Comparison of Several Space Target Tracking Filters

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ABSTRACT

In this paper, we present a comparative study of several nonlinear filters, namely, extended Kalman Filter (EKF), unscented KF (UKF), particle filter (PF), and recursive linear minimum mean square error (LMMSE) filter for the problem of satellite trajectory estimation. We evaluate the tracking accuracy of the above filtering algorithms and obtain the posterior Cramer-Rao lower bound (PCRLB) of the tracking error for performance comparison. Based on the simulation results, we provide recommendations on the practical tracking filter selection and guidelines for the design of observer configurations.

1. INTRODUCTION

The problem of estimating the continuous state of a dynamic system based on discrete time measurements arises in many applications. For the case of a linear system with a known state transition matrix and additive white Gaussian noise, the classical Kalman filter provides the optimal solution in terms of minimum means square error (MMSE) of the estimated state [1,3]. However, the state estimation for a nonlinear system remains a challenging problem which attracts intense research interest [2,23]. Optimal nonlinear filters are often infinite dimensional and thus difficult to implement [12]. Within a deterministic setting, optimal nonlinear observers are available for systems of special structure [22,25]. However, for general nonlinear filtering cases, approximate solutions are often implemented in practice [13].

Approximate nonlinear filters can be largely categorized into two classes. The first class is based on analytically or numerically linearizing the system dynamics and observation equation. Then standard Kalman filter is applied to the linearized state and measurement equation evaluated at some operating point, which is usually chosen to be the best estimate based on the linearized system and believed to be close to the true state [3]. Various types of extended Kalman filter (EKF) belong to this class, which is computationally effective and finds applications in many areas from motor control [6], target tracking [1] to weather forecasting [7].

Another class of nonlinear filters uses a collection of samples to approximate the whole posterior distribution of the state. Thus the posterior mean and error covariance of the state estimate are readily available from the approximate distribution. A particle filter (PF) usually propagates random samples through the nonlinear system and resamples those particles based on the weights derived from the likelihood function of the measurement [17,14]. In some areas such as target tracking [16,4] and computer vision [18], specific sampling techniques have been developed for PF under the umbrella of the so called ensemble Kalman filters [15] where only certain state components with highly nonlinear dynamics are approximated by particles in order to save the computational requirement for high dimensional nonlinear filtering problem. Alternatively, a deterministic approach to propagation the samples through nonlinear system dynamics has been proposed via unscented transform (UT) [20]. It preserves the exact moments up to certain order for a class of nonlinear systems with special structure. The filter based on linear minimum mean square error (LMMSE) criterion using UT is often called the unscented Kalman filter (UKF) [21]. In fact, exact moment matching for the state estimate and its error covariance to approximate the nonlinear system should be called recursive LMMSE filter. In the case that the system has linear dynamics while the measurement equation is nonlinear, UT is unnecessary while the recursive LMMSE filter is still applicable [24,31].

The primary goal of this paper is to compare various nonlinear filtering methods for the problem of space target trajectory estimation. We consider EKF, UKF, PF and recursive LMMSE filter for tracking a single space

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target. Both the orbital dynamics and measurement equations are nonlinear. The nonlinear filtering problem in itself is challenging and has practical importance to space awareness and sensor management. To our best knowledge, such comprehensive comparisons of the state-of-the-art nonlinear filters have not been thoroughly studied in space target tracking except a recent attempt in [27]. In addition to implementing PF and recursive LMMSE filter not studied in [27], we also compute the posterior Cramer-Rao lower bound (PCRLB) [29] which serves as the ideal estimation error performance limit to quantify how much performance improvement is possible from the existing nonlinear filters. Thus our study not only provides a benchmark test of nonlinear filtering algorithms but also guides the development of sensor constellation and management algorithms to meet certain space awareness requirements. A companion paper on the awareness based sensor management for space target tracking can be found in [8].

The rest of the paper is organized as follows. Section 2 presents the basic motion equation for a space target. Section 4 provides the implementation details of various nonlinear filtering algorithms and Section 5 presents the calculation of the PCRLB of the state estimation error. Section 6 compares the tracking accuracy of the state-of-the-art nonlinear filtering algorithms. Concluding remarks are in Section 7.

2. BASIC MOTION EQUATIONS OF SPACE TARGETS

2.1. Ideal Orbit State Equations

We consider a single space target orbiting the earth. The reference coordinate system uses the right-handed convention with x-axis pointing to the vernal equinox and z-axis pointing upwards, meaning that the x-y plane is the reference plane. Under the assumptions that the earth is spherical and uniform, without any perturbing force, the target position $\vec{r}$ relative to the center of the earth in earth-centered inertial coordinates should satisfy [27]

$$\ddot{\vec{r}} = -\frac{\mu}{||\vec{r}||^3}\vec{r} \tag{1}$$

where $\mu \triangleq 398,600\text{km}^3/\text{s}^2$ is the earth's gravitational parameter. The target velocity is $\vec{v} \triangleq \dot{\vec{r}}$ and the radial velocity is $v_r \triangleq \frac{\vec{v}\cdot\vec{r}}{r}$ where $r \triangleq ||\vec{r}||$ is the distance from the target to the center of the earth.

From the target position and velocity state vector, we can derive the following orbital elements used in astrodynamics [11]. The *specific angular momentum* lies normal to the orbital plane given by $\vec{h} = \vec{r} \times \vec{v}$ with magnitude $h \triangleq ||\vec{h}||$. *Inclination* is the angle between the equatorial plane and the orbital plane given by $i \triangleq \cos^{-1}\left(\frac{h_z}{h}\right)$ where h_z is the z-component of $\vec{h}$. *Eccentricity* of the orbit is given by

$$\vec{e} \triangleq \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{r} - rv_r\vec{v}\right] \tag{2}$$

with magnitude $e \triangleq ||\vec{e}||$. The longitude of the ascending node is given by

$$\Omega \triangleq \begin{cases} \cos^{-1}\left(\frac{n_x}{n}\right) & n_y \geq 0 \\ 2\pi - \cos^{-1}\left(\frac{n_x}{n}\right) & n_y < 0 \end{cases} \tag{3}$$

where $\vec{n}$ is the vector pointing towards the ascending node with magnitude $n \triangleq ||\vec{n}||$. The *argument of perigee* is angle between the node line and the eccentricity vector given by

$$\omega \triangleq \begin{cases} \cos^{-1}\left(\frac{\vec{n}\vec{e}}{ne}\right) & e_z > 0 \\ 2\pi - \cos^{-1}\left(\frac{\vec{n}\vec{e}}{ne}\right) & e_z < 0 \end{cases} \tag{4}$$

with the convention that

$$\omega = \cos^{-1}\left(\frac{e_x}{e}\right) \tag{5}$$

for an equatorial orbit. The *true anomaly* ν is the angle between the eccentricity vector and and the target's position vector given by

$$\nu \triangleq \begin{cases} \cos^{-1}\left(\frac{\vec{e}\vec{r}}{er}\right) & v_r > 0 \\ 2\pi - \cos^{-1}\left(\frac{\vec{e}\vec{r}}{er}\right) & v_r < 0 \end{cases} \tag{6}$$

with the convention that $\nu = \cos^{-1}\left(\frac{r_x}{r}\right)$ for a circular orbit. The *eccentric anomaly* is the angle between apogee and the current position of the target given by

$$E = \cos^{-1}\left(\frac{1 - r/a}{e}\right) \tag{7}$$

where a is the orbit's semi-major axis given by $a = \frac{1}{\frac{2}{r} - \frac{v^2}{\mu}}$. The *mean anomaly* is $M = E - e \sin E$. The orbital period is given by $T = 2\pi\sqrt{\frac{a^3}{\mu}}$. Note that the orbit of the target can be fully determined by the parameter set $\{i, \Omega, \omega, T, e\}$ with initial condition given by the target position at any particular time [28]. The angles $\{i, \Omega, \omega\}$ transform the inertial frame to the orbital frame while T and e specify the size and shape of the ellipsoidal orbit. The time dependent parameter $\nu(t)$ represents the position of the target along its orbit in the polar coordinate system.

2.2. State Propagation Model

Denote by $\mathbf{x}(t)$ the continuous time target state given by

$$\mathbf{x}(t) \triangleq \begin{bmatrix} \vec{r}(t) \\ \dot{\vec{r}}(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix} \tag{8}$$

For convenience, we omit the argument t and write the nonlinear state equation as follows.

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \mathbf{w} \tag{9}$$

where

$$f(\mathbf{x}) = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -(\mu/r^3)x \\ -(\mu/r^3)y \\ -(\mu/r^3)z \end{bmatrix} \tag{10}$$

and

$$\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ w_x \\ w_y \\ w_z \end{bmatrix} \tag{11}$$

The discrete time state propagation model is usually based on relative target motion

$$\Delta\vec{r}(k) \triangleq \vec{r}(t_k) - \vec{r}(t_{k-1}) \tag{12}$$

and can be computed efficiently using the numerical routine described in Appendix B of [10].

3. MEASUREMENT MODELS

3.1. Kinematic Measurement Model

We consider the case that a space satellite in low-earth orbit (LEO) observes a target in geostationary orbit (GEO). A radar onboard the space satellite can provide the following type of measurements: range, azimuth, elevation and range rate. The range between the i-th observer located at (x_i, y_i, z_i) and the space target located at (x, y, z) is given by

$$d_r(i) = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}. \tag{13}$$

The azimuth is

$$d_a(i) = \tan^{-1}\left(\frac{y - y_i}{x - x_i}\right). \tag{14}$$

The elevation is

$$d_e(i) = \tan^{-1}\left(\frac{z - z_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}}\right). \tag{15}$$

The range rate is

$$d_{\dot{r}}(i) = \frac{(x - x_i)(\dot{x} - \dot{x}_i) + (y - y_i)(\dot{y} - \dot{y}_i) + (z - z_i)(\dot{z} - \dot{z}_i)}{d_r}. \tag{16}$$

3.2. Earth Blockage Condition

Measurements from the i-th observer will be unavailable when the line-of-sign path between the observer and the target is blocked by the earth. Thus the constellation of multiple observers is important to maintain consistent coverage of the target of interest.

The condition of earth blockage is examined as follows. If there exist $\alpha \in [0, 1]$ such that $D_\alpha(i) < R_E$, where

$$D_\alpha(i) = \sqrt{[(1 - \alpha)x_i + \alpha x]^2 + [(1 - \alpha)y_i + \alpha y]^2 + [(1 - \alpha)z_i + \alpha z]^2}, \tag{17}$$

then the measurement from the i-th observer to the target will be unavailable [27]. In simulation, we choose R_E =6378km for the earth's equatorial radius. The minimum of $D_\alpha(i)$ is achieved at $\alpha = \alpha^*$ given by

$$\alpha^* = -\frac{x_i(x - x_i) + y_i(y - y_i) + z_i(z - z_i)}{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}. \tag{18}$$

Thus we first examine whether $\alpha^* \in [0, 1]$ and then check the earth blockage condition $D_{\alpha^*}(i) < R_E$.

4. NONLINEAR FILTERING METHODS

4.1. Extended Kalman Filter [3]

Denote by $\hat{\mathbf{x}}_k^-$ the state prediction from time t_{k-1} to time t_k based on the state estimate $\hat{\mathbf{x}}_{k-1}^+$ at time t_{k-1} with all measurements up to t_{k-1}. The prediction is made by numerically integrating the state equation given by

$$\dot{\hat{\mathbf{x}}}(t) = f(\hat{\mathbf{x}}(t)) \tag{19}$$

without process noise. The mean square error (MSE) of the state prediction is obtained by numerically integrate the following matrix equation

$$\dot{P}(t) = F(\hat{\mathbf{x}}_k^-)P(t) + P(t)F(\hat{\mathbf{x}}_k^-)^T + Q(t) \tag{20}$$

where $F(\hat{\mathbf{x}}_k^-)$ is the Jacobian matrix given by

$$F(\mathbf{x}) = \begin{bmatrix} 0_{3\times3} & I_3 \\ F_0(\mathbf{x}) & 0_{3\times3} \end{bmatrix}, \tag{21}$$

$$F_0(\mathbf{x}) = \mu \begin{bmatrix} \frac{3x^2}{r^5} - \frac{1}{r^3} & \frac{3xy}{r^5} & \frac{3xz}{r^5} \\ \frac{3xy}{r^5} & \frac{3y^2}{r^5} - \frac{1}{r^3} & \frac{3yz}{r^5} \\ \frac{3xz}{r^5} & \frac{3yz}{r^5} & \frac{3z^2}{r^5} - \frac{1}{r^3} \end{bmatrix}, \tag{22}$$

$$r = \sqrt{x^2 + y^2 + z^2} \tag{23}$$

and evaluated at $\mathbf{x} = \hat{\mathbf{x}}_k^-$. The measurement $\mathbf{z}_k$ obtained at time t_k is given by

$$\mathbf{z}_k = h(\mathbf{x}_k) + \mathbf{v}_k \tag{24}$$

where

$$\mathbf{v}_k \sim \mathcal{N}(0, R_k) \tag{25}$$

is the measurement noise, which is assumed independent of each other and independent to the initial state as well as process noise. If the measurement is target originated, then the state estimate can be updated by

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K(\hat{\mathbf{x}}_k^-)[\mathbf{z}_k - h(\hat{\mathbf{x}}_k^-)] \tag{26}$$

with the estimation error covariance given by

$$P_k^+ = \left[I - K(\hat{\mathbf{x}}_k^-)H(\hat{\mathbf{x}}_k^-)\right] P_k^- \tag{27}$$

where the filter gain matrix is

$$K(\hat{\mathbf{x}}_k^-) = P_k^- H(\hat{\mathbf{x}}_k^-)^T \left[H(\hat{\mathbf{x}}_k^-)P_k^- H(\hat{\mathbf{x}}_k^-)^T + R_k\right]^{-1} \tag{28}$$

and $H(\hat{\mathbf{x}}_k^-)$ is the Jacobian matrix given by

$$H(\mathbf{x}) = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \tag{29}$$

and evaluated at $\mathbf{x} = \hat{\mathbf{x}}_k^-$.

4.2. Unscented Kalman Filter [21]

Given the mean $\bar{\mathbf{x}}$ and covariance $\mathbf{C_x}$ of a random state $\mathbf{x}$ of dimension $n_\mathbf{x}$, the unscented transform (UT) of $\mathbf{y} = f(\mathbf{x})$ can be obtained by the following steps.

- Compute sigma points χ_i and their weights W_i
  $$\begin{array}{ll} \chi_0 = \bar{\mathbf{x}} & \\ W_0 = \kappa/(n_\mathbf{x} + \kappa) & i = 0 \\ \chi_i = \bar{\mathbf{x}} + (\sqrt{(n_\mathbf{x} + \kappa)\mathbf{C_x}})_i & \\ W_i = 1/\left[2(n_\mathbf{x} + \kappa)\right] & i = 1, \cdots, n_\mathbf{x} \\ \chi_i = \bar{x} - (\sqrt{(n_\mathbf{x} + \kappa)\mathbf{C_x}})_{i-n_\mathbf{x}} & \\ W_i = 1/\left[2(n_\mathbf{x} + \kappa)\right] & i = n_\mathbf{x} + 1, \cdots, 2n_\mathbf{x} \end{array}$$
  where κ is a scaling parameter and $(\sqrt{(n_\mathbf{x} + \kappa)C_\mathbf{x}})_i$ is the i-th row or column of the matrix square root of $(n_\mathbf{x} + \kappa)\mathbf{C_x}$.
- Propagate sigma points through the nonlinear function $\mathbf{y}_i = f(\mathbf{x}_i)$, $i = 0, \cdots, 2n_\mathbf{x}$.
- The mean and covariance of $\mathbf{y}$ and cross covariance between $\mathbf{x}$ and $\mathbf{y}$ are computed as follows.
  $\bar{\mathbf{y}} = \sum_{i=0}^{2n_\mathbf{x}} W_i \mathbf{y}_i$, $\mathbf{C_y} = \sum_{i=0}^{2n_\mathbf{x}} W_i \left(\mathbf{y}_i - \bar{\mathbf{y}}\right)\left(\mathbf{y}_i - \bar{\mathbf{y}}\right)^T$, $\mathbf{C_{xy}} = \sum_{i=0}^{2n_\mathbf{x}} W_i \left(\mathbf{x}_i - \bar{\mathbf{x}}\right)\left(\mathbf{y}_i - \bar{\mathbf{y}}\right)^T$.

With state propagation by the numerical integral via UT, we can obtain $\bar{\mathbf{x}}_k^-$ through the sigma points from time t_{k-1} to t_k. Similarly, we can obtain $\bar{\mathbf{z}}_k$ through the sigma points obtained by the UT based on the measurement equation. The state estimate is updated by

$$\hat{\mathbf{x}}_k^+ = \bar{\mathbf{x}}_k^- + \mathbf{K}[\mathbf{z}_k - \bar{\mathbf{z}}_k] \tag{30}$$

with the estimation error covariance given by

$$P_k^+ = P_k^- - \mathbf{K}P_k^-\mathbf{K}^T \tag{31}$$

where the filter gain matrix is

$$\mathbf{K} = \mathbf{C_{xy}}\mathbf{C_y}^{-1} \tag{32}$$

4.3. Recursive LMMSE Filter

Denote by $\mathbf{Z}^k$ the set of measurements up to time t_k. Denote by $E^*\left[\mathbf{x}_k|\mathbf{Z}^k\right]$ the best linear estimate of the state at time t_k in terms of minimum mean square error. The recursive LMMSE filter applies the following update equation [3]

$$\hat{\mathbf{x}}_{k|k} = E^*\left[\mathbf{x}_k|\mathbf{Z}^k\right] = \hat{\mathbf{x}}_{k|k-1} + K_k\tilde{\mathbf{z}}_{k|k-1} \tag{33}$$

$$P_{k|k} = P_{k|k-1} - K_k S_k K_k^{'} \tag{34}$$

where

$$\begin{aligned}
\hat{\mathbf{x}}_{k|k-1} &= E^*\left[\mathbf{x}_k|\mathbf{Z}^{k-1}\right] \\
\hat{\mathbf{z}}_{k|k-1} &= E^*\left[\mathbf{z}_k|\mathbf{Z}^{k-1}\right] \\
\tilde{\mathbf{x}}_{k|k-1} &= \mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1} \\
\tilde{\mathbf{z}}_{k|k-1} &= \mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1} \\
P_{k|k-1} &= E\left[\tilde{\mathbf{x}}_{k|k-1}\tilde{\mathbf{x}}'_{k|k-1}\right] \\
S_k &= E\left[\tilde{\mathbf{z}}_{k|k-1}\tilde{\mathbf{z}}'_{k|k-1}\right] \\
K_k &= C_{\tilde{\mathbf{x}}_k\tilde{\mathbf{z}}_k} S_k^{-1} \\
C_{\tilde{\mathbf{x}}_k\tilde{\mathbf{z}}_k} &= E\left[\tilde{\mathbf{x}}_{k|k-1}\tilde{\mathbf{z}}'_{k|k-1}\right].
\end{aligned}$$

Given the state estimate $\hat{\mathbf{x}}_{k-1|k-1}$ and its error covariance $P_{k-1|k-1}$ at time t_{k-1}, if the state prediction $\hat{\mathbf{x}}_{k|k-1}$, the corresponding error covariance $P_{k|k-1}$, the measurement prediction $\hat{\mathbf{z}}_{k|k-1}$, the corresponding error covariance S_k, and the crosscovariance $E\left[\tilde{\mathbf{x}}_{k|k-1}\tilde{\mathbf{z}}'_{k|k-1}\right]$ in (33) and (34) can be expressed as a function only through $\hat{\mathbf{x}}_{k-1|k-1}$ and $P_{k-1|k-1}$, then the above formula is truly recursive. However, for general nonlinear system dynamics (9) and measurement equation (24) , we have

$$\hat{\mathbf{x}}_{k|k-1} = E^*\left[\int_{t_{k-1}}^{t_k} f(\mathbf{x}(t),\mathbf{w}(t))dt + \mathbf{x}_{k-1}|\mathbf{Z}^{k-1}\right] \tag{35}$$

$$\hat{\mathbf{z}}_{k|k-1} = E^*\left[h(\mathbf{x}_k,\mathbf{v}_k)|\mathbf{Z}^{k-1}\right]. \tag{36}$$

Both $\hat{\mathbf{x}}_{k|k-1}$ and $\hat{\mathbf{z}}_{k|k-1}$ will depend on the measurement history $\mathbf{Z}^{k-1}$ and the corresponding moments in the LMMSE formula. In order to have a truly recursive filter, the required terms at time t_k can be obtained *approximately* through $\hat{\mathbf{x}}_{k-1|k-1}$ and $P_{k-1|k-1}$, i.e.,

$$\left\{\hat{\mathbf{x}}_{k|k-1}, P_{k|k-1}\right\} \approx \text{Pred}\left[f(\cdot), \hat{\mathbf{x}}_{k-1|k-1}, P_{k-1|k-1}\right]$$

$$\left\{\hat{\mathbf{z}}_{k|k-1}, S_k, C_{\tilde{\mathbf{x}}_k\tilde{\mathbf{z}}_k}\right\} \approx \text{Pred}\left[h(\cdot), \hat{\mathbf{x}}_{k|k-1}, P_{k|k-1}\right]$$

where $\text{Pred}\left[f(\cdot), \hat{\mathbf{x}}_{k-1|k-1}, P_{k-1|k-1}\right]$ denotes that $\left\{\hat{\mathbf{x}}_{k-1|k-1}, P_{k-1|k-1}\right\}$ propagates through the nonlinear function $f(\cdot)$ to approximate $E^*\left[f(\cdot)|\mathbf{Z}^{k-1}\right]$ and the corresponding error covariance $P_{k|k-1}$.

Similarly, $\text{Pred}\left[h(\cdot), \hat{\mathbf{x}}_{k|k-1}, P_{k|k-1}\right]$ predicts the measurement and the corresponding error covariance only through the approximated state prediction. This poses difficulties for the implementation of the recursive LMMSE filter due to insufficient information. The prediction of a random variable going through a nonlinear function, most often, can not be completely determined using only the first and second moments. Two remedies are often used: One is to approximate the system to the best extent such that the prediction based on the approximated system can be carried out only through $\left\{\hat{\mathbf{x}}_{k-1|k-1}, P_{k-1|k-1}\right\}$ [31]. Another approach is to replace the first and second moments with a reasonable distribution so that the prediction is made by carrying out the fully specified distribution through the nonlinear system [19,32]. Note that inaccurate approximation often leads to unexpected filter behavior and tracking performance being highly scenario dependent. Once the predicted first and second moments are obtained, the filter uses the LMMSE estimation formula (33)–(34) to update the state and error covariance.

4.4. Particle Filter [26]

The recursive Bayesian filtering for a continuous-time random process $\mathbf{x}(t)$ based on discrete-time measurement sequence $\mathbf{Z}^k \triangleq \{\mathbf{z}_i\}_{i=1,2,\ldots k}$ requires the knowledge of state propagation model $p(\mathbf{x}(t_k)|\mathbf{x}(t_{k-1}))$ and the likelihood function $p(\mathbf{z}_k|\mathbf{x}(t_k))$. The recursive Bayesian filtering paradigm provides the complete posterior distribution $p(\mathbf{x}_k|\mathbf{Z}^k)$ via the following *prediction – update* recursions:

$$p(\mathbf{x}(t_{k-1})|\mathbf{Z}^{k-1}) \to p(\mathbf{x}(t_k)|\mathbf{Z}^{k-1}) \to p(\mathbf{x}(t_k)|\mathbf{Z}^k) \tag{37}$$

given the initial distribution $p(\mathbf{x}(t_0))$ and measurements $\mathbf{Z}^k = \{\mathbf{z}_1, ..., \mathbf{z}_k\}$. In a general setting, the state and measurement models are given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= f\left(\mathbf{x}(t), \mathbf{w}(t), t\right), \\ \mathbf{z}_k &= h\left(\mathbf{x}(t_k), \mathbf{v}_k\right). \end{aligned}$$

Provided that the process noise $\mathbf{w}(t)$ and measurement noise $\mathbf{v}_k$ are zero mean, white, independent of the past and present states and mutually independent with known distributions $p(\mathbf{w})$ and $p(\mathbf{v}_k)$, respectively, then the corresponding probability distributions are given by [3]

$$p\left(\mathbf{x}(t_k)|\mathbf{x}(t_{k-1})\right) = \int \delta\left(\mathbf{x}(t_k) - \mathbf{x}(t_{k-1}) - \int_{t_{k-1}}^{t_k} f\left(\mathbf{x}(t), \mathbf{w}(t), t\right) dt\right) p\left(\mathbf{w}\right) d\mathbf{w} \tag{38}$$

$$p(\mathbf{z}_k|\mathbf{x}(t_k)) = \int \delta\left(\mathbf{z}_k - h\left(\mathbf{x}_k, \mathbf{v}_k\right)\right) p\left(\mathbf{v}_k\right) d\mathbf{v}_k \tag{39}$$

where $\delta(\cdot)$ is the Dirac delta function. Bayesian Monte Carlo filtering represents the posterior distribution of the state by random samples and implements the Bayesian filter recursion (37) directly on these samples instead of trying to obtain the closed form of the posterior distribution. Many existing particle filtering algorithms are naturally comprised by such a framework [14]. Because of its generality, here we only present a particle filter implementation based on *Sequential Importance Sampling and Resampling* (SIS/R). The marginal distribution $p\left(\mathbf{x}(t_k)|\mathbf{Z}^k\right)$ is approximated by a weighted sample set $\left\{\mathbf{x}_k^i, w_k^i\right\}_{i=1}^N$, which is recursively propagated and updated. It is also assumed that an *importance* (also called *proposal*) distribution $q(\cdot)$, containing the support of the posterior distribution, is available for sampling.

Algorithm SIS/R

Initialization $(k = 0)$

For $i = 1, \ldots, N$

Set $w_0^i = \frac{1}{N}$ and draw $\mathbf{x}_0^i \sim q\left(\mathbf{x}_0\right)$.

For $k = 1, 2, \ldots$

IS-step

For $i = 1, \ldots, N$

– Draw a sample $\tilde{\mathbf{x}}_k^i \sim q\left(\mathbf{x}(t_k)|\mathbf{x}_{k-1}^i, \mathbf{Z}^{k-1}\right)$

– Evaluate the importance weights

$$\widetilde{w}_k^i = w_{k-1}^i \frac{p\left(\mathbf{z}_k|\tilde{\mathbf{x}}_k^i\right) p\left(\tilde{\mathbf{x}}_k^i|\mathbf{x}_{k-1}^i\right)}{q\left(\tilde{\mathbf{x}}_k^i|\mathbf{x}_{k-1}^i, \mathbf{Z}^{k-1}\right)}$$

For $i = 1, \ldots, N$

– Normalize the importance weights

$$w_k^i = \frac{\tilde{w}_k^i}{\sum_{j=1}^N \tilde{w}_k^j}.$$

R-step

Effective sample size estimation $\hat{N}_{\text{eff}} = \frac{1}{\sum_{j=1}^{N} \left(w_k^j\right)^2}$

If $\hat{N}_{\text{eff}} < N_{\text{th}}$

– Obtain a new sample $\left\{\mathbf{x}_k^j(i)\right\}_{i=1}^{N}$ by resampling N times with *replacement* from $\left\{\mathbf{x}_k^j\right\}_{j=1}^{N}$ such that $\Pr\left\{\mathbf{x}_k^j(i) = \mathbf{x}_k^j\right\} = w_k^j$.

– Reset $w_k^i = \frac{1}{N}$.

Output (optional)

$\hat{\mathbf{x}}(t_k) = \sum_{i=1}^{N} w_k^i \mathbf{x}_k^i$

Note that the importance distribution $q(\cdot)$ is a "design parameter" and it permits quite a lot of choices to influence the performance of any specific application. Another important feature of the above scheme is the on-line detection of degeneracy and the use of resampling for its mitigation. In the particular case of choosing q to be the prior $p(\mathbf{x}(t_k)|\mathbf{x}_{k-1}^i)$ and with resampling at every time, the above particle filter reduces to the original SIR algorithm of [17]. Resampling on the other hand usually leads to another problem, known as sample impoverishment, which simply put, means replicating high probability sample points and thus loosing low probability sample points. One option to diversify the samples is to use a random move called Metropolis-Hasting step after an eventual resampling in the above SIS/R scheme [16].

5. POSTERIOR CRAMER-RAO LOWER BOUND

Denote by $J(t)$ the Fisher information matrix and the posterior Cramer-Rao lower bound (PCRLB) is given by [29]

$$B(t) = J(t)^{-1} \tag{40}$$

which quantifies the ideal mean square error of any filtering algorithm, i.e.,

$$E\left[(\hat{\mathbf{x}}(t_k) - \mathbf{x}(t_k))(\hat{\mathbf{x}}(t_k) - \mathbf{x}(t_k))^T|\mathbf{Z}^k\right] \geq B(t_k). \tag{41}$$

Under additive white Gaussian process noise model, the Fisher information matrix satisfies the following differential equation

$$\dot{J}(t) = -J(t)F(\mathbf{x}) - F(\mathbf{x})^T J(t) - J(t)Q(t)J(t) \tag{42}$$

for $t_{k-1} \leq t \leq t_k$ where F is the Jacobian matrix given by

$$F(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}. \tag{43}$$

When a measurement is obtained at time t_k with additive Gaussian noise $\mathcal{N}(0, R_k)$, the new Fisher information matrix is

$$J(t_k^+) = J(t_k^-) + E_{\mathbf{x}}\left[H(\mathbf{x})^T R_k^{-1} H(\mathbf{x})\right] \tag{44}$$

where H is the Jacobian matrix given by

$$H(\mathbf{x}) = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}. \tag{45}$$

The initial condition for the recursion is $J(t_0)$ and the PCRLB can be obtained with respect to the true distribution of the state $\mathbf{x}(t)$.

Note that unlike parameter estimation, PCRLB may not be achievable even asymptotically [29]. On the other hand, exact conditional mean and conditional covariance from the recursive Bayesian filter often require intense computation such as using large amount of particles to approximate the posterior distribution. Thus PCRLB is popularly used as a viable performance evaluation metric for nonlinear filtering problem. Other Bayesian bounds such as Bobrovsky-Zakai bound [5] and Weiss-Weinstein bound [30] are often sharper than the PCRLB, however, their heavy computational requirements prohibit them from being applied to orbital determination problem, i.e., the bound calculation is as expensive as running a particle filter with a large number of particles.

6. SIMULATION STUDY

6.1. Experiment Setup

We consider a single LEO observer that tracks a GEO satellite with possible maneuver. The LEO orbit is a nearly circular one with a radius around 6600km and its position is assumed to be accurately calibrated by the GPS.[2] The target being tracked is in a GEO with radius approximately 42164km. We consider range, azimuth and elevation measurements of the target without false alarm or missed detection except when the line of sight between the observer and the target is blocked by the earth. The standard deviations of measurement error are 0.1km, 2mrad/sec, 2mrad/sec, for range, azimuth, elevation, respectively. The sensor onboard the observer has a fixed sampling interval of 50s. The target has white process noise with the magnitude of random acceleration at $0.01\mathrm{m/s}^2$.

6.2. Comparison of Tracking Accuracy

We implemented EKF, UKF, PF and recursive LMMSE filter for orbital trajectory estimation assuming that the target does not maneuver. All filters were initialized by the first measurement with the same initial covariance matrix. We used 100 Monte Carlo runs for filter comparison and the same target trajectory and measurement sequence were used to evaluate all filters in each run. We consider the case that both the observer and the target share the same orbit plane. The RMS position errors obtained by various nonlinear filters and the PCRLB are shown in Figure 1. The RMS velocity errors obtained by various nonlinear filters and the PCRLB are shown in Figure 2. We can see the increase of RMS position error in three segments due to the earth blockage where state prediction has increasing error covariance over time due to the process noise. The EKF yields the largest RMS position error. Next to it is the UKF. LMMSE filter has the best position estimation accuracy and PF has the performance very close to it. Note that PF should approach the best achievable accuracy as the number of particles go to infinity. However, from the practical view point, LMMSE filter is the best choice among these trackers in terms of the tradeoff between tracking accuracy and computational requirement. Note that there exists a significant gap between the RMS position error by the LMMSE filter and that obtained by PCRLB. We suspect that the PCRLB may not be a tight bound. Obtaining tighter bounds for filter comparison deserves further research.

We also studied the effect of missed detection and false alarms on the tracking accuracy of various nonlinear filtering algorithms. We found that the recursive LMMSE with probabilistic data association filter (PDAF) has the best performance under mild random clutter. In addition, the LMMSE filter can be extended to update with the delayed measurement efficiently which yields slightly larger tracking error compared with the in-sequence filter update. Numerous comparisons have been omitted due to page limit but can be found in [9].

7. SUMMARY AND CONCLUSIONS

In this paper, we have compared the orbital state estimation accuracy among several nonlinear filters, namely, the extended Kalman Filter (EKF), the unscented KF (UKF), the particle filter (PF), and recursive linear minimum mean square error (LMMSE) filter. Various tracking configurations are considered based on the low orbital observers with different types of available measurements, including range, angle, and range rate. We found that the LMMSE filter has close to the best performance with moderate computational complexity. The tracking accuracy of PF depends highly on the number of particles and in our study is slightly worse than the LMMSE filter but superior to the UKF. The EKF has the largest RMS position and velocity error but can maintain the track during the time segments of earth blockage if the target does not maneuver. There exists a performance gap between the achievable RMS position and velocity error and the corresponding posterior Cramer-Rao lower bound (PCRLB). We suspect that PCRLB is not tight.

[2]The observer has eccentricity smaller than 0.01 and its mean motion is 16.2 revolutions per day. Most of the existing LEO satellites such as IRIDIUM and ORBCOMM satellites have circular orbits with radius in the 7000km to 7200km range and larger periods in rotation.

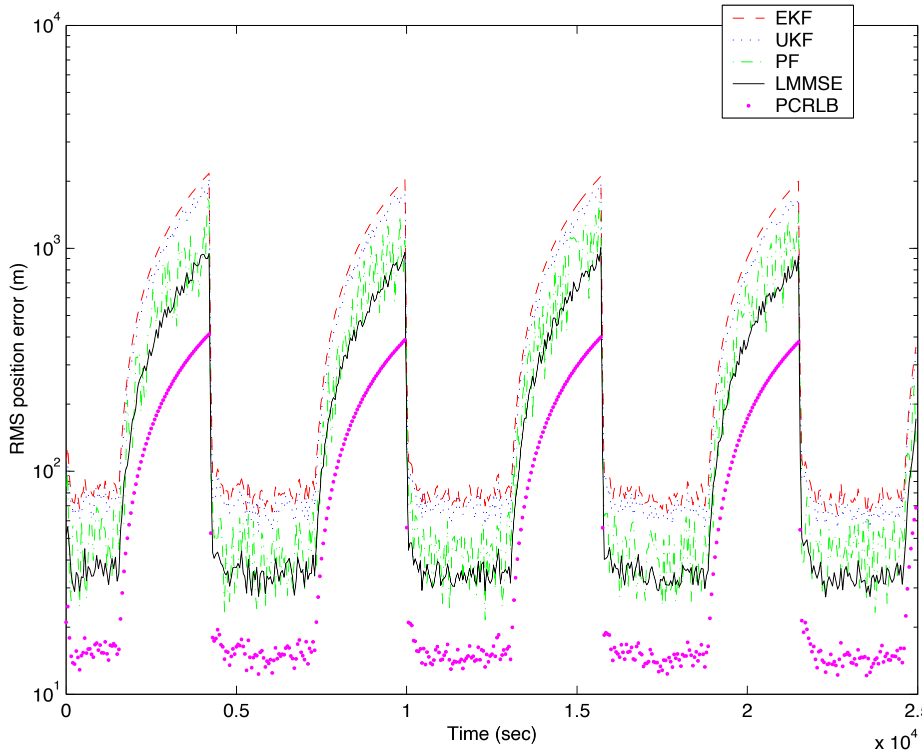

Figure 1. Comparison of RMS position error among various nonlinear filters, 100 runs, same orbit plane.

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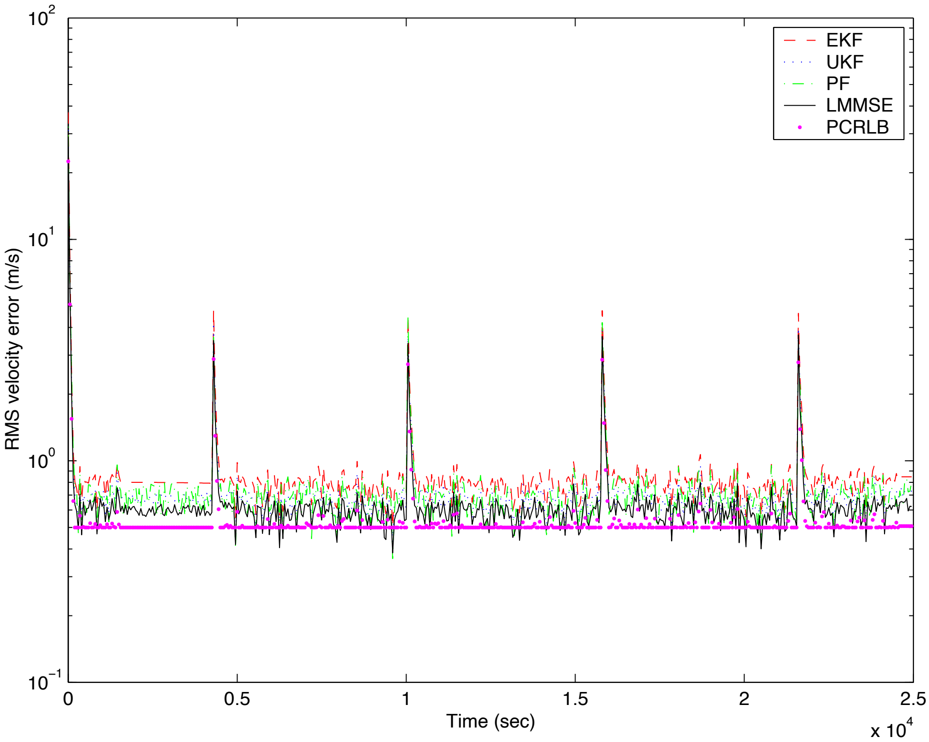

Figure 2. Comparison of RMS velocity error among various nonlinear filters, 100 runs, same orbit plane.

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