Calculus - P56

如 $\int \frac{x+2}{(2x+1)(x^2+x+1)}dx$，可以设 $\frac{x+2}{(2x+1)(x^2+x+1)} = \frac{A}{(2x+1)} + \frac{Cx+D}{(x^2+x+1)}$

则 $x+2 = (A+2C)x^2 + (A+C+2D)x + (A+D)$

比较系数可得方程组 $\begin{cases} A+2C=0 \\ A+C+2D=1 \\ A+D=2 \end{cases}$，解方程组得 $\begin{cases} A=2 \\ C=-1 \\ D=0 \end{cases}$

于是有 $\int \frac{x+2}{(2x+1)(x^2+x+1)}dx = \int (\frac{2}{2x+1} + \frac{-x}{x^2+x+1})dx$

$= \int \frac{1}{2x+1}d(2x+1) - \frac{1}{2}\int \frac{(2x+1)-1}{x^2+x+1}dx$

$= \int \frac{d(2x+1)}{2x+1} - \frac{1}{2}\int \frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{2}\int \frac{1}{(x+1/2)^2+3/4}d(x+\frac{1}{2})$

取 $a = \frac{\sqrt{3}}{2}$，则有 $\int \frac{d(x+1/2)}{(x+1/2)^2+a^2} = \frac{1}{a}\arctan\frac{x+1/2}{a}$

于是有 $\int \frac{x+2}{(2x+1)(x^2+x+1)}dx = \ln|2x+1| - \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$

对于某些积分函数，可以转换为有理分式的形式进行积分运算

如 $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$，可以利用三角函数的万能公式

即 $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$，$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ 来替换原式中的 $\sin x$ 和 $\cos x$

令 $t = \tan\frac{x}{2}$，其中 $(-\pi < x < \pi)$，而对任意 $x \in (-\pi+2k\pi, \pi+2k\pi)$

都可以变换为 $\tan\frac{x-2k\pi}{2} = \tan\frac{x}{2} = t$，即 $x = 2k\pi + 2\arctan t$

于是有 $\sin x = \frac{2t}{1+t^2}$，$\cos x = \frac{1-t^2}{1+t^2}$，而 $x = 2\arctan t$，则 $dx = \frac{2}{1+t^2}dt$

于是有 $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx = \int [1+\frac{2t}{1+t^2}]/[\frac{2t}{1+t^2}\cdot(\frac{1-t^2}{1+t^2}+1)]\cdot\frac{2}{1+t^2}dt$

$= \int \frac{(1+t)^2}{2t}dt = \frac{1}{2}\int(t+2+\frac{1}{t})dt$

$= \frac{1}{2}[\frac{1}{2}t^2 + 2t + \ln|t|] + C$

又 $t = \tan\frac{x}{2}$，则 $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx = \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + \frac{1}{2}\ln|\tan\frac{x}{2}| + C$

如 $\int \frac{\sqrt{x-1}}{x}dx$，可令 $u = \sqrt{x-1}$，于是有 $u^2 = x-1$，即 $2u\,du = dx$

则 $\int \frac{\sqrt{x-1}}{x}dx = \int \frac{u}{u^2+1}\cdot 2u\,du = \int(2-\frac{2}{u^2+1})du$

$= 2u - 2\arctan u + C$

又 $u = \sqrt{x-1}$，则有 $\int \frac{\sqrt{x-1}}{x}dx = 2(\sqrt{x-1} - \arctan\sqrt{x-1}) + C$

如 $\int \frac{1}{1+\sqrt[3]{x+2}}dx$，可令 $u = \sqrt[3]{x+2}$，于是有 $x+2 = u^3$

则有 $x = u^3 - 2$，$dx = 3u^2du$

于是有 $\int \frac{dx}{1+\sqrt[3]{x+2}} = \int \frac{1}{1+u}\cdot 3u^2du = 3\int \frac{(u^2-1)+1}{1+u}du$

$= 3\int[(u-1) + \frac{1}{u+1}]du$

$= 3(\frac{1}{2}u^2 - u + \ln|u+1|) + C$

又 $u = (x+2)^{1/3}$，则有 $\int \frac{dx}{1+\sqrt[3]{x+2}} = 3[\frac{1}{2}(x+2)^{2/3} - (x+2)^{1/3} + \ln|(x+2)^{1/3}+1|] + C$

Calculus - P57

如 $\int \frac{1}{(1+\sqrt[3]{x})\sqrt{x}}dx$，其中有两种 x 的根式 $\sqrt{x}$ 与 $\sqrt[3]{x}$，可以取根式次数的最小公倍数

令 $x=u^6$，则有 $\sqrt[3]{x}=u^2$，$\sqrt{x}=u^3$，$dx=6u^5du$

于是 $\int \frac{1}{(1+\sqrt[3]{x})\sqrt{x}}dx = \int \frac{1}{(1+u^2)u^3}\cdot 6u^5du = 6\int \frac{u^2}{1+u^2}du$

$= 6\int(1-\frac{1}{1+u^2})du = 6(u-\arctan u)+C$

又 $u=\sqrt[6]{x}$，于是有 $\int \frac{1}{(1+\sqrt[3]{x})\sqrt{x}}dx = 6(u-\arctan u)+C$

$= 6(\sqrt[6]{x}-\arctan\sqrt[6]{x})+C$

如 $\int \frac{1}{x}\sqrt{\frac{1+x}{x}}dx$，注意如果令 $u=\sqrt{x}$ 或 $u=\sqrt{1+x}$ 则在换元过程中可能产生新的根式

于是令 $u=\sqrt{\frac{1+x}{x}}$，则有 $u^2=\frac{1+x}{x}$，$x=\frac{1}{u^2-1}$，即 $dx=\frac{-2u}{(u^2-1)^2}du$

于是 $\int \frac{1}{x}\sqrt{\frac{1+x}{x}}dx = \int (u^2-1)\cdot u\cdot\frac{-2u}{(u^2-1)^2}du = -2\int\frac{u^2}{u^2-1}du$

$= -2\int(1+\frac{1}{u^2-1})du = -2(u+\frac{1}{2}\ln|\frac{u-1}{u+1}|)+C$

$= -2u - [\ln(|u-1|\cdot|u+1|) - \ln(u+1)^2]+C$

$= -2u+2\ln(u+1)-\ln|u^2-1|+C$

又 $u^2-1=\frac{1}{x}$，则 $\int\frac{1}{x}\sqrt{\frac{1+x}{x}}dx = -2u+2\ln(u+1)-\ln|u^2-1|+C$

$= -2\sqrt{\frac{1+x}{x}}+2\ln(\sqrt{\frac{1+x}{x}}+1)+\ln|x|+C$

对于形如 $\sqrt[n]{ax+b}$ 或 $\sqrt[n]{\frac{ax+b}{cx+b}}$ 的根式，可以令 u 等于根式，

由于 $u=\sqrt[n]{ax+b}$ 或 $u=\sqrt[n]{\frac{ax+b}{cx+b}}$ 都具有反函数，且反函数为有理分式 $f(u)$

则可以将对 x 的积分函数积分转换为对 u 的积分函数积分

积分表 (list of integral)，用于将被积函数直接或经过简单变形后得到表中常用的积分公式

ax+b型

$\int\frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|+C$

$\int(ax+b)^\mu dx = \frac{1}{a(\mu+1)}(ax+b)^{\mu+1}+C$，其中 $\mu\neq-1$

$\int\frac{x}{ax+b}dx = \frac{1}{a^2}(ax+b-b\ln|ax+b|)+C$

$\int\frac{x^2}{ax+b}dx = \frac{1}{a^3}[\frac{1}{2}(ax+b)^2-2b(ax+b)+b^2\ln|ax+b|]+C$

$\int\frac{1}{x(ax+b)}dx = -\frac{1}{b}\ln|\frac{ax+b}{x}|+C$

$\int\frac{1}{x^2(ax+b)}dx = -\frac{1}{bx}+\frac{a}{b^2}\ln|\frac{ax+b}{x}|+C$

$\int\frac{x}{(ax+b)^2}dx = \frac{1}{a^2}(\ln|ax+b|+\frac{b}{ax+b})+C$

$\int\frac{x^2}{(ax+b)^2}dx = \frac{1}{a^3}(ax+b-2b\ln|ax+b|-\frac{b^2}{ax+b})+C$

$\int\frac{1}{x(ax+b)^2}dx = \frac{1}{b(ax+b)}-\frac{1}{b^2}\ln|\frac{ax+b}{x}|+C$

其中 $a,b\in R$，C 为常数项

Calculus - P58

$\sqrt{ax+b}$ 型

$$\int \sqrt{ax+b}\,dx = \frac{2}{3a}\sqrt{(ax+b)^3} + C$$

$$\int x\sqrt{ax+b}\,dx = \frac{2}{15a^2}(3ax-2b)\sqrt{(ax+b)^3} + C$$

$$\int x^2\sqrt{ax+b}\,dx = \frac{2}{105a^3}(15a^2x^2-12abx+8b^2)\sqrt{(ax+b)^3} + C$$

$$\int \frac{x}{\sqrt{ax+b}}\,dx = \frac{2}{3a^2}(ax-2b)\sqrt{ax+b} + C$$

$$\int \frac{x^2}{\sqrt{ax+b}}\,dx = \frac{2}{15a^3}(3a^2x^2-4abx+8b^2)\sqrt{ax+b} + C$$

$$\int \frac{1}{x\sqrt{ax+b}}\,dx \ (b>0) = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C$$

$$\int \frac{1}{x\sqrt{ax+b}}\,dx \ (b<0) = \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}} + C$$

$$\int \frac{1}{x^2\sqrt{ax+b}}\,dx = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int\frac{1}{x\sqrt{ax+b}}\,dx$$

$$\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int\frac{1}{x\sqrt{ax+b}}\,dx$$

$$\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int\frac{1}{x\sqrt{ax+b}}\,dx$$

$x^2 \pm a^2$ 型

当$n\in Z^+$时

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$ 递归定义的基础步骤

$$\int \frac{1}{(x^2+a^2)^n}\,dx = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int\frac{1}{(x^2+a^2)^{n-1}}\,dx$$ 递归定义的递归步骤

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

ax^2+b 型 $(a>0)$

$$\int \frac{1}{ax^2+b}\,dx \ (b>0) = \frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{a}{b}}\,x + C$$

$$\int \frac{1}{ax^2+b}\,dx \ (b<0) = \frac{1}{2\sqrt{-ab}}\ln\left|\frac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right| + C$$

$$\int \frac{x}{ax^2+b}\,dx = \frac{1}{2a}\ln|ax^2+b| + C$$

$$\int \frac{x^2}{ax^2+b}\,dx = \frac{x}{a} - \frac{b}{a}\int\frac{1}{ax^2+b}\,dx$$

$$\int \frac{1}{x(ax^2+b)}\,dx = \frac{1}{2b}\ln\left|\frac{x^2}{ax^2+b}\right| + C$$

$$\int \frac{1}{x^2(ax^2+b)}\,dx = -\frac{1}{bx} - \frac{a}{b}\int\frac{1}{ax^2+b}\,dx$$

$$\int \frac{1}{x^3(ax^2+b)}\,dx = \frac{a}{2b^2}\ln\left|\frac{ax^2+b}{x^2}\right| - \frac{1}{2bx^2} + C$$

$$\int \frac{1}{(ax^2+b)^2}\,dx = \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int\frac{1}{ax^2+b}\,dx$$

ax^2+bx+c 型 $(a>0)$

$$\int \frac{1}{ax^2+bx+c}\,dx \ (b^2-4ac<0) = \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C$$

$$\int \frac{1}{ax^2+bx+c}\,dx \ (b^2-4ac>0) = \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C$$

$$\int \frac{x}{ax^2+bx+c}\,dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{2a}\int\frac{1}{ax^2+bx+c}\,dx$$

$\sqrt{x^2+a^2}$ 型 $(a>0)$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2}) + C = \operatorname{arsh}\frac{x}{a} + C_1$$

$$\int \frac{1}{\sqrt{(x^2+a^2)^3}}\,dx = \frac{x}{a^2\sqrt{x^2+a^2}} + C$$

$$\int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2} + C$$

$$\int \frac{x}{\sqrt{(x^2+a^2)^3}}\,dx = -\frac{1}{\sqrt{x^2+a^2}} + C$$

Calculus - P59

$\sqrt{x^2+a^2}$ 型 $(a>0)$

$$\int \frac{x^2}{\sqrt{x^2+a^2}}\,dx = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + C$$

$$\int \frac{x^2}{\sqrt{(x^2+a^2)^3}}\,dx = -\frac{x}{\sqrt{x^2+a^2}} + \ln(x+\sqrt{x^2+a^2}) + C$$

$$\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = \frac{1}{a}\ln\frac{\sqrt{x^2+a^2}-a}{|x|} + C$$

$$\int \frac{1}{x^2\sqrt{x^2+a^2}}\,dx = -\frac{\sqrt{x^2+a^2}}{a^2x} + C$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + C$$

$$\int \sqrt{(x^2+a^2)^3}\,dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3}{8}a^4\ln(x+\sqrt{x^2+a^2}) + C$$

$$\int x\sqrt{x^2+a^2}\,dx = \frac{1}{3}\sqrt{(x^2+a^2)^3} + C$$

$$\int x^2\sqrt{x^2+a^2}\,dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\ln(x+\sqrt{x^2+a^2}) + C$$

$$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} + a\ln\frac{\sqrt{x^2+a^2}-a}{|x|} + C$$

$$\int \frac{\sqrt{x^2+a^2}}{x^2}\,dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln(x+\sqrt{x^2+a^2}) + C$$

$\sqrt{x^2-a^2}$ 型 $(a>0)$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \frac{x}{|x|}\operatorname{arch}\frac{|x|}{a} + C_1 = \ln|x+\sqrt{x^2-a^2}| + C$$

$$\int \frac{1}{\sqrt{(x^2-a^2)^3}}\,dx = -\frac{x}{a^2\sqrt{x^2-a^2}} + C$$

$$\int \frac{x}{\sqrt{x^2-a^2}}\,dx = \sqrt{x^2-a^2} + C$$

$$\int \frac{x}{\sqrt{(x^2-a^2)^3}}\,dx = -\frac{1}{\sqrt{x^2-a^2}} + C$$

$$\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$$

$$\int \frac{x^2}{\sqrt{(x^2-a^2)^3}}\,dx = -\frac{x}{\sqrt{x^2-a^2}} + \ln|x+\sqrt{x^2-a^2}| + C$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\arccos\frac{a}{|x|} + C$$

$$\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx = \frac{\sqrt{x^2-a^2}}{a^2x} + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$$

$$\int \sqrt{(x^2-a^2)^3}\,dx = \frac{x}{8}(2x^2-5a^2)\sqrt{x^2-a^2} + \frac{3}{8}a^4\ln|x+\sqrt{x^2-a^2}| + C$$

$$\int x\sqrt{x^2-a^2}\,dx = \frac{1}{3}\sqrt{(x^2-a^2)^3} + C$$

$$\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln|x+\sqrt{x^2-a^2}| + C$$

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\cdot\arccos\frac{a}{|x|} + C$$

$$\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln|x+\sqrt{x^2-a^2}| + C$$

$\sqrt{a^2-x^2}$ 型 $(a>0)$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{(a^2-x^2)^3}}\,dx = \frac{x}{a^2\sqrt{a^2-x^2}} + C$$

$$\int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2} + C$$

$$\int \frac{x}{\sqrt{(a^2-x^2)^3}}\,dx = \frac{1}{\sqrt{a^2-x^2}} + C$$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

$$\int \frac{x^2}{\sqrt{(a^2-x^2)^3}}\,dx = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + C$$

Calculus - P60

$\sqrt{a^2-x^2}$ 型 $(a>0)$

$$\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = \frac{1}{a}\ln\frac{a-\sqrt{a^2-x^2}}{|x|} + C$$

$$\int \frac{1}{x^2\sqrt{a^2-x^2}}\,dx = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

$$\int \sqrt{(a^2-x^2)^3}\,dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3}{8}a^4\arcsin\frac{x}{a} + C$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3} + C$$

$$\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a} + C$$

$$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} + a\ln\frac{a-\sqrt{a^2-x^2}}{|x|} + C$$

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\frac{x}{a} + C$$

$\sqrt{\pm ax^2+bx+c}$ 型 $(a>0)$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right| + C$$

$$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right| + C$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}}\,dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right| + C$$

$$\int \frac{1}{\sqrt{-ax^2+bx+c}}\,dx = \frac{1}{\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}} + C$$

$$\int \sqrt{-ax^2+bx+c}\,dx = \frac{2ax-b}{4a}\sqrt{-ax^2+bx+c} + \frac{b^2+4ac}{8\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}} + C$$

$$\int \frac{x}{\sqrt{-ax^2+bx+c}}\,dx = -\frac{1}{a}\sqrt{-ax^2+bx+c} + \frac{b}{2\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}} + C$$

$\sqrt{\pm\frac{x-a}{x-b}}$ 型

$$\int \sqrt{\frac{x-a}{x-b}}\,dx = (x-b)\sqrt{\frac{x-a}{x-b}} + (b-a)\ln\left(\sqrt{|x-a|}+\sqrt{|x-b|}\right) + C$$

$$\int \sqrt{\frac{x-a}{b-x}}\,dx = (x-b)\sqrt{\frac{x-a}{b-x}} + (b-a)\arcsin\sqrt{\frac{x-a}{b-x}} + C$$

$\sqrt{(x-a)(b-x)}$ 型

$$\int \frac{1}{\sqrt{(x-a)(b-x)}}\,dx \quad (a<b) = 2\arcsin\sqrt{\frac{x-a}{b-x}} + C$$

$$\int \sqrt{(x-a)(b-x)}\,dx \quad (a<b) = \frac{2x-a-b}{4}\sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{4}\arcsin\sqrt{\frac{x-a}{b-x}} + C$$

三角函数型

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x+\tan x| + C = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$

$$\int \csc x\,dx = \ln|\csc x-\cot x| + C = \ln\left|\tan\frac{x}{2}\right| + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x\tan x\,dx = \sec x + C$$

$$\int \csc x\cot x\,dx = -\csc x + C$$

Calculus - P61

三角函数型

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$\rightarrow \int \sin x\,dx$, $\int \cos x\,dx$

$$\int \frac{1}{\sin^n x}\,dx = -\frac{1}{n-1}\cdot\frac{\cos x}{\sin^{n-1}x} + \frac{n-2}{n-1}\int\frac{1}{\sin^{n-2}x}\,dx$$

$$\int \frac{1}{\cos^n x}\,dx = \frac{1}{n-1}\cdot\frac{\sin x}{\cos^{n-1}x} + \frac{n-2}{n-1}\int\frac{1}{\cos^{n-2}x}\,dx$$

$\rightarrow \int \csc^2 x\,dx$, $\int \csc x\,dx$; $\int \sec^2 x\,dx$, $\int \sec x\,dx$

$$\int \cos^m x\sin^n x\,dx = \begin{cases} \frac{1}{m+n}\cos^{m-1}x\sin^{n+1}x + \frac{m-1}{m+n}\int \cos^{m-2}x\sin^n x\,dx \\ -\frac{1}{m+n}\cos^{m+1}x\sin^{n-1}x + \frac{n-1}{m+n}\int \cos^m x\sin^{n-2}x\,dx \end{cases}$$

$$\int \sin ax\cos bx\,dx = -\frac{1}{2(a+b)}\cos(a+b)x - \frac{1}{2(a-b)}\cos(a-b)x + C$$

$$\int \sin ax\sin bx\,dx = -\frac{1}{2(a+b)}\sin(a+b)x + \frac{1}{2(a-b)}\sin(a-b)x + C$$

$$\int \cos ax\cos bx\,dx = \frac{1}{2(a+b)}\sin(a+b)x + \frac{1}{2(a-b)}\sin(a-b)x + C$$

$$\int \frac{1}{a+b\sin x}\,dx\ (a^2>b^2) = \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}} + C$$

$$\int \frac{1}{a+b\sin x}\,dx\ (a^2<b^2) = \frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right| + C$$

$$\int \frac{1}{a+b\cos x}\,dx\ (a^2>b^2) = \frac{1}{a+b}\sqrt{\frac{a+b}{a-b}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) + C$$

$$\int \frac{1}{a+b\cos x}\,dx\ (a^2<b^2) = \frac{1}{a+b}\sqrt{\frac{a+b}{a-b}}\ln\left|\frac{\tan\frac{x}{2}+\sqrt{(a+b)/(b-a)}}{\tan\frac{x}{2}-\sqrt{(a+b)/(b-a)}}\right| + C$$

$$\int \frac{1}{a^2\cos^2 x+b^2\sin^2 x}\,dx = \frac{1}{ab}\arctan\left(\frac{b}{a}\tan x\right) + C$$

$$\int \frac{1}{a^2\cos^2 x-b^2\sin^2 x}\,dx = \frac{1}{2ab}\ln\left|\frac{b\tan x+a}{b\tan x-a}\right| + C$$

$$\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax + C$$

$$\int x^2\sin ax\,dx = -\frac{1}{a}x^2\cos ax + \frac{2}{a^2}x\sin ax + \frac{2}{a^3}\cos ax + C$$

$$\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax + C$$

$$\int x^2\cos ax\,dx = \frac{1}{a}x^2\sin ax + \frac{2}{a^2}x\cos ax - \frac{2}{a^3}\sin ax + C$$

反三角函数类型 $(a>0)$

$$\int \arcsin\frac{x}{a}\,dx = x\arcsin\frac{x}{a} + \sqrt{a^2-x^2} + C$$

$$\int x\arcsin\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2-x^2} + C$$

$$\int x^2\arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C$$

$$\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2} + C$$

$$\int x\arccos\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2-x^2} + C$$

$$\int x^2\arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C$$

$$\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2+x^2) + C$$

$$\int x\arctan\frac{x}{a}\,dx = \frac{1}{2}(a^2+x^2)\arctan\frac{x}{a} - \frac{a}{2}x + C$$

$$\int x^2\arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{6}x^2 + \frac{a^3}{6}\ln(a^2+x^2) + C$$

Calculus - P62

1)

指数型

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$
$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$
$$\int x e^{ax} dx = \frac{1}{a^2}(ax-1)e^{ax} + C$$
$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax} dx$$
$$\int x a^x dx = \frac{x}{\ln a} a^x - \frac{1}{(\ln a)^2} a^x + C$$
$$\int x^n a^x dx = \frac{1}{\ln a} x^n a^x - \frac{n}{\ln a}\int x^{n-1} a^x dx$$
$$\int e^{ax} \sin bx\, dx = \frac{1}{a^2+b^2} e^{ax}(a \sin bx - b \cos bx) + C$$
$$\int e^{ax} \cos bx\, dx = \frac{1}{a^2+b^2} e^{ax}(b \sin bx + a \cos bx) + C$$
$$\int e^{ax} \sin^n bx\, dx = \frac{1}{a^2+b^2n^2} e^{ax} \sin^{n-1} bx (a \sin bx - nb \cos bx) + \frac{n(n-1)b^2}{a^2+b^2n^2}\int e^{ax} \sin^{n-2} bx\, dx$$
$$\int e^{ax} \cos^n bx\, dx = \frac{1}{a^2+b^2n^2} e^{ax} \cos^{n-1} bx (a \cos bx + nb \sin bx) + \frac{n(n-1)b^2}{a^2+b^2n^2}\int e^{ax} \cos^{n-2} bx\, dx$$

对数型

$$\int \ln x\, dx = x \ln x - x + C$$
$$\int \frac{1}{x \ln x} dx = \ln|\ln x| + C$$
$$\int x^n \ln x\, dx = \frac{1}{n+1} x^{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$$
$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$$
$$\int x^m (\ln x)^n dx = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx \rightarrow \left\{\int x^m dx\right.$$

双曲函数型

$$\int \mathrm{sh}\,x\, dx = \mathrm{ch}\,x + C$$
$$\int \mathrm{ch}\,x\, dx = \mathrm{sh}\,x + C$$
$$\int \mathrm{th}\,x\, dx = \ln(\mathrm{ch}\,x) + C$$
$$\int \mathrm{sh}^2 x\, dx = -\frac{x}{2} + \frac{1}{4}\mathrm{sh}\,2x + C$$
$$\int \mathrm{ch}^2 x\, dx = \frac{x}{2} + \frac{1}{4}\mathrm{sh}\,2x + C$$

定积分部分

$$\int_{-\pi}^{\pi} \cos nx\, dx = \int_{-\pi}^{\pi} \sin nx\, dx = 0$$
$$\int_{-\pi}^{\pi} \cos mx \sin nx\, dx = 0$$
$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$
$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$
$$\int_0^{\pi} \sin mx \sin nx\, dx = \int_0^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \end{cases}$$
$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$$
$$I_n = \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{4}{5} \times \frac{2}{3}, & n\text{为奇数且}n>1, \quad I_1 = 1 \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}, & n\text{为偶数}, \quad I_0 = \frac{\pi}{2} \end{cases}$$

Calculus - P63

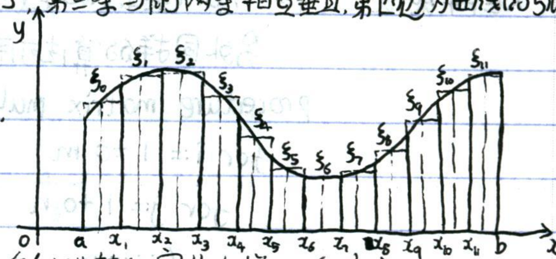
曲边梯形 (curved trapezoid)，"四边形的三条边为直线，其中两条互相平行，第三条与前两条相互垂直，第四边为曲线的弧

对于函数 $y=f(x)$，在 $[a,b]$ 上非负，连续

则由 $x=a$，$x=b$，$y=0$，$y=f(x)$ 围成的图形为曲边梯形，

注意曲边梯形的高在 $[a,b]$ 内是连续变化的

在很小的区间上，高的变化很小，近似于不变



把区间 $[a,b]$ 划分成许多小区间，可以用区间上某一点处的高，近似地替代窄曲边梯形的变高

于是每一个窄曲边梯形都可以近似地看作窄矩形，则窄矩形的面积是窄曲边梯形的近似值

当区间 $[a,b]$ 被无限细分，则小区间的宽度趋于零，窄矩形的面积趋于窄曲边梯形的面积

而所有的窄曲边梯形的面积之和为曲边梯形的面积

在区间 $[a,b]$ 插入任意多个分点，即 $a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b$，其中 $n\in Z^+$

分成 n 个小区间 $[x_0,x_1]$，$[x_1,x_2]$，$\cdots$，$[x_{n-1},x_n]$

则其长度分别为 $\Delta x_1=x_1-x_0$，$\Delta x_2=x_2-x_1$，$\cdots$，$\Delta x_n=x_n-x_{n-1}$

于是 $x=x_0$，$x=x_1$，$\cdots$，$x=x_{n-1}$，$x=x_n$ 划分出 n 个窄曲边梯形

在 $[x_{i-1},x_i]$ $(1\le i\le n)$ 上任取一点 ξ_i，则有以 $f(\xi_i)$ 为高的窄矩形

由于第 i 个窄曲边梯形的面积可近似地看作第 i 个窄矩形的面积

即曲边梯形面积 $A\approx f(\xi_1)\Delta x_1+f(\xi_2)\Delta x_2+\cdots+f(\xi_n)\Delta x_n=\sum_{i=1}^{n}f(\xi_i)\Delta x_i$

如果令 $\lambda=\max\{\Delta x_1,\Delta x_2,\cdots,\Delta x_n\}$，则当 $n\to\infty$ 时，$\lambda\to 0$

则有 $A=\lim_{\lambda\to 0}\sum_i f(\xi_i)\Delta x_i=\lim_{n\to\infty}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$

定积分 (definite integral)，指对于函数 $f(x)$ 在 $[a,b]$ 上有界，

则在 $[a,b]$ 插入任意多个分点，$a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b$

将 $[a,b]$ 分成 n 个小区间，$[x_0,x_1]$，$[x_1,x_2]$，$\cdots$，$[x_{n-1},x_n]$

区间长度分别为 $\Delta x_1=x_1-x_0$，$\Delta x_2=x_2-x_1$，$\cdots$，$\Delta x_n=x_n-x_{n-1}$

则在每个小区间 $[x_{i-1},x_i]$ 上任取一点，ξ_i $(x_{i-1}\le\xi_i\le x_i)$，$i=1,2,\cdots,n$

注意这里不要求 $f(\xi_i)\Delta x$ 与窄曲边梯形面积相等，

作窄矩形面积和 $S=\sum_{i=1}^{n}f(\xi_i)\Delta x_i$，

令 $\lambda=\max\{\Delta x_1,\Delta x_2,\cdots,\Delta x_n\}$，则当 $\lambda\to 0$ 时，有 $n\to\infty$

如果当 $\lambda\to 0$ 时，极限 $\lim_{\lambda\to 0}S=\lim_{n\to\infty}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$ 存在

且对 $[a,b]$ 的划分与其中点 ξ_i 的选择与极限存在与否无关

则称极限 $\lim_{n\to\infty}\sum_{i=1}^{n}f(\xi_i)\Delta x_i=\int_a^b f(x)dx=I$，

即极限 I 为函数 $f(x)$ 在区间 $[a,b]$ 上的定积分，$f(x)$ 称为被积函数 (integrand)

$f(x)dx$ 为被积表达式 (integrable expression)，x 为积分变量 (integration variable)

$[a,b]$ 为积分区间 (interval)，a,b 为积分下限/上限