Aristotle University of Thessaloniki
Department of Electrical and Computer Engineering

Advanced Signal Processing

1st Assignment – Summer Semester 2020/2021

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Exercise:

Consider $\boldsymbol{X(k)}$ given by

$$\boldsymbol{X(k) = W(k) - W(k-1)\,, k = \pm 1, \pm 2, \ldots,}$$

where $\{\boldsymbol{W(k)}\}$ is a stationary stochastic process with independent, identically distributed(i.i.d.) stochastic variables and $\boldsymbol{E\{W(k)\} = 0}$, $\boldsymbol{E\{W^2(k)\} = 1}$ $\boldsymbol{\kappa\alpha\iota}$ $\boldsymbol{E\{W^3(k)\} = 1.}$ The covariance sequence of $\{\boldsymbol{X(k)}\}$ is given by:

$$\boldsymbol{c_2^x(\tau) = m_2^x(\tau) = E\{X(k)X(k+\tau)\} = E\{\big(W(k) - W(k-1)\big)\big(W(k+\tau) - W(k+\tau-1)\big)\}}$$
$$\boldsymbol{= 2\delta(\tau) - \delta(\tau-1) - \delta(\tau+1)}$$

where $\delta(\tau)$ is the delta Kronecker function; hence,

$$\boldsymbol{c_2^x(\tau) = \begin{cases} 2, & \tau = 0 \\ -1, & \tau = 1, \tau = -1 \\ 0, & elsewhere \end{cases}}$$

The corresponding Power Spectrum is given by:

$$\boldsymbol{C_2^x(\omega) = \sum_{\tau=-1}^{1} c_2^x(\tau)\, e^{-j\omega\tau} = (2 - 2cos\omega)}$$

1. Find the 3rd order cumulants of $\{X(k)\}$, i.e., $\boldsymbol{c_3^x(\tau_1, \tau_2)}$
2. Find the skewness $\boldsymbol{\gamma_3^x = c_3^x(0,0).}$ What do you observe?
3. Find the Bispectrum $\boldsymbol{C_3^x(\omega_1, \omega_2).}$ Is it complex, real or imaginary?
4. How the result of 2 affects the result of 3? Can you draw a general comment?

Solution:

1. $$c_3^x(\tau_1,\tau_2) = m_3^x(\tau_1,\tau_2) - m_1^x[m_2^x(\tau_1) + m_2^x(\tau_2) + m_2^x(\tau_2 - \tau_1)] + 2(m_1^x)^3 \qquad \textbf{(A)}$$

Given that

$$m_1^x = EX(k) = E\{(W(k) - W(k-1))\} = E\{W(k)\} - E\{W(k-1)\} = 0 - 0 = 0$$

(A) becomes:

$$c_3^x(\tau_1,\tau_2) = m_3^x(\tau_1,\tau_2) = E\{X(k)\,X(k+\tau_1)\,X(k+\tau_2)\}$$
$$= E\{(W(k) - W(k-1))(W(k+\tau_1) - W(k+\tau_1-1))(W(k+\tau_2) - W(k+\tau_2-1))\}$$
$$= E\{(W(k) - W(k-1))[W(k+\tau_1)W(k+\tau_2) - W(k+\tau_1-1)W(k+\tau_2) - W(k+\tau_1)\,W(k+\tau_2-1) + W(k+\tau_1-1)W(k+\tau_2-1)]\}$$
$$\begin{aligned} = E\{(&W(k)W(k+\tau_1)W(k+\tau_2) - W(k)W(k+\tau_1-1)W(k+\tau_2) \\ &- W(k)W(k+\tau_1)W(k+\tau_2-1) + W(k)W(k+\tau_1-1)W(k+\tau_2-1) \\ &- W(k)W(k+\tau_1)W(k+\tau_2) + W(k)W(k+\tau_1-1)W(k+\tau_2) \\ &+ W(k)W(k+\tau_1)W(k+\tau_2-1) - W(k)W(k+\tau_1-1)W(k+\tau_2-1))\} \end{aligned}$$
$$\begin{aligned} = &E\{W(k)W(k+\tau_1)W(k+\tau_2)\} - E\{W(k)W(k+\tau_1-1)W(k+\tau_2)\} \\ &- E\{W(k)W(k+\tau_1)W(k+\tau_2-1)\} \\ &+ E\{W(k)W(k+\tau_1-1)W(k+\tau_2-1)\} - E\{W(k)W(k+\tau_1)W(k+\tau_2)\} \\ &+ E\{W(k)W(k+\tau_1-1)W(k+\tau_2)\} + E\{W(k)W(k+\tau_1)W(k+\tau_2-1)\} \\ &- E\{W(k)W(k+\tau_1-1)W(k+\tau_2-1)\} \qquad (B) \end{aligned}$$

Each term takes non zero value if all 3 inputs are the same. For example,

$$E\{W(k)W(k+\tau_1)W(k+\tau_2)\}$$

takes zero value for every combination that at least one of $\tau_1 . \tau_2$ is different than zero, while in other case it is simply

$E\{W(k)W(k)W(k)\} = 1$

Thus, we can express it as

$$E\{W(k)W(k+\tau_1)W(k+\tau_2)\} = \delta(\tau_1)\delta(\tau_2)$$

Therefore,

(B) :

$$\begin{aligned} c_3^x(\tau_1,\tau_2) = \;&\delta(\tau_1)\delta(\tau_2) - \delta(\tau_1-1)\delta(\tau_2) - \delta(\tau_1)\delta(\tau_2-1) + \delta(\tau_1-1)\delta(\tau_2-1) \\ &- \delta(\tau_1+1)\delta(\tau_2+1) + \delta(\tau_1)\delta(\tau_2+1) + \delta(\tau_1+1)\delta(\tau_2) - \delta(\tau_1)\delta(\tau_2) \end{aligned}$$

$$\Leftrightarrow c_3^x(\tau_1,\tau_2) = \begin{cases} -1, & (\tau_1,\tau_2) = (1,0) \text{ or } (\tau_1,\tau_2) = (0,1) \text{ or } (\tau_1,\tau_2) = (-1,-1) \\ 1, & (\tau_1,\tau_2) = (1,1) \text{ or } (\tau_1,\tau_2) = (0,-1) \text{ or } (\tau_1,\tau_2) = (-1,0) \\ 0, & \text{elsewhere} \end{cases}$$

2. **Skewness:**

$$\boldsymbol{\gamma_3^x = c_3^x(0,0) = 0 \quad (from\ 1), there\ is\ no\ skewness}$$

3. **Bispectrum:**

$$\begin{aligned}\boldsymbol{C_3^x(\omega_1,\omega_2)} &= \sum_{\tau_1=-\infty}^{+\infty}\sum_{\tau_2=-\infty}^{+\infty} \boldsymbol{c_3^x(\tau_1,\tau_2)\, exp\{-j(\omega_1\tau_1+\omega_2\tau_2)\}} \\ &= \sum_{\tau_1=-1}^{+1}\sum_{\tau_2=-1}^{+1} \boldsymbol{c_3^x(\tau_1,\tau_2)\, exp\{-j(\omega_1\tau_1+\omega_2\tau_2)\}} \\ &= -e^{j(\omega_1+\omega_2)} - e^{-j\omega_1} - e^{-j\omega_2} + e^{j\omega_1} + e^{j\omega_2} + e^{-j(\omega_1+\omega_2)} \\ &= 2j\cdot sin(\omega_1) + 2j\cdot sin(\omega_2) - 2j\cdot sin(\omega_1+\omega_2)\end{aligned}$$

Thus, the Bispectrum is imaginary

4. **Comments:**

It is clear that the result of 2 affects the result of 3 because the term that we get for

$$\boldsymbol{(\tau_1,\tau_2) = (0,0)}$$

shows up at the sum, affecting whether the Bispectrum is imaginary (if skewness is 0) or if it's complex (if skewness is ≠ 0). In 3 we observe symmetry. We can see the terms $\boldsymbol{E\{W(k)W(k+\tau_1)W(k+\tau_2)\}}$ and $\boldsymbol{-E\{W(k)W(k+\tau_1-1)W(k+\tau_2-1)}$ in 1, cancel each other and thus we get

$$\boldsymbol{\gamma_3^x = c_3^x(0,0) = 0}$$

which is by itself a symmetry factor.