Graphs and Graph Algorithms

Fundamentals, Terminology, Traversal and Algorithms

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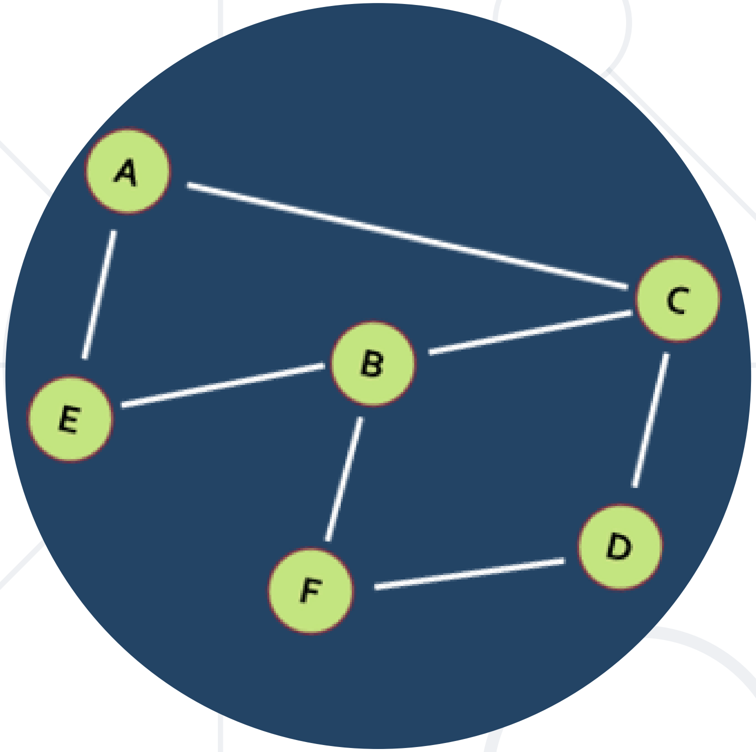
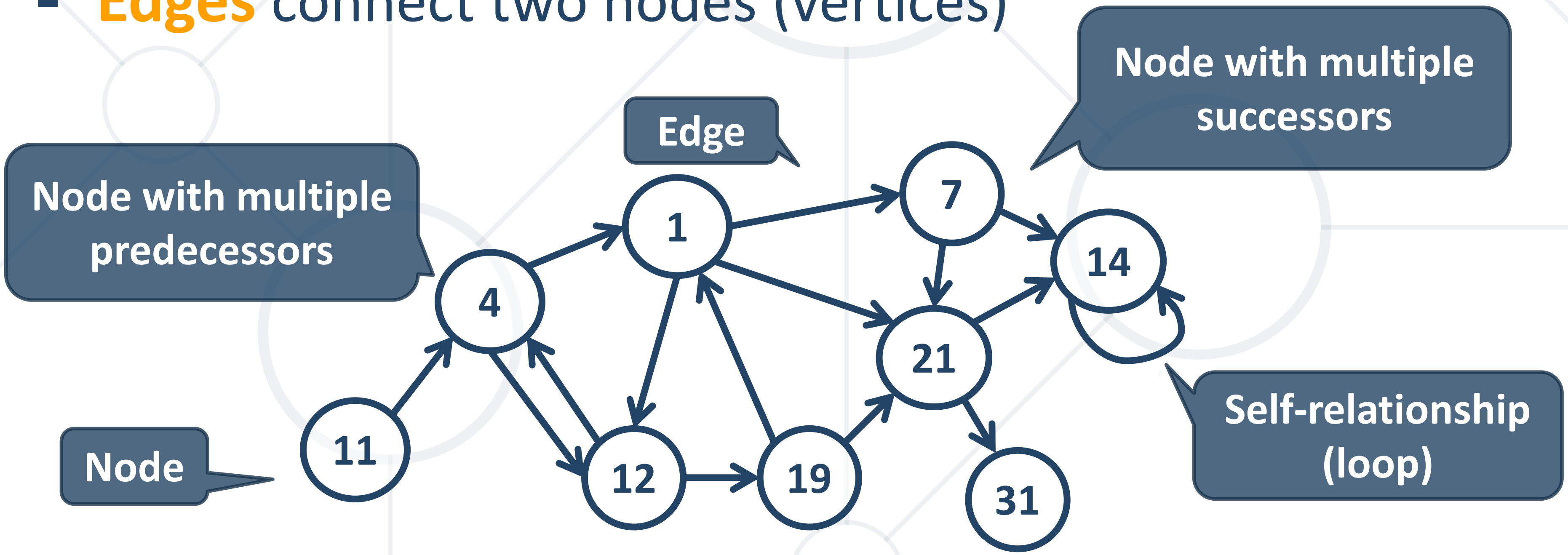


Graphs

Definitions and Terminology

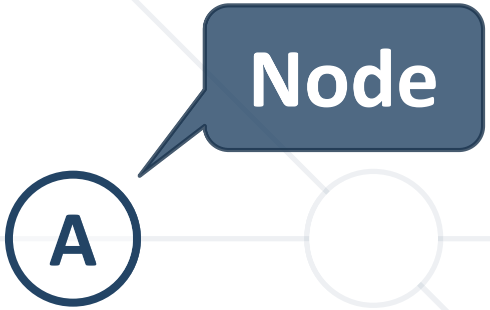
Graph Data Structure

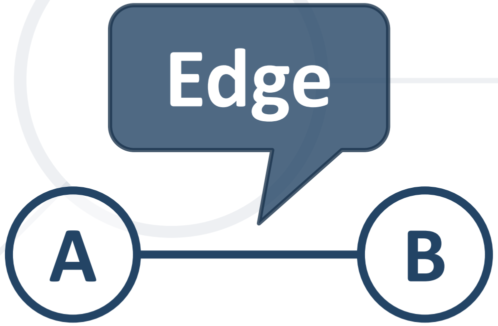
- **Graph**, denoted as **G(V, E)**
  - Set of nodes **V** with many-to-many relationship between them (edges **E**)
  - Each **node** (**vertex**) has **multiple** predecessors and **multiple** successors
  - **Edges** connect two nodes (vertices)



Graph Definitions (1)

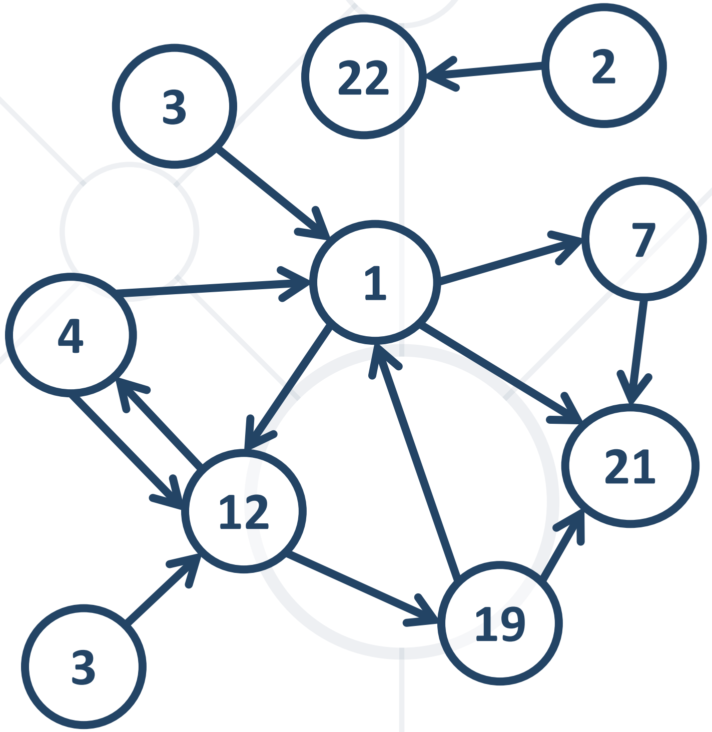
- **Node** (vertex)
  - Element of a graph
  - Can have name / value
  - Keeps a list of adjacent nodes
- **Edge**
  - Connection between two nodes
  - Can be directed / undirected
  - Can be weighted / unweighted



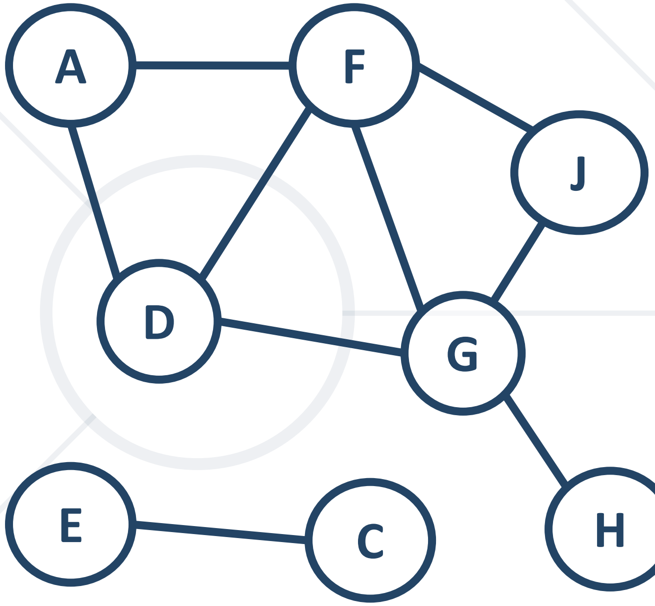


Graph Definitions (2)

- **Directed graph**
  - Edges have direction

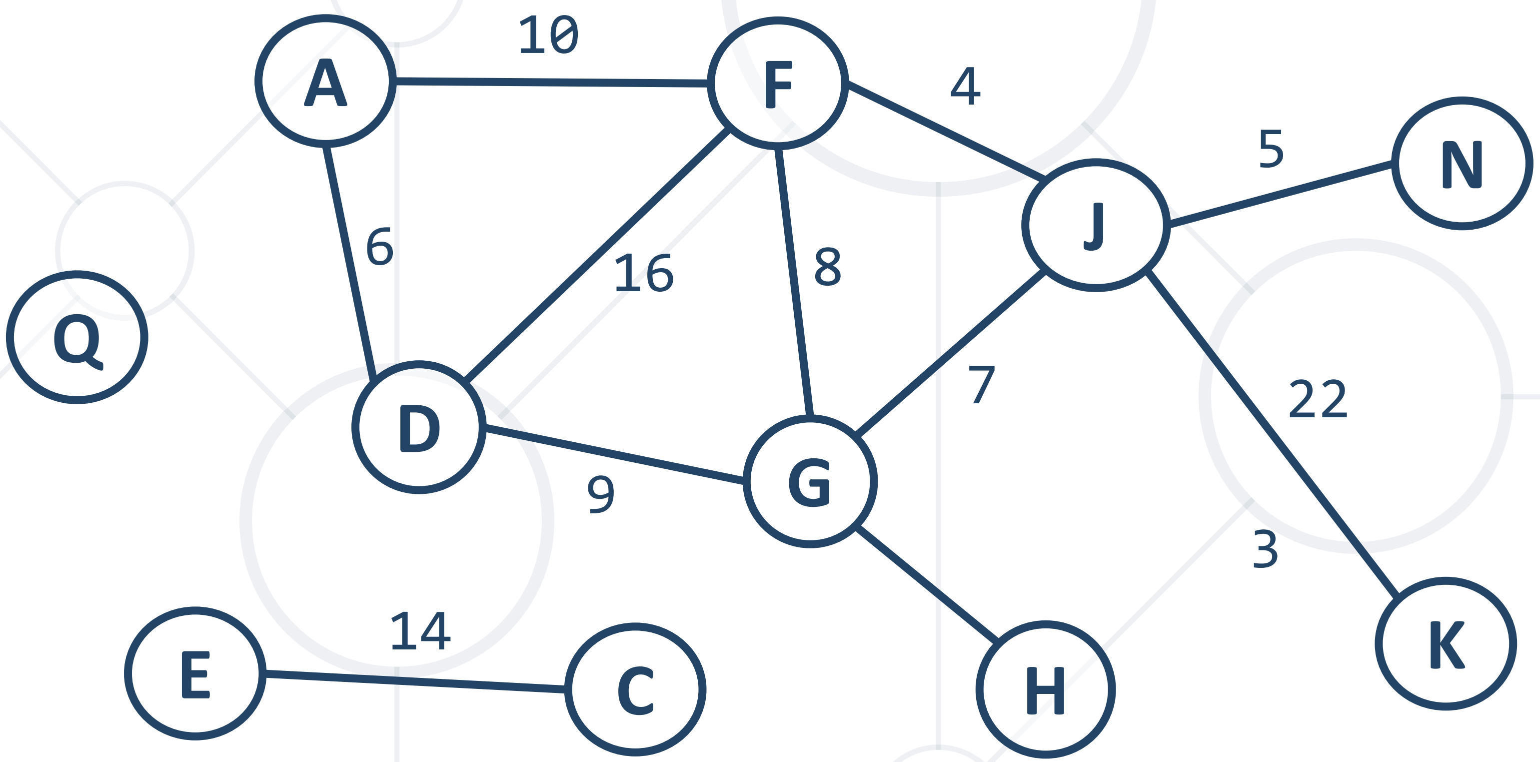


- **Undirected graph**
  - Undirected edges



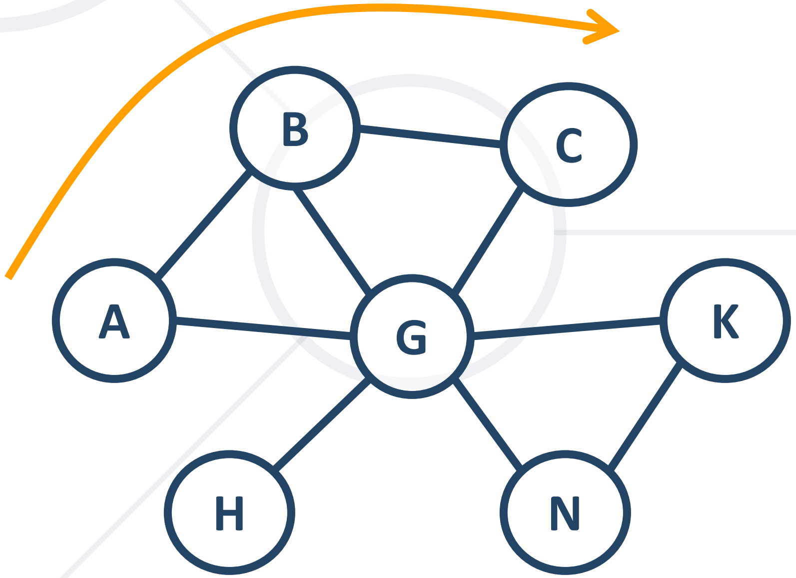
Graph Definitions (3)

- **Weighted graph**
  - Weight (cost) is associated with each edge



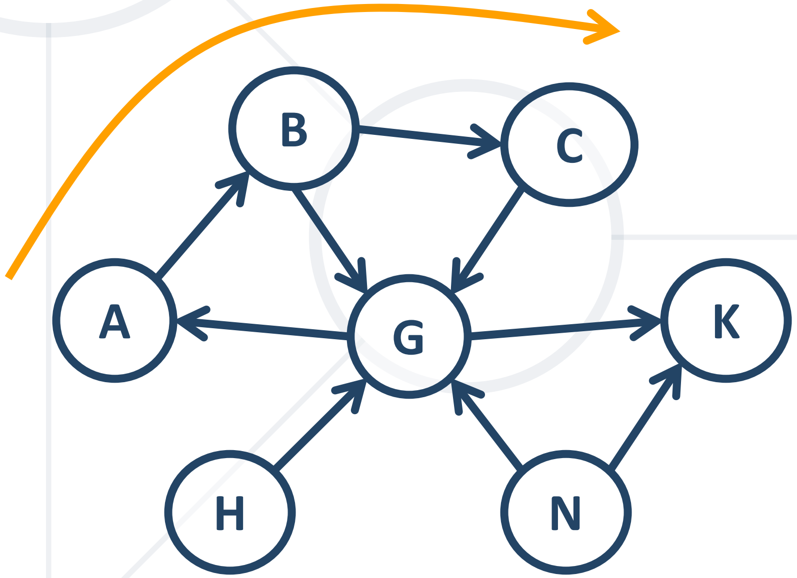
Graph Definitions (4)

- **Path** (in undirected graph)
  - Sequence of nodes n_1, n_2, … n_k
  - Edge exists between each pair of nodes n_i, n_{i+1}
  - Examples:
    - A, B, C is a path
    - A, B, G, N, K is a path
    - H, K, C is not a path
    - H, G, B, G, N is a path



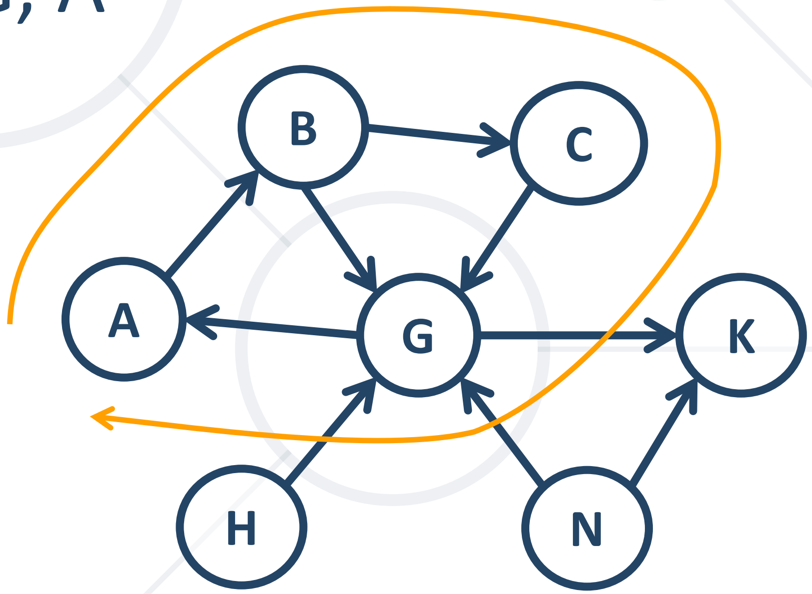
Graph Definitions (5)

- **Path** (in directed graph)
  - Sequence of nodes $n_1, n_2, \ldots n_k$
  - Directed edge exists between each pair of nodes n_i, n_{i+1}
  - Examples:
    - A, B, C is a path
    - N, G, A, B, C is a path
    - A, G, K is not a path
    - H, G, K, N is not a path



Graph Definitions (6)

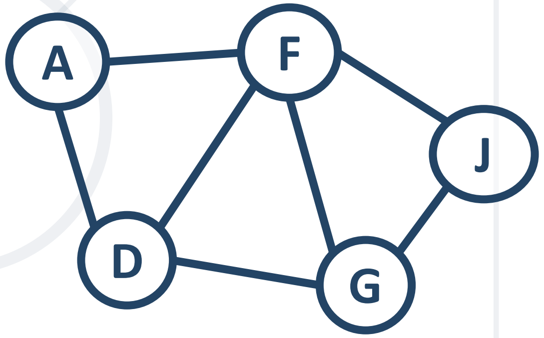
- **Cycle**
  - Path that ends back at the starting node
  - Example of cycle: A, B, C, G, A
- **Simple path**
  - No cycles in path
- **Acyclic graph**
  - Graph with no cycles
  - Acyclic undirected graphs are trees



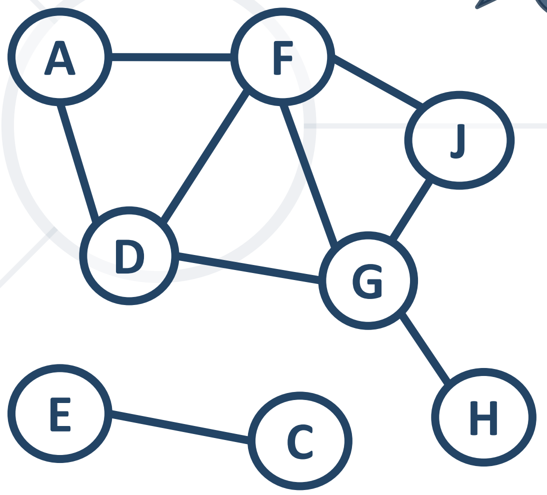
Graph Definitions (7)

- Two nodes are **reachable** if a path exists between them
- **Connected graph**
  - Every two nodes are reachable from each other

Connected graph

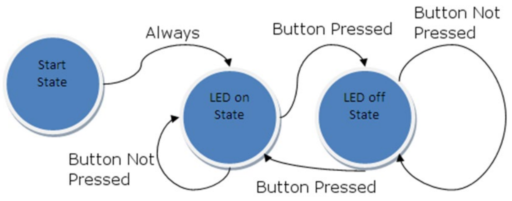


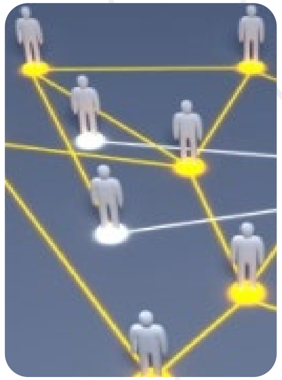
Unconnected graph holding two connected components



Graphs and Their Applications

- Graphs have many real-world applications
  - Modeling a **computer network** like the Internet
    - Routes are simple paths in the network
  - Modeling a city **map**
    - Streets are edges, crossings are vertices
  - **Social networks**
    - People are nodes and their connections are edges
  - **State machines**
    - States are nodes, transitions are edges





```
{1,2} {1,4}
{2,3} {3,1}
   {4,2}
```

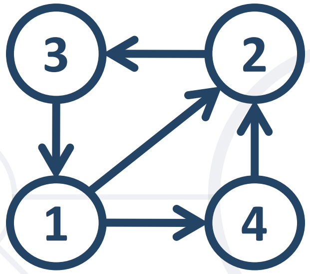
Representing Graphs

Classic and OOP Ways

Representing Graphs

- **Adjacency list**
  - Each node holds a list of its neighbors

```
1 -> {2, 4}
2 -> {3}
3 -> {1}
4 -> {2}
```



- **Adjacency matrix**
  - Each cell keeps whether and how two nodes are connected

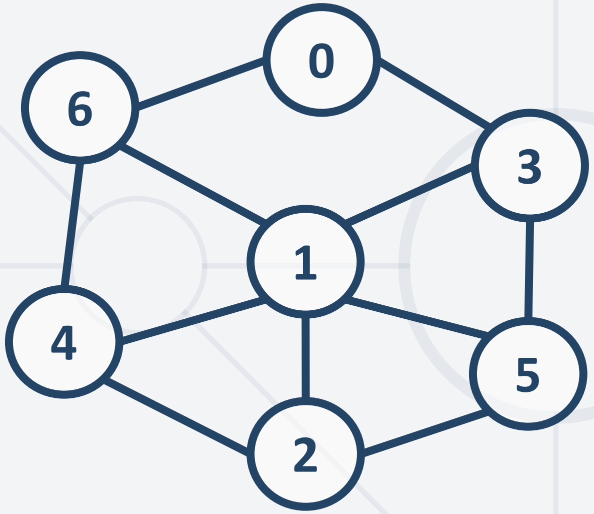
	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	1	0	0	0
4	0	1	0	0

- **List of edges**

```
{1,2}, {1,4}, {2,3}, {3,1}, {4,2}
```

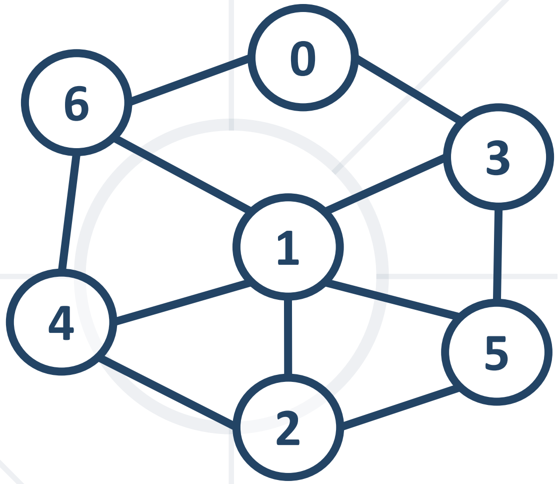
Graph Representation: Adjacency List

```
g = [
    [3, 6],
    [2, 3, 4, 5, 6],
    [1, 4, 5],
    [0, 1, 5],
    [1, 2, 6],
    [1, 2, 3],
    [0, 1, 4],
]
// Add an edge { 3 <-> 6 }
g[3].append(6)
g[6].append(3)
child_nodes = g[1] # List the children of node 1
```



Graph Representation: Adjacency Matrix

```
graph = [
    [0, 0, 0, 1, 0, 0, 1],  # node 0
    [0, 0, 1, 1, 1, 1, 1],  # node 1
    [0, 1, 0, 0, 1, 1, 0],  # node 2
    [1, 1, 0, 0, 0, 1, 0],  # node 3
    [0, 1, 1, 0, 0, 0, 1],  # node 4
    [0, 1, 1, 1, 0, 0, 0],  # node 5
    [1, 1, 0, 0, 1, 0, 0],  # node 6
]
# Add an edge { 3 -> 6 }
graph[3][6] = 1
# List the children of node 1
child_nodes = graph[1]
```



Graph Representation: List of Edges

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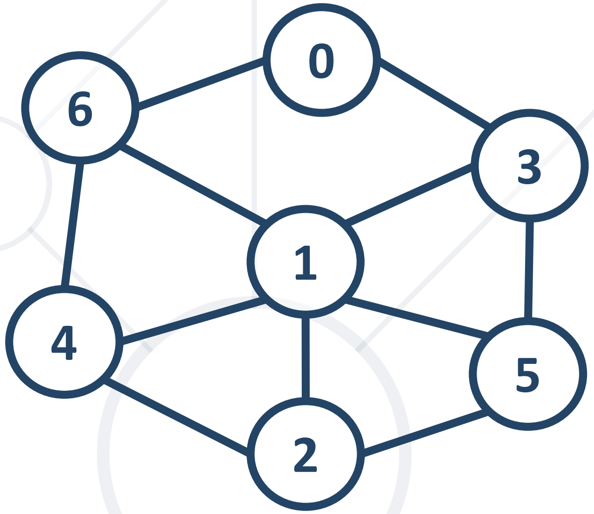
```python
class Edge:
    def __init__(self, parent, child):
        self.parent = parent
        self.child = child
graph = [
    Edge(0, 3),
    Edge(0, 6),
]
# Add an edge { 3 -> 6 }
graph.append(Edge(3, 6))
# List the children of node 1
child_nodes = [e for e in graph if e.parent == 1]
```

Graph Representation: Dictionary

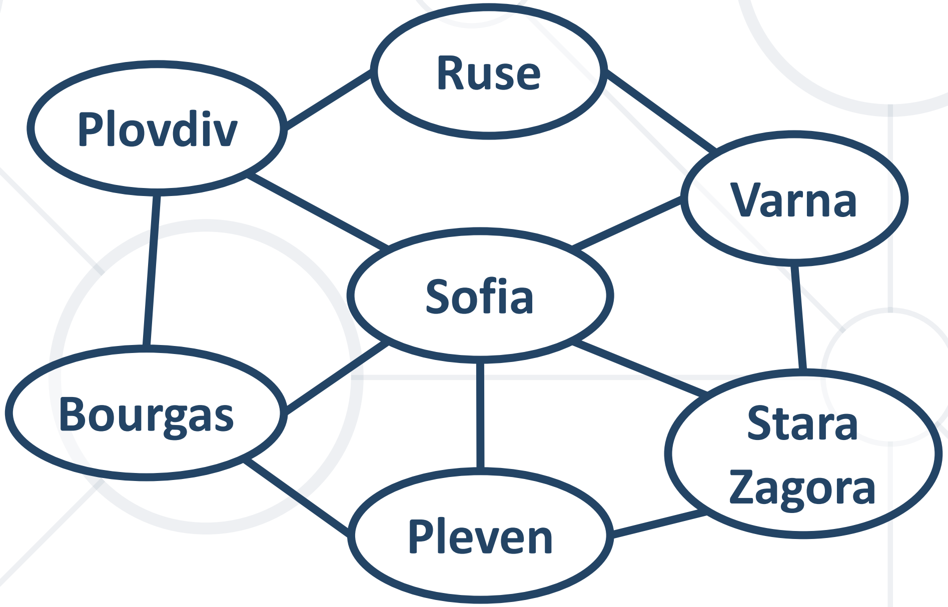
```python
graph = {
    'Sofia': ['Plovdiv', 'Ruse', 'Varna'],
    'Plovdiv': ['Ruse', 'Sofia'],
    'Ruse': ['Plovdiv', 'Varna'],
    'Varna': ['Ruse', 'Sofia']
}

# Adding a new edge
graph['Varna'].append('Plovdiv')
graph['Plovdiv'].append('Varna’)

# All neighbours of node with id 'Sofia'
child_nodes = graph['Sofia']
```

Numbering Graph Nodes

- A common technique to **speed up** working with graphs
  - **Numbering the nodes** and accessing them by index (not by name)



Graph of **numbered nodes**: [0…6]



Graph of **named nodes**

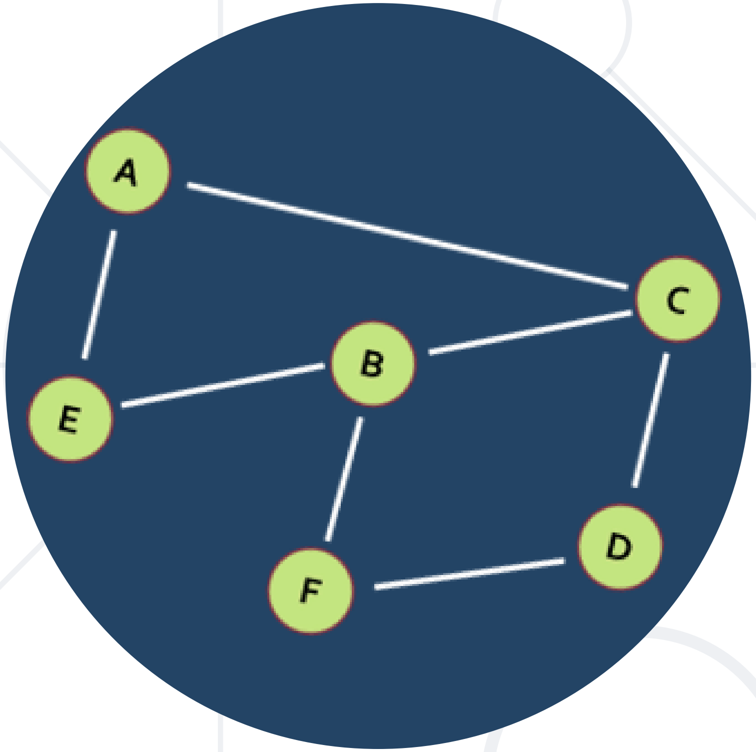
Numbering Graph Nodes – How?

- Suppose we have a **graph of *n* nodes**
  - We can assign a number for each node in the range [0…**n**-1]

#	Node
0	Ruse
1	Sofia
2	Pleven
3	Varna
4	Bourgas
5	Stara Zagora
6	Plovdiv

OOP-Based-Graph Representation

- Using OOP:
  - Class **Node**
  - Class **Edge** (**Connection**)
  - Class **Graph**
  - Optional classes
  - Algorithm classes



Graphs Traversals

Depth-First Search and Breadth-First Search

Graph Traversal Algorithms

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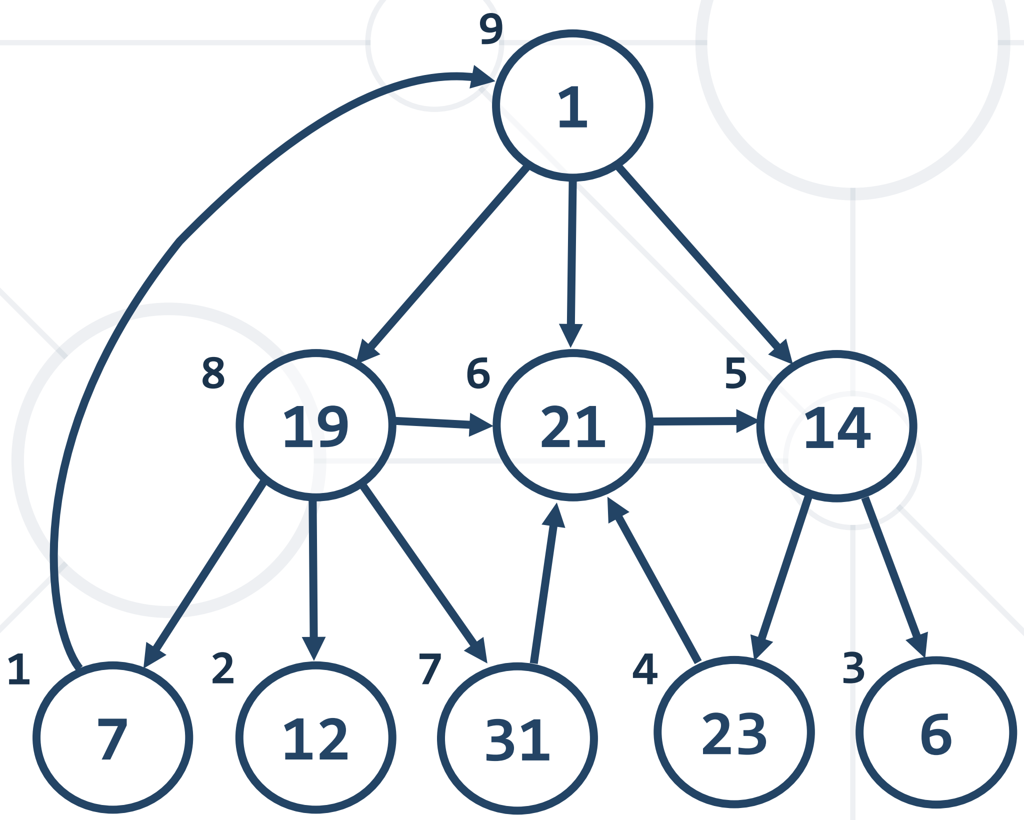
- **Traversing a graph** means to visit each of its nodes exactly once
  - The order of visiting nodes may vary on the traversal algorithm
  - **Depth-First Search** (DFS)
    - Visit node's successors first
    - Usually implemented by **recursion**
  - **Breadth-First Search** (BFS)
    - Nearest nodes visited first
    - Implemented with a **queue**

Depth-First Search (DFS)

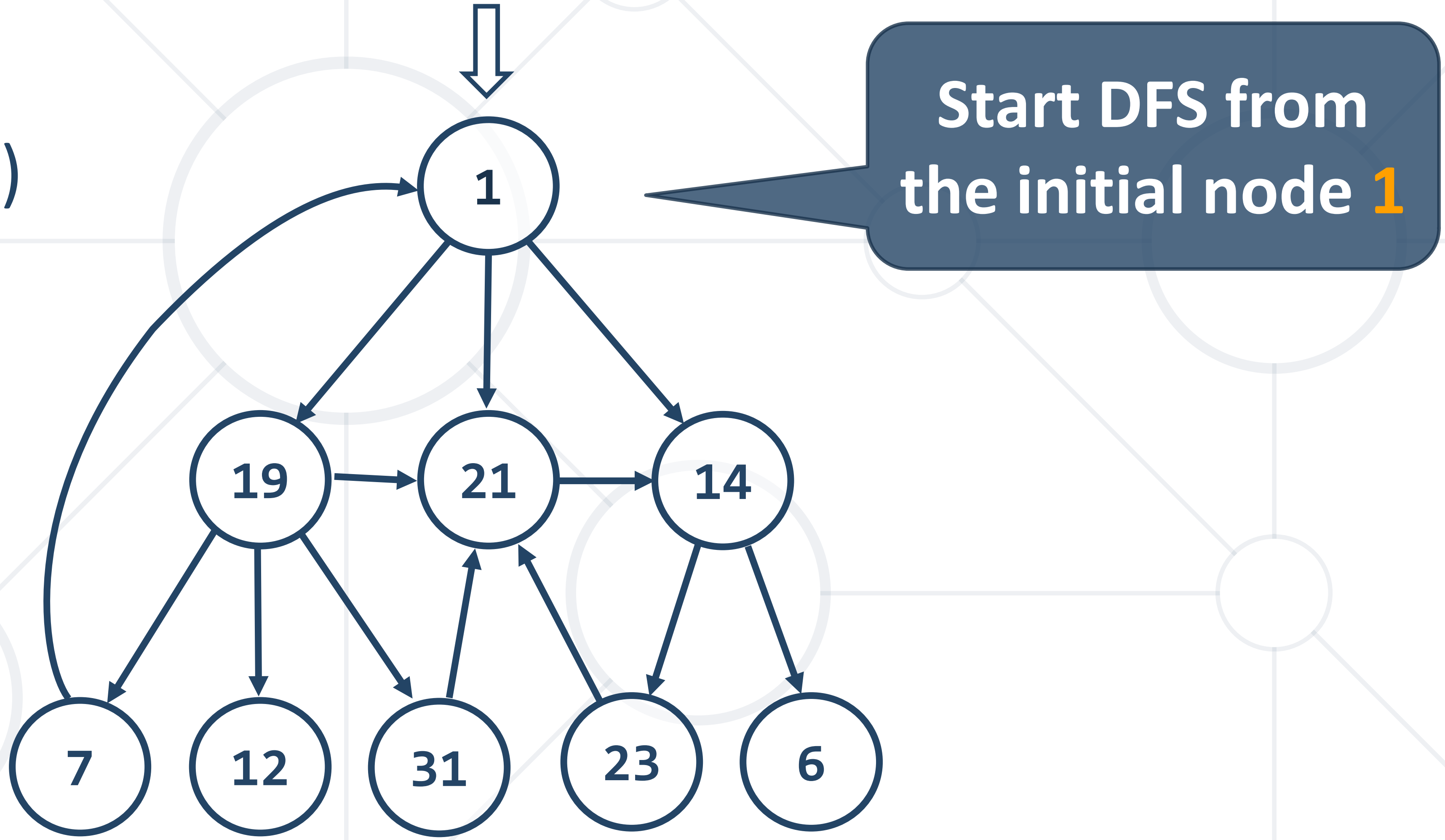
- **Depth-First Search** (**DFS**) first visits all descendants of given node recursively, finally visits the node itself

```
visited[0 … n-1] = false;
for (v = 0 … n-1) dfs(v)
dfs (node) {
  if not visited[node] {
    visited[node] = true;
    for each child c of node
      dfs(c);
    print node;
  }
}
```



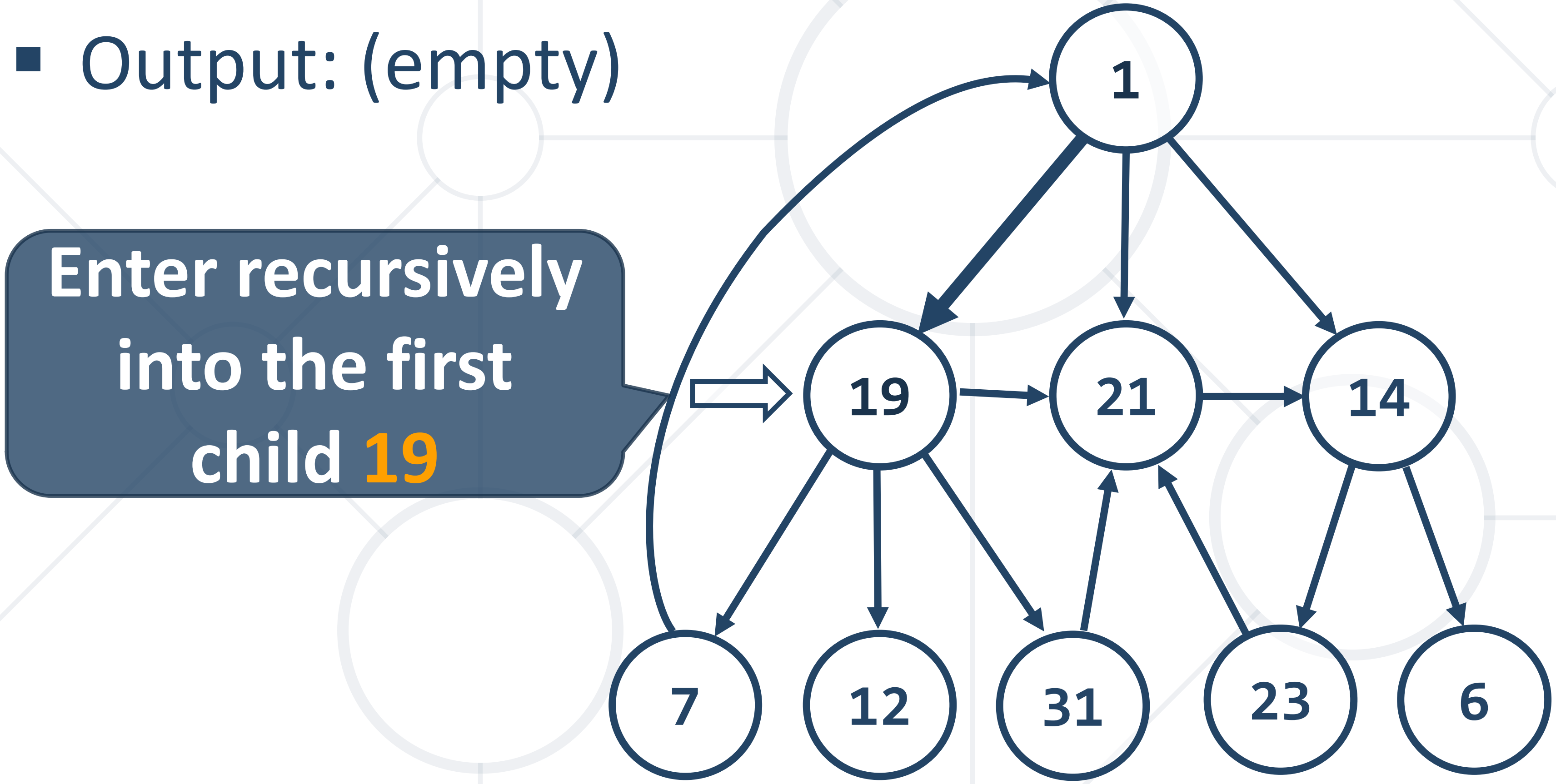
DFS in Action (Step 1)

- Stack: 1
- Output: (empty)



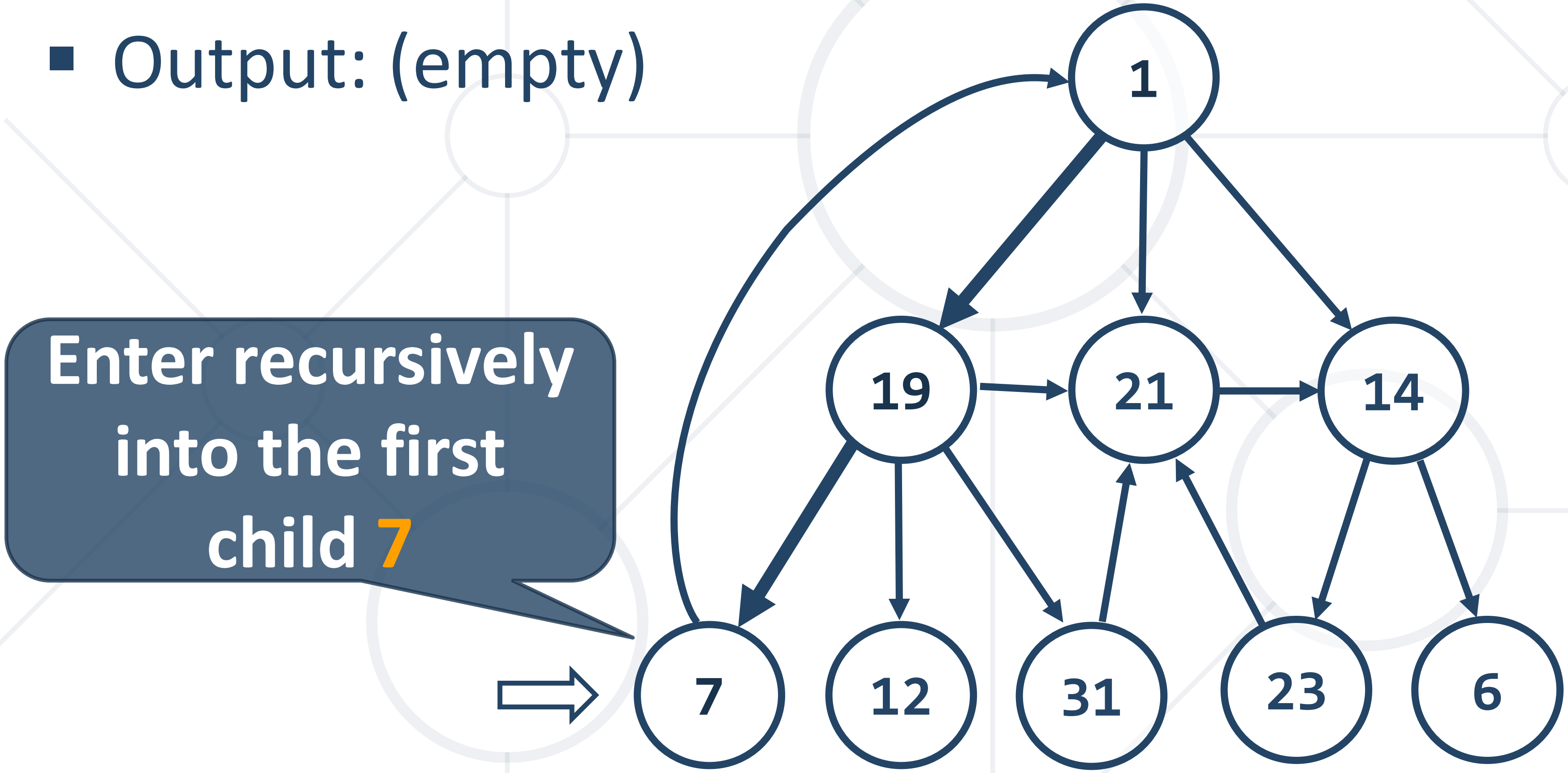
DFS in Action (Step 2)

- Stack: 1, 19
- Output: (empty)



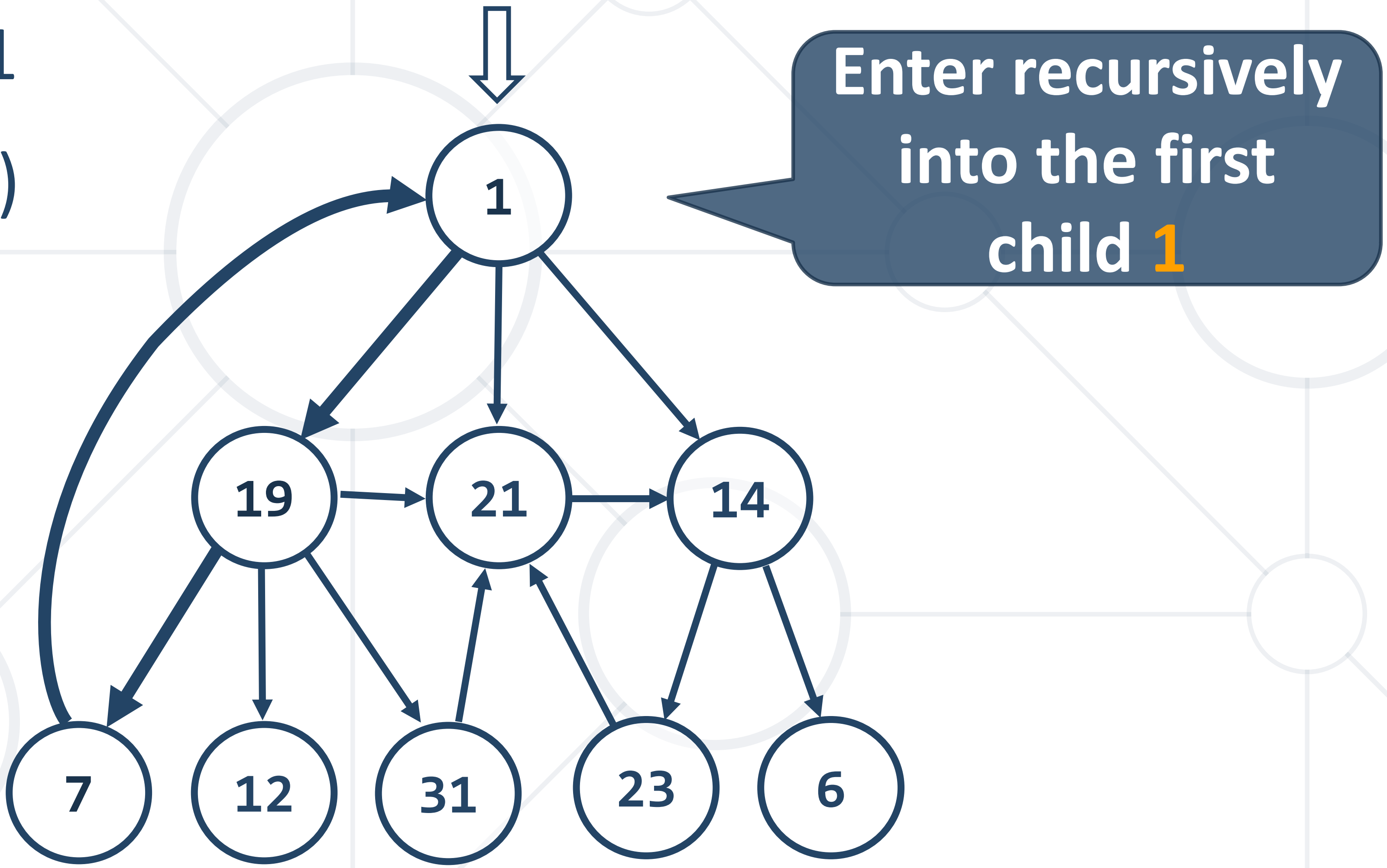
DFS in Action (Step 3)

- Stack: 1, 19, 7
- Output: (empty)



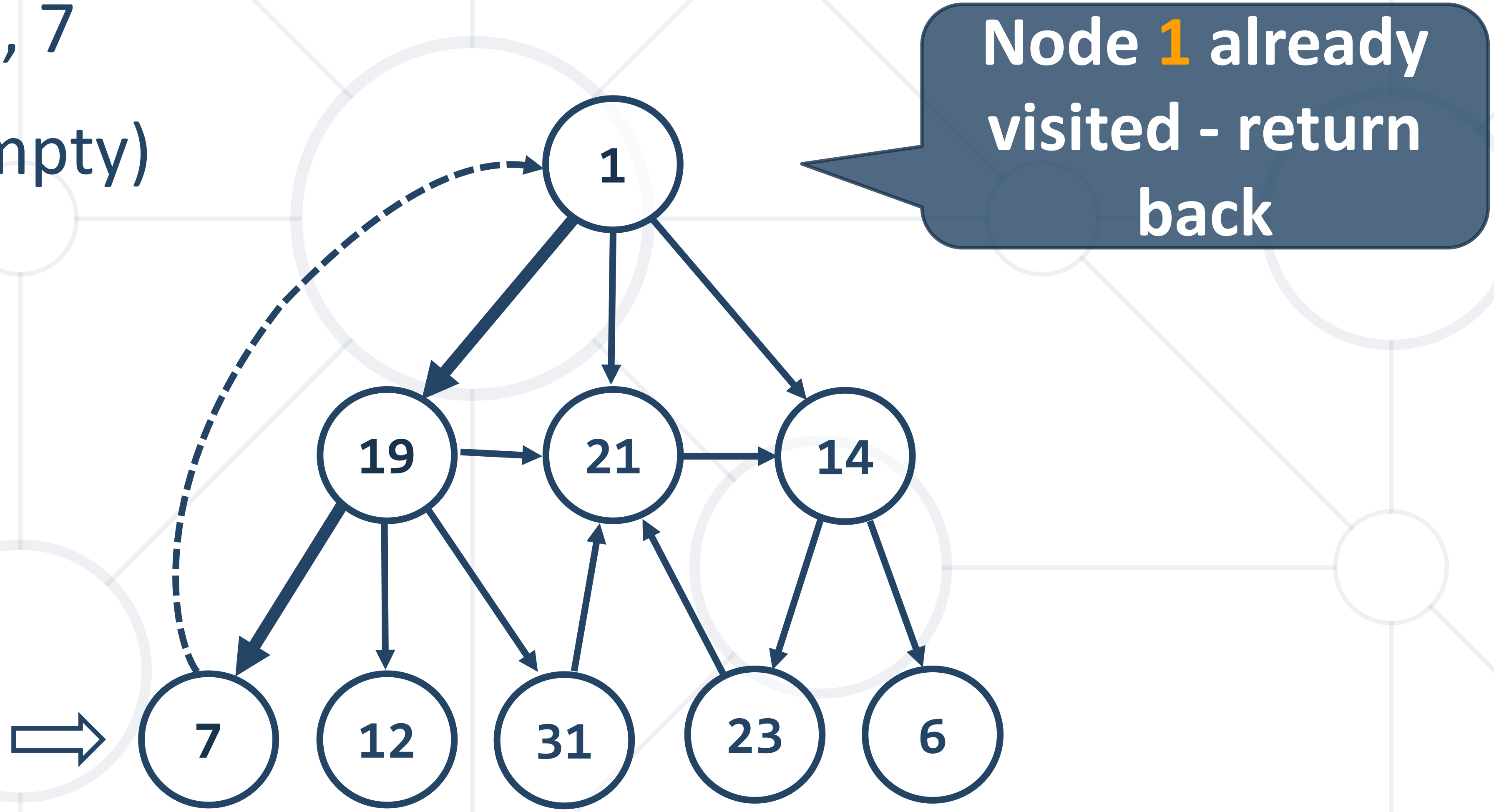
DFS in Action (Step 4)

- Stack: 1, 19, 7, 1
- Output: (empty)



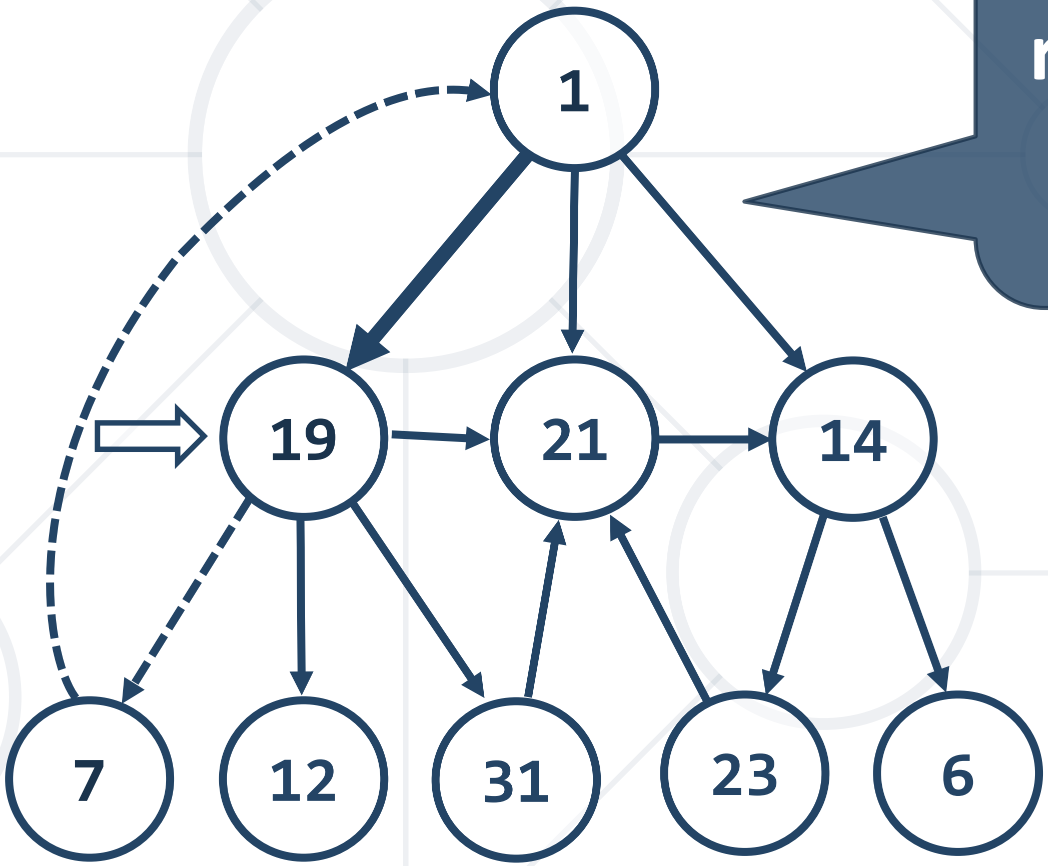
DFS in Action (Step 5)

- Stack: 1, 19, 7
- Output: (empty)



DFS in Action (Step 6)

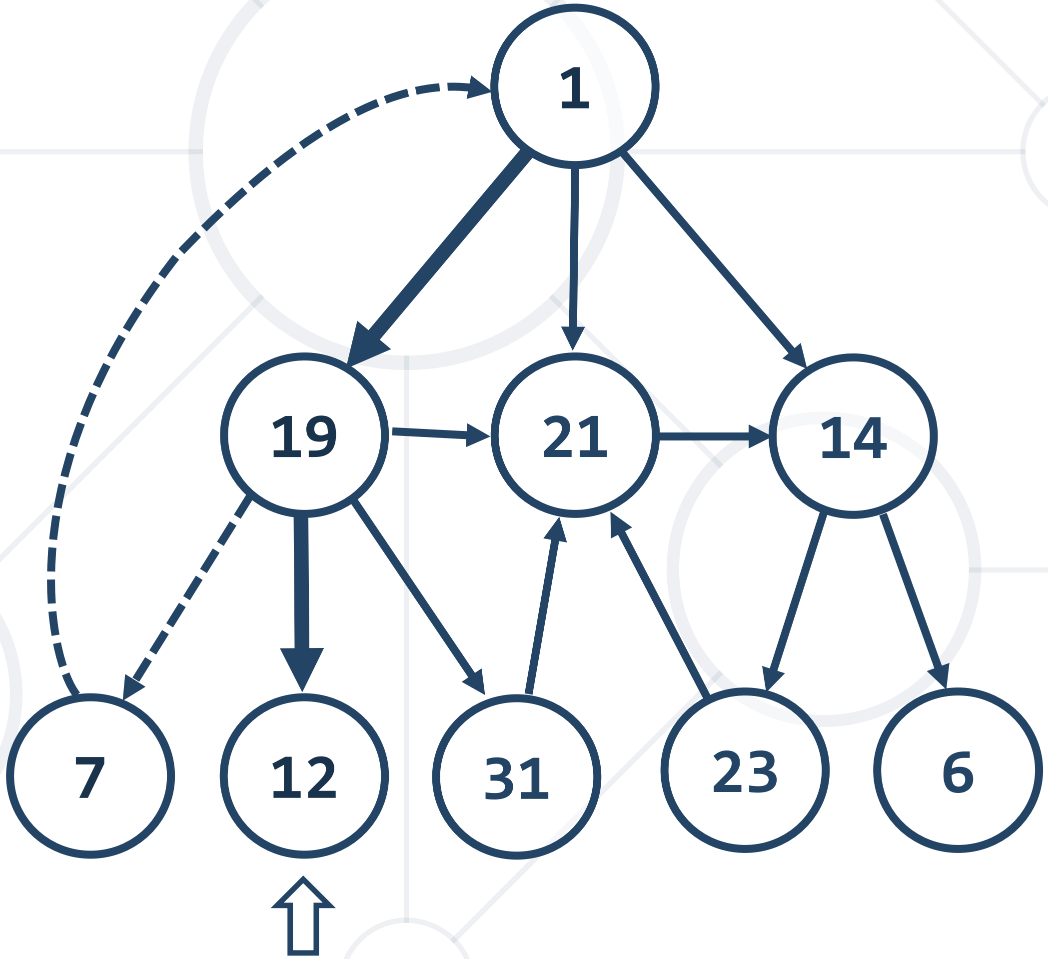
- Stack: 1, 19
- Output: 7



Return back from recursion and print the last visited node - 7

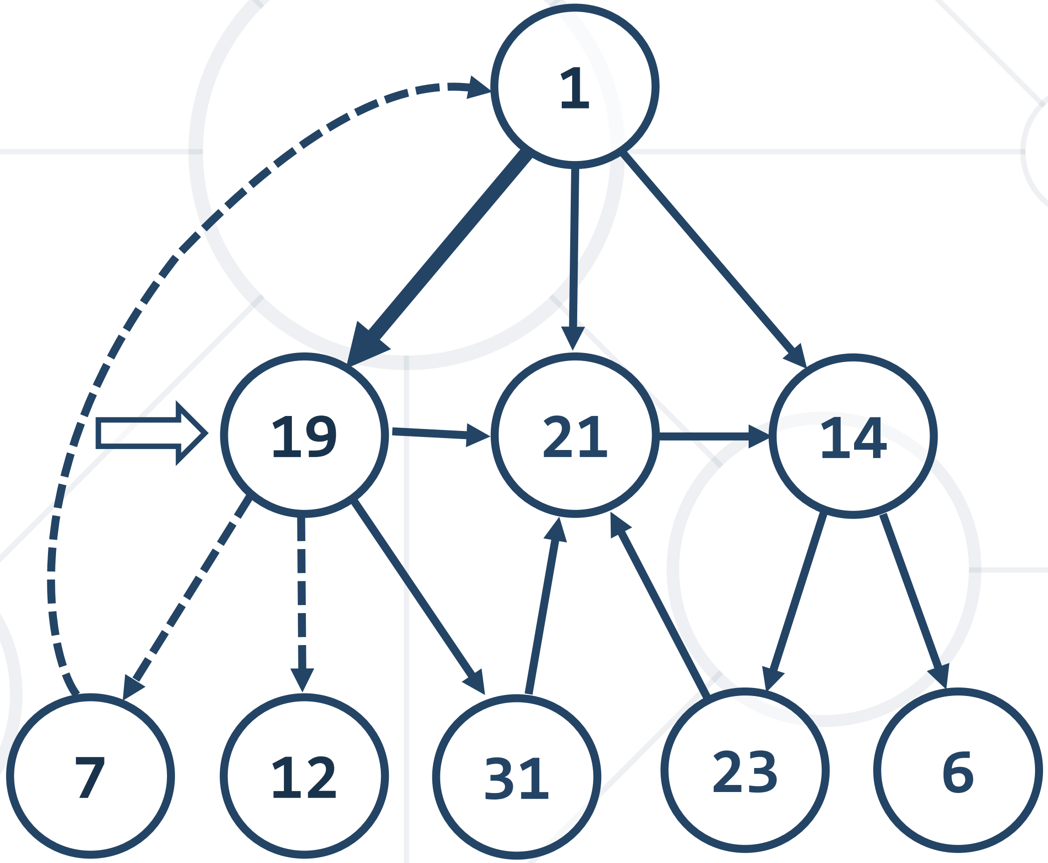
DFS in Action (Step 7)

- Stack: 1, 19, 12
- Output: 7



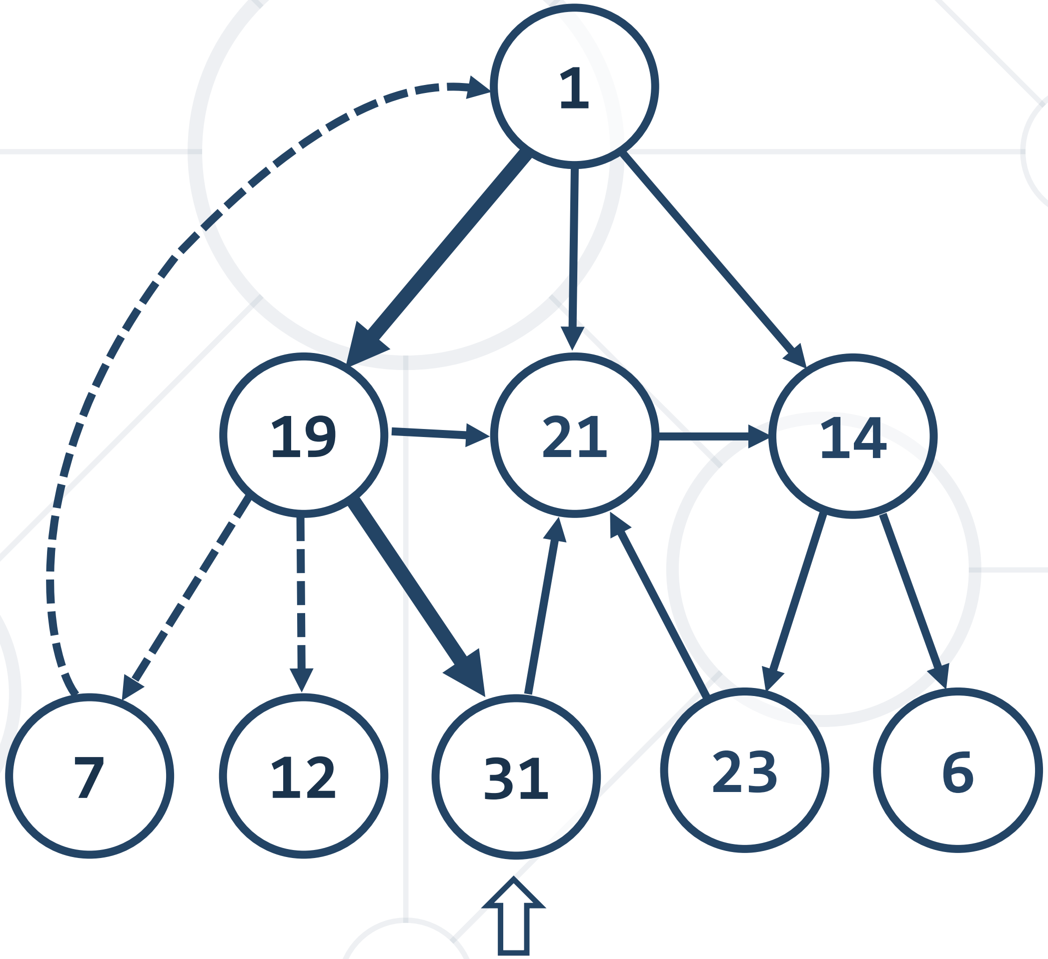
DFS in Action (Step 8)

- Stack: 1, 19
- Output: 7, 12



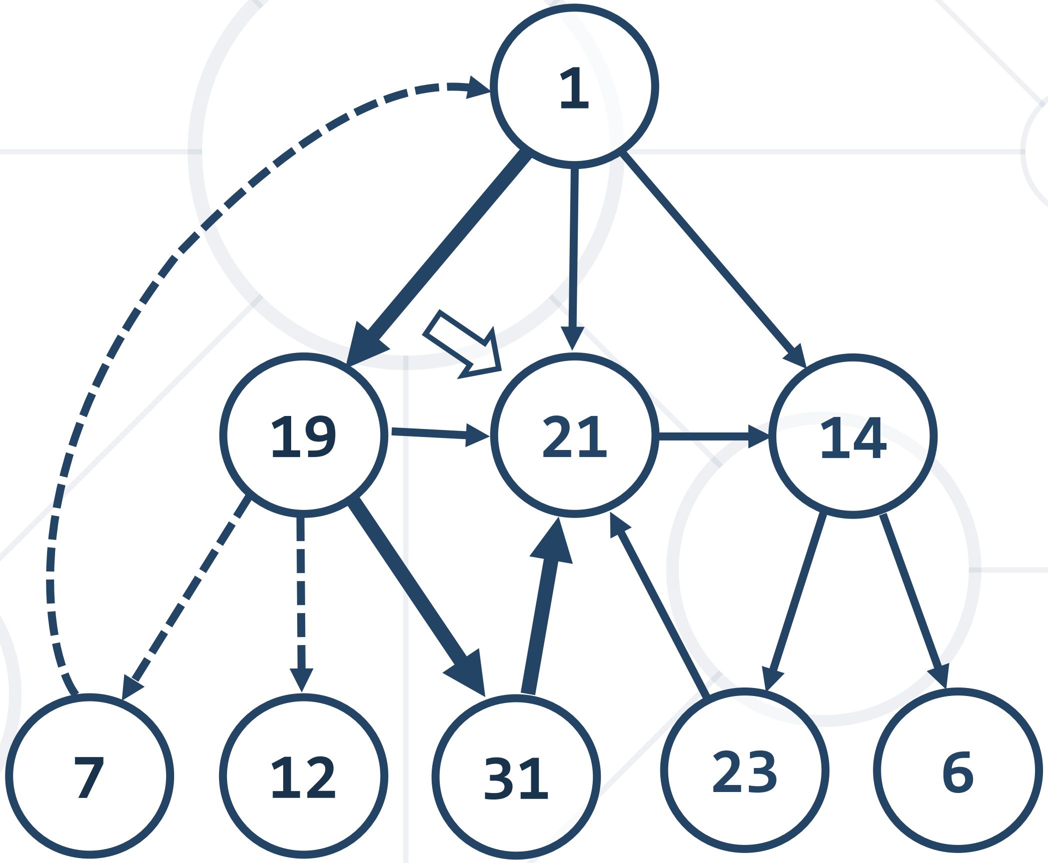
DFS in Action (Step 9)

- Stack: 1, 19, 31
- Output: 7, 12



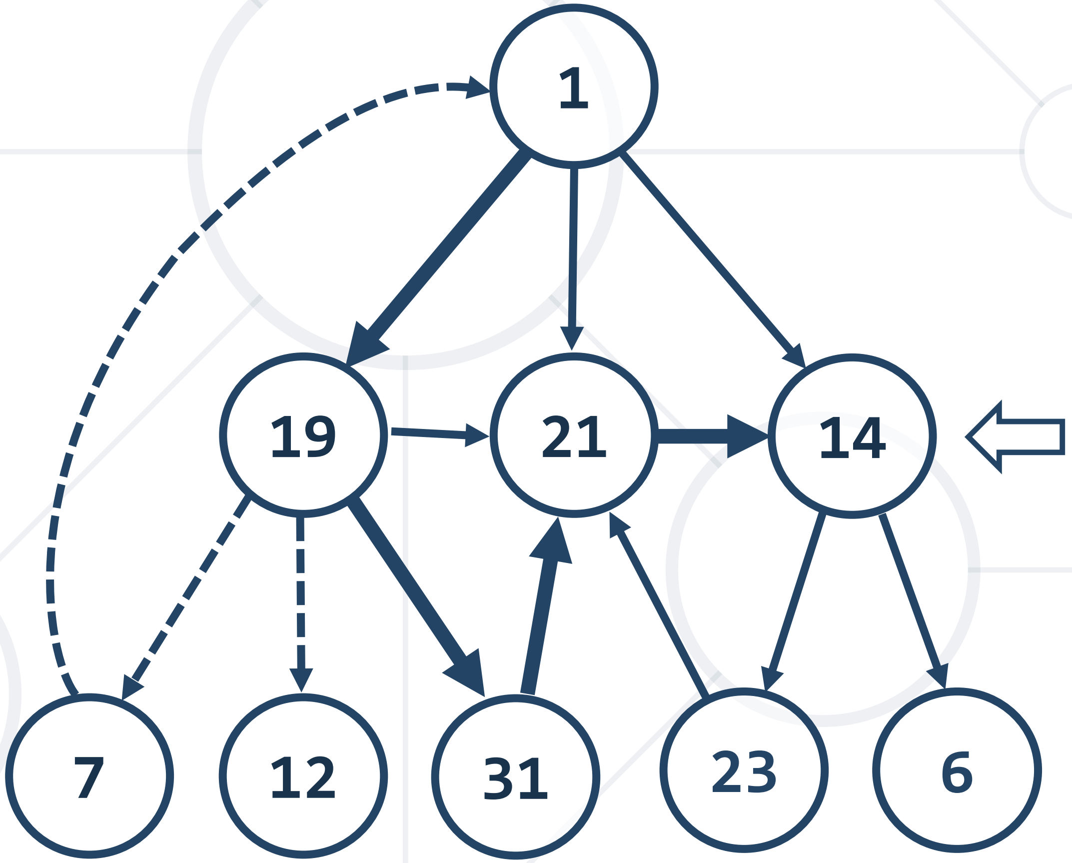
DFS in Action (Step 10)

- Stack: 1, 19, 31, 21
- Output: 7, 12



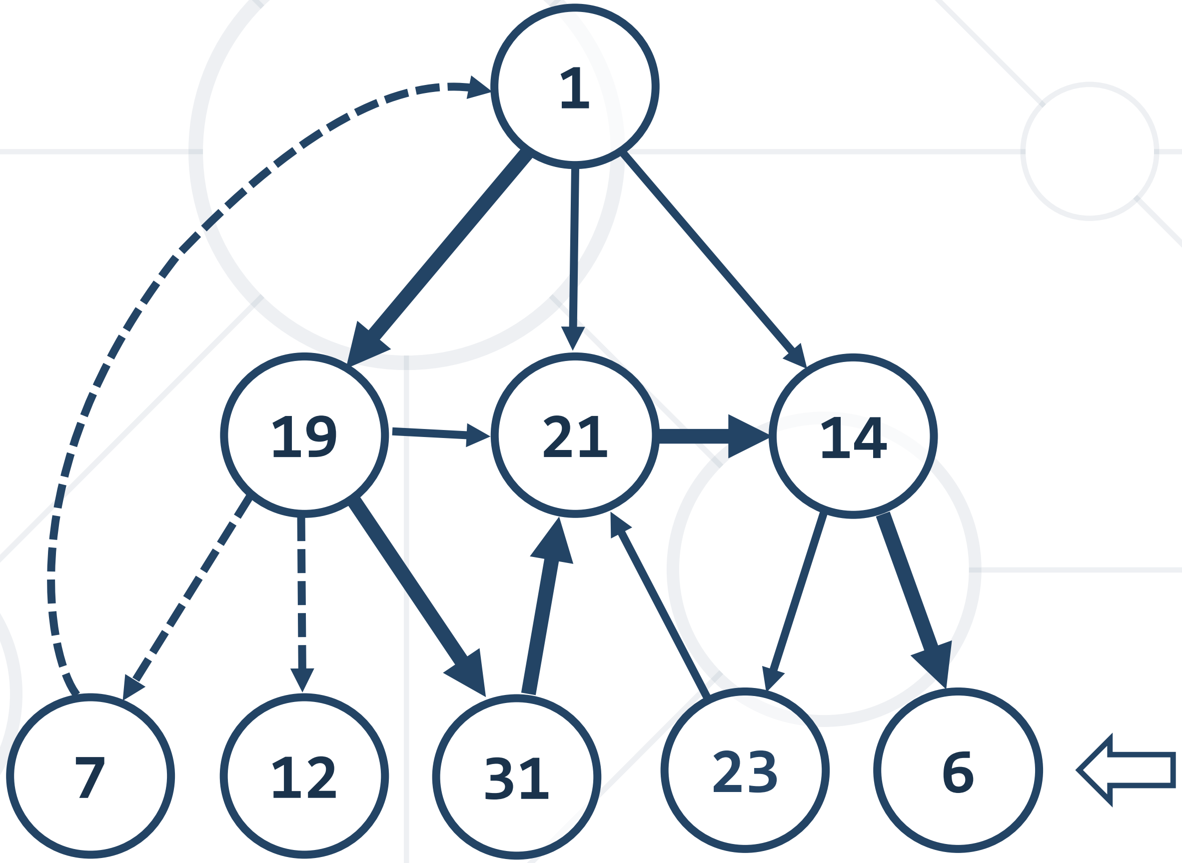
DFS in Action (Step 11)

- Stack: 1, 19, 31, 21, 14
- Output: 7, 12



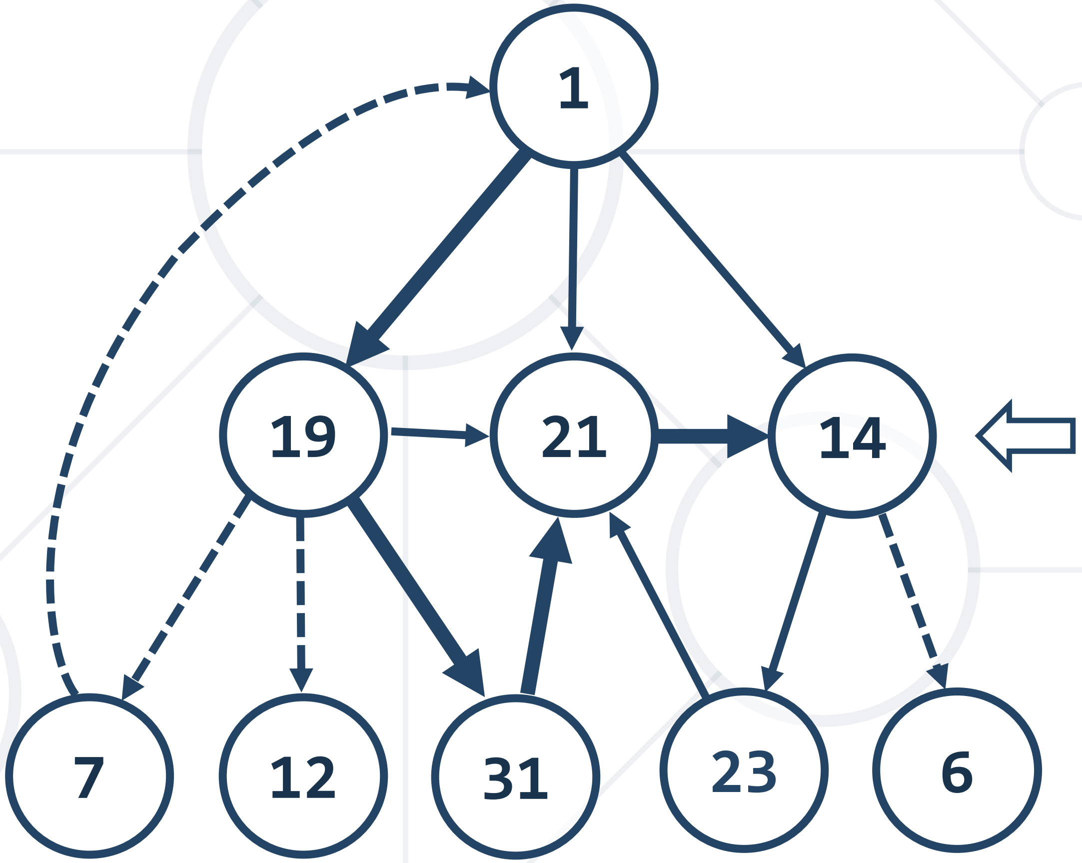
DFS in Action (Step 12)

- Stack: 1, 19, 31, 21, 14, 6
- Output: 7, 12



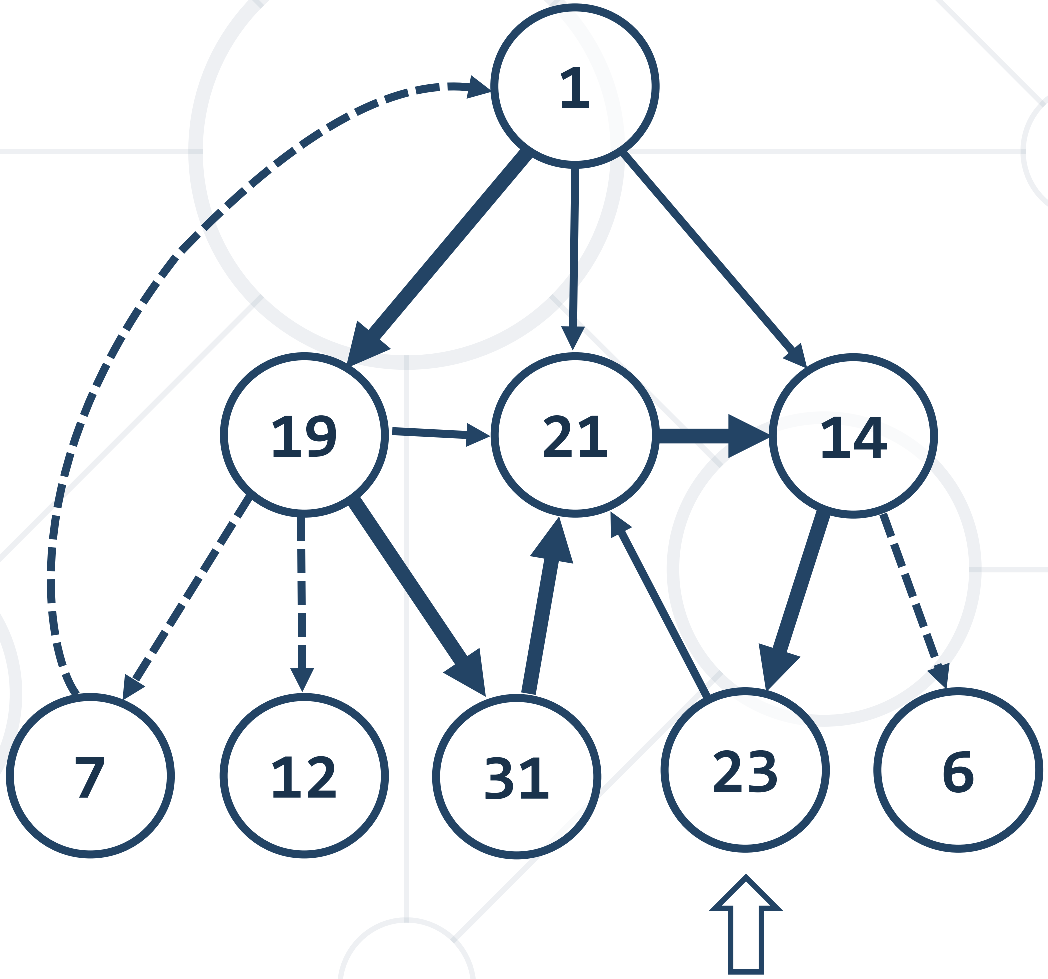
DFS in Action (Step 13)

- Stack: 1, 19, 31, 21, 14
- Output: 7, 12, 6



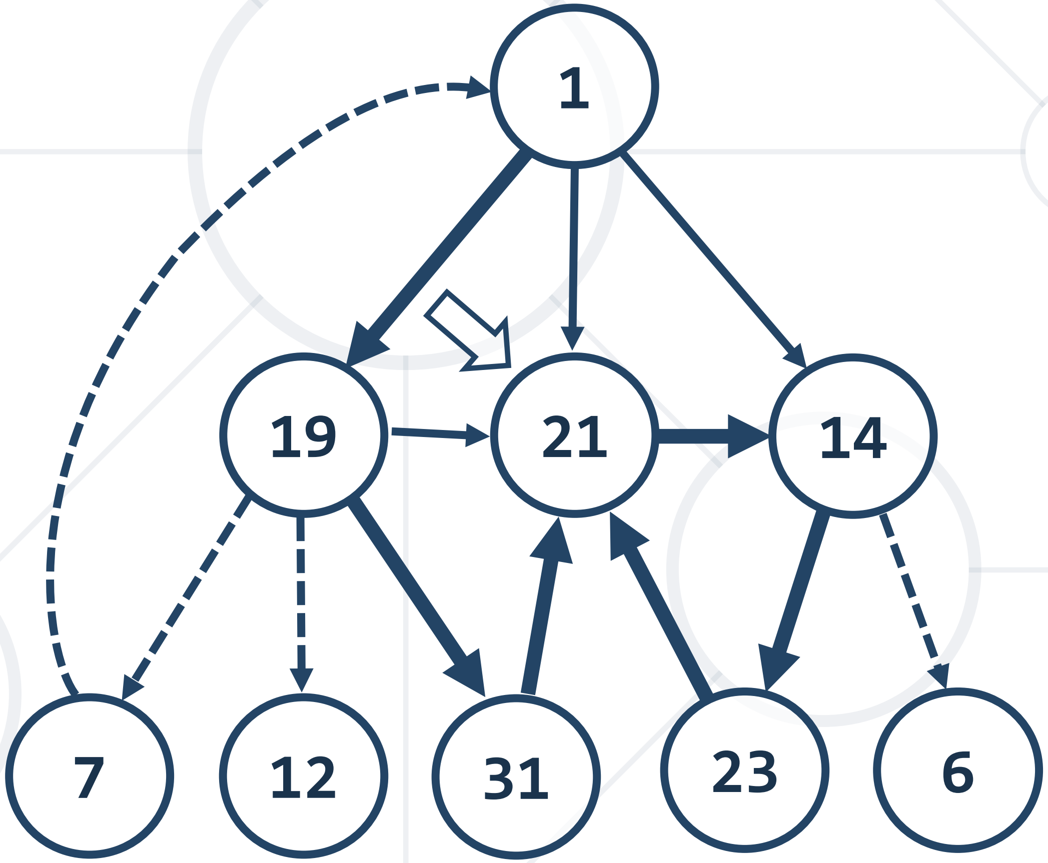
DFS in Action (Step 14)

- Stack: 1, 19, 31, 21, 14, 23
- Output: 7, 12, 6



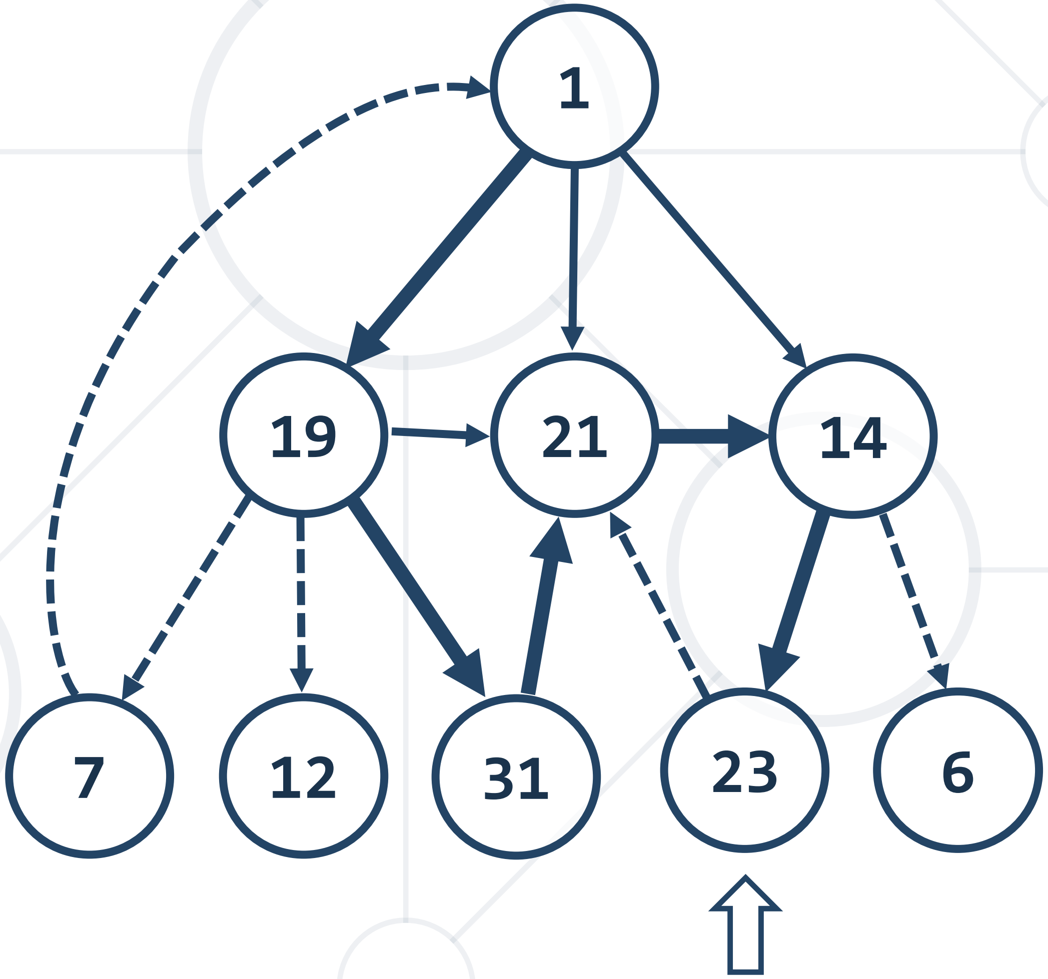
DFS in Action (Step 15)

- Stack: 1, 19, 31, 21, 14, 23, 21
- Output: 7, 12, 6



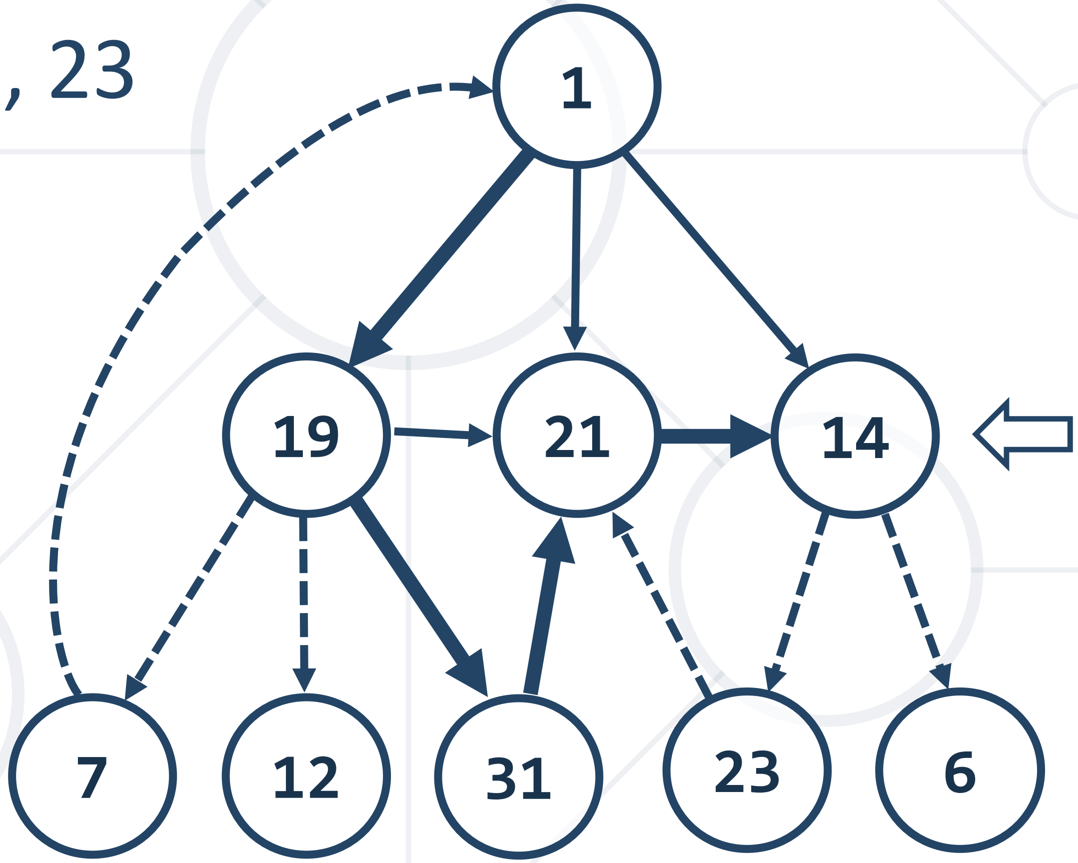
DFS in Action (Step 16)

- Stack: 1, 19, 31, 21, 14, 23
- Output: 7, 12, 6



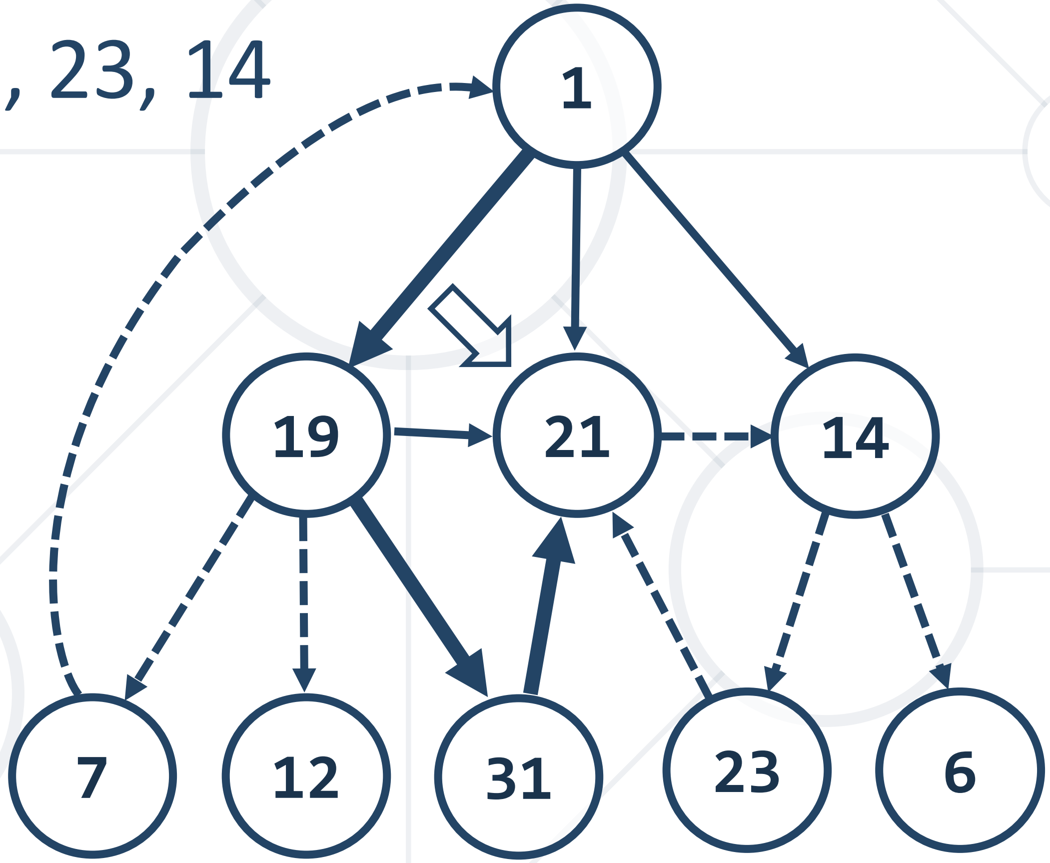
DFS in Action (Step 17)

- Stack: 1, 19, 31, 21, 14
- Output: 7, 12, 6, 23



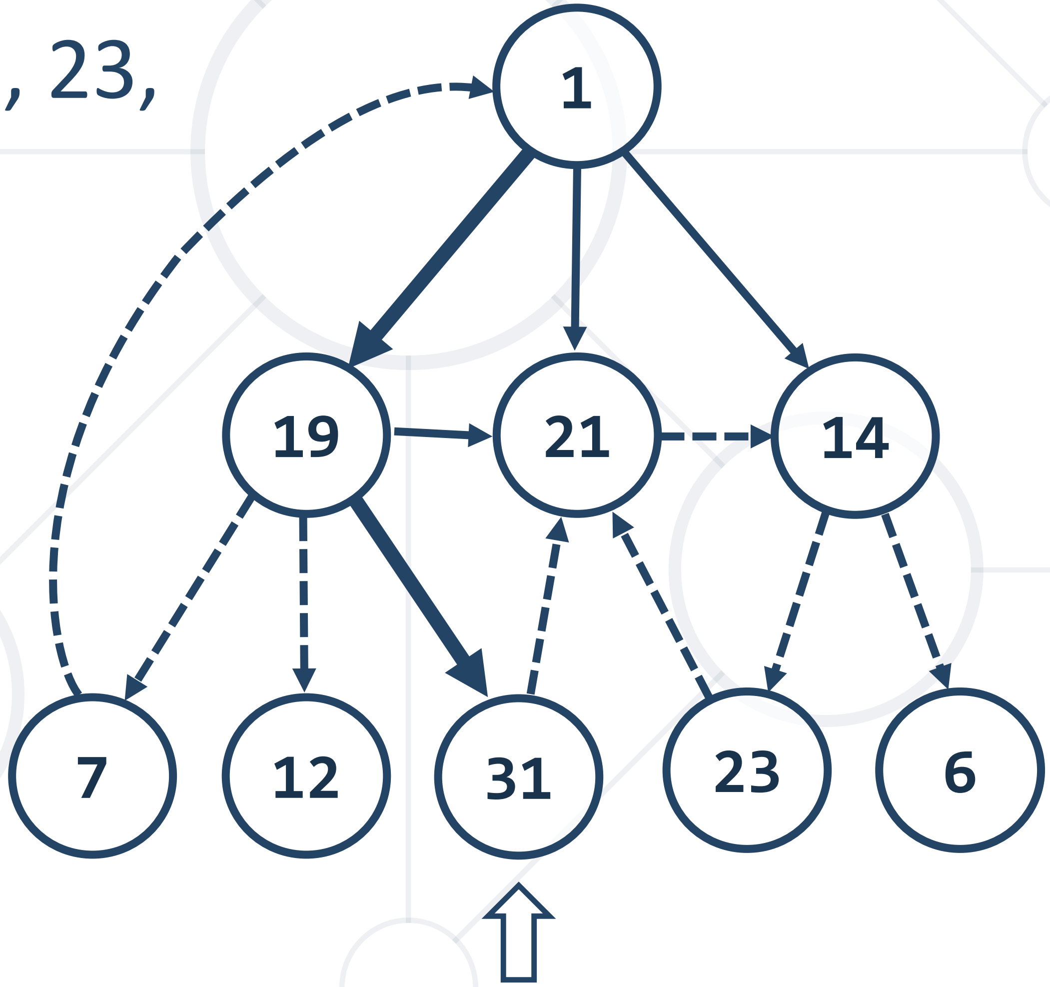
DFS in Action (Step 18)

- Stack: 1, 19, 31, 21
- Output: 7, 12, 6, 23, 14



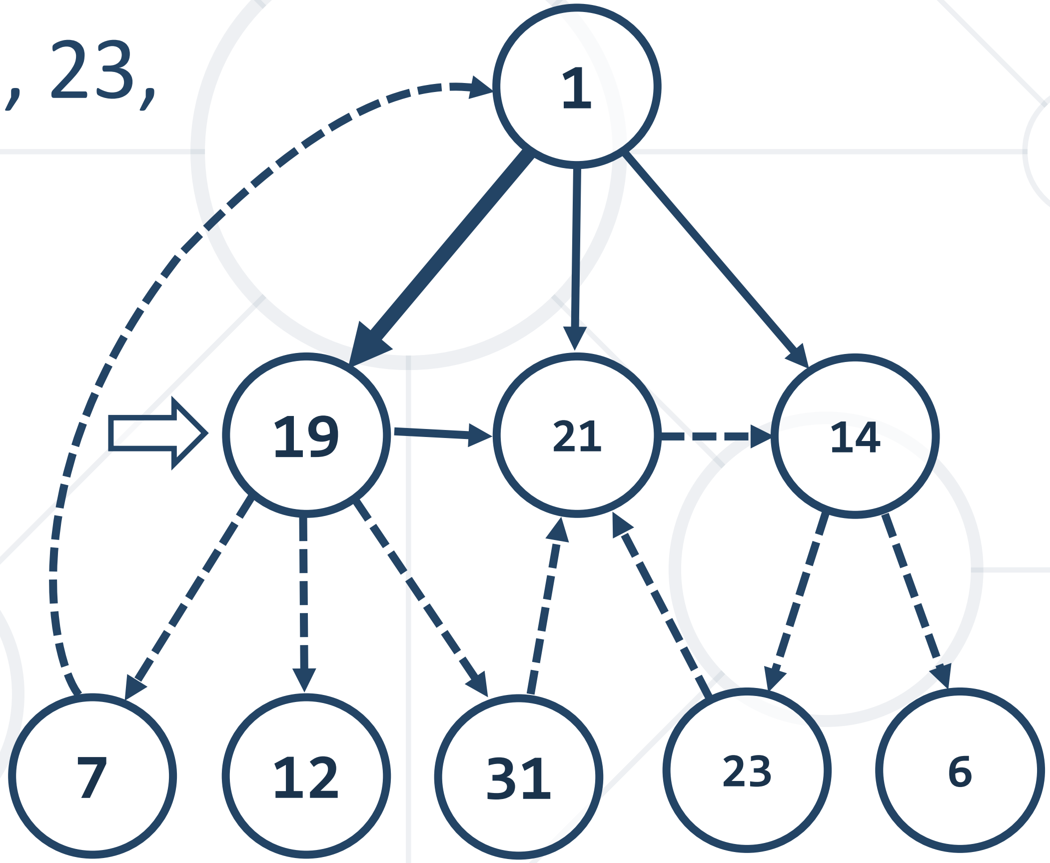
DFS in Action (Step 19)

- Stack: 1, 19, 31
- Output: 7, 12, 6, 23, 14, 21



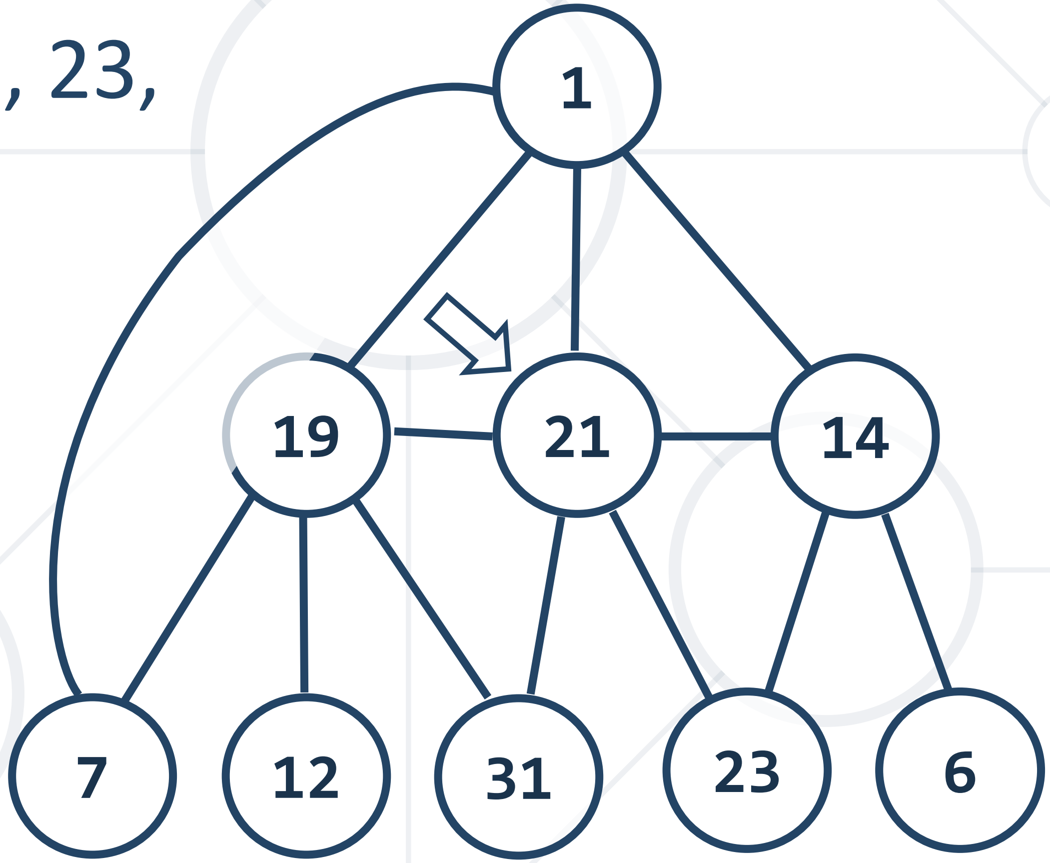
DFS in Action (Step 20)

- Stack: 1, 19
- Output: 7, 12, 6, 23, 14, 21, 31



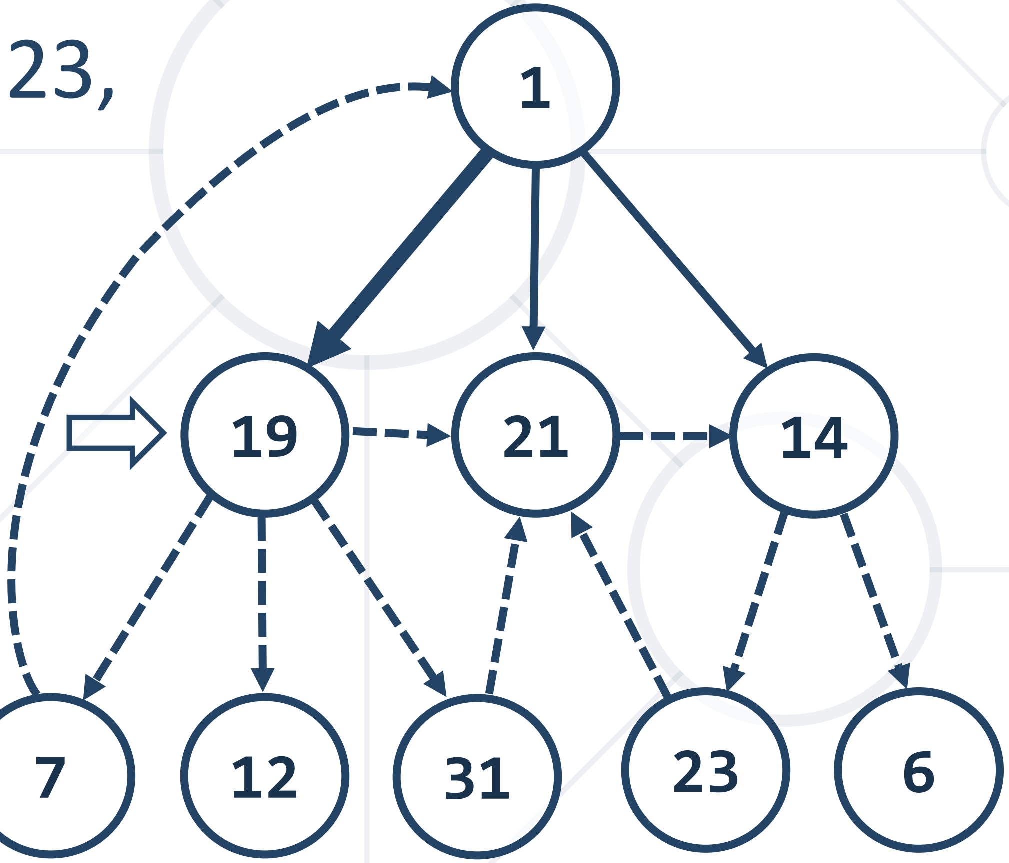
DFS in Action (Step 21)

- Stack: 1, 19, 21
- Output: 7, 12, 6, 23, 14, 21, 31



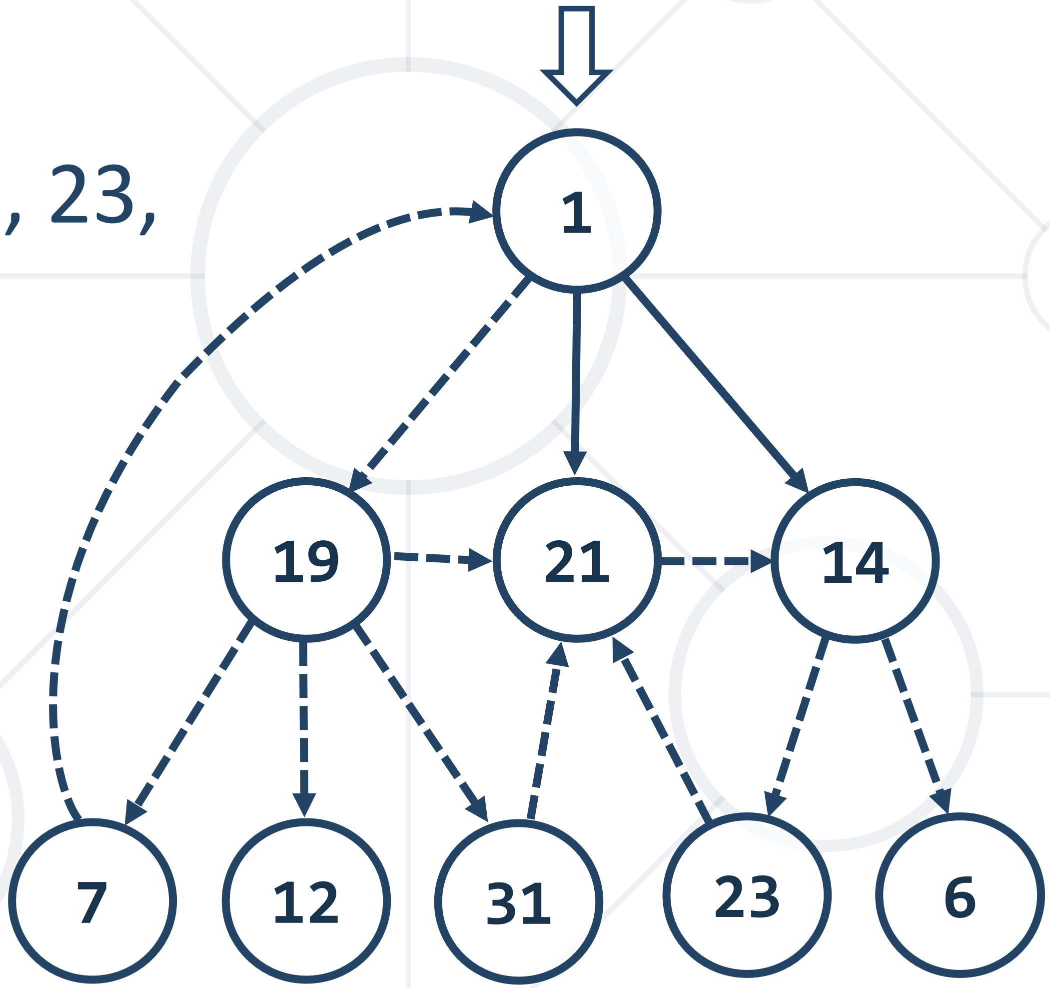
DFS in Action (Step 22)

- Stack: 1, 19
- Output: 7, 12, 6, 23, 14, 21, 31



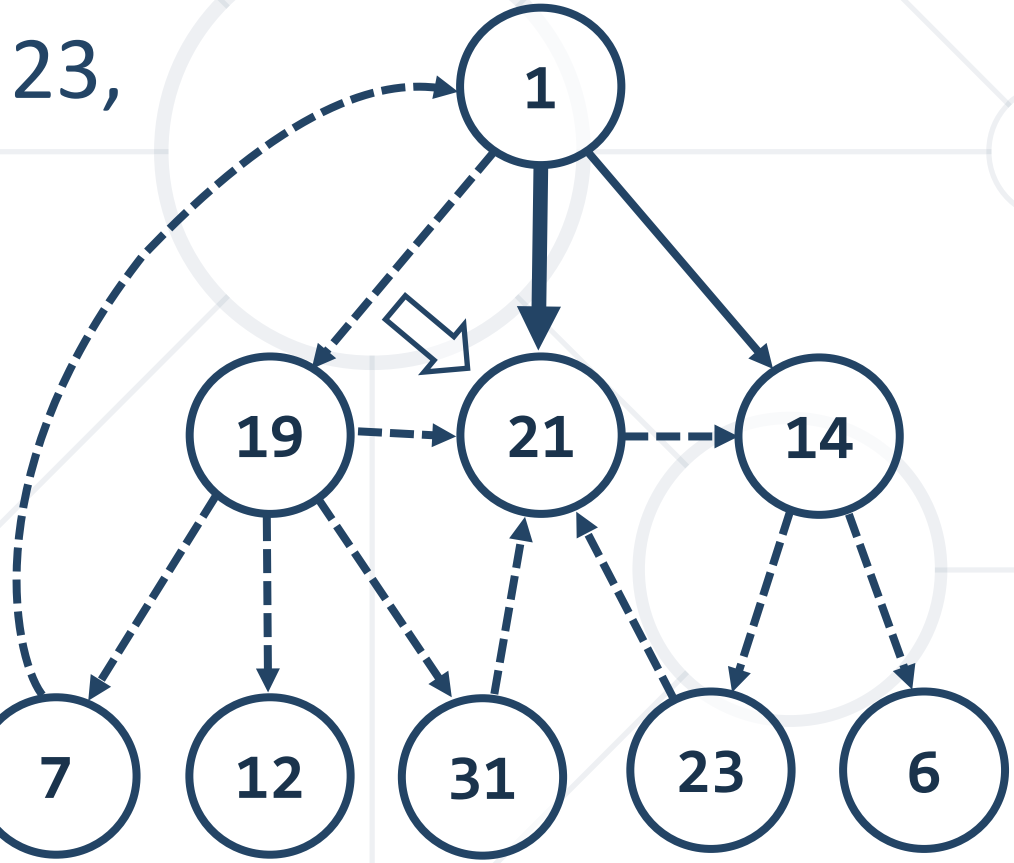
DFS in Action (Step 23)

- Stack: 1
- Output: 7, 12, 6, 23, 14, 21, 31, 19



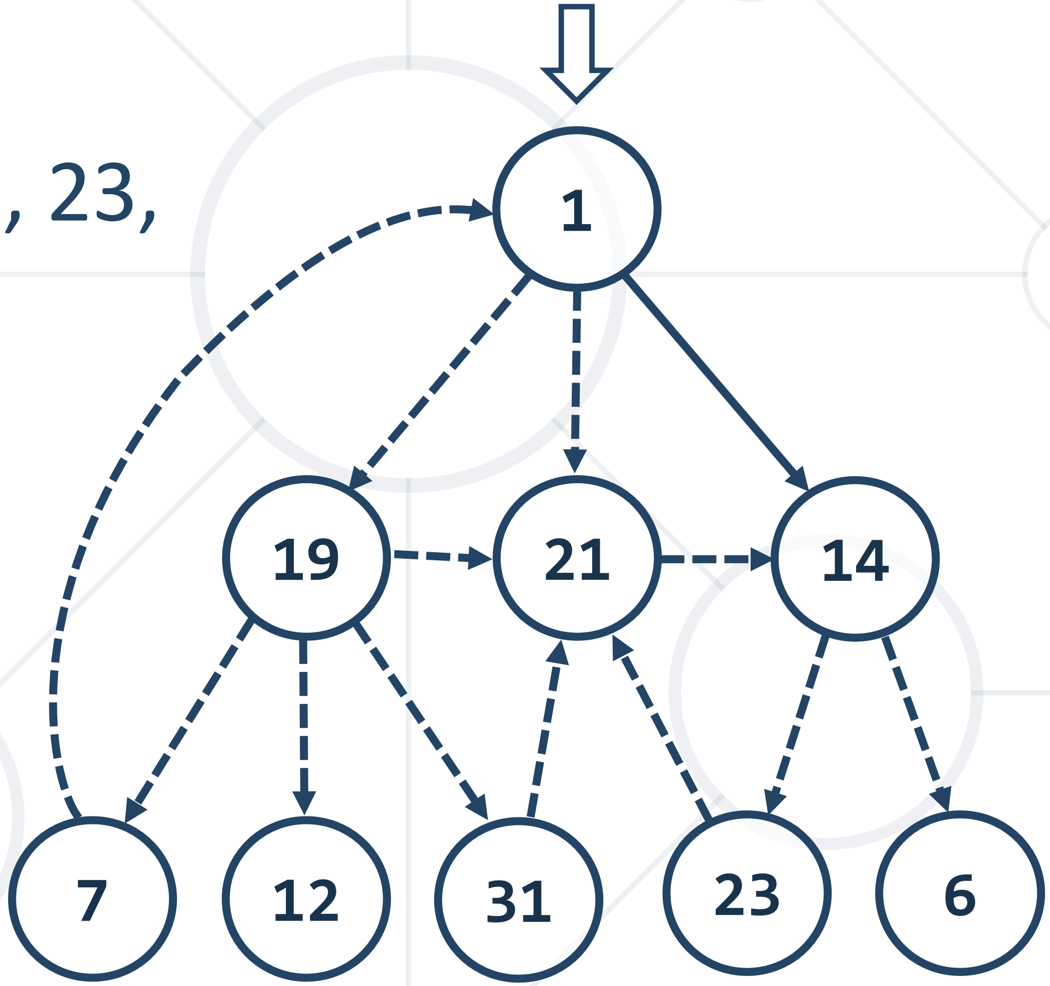
DFS in Action (Step 24)

- Stack: 1, 21
- Output: 7, 12, 6, 23, 14, 21, 31, 19



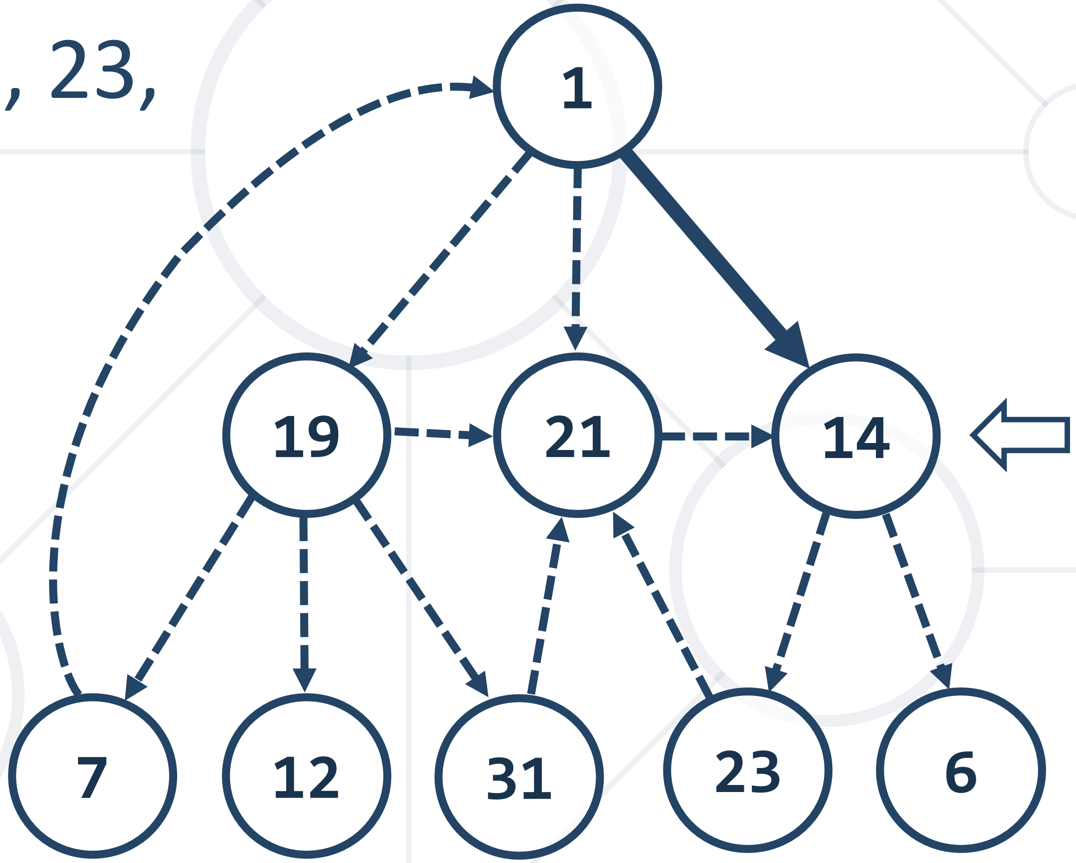
DFS in Action (Step 25)

- Stack: 1
- Output: 7, 12, 6, 23, 14, 21, 31, 19



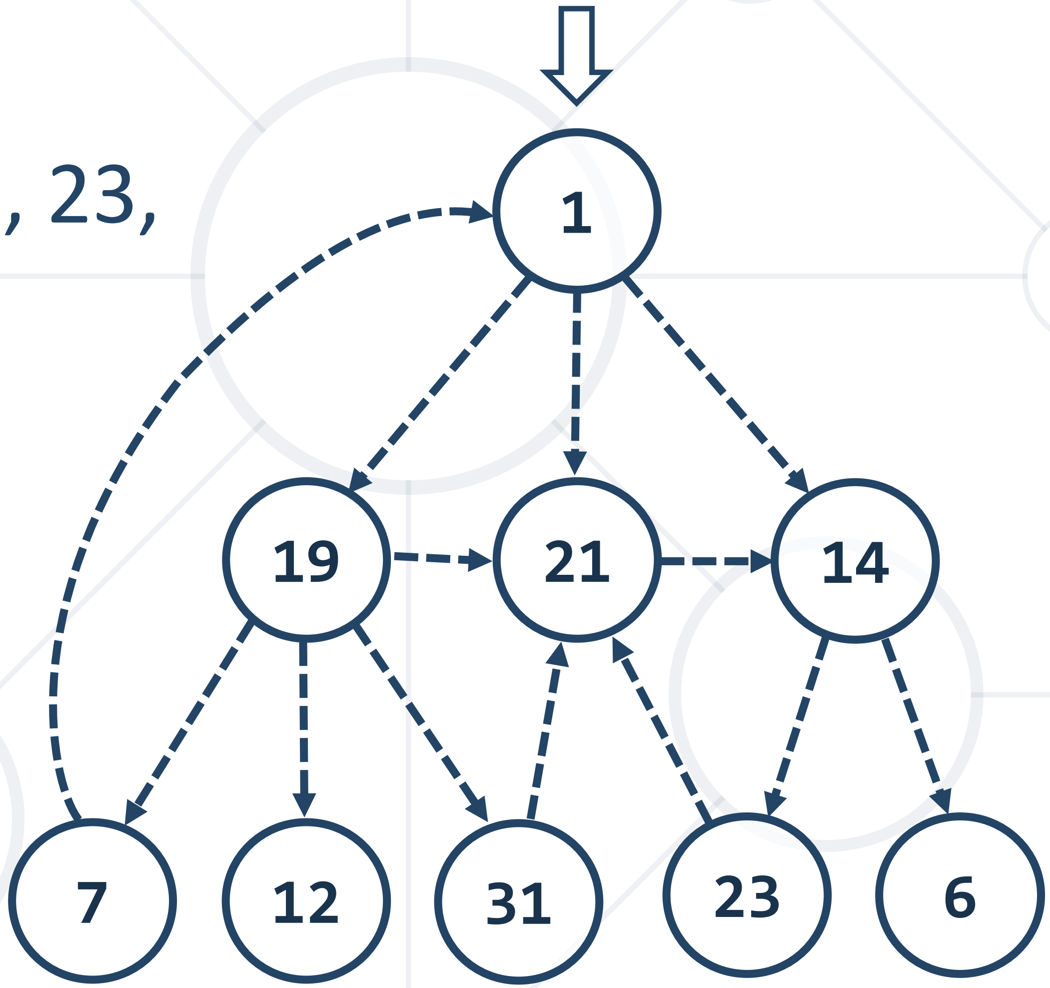
DFS in Action (Step 26)

- Stack: 1, 14
- Output: 7, 12, 6, 23, 14, 21, 31, 19



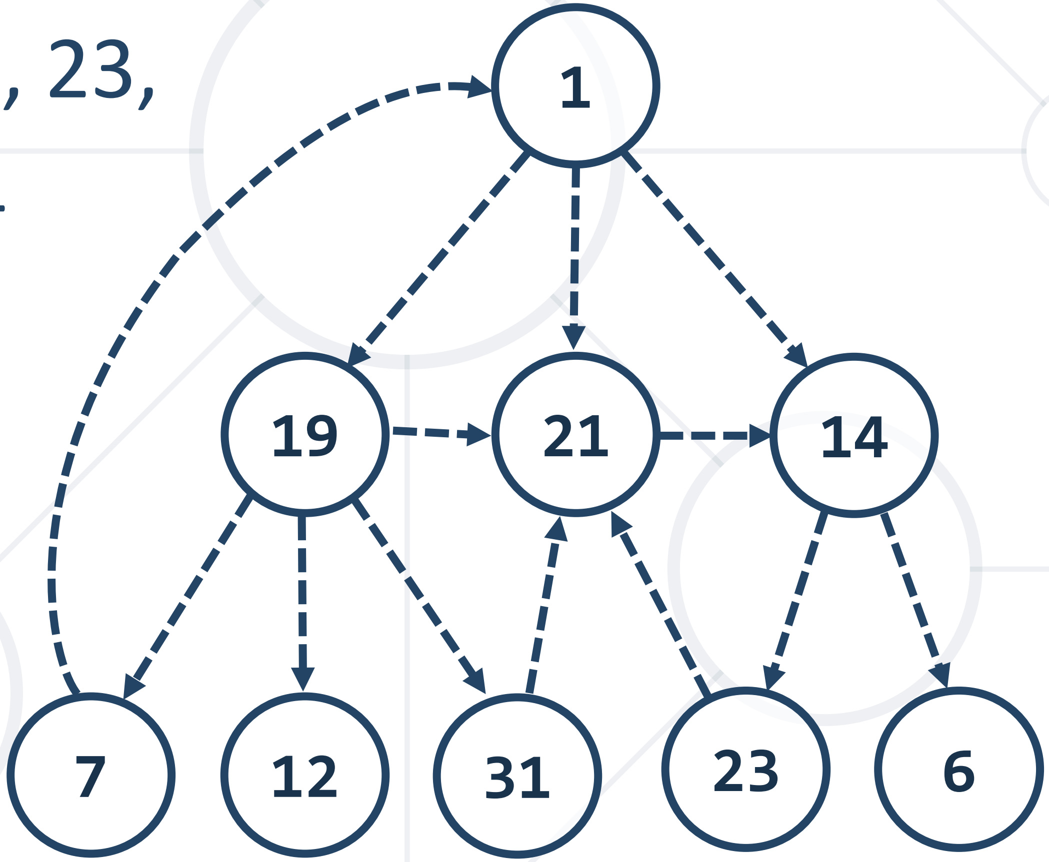
DFS in Action (Step 27)

- Stack: 1
- Output: 7, 12, 6, 23, 14, 21, 31, 19



DFS in Action (Step 28)

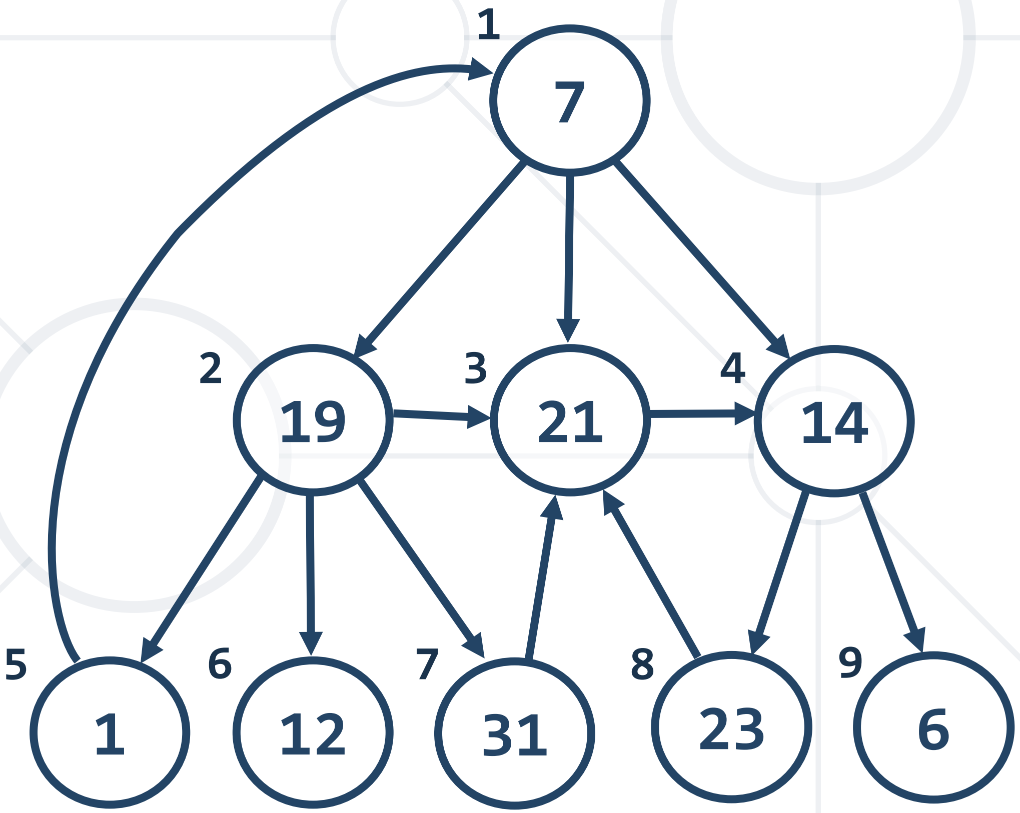
- Stack: (empty)
- Output: 7, 12, 6, 23, 14, 21, 31, 19, 1



Breadth-First Search (BFS)

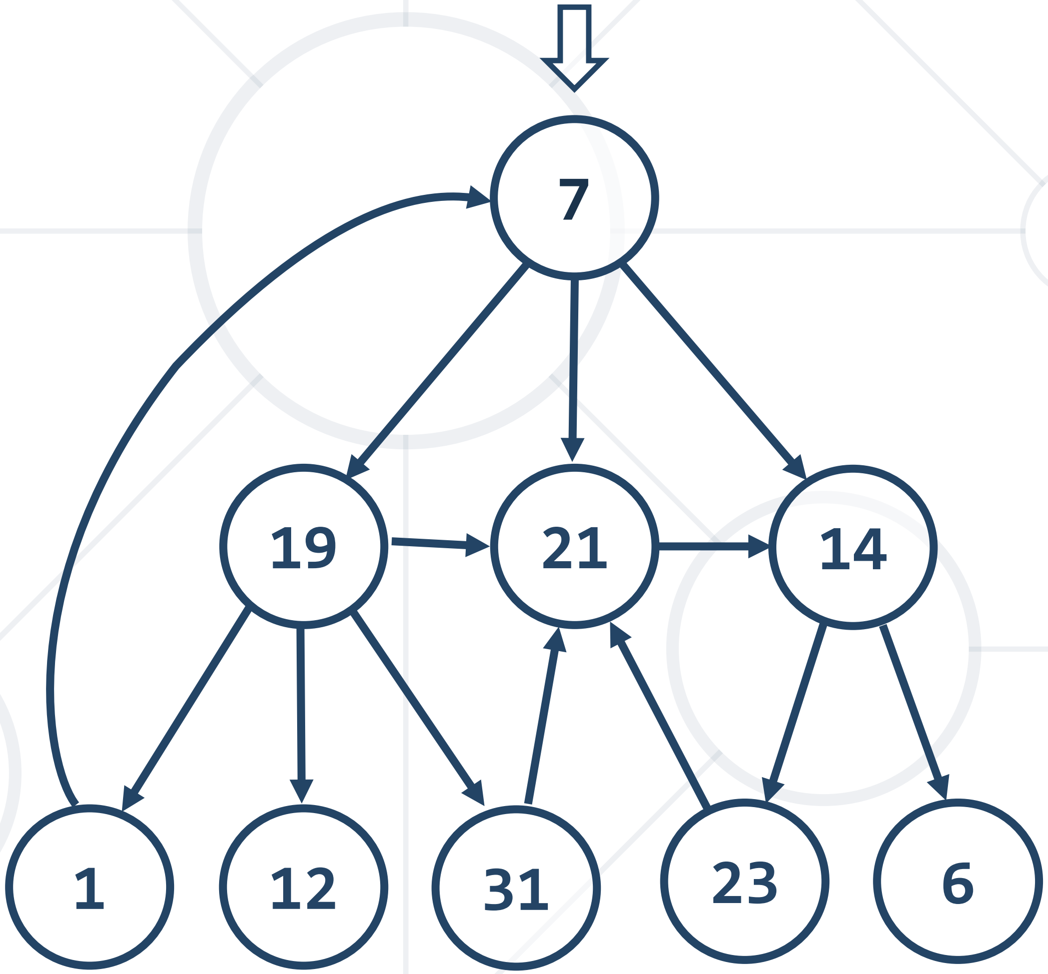
- **Breadth-First Search** (**BFS**) first visits the neighbor nodes, then the neighbors of neighbors, then their neighbors, etc.

```
bfs(node) {
  queue ← node
  visited[node] = true
  while queue not empty
    v ← queue
    print v
    for each child c of v
      if not visited[c]
        queue ← c
        visited[c] = true
}
```



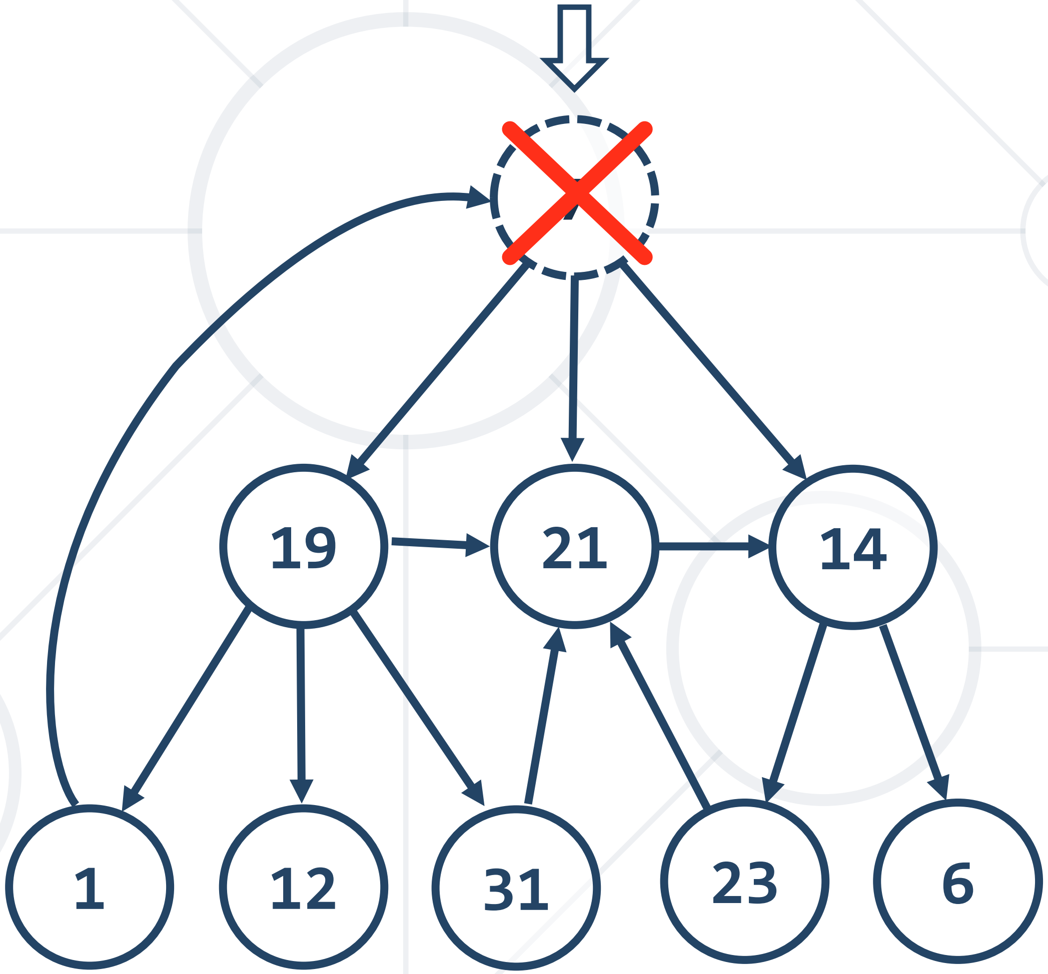
BFS in Action (Step 1)

- Queue: 7
- Output:



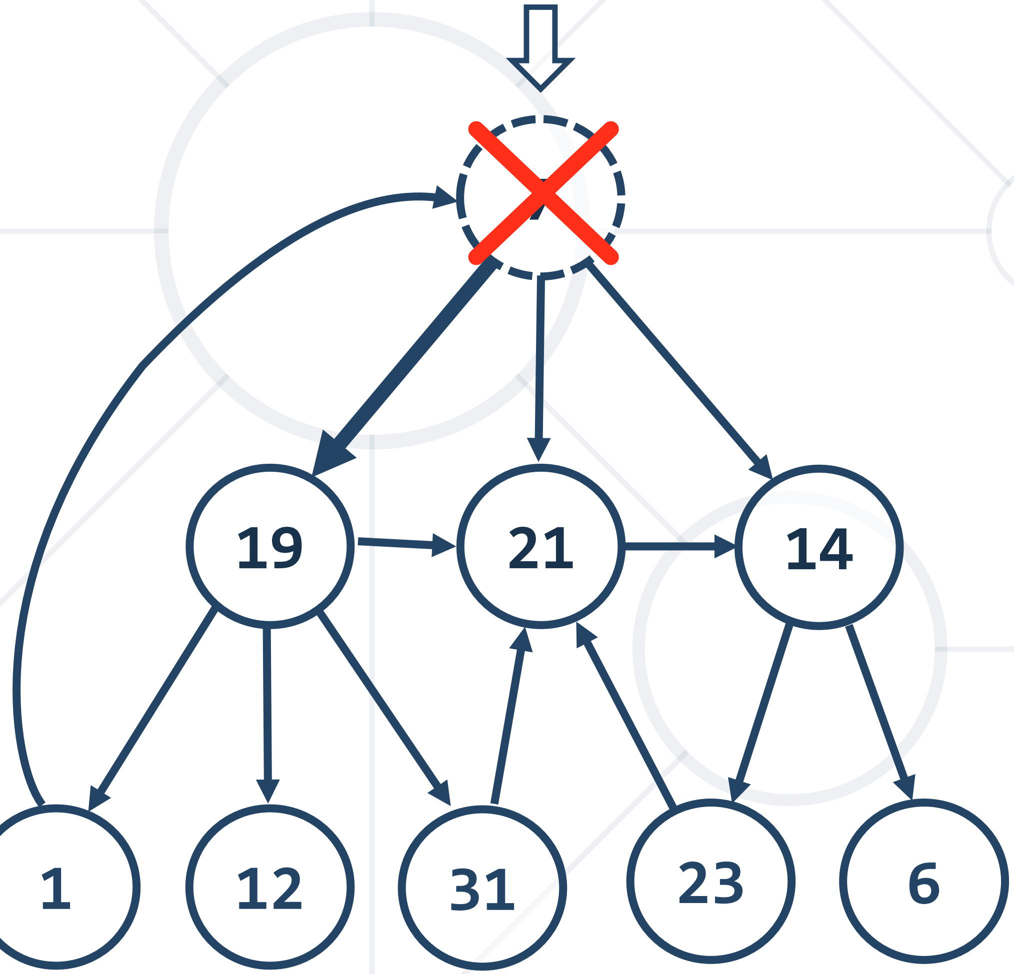
BFS in Action (Step 2)

- Queue: ~~7~~
- Output: 7



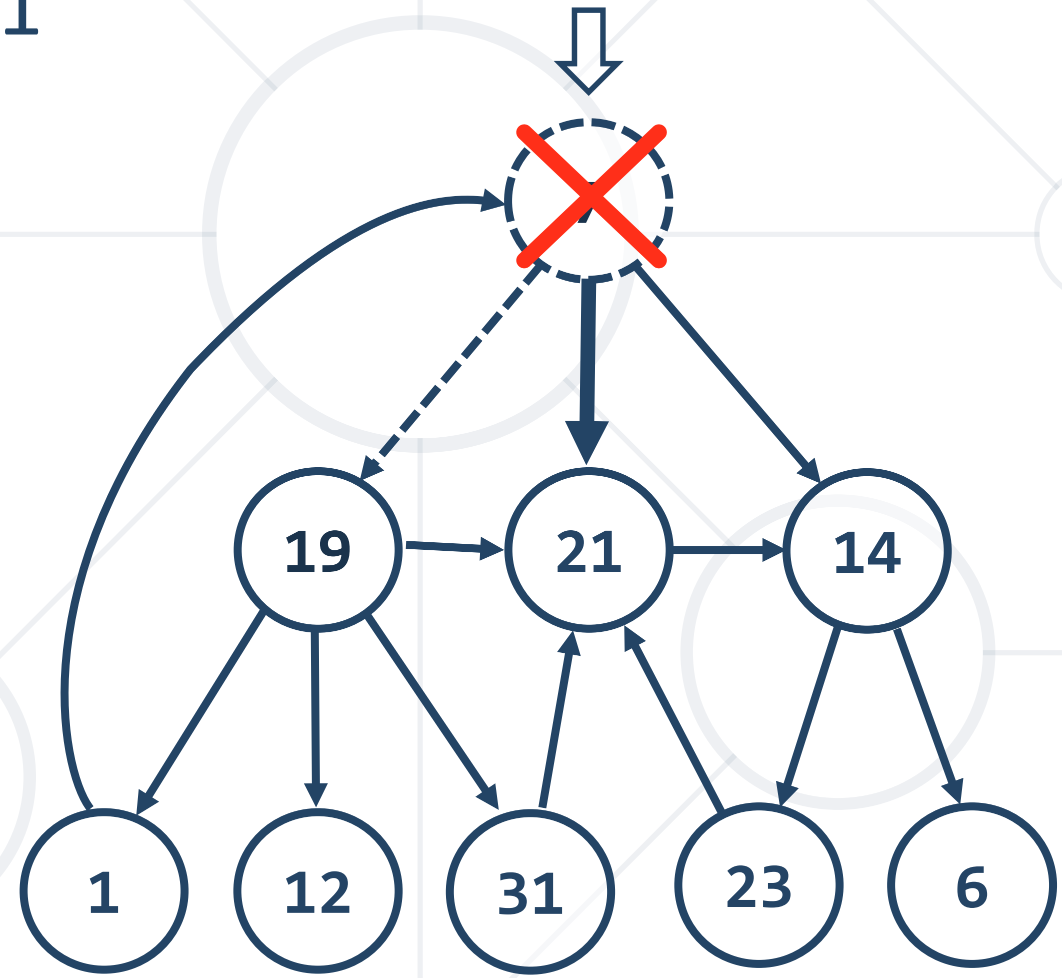
BFS in Action (Step 3)

- Queue: ~~7~~, 19
- Output: 7



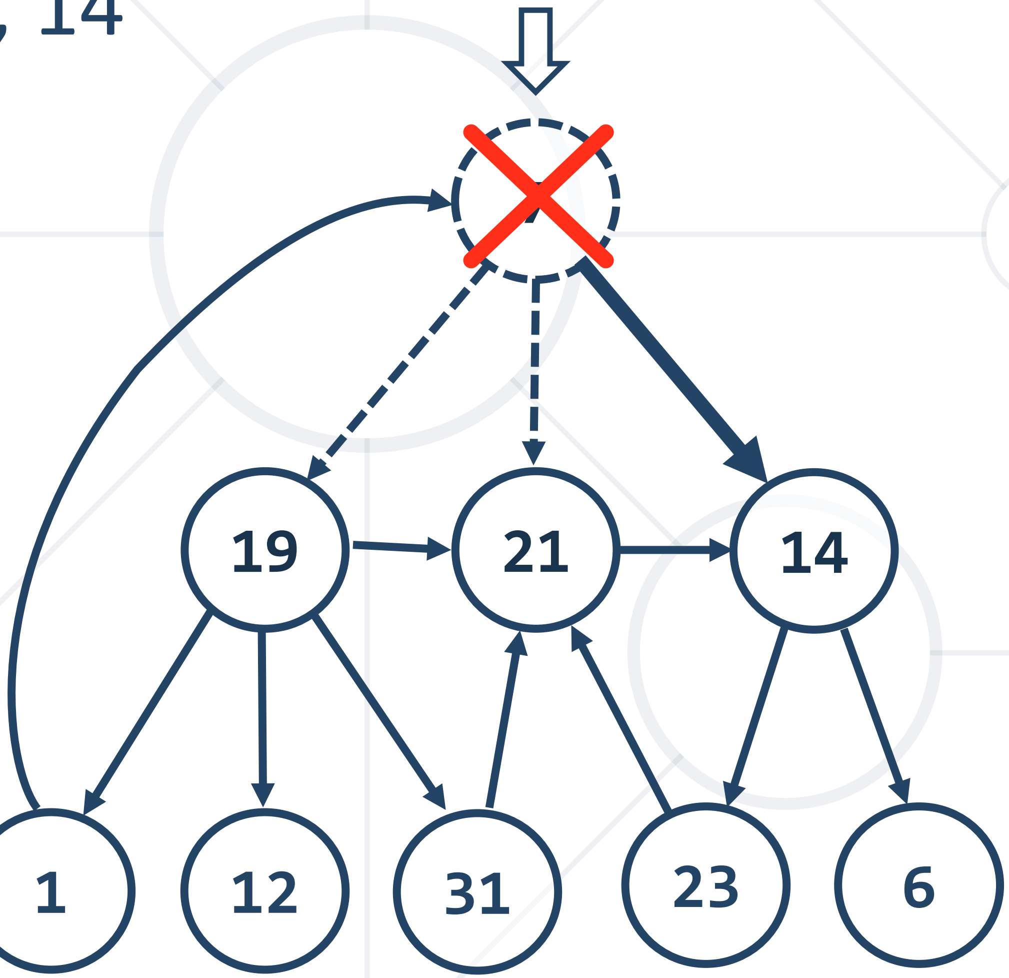
BFS in Action (Step 4)

- Queue: ~~7~~, 19, 21
- Output: 7



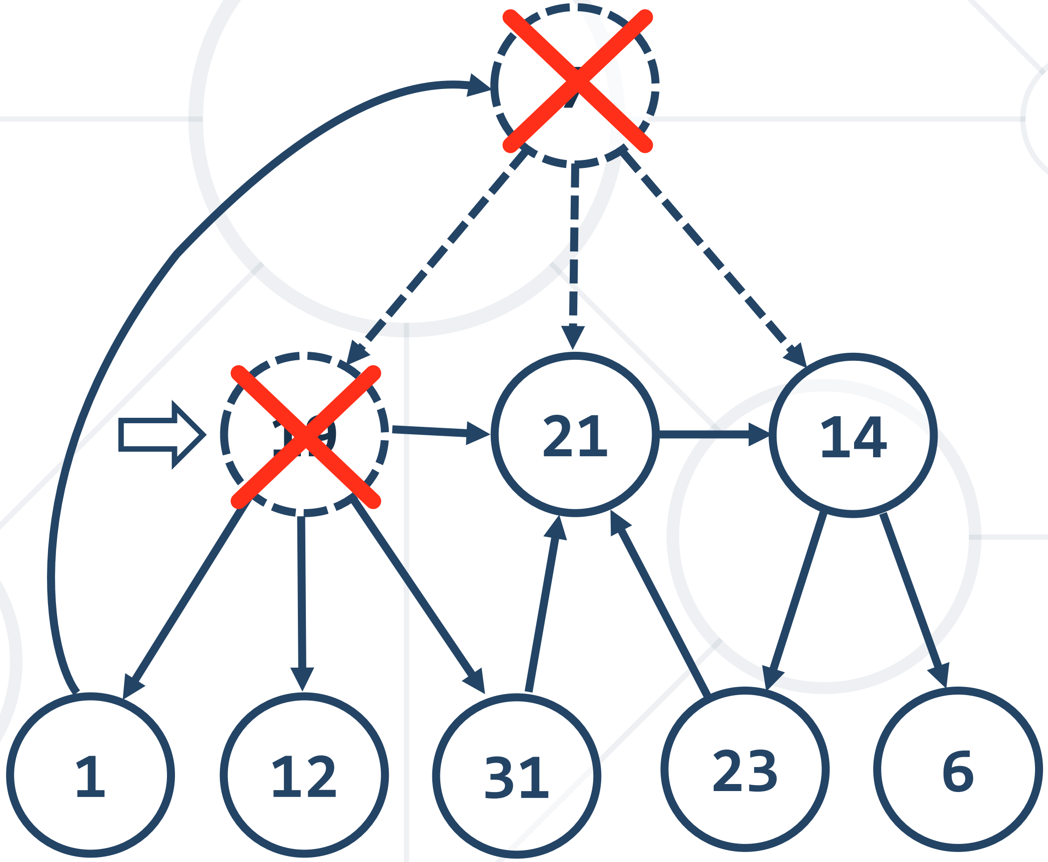
BFS in Action (Step 5)

- Queue: ~~7~~, 19, 21, 14
- Output: 7



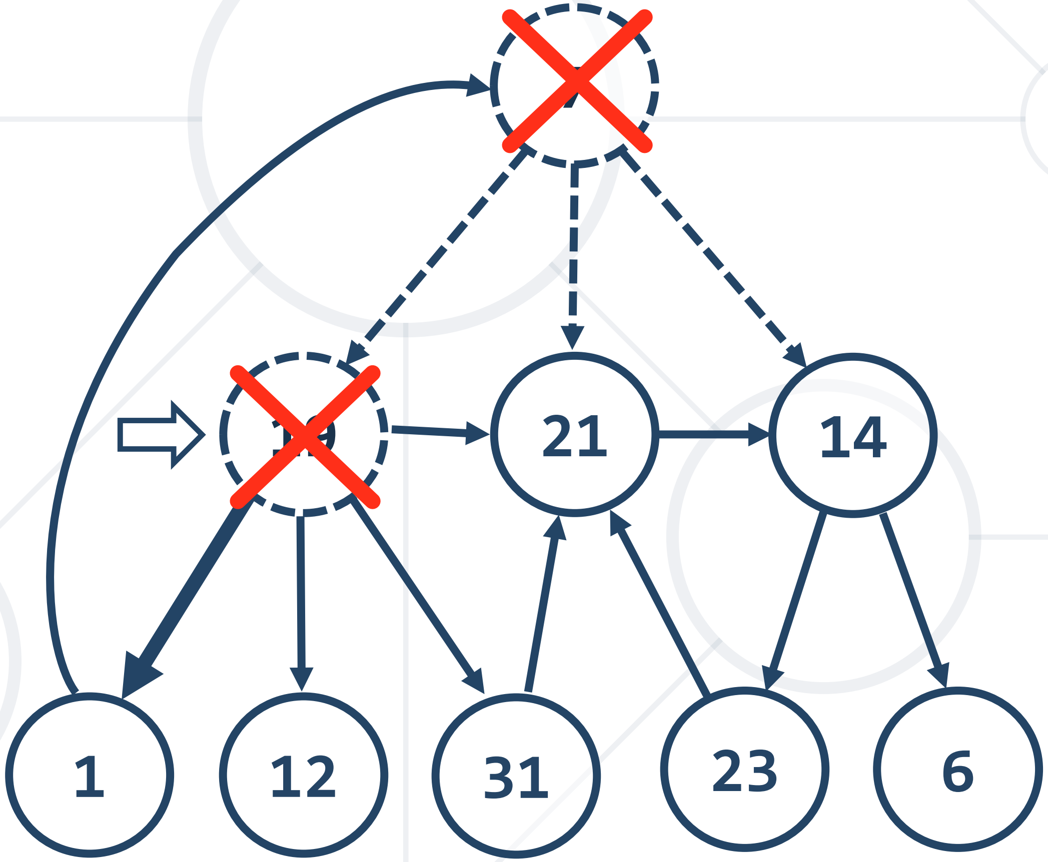
BFS in Action (Step 6)

- Queue: ~~7~~, ~~19~~, 21, 14
- Output: 7, 19



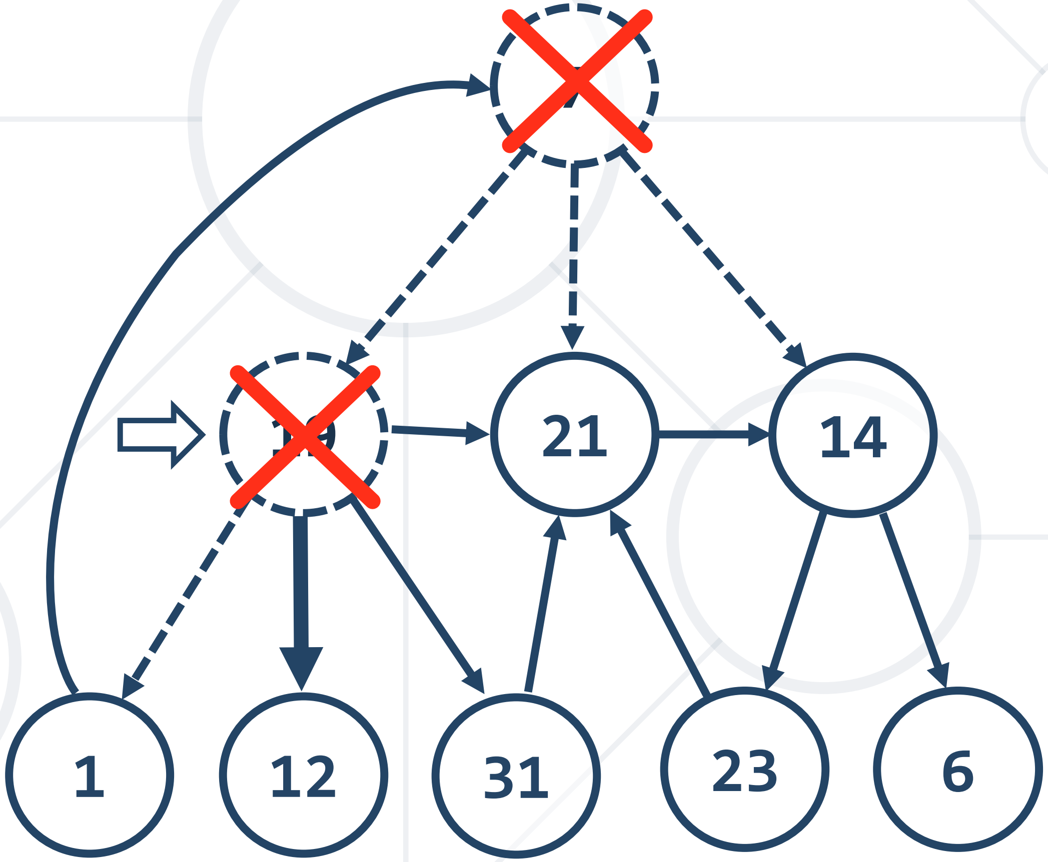
BFS in Action (Step 7)

- Queue: ~~7~~, ~~19~~, 21, 14, 1
- Output: 7, 19



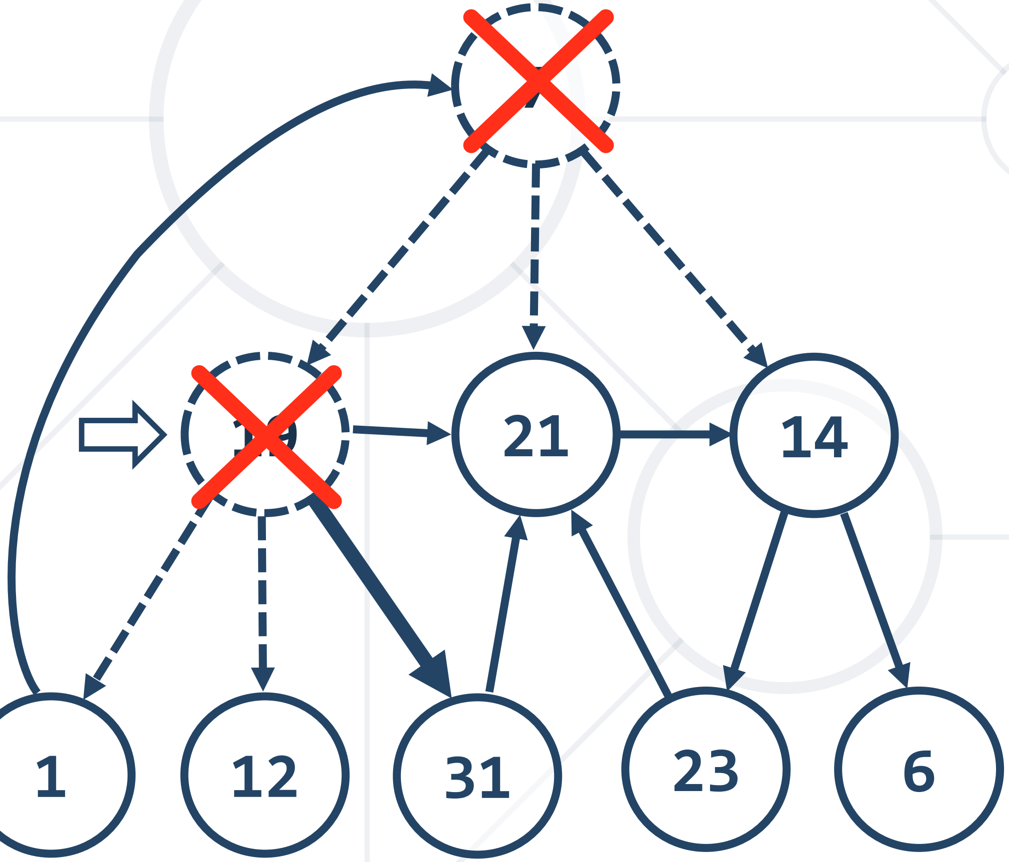
BFS in Action (Step 8)

- Queue: ~~7~~, ~~19~~, 21, 14, 1, 12
- Output: 7, 19



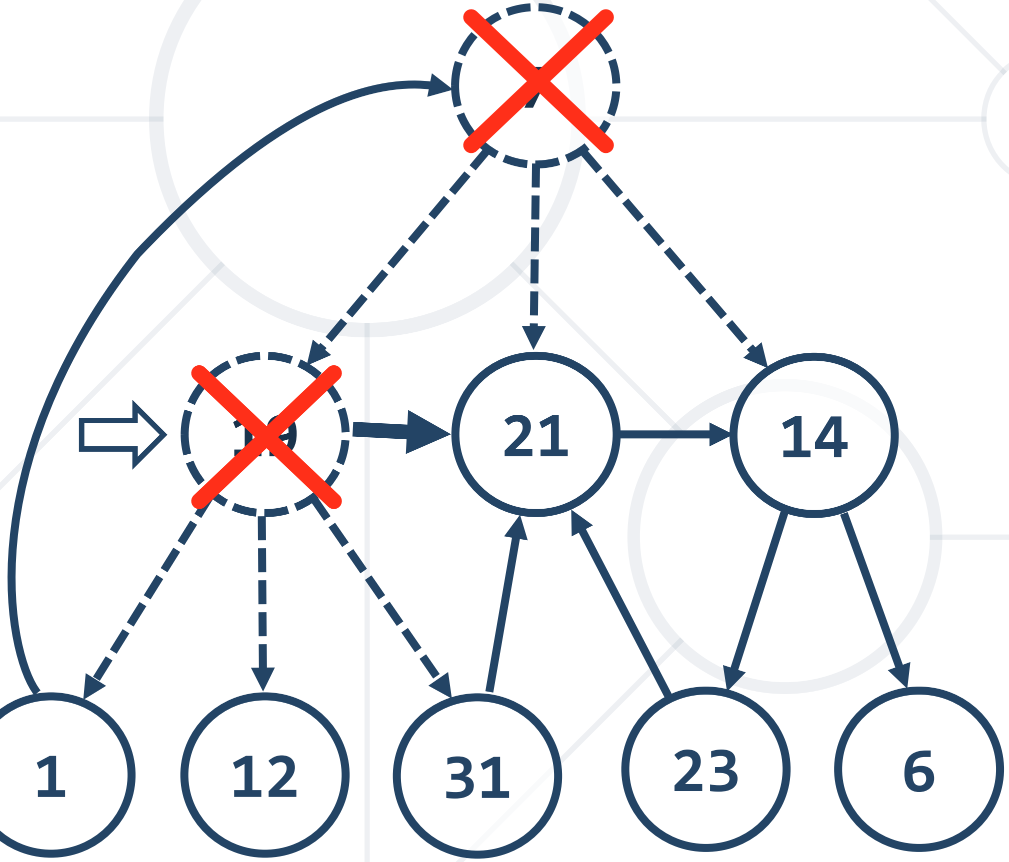
BFS in Action (Step 9)

- Queue: ~~7~~, ~~19~~, 21, 14, 1, 12, 31
- Output: 7, 19



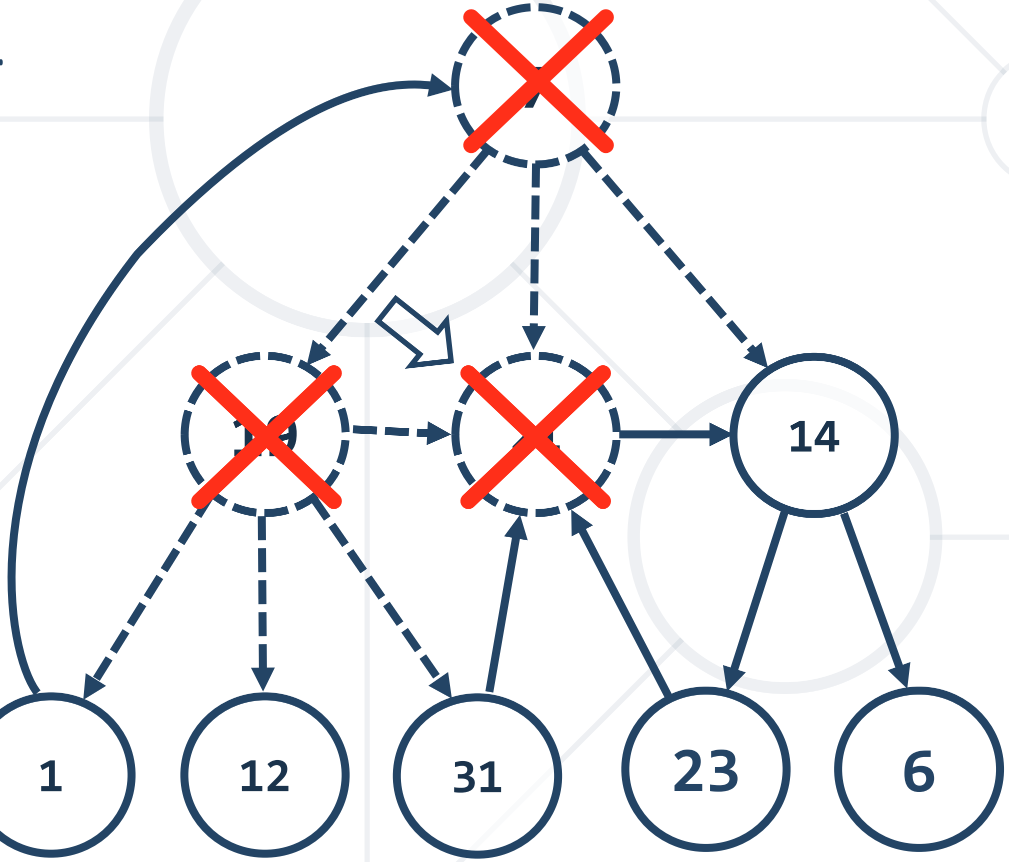
BFS in Action (Step 10)

- Queue: ~~7~~, ~~19~~, 21, 14, 1, 12, 31
- Output: 7, 19



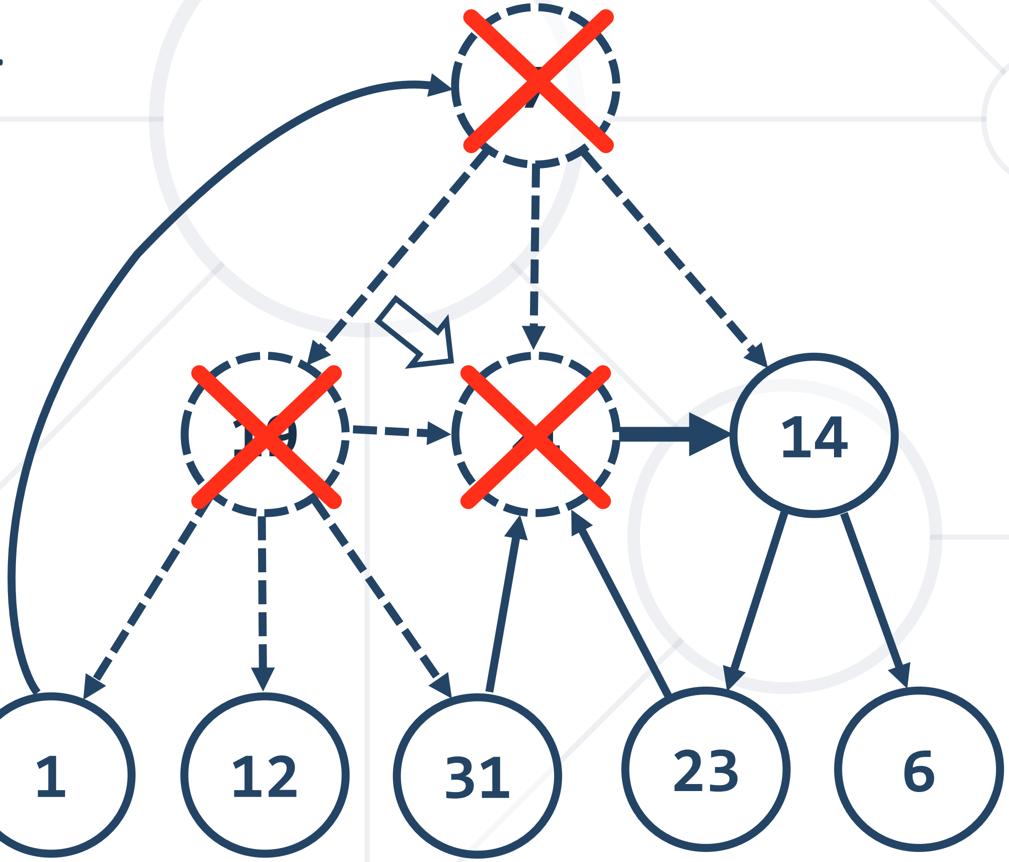
BFS in Action (Step 11)

- Queue: ~~7~~, ~~19~~, ~~21~~, 14, 1, 12, 31
- Output: 7, 19, 21



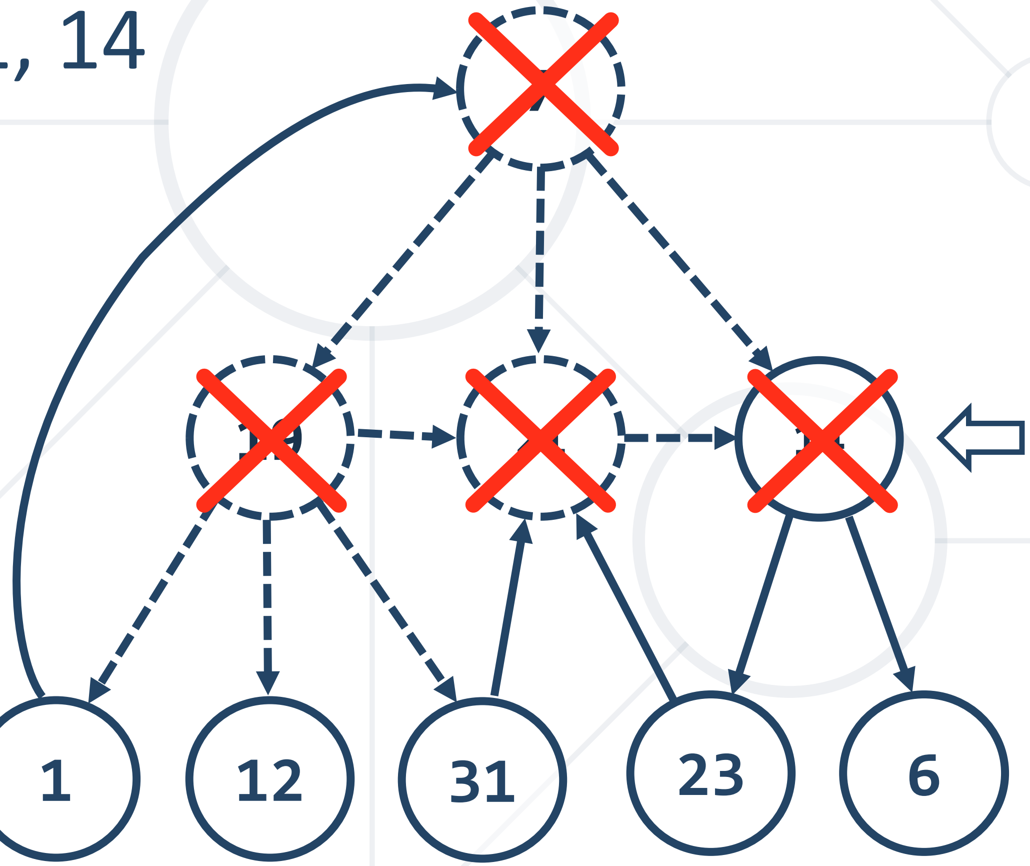
BFS in Action (Step 12)

- Queue: ~~7~~, ~~19~~, ~~21~~, 14, 1, 12, 31
- Output: 7, 19, 21



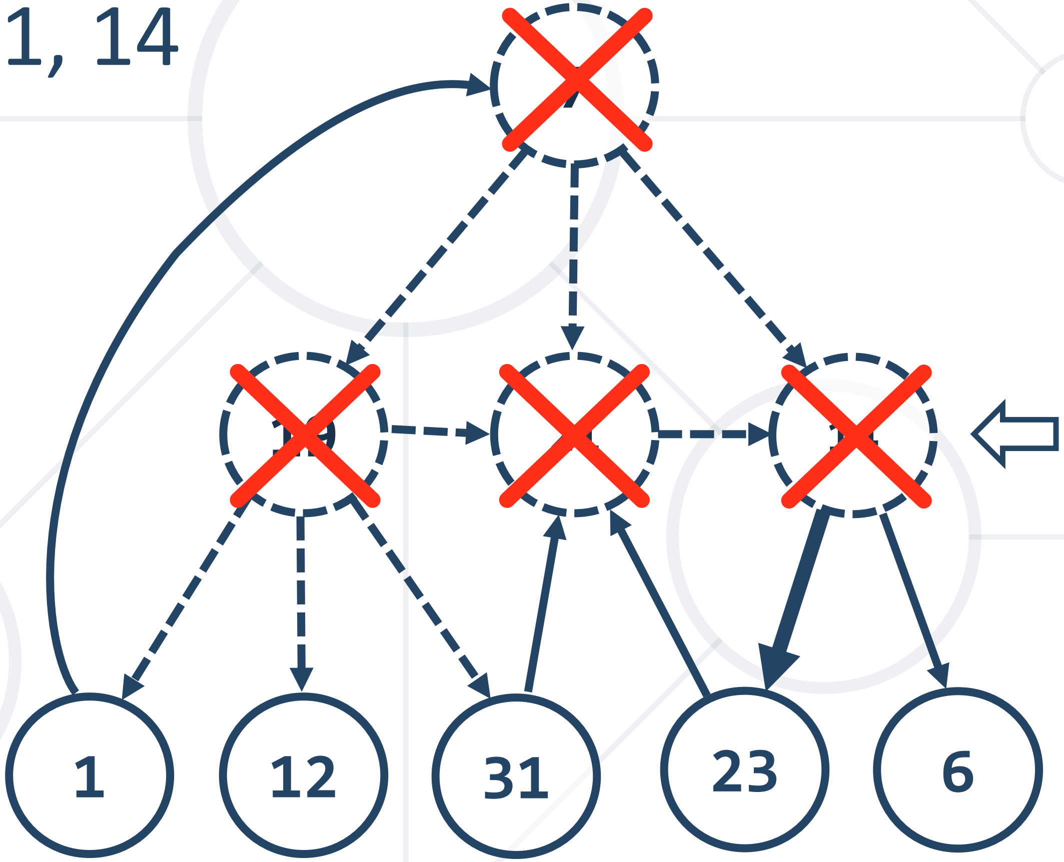
BFS in Action (Step 13)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, 1, 12, 31
- Output: 7, 19, 21, 14



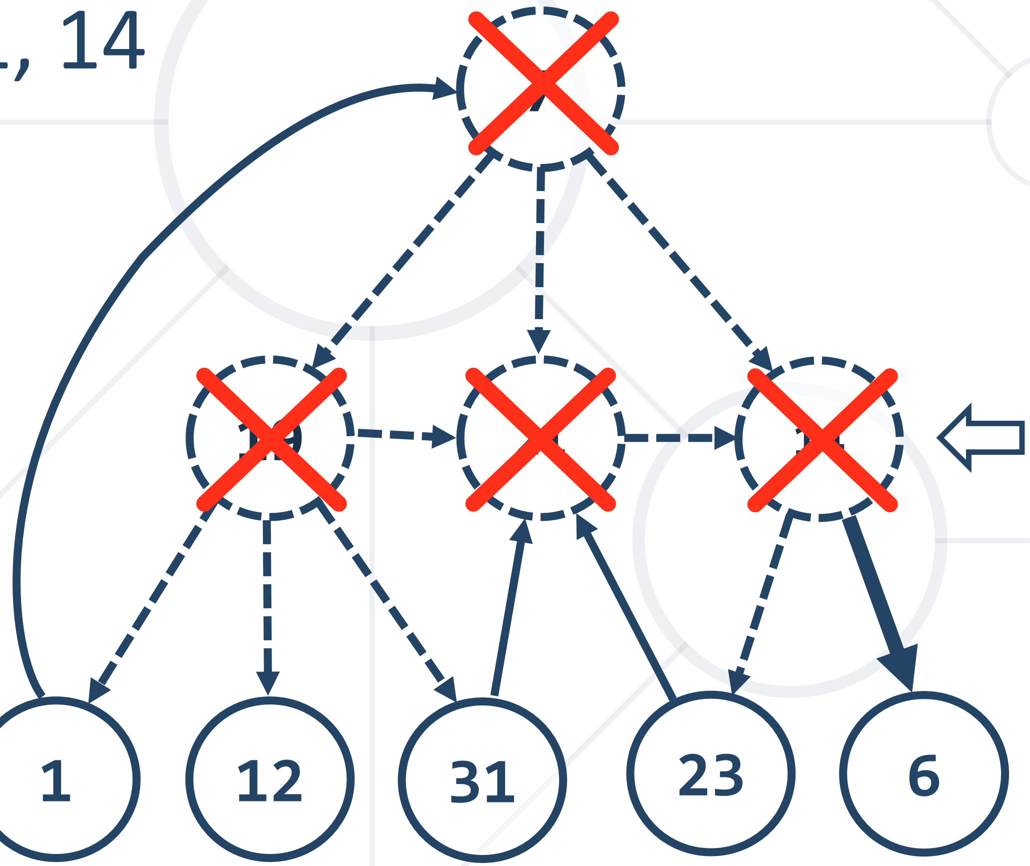
BFS in Action (Step 14)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, 1, 12, 31, 23
- Output: 7, 19, 21, 14



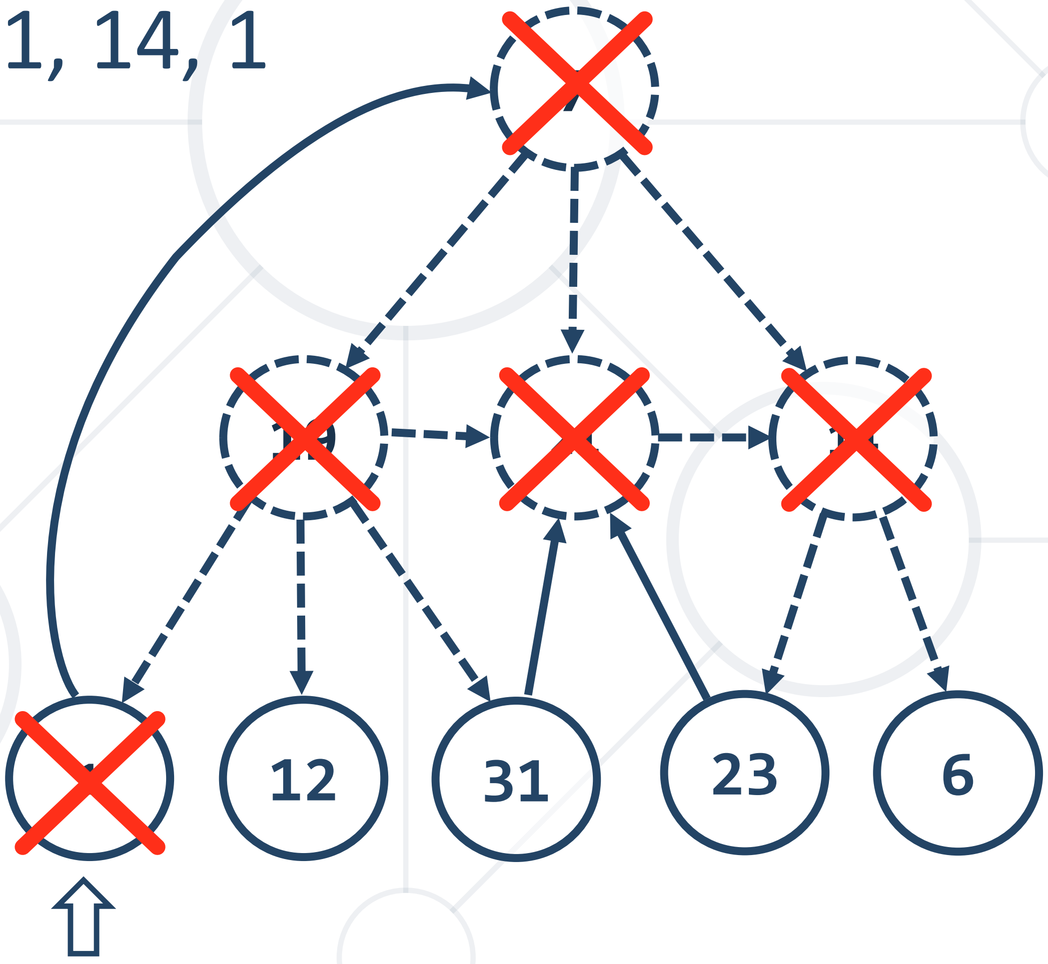
BFS in Action (Step 15)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, 1, 12, 31, 23, 6
- Output: 7, 19, 21, 14



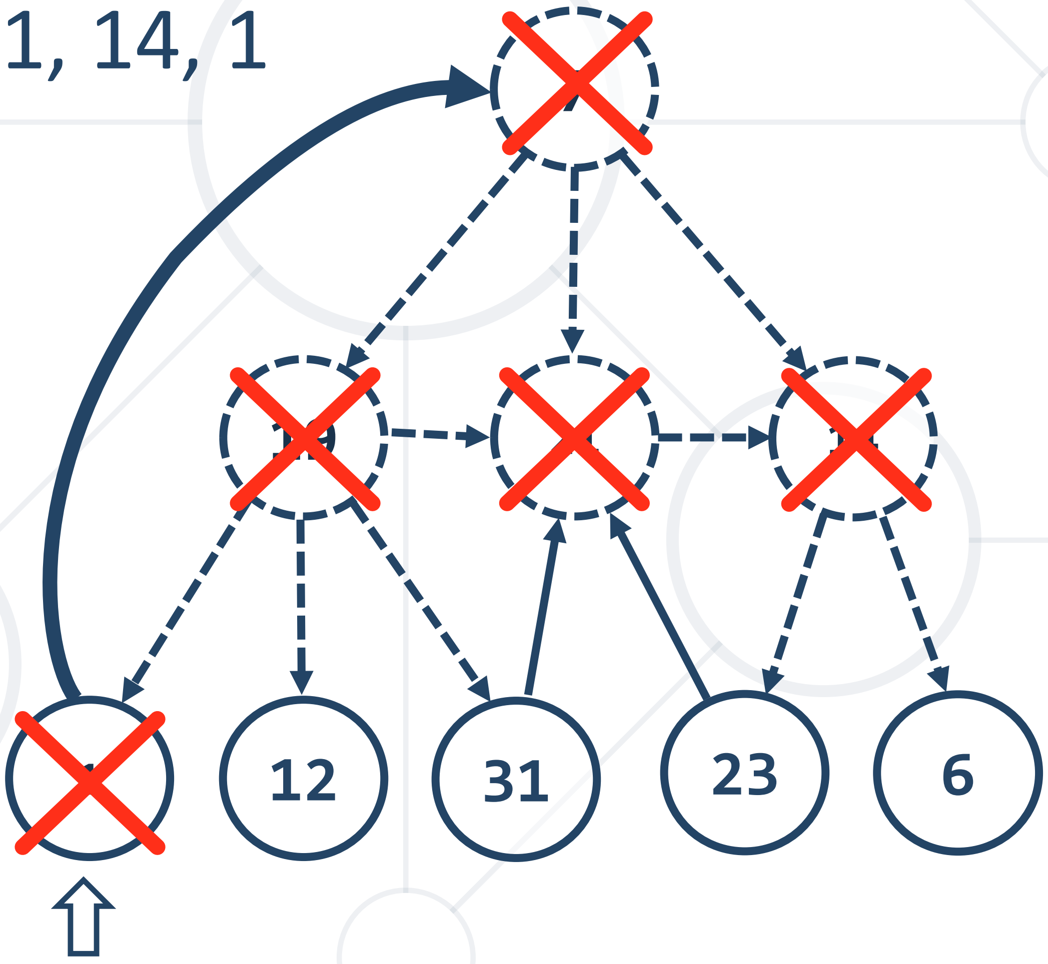
BFS in Action (Step 16)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, 12, 31, 23, 6
- Output: 7, 19, 21, 14, 1



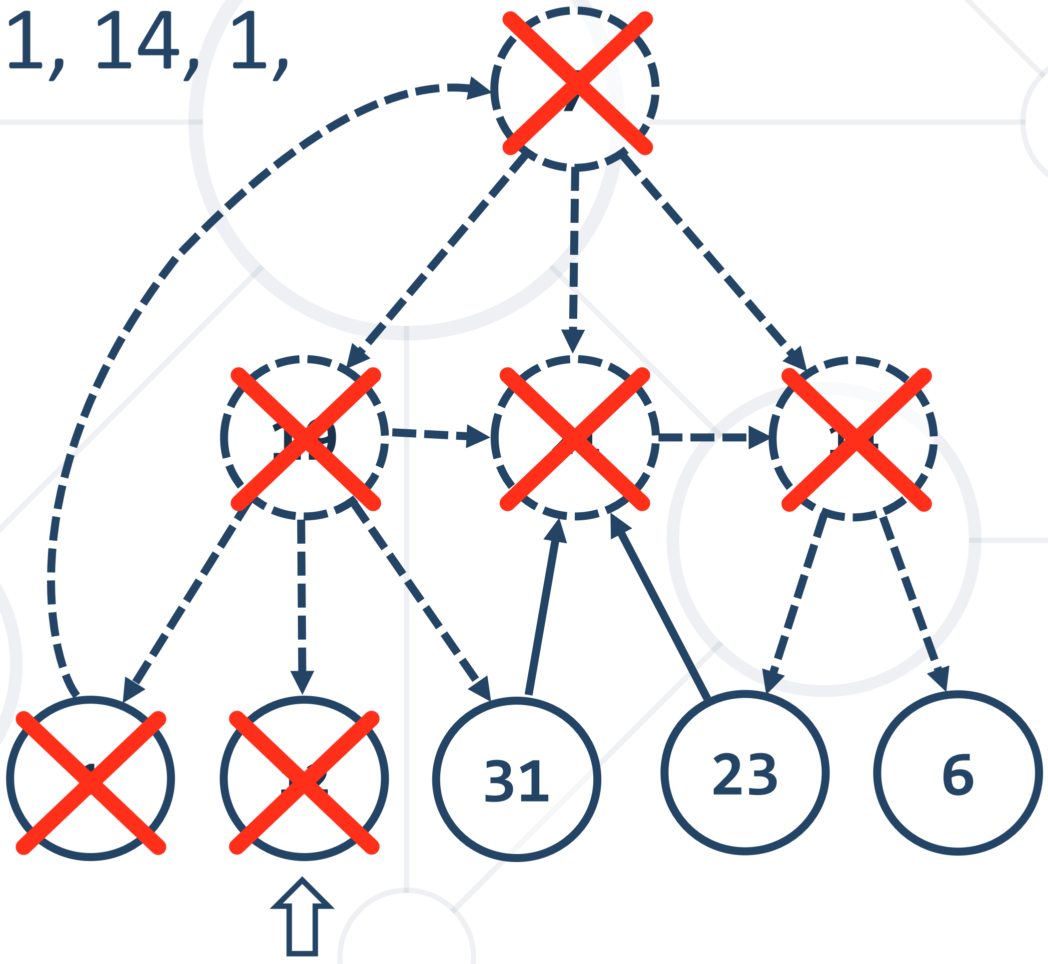
BFS in Action (Step 17)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, 12, 31, 23, 6
- Output: 7, 19, 21, 14, 1



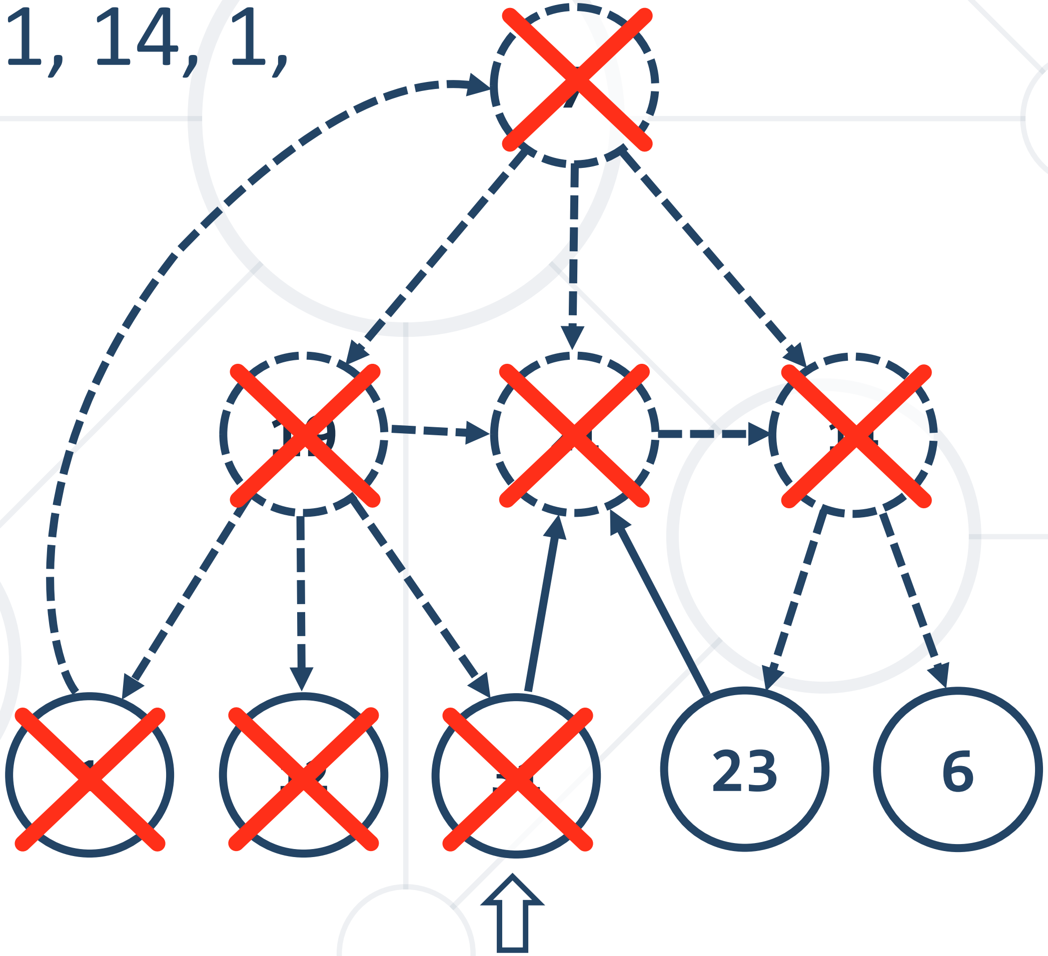
BFS in Action (Step 18)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, ~~12~~, 31, 23, 6
- Output: 7, 19, 21, 14, 1, 12



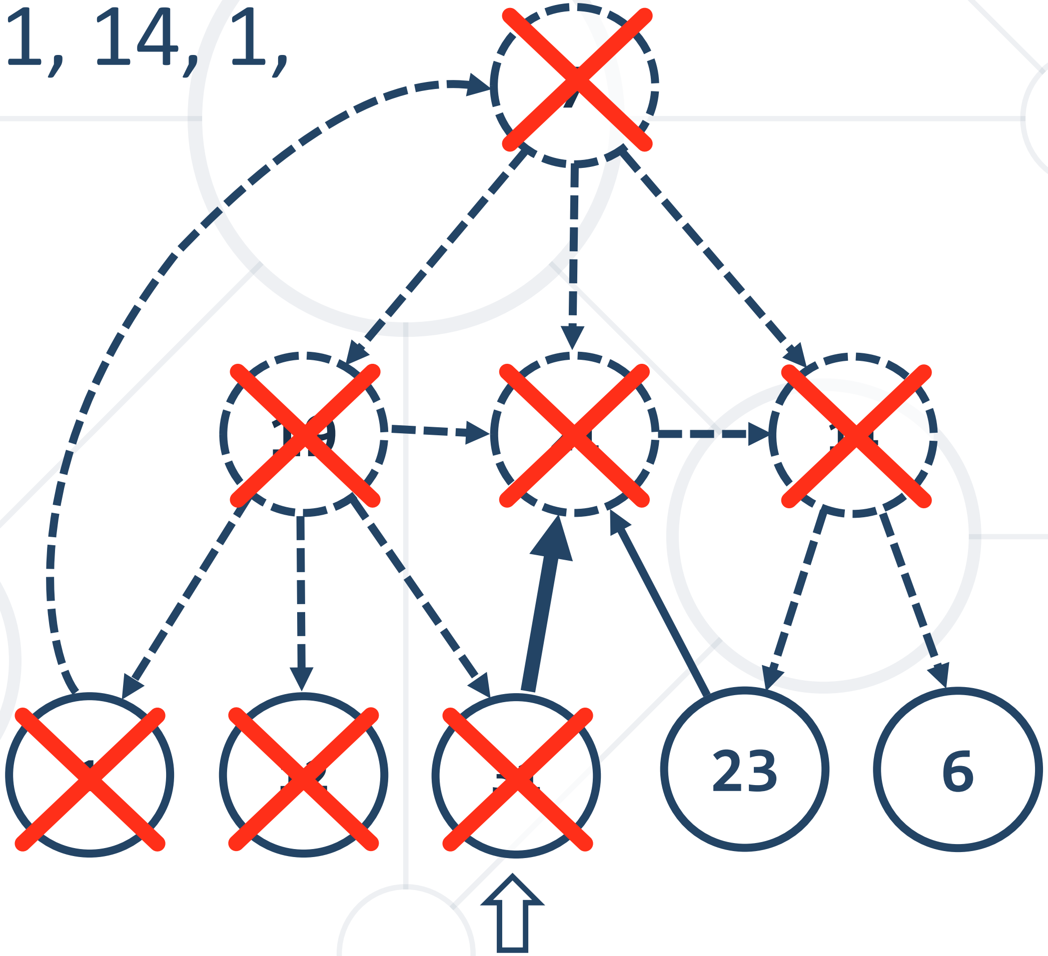
BFS in Action (Step 19)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, ~~12~~, ~~31~~, 23, 6
- Output: 7, 19, 21, 14, 1, 12, 31



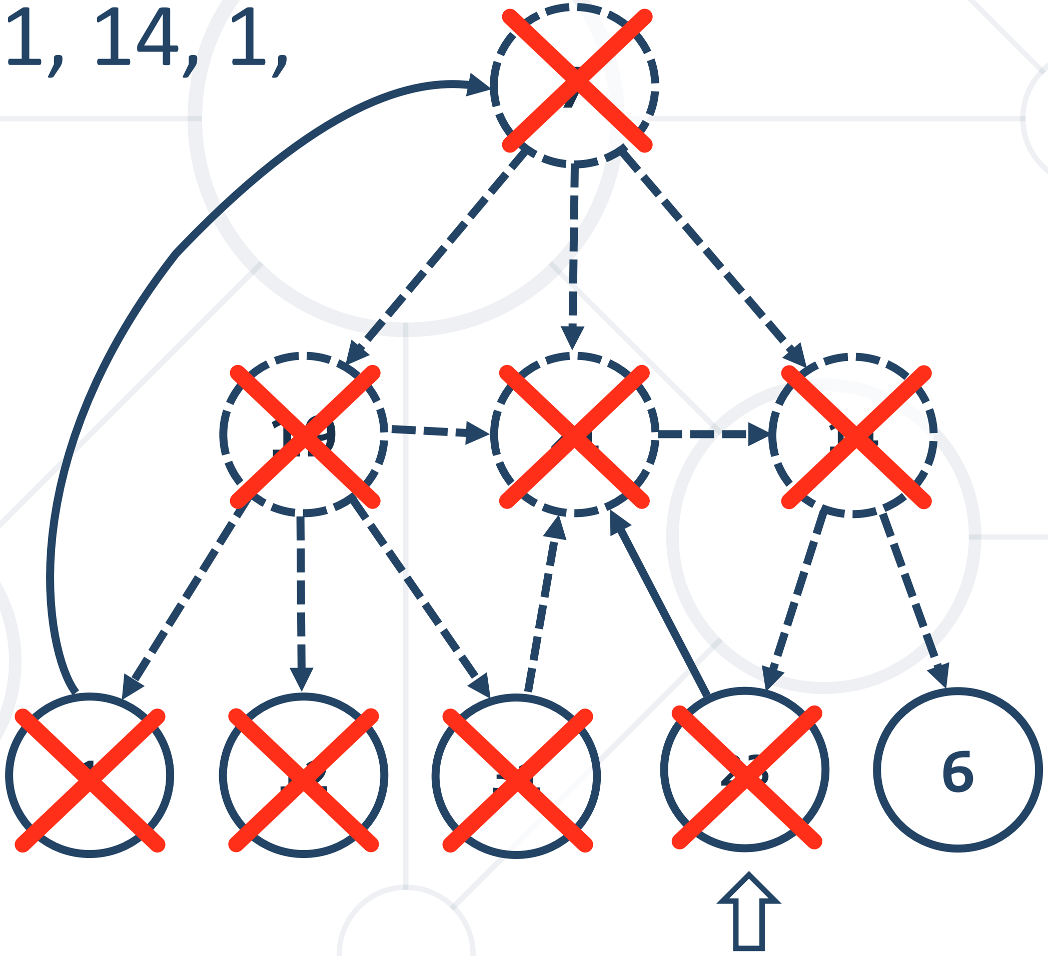
BFS in Action (Step 20)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, ~~12~~, ~~31~~, 23, 6
- Output: 7, 19, 21, 14, 1, 12, 31



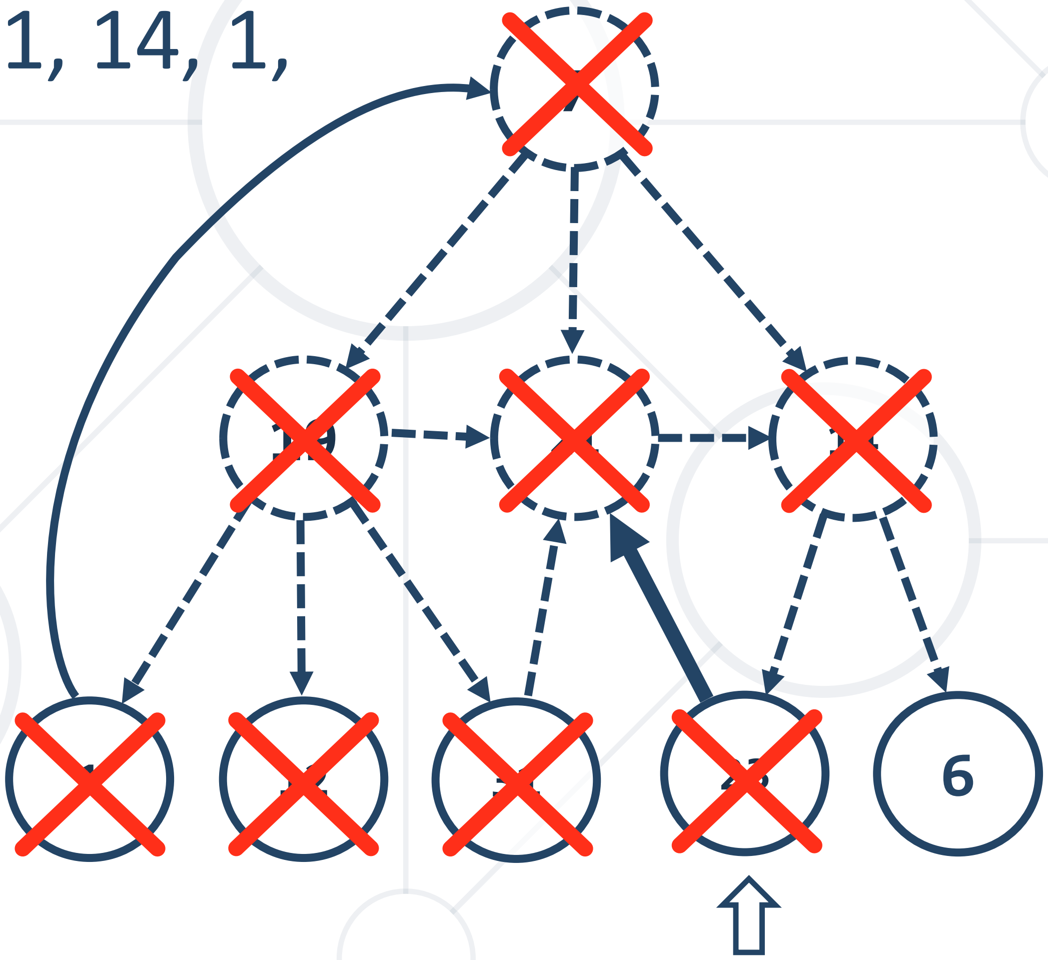
BFS in Action (Step 21)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, ~~12~~, ~~31~~, ~~23~~, 6
- Output: 7, 19, 21, 14, 1, 12, 31, 23



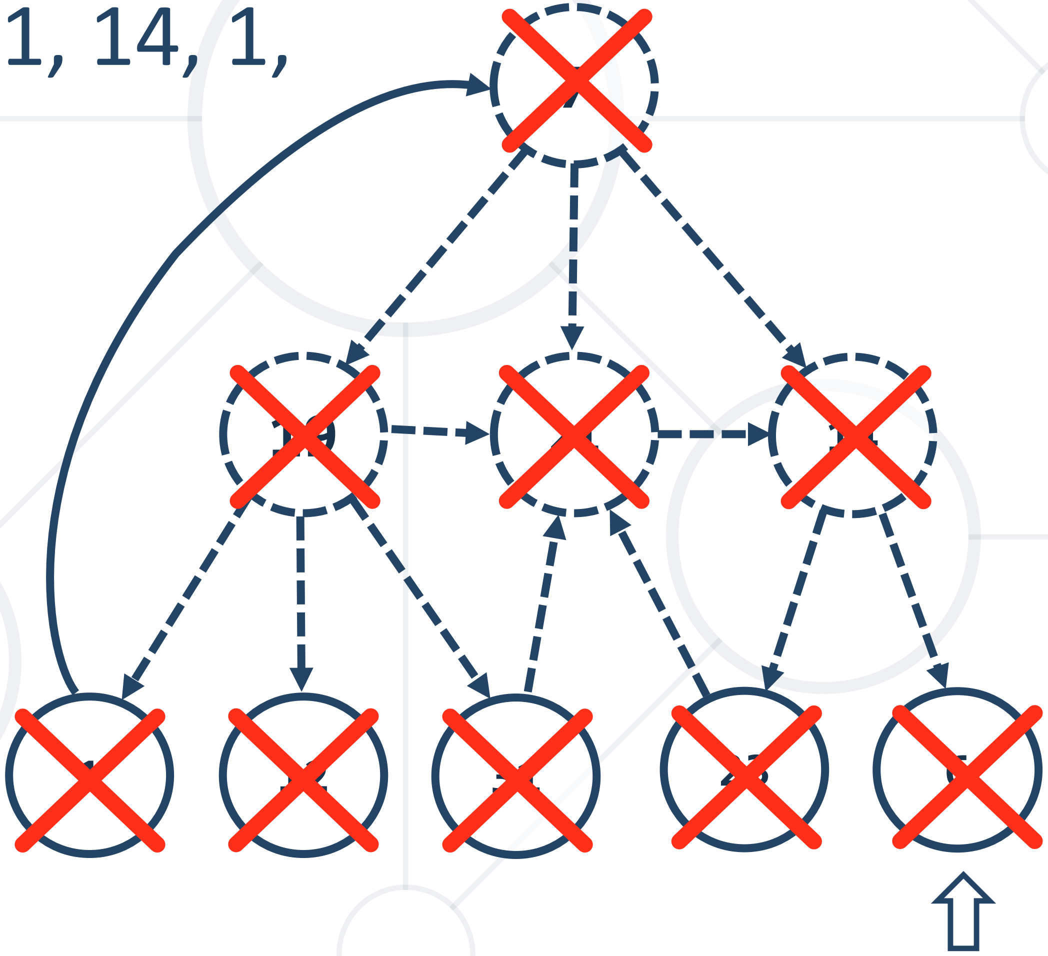
BFS in Action (Step 22)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, ~~12~~, ~~31~~, ~~23~~, 6
- Output: 7, 19, 21, 14, 1, 12, 31, 23



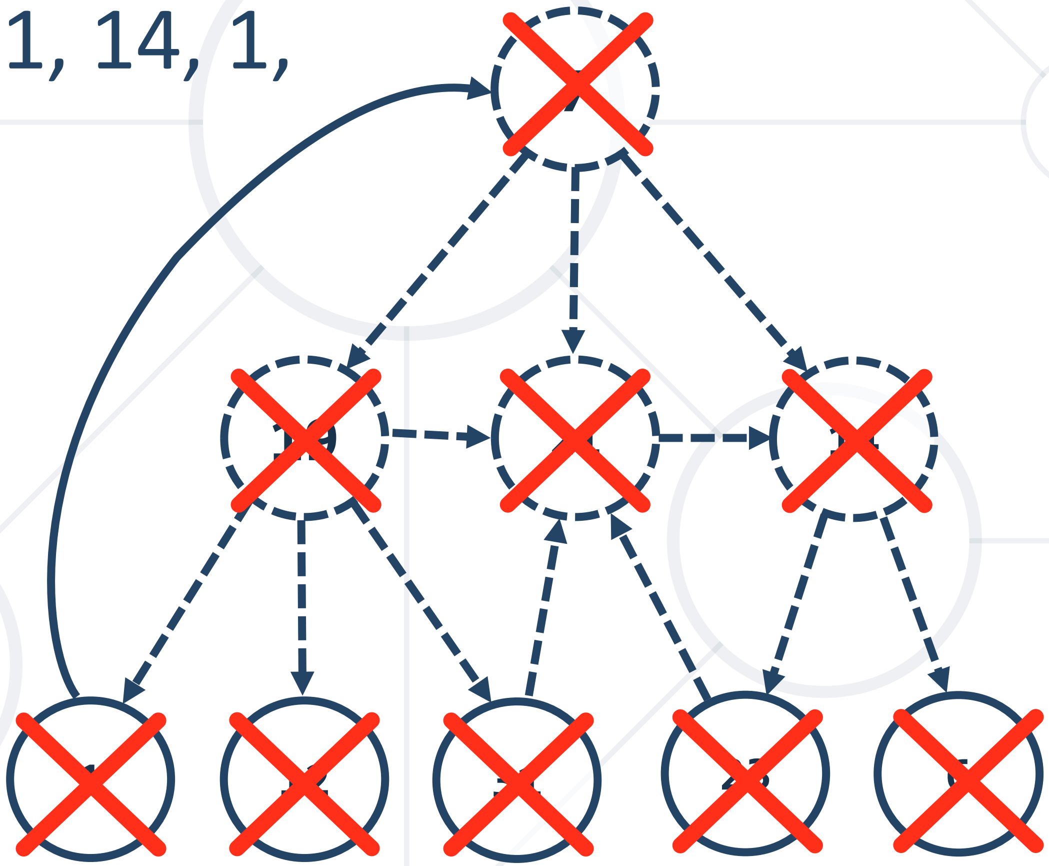
BFS in Action (Step 23)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, ~~12~~, ~~31~~, ~~23~~, ~~6~~
- Output: 7, 19, 21, 14, 1, 12, 31, 23, 6



BFS in Action (Step 24)

- Queue: ~~7~~, ~~19~~, ~~21~~, ~~14~~, ~~1~~, ~~12~~, ~~31~~, ~~23~~, ~~6~~
- Output: 7, 19, 21, 14, 1, 12, 31, 23, 6

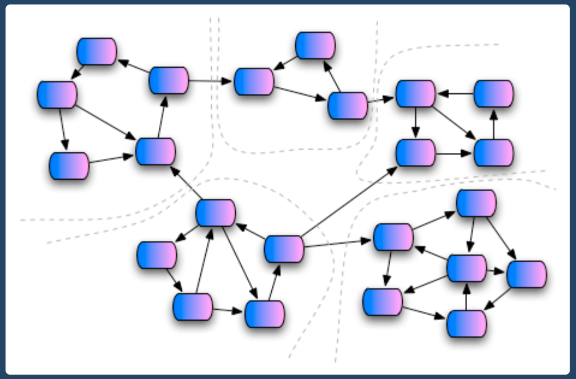


Iterative DFS and BFS

- What will happen if in the **Breadth-First Search** (**BFS**) algorithm we change the **queue** with a **stack**?
  - An iterative stack-based **Depth-First Search** (**DFS**)

```
bfs(node) {
  queue ⇐  node
  visited[node] = true
  while queue not empty
    v ⇐ queue
    print    v
    for each child c of v
      if not visited[c]
        queue ⇐ c
        visited[c] = true
}
```

```
dfs(node) {
  stack ⇐ node
  visited[node] = true
  while stack not empty
    v  ⇐ stack
    print v
    for each child c of v
      if not visited[c]
        stack ⇐ c
        visited[c] = true
}
```

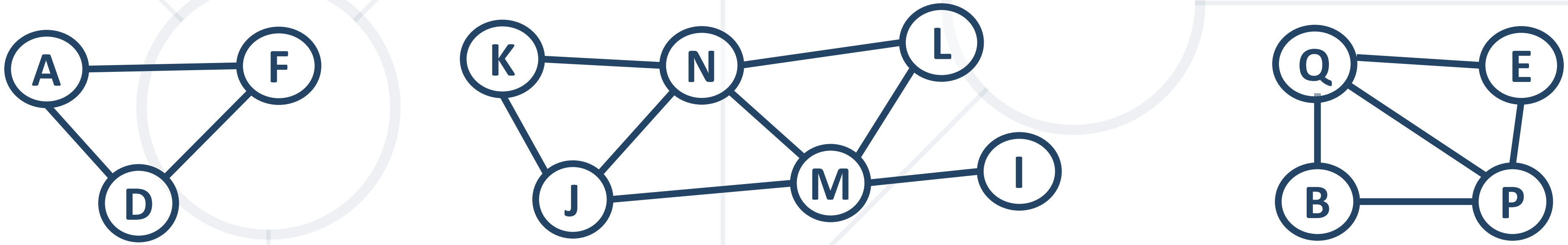


Graph Connectivity

Finding the Connected Components

Graph Connectivity

- Connected component of undirected graph
  - A sub-graph in which any two nodes are connected to each other by paths
  - E.g., the graph below consists of 3 connected components



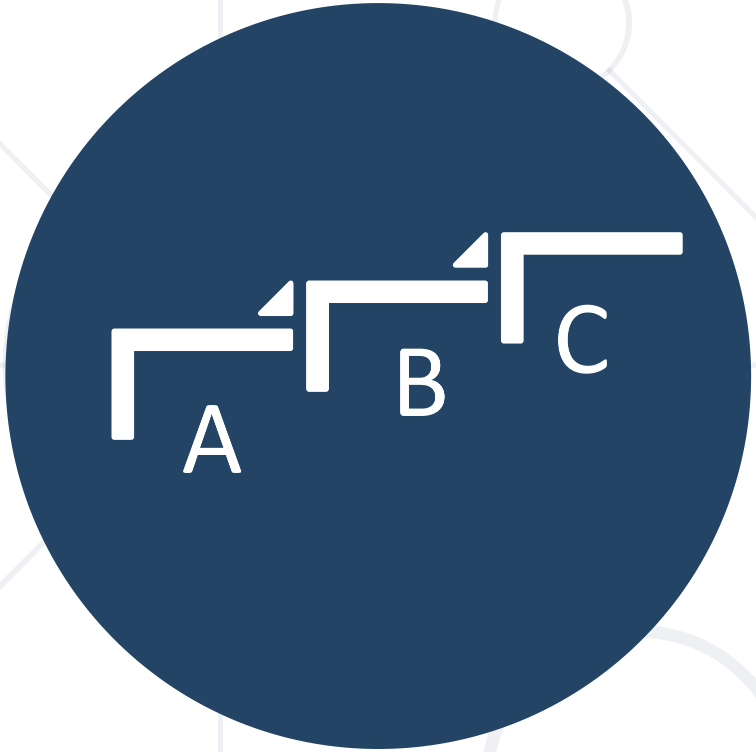
Finding All Graph Connected Components

- Finding the connected components in a graph
  - Loop through all nodes and start a **DFS** / **BFS** traversing from any **unvisited** node
- Each time you start a new traversal
  - You find a new connected component

Graph Connected Components: Algorithm

```
visited[] = false;
foreach node from graph G {
   if (not visited[node]) {
      dfs(node);
      countOfComponents++;
   }
}
```

```
dfs(node) {
  if (not visited[node]) {
    visited[node] = true;
    foreach c in node.children
      dfs(c);
  }
}
```

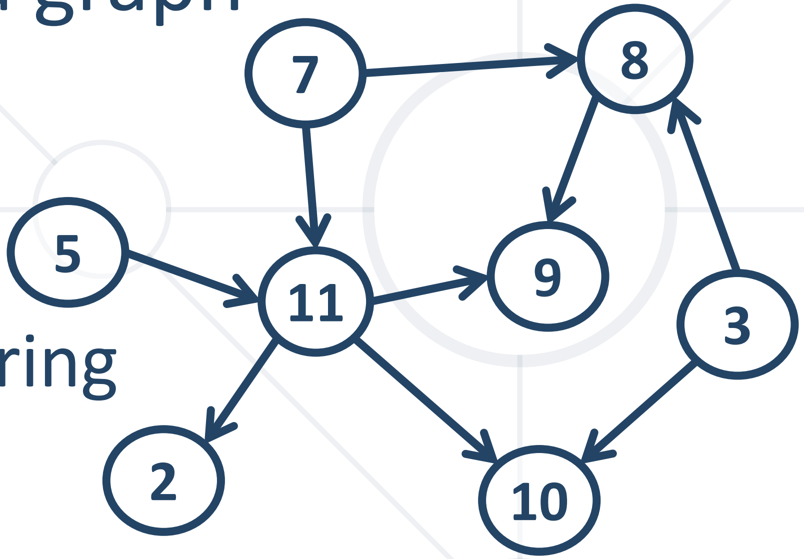


Topological Sorting

Ordering a Graph by Set of Dependencies

Topological Sorting

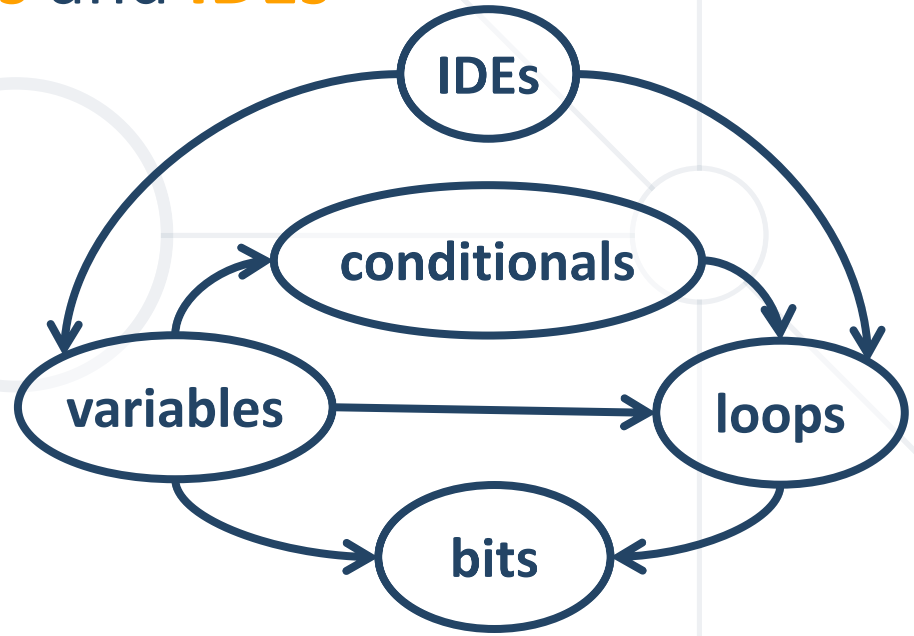
- **Topological sorting** (ordering) of a directed graph
  - Linear ordering of its vertices, such that:
    - For every directed edge from vertex ***u*** to vertex ***v***, ***u*** comes before ***v*** in the ordering



- Example:
  - 7 → 5 → 3 → 11 → 8 → 2 → 9 → 10
  - 3 → 5 → 7 → 8 → 11 → 2 → 9 → 10
  - 5 → 7 → 3 → 8 → 11 → 10 → 9 → 2

Topological Sorting – Example

- We have a set of learning **topics** with **dependencies**
  - Order the topics in such order that all dependencies are met
- Example:
  - **Loops** depend on **variables**, **conditionals** and **IDEs**
  - **Variables** depend on **IDEs**
  - **Bits** depend on **variables** and **loops**
  - **Conditionals** depend on **variables**
- Ordering:
  - **IDEs** → **variables** → **loops** → **bits**



Topological Sorting – Rules

- Rules
  - Undirected graphs cannot be sorted
  - Graphs with cycles cannot be sorted
  - Sorting is not unique
  - Various sorting algorithms exists, and they give different results

Topological Sorting: Source Removal Algorithm

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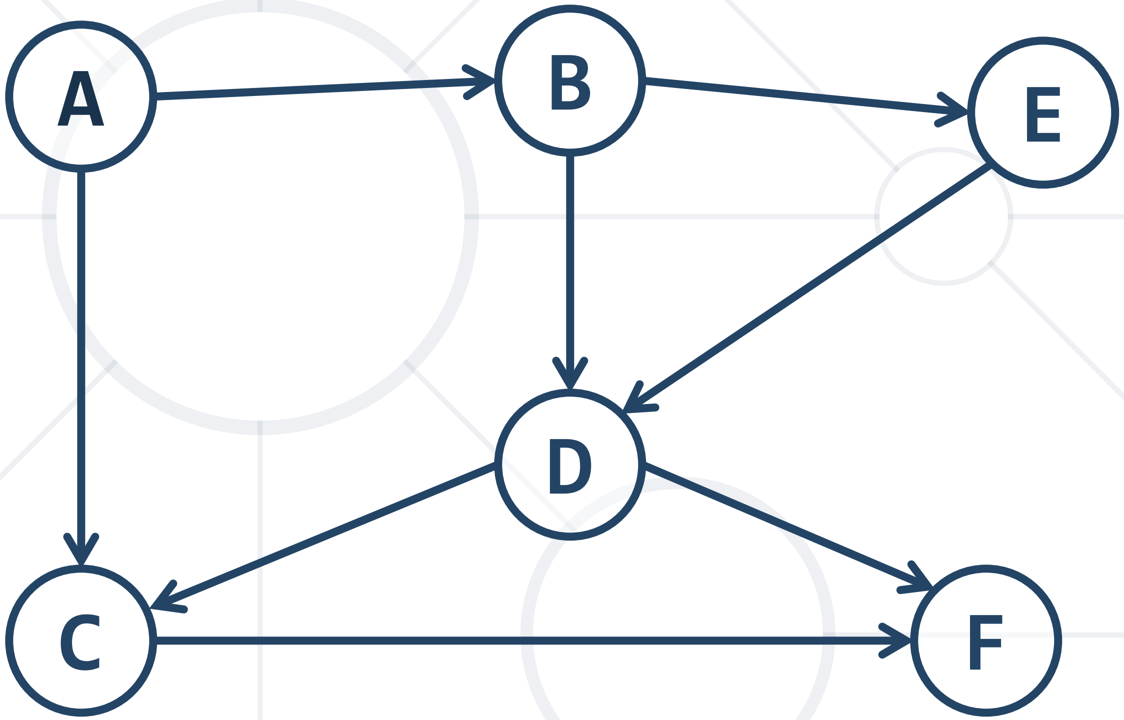
- **Source removal top-sort algorithm**
  - Create an empty list
  - Repeat until the graph is empty:
    - Find a node without incoming edges
    - Add this node to the list
    - Remove the edge from the graph

Source Removal Algorithm

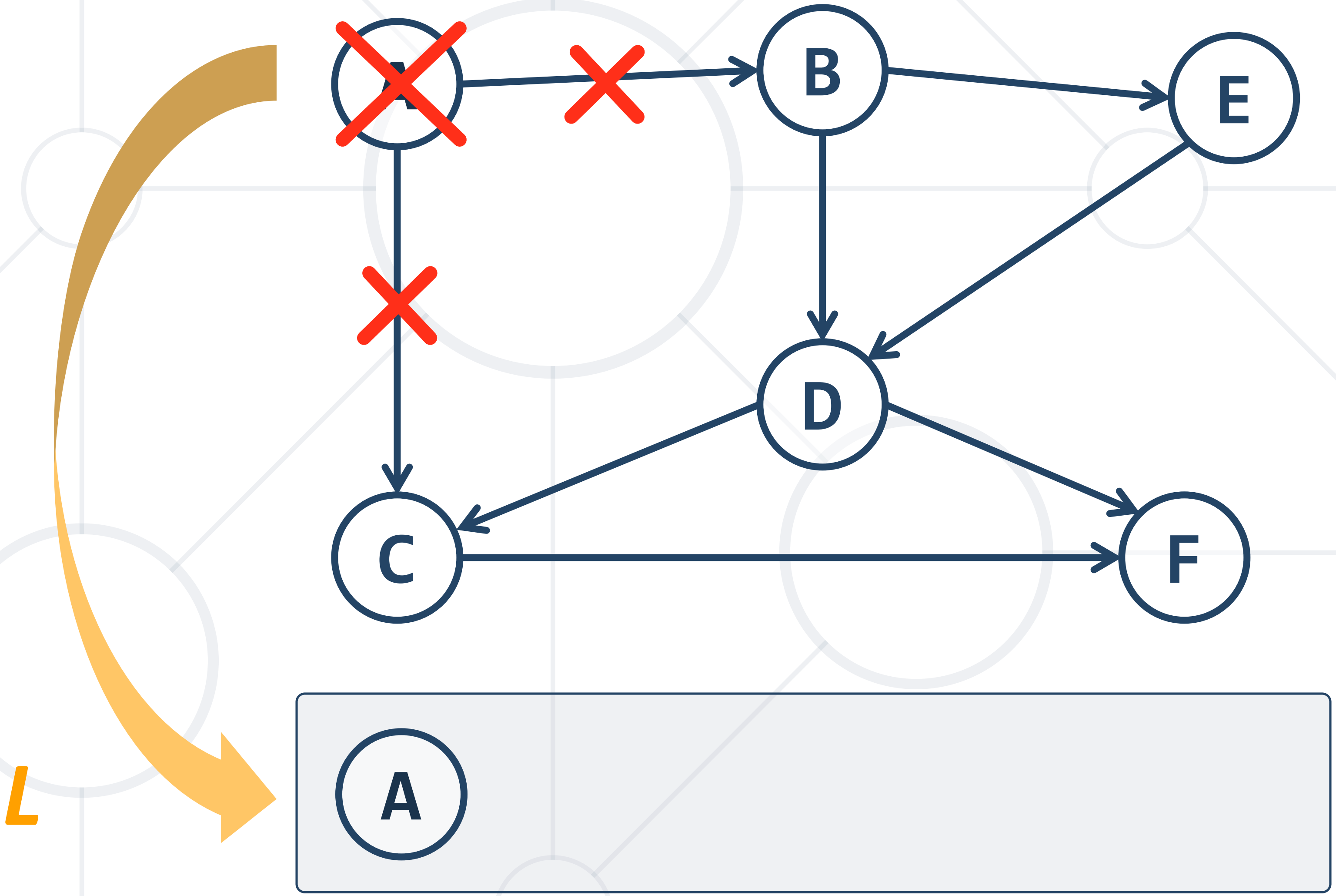
```
L ← empty list that will hold the sorted elements (output)
S ← set of all nodes with no incoming edges
while S is non-empty do
    remove some node n from S
    append n to L
    for each node m with an edge e: { n through m }
        remove edge e from the graph
        if m has no other incoming edges then
            insert m into S
if graph is empty
   return L (a topologically sorted order)
else
   return "Error: graph has at least one cycle"
```

Step #1: Find a Node with No Incoming Edges

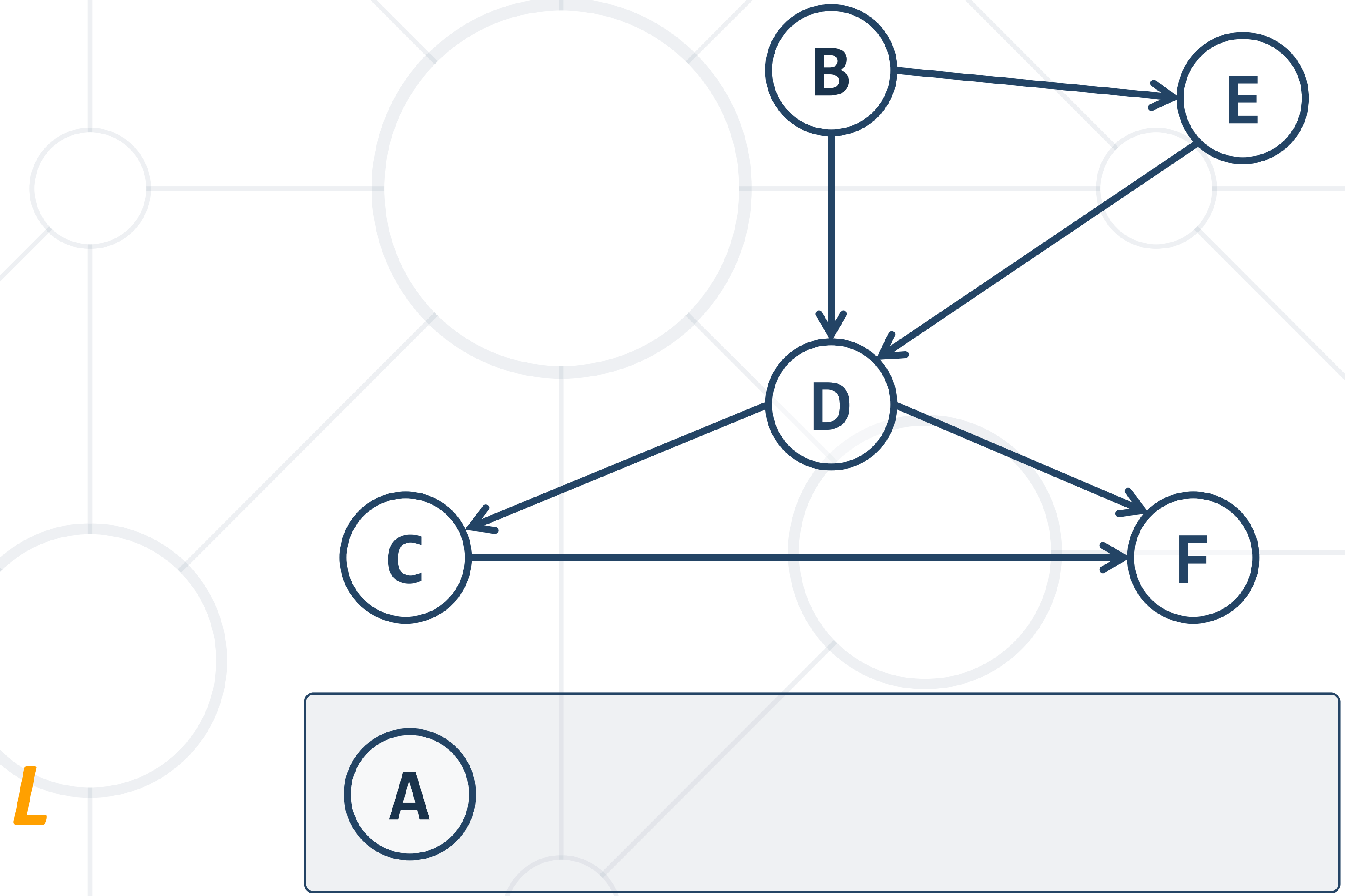


L

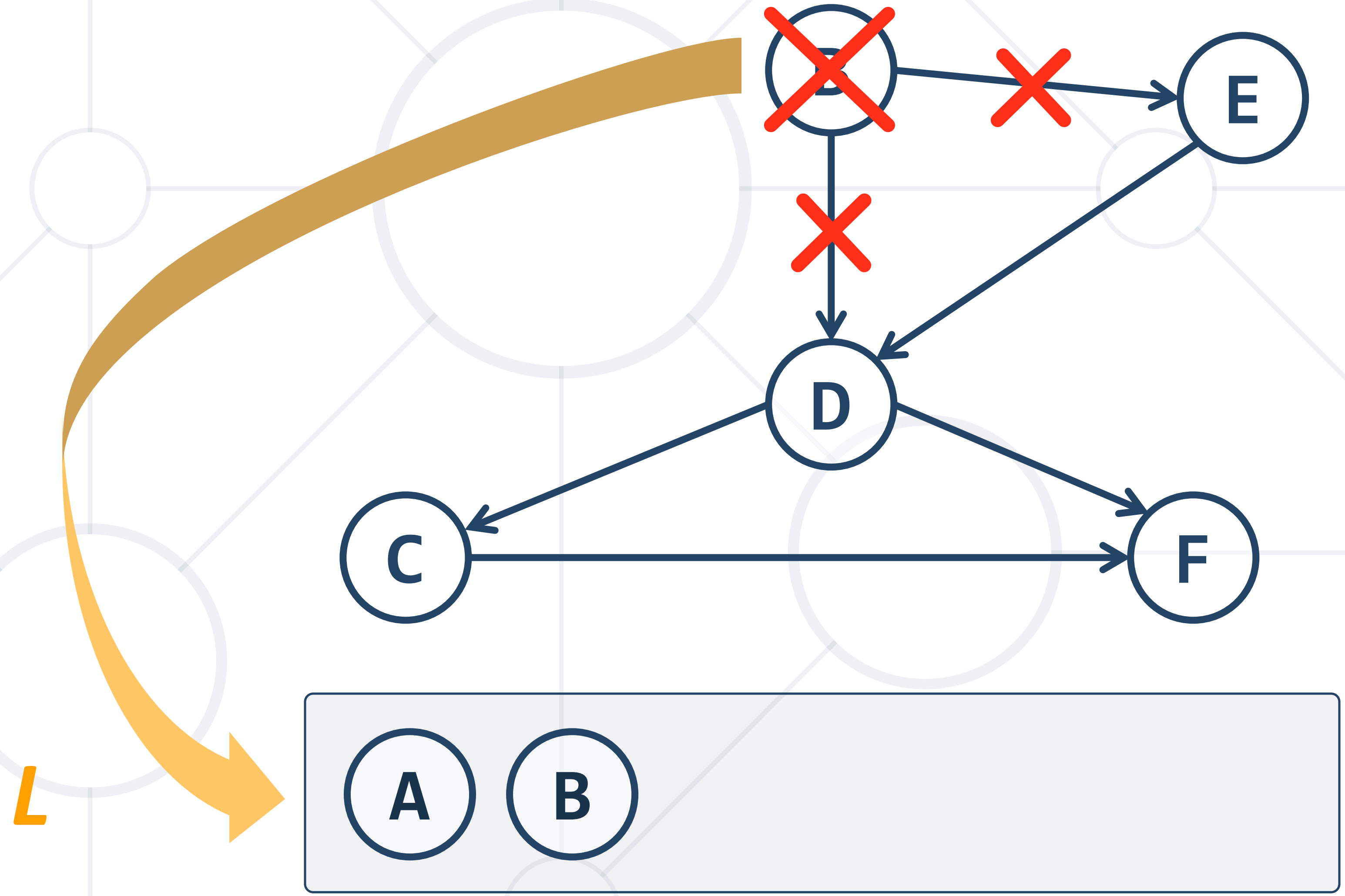
Step #2: Remove Node A with Its Edges



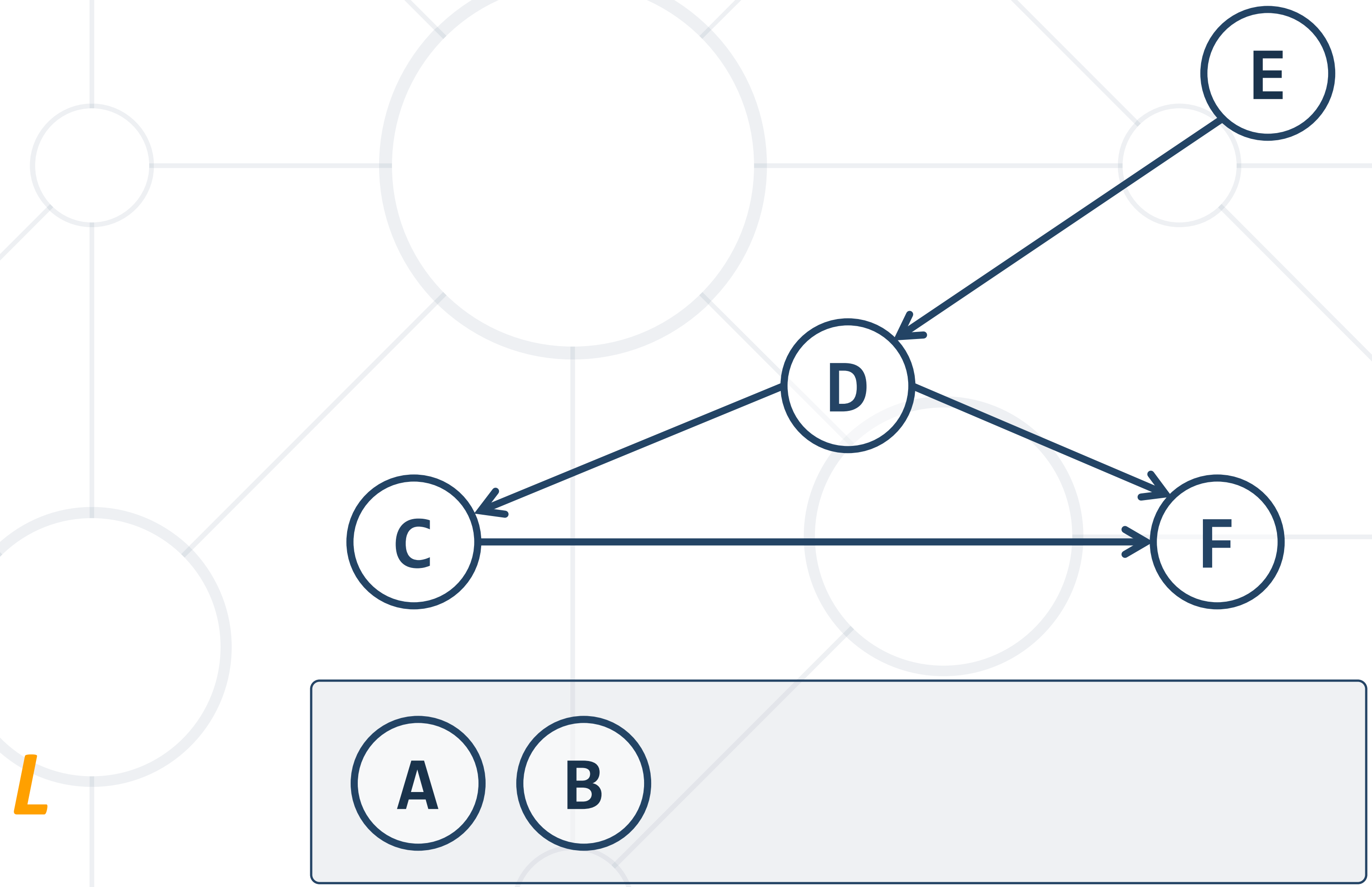
Step #3: Find a Node with No Incoming Edges



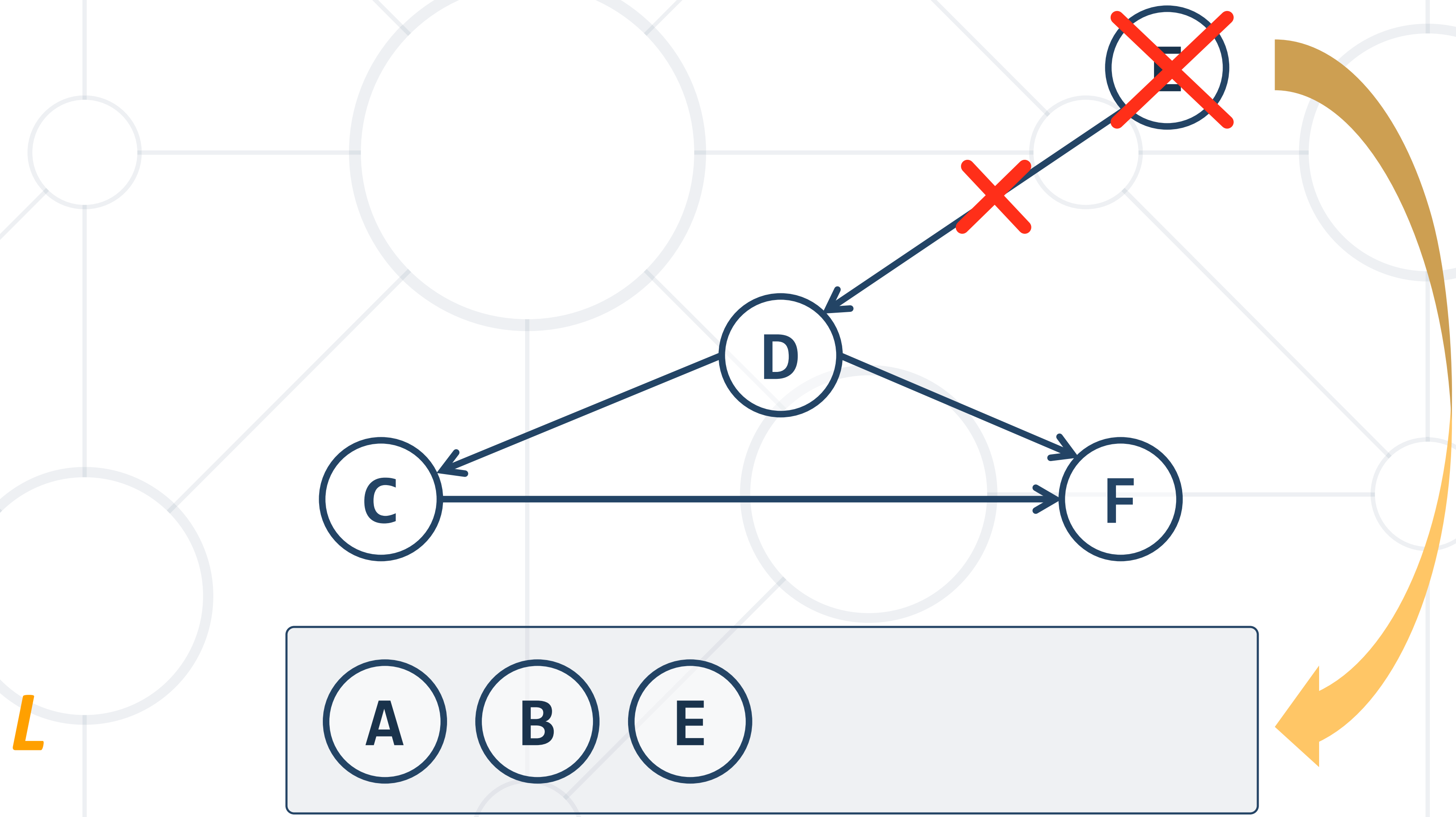
Step #4: Remove Node B with Its Edges



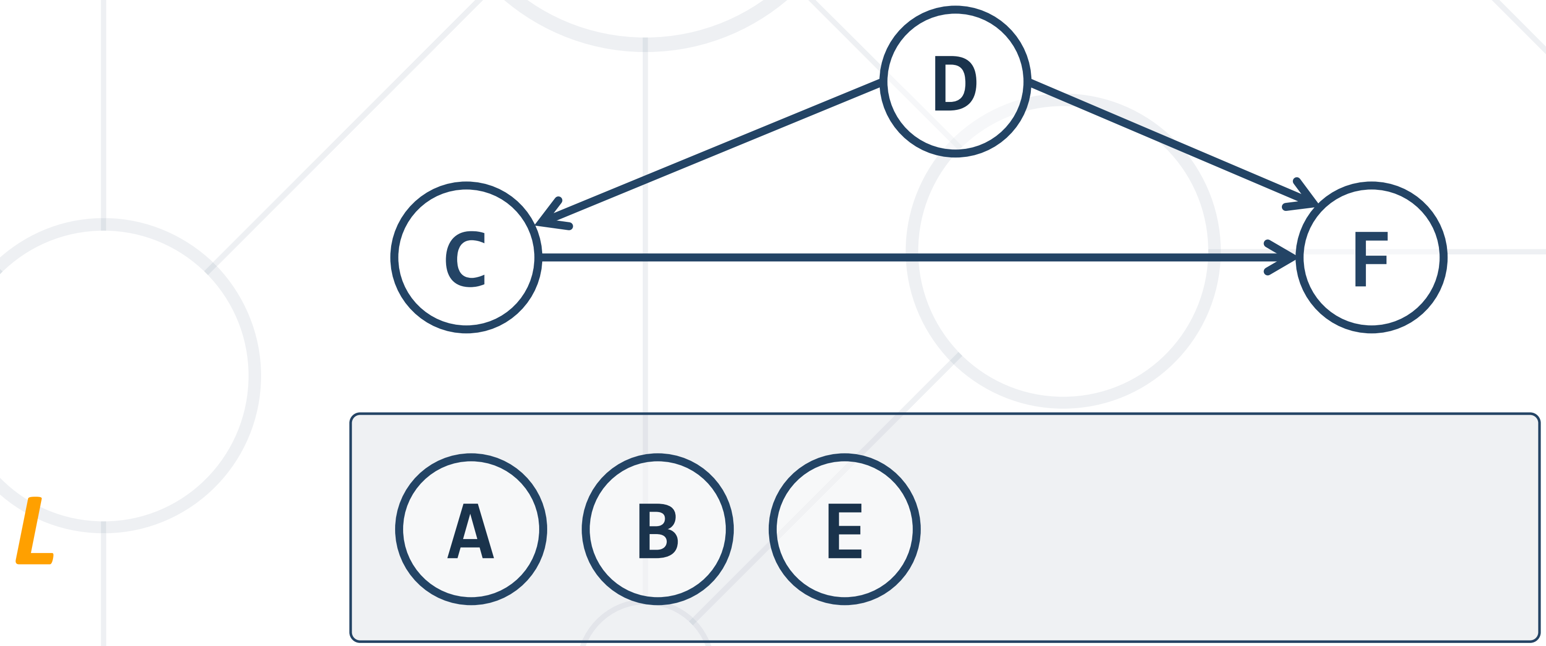
Step #5: Find a Node with No Incoming Edges



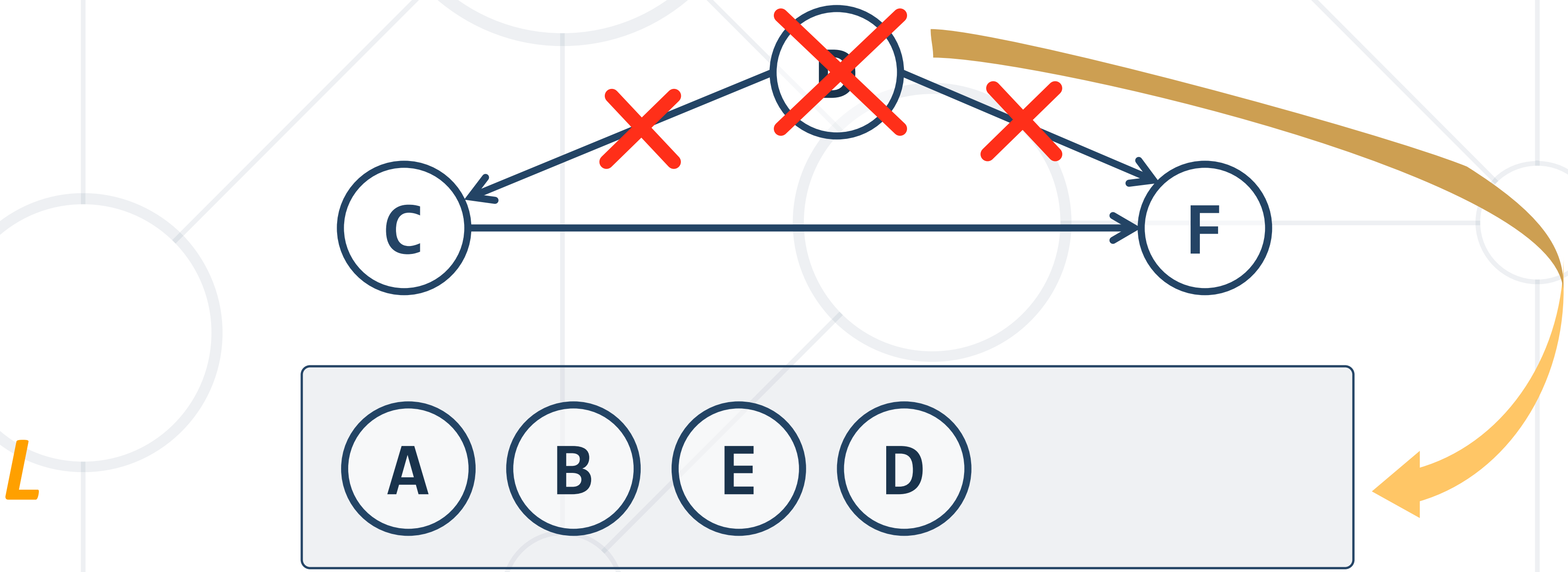
Step #6: Remove Node E with Its Edges



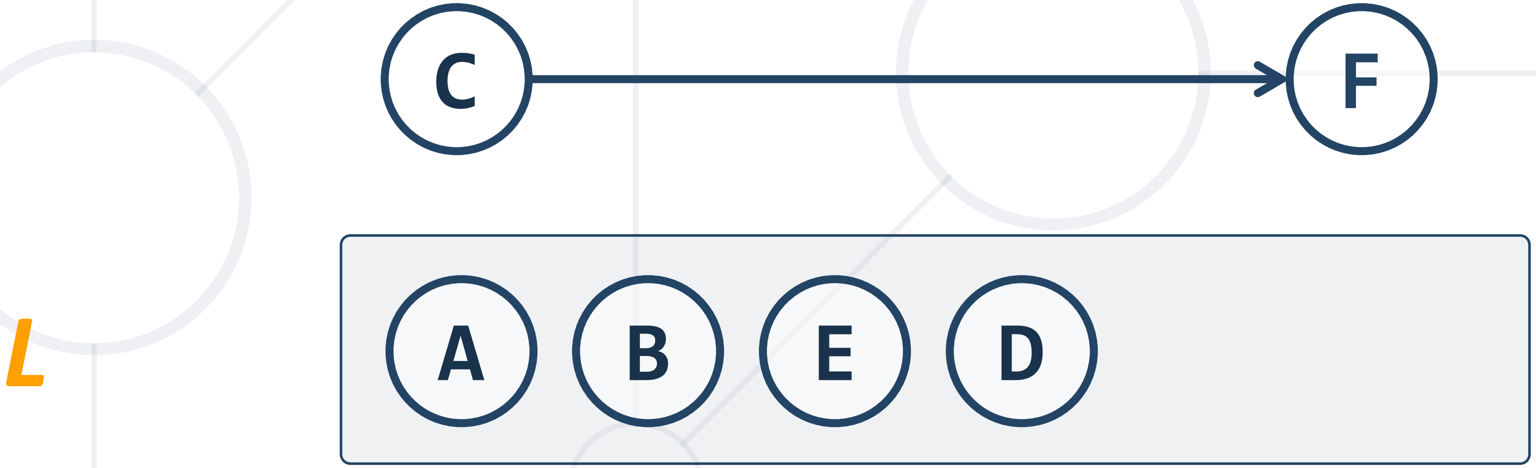
Step #7: Find a Node with No Incoming Edges



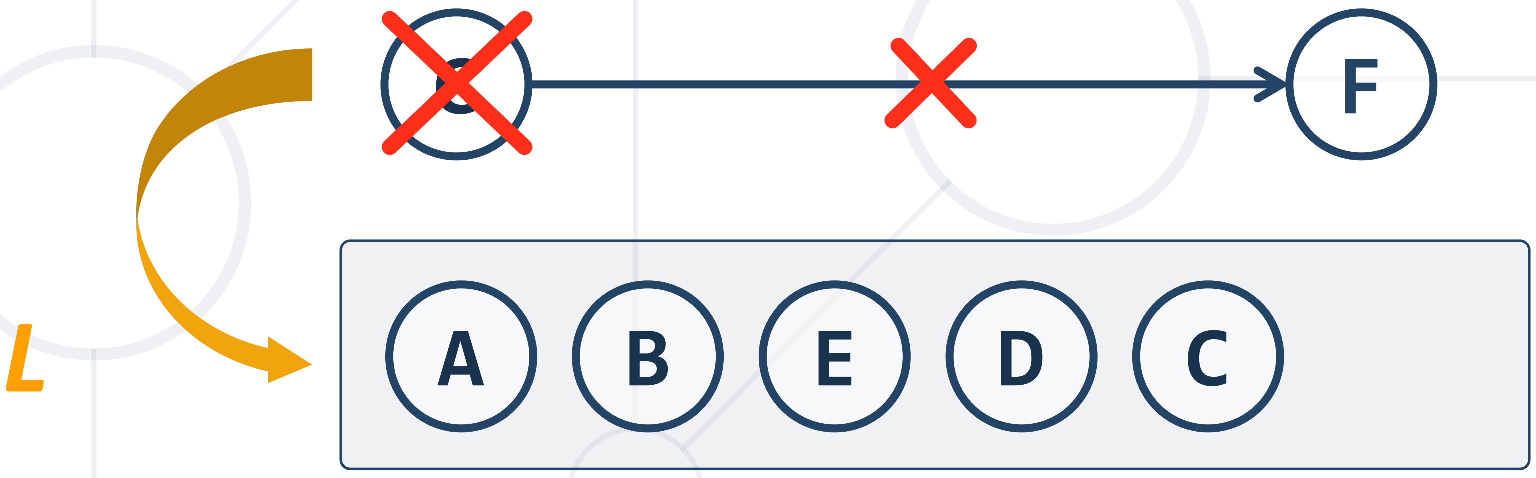
Step #8: Remove Node D with Its Edges



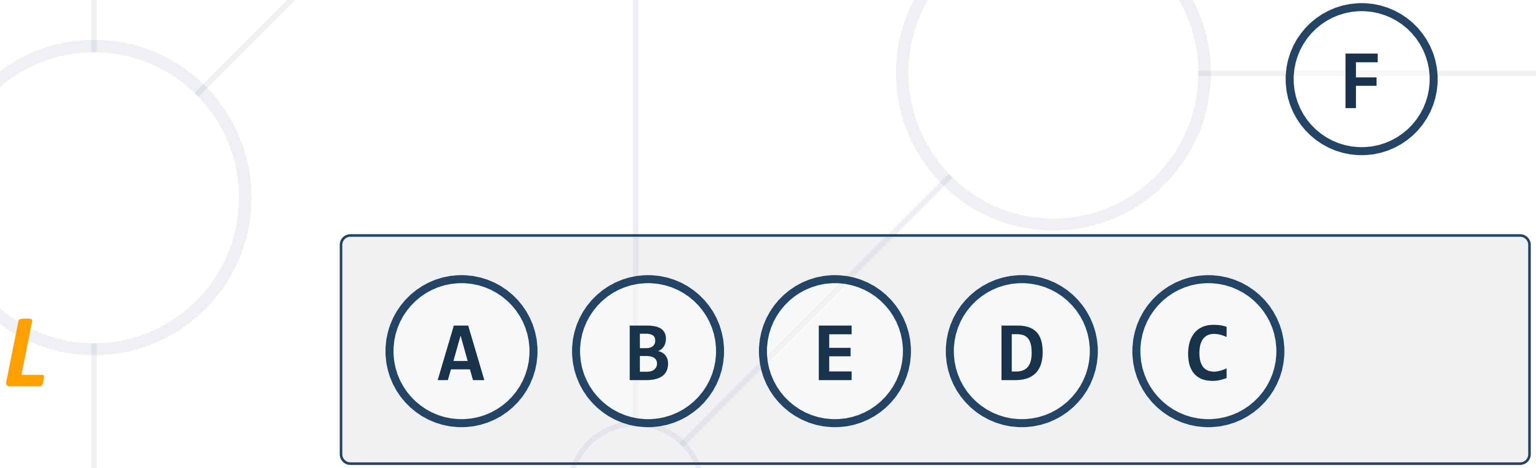
Step #9: Find a Node with No Incoming Edges



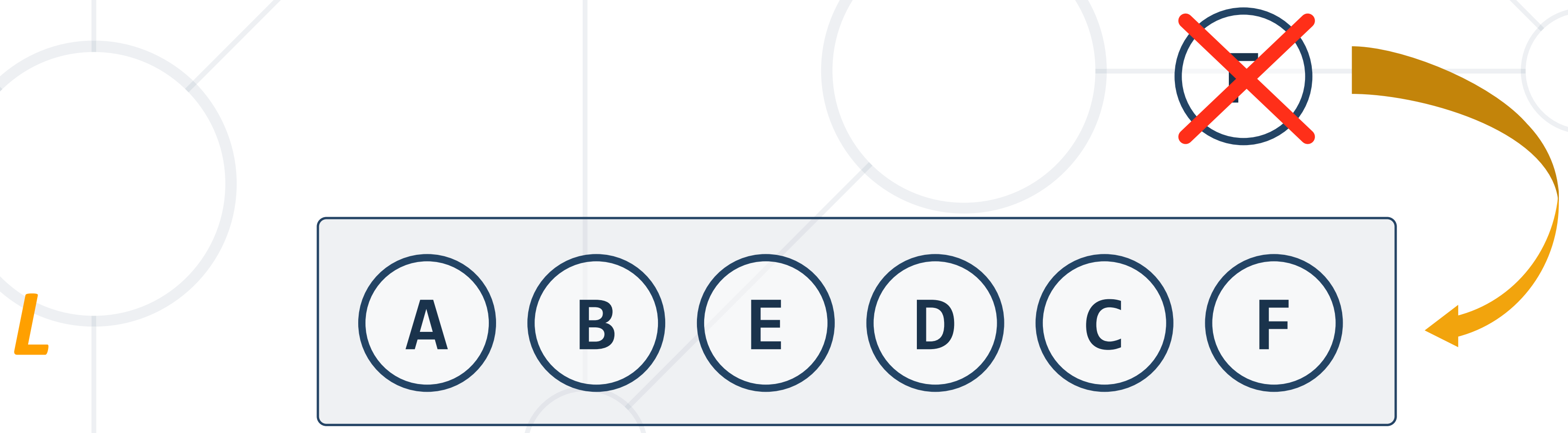
Step #10: Remove Node C with Its Edges



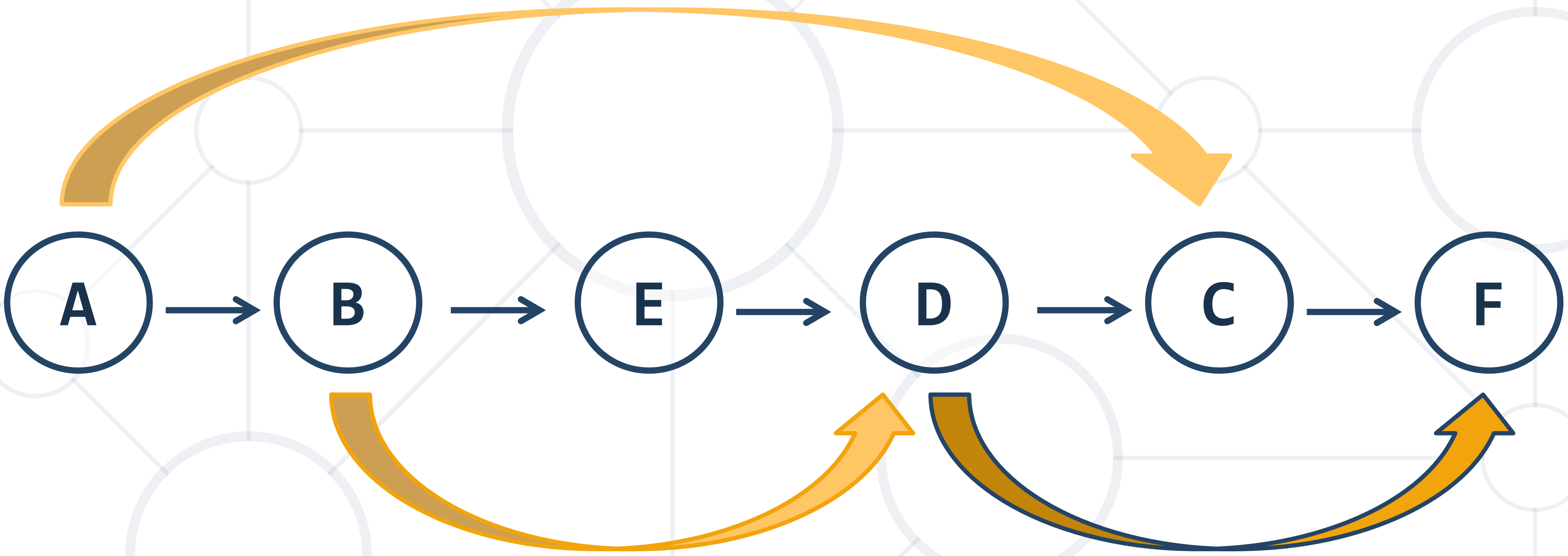
Step #11: Find a Node with No Incoming Edges



Step #12: Remove Node F with Its Edges



Result: Topological Sorting



Topological Sorting: DFS Algorithm

```
sortedNodes = { }   // linked list to hold the result
visitedNodes = { }  // set of already visited nodes
foreach node in graphNodes
    topSortDFS(node)
topSortDFS(node)
    if node ∉ visitedNodes
        visitedNodes ← node
        for each child c of node
            TopSortDFS(c)
        insert node upfront in the sortedNodes
```

TopSort: DFS Algorithm + Cycle Detection

Software University

```
sortedNodes = { }   // linked list to hold the result
visitedNodes = { }  // set of already visited nodes
cycleNodes = { }    // set of nodes in the current DFS cycle
foreach node in graphNodes
    topSortDFS(node)
topSortDFS(node)
    if node ∊ cycleNodes
        return "Error: cycle detected"
    if node ∉ visitedNodes
        visitedNodes ← node
        cycleNodes ← node
        for each child c of node
            topSortDFS(c)
        remove node from cycleNodes
        insert node upfront in the sortedNodes
```

Summary

- Representing graphs in memory
  - **Adjacency list** holding the children for each node
  - **Adjacency matrix**
  - **List of edges**
  - Numbering the nodes for faster access
- Depth-First Search (**DFS**) – recursive in-depth traversal
- Breadth-First Search (**BFS**) – in-width traversal with a queue
- **Topological sorting**

Questions?

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