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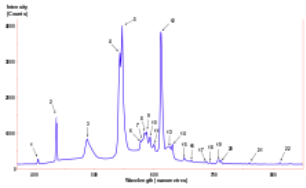
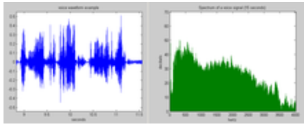
Spectral density



The spectral density of a fluorescent light as a function of optical wavelength shows peaks at atomic transitions, indicated by the numbered arrows.

The power spectrum $S_{xx}(f)$ of a time series $x(t)$ describes the distribution of power into frequency components composing that signal.[1] According to Fourier analysis, any physical signal can be decomposed into a number of discrete frequencies, or a spectrum of frequencies over a continuous range. The statistical average of a certain signal or sort of signal (including noise) as analyzed in terms of its frequency content, is called its spectrum.

When the energy of the signal is concentrated around a finite time interval, especially if its total energy is finite, one may compute the **energy spectral density**. More commonly used is the **power spectral density** (or simply **power spectrum**), which applies to signals existing over *all* time, or over a time period large enough (especially in relation to the duration of a measurement) that it could as well have been over an infinite time interval. The power spectral density (PSD) then refers to the spectral energy distribution that would be found per unit time, since the total energy of such a signal over all time would generally be infinite. Summation or integration of the spectral components yields the total power (for a physical process) or variance (in a statistical process), identical to what would be obtained by integrating $x^2(t)$ over the time domain, as dictated by Parseval's theorem.[2]



The voice waveform over time (left) has a broad audio power spectrum (right).

The spectrum of a physical process $x(t)$ often contains essential information about the nature of x. For instance, the pitch and timbre of a musical instrument are immediately determined from a spectral analysis. The color of a light source is determined by the spectrum of the electromagnetic wave's electric field $E(t)$ as it fluctuates at an extremely high frequency. Obtaining a spectrum from time series such as these involves the Fourier transform, and generalizations based on Fourier analysis. In many cases the time domain is not specifically employed in practice, such as when a dispersive prism is used to obtain a spectrum of light in a spectrograph, or when a sound is perceived through its effect on the auditory receptors of the inner ear, each of which is sensitive to a particular frequency.

However this article concentrates on situations in which the time series is known (at least in a statistical sense) or directly measured (such as by a microphone sampled by a computer). The power spectrum is important in statistical signal processing and in the statistical study of stochastic processes, as well as in many other branches of physics and engineering. Typically the process is a function of time, but one can similarly discuss data in the spatial domain being decomposed in terms of spatial frequency.[3]

Contents

Explanation

Any signal that can be represented as a variable that varies in time has a corresponding frequency spectrum. This includes familiar entities such as visible light (perceived as color), musical notes (perceived as pitch), radio/TV (specified by their frequency, or sometimes wavelength) and even the regular rotation of the earth. When these signals are viewed in the form of a frequency spectrum, certain aspects of the received signals or the underlying processes producing them are revealed. In some cases the frequency spectrum may include a distinct peak corresponding to a sine wave component. And additionally there may be peaks corresponding to harmonics of a fundamental peak, indicating a periodic signal which is *not* simply sinusoidal. Or a continuous spectrum may show narrow frequency intervals which are strongly enhanced corresponding to resonances, or frequency intervals containing almost zero power as would be produced by a notch filter.

In physics, the signal might be a wave, such as an electromagnetic wave, an acoustic wave, or the vibration of a mechanism. The *power spectral density* (PSD) of the signal describes the power present in the signal as a function of frequency, per unit frequency. Power spectral density is commonly expressed in watts per hertz (W/Hz).[4]

When a signal is defined in terms only of a voltage, for instance, there is no unique power associated with the stated amplitude. In this case "power" is simply reckoned in terms of the square of the signal, as this would always be *proportional* to the actual power delivered by that signal into a given impedance. So one might use units of $V^2\ Hz^{-1}$ for the PSD and $V^2\ s\ Hz^{-1}$ for the ESD (*energy spectral density*)[5] even though no actual "power" or "energy" is specified.

Sometimes one encounters an *amplitude spectral density* (ASD), which is the square root of the PSD; the ASD of a voltage signal has units of V $Hz^{-1/2}$.[6] This is useful when the *shape* of the spectrum is rather constant, since variations in the ASD will then be proportional to variations in the signal's voltage level itself. But it is mathematically preferred to use the PSD, since only in that case is the area under the curve meaningful in terms of actual power over all frequency or over a specified bandwidth.

In the general case, the units of PSD will be the ratio of units of variance per unit of frequency; so, for example, a series of displacement values (in meters) over time (in seconds) will have PSD in units of m^2/Hz. For random vibration analysis, units of g^2 Hz^{-1} are frequently used for the PSD of acceleration. Here *g* denotes the g-force.[7]

Mathematically, it is not necessary to assign physical dimensions to the signal or to the independent variable. In the following discussion the meaning of *x(t)* will remain unspecified, but the independent variable will be assumed to be that of time.

Definition

Energy spectral density

Energy spectral density describes how the energy of a signal or a time series is distributed with frequency. Here, the term energy is used in the generalized sense of signal processing;[8] that is, the energy E of a signal $x(t)$ is**:**

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \ dt.$$

The energy spectral density is most suitable for transients—that is, pulse-like signals—having a finite total energy. Finite or not, Parseval's theorem [9] (or Plancherel's theorem) gives us an alternate expression for the energy of the signal**:**

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 \, df,$$

where**:**

$$\hat{x}(f) \triangleq \int_{-\infty}^{\infty} e^{-i2\pi ft} x(t) \ dt$$

is the value of the Fourier transform of $x(t)$ at frequency f (in Hz). The theorem also holds true in the discrete-time cases. Since the integral on the right-hand side is the energy of the signal, the integrand $|\hat{x}(f)|^2$ can be interpreted as a density function describing the energy contained in the signal at the frequency f. Therefore, the **energy spectral density** of $x(t)$ is defined as**:**[9]

$$\bar{S}_{xx}(f) \triangleq |\hat{x}(f)|^2 \qquad \textbf{(Eq.1)}$$

The function $\bar{S}_{xx}(f)$ and the autocorrelation of $x(t)$ form a Fourier transform pair, a result is known as Wiener–Khinchin theorem. (also see Periodogram)

As a physical example of how one might measure the energy spectral density of a signal, suppose $V(t)$ represents the potential (in volts) of an electrical pulse propagating along a transmission line of impedance Z, and suppose the line is terminated with a matched resistor (so that all of the pulse energy is delivered to the resistor and none is reflected back). By Ohm's law, the power delivered to the resistor at time t is equal to $V(t)^2/Z$, so the total energy is found by integrating $V(t)^2/Z$ with respect to time over the duration of the pulse. To find the value of the energy spectral density $\bar{S}_{xx}(f)$ at frequency f, one could insert between the transmission line and the resistor a bandpass filter which passes only a narrow range of frequencies (Δf, say) near the frequency of interest and then measure the total energy $E(f)$ dissipated across the resistor. The value of the energy spectral density at f is then estimated to be $E(f)/\Delta f$. In this example, since the power $V(t)^2/Z$ has units of V^2 Ω^{-1}, the energy $E(f)$ has units of V^2 s Ω^{-1} = J, and hence the estimate $E(f)/\Delta f$ of the energy spectral density has units of J Hz^{-1}, as required. In many situations, it is common to forgo the step of dividing by Z so that the energy spectral density instead has units of V^2 Hz^{-1}.

This definition generalizes in a straightforward manner to a discrete signal with an infinite number of values x_n such as a signal sampled at discrete times $x_n = x(n\Delta t)$**:**

$$\bar{S}_{xx}(f) = \lim_{N\to\infty} (\Delta t)^2 \underbrace{\left|\sum_{n=-N}^{N} x_n e^{-i2\pi fn\Delta t}\right|^2}_{|\hat{x}_d(f)|^2},$$

where $\hat{x}_d(f)$ is the discrete-time Fourier transform of x_n. The sampling interval Δt is needed to keep the correct physical units and to ensure that we recover the continuous case in the limit $\Delta t \to 0$. But in the mathematical sciences the interval is often set to 1, which simplifies the results at the expense of generality. (also see Normalized frequency)

Power spectral density

The above definition of energy spectral density is suitable for transients (pulse-like signals) whose energy is concentrated around one time window; then the Fourier transforms of the signals generally exist. For continuous signals over all time, one must rather define the *power spectral density* (PSD) which exists for stationary processes; this describes how power of a signal or time series is distributed over frequency, as in the simple example given previously. Here, power can be the actual physical power, or more often, for convenience with abstract signals, is simply identified with the squared value of the signal. For example, statisticians study the variance of a function over time $x(t)$ (or over another independent variable), and using an analogy with electrical signals (among other physical processes), it is customary to refer to it as the *power spectrum* even when there is no physical power involved. If one were to create a physical voltage source which followed $x(t)$ and applied it to the terminals of a 1 ohm resistor, then indeed the instantaneous power dissipated in that resistor would be given by $x(t)^2$ watts.

The average power P of a signal $x(t)$ over all time is therefore given by the following time average, where the period T is centered about some arbitrary time $t = t_0$:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} |x(t)|^2 \, dt$$

However, for the sake of dealing with the math that follows, it is more convenient to deal with time limits in the signal itself rather than time limits in the bounds of the integral. As such, we have an alternative representation of the average power, where $x_T(t) = x(t)w_T(t)$ and $w_T(t)$ is unity within the arbitrary period and zero elsewhere.

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x_T(t)|^2 \, dt$$

Clearly in cases where the above expression for P is non-zero (even as T grows without bound) the integral itself must also grow without bound. That is the reason that we cannot use the energy spectral density itself, which *is* that diverging integral, in such cases.

In analyzing the frequency content of the signal $x(t)$, one might like to compute the ordinary Fourier transform $\hat{x}(f)$; however, for many signals of interest the Fourier transform does not formally exist.[N 1] Regardless, Parseval's Theorem tells us that we can re-write the average power as follows.

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |\hat{x}_T(f)|^2 \, df$$

Then the power spectral density is simply defined as the integrand above.[11][12]

$$S_{xx}(f) = \lim_{T\to\infty} \frac{1}{T} |\hat{x}_T(f)|^2 \qquad \text{(Eq.2)}$$

From here, we can also view $|\hat{x}_T(f)|^2$ as the Fourier transform of the time convolution of $x_T^*(-t)$ and $x_T(t)$

$$|\hat{x}_T(f)|^2 = \mathcal{F}\left\{x_T^*(-t) * x_T(t)\right\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T^*(t-\tau)x_T(t)dt\right] e^{-i2\pi f\tau} \, d\tau$$

Now, if we divide the time convolution above by the period T and take the limit as $T \to \infty$, it becomes the autocorrelation function of the non-windowed signal $x(t)$, which is denoted as $R_{xx}(\tau)$, provided that $x(t)$ is ergodic, which is true in most, but not all, practical cases.[13].

$$\lim_{T\to\infty} \frac{1}{T} |\hat{x}_T(f)|^2 = \int_{-\infty}^{\infty} \left[\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} x_T^*(t-\tau)x_T(t)dt\right] e^{-i2\pi f\tau} \, d\tau = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i2\pi f\tau} d\tau$$

From here we see, again assuming the ergodicity of $x(t)$, that the power spectral density can be found as the Fourier transform of the autocorrelation function (Wiener–Khinchin theorem).

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i2\pi f\tau} \, d\tau = \hat{R}_{xx}(f) \qquad \text{(Eq.3)}$$

Many authors use this equality to actually *define* the power spectral density.[14]

The power of the signal in a given frequency band $[f_1, f_2]$, where $0 < f_1 < f_2$, can be calculated by integrating over frequency. Since $S_{xx}(-f) = S_{xx}(f)$, an equal amount of power can be attributed to positive and negative frequency bands, which accounts for the factor of 2 in the following form (such trivial factors dependent on conventions used):

$$P_{\textsf{bandlimited}} = 2 \int_{f_1}^{f_2} S_{xx}(f) \, df$$

More generally, similar techniques may be used to estimate a time-varying spectral density. In this case the time interval T is finite rather than approaching infinity. This results in decreased spectral coverage and resolution since frequencies of less than $1/T$ are not sampled, and results at frequencies which are not an integer multiple of $1/T$ are not independent. Just using a single such time series, the estimated power spectrum will be very "noisy"; however this can be alleviated if it is possible to evaluate the expected value (in the above equation) using a large (or infinite) number of short-term spectra corresponding to statistical ensembles of realizations of $x(t)$ evaluated over the specified time window.

Just as with the energy spectral density, the definition of the power spectral density can be generalized to discrete time variables x_n. As before, we can consider a window of $-N \le n \le N$ with the signal sampled at discrete times $x_n = x(n\Delta t)$ for a total measurement period $T = (2N+1)\Delta t$.

$$S_{xx}(f) = \lim_{N\to\infty} \frac{(\Delta t)^2}{T} \left| \sum_{n=-N}^{N} x_n e^{-i2\pi f n \Delta t} \right|^2$$

Note that a single estimate of the PSD can be obtained through a finite number of samplings. As before, the actual PSD is achieved when N (and thus T) approach infinity and the expected value is formally applied. In a real-world application, one would typically average a finite-measurement PSD over many trials to obtain a more accurate estimate of the theoretical PSD of the physical process underlying the individual measurements. This computed PSD is sometimes called a periodogram. This periodogram converges to the true PSD as the number of estimates as well as the averaging time interval T approach infinity (Brown & Hwang).[15]

If two signals both possess power spectral densities, then the cross-spectral density can similarly be calculated; as the PSD is related to the autocorrelation, so is the cross-spectral density related to the cross-correlation.

Properties of the power spectral density

Some properties of the PSD include:[16]

- The power spectrum is always real and non-negative, and the spectrum of a real valued process is also an even function of frequency: $S_{xx}(-f) = S_{xx}(f)$.
- For a continuous stochastic process x(t), the autocorrelation function $R_{xx}(t)$ can be reconstructed from its power spectrum $S_{xx}(f)$ by using the inverse Fourier transform
- Using Parseval's theorem, one can compute the variance (average power) of a process by integrating the power spectrum over all frequency:

$$P = \operatorname{Var}(x) = \int_{-\infty}^{\infty} S_{xx}(f)\, df$$

- For a real process x(t) with power spectral density $S_{xx}(f)$, one can compute the *integrated spectrum* or *power spectral distribution* $F(f)$, which specifies the average *bandlimited* power contained in frequencies from DC to f using:[17]

$$F(f) = 2\int_0^f S_{xx}(f')\, df'.$$

  Note that the previous expression for total power (signal variance) is a special case where $f \to \infty$.

Cross power spectral density

Given two signals $x(t)$ and $y(t)$, each of which possess power spectral densities $S_{xx}(f)$ and $S_{yy}(f)$, it is possible to define a **cross power spectral density** (**CPSD**) or **cross spectral density** (**CSD**). To begin, let us consider the average power of such a combined signal.

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} \left[x_T(t) + y_T(t)\right]^* \left[x_T(t) + y_T(t)\right] dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |x_T(t)|^2 + x_T^*(t) y_T(t) + y_T^*(t) x_T(t) + |y_T(t)|^2 dt$$

Using the same notation and methods as used for the power spectral density derivation, we exploit Parseval's theorem and obtain

$$S_{xy}(f) = \lim_{T\to\infty} \frac{1}{T}\left[\hat{x}_T^*(f)\hat{y}_T(f)\right] \qquad S_{yx}(f) = \lim_{T\to\infty} \frac{1}{T}\left[\hat{y}_T^*(f)\hat{x}_T(f)\right]$$

where, again, the contributions of $S_{xx}(f)$ and $S_{yy}(f)$ are already understood. Note that $S_{xy}^*(f) = S_{yx}(f)$, so the full contribution to the cross power is, generally, from twice the real part of either individual **CPSD**. Just as before, from here we recast these products as the Fourier transform of a time convolution, which when divided by the period and taken to the limit $T \to \infty$ becomes the Fourier transform of a cross-correlation function.[18]

$$S_{xy}(f) = \int_{-\infty}^{\infty} \left[\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} x_T^*(t-\tau) * y_T(t) dt\right] e^{-i2\pi f\tau} d\tau = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i2\pi f\tau} d\tau$$
$$S_{yx}(f) = \int_{-\infty}^{\infty} \left[\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} y_T^*(t-\tau) * x_T(t) dt\right] e^{-i2\pi f\tau} d\tau = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-i2\pi f\tau} d\tau$$

where $R_{xy}(\tau)$ is the cross-correlation of $x(t)$ with $y(t)$ and $R_{yx}(\tau)$ is the cross-correlation of $y(t)$ with $x(t)$. In light of this, the PSD is seen to be a special case of the CSD for $x(t) = y(t)$. For the case that $x(t)$ and $y(t)$ are voltage or current signals, their associated amplitude spectral densities $\hat{x}(f)$ and $\hat{y}(f)$ are strictly positive by convention. Therefore, in typical signal processing, the full **CPSD** is just one of the **CPSD**s scaled by a factor of two.

$$CPSD_{Full} = 2S_{xy}(f) = 2S_{yx}(f)$$

For discrete signals x_n and y_n, the relationship between the cross-spectral density and the cross-covariance is

$$S_{xy}(f) = \sum_{n=-\infty}^{\infty} R_{xy}(\tau_n) e^{-i2\pi f\tau_n}\, \Delta\tau$$

Estimation

The goal of spectral density estimation is to estimate the spectral density of a random signal from a sequence of time samples. Depending on what is known about the signal, estimation techniques can involve parametric or non-parametric approaches, and may be based on time-domain or frequency-domain analysis. For example, a common parametric technique involves fitting the observations to an autoregressive model. A common non-parametric technique is the periodogram.

The spectral density is usually estimated using Fourier transform methods (such as the Welch method), but other techniques such as the maximum entropy method can also be used.

Related concepts

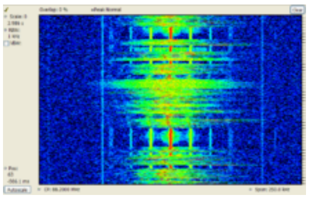
- The spectral centroid of a signal is the midpoint of its spectral density function, i.e. the frequency that divides the distribution into two equal parts.
- The spectral edge frequency of a signal is an extension of the previous concept to any proportion instead of two equal parts.
- The spectral density is a function of frequency, not a function of time. However, the spectral density of small windows of a longer signal may be calculated, and plotted versus time associated with the window. Such a graph is called a *spectrogram*. This is the basis of a number of spectral analysis techniques such as the short-time Fourier transform and wavelets.
- A "spectrum" generally means the power spectral density, as discussed above, which depicts the distribution of signal content over frequency. This is not to be confused with the frequency response of a transfer function which also includes a phase (or equivalently, a real and imaginary part as a function of frequency). For transfer functions, (e.g., Bode plot, chirp) the complete frequency response may be graphed in two parts, amplitude versus frequency and phase versus frequency (or less commonly, as real and imaginary parts of the transfer function). The impulse response (in the time domain) $h(t)$, cannot generally be uniquely recovered from the amplitude spectral density part alone without the phase function. Although these are also Fourier transform pairs, there is no symmetry (as there is for the autocorrelation) forcing the Fourier transform to be real-valued. See spectral phase and phase noise.

Applications

Electrical engineering

The concept and use of the power spectrum of a signal is fundamental in electrical engineering, especially in electronic communication systems, including radio communications, radars, and related systems, plus passive remote sensing technology. Electronic instruments called spectrum analyzers are used to observe and measure the ***power spectra*** of signals.

The spectrum analyzer measures the magnitude of the short-time Fourier transform (STFT) of an input signal. If the signal being analyzed can be considered a stationary process, the STFT is a good smoothed estimate of its power spectral density.



Spectrogram of an FM radio signal with frequency on the horizontal axis and time increasing upwards on the vertical axis.

Cosmology

Primordial fluctuations, density variations in the early universe, are quantified by a power spectrum which gives the power of the variations as a function of spatial scale.

See also

- Noise spectral density
- Spectral density estimation
- Spectral efficiency
- Spectral power distribution
- Brightness temperature
- Colors of noise
- Spectral leakage
- Window function
- Bispectrum
- Whittle likelihood

Notes

1. Some authors (e.g. Risken[10]) still use the non-normalized Fourier transform in a formal way to formulate a definition of the power spectral density

$$\langle \hat{x}(\omega)\hat{x}^*(\omega')\rangle = 2\pi f(\omega)\delta(\omega-\omega'),$$

where $\delta(\omega-\omega')$ is the Dirac delta function. Such formal statements may sometimes be useful to guide the intuition, but should always be used with utmost care.

References

1. P Stoica & R Moses (2005). "Spectral Analysis of Signals" (http://user.it.uu.se/~ps/SAS-new.pdf) (PDF).
2. P Stoica & R Moses (2005). "Spectral Analysis of Signals" (http://user.it.uu.se/~ps/SAS-new.pdf) (PDF).
3. P Stoica & R Moses (2005). "Spectral Analysis of Signals" (http://user.it.uu.se/~ps/SAS-new.pdf) (PDF).
4. Gérard Maral (2003). *VSAT Networks* (https://books.google.com/books?id=CMx5HQ1Mr_UC&q=%22power+spectral+density%22+W/Hz&pg=PR20). John Wiley and Sons. ISBN 978-0-470-86684-9.
5. Michael Peter Norton & Denis G. Karczub (2003). *Fundamentals of Noise and Vibration Analysis for Engineers* (https://books.google.com/books?id=jDeRCSqtev4C&q=%22power+spectral+density%22+%22energy+spectral+density%22&pg=PA352). Cambridge University Press. ISBN 978-0-521-49913-2.
6. Michael Cerna & Audrey F. Harvey (2000). "The Fundamentals of FFT-Based Signal Analysis and Measurement" (http://www.lumerink.com/courses/ece697/docs/Papers/The%20Fundamentals%20of%20FFT-Based%20Signal%20Analysis%20and%20Measurements.pdf) (PDF).

7. Alessandro Birolini (2007). *Reliability Engineering* (https://books.google.com/books?id=xPIW3AI9tdAC&q=acceleration-spectral-density+g+hz&pg=PA83). Springer. p. 83. ISBN 978-3-540-49388-4.
8. Oppenheim; Verghese. *Signals, Systems, and Inference*. pp. 32–4.
9. Stein, Jonathan Y. (2000). *Digital Signal Processing: A Computer Science Perspective*. Wiley. p. 115.
10. Hannes Risken (1996). *The Fokker–Planck Equation: Methods of Solution and Applications* (https://books.google.com/books?id=MG2V9vTgSgEC&pg=PA30) (2nd ed.). Springer. p. 30. ISBN 9783540615309.
11. Fred Rieke; William Bialek & David Warland (1999). *Spikes: Exploring the Neural Code (Computational Neuroscience)*. MIT Press. ISBN 978-0262681087.
12. Scott Millers & Donald Childers (2012). *Probability and random processes*. Academic Press. pp. 370–5.
13. The Wiener–Khinchin theorem makes sense of this formula for any wide-sense stationary process under weaker hypotheses: R_{xx} does not need to be absolutely integrable, it only needs to exist. But the integral can no longer be interpreted as usual. The formula also makes sense if interpreted as involving distributions (in the sense of Laurent Schwartz, not in the sense of a statistical Cumulative distribution function) instead of functions. If R_{xx} is continuous, Bochner's theorem can be used to prove that its Fourier transform exists as a positive measure, whose distribution function is F (but not necessarily as a function and not necessarily possessing a probability density).
14. Dennis Ward Ricker (2003). *Echo Signal Processing* (https://books.google.com/books?id=NF2Tmty9nugC&q=%22power+spectral+density%22+%22energy+spectral+density%22&pg=PA23). Springer. ISBN 978-1-4020-7395-3.
15. Robert Grover Brown & Patrick Y.C. Hwang (1997). *Introduction to Random Signals and Applied Kalman Filtering*. John Wiley & Sons. ISBN 978-0-471-12839-7.
16. Storch, H. Von; F. W. Zwiers (2001). *Statistical analysis in climate research*. Cambridge University Press. ISBN 978-0-521-01230-0.
17. An Introduction to the Theory of Random Signals and Noise, Wilbur B. Davenport and Willian L. Root, IEEE Press, New York, 1987, ISBN 0-87942-235-1
18. William D Penny (2009). "Signal Processing Course, chapter 7" (http://www.fil.ion.ucl.ac.uk/~wpenny/course/course.html).

External links

- Power Spectral Density Matlab scripts (http://vibrationdata.wordpress.com/category/power-spectral-density/)