Introduction to Numerical Optimization – Project
Compressed Sensing

1 Introduction

In the digital age, sampling methods have come to form the cornerstone of data acquisition protocols used in a broad range of fields, from consumer audio and visual electronics to medical imaging and astronomy. Roughly speaking, a sampling method can be viewed as a means of acquiring measurements of an object of interest, usually in the form of projections on sensing waveforms that characterise the *sensing modality* of a sensor, along with an estimate of the number of measurements required to achieve a good approximation of the original signal. Shannon's theory underpins most of the standard methods and provides some bounds on the number of samples required to properly reconstruct a signal via the Shannon-Nyquist sampling theorem. Several data acquisition technologies have reached levels of maturity enabling the collection of very high resolution data in a cost-effective way, e.g., megapixel cameras typically found in smartphones.

On the other hand, the advent of cheap, very high resolution data acquisition techniques has also led to increased transmission and storage requirements. Signal compression algorithms have been developed to alleviate such issues. At its core, a compression algorithm seeks a parsimonious representation of the data at hand. In the language of linear algebra, this consists in identifying a basis in which the original signal has a sparse representation (i.e. it can be represented by a weighted combination of orthonormal vectors with (very) few nonzero weights) and then finding its coefficients in this basis. Traditionally, sampling and compression have been performed in a sequential manner and were thus essentially decoupled from one another. Hence, standard compression algorithms were designed under the assumption that high resolution data was available.

In some cases, however, high resolution data acquisition may be expensive or impossible, and undersampling the signal may therefore be the only option available. In such settings, Shannon's theory provides little guidance. Hence, a new paradigm combining the sampling and compression stages with the aim of accurately reconstructing signals from a few measurements only has been proposed. This approach, which is known as *compressed sensing* or *compressive sampling*, formulates conditions that must be jointly satisfied by the sensing and compression protocols in order to achieve exact signal recovery, provides an estimate of the number of measurements required, and relies on an optimisation framework for effective signal reconstruction.

This project will explore this framework, and apply it to the problem of reconstructing cell images from a limited number of uncorrupted and noisy measurements.

2 Problem Statement

In this project, the discrete compressed sensing framework is explored, which implies that one works with finite-length, discrete-time signals and attempts to reconstruct them from a small number of measurements.

More formally, let $r \in \mathbb{R}^N$ be the reference signal we wish to reconstruct. Then, let $\Phi \in \mathbb{R}^{M \times N}$ be a measurement matrix and let $m \in \mathbb{R}^M$ be a set of measurements of r, with $M < N$, such that $m = \Phi r$. In other words, m can be viewed as a set of projections of r onto a set of sensing waveforms (also known as *modes*), $\phi_i^T,\ i = 1, \ldots, M$, forming the rows of Φ. In addition, let $\Psi \in \mathbb{R}^{N \times N}$ be an orthonormal basis matrix in which r is assumed to have a sparse representation, with $\psi_j,\ j = 1, \ldots, N$, a set of basis functions forming the columns of Ψ. Finally, let $x \in \mathbb{R}^N$ denote the sparse representation r, such that $r = \Psi x$.

Given the measurement matrix Φ, the sparsifying basis Ψ and a set of measurements m, the compressive sampling protocol seeks to identify an (approximately) sparse vector $\hat{x}$ in the hope that $\hat{r} = \Psi\hat{x}$ is a faithful reconstruction of r. This is achieved using optimisation models specifically promoting sparse solutions.

3 Tasks

3.1 Modelling

1. Formulate the problem using the ℓ_0 "norm", which counts the number of nonzero entries in a given input vector. Show that the resulting problem is non-convex.
2. Formulate the problem using the ℓ_1 norm and show that it can be expressed as a linear program.
3. Formulate the problem using the ℓ_2 norm and show that it can be expressed as a second-order cone program.
4. Provide a closed-form solution to the ℓ_2-norm problem. *Hint: use optimality conditions for the primal and dual problems.*
5. Formulate at least two robust variants of the ℓ_1-norm problem, whereby the reconstructed signal may not exactly match the measurements, up to some prespecified tolerance ϵ.

3.2 Numerical Experiments

1. Code up the formulations proposed earlier in Julia JuMP.
2. Solve the ℓ_1 and ℓ_2-norm problems numerically for the set of uncorrupted measurements. Discuss the performance of each method.
3. Provide an interpretation of the dual variables associated with equality constraints in the ℓ_1-norm formulation.
4. Solve the ℓ_1-norm formulation and its robust variants numerically for the set of noisy measurements. Test your methods for different values of ϵ and discuss their performance.

3.3 Deliverables

Students will work on this project in pairs or individually. For organisational reasons, each group is expected to send an email to *mathias.berger@uliege.be* indicating the names of group members.

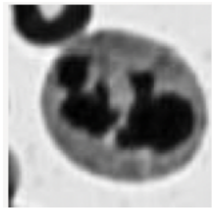
Each group will present its methods and findings on December 10. The exact format of the presentation will depend on the sanitary situation and updates will be given in due course. Each group is also expected to turn in a short report describing the problem formulations used and discussing findings by December 7.

4 Data

The data comes from a cytological analysis. More precisely, a 78×78 image of a cell will be used. The original RGB image is shown in Figure 1a, while its grayscale counterpart is shown in Figure 1b.

The grayscale image, which is represented by a matrix of floats with values between 0 and 1, can be transformed into a vector by stacking matrix columns on top of each other. The resulting vector has $N = 78 \times 78 = 6084$ entries. This is the signal r we wish to reconstruct from measurements.

(a) RGB image of a cell.



(b) Grayscale image of a cell.

Eight sets of M measurements are provided for the analysis. Four of them are uncorrupted, and the other four are noisy, with $M \in \{608, 1014, 1521, 3042\}$ in both cases. The vectors storing these measurements correspond to m in the compressed sensing framework.

Four $M \times N$ measurement matrices Φ corresponding to the four values of M mentioned above are also provided along with the $N \times N$ sparsifying basis Ψ, which remains the same for all experiments.

The various input vectors and matrices were pickled to reduce their size. The *unpickler* function provided in the *utilities.jl* script can be used to load them into Julia. Note that this script requires the installation of the *PyCall* package.

Once the signal has been reconstructed, the recovered vector must be transformed back into a 78×78 matrix in order to visualise it. The image can be plotted with the *imshow* function of the *ImageView* package. The *Images* package also provides useful utilities for image processing and analysis. Both packages can be installed with the Julia package manager *Pkg*.

Visually inspecting the reconstructed image and comparing it with the original grayscale image will provide a good idea of the performance of the various methods studied. Alternatively, the spectrum of the signals may be analysed.