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DECOMPOSITION OF PETRI NETS

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The problem of splitting any given Petri net into functional subnets is considered. The properties of functional subnets and sets that induce them are investigated. An algorithm of polynomial complexity is constructed for decomposition of nets.

Keywords: *Petri net, subnet, decomposition, algorithm.*

INTRODUCTION

At the present time, the significance of the theory of Petri nets increases in connection with the extension of application domains that include the organization of parallel computations, design of electronic devices, verification of network protocols, control of production processes and robotic systems, and traffic control. This list, which is far from complete, can be extended by such exotic object domains as the planning of military operations and simulation of nuclear and chemical reactions.

During the last three decades, about ten monographs were published in which fundamental theoretical results are presented and numerous object domains and also models of actual systems and processes are analyzed. In our brief review of the most well-known works, it is relevant to note that [1] is remarkable for the breadth of coverage of object domains and simplicity of presentation of material, the monograph [2] is more theoretical and is written in a rigorous mathematical style, in [3], specific results connected with finding net invariants and reducing nets are presented, and, in the monograph [4], a generalized class of loaded nets is introduced, their properties are investigated, and methods of construction of models and algorithms of control over sophisticated production systems are considered.

The problem of construction of models of systems and processes by integration (composition) of models of basic components was considered practically in all the above-mentioned works. In [1], it has been suggested that nets be united by superposition (matching) of their places. In [2], regular nets constructed from elementary nets by uniting special contact places are investigated. In [4, 7], external (contact) places of a net are subdivided into two classes, namely, input and output ones, and special rules of uniting places of opposite classes in composing models and algorithms are introduced. Thus, the problem of composition of nets is sufficiently thoroughly investigated.

The necessity of development of inverse methods that make it possible to partition a general model into component parts was noted in [2, 4]. And though the problem of decomposition is trivial for nets with contact places (which is shown in [2]) since an arbitrary subset of transitions generates a subnet with contact places, a special investigation is required for nets with input and output places [4–6]. The results of such an investigation are presented in this article.

It is relevant to note that the partibility into subnets with input and output places is formulated as a structural property of a Petri net. Classical examples of structural properties are p- and t-invariance and the presence of siphons or traps [1–3]. These properties are inherent in nets irrespective of their labeling. Thus, the results obtained are valid for an arbitrary bipartite oriented graph [8].

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BASIC CONCEPTS AND DEFINITIONS

A Petri net is a triple $N=(P,T,F)$, where $P=\{p\}$ is a finite set of nodes called places, $T=\{t\}$ is a finite set of nodes called transitions, and the adjacency relation of nodes $F\subseteq P\times T\cup T\times P$ specifies the set of edges that connect places and transitions. Thus, a Petri net is a bipartite oriented graph [8] in which one part of nodes consists of places and the other consists of transitions. An example of a Petri net N_1 that has five places and six transitions is presented in Fig. 1.

In the theory of Petri nets, the introduced graph N is supplemented with dynamic elements called tokens; tokens fill places and move along the net as a result of firing transitions [1, 2]. A lot of classes of Petri nets are well known in which the multiplicities of edges, operating times of transitions, colors of tokens [4], and other characteristics of elements of a net are used. In this article, the concept of dynamics of nets is not used but their structural properties are investigated that are independent of labeling and other additional characteristics. Therefore, the results obtained are valid for an arbitrary bipartite oriented graph and can be applied to all classes of nets in which the graph N is used.

Let us introduce the following special notations for sets of input and output nodes for places and transitions of a net:

$$^{\bullet}p=\{t\,|\,\exists(t,p)\in F\},\quad p^{\bullet}=\{t\,|\,\exists(p,t)\in F\},$$

$$^{\bullet}t=\{p|\exists\{p,t\}\in F\},\ t^{\bullet}=\{p\,|\,\exists(t,p)\in F\}.$$

We can similarly define the sets of input, output, and adjacent nodes for an arbitrary subset of places or transitions; such notations will be used in what follows.

A Petri net in which special subsets of input and output places are specified is called a net with input and output places. In investigating such a net, external actions are applied only to input places, and the result of functioning of the net is considered only at its output places. Input and output places are also called contact places. Various methods of definition of such nets are well known. These methods are connected with the presence or absence of additional constraints on edges of contact places and also with the form of these constraints [4–7]. The most regular structure, which is convenient for construction of models of systems, is inherent in the so-called functional nets [6]. Such nets are also investigated in this article.

By a functional net we understand a triple $Z=(N,X,Y)$, where N is a Petri net, $X\subseteq P$ are its input places, and $Y\subseteq P$ are its output places; in this case, the sets of input and output places do not intersect, i.e., we have $X\cap Y=\varnothing$, and, moreover, input places have no incoming edges and output places have no outgoing edges, i.e., we have $\forall p\in X:{}^{\bullet}p=\varnothing$ and $\forall p\in Y:p^{\bullet}=\varnothing$. The places from the set $Q=P\setminus(X\cup Y)$ are called internal. Thus, according to the terminology of graph theory [8], the input places are sources and output ones are drains. We can also write a functional net in the form $Z=(X,Q,Y,T,F)$ to indicate structural elements of the Petri net N. We omit here details of construction and note that an example of a functional net with sets of input and output places $X=\{p_1\}$ and $Y=\{p_4,p_5\}$, respectively, is presented in Fig. 2 (the subnets Z_1 and Z_2 are presented in Fig. 3).

Statement 1. An arbitrary Petri net N can be considered as functional if we consider that the set X consists of sources of the net N and the set Y consists of its drains.

Thus, without loss of generality, we will henceforth consider functional Petri nets, assuming that their sets of contact places can be empty.

A Petri net $N'=(P',T',F')$ is a subnet of a net N if we have $P'\subseteq P$, $T'\subseteq T$, and $F'\subseteq F$. By a subnet generated by a specified set of nodes $B(P',T')$ we understand a subnet $N'=(P',T',F')$, where F' contains all the edges that connect the nodes P' and T' in the initial net, i.e., we have

$$F'=\{(p,t)|\,p\in P',\ t\in T',\ (p,t)\in F\}\cup\{(t,p)|\,p\in P',\ t\in T',\ (t,p)\in F\}.$$

By the subnet generated by a specified set of transitions $B(T')$ we understand the subnet $B(P',T')$, where $P'=\{p\,|\,p\in P\ \exists t\in T':(t,p)\in F\vee(p,t)\in F\}$. In other words, together with transitions from T', the net contains all the places adjacent to them and is generated by these nodes.

In what follows, in considering subnets, we will use, as a rule, all the edges that connect selected nodes in the initial net, i.e., consider the subnets generated by sets of nodes. Therefore, for brevity, we omit the adjacency relation in notations and use the initial adjacency relation F between nodes of a net.

We call a functional net $Z=(N',X,Y)$ a functional subnet of a net N and denote $Z\succ N$ if N' is a subset of N and, moreover, Z is connected with the remaining part of the net only by edges incident to input or output places and, in this case,

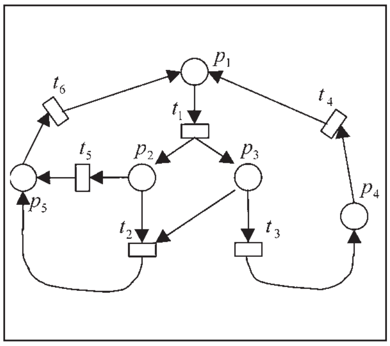

Fig. 1. A Petri net N_1.

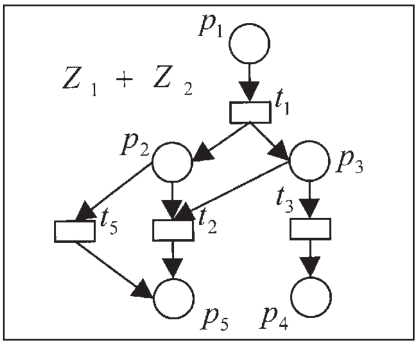

Fig. 2. A functional subnet $Z_1 + Z_2$ of the Petri net N_1.

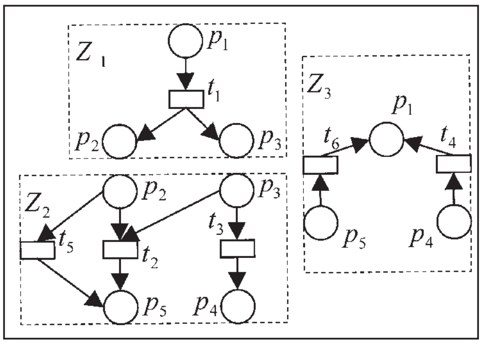

Fig. 3. Decomposition of the Petri net N_1 into functional subnets.

the input places can have only incoming edges and output places can have only outgoing edges. Thus, we have

$$\forall p \in X : \{(p, t) | t \in T \setminus T'\} = \varnothing, \quad \forall p \in Y : \{(t, p) | t \in T \setminus T'\} = \varnothing,$$

$$\forall \in Q : \{(p, t) | t \in T \setminus T'\} = \varnothing \wedge \{(t, p) | t \in T \setminus T'\} = \varnothing.$$

Statement 2. A functional subnet is generated by the set of its transitions.

It makes sense to prove the statement by contradiction, assuming that the subnet contains a nonisolated place incident to some external transition or that the net does not contain some place incident to an internal transition. The obtaining of a contradiction with the definition of a functional net completes the proof in both cases.

The introduced definition of a functional subnet requires additional explanations. First, the specification of the set of transitions that generate a subnet makes it possible to uniquely define it. Second, the introduced constraints on the edges of input and output places imply that the remaining part of the net is also a functional subnet and the initial net can be obtained by uniting subnets by merging places of opposite classes, i.e., by merging input places with output ones and vice versa [4, 7]. In Fig. 3, functional subnets of the Petri net N_1 depicted in Fig. 1 are presented, and a method of construction of these subnets is described below.

The net that is the difference of the initial Petri net and its functional subnet $Z'' = N - Z'$ is defined as follows:

$$Z'' = (Y, P \setminus (X \cup Y \cup Q), X, T \setminus T').$$

Statement 3. If we have $Z' \succ N$, then we obtain $N - Z' \succ N$.

The statement is proved by the fact that the constraints on the edges in the definition of a functional subnet satisfy the requirements of the definition of a functional net and, moreover, only contact places are common ones.

A functional subnet $Z' \succ N$ is minimal if and only if it does not contain another functional subnet of the initial Petri net N. Note that, as is shown below, the functional subnets presented in Fig. 3 are minimal functional subnets of the net N_1.

PROPERTIES OF FUNCTIONAL SUBNETS

THEOREM 1. The sets of transitions of two arbitrary minimal functional subnets Z' and Z'' of a Petri net N do not intersect.

Proof. Suppose the contrary, i.e., that the set $R = T' \cap T''$ is nonempty. Let $S = T' \setminus R$. We will construct the net $B(S)$ that is generated by this set of transitions. We will show that $B(S)$ is a functional subnet of the Petri net N. Since Z'' is a functional subnet of N, $B(S)$ is connected with nodes of the subnet Z'' only by contact places and, moreover, since Z' is a functional subnet of N, $B(S)$ is also connected with nodes of the subnet $N - Z'$ only by contact places. We note that, for the input and output places of $B(S)$, all the constraints of a functional subnet are satisfied. Hence, the functional subnet Z' contains the functional subnet $B(S)$, contrary to the minimality of Z'. The contradiction obtained proves the falsity of the initial assumption on the intersection of the subsets T' and T''.

COROLLARY 1. The sets of internal places of two arbitrary minimal functional subnets Z' and Z'' of a Petri net N do not intersect.

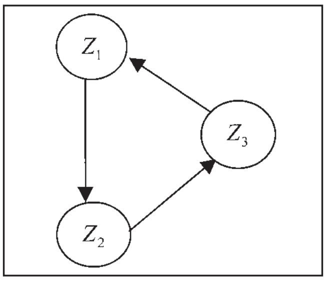

Fig. 4. The graph of functional subnets G_1 of the Petri net N_1.

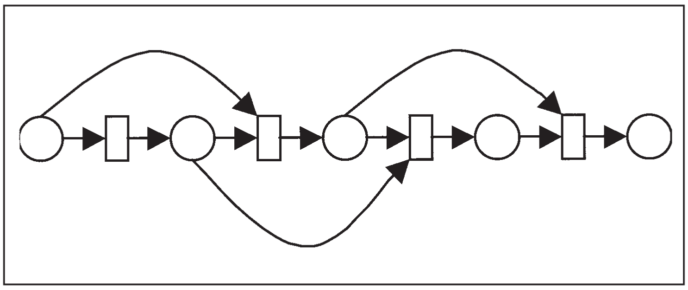

Fig. 5. An indecomposable network N_2.

COROLLARY 2. The set of minimal functional nets $\mathfrak{J} = \{Z^j\}$, $Z^j \succ N$, determines a partition of the set T into disjoint subsets T^j such that we have $T = \bigcup_j T^j$, $T^j \cap T^k = \varnothing$, $j \neq k$.

By the graph of functional subnets of a given Petri net N we understand an oriented graph $G = (Z, E)$ in which the set of nodes Z consists of minimal functional subnets of the net N and the edges E connect nodes if the corresponding subnets have common contact places in such a manner that we have $E = \{(Z_j, Z_k) | \exists p : p \in Y^j, p \in X^k\}$. The graph makes it possible to schematically represent interrelations between functional subnets of the initial net. In Fig. 4, the graph of functional subnets of the Petri net N_1 is presented.

It is relevant to note that, in general, the minimality does not mean that there are some small number of places and transitions and only implies the further indecomposability into internal functional subnets. In this case, an indecomposable net can be of a however large size. An example of an indecomposable net is presented in Fig. 5; the chain of places and transitions connected by edges of the specified configuration can contain an unlimited number of nodes.

THEOREM 2. Any functional subnet Z' of an arbitrary Petri net N is the sum (union) of a finite number of minimal functional subnets.

Proof. Suppose the contrary, namely, that there exists a functional net Z' that is a subnet of the Petri net N and is not a union of minimal subnets. Since $\mathfrak{J}$ specifies a partition of the set T, T' contains parts of subsets T^j. This fact can be formally formulated as follows: we have $T' = \bigcup_{i \in I} R^i$, where I is the set of numbers of the subnets whose transitions are contained in T', $R^i \subseteq T^i$ and, moreover, there exists at least one set $R^j \subset T^j$ for some $j \in I$. Let us consider the set of transitions $S = T^j \setminus R^j$ and show that it generates a functional subnet $B(S)$ of the Petri net N. Since Z' is a functional subnet of N, $B(S)$ is connected with nodes of the subnet Z' only by contact places, and, moreover, since Z^j is a functional subnet of N, $B(S)$ is also connected with nodes of the subnet $N - Z^j$ only by means of contact places. In this case, for the input and output places of $B(S)$, all the constraints imposed on a functional subnet are satisfied. Consequently, the functional subnet Z^j contains the functional subnet $B(S)$, contrary to the minimality of Z^j. The obtained contradiction proves the falsity of the initial assumption that $S \neq \varnothing$. Hence, T' contains the entire set T^j, which proves the theorem by virtue of the arbitrariness of choice of T^j.

The theorem can be illustrated by Fig. 2, in which the net presented is the sum of two minimal functional subnets of the Petri net N_1.

COROLLARY. If a partition of the set T is determined by the set of minimal functional subnets, then it is the generating family of the set of functional subnets of the Petri net N. The objective of this article is the development of efficient algorithms of construction of such a generating family for a given arbitrary Petri net.

DECOMPOSITION OF A PETRI NET WITH THE HELP OF LOGICAL EQUATIONS

Let us consider a functional subnet $Z^0 = (P^2, P^0, P^3, T^0)$ of a Petri net $N = (P, T, F)$. Let $N - Z^0$ form a functional subnet $Z^1 = (P^3, P^1, P^2, T^1)$. The interrelation of these subnets is illustrated by Fig. 6. We construct the following

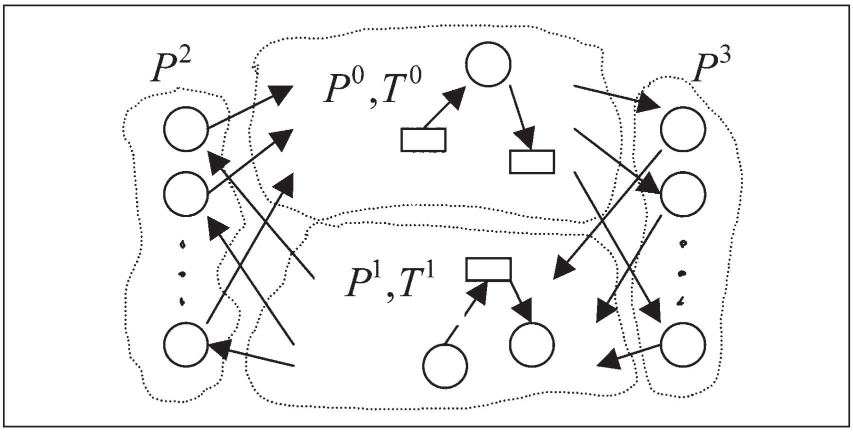

Fig. 6. Interrelation of functional subnetworks.

equations that specify the belonging of places and transitions of the net to different subsets in the first-order predicate calculus, using the definition of a functional subnet and also the fact that, according to Statement 2, Z^1 is also a functional subnet of N:

$$\begin{cases} (t \in T^0) \equiv (\forall p \in {}^\bullet t)((p \in P^0) \vee (P \in P^2)) \wedge (\forall p \in t^\bullet)((p \in P^0) \vee (p \in P^3)), \\ (t \in T^1) \equiv (\forall p \in {}^\bullet t)((p \in P^1) \vee (P \in P^3)) \wedge (\forall p \in t^\bullet)((p \in P^1) \vee (p \in P^2)). \end{cases} \quad (1)$$

We can similarly construct the following equations that specify the belonging of places of the net:

$$\begin{cases} (p \in P^0) \equiv (\forall t \in {}^\bullet p)(t \in T^0) \wedge (\forall t \in p^\bullet)(t \in T^0), \\ (p \in P^1) \equiv (\forall t \in {}^\bullet p)(t \in T^1) \wedge (\forall t \in p^\bullet)(t \in T^1), \\ (p \in P^2) \equiv (\forall t \in {}^\bullet p)(t \in T^1) \wedge (\forall t \in p^\bullet)(t \in T^0), \\ (p \in P^3) \equiv (\forall t \in {}^\bullet p)(t \in T^0) \wedge (\forall t \in p^\bullet)(t \in T^1). \end{cases} \quad (2)$$

We substitute Eqs. (2) in Eqs. (1) and note that since we have $T^0 \cup T^1 = T$ and $T^0 \cap T^1 = \varnothing$, it suffices to consider only one of Eqs. (1), for example, the equation that determine the belonging of transitions to T^1. We obtain the following system:

$$(t \in T^1) \equiv (\forall p \in {}^\bullet t)(((\forall s \in {}^\bullet p)(s \in T^1) \wedge (\forall s \in p^\bullet)(s \in T^1))$$

$$\vee ((\forall s \in {}^\bullet p)(s \in T^0) \wedge (\forall s \in p^\bullet)(t \in T^1))) \wedge (\forall p \in t^\bullet)(((\forall s \in {}^\bullet p)(s \in T^1) \wedge (\forall s \in p^\bullet)(s \in T^1))$$

$$\vee ((\forall s \in {}^\bullet p)(s \in T^1) \wedge (\forall s \in p^\bullet)(t \in T^0))). \quad (3)$$

Using the finiteness of the sets of places and transitions of a net, we replace the generality quantifiers by the conjunction of the corresponding subsets of elements and also introduce indicators τ_t of belonging of transitions to subsets in such a manner that we have $\tau_t = j \Leftrightarrow t \in T^j$. We note that $\tau_t \in \{0, 1\}$, and hence, these quantities can be used in logical equations. And since $T^0 \cap T^1 = \varnothing$, we have $\tau_t \Leftrightarrow (t \in T^1)$ and also $\bar{\tau}_t \Leftrightarrow (t \in T^0)$. Therefore, Eqs. (3) can be represented in binary logic in the form

$$\tau_t \equiv \underset{p \in {}^\bullet t}{\&} \left(\left(\underset{s \in {}^\bullet p}{\&} \tau_s \wedge \underset{s \in p^\bullet}{\&} \tau_s \right) \vee \left(\underset{s \in {}^\bullet p}{\&} \bar{\tau}_s \wedge \underset{s \in p^\bullet}{\&} \tau_s \right) \right)$$

$$\wedge \underset{p \in t^\bullet}{\&} \left(\left(\underset{s \in {}^\bullet p}{\&} \tau_s \wedge \underset{s \in p^\bullet}{\&} \tau_s \right) \vee \left(\underset{s \in {}^\bullet p}{\&} \tau_s \wedge \underset{s \in p^\bullet}{\&} \bar{\tau}_s \right) \right). \quad (4)$$

Thus, the theorem formulated below is proved.

THEOREM 3. A partition of an arbitrary Petri net into functional subnets is completely determined by the system of logical equations (4).

In process of solution, system (4) can be replaced by one equation in the form of the conjunction of equations corresponding to every transition of the net. Methods of solution of logical equations are sufficiently thoroughly investigated in [9].

Let us consider an example of decomposition of the net N_1 (see Fig. 1). We construct the following system of logical equations of the form (4):

$$\begin{cases} \tau_1 \equiv (\tau_6\tau_4\tau_1 \vee \bar{\tau}_6\bar{\tau}_4\tau_1)(\tau_1\tau_5\tau_2 \vee \tau_1\bar{\tau}_5\bar{\tau}_2)(\tau_1\tau_2\tau_3 \vee \tau_1\bar{\tau}_2\bar{\tau}_3), \\ \tau_2 \equiv (\tau_1\tau_5\tau_2 \vee \bar{\tau}_1\tau_5\tau_2)(\tau_1\tau_2\tau_3 \vee \bar{\tau}_1\tau_2\tau_3)(\tau_5\tau_2\tau_6 \vee \tau_5\tau_2\bar{\tau}_6), \\ \tau_3 \equiv (\tau_1\tau_2\tau_3 \vee \bar{\tau}_1\tau_2\tau_3)(\tau_3\tau_4 \vee \tau_3\bar{\tau}_4), \\ \tau_4 \equiv (\tau_3\tau_4 \vee \bar{\tau}_3\tau_4)(\tau_1\tau_4\tau_6 \vee \bar{\tau}_1\tau_4\tau_6), \\ \tau_5 \equiv (\tau_1\tau_5\tau_2 \vee \bar{\tau}_1\tau_5\tau_2)(\tau_5\tau_2\tau_6 \vee \tau_5\tau_2\bar{\tau}_6), \\ \tau_6 \equiv (\tau_5\tau_2\tau_6 \vee \bar{\tau}_5\bar{\tau}_2\tau_6)(\tau_1\tau_4\tau_6 \vee \bar{\tau}_1\tau_4\tau_6). \end{cases}$$

We take the conjunction of the equations of the system, simplify the expression obtained, and reduce it to the disjunctive perfect normal form

$$\bar{\tau}_1\bar{\tau}_2\bar{\tau}_3\bar{\tau}_4\bar{\tau}_5\bar{\tau}_6 \vee \tau_1\bar{\tau}_2\bar{\tau}_3\bar{\tau}_4\bar{\tau}_5\bar{\tau}_6 \vee \bar{\tau}_1\bar{\tau}_2\bar{\tau}_3\tau_4\bar{\tau}_5\tau_6 \vee \bar{\tau}_1\tau_2\tau_3\bar{\tau}_4\tau_5\bar{\tau}_6$$

$$\vee\, \tau_1\bar{\tau}_2\bar{\tau}_3\tau_4\bar{\tau}_5\tau_6 \vee \tau_1\tau_2\tau_3\bar{\tau}_4\tau_5\bar{\tau}_6 \vee \bar{\tau}_1\tau_2\tau_3\tau_4\tau_5\tau_6 \vee \tau_1\tau_2\tau_3\tau_4\tau_5\tau_6.$$

Note that the first term corresponds to the empty subnet, the next three terms correspond to the minimal subnets presented in Fig. 3, and the remaining terms describe the sums of minimal subnets. For example, the sixth term describes the subnet presented in Fig. 2.

It is relevant to note that, in general, the algorithmic complexity of decomposition of an arbitrary Petri net into functional subnets with the help of described logical equations is asymptotically exponential, which is connected with estimates of complexity of solution of logical equations [9]. Therefore, special efficient algorithms of decomposition of nets should be constructed.

AN ALGORITHM OF DECOMPOSITION OF PETRI NETS

A subnet $Z = B(R) = (X, Q, Y, R)$ of a Petri net N is called complete in N if the relations $X^\bullet \subseteq R$, ${}^\bullet Y \subseteq R$, and ${}^\bullet Q^\bullet \subseteq R$ are true in N.

Algorithm 1. Step 0. Choose an arbitrary transition $t \in T$ of the net N and add it to the set of selected transitions $R := \{t\}$.

Step 1. Construct the subnet Z generated by the set R: $Z = B(R) = (X, Q, Y, R)$.

Step 2. If Z is complete in N, then Z is the sought-for subnet.

Step 3. Form the set of absorbed transitions

$$S = \{t \mid t \in X^\bullet \wedge t \notin R \vee t \in {}^\bullet Y \wedge t \notin R \vee t \in {}^\bullet Q^\bullet \wedge t \notin R\}.$$

Step 4. Assume that $R := R \cup S$ and go to step 1.

THEOREM 4. A subnet Z is complete in a Petri net N if and only if it is a functional subnet of N.

Proof. We begin with the proof of the necessity of completeness. Let Z be a functional subnet of N, i.e., we have $Z \succ N$. Then the completeness conditions are fulfilled for the adjacent transitions of the places from Q by virtue of the definition of internal places, and these conditions are fulfilled for the output transitions of the input places and input transitions of the output places by virtue of constraints on edges in the definition of a functional subnet.

Let us prove the sufficiency. It is well known that Z is a functional net and also a subnet of N and is generated by the set of transitions R. It remains to prove that the constraints on the edges that connect the places of the subnet Z with the remaining part of the net are fulfilled. We denote the remaining part of the net by $Z' = N - B(R) = (Y, Q', X, R')$, where

$Q' = P \setminus (X \cup Q \cup Y)$ and $R' = T \setminus R$. Suppose the contrary. Let N contain one or several forbidden edges of one of the following possible six types: (a) (x, r'); (b) (r', y); (c) (r, q'); (d) (q', r); (e) (q, r'); (f) (r', q), where $x \in X$, $y \in Y$, $q \in Q$, $q' \in Q'$, $r \in R$, and $r' \in R'$. Let us consider each of the selected types of edges:

(a) if $(x, r') \in F$, then we have $r' \in x^{\bullet}$ and, hence, we obtain $X^{\bullet} \not\subset R$;

(b) if $(r', y) \in F$, then we have $r' \in {}^{\bullet}y$ and, hence, we obtain ${}^{\bullet}Y \not\subset R$;

(c) if $(r, q') \in F$, then we have $q' \in Y$;

(d) if $(q', r) \in F$, then we have $q' \in X$;

(e) if $(q, r') \in F$, then we have $r' \in q^{\bullet}$ and, hence, we obtain ${}^{\bullet}Q^{\bullet} \not\subset R$;

(f) if $(r', q) \in F$, then we have $r' \in {}^{\bullet}q$ and, hence, we obtain ${}^{\bullet}Q^{\bullet} \not\subset R$.

Thus, in each above-mentioned case, we have a contradiction, which proves the sufficiency of the completeness of the subnet.

THEOREM 5. The subnet Z constructed by Algorithm 1 is the minimal functional subnet of the Petri net N.

Proof. Suppose the contrary, i.e., that Z is not minimal. Then there exists a net Z' that is a minimal functional subnet of the Petri net N and is such that we have $Z' = B(T'')$ and $T'' \subset T'$, i.e., Z contains Z'. Let us consider two possible variants of execution of Algorithm 1: (a) after starting from a transition $t \in T'$ such that we have $t \in T''$ and (b) after starting from a transition $t \in T'$ such that we have $t \notin T''$. Let us consider each of the cases mentioned.

1. Let $t \in T''$. Let us consider the first transition v that belongs to the set $T' \setminus T''$ and is assigned to the set S during some passage of the basic loop of Algorithm 1. In this case, according to the description of step 3, one of the following three variants is possible: $v \in X^{\bullet}$, $v \in {}^{\bullet}Y$, or $v \in {}^{\bullet}S^{\bullet}$. In the first case, a place $x \in X$ such that $v \in x^{\bullet}$ cannot be an input, an output, or an internal place of the net Z'. We obtain a contradiction. Similarly, we arrive at a contradiction in considering the second and third cases.

2. Let $t \notin T''$. Let us consider the first transition v that belongs to the set T'' and is assigned to the set S during some passage of the basic loop of Algorithm 1. In this case, according to the description of step 3, one of the following three variants is possible: $v \in X^{\bullet}$, $v \in {}^{\bullet}Y$, or $v \in {}^{\bullet}S^{\bullet}$. In the first case, a place $x \in X$ such that we have $v \in x^{\bullet}$ cannot be an input, an output, or an internal place of the net Z'. We obtain a contradiction. We similarly arrive at a contradiction in considering the second and third cases.

Thus, Algorithm 1 makes it possible to construct a minimal functional subnet Z of the Petri net N. We put $i := 1$ and $Z^i := Z$. Then we put $N := N - Z$ and repeat the execution of Algorithm 1 if the set T is nonempty. Continuing the process and choosing $i := i + 1$, we construct the set of minimal functional subnets $Z^1, Z^2, \ldots, Z^k$ of the net N that represent the sought-for partition of the initial net.

The result of application of the algorithm to the net presented in Fig. 1 coincides with that obtained in the previous section and is presented in Fig. 3.

We will estimate the complexity of the algorithm, assuming that a net contains n nodes. At each passage of the algorithm, the set R contains at most n transitions and, for each transition, at most n places are enumerated at step 1. Thus, the complexity of step 1 is equal to $o(n^2)$. The estimates for steps 2, 3, and 4 are similar. And since the basic loop of the algorithm is performed at most n times, the above reasoning proves the theorem presented below.

THEOREM 6. The complexity of Algorithm 1 is polynomial and does not exceed $o(n^3)$, where n is the number of nodes of a net.

Note that since algorithm processes each transition of a net only once, the highest degree of the polynomial is determined in the general case by the number of edges that connect nodes of the net. And their upper bound in the estimate is chosen so that it is equal to n^2. Practical investigation of a few tens of actual net models with the help of a program realization of Algorithm 1 showed that network models have a low density of edges that frequently does not exceed some constant c for each node of a net. In this case, the complexity of the algorithm is linear.

CONCLUSIONS

In this article, the properties of the set of functional subnets of an arbitrary given Petri net are investigated. Two different algorithms for construction of generating families of functional subnets are constructed and justified. The complexity of the basic algorithm is demonstrated to be polynomial, namely, $o(n^3)$, where n is the number of nodes of a net, and it is linear for nets with a low fixed density of edges.

A program realization of the algorithm was used to decompose a number of models of systems and processes described in different sources. As a rule, nets are easily decomposed, which makes it possible to subdivide them into subnets consisting of several tens of nodes.

Together with [6], in which a formal description of the transfer function of a net with input and output places is obtained, this article gives a closed processing chain for analyzing properties of nets, and this chain is based on the selection of functional subnets, formal representation of their transfer functions, and succeeding equivalent transformations. In contrast to [3], such a reduction of a net retains all its properties rather than a given set of them.

Since the concept of dynamics of a net is not used in this article, the results obtained are valid for arbitrary bipartite oriented graphs [8].

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