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Υπολογιστών

Προηγμένες Τεχνικές Επεξεργασίας Σήματος

Εργασία 1^η – Εαρινό Εξάμηνο 2020/2021



Exercise:

Consider $\mathbf{X}(k)$ given by

$$\mathbf{X}(k) = \mathbf{W}(k) - \mathbf{W}(k-1), k = \pm 1, \pm 2, \dots,$$

where $\{\mathbf{W}(k)\}$ is a stationary stochastic process with independent, identically distributed (i.i.d.) stochastic variables and $\mathbf{E}\{\mathbf{W}(k)\} = \mathbf{0}$, $\mathbf{E}\{\mathbf{W}^2(k)\} = \mathbf{1}$ and $\mathbf{E}\{\mathbf{W}^3(k)\} = \mathbf{0}$. The covariance sequence of $\{\mathbf{X}(k)\}$ is given by:

$$\begin{aligned} c_2^x(\tau) &= m_2^x(\tau) = \mathbf{E}\{\mathbf{X}(k)\mathbf{X}(k+\tau)\} = \mathbf{E}\{(\mathbf{W}(k) - \mathbf{W}(k-1))(\mathbf{W}(k+\tau) - \mathbf{W}(k+\tau-1))\} \\ &= 2\delta(\tau) - \delta(\tau-1) - \delta(\tau+1) \end{aligned}$$

where $\delta(\tau)$ is the delta Kronecker function; hence,

$$c_2^x(\tau) = \begin{cases} 2, & \tau = 0 \\ -1, & \tau = 1, \tau = -1 \\ 0, & \text{elsewhere} \end{cases}$$

The corresponding Power Spectrum is given by:

$$C_2^x(\omega) = \sum_{\tau=-1}^1 c_2^x(\tau) e^{-j\omega\tau} = (2 - 2\cos\omega)$$

1. Find the 3rd order cumulants of $\{\mathbf{X}(k)\}$, i.e., $c_3^x(\tau_1, \tau_2)$
2. Find the skewness $\gamma_3^x = c_3^x(0, 0)$. What do you observe?
3. Find the Bispectrum $C_3^x(\omega_1, \omega_2)$. Is it complex, real or imaginary?
4. How the result of 2 affects the result of 3? Can you draw a general comment?

Solution:

$$1. \quad c_3^x(\tau_1, \tau_2) = m_3^x(\tau_1, \tau_2) - m_1^x[m_2^x(\tau_1) + m_2^x(\tau_2) + m_2^x(\tau_2 - \tau_1)] + 2(m_1^x)^3 \quad (\text{A})$$

Given that

$$m_1^x = EX(k) = E\{(W(k) - W(k-1))\} = E\{W(k)\} - E\{W(k-1)\} = 0 - 0 = 0$$

(A) becomes:

$$\begin{aligned} c_3^x(\tau_1, \tau_2) &= m_3^x(\tau_1, \tau_2) = E\{X(k) X(k + \tau_1) X(k + \tau_2)\} \\ &= E\{(W(k) - W(k-1))(W(k + \tau_1) - W(k + \tau_1 - 1))(W(k + \tau_2) - W(k + \tau_2 - 1))\} \\ &= E\{(W(k) - W(k-1))[W(k + \tau_1)W(k + \tau_2) - W(k + \tau_1 - 1)W(k + \tau_2) - \\ &\quad W(k + \tau_1)W(k + \tau_2 - 1) + W(k + \tau_1 - 1)W(k + \tau_2 - 1)]\} \\ &= E\{(W(k)W(k + \tau_1)W(k + \tau_2) - W(k)W(k + \tau_1 - 1)W(k + \tau_2) \\ &\quad - W(k)W(k + \tau_1)W(k + \tau_2 - 1) + W(k)W(k + \tau_1 - 1)W(k + \tau_2 - 1) \\ &\quad - W(k)W(k + \tau_1)W(k + \tau_2) + W(k)W(k + \tau_1 - 1)W(k + \tau_2) \\ &\quad + W(k)W(k + \tau_1)W(k + \tau_2 - 1) - W(k)W(k + \tau_1 - 1)W(k + \tau_2 - 1))\} \\ &= E\{W(k)W(k + \tau_1)W(k + \tau_2)\} - E\{W(k)W(k + \tau_1 - 1)W(k + \tau_2)\} \\ &\quad - E\{W(k)W(k + \tau_1)W(k + \tau_2 - 1)\} \\ &\quad + E\{W(k)W(k + \tau_1 - 1)W(k + \tau_2 - 1)\} - E\{W(k)W(k + \tau_1)W(k + \tau_2)\} \\ &\quad + E\{W(k)W(k + \tau_1 - 1)W(k + \tau_2)\} + E\{W(k)W(k + \tau_1)W(k + \tau_2 - 1)\} \\ &\quad - E\{W(k)W(k + \tau_1 - 1)W(k + \tau_2 - 1)\} \quad (\text{B}) \end{aligned}$$

Each term takes non zero value if all 3 inputs are the same. For example,

$$E\{W(k)W(k + \tau_1)W(k + \tau_2)\}$$

takes zero value for every combination that at least one of τ_1, τ_2 is different than zero, while in other case it is simply

$$E\{W(k)W(k)W(k)\} = 1$$

Thus, we can express it as

$$E\{W(k)W(k + \tau_1)W(k + \tau_2)\} = \delta(\tau_1)\delta(\tau_2)$$

Therefore,

(B):

$$\begin{aligned} c_3^x(\tau_1, \tau_2) &= \delta(\tau_1)\delta(\tau_2) - \delta(\tau_1 - 1)\delta(\tau_2) - \delta(\tau_1)\delta(\tau_2 - 1) + \delta(\tau_1 - 1)\delta(\tau_2 - 1) \\ &\quad - \delta(\tau_1 + 1)\delta(\tau_2 + 1) + \delta(\tau_1)\delta(\tau_2 + 1) + \delta(\tau_1 + 1)\delta(\tau_2) - \delta(\tau_1)\delta(\tau_2) \end{aligned}$$

$$\Leftrightarrow c_3^x(\tau_1, \tau_2) = \begin{cases} -1, & (\tau_1, \tau_2) = (1, 0) \text{ or } (\tau_1, \tau_2) = (0, 1) \text{ or } (\tau_1, \tau_2) = (-1, -1) \\ 1, & (\tau_1, \tau_2) = (1, 1) \text{ or } (\tau_1, \tau_2) = (0, -1) \text{ or } (\tau_1, \tau_2) = (-1, 0) \\ 0, & \text{elsewhere} \end{cases}$$

2. Skewness:

$$\gamma_3^x = c_3^x(0,0) = 0 \quad (\text{from 1}), \text{there is no skewness}$$

3. Bispectrum:

$$\begin{aligned} C_3^x(\omega_1, \omega_2) &= \sum_{\tau_1=-\infty}^{+\infty} \sum_{\tau_2=-\infty}^{+\infty} c_3^x(\tau_1, \tau_2) \exp\{-j(\omega_1\tau_1 + \omega_2\tau_2)\} \\ &= \sum_{\tau_1=-1}^{+1} \sum_{\tau_2=-1}^{+1} c_3^x(\tau_1, \tau_2) \exp\{-j(\omega_1\tau_1 + \omega_2\tau_2)\} \\ &= -e^{j(\omega_1+\omega_2)} - e^{-j\omega_1} - e^{-j\omega_2} + e^{j\omega_1} + e^{j\omega_2} + e^{-j(\omega_1+\omega_2)} \\ &= 2j \cdot \sin(\omega_1) + 2j \cdot \sin(\omega_2) - 2j \cdot \sin(\omega_1 + \omega_2) \end{aligned}$$

Thus, the Bispectrum is imaginary

4. Comments:

It is clear that the result of 2 affects the result of 3 because the term that we get for

$$(\tau_1, \tau_2) = (0, 0)$$

shows up at the sum, affecting whether the Bispectrum is imaginary (if skewness is 0) or if it's complex (if skewness is $\neq 0$). In 3 we observe symmetry. We can see the terms $E\{W(k)W(k + \tau_1)W(k + \tau_2)\}$ and $-E\{W(k)W(k + \tau_1 - 1)W(k + \tau_2 - 1)\}$ in 1, cancel each other and thus we get

$$\gamma_3^x = c_3^x(0,0) = 0$$

which is by itself a symmetry factor.