was mentioned in section 4, matrix G may be composed in different ways. Two variants of this matrix are represented as follows

$$G = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix} \text{ or}$$

$$G = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix}.$$

And finally the result matrix of basis solutions of the system (1) has the form

$$H = R \cdot G = \begin{Vmatrix} 1 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix}.$$

The result obtained coincides with the basis solutions calculated in usual manner with Toudic method.

7. Conclusions

Present paper introduced new technique of linear system solution with decomposition into clans. It is efficient in the combination with methods of system solution more hard than a polynomial of third degree and in the case the system may be decomposed in more than one clan.

As a result of this technique application to various matrixes it may be concluded that a good opportunity to decomposition have sparse matrixes. This is realistic enough situation as large-scale real-life models contain well-localized interconnections of elements.

It should be noted also the relationship of technique proposed with the decomposition of Petri nets [11]. We may construct the matrix of directions D for an arbitrary matrix A of linear system as $D = sign(A)$. Matrix D may be considered further as incidence matrix of Petri net.

The most significant acceleration of computations was obtained for Diophantine (integer) systems especially solving over the set of nonnegative numbers. There are only exponential methods for solution of such systems. In this case the acceleration of computation obtained is exponential.

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