Chapter 4

# Representations of the Symmetric Groups

## 4.1 Conjugacy Classes in $S_n$

The symmetric groups occur as symmetry groups of quantum mechanical systems which contain $n$ identical particles. Furthermore, the irred reps of $S_n$ are intimately bound up with the irred reps of certain Lie groups, most notably $GL(n, \mathfrak{C})$ and $O(n, \mathfrak{C})$. (See Section 4.3, where we discuss the relation between $S_n$ and symmetry classes of tensors.) For these reasons a knowledge of the rep theory of $S_n$ is indispensable for an understanding of the role of groups in modern physical theories.

For low values of $n$, say $n \leq 5$, we could use the methods of Section 3.6 to compute the character tables and rep matrices of $S_n$. For example $S_4$ is isomorphic to the octahedral group $O$ and has the character table (6.22). (Every $g \in O$ is uniquely determined by a permutation of the four threefold axes.) However, the construction of character tables becomes rapidly more difficult as $n$ increases.

To obtain the irred reps of $S_n$ for all $n$ simultaneously, we develop new tools which exploit the structure of these groups. In distinction to the general methods of Chapter 3, the methods introduced in this chapter apply to the symmetric groups alone. Furthermore, the proofs of the basic facts about the rep theory of $S_n$ are somewhat complicated, although the final results are not difficult to state. We will not give a complete coverage of the symmetric groups, but merely determine the primitive idempotents in the group ring of $S_n$, compute the simple characters, and study the relation between $S_n$ and symmetry classes of tensors. The principal omissions, which the reader can fill in by consulting Boerner [1], Hamermesh [1], Robinson [1], or Rutherford [1], are a construction of the matrix elements of irred reps and a detailed study of the computational problems involved in construction and decomposition of reps. The theory developed here is sufficient for all subsequent applications of $S_n$ which occur in this book and for the majority of applications to modern physical theories.

To begin we investigate the structure of $S_n$ in more detail. We will use the notation for permutations introduced in Example 5, Section 1.1. Let

$$s = \begin{pmatrix} 1 & 2 & \cdots & n \\ s(1) & s(2) & \cdots & s(n) \end{pmatrix} \tag{1.1}$$

be an element of $S_n$ and

$$h(\mathbf{x}) = \prod_{1 \le \mu < \nu \le n} (x_\nu - x_\mu), \qquad x = (x_1, \ldots, x_n), \tag{1.2}$$

where the $x_j$ are arbitrary variables. If $f$ is any function of $\mathbf{x}$ we define the new function $T_s f$ by

$$T_s f(\mathbf{x}) = f(\mathbf{x}_s), \qquad \mathbf{x}_s = (\mathbf{x}_{s(1)}, \ldots, \mathbf{x}_{s(n)}). \tag{1.3}$$

This mapping satisfies the homomorphism property since

$$[T_{st} f](\mathbf{x}) = f(\mathbf{x}_{st}) = \{T_s[T_t f]\}(\mathbf{x}) \tag{1.4}$$

for $s, t \in S_n$. Indeed $s[t(i)] = st(i)$ and $[T_t f](\mathbf{x}_s) = [T_t f](\mathbf{y}) = f(\mathbf{y}_t) = f(\mathbf{x}_{st})$ since $y_i = x_{s(i)}$ and $y_{t(i)} = x_{st(i)}$. Now $[T_s h](\mathbf{x}) = \pm h(\mathbf{x})$, for every $s \in S_n$, where $h$ is the function (1.2). That is, an arbitrary permutation of the indices of $\mathbf{x}$ either leaves $h$ fixed or changes its sign. The restriction of the operators $T_s$ to the one-dimensional vector space generated by $h$ yields an irred rep of $S_n$ called the **alternating** rep. We can regard this rep as a homomorphism $\mu$ of $S_n$ into the cyclic group of order two containing the elements $\{\pm 1\}$. The permutation $s$ is **even** if $\mu$ maps $s$ into $+1$ and **odd** if $s$ is mapped into $-1$. The reader can verify that the permutation $s = (12)$ in cycle notation is odd. This proves that $\mu$ is onto for $n \ge 2$. Let $A_n$ be the kernel of $\mu$, i.e., the set of even permutations. By Theorem 1.3, $A_n$ is a normal subgroup of index two in $S_n$. Thus $\{A_n, (12)A_n\}$ is a coset decomposition of $S_n$.

A two-cycle $(i_1\, i_2) \in S_n$ is called a **transposition.** If $s$ is the permutation (1.1) then $s(12)s^{-1} = (s(1), s(2))$. Thus, any two transpositions in $S_n$ are conjugate. Since $(12) \notin A_n$ all transpositions are odd. It follows that a product of an odd number of transpositions is odd while a product of an even number of transpositions is even. Every permutation $s$ is a product of transpositions. Indeed $s$ is a product of cycles and any cycle

$$(i_1 i_2 \cdots i_j) = (i_1 i_2)(i_2 i_3) \cdots (i_{j-1} i_j) \tag{1.5}$$

is a product of transpositions. A permutation can be written as a product of transpositions in many ways but the number of factors is always even or odd depending on the parity of the permutation.

The conjugacy classes of $S_n$ are easily described. If $s, t \in S_n$ then

$$[sts^{-1}](s(j)) = sts^{-1}s(j) = s[t(j)], \qquad 1 \leq j \leq n,$$

so

$$sts^{-1} = \begin{pmatrix} s(1) & s(2) & \cdots & s(n) \\ st(1) & st(2) & \cdots & st(n) \end{pmatrix}. \tag{1.6}$$

Thus $sts^{-1}$ is obtained from $t$ by applying $s$ to the numbers in the two rows of

$$t = \begin{pmatrix} 1 & 2 & \cdots & n \\ t(1) & t(2) & \cdots & t(n) \end{pmatrix}.$$

In terms of cycle notation the results are even more transparent. For example, if $t = (13642)(57)(8) \in S_8$ and $s$ is given by (1.1) with $n = 8$, then

$$sts^{-1} = (s_1 s_3 s_6 s_4 s_2)(s_5 s_7)(s_8), \qquad s(j) = s_j. \tag{1.7}$$

Two elements of $S_n$ are conjugate if and only if they have the same cycle structure. Furthermore, the elements of a conjugacy class are either all even or all odd.

As an illustration we list the five conjugacy classes of $S_4$:

$$\begin{aligned} &\{e\}, \quad \{(12), (13), (14), (23), (24), (34)\}, \\ &\{(12)(34), (13)(24), (14)(23)\}, \\ &\{(123), (124), (132), (134), (142), (143), (234), (243)\} \\ &\{(1234), (1243), (1324), (1342), (1423), (1432)\}. \end{aligned}$$

To each set of nonnegative integers $(\nu_1, \nu_2, \ldots, \nu_n)$ such that

$$n = \nu_1 + 2\nu_2 + \cdots + n\nu_n \tag{1.8}$$

there corresponds a conjugacy class in $S_n$. This class consists of those elements with $\nu_1$ one-cycles, $\nu_2$ two-cycles, $\ldots$, and $\nu_n$ $n$-cycles. According to Section 1.2, the number of elements in the conjugacy class $(\nu_i)$ is $m_\nu = n!/n_\nu$, where $n_\nu$ is the order of the group

$$H^s = \{t \in S_n : tst^{-1} = s\}$$

and $s$ is an element in the conjugacy class $(\nu_i)$. We compute the number of possible permutations $t$. Any cycle of length $i$ in $s$ remains invariant under any one of the $i$ cyclic permutations of its digits. Each of the $\nu_i$ $i$-cycles can be acted on independently in this fashion and the $i$-cycles can also be permuted among themselves. Thus, there are a total of $i^{\nu_i}\nu_i!$ permutations which preserve the cycles of length $i$ in $s$. Since cycles of different length can be considered independently, we find

$$n_\nu = 1^{\nu_1}\nu_1!\, 2^{\nu_2}\nu_2! \cdots n^{\nu_n}\nu_n!, \qquad m_\nu = n!/n_\nu. \tag{1.9}$$

The number of conjugacy classes in $S_n$, hence the number of nonequivalent irred reps is just the number of sets of nonnegative integers $(\nu_i)$ satisfying (1.8). The structure of such solutions $(\nu_i)$ is more easily comprehended in terms of the nonnegative integers $\lambda_i$,

$$\begin{aligned}
\lambda_1 &= \nu_1 + \nu_2 + \cdots \quad + \nu_n \\
\lambda_2 &= \nu_2 + \nu_3 + \cdots \quad + \nu_n \\
\lambda_3 &= \nu_3 + \nu_4 + \cdots + \nu_n \\
&\ \vdots \\
\lambda_n &= \nu_n.
\end{aligned} \tag{1.10}$$

Clearly,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = n, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0. \tag{1.11}$$

The integers $\{\lambda_i\}$ satisfying (1.11) are said to form a **partition** of $n$. We have shown that each conjugacy class $(\nu_i)$ corresponds to a partition of $n$. Conversely if $\{\lambda_1, \ldots, \lambda_n\}$ form a partition of $n$ then the integers

$$\nu_i = \lambda_i - \lambda_{i+1}, \qquad 1 \leq i \leq n-1, \quad \nu_n = \lambda_n, \tag{1.12}$$

determine a conjugacy class $(\nu_i)$ in $S_n$. Thus the number of conjugacy classes of $S_n$ is equal to the number of partitions (1.11) of $n$.

Ordinarily a partition $\{\lambda_1, \ldots, \lambda_r, 0, \ldots, 0\}$ of $n$ is written $\{\lambda_1, \ldots, \lambda_r\}$, i.e., we leave out the $\lambda_i$ that are 0. Also, if several of the $\lambda_i$ are equal we use exponents to shorten the notation. Thus, the partitions $\{22100\}$, $\{21110\}$, $\{31100\}$ of 5 are usually written in the abbreviated forms

$$\{2^2 1\}, \quad \{21^3\}, \quad \{31^2\},$$

respectively. The cycle structure of the conjugacy classes corresponding to $\{\lambda_i\}$ can be recovered from (1.12). As an example we list the five partitions of 4, or what is the same thing, the five conjugacy classes of $S_4$:

$$\{4\}, \quad \{31\}, \quad \{2^2\}, \quad \{21^2\}, \quad \{1^4\}.$$

In this example as in the rest of this chapter we adopt a dictionary ordering of partitions. That is, the partition $\{\lambda_1, \ldots, \lambda_n\}$ precedes (or is greater than) the partition $\{\lambda_1', \ldots, \lambda_n'\}$ if the first nonzero difference $\lambda_i - \lambda_i'$, $i = 1, \ldots, n$, is positive.

## 4.2 Young Tableaux

We now proceed to determine the irred reps of $S_n$ by the method of Young as simplified by Von Neumann (Boerner [1], Weyl [3]). In this approach one computes the primitive idempotents in the group ring. As we have shown in

Section 3.7 each such idempotent generates an irred rep of $S_n$. We will obtain the simple characters indirectly via an examination of symmetry classes of tensors.

There is another approach to this theory, due to Frobenius, in which the method of induced reps is employed to compute the simple characters directly. The primitive idempotents and matrix elements of irred reps are then derived from the characters. See Hamermesh [1] and Littlewood [1] for an exposition of this method.

Let $R_n = R_{S_n}$ be the group ring of $S_n$. Every $x \in R_n$ can be written uniquely in the form $x = \sum x(s) \cdot s$, where $s$ runs over the $n!$ elements of $S_n$. According to the results of Section 3.7 any primitive idempotent in $R_n$ generates an irred rep of $S_n$ and every irred rep can be so generated. We already know two irred reps: the one-dimensional identity and alternating reps. Each of these reps is contained exactly once in the decomposition of the left regular rep $\mathbf{L}$ of $S_n$ on $R_n$. The corresponding idempotents are easily constructed. Consider the element $c = (n!)^{-1} \sum s$, where the sum extends over $S_n$. Clearly, $sc = cs = c$ for all $s \in S_n$. It follows that $c^2 = c$, so $c$ is idempotent. Furthermore, $c$ is primitive idempotent because the invariant subspace it generates consists of the elements $\lambda c$, $\lambda \in \mathfrak{C}$. Since $\mathbf{L}(s)c = sc = c$, the restriction of $\mathbf{L}$ to $\{\lambda c\}$ is equivalent to the identity rep of $S_n$.

Similarly the element $c' = (n!)^{-1} \sum \delta_s s$, where $\delta_s = +1$ if $s$ is even and $\delta_s = -1$ if $s$ is odd, satisfies $sc' = c's = \delta_s c'$ for all $s \in S_n$. Thus, $(c')^2 = c'$ and $c'$ is idempotent. The reader can check that $c'$ generates an invariant subspace under $\mathbf{L}$ which transforms according to the alternating rep of $S_n$.

Unfortunately the remaining idempotents are not so easy to find. To simplify our discussion slightly we introduce the concept of essential idempotence. An element $c$ is **essentially idempotent** if there exists a nonzero constant $\lambda$ such that $c^2 = \lambda c$. If $c$ is essentially idempotent then $c' = \lambda^{-1}c$ is idempotent since $(c')^2 = \lambda^{-2}c^2 = \lambda^{-1}c = c'$. We shall find it convenient to work with essential idempotents $c$ and there is no loss of generality in doing so since $c$ can be normalized to an idempotent element.

There are exactly as many irred reps of $S_n$ as there are partitions $\{\lambda_j\}$ of $n$, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, so it seems reasonable that each partition should be related to an irred rep. We shall describe this relationship first and then verify its validity.

Consider the partition $\{\lambda_j\}$ of $n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, $\lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0$. To this partition we associate a **frame** consisting of $n$ squares arranged in $r$ rows. The first row consists of $\lambda_1$ squares, the second of $\lambda_2$ squares, ..., and the $r$th of $\lambda_r$ squares. For example the partition $\{3, 2^2, 1\} = \{3, 2, 2, 1, 0, 0, 0, 0\}$ of $n = 8$ is associated with the frame shown in Fig. 4.1. A **Young tableau** is obtained by filling in the $n$ squares of the frame

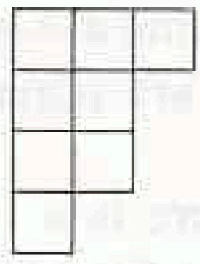

FIGURE 4.1

with the digits 1, 2, . . . , $n$ taken in any order. Each digit is used exactly once. As an example Fig. 4.2 shows two tableaux each of whose frame is that of Fig. 4.1. Clearly, there are $n!$ tableaux associated with the frame $\{\lambda_j\}$.

$$\text{a.}\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \qquad\qquad \text{b.}\quad \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 8 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

FIGURE 4.2

Given a tableau $T$ we define two sets of permutations $R(T)$ and $C(T)$. Here $R(T)$ consists of all $p \in S_n$ that permute the digits in each row of $T$ among themselves without altering the row in which a digit lies. The elements $p$ are called **row permutations**. The set $C(T)$ consists of all $q \in S_n$ that permute the digits in each column of $T$. The $q$ are called **column permutations**. It is easy to verify that $R(T)$ and $C(T)$ are subgroups of $S_n$.

For example, the tableau shown in Fig. 4.3 is associated with

$$R(T) = \{(146), (164), (14), (16), (46), (1), (146)(25), (164)(25), (14)(25), (16)(25), (46)(25), (25)\}$$

$$C(T) = \{(123), (132), (12), (13), (23), (1), (123)(45), (132)(45), (12)(45), (13)(45), (23)(45), (45)\}.$$

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

FIGURE 4.3

Corresponding to the tableau $T$ we construct the elements

(2.1) $$P = \sum_{p \in R(T)} p, \qquad Q = \sum_{q \in C(T)} \delta_q q$$

in the group ring $R_n$, where $\delta_q = +1$ if $q$ is even and $-1$ if $q$ is odd.

**Theorem 4.1.** The ring element $c = PQ$ is essentially idempotent and the invariant subspace $R_n c$ determines an irred rep of $S_n$. Reps determined by different tableaux with the same frame are equivalent, while those determined by tableaux with different frames are nonequivalent.

According to this theorem there is a 1–1 correspondence between irred reps of $S_n$ and frames $\{\lambda_j\}$. The proof is complicated and relies heavily on the methods of Section 3.7. We verify the theorem through a series of lemmas.

Note that $R(T) \cap C(T) = \{e\}$ since only the identity element is simultaneously a row and a column permutation. If $pq = p'q'$, where $p, p' \in R(T)$ and $q, q' \in C(T)$, then $q(q')^{-1} = p^{-1}p' = e$, so $q = q'$, $p = p'$. Thus, each of the terms $pq$ in

$$c = \sum_{pq} \delta_q pq \tag{2.2}$$

is a distinct element of $S_n$, so $c \neq 0$.

If $T$ is a tableau and $s \in S_n$, let $T' = sT$ be the tableau obtained by applying $s$ to the digits of $T$. Thus, if $T$ is the tableau pictured in Fig. 4.2(b) and $s = (257)(34)(16)$, then we have the situation shown in Fig. 4.4.

$$sT = \begin{array}{ccc} 6 & 3 & 1 \\ 5 & 7 & \\ 4 & 8 & \\ 2 & & \end{array}$$

FIGURE 4.4

We say that a digit $m$ is at position $(i, j)$ in $T$ if $m$ lies in the $i$th row and $j$th column of $T$. In Figure 4.3, 4 is in position $(1, 2)$ and 5 is in position $(2, 2)$.

**Lemma 4.1.** Let $r, s \in S_n$ and $T' = sT$. If the digit $m$ at position $(i, j)$ in $T$ lies at position $(i_1, j_1)$ in $rT$ then the digit $s(m)$ at position $(i, j)$ in $T'$ lies at position $(i_1, j_1)$ in $r'T'$, where $r' = srs^{-1}$. We say that $r'$ is the permutation **corresponding** to $r$ for $T'$.

***Proof.*** If the digit $m$ lies at position $(i, j)$ in $T$ and $(i_1, j_1)$ in $rT$ then the digit $k$ at $(i_1, j_1)$ in $T$ satisfies $r(k) = m$. Therefore, the digit at position $(i_1, j_1)$ of $r'T' = (srs^{-1})sT$ is $srs^{-1} \cdot s(k) = s[r(k)] = s(m)$, which is the digit at position $(i, j)$ of $T'$. Q.E.D.

**Example.** Let $s = (257)(34)(16)$, $r = (135)$ and let $T$ be the tableau in Fig. 4.2(b). Then $r' = srs^{-1} = (647)$.

$$T = \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & 8 & \\ 7 & & \end{array}, \qquad rT = \begin{array}{ccc} 3 & 4 & 6 \\ 2 & 1 & \\ 5 & 8 & \\ 7 & & \end{array},$$

$$T' = sT = \begin{array}{|c|c|c|}\hline 6 & 3 & 1 \\ \hline 5 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}\ , \qquad r'T' = srT = \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 6 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}\ .$$

Now the digit 3 at position (3, 1) of $T$ goes to position (1, 1) of $rT$, while the digit $4 = s(3)$ at position (3, 1) of $T'$ goes to position (1, 1) of $r'T'$. Similarly, the reader can check the validity of the lemma for all entries of $T$.

**Corollary 4.1.** If $T' = sT$ then $R(T') = sR(T)s^{-1}$, $C(T') = sC(T)s^{-1}$, $P' = sPs^{-1}$, $Q' = sQs^{-1}$, $c' = scs^{-1}$.

***Proof.*** By the lemma, if $p$ is a row permutation of $T$ then $p' = sps^{-1}$ is a row permutation of $T'$. Similarly, if $q \in C(T)$ then $q' = sqs^{-1} \in C(T')$. The corollary follows easily from these remarks and the elementary fact that $\delta_{q'} = \delta_q$. Q.E.D.

**Lemma 4.2.** An element $s$ of $S_n$ can be written $s = pq$, where $p, q$ are row and column permutations of the tableau $T$, if and only if no two digits in the same row of $T$ lie in the same column of $T' = sT$.

***Proof.*** Suppose $s = pq$ and $m_1, m_2$ are distinct digits in the $i$th row of $T$. Then $sT = (pqp^{-1})pT = q'pT$. According to Lemma 4.1 we can obtain the tableau $sT$ from $T$ by performing a row permutation $p$ on $T$ followed by a column permutation $q' = pqp^{-1}$ on the resultant tableau $pT$. Clearly, $m_1$ and $m_2$ are still in the $i$th row of $pT$, so they must lie in different columns of $q'pT$.

***Note.*** The tableau $pqT$ may *not* be obtained by applying a row permutation to $qT$. The element $p$ is a row permutation of $T$ but not necessarily an element of $R(qT)$.

Conversely, suppose no two digits in the same row of $T$ lie in the same column of $T'$. Then the digits in the first column of $T'$ lie in different rows of $T$. Applying a suitable row permutation to $T$ we can move these digits into the first column. Leaving the first column fixed, we can apply the same procedure to the second column, and so on. Thus there is a $p \in R(T)$ such that the digits in each column of $T'$ are the same as the digits in the corresponding column of $pT$, though not necessarily in the same order. Finally we can apply a column permutation $q'$ to $pT$ such that $T' = q'pT$. Writing $q = p^{-1}q'p$ we note from Corollary 4.1 that $q \in C(T)$. Therefore, $T' = pqT$. Q.E.D.

**Example.** The permutation $s = (257)(34)(16)$ is not a $pq$ for the tableau of Fig. 4.2(b) since the digits 2, 5 which lie in row two of $T$ also lie in column one of $sT$ (Fig. 4.4).

**Lemma 4.3.** If $T$ belongs to the frame $\{\lambda_j\}$ and $T'$ belongs to the frame $\{\lambda_j'\}$ with $\{\lambda_j\} > \{\lambda_j'\}$, then there exist two digits which lie in the same row of $T$ and the same column of $T'$.

***Proof.*** If such a pair of digits did not exist then the $\lambda_1$ digits in the first row of $T$ would lie in different columns of $T'$. Thus, $\lambda_1' \geq \lambda_1$ but since $\{\lambda_j\} > \{\lambda_j'\}$ it follows that $\lambda_1' = \lambda_1$. By means of a column permutation of $T'$ we can transform $T'$ to a tableau $T''$ with the same first row as $T$. Since this permutation does not change the distribution of digits among the columns, we can repeat our argument on the second row of $T''$ to obtain $\lambda_2' = \lambda_2$. Similarly $\lambda_j' = \lambda_j$ for all $j$ and $\{\lambda_j'\} = \{\lambda_j\}$, which is impossible. Q.E.D.

Let $T$ and $T'$ be two tableaux associated with $S_n$, not necessarily with the same frame. Let $P$, $Q$, $c$, respectively $P'$, $Q'$, $c'$, be the ring elements defined by (2.1) and (2.2).

**Lemma 4.4.** If there exist two digits which lie in one row of $T$ and one column of $T'$ then $c'c = 0$.

***Proof.*** First we remark that

$$pP = Pp = P, \qquad qQ = Qq = \delta_q Q \tag{2.3}$$

$$pcq = pPQq = \delta_q c \tag{2.4}$$

for all $p \in R(T)$ and $q \in C(T)$. The occurrence of the parity $\delta_q$ follows from $\delta_{s_1 s_2} = \delta_{s_1}\delta_{s_2}$.

By hypothesis there exist two digits $m, k$ such that the transposition $t = (mk) \in R(T) \cap C(T')$. Thus, $Q'P = Q't \cdot tP = -Q'P$, so $Q'P = 0$, since $t^2 = e$ and $\delta_t = -1$. Therefore $c'c = P'Q'PQ = 0$. Q.E.D.

We shall now show that property (2.4) characterizes $c$ up to a scalar multiple.

**Lemma 4.5.** If $x \in R_n$ such that $pxq = \delta_q x$ for all $p \in R(T)$ and $q \in C(T)$ then $x = \lambda c$ for some $\lambda \in \mathfrak{C}$.

***Proof.*** Let $x = \sum x(s) \cdot s$. Then

$$x = \delta_q p^{-1} x q^{-1} = \delta_q \sum_s x(s) \cdot p^{-1} s q^{-1} = \delta_q \sum_s x(psq) \cdot s$$

so

$$x(s) = \delta_q x(psq) \tag{2.5}$$

for every $s \in S_n, p \in R(T), q \in C(T)$. Setting $s = e$ we see that $x(pq) = \delta_q \lambda$, where $\lambda = x(e)$. A comparison of this result with (2.2) shows that the lemma is true provided we can show $x(s) = 0$ whenever $s$ is not a $pq$.

If $s$ is not a $pq$ there exist two digits in the same row of $T$ and the same column of $sT$. The transposition $p$ of these digits belongs to $R(T) \cap C(sT)$. Similarly the transposition $q^{-1} = s^{-1}ps \in C(T)$ by Corollary 4.1. Thus $s = psq$ and $x(s) = x(psq) = -x(s)$ by (2.5), so $x(s) = 0$. Q.E.D.

**Lemma 4.6.** The ring element $c = PQ$ corresponding to the tableau $T$ is essentially idempotent, and the invariant subspace $R_n c$ yields an irred rep of $S_n$ whose degree divides $n!$.

***Proof.*** Since $pc^2q = pccq = \delta_q c^2$ for all $p \in R(T)$, $q \in C(T)$, it follows from Lemma 4.5 that $c^2 = \lambda c$ for some $\lambda \in \mathfrak{C}$. Thus, $c$ is essentially idempotent if $\lambda \neq 0$. [Since the coefficients $c(pq)$ of $c$ are $\pm 1$ it follows that $\lambda$ is an integer.]

Consider the linear transformation $\mathbf{A}$ on $R_n$ defined by $\mathbf{A}x = xc$, $x \in R_n$. We will compute the trace of $\mathbf{A}$ in the natural basis $\{s_j\}$ of elements of $S_n$. Writing $c = \sum_{j=1}^{n!} c(s_j)\cdot s_j = \sum \delta_q \cdot pq$, we find

$$[\mathbf{A}s_j](s_j) = [s_j c](s_j) = c(e) = 1, \qquad 1 \leq j \leq n!.$$

Therefore the trace of the matrix describing $\mathbf{A}$ is $n!$.

Next we compute the trace of $\mathbf{A}$ with respect to a basis $v_1, \ldots, v_{n!}$, where $v_1, \ldots, v_f$ form a basis for the $f$-dimensional space $R_n c$. If $x = yc \in R_n c$ then $\mathbf{A}x = xc = yc^2 = \lambda yc = \lambda x$, so $\mathbf{A}v_j = \lambda v_j$, $1 \leq j \leq f$. Furthermore $v_j \notin R_n c$ for $f+1 \leq j \leq n!$ and $\mathbf{A}v_j \in R_n c$ for all $j$. Thus the trace of $\mathbf{A}$ in the $v$-basis is $\lambda f$. We conclude that $\lambda = n!/f > 0$. Since $\lambda$ is an integer, $f$ divides $n!$.

By Theorem 3.14, to show that $\lambda^{-1}c$ is a primitive idempotent, it is enough to verify that $\tilde{x} = cxc$ is a multiple of $c$ for every $x \in R_n$. For any $p \in R(T)$ and $q \in C(T)$ we have

$$p\tilde{x}q = pcxcq = \delta_q cxc = \delta_q \tilde{x}.$$

Therefore, by Lemma 4.5, $\tilde{x}$ is a multiple of $c$. Q.E.D.

**Lemma 4.7.** Tableaux corresponding to different frames yield nonequivalent reps of $S_n$, while those corresponding to the same frame yield equivalent reps.

***Proof.*** Suppose $T$ is a tableau with frame $\{\lambda_j\}$ and $T'$ is a tableau with different frame $\{\lambda_j'\}$. Without loss of generality we can assume $\{\lambda_j\} > \{\lambda_j'\}$. By Theorem 3.15, to prove that the reps determined by $T$ and $T'$ are nonequivalent it is enough to show $c'xc = 0$ for all $x \in R_n$. From Lemmas 4.3 and 4.4 there follows $c'c = 0$. Furthermore $sT$ has the same frame as $T$ and essential idempotent $scs^{-1}$ for each $s \in S_n$. Thus, $c'scs^{-1} = 0$ or $c'sc = 0$ for all $s \in S_n$. Therefore $c'xc = \sum x(s)\cdot c'sc = 0$ for all $x \in R_n$.

If $T$ and $T'$ have the same frame then $T' = sT$ for some $s \in S_n$ and

$c' = scs^{-1}$. Thus $c'sc = (scs^{-1})sc = sc^2 = \lambda sc \neq 0$ since $\lambda c \neq 0$. By Theorem 3.15, $c$ and $c'$ generate equivalent reps of $S_n$. Q.E.D.

Lemmas 4.1–4.7 constitute a proof of Theorem 4.1. Note that the identity rep of $S_n$ corresponds to the frame $\{n\}$, i.e., the frame with one row of $n$ squares. The alternating rep corresponds to the frame $\{1^n\}$, i.e., the frame with one column of $n$ squares.

Since there are $n!$ tableaux with the same frame, Theorem 4.1 enables us to construct $n!$ subspaces in the $n!$-dimensional space $R_n$ which transform under a given $f$-dimensional irred rep of $S_n$. The multiplicity of this irred rep in $R_n$ is only $f$, so these subspaces are not all independent of one another. We shall show that it is possible to select $f$ generating idempotents $c_1, \ldots, c_f$ corresponding to the given frame such that $R_nc_1, \ldots, R_nc_f$ are linearly independent. Then, according to the general theory of Section 3.7, every irred subspace $R_nc$ corresponding to this frame is a subspace of

$$R_nc_1 \oplus \cdots \oplus R_nc_f.$$

As an example, we already know that the one-dimensional identity rep has multiplicity one in $R_n$. Therefore, the $n!$ generating idempotents $c_j$ corresponding to the frame $\{n\}$ must all generate the same one-dimensional subspace of $R_n$. The reader can easily show that $c_1 = c_2 = \cdots = c_{n!}$.

Let $\{\lambda_1, \ldots, \lambda_n\}$ be a frame with corresponding (essential) generating idempotents $c_1, \ldots, c_{n!}$. We shall first show that the $n!$ left ideals $R_nc_j$ span the minimal two sided ideal $U$ which contains all left ideals transforming under the irred rep $\{\lambda_j\}$.

**Lemma 4.8.** Let $R_nc$ be an irred subspace of $R_n$ transforming according to the irred rep $\{\lambda_j\}$ under **L**. Then

$$R_nc \subseteq R_nc_1 + R_nc_2 + \cdots + R_nc_{n!},$$

i.e., each $x \in R_nc$ is a linear combination of elements in the $R_nc_j$.

***Proof.*** Since $R_nc$ and $R_nc_1$ correspond to equivalent reps of $S_n$ under **L** it follows from Theorem 3.15 that there exists $y \in R_n$ such that $R_nc = R_nc_1y$ (as vector spaces). Now $y = \sum y(s)\cdot s$, so the lemma will be proved if we can show that for each $s$, $R_nc_1s = R_nc_j$, where $c_j$ is one of the generating idempotents. If $T_1$ is the tableau with idempotent $c_1$ then $s^{-1}T_1 = T_j$ is a tableau with the same frame corresponding to the idempotent (say) $c_j$. Thus $c_j = s^{-1}c_1s$, so $xc_1s = xsc_j \subseteq R_nc_j$ for all $x \in R_n$. Since both $R_nc_1s$ and $R_nc_j$ are minimal left ideals it follows that $R_nc_1s = R_nc_j$. Q.E.D.

The left ideals $R_nc_j$ span the two-sided ideal $U$ but they are not linearly independent. We can obtain a linearly independent set of left ideals which span $U$ by considering the standard tableaux. A tableau $T$ is called a **standard tableau** if the digits in each row of $T$ increase from left to right and the digits in each column increase from top to bottom. For example, the tableaux in Fig. 4.2 and 4.3 are standard, while the tableau in Fig. 4.4 is not.

We have already defined a dictionary ordering for frames. Similarly we can define a dictionary ordering for the standard tableaux belonging to a given frame. Given two such tableaux $T, T'$ we compare their corresponding digits, starting at the left end of the first row and going from left to right. If the first nonzero difference $m - m'$ is positive for corresponding digits $m$ in $T$ and $m'$ in $T'$ we say $T > T'$. If all corresponding digits in the first row are equal we compare digits in the second row, etc. As an example we list the standard tableaux in increasing order of the frame $\{3, 1^2\}$:

$$\begin{matrix} 1\,2\,3 & & 1\,2\,4 & & 1\,2\,5 & & 1\,3\,4 & & 1\,3\,5 & & 1\,4\,5 \\ 4 & & 3 & & 3 & & 2 & & 2 & & 2 \\ 5 & & 5 & & 4 & & 5 & & 4 & & 3 \end{matrix}$$

**Theorem 4.2.** The dimension $f$ of the irred rep corresponding to the frame $\{\lambda_j\}$ is equal to the number of standard tableaux $T_1, \ldots, T_f$ belonging to this frame.

A proof of this theorem will be given in Section 4.4. The theorem implies, for example, that the dimension of the rep $\{3, 1^2\}$ is six. More important, it says that the multiplicity of the rep $\{\lambda_j\}$ is equal to the number $f$ of standard tableaux belonging to the frame $\{\lambda_j\}$. We shall show that the generating idempotents $c_1, \ldots, c_f$ of the standard tableaux generate linearly independent left ideals $R_n c_1, \ldots, R_n c_f$.

**Lemma 4.9.** If $T_i < T_l$ then $c_l c_i = 0$.

***Proof.*** By Lemma 4.4 it is enough to show that there exist two digits in the same row of $T_l$ and the same column of $T_i$. Consider the first space $(j, k)$ (row $j$, column $k$) which is occupied by different digits, $m$ in $T_i$ and $m'$ in $T_l$. Clearly, $m' > m$ since $T_l > T_i$. If $k = 1$ then the entry at $(j, 1)$ is the smallest integer not in the first $j - 1$ rows. Thus the entries $m, m'$ at $(j, 1)$ would be the same for the two tableaux, which is impossible. Thus, $k > 1$. The digit $m$ lies at $(j, k)$ in $T_i$. We will determine the position $(a, b)$ of $m$ in $T_l$. Since $m < m'$ we cannot have $a \geq j$ and $b \geq k$, i.e., $m$ cannot lie below and to the right of $(j, k)$. Also, by definition of $(j, k)$, $m$ cannot lie at a position which comes before $(j, k)$ in the dictionary ordering. Thus the only possibility is $a < j, b < k$; so $m$ lies below and to the left of $(j, k)$ (Fig. 4.5). The position $(j, b)$ comes before $(j, k)$ in the dictionary ordering, so the digit $q$ at this

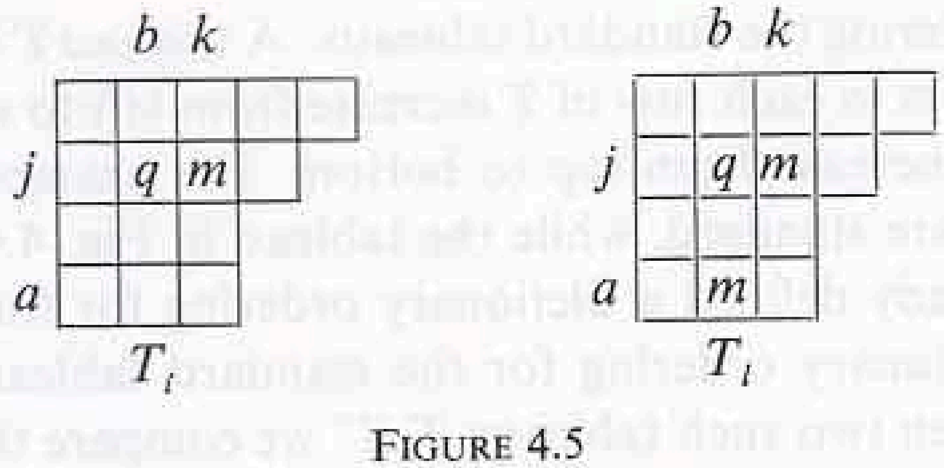


FIGURE 4.5

position is the same for both $T_i$ and $T_l$. Thus, the digits $q, m$ lie in the $j$th row of $T_i$ and the $b$th column of $T_l$. Q.E.D.

**Theorem 4.3.** The left ideals $R_n c_1, \ldots, R_n c_f$ corresponding to the standard tableaux are linearly independent and

$$U = R_n c_1 \oplus \cdots \oplus R_n c_f.$$

***Proof.*** By Theorem 4.2 it is enough to show that if

$$x_1 c_1 + \cdots + x_f c_f = 0, \qquad x_j \in R_n, \tag{2.6}$$

then each term $x_j c_j = 0$. If we multiply on the right by $c_1$, then Lemma 4.9 implies that all terms $c_j c_1$ are zero unless $j = 1$. Thus $x_1 c_1{}^2 = 0$ or $\lambda x_1 c_1 = 0$, with $\lambda \neq 0$. Similarly, multiplication of (2.6) on the right by $c_2$ yields $x_2 c_2 = 0$. Continuing in this way, we can show that $x_j c_j = 0$, $1 \leq j \leq f$. Q.E.D.

## 4.3 Symmetry Classes of Tensors

An important application of $S_n$ rep theory to physics is in the construction of symmetry classes of tensors. As we shall see, this subject is closely related to the rep theory of the general linear groups. A formula for the simple characters of $S_n$ will arise as a by-product of our analysis.

Let $V$ be a complex $m$-dimensional vector space and consider the rep defined on $V$ by the group $GL(m, \mathfrak{C})$ of all invertible linear operators, $\mathbf{g}: V \to V$. In terms of a basis $\{\mathbf{v}_i\}$ for $V$ the rep matrices $g = (g_i{}^j)$ are defined by

$$\mathbf{g}\mathbf{v}_i = \sum_{j=1}^{m} g_i{}^j \mathbf{v}_j, \qquad 1 \leq i \leq m. \tag{3.1}$$

(It will prove convenient to adopt this superscript–subscript notation for indices.) As $\mathbf{g}$ runs over the group of invertible operators on $V$, $g$ runs over the group of complex $m \times m$ nonsingular matrices. We will refer to either of these isomorphic groups as $GL(m, \mathfrak{C})$.

Consider the tensor product rep $\mathbf{T}$ of $GL(m, \mathfrak{C})$ on $V^{\otimes \alpha}$ defined by

$$\mathbf{T}(g)[\mathbf{w}_1 \otimes \cdots \otimes \mathbf{w}_\alpha] = \mathbf{g}\mathbf{w}_1 \otimes \cdots \otimes \mathbf{g}\mathbf{w}_\alpha \tag{3.2}$$

for any $\mathbf{w}_j \in \mathbf{V}$ (see Section 3.5). In terms of the basis $\{\mathbf{v}_i\}$ for $V$,

$$(3.3)\qquad \mathbf{w} = \sum_{j_1 \cdots j_\alpha = 1}^{m} a^{j_1 \cdots j_\alpha} \mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_\alpha} \in V^{\otimes \alpha},$$

$$(3.4)\qquad \mathbf{T}(g)\mathbf{w} = \sum_{i_1 \cdots i_\alpha = 1}^{m} [T(g)a]^{i_1 \cdots i_\alpha} \mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_\alpha},$$

$$(3.5)\qquad [T(g)a]^{i_1 \cdots i_\alpha} = \sum_{j_1 \cdots j_\alpha = 1}^{m} a^{j_1 \cdots j_\alpha} g^{i_1}_{j_1} \cdots g^{i_\alpha}_{j_\alpha}.$$

We can view this rep either as acting on tensors of rank $\alpha$ [Eq. (3.2)], or on the tensor components [Eq. (3.5)]. It will be convenient to shift back and forth between these equivalent interpretations.

The definitions of reducible and irred reps given in Chapter 3 hold for the infinite groups $GL(m, \mathfrak{C})$ as well as for finite groups. Also, the Schur lemmas are immediately applicable to infinite groups. However, those results which explicitly use the finiteness property of a group, such as the character theory, cannot be directly applied to $GL(m, \mathfrak{C})$. Indeed, Theorem 3.3, which states that any rep of a finite group can be decomposed into a direct sum of irred reps, is *not* true for $GL(m, \mathfrak{C})$. Fortunately, we can show that the tensor product rep $\mathbf{T}$ of $GL(m, \mathfrak{C}) = G_m$ on $V^{\otimes \alpha}$ is decomposable into a direct sum of irred reps. The symmetric group $S_\alpha$ figures strongly in this decomposition.

To clarify the relationship between $S_\alpha$ and $G_m = GL(m, \mathfrak{C})$ we define a rep of $S_\alpha$ on $V^{\otimes \alpha}$. For any $s \in S_\alpha$,

$$(3.6)\qquad s = \begin{pmatrix} 1 & \cdots & \alpha \\ s(1) & \cdots & s(\alpha) \end{pmatrix},$$

let $\mathbf{s}$ be the linear operator on $V^{\otimes \alpha}$ defined by

$$\mathbf{s}\mathbf{w} = \mathbf{w}_{s^{-1}(1)} \otimes \cdots \otimes \mathbf{w}_{s^{-1}(\alpha)}$$

for $\mathbf{w} = \mathbf{w}_1 \otimes \cdots \otimes \mathbf{w}_\alpha$ any indecomposable element of $V^{\otimes \alpha}$. These operators are well-defined and yield a rep of $S_\alpha$. If an arbitrary tensor $\mathbf{w}$ is given by (3.3) then the action of $s$ on the tensor components $a^{j_1 \cdots j_\alpha}$ of $\mathbf{w}$ is

$$(3.7)\qquad (sa)^{j_1 \cdots j_\alpha} = a^{j_{s(1)} \cdots j_{s(\alpha)}}.$$

(Verify this.) For example, if $\alpha = 4$, $s = (12)(34)$, then $(sa)^{2331} = a^{3213}$, $(sa)^{1111} = a^{1111}$. If $s = (123)$ then $(sa)^{2331} = a^{3321}$. The reader should check these examples carefully to make sure he understands Eq. (3.7).

The **symmetric tensors** are those $\mathbf{w} \in V^{\otimes \alpha}$ such that $\mathbf{s}\mathbf{w} = \mathbf{w}$ for all $s \in S_\alpha$. Clearly, these tensors form a subspace $\mathfrak{S}$ of $V^{\otimes \alpha}$. It follows from (3.7) that with respect to the fixed basis $\{\mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_\alpha}\}$, the elements of $\mathfrak{S}$ are those tensors $\mathbf{w}$ whose components differing only in the order of the indices are equal. Thus $\mathbf{w} \in \mathfrak{S}$ is uniquely determined by the independent components $a^{j_1 \cdots j_\alpha}$ with $j_1 \le j_2 \le \cdots \le j_\alpha$. To compute the dimension of $\mathfrak{S}$

note that the integers $j_1, j_2+1, j_3+2, \ldots, j_\alpha+\alpha-1$ are $\alpha$ distinct numbers chosen from $1, 2, \ldots, m+\alpha-1$, and every such choice labels a component. Therefore $\dim \mathcal{S} = (m+\alpha-1)!/[\alpha!(m-1)!]$, the number of combinations of $m+\alpha-1$ objects taken $\alpha$ at a time.

The tensors $\mathbf{v}\otimes\mathbf{v}\otimes\cdots\otimes\mathbf{v}$, $\mathbf{v}\in V$, obviously lie in $\mathcal{S}$. We shall show that every $\mathbf{w}\in\mathcal{S}$ is a linear combination of such tensors.

**Lemma 4.10.** The set of all tensors $\mathbf{v}\otimes\mathbf{v}\otimes\cdots\otimes\mathbf{v}$ spans $\mathcal{S}$.

***Proof.*** If $\mathbf{v}=\sum a^j\mathbf{v}_j$ then $\mathbf{v}\otimes\cdots\otimes\mathbf{v}$ has tensor components $a^{j_1}a^{j_2}\cdots a^{j_\alpha}$. By symmetry we can restrict ourselves to the components for which $1\le j_1\le j_2\le\cdots\le j_\alpha\le m$. If the set of all such tensors does not span $\mathcal{S}$ then there must exist constants $C_{j_1\cdots j_\alpha}$, not all zero, such that

$$\sum_{j_1\le j_2\le\cdots\le j_\alpha} C_{j_1\cdots j_\alpha}a^{j_1}\cdots a^{j_\alpha}\equiv 0 \tag{3.8}$$

for all numbers $a^j$, $1\le j\le m$. (Prove it!) In each term $a^{j_1}\cdots a^{j_\alpha}$ let $k_1$ be the number of $j_l$ equal to one, $k_2$ the number of $j_l$ equal to two, etc., and write

$$a^{j_1}\cdots a^{j_\alpha}=(a^1)^{k_1}\cdots(a^m)^{k_m},\qquad C_{j_1\cdots j_\alpha}=C_{k_1\cdots k_m}.$$

Then (3.8) becomes

$$\sum_{k_1\cdots k_m} C_{k_1\cdots k_m}(a^1)^{k_1}\cdots(a^m)^{k_m}\equiv 0,\qquad k_1+\cdots+k_m=\alpha. \tag{3.9}$$

It is a well-known result from algebra (Van der Waerden [1]) that this homogeneous polynomial of degree $\alpha$ can be identically zero for all $a^1, \ldots, a^m$ only if all the coefficients $C_{k_1\cdots k_m}=0$. Q.E.D.

Expression (3.2) for the operators $\mathbf{T}(g)$ makes sense for all linear operators $g$ on $V$, invertible or not. Furthermore, the homomorphism property $\mathbf{T}(g_1g_2) = \mathbf{T}(g_1)\mathbf{T}(g_2)$ holds even if $g_1, g_2$ are not invertible. The set $\tilde{G}_m$ of all linear operators on $V$ is said to form a **semigroup**, that is, $\tilde{G}_m$ satisfies the group axioms except that an element of $\tilde{G}_m$ need not have an inverse. Thus, we can define a rep of $\tilde{G}_m$ on $V$ by means of the operators $\mathbf{T}(g)$.

Note that $\mathbf{s}\mathbf{T}(g)\mathbf{w}=\mathbf{T}(g)\mathbf{s}\mathbf{w}$ for all $s\in S_\alpha$, $g\in\tilde{G}_m$, $\mathbf{w}\in V^{\otimes\alpha}$, i.e., the operators $\mathbf{s}$ and $\mathbf{T}(g)$ always commute. The proof follows from

$$\begin{aligned}\mathbf{s}\mathbf{T}(g)[\mathbf{v}_{j_1}\otimes\cdots\otimes\mathbf{v}_{j_\alpha}] &= \sum_{i_1\cdots i_\alpha} g^{i_1}_{j_1}\cdots g^{i_\alpha}_{j_\alpha}\mathbf{v}_{i_{s^{-1}(1)}}\otimes\cdots\otimes\mathbf{v}_{i_{s^{-1}(\alpha)}}\\ &= \sum g^{i_{s(1)}}_{j_1}\cdots g^{i_{s(\alpha)}}_{j_\alpha}\mathbf{v}_{i_1}\otimes\cdots\otimes\mathbf{v}_{i_\alpha}\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\mathbf{T}(g)\mathbf{s}[\mathbf{v}_{j_1}\otimes\cdots\otimes\mathbf{v}_{j_\alpha}] &= \sum_{i_1\cdots i_\alpha} g^{i_1}_{j_{s^{-1}(1)}}\cdots g^{i_\alpha}_{j_{s^{-1}(\alpha)}}\mathbf{v}_{i_1}\otimes\cdots\otimes\mathbf{v}_{i_\alpha}\\ &= \sum_{i_1\cdots i_\alpha} g^{i_{s(1)}}_{j_1}\cdots g^{i_{s(\alpha)}}_{j_\alpha}\mathbf{v}_{i_1}\otimes\cdots\otimes\mathbf{v}_{i_\alpha}.\end{aligned} \tag{3.11}$$

We now determine the largest set $A_\alpha$ of linear operators on $V^{\otimes\alpha}$ which commute with all permutations $\mathbf{s}$. Each such operator $\mathfrak{A}$ has matrix elements defined by

(3.12) $$\mathfrak{A}[\mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_\alpha}] = \sum_{i_1 \cdots i_\alpha} \mathfrak{A}^{i_1 \cdots i_\alpha}_{j_1 \cdots j_\alpha} \mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_\alpha}.$$

The requirement $\mathbf{s}\mathfrak{A} = \mathfrak{A}\mathbf{s}$ for all $s \in S_\alpha$ implies

(3.13) $$\mathfrak{A}^{i_{s(1)} \cdots i_{s(\alpha)}}_{j_{s(1)} \cdots j_{s(\alpha)}} = \mathfrak{A}^{i_1 \cdots i_\alpha}_{j_1 \cdots j_\alpha}$$

as the reader can verify. The elements of $A_\alpha$ are called **bisymmetric transformations**. Clearly, the operators $\mathbf{T}(g)$ are elements of $A_\alpha$. It is evident that $A_\alpha$ is a vector space since linear combinations of bisymmetric transformations are bisymmetric. Moreover, $A_\alpha$ is an algebra, i.e., if $\mathfrak{A}, \mathfrak{B} \in A_\alpha$ then $\mathfrak{A}\mathfrak{B} \in A_\alpha$.

We now show that the relation between $V^{\otimes\alpha}$ and the symmetric tensors provided by Lemma 4.10 is analogous to the relation between the bisymmetric transformations and the operators $\mathbf{T}(g)$.

**Theorem 4.4.** The set of all operators $\mathbf{T}(g)$, $g \in \tilde{G}_m$, spans $A_\alpha$.

***Proof.*** Designate the matrix elements of $\mathfrak{A} \in A_\alpha$ by $\mathfrak{A}^{i_1 \cdots i_\alpha}_{j_1 \cdots j_\alpha} = \mathfrak{A}_{\mu_1 \cdots \mu_\alpha}$ where the pair of indices $(i_k, j_k)$ is considered as a single index $\mu_k$ which takes $m^2$ values. According to (3.13) we can consider $A_\alpha$ as the subspace of all symmetric tensors in $W^{\otimes\alpha}$, where $\dim W = m^2$. The totality of all operators $\mathbf{T}(g)$ forms a subset of $A_\alpha$ with matrix elements $g^{i_1}_{j_1} \cdots g^{i_\alpha}_{j_\alpha} = g_{\mu_1} \cdots g_{\mu_\alpha}$, where the $m^2$ values $g_\mu$ range over all complex numbers. By Lemma 4.10 the tensors $g_{\mu_1} \cdots g_{\mu_\alpha}$ span the subspace of symmetric tensors in $W^{\otimes\alpha}$. Hence, they span $A_\alpha$. Q.E.D.

We have shown that the algebra of bisymmetric transformations $A_\alpha$ is generated by the operators $\mathbf{T}(g)$, $g \in \tilde{G}_m$. Similarly, we can consider the algebra $B_\alpha$ of operators $\mathbf{x}$ generated by the permutations $s$:

(3.14) $$\mathbf{x}\mathbf{w} = \sum_{s \in S_\alpha} x(s) \cdot \mathbf{s}\mathbf{w}, \qquad x(s) \in \mathfrak{C}, \quad \mathbf{w} \in V^{\otimes\alpha}.$$

Clearly, $B_\alpha$ is a homomorphic image of the group ring $R_\alpha$. The mapping

$$x = \sum x(s) \cdot s \longrightarrow \mathbf{x}$$

is not only a vector space homomorphism, but a ring homomorphism. That is, the product transformation $\mathbf{x}(\mathbf{y}\mathbf{w}) = (\mathbf{x}\mathbf{y})\mathbf{w}$ corresponds to the convolution product for $x, y \in R_\alpha$. The mapping may not be an isomorphism since a nonzero element $x$ in $R_\alpha$ may be mapped into the zero operator on $V^{\otimes\alpha}$.

Any operator on $V^{\otimes\alpha}$ which commutes with all permutations $\mathbf{s}$ must commute with every element of $B_\alpha$. This proves the following theorem.

**Theorem 4.5.** The linear operator $\mathbf{C}$ on $V^{\otimes\alpha}$ commutes with all elements of $B_\alpha$ if and only if $\mathbf{C} \in A_\alpha$.

Let $A$ be an associative algebra with multiplicative identity $e$ and let $W$ be a complex vector space. A **representation** $\mathbf{T}$ of $A$ on $W$ is determined by a set of linear operators $\mathbf{T}(a)$ on $W$ such that

(1) $\mathbf{T}(\gamma a + \mu b) = \gamma\mathbf{T}(a) + \mu\mathbf{T}(b), \quad a, b \in A, \quad \gamma, \mu \in \mathfrak{C};$
(2) $\mathbf{T}(ab) = \mathbf{T}(a)\mathbf{T}(b);$
(3) $\mathbf{T}(e) = \mathbf{E}.$

The notions of reducibility, irreducibility, and equivalence of reps of $A$ are analogous to those for group reps.

The rep of $S_\alpha$ on $V^{\otimes\alpha}$ defined by the operators $\mathbf{s}$ induces a rep of the group ring $R_\alpha$ by the operators in $B_\alpha$. We know that $V^{\otimes\alpha}$ can be decomposed into a direct sum of subspaces such that each subspace is irred under $S_\alpha$.

The reader can verify the following facts: (1) Every rep $\mathbf{T}$ of $S_\alpha$ determines a rep $\mathbf{T}$ of $R_\alpha$. [Set $\mathbf{T}(x) = \sum x(s)\mathbf{T}(s)$.] (2) Every rep of $R_\alpha$ determines a rep of $S_\alpha$. [Restrict the operators $\mathbf{T}(x)$ to $x = 1 \cdot s$.] (3) Equivalent reps $\mathbf{T}, \mathbf{T}'$ of $S_\alpha$ correspond to equivalent reps of $R_\alpha$. [$\mathbf{U}\mathbf{T}(x)\mathbf{U}^{-1} = \mathbf{T}'(x)$ for all $x = \sum x(s)\cdot s$ if and only if $\mathbf{U}\mathbf{T}(s)\mathbf{U}^{-1} = \mathbf{T}'(s)$ for all $s \in S_\alpha$.]

According to the above remarks, the rep of $R_\alpha$ provided by the operators in $B_\alpha$ can also be decomposed into a direct sum of irred reps. The irred subspaces of $V^{\otimes\alpha}$ under $R_\alpha$ are just the irred subspaces under $S_\alpha$. We will show that this decomposition of $V^{\otimes\alpha}$ into a direct sum of irred subspaces induces a similar decomposition for the rep of the algebra $A_\alpha$ defined by (3.12).

Let $D^{(1)}, \ldots, D^{(\mu)}$ be a complete set of nonequivalent irred matrix reps of $S_\alpha$. Then with respect to a suitable basis for $V^{\otimes\alpha}$ the $m^\alpha \times m^\alpha$ matrix corresponding to each permutation $\mathbf{s}$ is

$$
\left(\begin{array}{cccccccccc}
D^{(1)}(s) & a_1 & & & & & & & & \\
& \ddots & & & & & & Z & & \\
& & D^{(1)}(s) & & & & & & & \\
& & & D^{(2)}(s) & a_2 & & & & & \\
& & & & \ddots & & & & & \\
& & & & & D^{(2)}(s) & & & & \\
& & & & & & \ddots & & & \\
& & & & & & & D^{(\mu)}(s) & a_\mu & \\
& Z & & & & & & & \ddots & \\
& & & & & & & & & D^{(\mu)}(s)
\end{array}\right) \tag{3.15}
$$

That is, the permutation rep decomposes into a direct sum of irred reps,

$$a_1 D^{(1)} \oplus a_2 D^{(2)} \oplus \cdots \oplus a_\mu D^{(\mu)}.$$

The matrices corresponding to $A_\alpha$ are just those which commute with the matrices (3.15) for all $s \in S_\alpha$. It is instructive to work out a simple example.

Suppose the matrices (3.15) take the form

$$(3.16)\qquad s = \begin{array}{cc} \begin{array}{ccc} \quad n_1 \quad & \quad n_2 \quad & \quad n_3 \quad \end{array} & \\ \begin{pmatrix} D^{(1)}(s) & Z & Z \\ Z & D^{(2)}(s) & Z \\ Z & Z & D^{(2)}(s) \end{pmatrix} & \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \end{array}$$

with $n_1 = \dim D^{(1)}$, $n_2 = n_3 = \dim D^{(2)}$. The matrix of a bisymmetric transformation $\mathfrak{A}$ can be written

$$(3.17)\qquad \mathfrak{A} = \begin{pmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} & \mathfrak{A}_{13} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} & \mathfrak{A}_{23} \\ \mathfrak{A}_{31} & \mathfrak{A}_{32} & \mathfrak{A}_{33} \end{pmatrix}$$

where $\mathfrak{A}_{jk}$ is an $n_j \times n_k$ matrix. The condition $\mathfrak{A} \in A_\alpha$ is just that the matrices $\mathfrak{A}$ and $s$ commute for all $s \in S_\alpha$. Now

$$(3.18)\qquad \begin{aligned} \mathfrak{A}s &= \begin{pmatrix} \mathfrak{A}_{11}D^{(1)}(s) & \mathfrak{A}_{12}D^{(2)}(s) & \mathfrak{A}_{13}D^{(2)}(s) \\ \mathfrak{A}_{21}D^{(1)}(s) & \mathfrak{A}_{22}D^{(2)}(s) & \mathfrak{A}_{23}D^{(2)}(s) \\ \mathfrak{A}_{31}D^{(1)}(s) & \mathfrak{A}_{32}D^{(2)}(s) & \mathfrak{A}_{33}D^{(2)}(s) \end{pmatrix} \\ s\mathfrak{A} &= \begin{pmatrix} D^{(1)}(s)\mathfrak{A}_{11} & D^{(1)}(s)\mathfrak{A}_{12} & D^{(1)}(s)\mathfrak{A}_{13} \\ D^{(2)}(s)\mathfrak{A}_{21} & D^{(2)}(s)\mathfrak{A}_{22} & D^{(2)}(s)\mathfrak{A}_{23} \\ D^{(2)}(s)\mathfrak{A}_{31} & D^{(2)}(s)\mathfrak{A}_{32} & D^{(2)}(s)\mathfrak{A}_{33} \end{pmatrix} \end{aligned}$$

The requirement $\mathfrak{A}s = s\mathfrak{A}$ leads to a series of relations of the type

$$\mathfrak{A}_{11}D^{(1)}(s) = D^{(1)}(s)\mathfrak{A}_{11}, \qquad \mathfrak{A}_{12}D^{(2)}(s) = D^{(1)}(s)\mathfrak{A}_{12}.$$

Since $D^{(1)}$ and $D^{(2)}$ are nonequivalent irred reps of $S_\alpha$, the Schur lemmas, Section 3.3, imply $\mathfrak{A}_{12} = Z$ and $\mathfrak{A}_{11} = \lambda_{11}E_{n_1}$, where $\lambda_{11}$ is any complex number and $E_{n_1}$ is the $n_1 \times n_1$ identity matrix. These considerations lead to the result

$$(3.19)\qquad \mathfrak{A} = \begin{pmatrix} \lambda_{11}E_{n_1} & Z & Z \\ Z & \lambda_{22}E_{n_2} & \lambda_{23}E_{n_2} \\ Z & \lambda_{32}E_{n_2} & \lambda_{33}E_{n_2} \end{pmatrix}, \qquad \lambda_{jk} \in \mathfrak{C}.$$

A simple rearrangement of rows and columns yields the matrix realization

$$(3.20)\qquad \mathfrak{A} = \begin{pmatrix} \lambda_{11} & n_1 & & & & \\ & \ddots & & & Z & \\ & & \lambda_{11} & & & \\ & & & \begin{pmatrix} \lambda_{22} & \lambda_{23} \\ \lambda_{32} & \lambda_{33} \end{pmatrix} & & n_2 \\ & & & & \ddots & \\ & Z & & & & \begin{pmatrix} \lambda_{22} & \lambda_{23} \\ \lambda_{32} & \lambda_{33} \end{pmatrix} \end{pmatrix}.$$

We shall see that (3.20) is an explicit decomposition of the rep of $A_\alpha$ on $V^{\otimes\alpha}$ into irred reps. Note that the multiplicities $n_1, n_2$ of the matrix blocks in (3.20) are just the dimensions of $D^{(1)}$ and $D^{(2)}$, while the multiplicities 1, 2 of $D^{(1)}$ and $D^{(2)}$ are the dimensions of the matrix blocks in (3.20).

The general case is now clear. If the permutation rep of $S_\alpha$ decomposes in the form (3.15),

$$s \sim a_1 D^{(1)}(s) \oplus \cdots \oplus a_\mu D^{(\mu)}(s),$$

where $a_j$ is the multiplicity of $D^{(j)}$ and $n_j = \dim D^{(j)}$, then the elements of $A_\alpha$ are all those of the form

$$\mathfrak{A} \sim n_1 C^{(1)}(\mathfrak{A}) \oplus \cdots \oplus n_\mu C^{(\mu)}(\mathfrak{A}) \tag{3.21}$$

where $C^{(k)}(\mathfrak{A})$ runs over all $a_k \times a_k$ matrices as $\mathfrak{A}$ runs over $A_\alpha$. The matrix block $C^{(k)}(\mathfrak{A})$ occurs $n_k$ times along the diagonal in the matrix expression for $\mathfrak{A}$ analogous to (3.20). Evidently, each of the matrix blocks $C^{(k)}$ is itself a matrix rep of $A_\alpha$. Furthermore, this rep is irred. Indeed any $a_k \times a_k$ matrix $B$ with the property $BC^{(k)}(\mathfrak{A}) = C^{(k)}(\mathfrak{A})B$ for all $\mathfrak{A} \in A_\alpha$ must be a multiple of $E_{a_k}$, since the $C^{(k)}(\mathfrak{A})$ run over *all* $a_k \times a_k$ matrices.

The irred reps $C^{(j)}$, $C^{(k)}$ for $j \neq k$ must be nonequivalent because $C^{(j)}(\mathfrak{A})$ and $C^{(k)}(\mathfrak{A})$ run over all $a_j \times a_j$ and $a_k \times a_k$ matrices completely independent of one another.

**Theorem 4.6.** The algebra $A_\alpha$ of bisymmetric transformations acting on $V^{\otimes\alpha}$ can be decomposed into the direct sum (3.21) of irred reps $C^{(k)}$.

According to Theorem 4.4 the irred matrix reps $C^{(k)}(\mathfrak{A})$ must remain irred when $\mathfrak{A}$ is restricted to elements of the form $\mathbf{T}(g), g \in \tilde{G}_m$. This is because an arbitrary $\mathfrak{A} \in A_\alpha$ can be written in the form $\mathfrak{A} = \sum \beta_i \mathbf{T}(g_i)$, $g_i \in \tilde{G}_m$. If the restriction of $C^{(k)}$ to $\tilde{G}_m$ were reducible then the property $C^{(k)}(\mathfrak{A}) = \sum \beta_i C^{(k)}(g_i)$ would imply that $C^{(k)}$ was itself reducible. [For simplicity we write $C^{(k)}(g)$ for $C^{(k)}(T(g))$.]

If there were a nonsingular matrix $B$ such that $C^{(k)}(g) = BC^{(j)}(g)B^{-1}$ for some $j \neq k$ and all $g \in \tilde{G}_m$, then by the argument in the preceding paragraph, $C^{(k)}$ and $C^{(j)}$ would be equivalent reps of $A_\alpha$. Since this is false, the restrictions of the $C^{(k)}$ to $\tilde{G}_m$ remain nonequivalent. Finally, we can restrict the $C^{(k)}$ to $G_m = GL(m, \mathfrak{C})$, or more precisely to the elements $\mathbf{T}(g)$, $g \in G_m$. The matrix elements of $C^{(k)}(g)$ are homogeneous polynomials of order $\alpha$ in the $g_i{}^j$. Let $B$ be an $a_k \times a_k$ matrix such that $BC^{(k)}(g) = C^{(k)}(g)B$ for all $g \in G_m$. This relation leads to a number of identities between the matrix elements of $C^{(k)}(g)$ which remain valid for singular matrices $g \in \tilde{G}_m$. However, the restriction of $C^{(k)}$ to $\tilde{G}_m$ is irred, so $B$ is a multiple of the identity and the $C^{(k)}(g)$ define an irred rep of $G_m$. A similar argument shows that $C^{(1)}, \ldots, C^{(\mu)}$ yield nonequivalent irred reps of $G_m$.

**Theorem 4.7.** The rep $\mathbf{T}$ of $GL(m, \mathfrak{C})$ on $V^{\otimes\alpha}$ can be decomposed into a direct sum of irred reps

$$\mathbf{T} \cong n_1 C^{(1)} \oplus \cdots \oplus n_\mu C^{(\mu)} \tag{3.22}$$

analogous to the decomposition

$$a_1 D^{(1)} \oplus \cdots \oplus a_\mu D^{(\mu)}$$

of the permutation rep of $S_\alpha$ on $V^{\otimes\alpha}$. Here $a_k = \dim C^{(k)}$ and $n_k = \dim D^{(k)}$.

It will be shown in Chapter 9 that the decomposition (3.22) is essentially unique, i.e., the irred reps $C^{(k)}$ occurring in the decompositon and their multiplicities are uniquely determined. This does not follow from the results of Chapter 3, since $GL(m, \mathfrak{C})$ is not a finite group. We could prove the uniqueness directly at this point by making use of the rep theory of complete matrix algebras, but the proof will be deferred to save space.

A proof of the following theorem will also be deferred to Section 9.1. (See Boerner [1, p. 137] for a direct proof.)

**Theorem 4.8.** Let $W$ be a subspace of $V^{\otimes\alpha}$ which is invariant under the rep $\mathbf{T}$ of $G_m$. Then there exists an invariant subspace $W'$ such that $V^{\otimes\alpha} = W \oplus W'$.

It is clear that $A_\alpha$ consists of all linear transformations that commute with every $\mathbf{x} \in B_\alpha$. On the other hand, we have the following result.

**Theorem 4.9.** $B_\alpha$ consists of all linear transformations on $V^{\otimes\alpha}$ that commute with each $\mathfrak{A} \in A_\alpha$.

The major steps in the proof of this theorem are provided by the following lemmas, which are of independent interest.

**Lemma 4.11.** Let $D$ be an irred $n \times n$ matrix rep of the finite group $H$. Then the matrices $D(h)$, $h \in H$, span the $n^2$-dimensional space of all $n \times n$ matrices, i.e., every matrix can be expressed as a linear combination of the matrices $D(h)$.

***Proof.*** If the matrices $D(h)$ do not span an $n^2$-dimensional space there must exist a relation

$$\sum_{i,j=1}^{n} c_{ij} D_{ij}(h) = 0, \qquad \text{all} \quad h \in H, \tag{3.23}$$

among the matrix elements of $D(h)$ which is satisfied for constants $c_{ij}$ not all zero. However, the orthogonality relations (3.7), Section 3.3, and (3.23) lead to

$$0 = n(H)^{-1} \sum_{h \in H} [\sum_{i,j} c_{ij} D_{ij}(h)]\, D_{lm}(h^{-1}) = c_{ml}/n$$

so $c_{ml} = 0$ for $m, l = 1, \ldots, n$. Q.E.D.

According to this result the matrix rep of the group ring $R_H$ determined by $D$ has the property that $D(x)$ runs over *all* $n \times n$ matrices as $x = \sum x(h) \cdot h$ runs over $R_H$. Indeed, $D(x) = \sum x(h) D(h)$ and the $x(h)$ range over all complex numbers as $x$ runs over $R_H$.

Let $H$ be a finite group and

$$R_H = U_1 \oplus U_2 \oplus \cdots \oplus U_\gamma$$

the decomposition of $R_H$ into minimal two-sided ideals as described at the end of Section 3.7. The ideal $U_\nu$ corresponds to the irred rep $\mathbf{D}^{(\nu)}$ of $H$. Indeed, under the left regular rep, $U_\nu$ decomposes into a direct sum of $n_\nu$ left ideals, each left ideal transforming according to $\mathbf{D}^{(\nu)}$. Let $\mathbf{D}$ be a rep of $R_H$ on the vector space $W$ and let $W_\nu$ be an irred subspace of $W$ such that $\mathbf{D} \mid W_\nu \cong \mathbf{D}^{(\nu)}$. (Recall that every irred rep of $R_H$ remains irred when restricted to $H$.)

**Lemma 4.12.** If $\nu \neq \mu$, then $\mathbf{D}(y)\mathbf{w} = \boldsymbol{\theta}$ for all $\mathbf{w} \in W_\nu$ and all $y \in U_\mu$.

***Proof.*** Let $W_\nu'$ be the subspace of $W_\nu$ spanned by all vectors of the form $\mathbf{D}(x)\mathbf{w}$, $x \in U_\nu$, $\mathbf{w} \in W_\nu$. Since $W_\nu$ is irred, either $W_\nu' = \{\boldsymbol{\theta}\}$ or $W_\nu' = W_\nu$. We shall show $W_\nu' \neq \{\boldsymbol{\theta}\}$.

From Section 3.7, $\mathbf{P}_\nu \mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in W_\nu$, where the projection operator is

$$\mathbf{P}_\nu = \frac{n_\nu}{n(H)} \sum_{h \in H} \overline{\chi^{(\nu)}(h)}\, \mathbf{D}(h) = \mathbf{D}(p_\nu), \tag{3.24}$$

$$p_\nu = \frac{n_\nu}{n(H)} \sum_{h \in H} \overline{\chi^{(\nu)}(h)} \cdot h \in R_H. \tag{3.25}$$

If we assume that the matrices $D^{(\nu)}(h)$ are unitary, the relation

$$R(g)\bar{D}^{(\nu)}_{jk}(h) = \bar{D}^{(\nu)}_{jk}(hg^{-1}) = \sum_l D^{(\nu)}_{kl}(g)\bar{D}^{(\nu)}_{jl}(h)$$

implies that the ring elements $\bar{D}^{(\nu)}_{jl} = \sum \bar{D}^{(\nu)}_{jl}(h) \cdot h$ for $1 \leq l \leq n_\nu$ and fixed $j$ form a basis for the rep $\mathbf{D}^{(\nu)}$. Thus each of these ring elements lies in $U_\nu$ and it follows that $p_\nu \in U_\nu$. Therefore, $W_\nu' = W_\nu$.

If $\mathbf{y} \in U_\mu$, $\mu \neq \nu$, and $x \in U_\nu$ then $yx \in U_\mu \cap U_\nu = \{0\}$, so $\mathbf{D}(y)\mathbf{D}(x)\mathbf{w} = \mathbf{D}(yx)\mathbf{w} = \boldsymbol{\theta}$ for all $\mathbf{w} \in W_\nu$. But then $\mathbf{D}(y)\mathbf{w}' = \boldsymbol{\theta}$ for all $\mathbf{w}' \in W_\nu$ since $\mathbf{w}'$ can be expressed as a finite linear combination of elements of the form $\mathbf{D}(x)\mathbf{w}$. Q.E.D.

Now we turn to the proof of Theorem 4.9. To clarify the argument we consider the example given by expressions (3.17)–(3.20). Suppose the bisymmetric transformations have the matrix realization (3.19). To find all operators which commute with every element of $A_\alpha$ it is enough to determine those matrices $B$ that commute with all matrices (3.19). The result is easily shown to be

$$B = \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_2 \end{array} \\ \begin{pmatrix} B^{(1)} & Z & Z \\ Z & B^{(2)} & Z \\ Z & Z & B^{(2)} \end{pmatrix} \end{array} \begin{array}{c} \\ n_1 \\ n_2 \\ n_2 \end{array} \tag{3.26}$$

where $B^{(1)}$ and $B^{(2)}$ independently range over all $n_1 \times n_1$ matrices, $n_2 \times n_2$ matrices, respectively. Comparing (3.26) with (3.17) and using the lemmas we see that each matrix $B$ corresponds to an element of $B_\alpha$. Indeed as $x = \sum x(s)\cdot s$ ranges over $R_\alpha$ the matrices $D^{(i)}(x) = \sum x(s)D^{(i)}(s)$, $i = 1, 2$, range over all $n_i \times n_i$ matrices $B^{(i)}$. Moreover, according to Lemma 4.12 the matrices $B^{(1)}$ and $B^{(2)}$ are independent. That is, given any two matrices $B^{(1)}$ and $B^{(2)}$ there is an $x \in R_\alpha$ with $D^{(1)}(x) = B^{(1)}$ and $D^{(2)}(x) = B^{(2)}$. Thus the matrix $B$, (3.26), corresponds to an element of $B_\alpha$. The argument for the general case proceeds exactly as in our example. Q.E.D.

We now resume the analysis of the rep $\mathbf{T}$ of $G_m = GL(m, \mathfrak{C})$ on $V^{\otimes\alpha}$. Our previous results have shown that $\mathbf{T}$ can be decomposed into a direct sum of irred reps and that this decomposition is closely related to the decomposition of $V^{\otimes\alpha}$ into subspaces irred under the permutation rep of $S_\alpha$. However, these results are of a theoretical character and do not lend themselves to a practical method for decomposing tensor reps of $G_m$.

Theorem 4.9 provides us with the proper tool to obtain such a practical decomposition. Let $W_1$ be a subspace of $V^{\otimes\alpha}$ which is invariant under $\mathbf{T}$. According to Theorem 4.8 there exists a $\mathbf{T}$-invariant subspace $W_2$ of $V^{\otimes\alpha}$ such that $V^{\otimes\alpha} = W_1 \oplus W_2$. Let $\mathbf{P}$ be the projection operator on $W_1$ defined by $R_\mathbf{P} = W_1$, $N_\mathbf{P} = W_2$. From the results of Section 3.7, $\mathbf{T}(g)\mathbf{P} = \mathbf{P}\mathbf{T}(g)$ for all $g \in G_m$, hence $\mathbf{P}$ commutes with all elements of $A_\alpha$. Theorem 4.9 yields the important conclusion: $\mathbf{P} \in B_\alpha$.

We can immediately apply Theorems 3.10–3.12 to the rep $\mathbf{T}$. Let $P(\mathbf{T})$ be the set of all projection operators on $V^{\otimes\alpha}$ that commute with the operators $\mathbf{T}(g)$.

**Theorem 4.10.** (1) There is a 1–1 relationship between projections $\mathbf{P}$ in $P(\mathbf{T})$ and decompositions $V^{\otimes\alpha} = W_1 \oplus W_2$ into $\mathbf{T}$-invariant subspaces, given by $R_\mathbf{P} = W_1$, $N_\mathbf{P} = W_2$.

(2) If $\mathbf{P} \in P(\mathbf{T})$ then $\mathbf{P} \in B_\alpha$. Conversely, if $\mathbf{Q} \in B_\alpha$ and $\mathbf{Q}^2 = \mathbf{Q} \neq \mathbf{Z}$ then $\mathbf{Q} \in P(\mathbf{T})$.

(3) Let $W_1$ be a T-invariant subspace of $V^{\otimes\alpha}$ and let $\mathbf{P} \in P(\mathbf{T})$ be a projection operator on $W_1$. Then $W_1$ is irred if and only if there do not exist nonzero operators $\mathbf{P}_1, \mathbf{P}_2 \in P(\mathbf{T})$ such that $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ and $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{Z}$. [Recall that the projection operators $\mathbf{P}$ corresponding to irred subspaces $W_1$ are the elements of the set $IP(\mathbf{T})$.]

This theorem implies that any $T$-irred subspace $W$ of $V^{\otimes\alpha}$ is given by

$$W = \{\mathbf{xv} : \mathbf{v} \in V^{\otimes\alpha}\} \tag{3.27}$$

where $\mathbf{x}$ is a primitive idempotent in the algebra $B_\alpha$. (Usually $\mathbf{x}$ is not uniquely determined by $W$.) Conversely, each primitive idempotent $\mathbf{x}$ in $B_\alpha$ uniquely determines a T-irred subspace $W$ by (3.27).

Next we investigate the relationship between $B_\alpha$ and the group ring $R_\alpha$. To clarify the discussion we introduce the notation $\mathbf{D}(x) = \mathbf{x}$, $x \in R_\alpha$, for the elements (3.14) of $B_\alpha$. The $\mathbf{D}(x)$ define a rep of $R_\alpha$ which may not be faithful. That is, we may have $\mathbf{D}(x) = \mathbf{D}(y)$ for $x \neq y$. Let $\mathcal{O}_\alpha$ be the set of all $x \in R_\alpha$ such that $\mathbf{D}(x) = \mathbf{Z}$.

**Lemma 4.13.** $\mathcal{O}_\alpha$ is a two-sided ideal in $R_\alpha$. There exists a two-sided ideal $\mathcal{R}_\alpha$ in $R_\alpha$ such that

$$R_\alpha = \mathcal{R}_\alpha \oplus \mathcal{O}_\alpha, \tag{3.28}$$

and the map $x \longrightarrow D(x)$ of $\mathcal{R}_\alpha$ into $B_\alpha$ is 1–1 and onto.

***Proof.*** If $x \in \mathcal{O}_\alpha$, $y \in R_\alpha$, then

$$\mathbf{D}(yx) = \mathbf{D}(y)\mathbf{D}(x) = \mathbf{Z}, \qquad \mathbf{D}(xy) = \mathbf{D}(x)\mathbf{D}(y) = \mathbf{Z}$$

since $\mathbf{D}(x) = \mathbf{Z}$. Thus $yx, xy \in \mathcal{O}_\alpha$ and $\mathcal{O}_\alpha$ is a two-sided ideal.

According to the results at the end of Section 3.7 the group ring can be expressed as a direct sum of minimal two-sided ideals

$$R_\alpha = U_1 \oplus \cdots \oplus U_k \tag{3.29}$$

where $k$ is the number of conjugacy classes in $S_\alpha$ and the number of partitions (frames) $\{\lambda_1, \ldots, \lambda_\alpha\}$ of $\alpha$. Taking the frames $\{\lambda_j\}$ in dictionary order, we label the two-sided ideals such that $U_i$ consists of those minimal left ideals (irred subspaces under $\mathbf{L}$) that correspond to the $i$th frame. The ideal $\mathcal{O}_\alpha$ can be written as a direct sum of minimal left ideals. It follows from the last paragraph of Section 3.7 that if $\mathcal{O}_\alpha$ contains a minimal left ideal corresponding to the $i$th frame then $\mathcal{O}_\alpha$ contains all minimal left ideals corresponding to the $i$th frame, i.e., $U_i \subseteq \mathcal{O}_\alpha$. Thus,

$$\mathcal{O}_\alpha = U_{j_1} \oplus U_{j_2} \oplus \cdots \oplus U_{j_t} \tag{3.30}$$

where $1 \leq j_1 < j_2 < \cdots < j_l \leq k$. Choosing integers $j_{l+1} < \cdots < j_k$ such that $\{j_1, \ldots, j_k\}$ is a permutation of $\{1, \ldots, k\}$ we see

$$R_\alpha = (U_{j_1} \oplus \cdots \oplus U_{j_l}) \oplus (U_{j_{l+1}} \oplus \cdots \oplus U_{j_k}) = \mathcal{O}_\alpha \oplus \mathcal{R}_\alpha$$

where

$$\mathcal{R}_\alpha = U_{j_{l+1}} \oplus \cdots \oplus U_{j_k} \tag{3.31}$$

is a two-sided ideal.

Let $x_1, x_2 \in \mathcal{R}_\alpha$ such that $\mathbf{D}(x_1) = \mathbf{D}(x_2)$. Then $\mathbf{D}(x_1 - x_2) = \mathbf{Z}$ so $x_1 - x_2 \in \mathcal{O}_\alpha$. Thus, $x_1 - x_2 \in \mathcal{O}_\alpha \cap \mathcal{R}_\alpha = \{0\}$ or $x_1 = x_2$. This proves that the map $x \longrightarrow \mathbf{D}(x)$ is an isomorphism of $\mathcal{R}_\alpha$ and $B_\alpha$. Q.E.D.

Because of the isomorphism between $\mathcal{R}_\alpha$ and $B_\alpha$ we can identify the operator $\mathbf{D}(x) \in B_\alpha$ with $x \in \mathcal{R}_\alpha$. Thus, there is a 1–1 correspondence between projection operators $\mathbf{P} \in IP(\mathbf{T})$ and primitive idempotents $c$ in $\mathcal{R}_\alpha$, given by $\mathbf{P} = \mathbf{D}(c)$. In the previous section we have already discussed the determination of primitive idempotents in $R_\alpha$. Our analysis of symmetry classes of tensors will be complete if we can develop a practical method for determining $\mathcal{R}_\alpha$ and for decomposing $V^{\otimes\alpha}$ into a direct sum of $\mathbf{T}$-irred subspaces $W_i$,

$$V^{\otimes\alpha} = W_1 \oplus \cdots \oplus W_\gamma.$$

The basic problem remaining to be solved is this: How do we choose the primitive idempotents $c_i \in \mathcal{R}_\alpha$ such that $W_i = \{\mathbf{D}(c_i)\mathbf{w}: \mathbf{w} \in V^{\otimes\alpha}\}$? A clue to the solution of this problem is the observation that there is a 1–1 relationship between $\mathbf{T}$-invariant subspaces $W$ of $V^{\otimes\alpha}$ and *right* ideals $\mathcal{I}$ in $\mathcal{R}_\alpha$.

Indeed, let $W$ be a nontrivial $\mathbf{T}$-invariant subspace and let

$$\mathcal{I}_W = \{x \in \mathcal{R}_\alpha : x\mathbf{v} \in W \quad \text{for all} \quad \mathbf{v} \in V^{\otimes\alpha}\} \tag{3.32}$$

[Recall that $\mathbf{D}(x)\mathbf{v} = x\mathbf{v}$.] If $x \in \mathcal{I}_W$ and $y \in \mathcal{R}_\alpha$ then $(xy)\mathbf{v} = x(y\mathbf{v}) \in W$ for all $\mathbf{v}$, so $xy \in \mathcal{I}_W$ and $\mathcal{I}_W$ is a right ideal. Moreover, if $\mathbf{P} = \mathbf{D}(z)$, $z \in \mathcal{R}_\alpha$, is a projection operator on $W$ which commutes with the $\mathbf{T}(g)$ operators, then $z \in \mathcal{I}_W$, so $\mathcal{I}_W \neq \{0\}$.

On the other hand, given a nontrivial right ideal $\mathcal{I}$ in $\mathcal{R}_\alpha$ we can define a $\mathbf{T}$-invariant subspace $W_\mathcal{I}$ of $V^{\otimes\alpha}$ generated by all elements of the form $x\mathbf{v}$, $x \in \mathcal{I}$, $\mathbf{v} \in V^{\otimes\alpha}$.

Before proceeding with this analysis we remark that the basic facts concerning left ideals proved in Section 3.7 have an obvious modification for right ideals in a group ring $R_G$. A right ideal is just an invariant subspace of the group ring under the right regular rep. If $x \in R_G$ then the set $xR_G$ is a right ideal. Conversely, every right ideal is of this form. If $\mathcal{I}$ is a right ideal then there is a **generating idempotent** $c \in R_G$ (not unique) such that

$\mathcal{I} = cR_G$. Also, $\mathcal{I}$ is a minimal right ideal if and only if $c$ is a primitive idempotent.

Let $\mathcal{I}$ be a right ideal in $\mathcal{R}_\alpha$ and $W$ a **T**-invariant subspace of $V^{\otimes\alpha}$.

**Theorem 4.11.** (1) $W_{(\mathcal{I}_W)} = W$.

(2) $\mathcal{I}_{(W_\mathcal{I})} = \mathcal{I}$.

That is, the relation between right ideals and **T**-invariant subspaces defined above is 1–1.

(3) Let $\mathcal{I} = \mathcal{I}_W$. Then $\mathcal{I}$ is minimal if and only if $W$ is **T**-irred.

(4) Let $\mathcal{I} = \mathcal{I}_W$, $\mathcal{I}' = \mathcal{I}_{W'}$. Then $\mathcal{I}$ and $\mathcal{I}'$ are equivalent right ideals in $\mathcal{R}_\alpha$ if and only if the **T**-invariant subspaces $W$ and $W'$ define equivalent reps of $G_m$.

***Proof.*** (1) Let $c$ be the generating idempotent of $\mathcal{I}_W$. Then $W = \mathbf{c}V^{\otimes\alpha}$ since every $x \in \mathcal{I}_W$ can be written $x = cx$. Now $W_{(\mathcal{I}_W)}$ is the space generated by all elements of the form $\mathbf{xv} = \mathbf{cxv}$, $x \in \mathcal{I}_W$, $\mathbf{v} \in V^{\otimes\alpha}$, which is obviously $\mathbf{c}V^{\otimes\alpha}$.

(2) If $c$ is a generating idempotent of $\mathcal{I}$, then $W_\mathcal{I} = \mathbf{c}V^{\otimes\alpha}$. If $x \in \mathcal{I}$, $\mathbf{v} \in V^{\otimes\alpha}$, then $\mathbf{xv} = (\mathbf{cx})\mathbf{v} = \mathbf{c}(\mathbf{xv}) \in W_\mathcal{I}$, so $\mathcal{I} \subseteq \mathcal{I}_{(W_\mathcal{I})}$. Conversely, suppose $y \in \mathcal{I}_{(W_\mathcal{I})}$. Then $\mathbf{yv} \in W_\mathcal{I}$, so $\mathbf{yv} = \mathbf{c}(\mathbf{yv}) = (\mathbf{cy})\mathbf{v}$ for all $\mathbf{v} \in V^{\otimes\alpha}$. This shows $y = cy \in \mathcal{I}$, so $\mathcal{I} = \mathcal{I}_{(W_\mathcal{I})}$.

(3) This follows immediately from Theorem 3.13 and the fact that $\mathbf{D}(c)$ is a projection on a **T**-irred subspace $W$ if and only if $c$ is a primitive idempotent.

(4) By property (3) it is sufficient to prove this assertion for primitive right ideals $\mathcal{I}, \mathcal{I}'$. Suppose $\mathcal{I}$ and $\mathcal{I}'$ are equivalent ideals with primitive generating idempotents $c, c'$. Then there is a nonzero $x = c'xc \in \mathcal{I}'$ such that $\mathcal{I}' = x\mathcal{I} = x\mathcal{R}_\alpha$. Thus, $W = \mathbf{c}V^{\otimes\alpha}$ and $W' = \mathbf{c}'V^{\otimes\alpha} = \mathbf{x}V^{\otimes\alpha} = \mathbf{xc}V^{\otimes\alpha} = \mathbf{D}(x)W$. Since $W$ and $W'$ are **T**-irred and the nonzero operator $\mathbf{D}(x)$ from $W$ to $W'$ commutes with the $\mathbf{T}(g)$, $W$ and $W'$ define equivalent reps. Conversely, if $W$ is equivalent to $W'$ there is a nonzero mapping $\mathbf{A}$ of $W$ onto $W'$ which commutes with the $\mathbf{T}(g)$. Let $W''$ be a **T**-invariant subspace such that $V^{\otimes\alpha} = W \oplus W''$. We can extend $\mathbf{A}$ to $V^{\otimes\alpha}$ by requiring $\mathbf{A}(\mathbf{w} + \mathbf{w}'') = \mathbf{Aw}$ for all $\mathbf{w} \in W$, $\mathbf{w}'' \in W''$. Clearly, the extended operator $\mathbf{A}$ commutes with the $\mathbf{T}(g)$, so $\mathbf{A} \in B_\alpha$ and $\mathbf{A} = \mathbf{D}(x)$, $x \in \mathcal{R}_\alpha$. This shows that $W' = \mathbf{A}W = \mathbf{x}W$. Since $\mathcal{I} = \{y \in \mathcal{R}_\alpha : \mathbf{y}V^{\otimes\alpha} \subseteq W\}$ with a similar definition for $\mathcal{I}'$, it follows that $\{0\} \subset x\mathcal{I} \subseteq \mathcal{I}'$. But $\mathcal{I}$ and $\mathcal{I}'$ are minimal, so $\mathcal{I}' = x\mathcal{I}$ and the ideals are equivalent. Q.E.D.

We can now explicitly decompose $V^{\otimes\alpha}$ into **T**-irred subspaces. Let $\mathcal{R}_\alpha = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_q$ be a decomposition of $\mathcal{R}_\alpha$ into minimal right ideals. Suppose $\mathcal{I}_j = c_j\mathcal{R}_\alpha$, $1 \leq j \leq q$, where the generators $c_j$ are primitive idempotents.

Then

$$V^{\otimes\alpha} = W_1 \oplus \cdots \oplus W_q \tag{3.34}$$

where $W_j = \mathbf{c}_j V^{\otimes\alpha}$ is a **T**-irred subspace. Indeed it is evident from Theorem 4.10 that the sum $W_1 \oplus \cdots \oplus W_q$ is direct. The spaces $W_1, \ldots, W_q$ span $V^{\otimes\alpha}$ since the identity operator **E** belongs to $B_\alpha \cong \mathcal{R}_\alpha$. Thus, $\mathbf{E} = \mathbf{D}(x)$, $x \in \mathcal{R}_\alpha$, where $x$ has the unique decomposition

$$x = x_1 + \cdots + x_q = c_1 x_1 + \cdots + c_q x_q,$$

$x_j \in \mathcal{I}_j$. If $\mathbf{v} \in V^{\otimes\alpha}$, then

$$\mathbf{v} = \mathbf{E}\mathbf{v} = \mathbf{x}\mathbf{v} = \mathbf{c}_1(\mathbf{x}_1\mathbf{v}) + \cdots + \mathbf{c}_q(\mathbf{x}_q\mathbf{v})$$

where $\mathbf{c}_j(\mathbf{x}_j\mathbf{v}) \in W_j$. The **T**-irred subspaces $W_j$ are called **symmetry classes** of tensors.

To conclude our analysis we determine the relationship between Theorems 4.7 and 4.11. The first of these theorems decomposes **T** in terms of minimal **left** ideals of $R_\alpha$, while the latter decomposition is in terms of minimal **right** ideals. In particular the **T**-irred subspaces $W_j$ are not in general invariant under the operators $\mathbf{D}(s)$, $s \in S_\alpha$.

Let us fix our attention on one of the **T**-irred subspaces $W_j = \mathbf{c}_j V^{\otimes\alpha}$ in (3.34). According to Theorem 4.7 the restriction of **T** to $W_j$ is equivalent to the irred rep $\mathbf{C}^{(\beta)}$ of $G_m$. Let $\mathbf{D}^{(\beta)}$ be the corresponding rep of $S_\alpha$ determined by this theorem and let $U_\beta$ be the minimal two-sided ideal in $R_\alpha$ consisting of all minimal left ideals (**L**-irred subspaces) that transform according to $\mathbf{D}^{(\beta)}$. By Lemma 4.12, $\mathbf{D}(x)\mathbf{w} = \boldsymbol{\theta}$ for all $x \in U_\nu$, $\nu \neq \beta$, and all $\mathbf{w} \in W_j$. Let $\mathcal{I}$ be the minimal right ideal in $\mathcal{R}_\alpha$ associated with $W_j$:

$$\mathcal{I} = \{x \in \mathcal{R}_\alpha : \mathbf{x}\mathbf{v} \in W_j \quad \text{for all} \quad \mathbf{v} \in V^{\otimes\alpha}\}$$

Clearly, $\mathbf{x}W_j \neq \{\boldsymbol{\theta}\}$ for nonzero $x \in \mathcal{I}$, so the minimal two-sided ideal in $\mathcal{R}_\alpha$ containing $\mathcal{I}$ must be $U_\beta$. Thus, the minimal left and right ideals associated with $W_j$ lie in the same two-sided ideal $U_\beta$.

Let $D^{(\beta)}$ be a unitary matrix rep corresponding to the operator rep $\mathbf{D}^{(\beta)}$. Then the matrix elements $\bar{D}^{(\beta)}_{jk}(s)$ considered as element of $R_\alpha$ satisfy the relations

$$L(t)\bar{D}^{(\beta)}_{jk}(s) = \bar{D}^{(\beta)}_{jk}(t^{-1}s) = \sum_{l=1}^{n_\beta} D^{(\beta)}_{lj}(t)\bar{D}^{(\beta)}_{lk}(s), \qquad 1 \le j, k \le n_\beta. \tag{3.35}$$

Thus, the ring elements $\bar{D}^{(\beta)}_{jk} = \sum \bar{D}^{(\beta)}_{jk}(s)\cdot s$ for fixed $k$ generate a minimal left ideal in $U_\beta$, and the totality of these $n_\beta{}^2$ elements form a basis for $U_\beta$. The minimal right ideals in $U_\beta$ all transform irreducibly under the right regular rep **R**, and the irred reps determined by these right ideals are equivalent. We will identify this irred rep. A simple computation yields

$$R(t)\bar{D}^{(\beta)}_{jk}(s) = \bar{D}^{(\beta)}_{jk}(st) = \sum_{l=1}^{n_\beta} \bar{D}^{(\beta)}_{lk}(t)\bar{D}^{(\beta)}_{jl}(s), \qquad 1 \le j, k \le n_\beta. \tag{3.36}$$

Thus, for fixed $j$ the ring elements $\bar{D}_{jk}^{(\beta)}$ form a basis for a right ideal transforming according to $\bar{D}^{(\beta)}$, i.e., the matrices of this rep are the complex conjugates $\bar{D}^{(\beta)}(t)$. The character $\bar{\chi}^{(\beta)}$ of this rep is the complex conjugate of the simple character $\chi^{(\beta)}$. Since $\langle\bar{\chi}^{(\beta)}, \bar{\chi}^{(\beta)}\rangle = 1$ it follows that $\bar{\chi}^{(\beta)}$ is also a simple character.

The computations in the preceding paragraph are valid for any finite group $H$. In the following section we will explicitly compute the simple characters of $S_\alpha$ and show that they are all real. That is, $\bar{\chi}^{(\beta)}(s) = \chi^{(\beta)}(s)$ for $s \in S_\alpha$. Anticipating this result we see that the minimal left and right ideals in $U_\beta$ all define irred reps of $S_\alpha$ equivalent to $\mathbf{D}^{(\beta)}$.

To clarify the relation between minimal left and right ideals in $U_\beta$ consider the mapping $x \longrightarrow \hat{x}$ of $R_\alpha$ onto $R_\alpha$ defined by

$$\hat{x} = \sum_{s \in S_\alpha} x(s) \cdot s^{-1} = \sum_s x(s^{-1}) \cdot s \tag{3.37}$$

where $x = \sum x(s) \cdot s$. This transformation is a vector space isomorphism. (Prove it.) However, it is not a homomorphism of the group ring onto itself since $\widehat{xy} = \hat{y}\hat{x}$. In particular $\widehat{st} = (st)^{-1} = t^{-1}s^{-1} = \hat{t}\hat{s}$ for $s, t \in S_\alpha$. Such a map is sometimes called an **inverted isomorphism**, since it inverts the order of ring multiplication. It is clear that each left ideal is transformed into a right ideal by this mapping. Since $(\hat{x})\hat{} = x$ it also follows that each right ideal is the image of some left ideal. The left ideal $R_\alpha c$ with generating idempotent $c$ is mapped onto the right ideal $\hat{c}\hat{R}_\alpha = \hat{c}R_\alpha$ with generating idempotent $\hat{c}$. [Note that $\hat{c}^2 = (c^2)\hat{} = \hat{c}$.] The idempotent $c$ is primitive if and only if $\hat{c}$ is primitive.

Let $\mathfrak{L}$ be a minimal left ideal in $U_\beta$. There exists a basis $\{x_i\}$ for $\mathfrak{L}$ such that

$$sx_i = \sum_{j=1}^{n_\beta} D_{ji}^{(\beta)}(s)x_j, \qquad 1 \le i \le n_\beta. \tag{3.38}$$

Under the transformation $x \longrightarrow \hat{x}$, $\mathfrak{L}$ is mapped onto a minimal right ideal $\hat{\mathfrak{L}}$ with basis $\{\hat{x}_i\}$ such that

$$\hat{x}_i s^{-1} = \sum_{j=1}^{n_\beta} D_{ji}^{(\beta)}(s)\hat{x}_j, \tag{3.39}$$

[obtained by applying the inverted isomorphism to (3.38)]. Since the right ideal $\hat{\mathfrak{L}}$ yields the irred rep $\mathbf{D}^{(\beta)}$ it follows that $\hat{\mathfrak{L}} \subseteq U_\beta$. Thus, $x \longrightarrow \hat{x}$ maps the two-sided ideal $U_\beta$ onto itself.

According to Section 4.2, the irred rep $\mathbf{D}^{(\beta)}$ of $S_\alpha$ corresponds to a partition $\{\lambda_1, \ldots, \lambda_\alpha\}$ of $\alpha$. Recall that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\alpha \ge 0$ and $\lambda_1 + \cdots + \lambda_\alpha = \alpha$. We use the same notation $\{\lambda_1, \ldots, \lambda_\alpha\}$ to denote the frame corresponding to this partition. Let $T_1, \ldots, T_f$ be the $f$ standard tableaux of the frame in dictionary order, and let $c_1, \ldots, c_f$ be the corresponding essential idempotents. According to Theorem 4.3,

$$U_\beta = R_\alpha c_1 \oplus \cdots \oplus R_\alpha c_f$$

is a decomposition of $U_\beta$ as a direct sum of minimal left ideals. Applying the inverted isomorphism we immediately conclude that

$$U_\beta = \hat{c}_1 R_\alpha \oplus \cdots \oplus \hat{c}_f R_\alpha$$

is a decomposition of $U_\beta$ into a direct sum of minimal right ideals. Thus, $W_j = \hat{c}_j V^{\otimes\alpha}$, $1 \leq j \leq f$, are **T**-irred subspaces of $V^{\otimes\alpha}$ all transforming according to equivalent reps of $G_m$. Applying this process to each minimal two-sided ideal in $R_\alpha$, we obtain a decomposition of $V^{\otimes\alpha}$ into a direct sum of **T**-irred subspaces. (Note that an essential idempotent determines the same **T**-invariant subspace as does the corresponding idempotent.)

Let $T$ be a standard tableau belonging to the frame $\{\lambda_1, \ldots, \lambda_\alpha\}$ and let $c$ be the essential idempotent corresponding to $T$. From Theorem 4.1, $c = PQ$, where

$$P = \sum_{p \in R(T)} p, \qquad Q = \sum_{q \in C(T)} \delta_q q.$$

It follows that $\hat{c} = \hat{Q}\hat{P}$, where

$$\hat{P} = \sum_p \hat{p} = \sum_p p^{-1}, \qquad \hat{Q} = \sum_q \delta_q \hat{q} = \sum_q \delta_q q^{-1}.$$

However, $p^{-1}$ ranges over $R(T)$ as $p$ does, $q^{-1}$ ranges over $C(T)$ as $q$ does, and $\delta_{q^{-1}} = \delta_q$. Thus, $\hat{P} = P$, $\hat{Q} = Q$, and

$$\hat{c} = QP = \sum_{q,p} \delta_q qp. \tag{3.40}$$

The essential idempotent $\hat{c}$ is obtained from $c = PQ$ by interchanging the ring elements $P$ and $Q$.

Finally, we note from Theorem 4.7 that there is a 1–1 relationship between frames $\{\lambda_1, \ldots, \lambda_\alpha\}$, $\lambda_1 + \cdots + \lambda_\alpha = \alpha$, and equivalence classes of irred reps of $G_m$. Therefore, we can use the frames to label the tensor irred reps of $G_m$.

**Examples.** Consider the case $\alpha = 2$, $m \geq 2$. There are two frames corresponding to $S_2$:

$$\begin{array}{|c|c|}\hline \phantom{1} & \phantom{1} \\ \hline \end{array}\,, \qquad \begin{array}{|c|}\hline \phantom{1} \\ \hline \phantom{1} \\ \hline \end{array}\,,$$

i.e., $\{2, 0\} = \{2\}$ and $\{1, 1\}$. The only standard tableaux are

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\,, \qquad T' = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\,,$$

so each frame defines a one-dimensional rep of $S_2$. The corresponding essential idempotents are

$$\hat{c} = QP = e[e + (12)] = e + (12), \qquad \hat{c}' = Q'P' = [e - (12)]e = e - (12).$$

The space of second-rank tensors is decomposed into two **T**-irred subspaces $W_1 = \hat{\mathbf{c}} V^{\otimes 2}$ and $W_2 = \hat{\mathbf{c}}' V^{\otimes 2}$. Introducing a basis $\{\mathbf{v}_j\}$ for $V$ and $\{\mathbf{v}_j \otimes \mathbf{v}_k\}$

for $V^{\otimes 2}$ we see from (3.7) that the elements of $W_1$ are those tensors $\mathbf{b}$ whose components can be represented in the form

$$b^{i_1i_2} = a^{i_1i_2} + a^{i_2i_1}$$

where the $a^{i_1i_2}$ are arbitrary, i.e., the symmetric tensors. The elements of $W_2$ are tensors $\mathbf{b}$ whose components take the form

$$b^{\substack{i_1\\i_2}} = a^{i_1i_2} - a^{i_2i_1},$$

the skew-symmetric tensors. (It is convenient to arrange the superscripts in the shape of the frame.) We have shown that $V^{\otimes 2} = W_1 \oplus W_2$, which we already knew, and that these two subspaces define irred reps of $G_m$.

Now consider the less trivial case $\alpha = 3$, $m \geq 2$. There are three frames corresponding to $\alpha = 3$: $\{3\}$, $\{2, 1\}$, $\{1^3\}$. The frames $\{3\}$, $\{1^3\}$ have one standard tableau each:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\,, \qquad T'' = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\,.$$

The frame $\{2, 1\}$ has two standard tableaux:

$$T_1' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \qquad T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,.$$

The essential idempotents are

$$\hat{c} = e + (12) + (13) + (23) + (123) + (132)$$

$$\hat{c}'' = e - (12) - (13) - (23) + (123) + (132)$$

$$\hat{c}_1{}' = e + (12) - (13) - (13)(12), \qquad \hat{c}_2{}' = e - (12) + (13) - (12)(13)$$

Thus,

$$V^{\otimes 3} = \hat{\mathbf{c}}V^{\otimes 3} \oplus \hat{\mathbf{c}}_1{}'V^{\otimes 3} \oplus \hat{\mathbf{c}}_2{}'V^{\otimes 3} \oplus \hat{\mathbf{c}}''V^{\otimes 3} \tag{3.41}$$

is a decomposition of $V^{\otimes 3}$ into $\mathbf{T}$-irred subspaces. The subspace corresponding to frame $\{3\}$ consists of those tensors whose tensor components

$$b^{i_1i_2i_3} \tag{3.42}$$

are completely symmetric with respect to the interchange of any two indices. Furthermore the subspace corresponding to $\{1^3\}$ consists of the completely skew-symmetric tensors

$$b^{\substack{i_1\\i_2\\i_3}} \tag{3.43}$$

which change sign upon the transposition of any pair of indices. There are two subspaces corresponding to the frame $\{2, 1\}$. One space consists of all tensors of the form

$$b^{\substack{i_1i_2\\i_3}} = a^{i_1i_2i_3} + a^{i_2i_1i_3} - a^{i_3i_2i_1} - a^{i_2i_3i_1}, \quad a^{ijk} \text{ arbitrary,} \tag{3.44}$$

and the other consists of all tensors

$$(b')^{i_1 i_3}_{i_2} = a^{i_1 i_2 i_3} - a^{i_2 i_1 i_3} + a^{i_3 i_2 i_1} - a^{i_3 i_1 i_2}. \tag{3.45}$$

Note that both of these tensor classes are skew-symmetric with respect to the transposition of indices in the same column. However, they are **not** symmetric with respect to the transposition of indices in the same row.

If $m \geq 3$ it is easy to see that each of the subspaces in the decomposition (3.41) is nonzero. Thus, $\mathcal{R}_3 = R_3$, $\mathcal{O}_3 = \{0\}$ and we have decomposed the tensor rep **T** of $G_m$ on $V^{\otimes 3}$ into three irred reps, one with multiplicity two. The corresponding tensor subspaces are called symmetry classes of tensors.

However, if $m = 2$, the tensors (3.43) are identically zero. Indeed, any tensor component must have at least two equal indices. A transposition of these indices obviously leads to the same component. On the other hand such a transposition changes the sign of the component since the tensors are completely skew-symmetric. Thus, the tensors (3.43) are identically zero and $\hat{c}''R_3 \subseteq \mathcal{O}_3$. The reader can easily check that the other symmetry classes of tensors are nonzero, so $\hat{c}''R_3 = \mathcal{O}_3$ and

$$\mathcal{R}_3 = \hat{c}R_3 \oplus \hat{c}_1{}'R_3 \oplus \hat{c}_2{}'R_3$$

for the case $m = 2$.

How do we determine the two-sided ideals $\mathcal{O}_\alpha$ and $\mathcal{R}_\alpha$ in the general case? Let $T$ be a Young tableau with frame $\{\lambda_1, \ldots, \lambda_\alpha\}$ and Young operator $\hat{c} = QP$,

$$T = \begin{array}{|c|c|c|c|c|}
\hline
j & k & \cdot & \cdot & \\
\hline
 & & & & \multicolumn{1}{c}{} \\
\cline{1-4}
 & & \multicolumn{3}{c}{} \\
\cline{1-2}
 & & \multicolumn{3}{c}{} \\
\cline{1-2}
 & \multicolumn{4}{c}{} \\
\cline{1-1}
l & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array} \tag{3.46}$$

Then the elements **b** of $\hat{\mathbf{c}}V^{\otimes\alpha}$ have tensor components

$$b^J = b^{\overset{i_j i_k \cdots}{i_i}} = QPa^{i_1 i_2 \cdots i_\alpha} \tag{3.47}$$

Since $qQ = \delta_q Q$ for any $q \in C(T)$ we have $\mathbf{qb} = \mathbf{qQPa} = \delta_q\mathbf{QPa} = \delta_q\mathbf{b}$. In particular, the tensor components of **b** change sign whenever two indices in the same column of $J$ are transposed. The tensors **b** are **skew-symmetric in the columns of** $J$. In general, the tensors **b** are not symmetric in the rows of $J$ unless $J$ has only one row.

These remarks enable us to determine $\mathcal{O}_\alpha$ and $\mathcal{R}_\alpha$ for the representation **T** of $G_m$ on $V^{\otimes\alpha}$ with dim $V = m$. Each minimal two-sided ideal $U_\beta$ in the group ring $R_\alpha$ is uniquely associated with a frame $\{\lambda_1, \ldots, \lambda_\alpha\}$ and $R_\alpha = \mathcal{R}_\alpha \oplus \mathcal{O}_\alpha$.

**Theorem 4.12.** The two-sided ideals $U_\beta$ corresponding to frames with $r$ rows, $r > m$, lie in $\mathcal{O}_\alpha$. If $r \leq m$ then $U_\beta \subseteq \mathcal{R}_\alpha$. Thus, the decomposition of $V^{\otimes\alpha}$ into **T**-irred subspaces is determined by those frames of $S_\alpha$ with $r \leq m$ rows.

***Proof.*** Let **b** be a tensor corresponding to a tableau $T$, (3.46), with $r > m$ rows. Then each tensor component $b^J$, (3.47), of **b** has $r > m$ indices in the first column of $J$ and at least two of these indices must be equal. A transposition $q \in C(T)$ of two equal indices obviously leaves $b^J$ fixed, $qb^J = b^J$. On the other hand $qb^J = -b^J$ since $b^J$ is skew-symmetric in the columns of $J$. Therefore, $b^J = 0$.

Now suppose the tableau has $r \leq m$ rows and consider the tensor $\mathbf{a} \in V^{\otimes\alpha}$ with components $a^{J_0} = 1$,

$$J_0 = \begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ r & r & \cdots & r & \end{matrix}$$

and $a^J = 0$ for all indices $J \neq J_0$. (Here we arrange the indices in the shape of the tableau $T$.) We will show that $\hat{\mathbf{c}}\mathbf{a} = \mathbf{QPa} \neq \boldsymbol{\theta}$, where $\hat{\mathbf{c}}$ is the Young operator corresponding to $T$. This proves $U_\beta \subseteq R_\alpha$. Clearly, $(Pa)^{J_0} = n > 0$, where $n$ is the order of $R(T)$ and $(Pa)^J = 0, J \neq J_0$. Thus, $\mathbf{Pa} = n\mathbf{a}$, so $\hat{\mathbf{c}}\mathbf{a} = n\mathbf{Qa} = n \sum \delta_q \mathbf{qa}$. The reader can easily check that each tensor $\mathbf{qa}$ has exactly one nonzero component and $\mathbf{qa}$, $\mathbf{q'a}$ have the same nonzero component if and only if $q = q'$. This follows from the fact that $r \leq m$. Thus, the sum $\sum \delta_q \mathbf{qa}$ is nonzero and $\hat{\mathbf{c}}\mathbf{a} \neq \boldsymbol{\theta}$. Q.E.D.

In summary, we have the following result.

**Theorem 4.13.** Let $T$ be a tableau of $S_\alpha$ with $r \leq m$ rows. Then the subspace $\hat{\mathbf{c}}_T V^{\otimes\alpha}$ transforms according to an irred rep of $G_m$, where $\hat{c}_T = QP$ and $Q, P$ are obtained from $T$ by (2.1). Tableaux $T$ and $T'$ determine equivalent reps of $G_m$ if and only if they belong to the same frame $\{\lambda_j\}$. Furthermore,

$$V^{\otimes\alpha} = \sum_T \oplus \hat{\mathbf{c}}_T V^{\otimes\alpha}$$

where $T$ runs over all standard tableaux of $S_\alpha$ with $r \leq m$ rows. The multiplicity of the irred rep $\{\lambda_j\}$ of $G_m$ in $\mathbf{T} \mid V^{\otimes\alpha}$ is equal to the number of standard tableaux with frame $\{\lambda_j\}$.

We shall examine this construction from another point of view in Chapter 9, where we study the irred reps of $G_m$ by Lie theory methods.

## 4.4 The Simple Characters of $S_\alpha$

We now use the results of Section 4.3 to compute the simple characters of $S_\alpha$. As usual we consider the reps $\mathbf{T}$ of $G_m$ and $\mathbf{D}$ of $S_\alpha$ on $V^{\otimes\alpha}$ defined by (3.2)–(3.7). The irred reps $\mathbf{C}^{[\lambda_j]}$ of $G_m$ and $\mathbf{D}^{[\lambda_j]}$ of $S_\alpha$ that occur with nonzero multiplicty in $\mathbf{T}$ and $\mathbf{D}$ are labeled by those frames $\{\lambda_1, \ldots, \lambda_\alpha\}$ of $S_\alpha$ that have $r \leq m$ rows. According to Theorem 4.7, the multiplicity of $\mathbf{C}^{[\lambda_j]}$ in $\mathbf{T}$ is equal to $\dim \mathbf{D}^{[\lambda_j]}$ and the multiplicity of $\mathbf{D}^{[\lambda_j]}$ in $\mathbf{D}$ is equal to $\dim \mathbf{C}^{[\lambda_j]}$.

According to (3.10) and (3.11), $\mathbf{sT}(g) = \mathbf{T}(g)\mathbf{s}$ for all $g \in G_m$, $s \in S_\alpha$. Thus we can define a rep $\mathbf{T}'$ of the direct product group $S_\alpha \times G_m$ on $V^{\otimes\alpha}$ by

$$\mathbf{T}'(sg) = \mathbf{sT}(g).$$

In terms of a basis $\{\mathbf{v}_j : 1 \leq j \leq m\}$ for $V$ we have

$$\text{(4.1)} \qquad \mathbf{sT}(g)[\mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_\alpha}] = \sum_{i_1 \cdots i_\alpha = 1}^{m} g^{i_{s(1)}}_{j_1} \cdots g^{i_{s(\alpha)}}_{j_\alpha} \mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_\alpha},$$

[see (3.10)]. We will compute the character $\chi$ of $\mathbf{T}'$ in two different ways.

First it follows from the proof of Theorem 4.7 that if $d = \dim \mathbf{D}^{[\lambda_j]}$ and $c = \dim \mathbf{C}^{[\lambda_j]}$ then there exist $cd$ linearly independent tensors

$$\text{(4.2)} \qquad \{\mathbf{v}_l^k : 1 \leq k \leq d, \quad 1 \leq l \leq c\}$$

such that

$$\text{(4.3)} \qquad \mathbf{T}(g)\mathbf{v}_l^k = \sum_{l'=1}^{c} C^{[\lambda_j]}_{l'l}(g)\mathbf{v}_{l'}^k, \qquad \mathbf{s}\mathbf{v}_l^k = \sum_{k'=1}^{d} D^{[\lambda_j]}_{k'k}(g)\mathbf{v}_l^{k'},$$

where the matrix reps $C^{[\lambda_j]}$, $D^{[\lambda_j]}$ correspond to the operator reps $\mathbf{C}^{[\lambda_j]}$, $\mathbf{D}^{[\lambda_j]}$. Choosing vectors $\{\mathbf{v}_l^k\}$ for each frame $\{\lambda_1, \ldots, \lambda_\alpha\}$ of $r \leq m$ rows we get a total of $m^\alpha$ linearly independent vectors, which form a basis for $V^{\otimes\alpha}$.

We compute the trace of the operator $\mathbf{T}'(sg)$ restricted to the subspace $W^{[\lambda_j]}$ spanned by the basis vectors (4.2). Since

$$\mathbf{sT}(g)\mathbf{v}_l^k = \sum_{l'=1}^{c} \sum_{k'=1}^{d} C^{[\lambda_j]}_{l'l}(g) D^{[\lambda_j]}_{k'k}(s)\mathbf{v}_{l'}^{k'}$$

we find

$$\text{(4.4)} \qquad \operatorname{tr}[\mathbf{sT}(g)]_{W^{[\lambda_j]}} = \sum_{l,k} C^{[\lambda_j]}_{ll}(g) D^{[\lambda_j]}_{kk}(s) = \varphi^{[\lambda_j]}(g)\chi^{[\lambda_j]}(s)$$

where $\varphi^{[\lambda_j]}$, $\chi^{[\lambda_j]}$ are the characters of $\mathbf{C}^{[\lambda_j]}$ and $\mathbf{D}^{[\lambda_j]}$, respectively. Thus,

$$\text{(4.5)} \qquad \chi(sg) = \operatorname{tr}[\mathbf{sT}(g)] = \sum \varphi^{[\lambda_j]}(g)\chi^{[\lambda_j]}(s)$$

where the sum is taken over all partitions $\{\lambda_1, \ldots, \lambda_m\}$ with $\lambda_1 + \cdots + \lambda_m = \alpha$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$.

Now we compute $\chi(sg)$ another way. From (4.1) there follows

$$\text{(4.6)} \qquad \chi(sg) = \sum_{j_1 \cdots j_\alpha = 1}^{m} g^{j_{s(1)}}_{j_1} \cdots g^{j_{s(\alpha)}}_{j_\alpha}.$$

We know that $\chi$ depends only on the conjugacy class in which $s$ lies, not on $s$ itself. Suppose $s$ belongs to the conjugacy class $(\nu_1, \nu_2, \ldots, \nu_\alpha)$, i.e., $\nu_1$ one-cycles, $\nu_2$ two-cycles, etc. Clearly, to each $\mu$-cycle $(klp \cdots z)$ in $s$ there corresponds a closed sum

$$\sum_{j_k \cdots j_z = 1}^{m} g^{j_k}_{j_z} g^{j_l}_{j_k} g^{j_p}_{j_l} \cdots g^{j_z}_{j_y} = \operatorname{tr}(g^\mu) = \sigma_\mu. \tag{4.7}$$

Therefore,

$$\chi(sg) = (\sigma_1)^{\nu_1}(\sigma_2)^{\nu_2} \cdots (\sigma_\alpha)^{\nu_\alpha}$$

where $\sigma_\mu$ is the trace of the matrix $g^\mu$. We have derived the formula

$$(\sigma_1)^{\nu_1}(\sigma_2)^{\nu_2} \cdots (\sigma_\alpha)^{\nu_\alpha} = \sum_{\{\lambda_j\}} \varphi^{\{\lambda_j\}}(g)\chi^{\{\lambda_j\}}(s). \tag{4.8}$$

Another identity is obtained by using the orthogonality relations for the characters $\chi^{\{\lambda_j\}}$. By (1.9) the conjugacy class $(\nu_1, \ldots, \nu_\alpha)$ contains

$$m_\nu = \alpha!/(1^{\nu_1}\nu_1! \cdots \alpha^{\nu_\alpha}\nu_\alpha!)$$

elements. Taking the inner product of (4.8) with $\chi^{\{\lambda_k\}}(s)$ we obtain

$$\varphi^{\{\lambda_j\}}(g) = \sum_{(\nu)} \frac{\chi_\nu^{\{\lambda_j\}}}{\nu_1! \cdots \nu_\alpha!}\left(\frac{\sigma_1}{1}\right)^{\nu_1} \cdots \left(\frac{\sigma_\alpha}{\alpha}\right)^{\nu_\alpha} \tag{4.9}$$

where the sum goes over all conjugacy classes $(\nu) = (\nu_1, \ldots, \nu_\alpha)$ in $S_\alpha$ and $\chi_\nu^{\{\lambda_j\}}$ is the value of the simple character $\chi^{\{\lambda_j\}}(s)$ for $s$ in the conjugacy class $(\nu)$.

From (4.9) the simple character $\varphi^{\{\lambda_j\}}(g)$ is a generating function for the characters $\chi_\nu^{\{\lambda_j\}}$ of $S_\alpha$. In Section 9.2 we shall compute $\varphi^{\{\lambda_j\}}(g)$ by a method entirely distinct from that used here. The result is as follows: Let $\varepsilon_1, \ldots, \varepsilon_m$ be the eigenvalues of $g \in G_m$. Then

$$\varphi^{\{\lambda_j\}}(\varepsilon_1, \ldots, \varepsilon_m) = \frac{|\varepsilon^{l_1}, \varepsilon^{l_2}, \ldots, \varepsilon^{l_{m-1}}, \varepsilon^{l_m}|}{|\varepsilon^{n-1}, \varepsilon^{n-2}, \ldots, \varepsilon, 1|}, \tag{4.10}$$

where $l_j = \lambda_j + m - j$, $j = 1, \ldots, m$, and

$$|\varepsilon^{l_1}, \ldots, \varepsilon^{l_m}| = \det\begin{pmatrix} \varepsilon_1^{l_1} & \varepsilon_1^{l_2} & \cdots & \varepsilon_1^{l_m} \\ \varepsilon_2^{l_1} & \varepsilon_2^{l_2} & \cdots & \varepsilon_2^{l_m} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \varepsilon_m^{l_1} & \varepsilon_m^{l_2} & \cdots & \varepsilon_m^{l_m} \end{pmatrix} \tag{4.11}$$

with a similar interpretation of the denominator in (4.10).

Thus (4.8) implies

$$\sigma_1^{\nu_1} \cdots \sigma_\alpha^{\nu_\alpha} |\varepsilon^{m-1}, \ldots, \varepsilon, 1| = \sum_{\{\lambda_j\}} \chi_\nu^{\{\lambda_j\}} |\varepsilon^{l_1}, \ldots, \varepsilon^{l_m}|, \tag{4.12}$$

$$\sigma_\mu = \operatorname{tr}(g^\mu) = \varepsilon_1{}^\mu + \varepsilon_2{}^\mu + \cdots + \varepsilon_m{}^\mu. \tag{4.13}$$

There is a similar formula for (4.9). These results are correct whether or not $g$ can be diagonalized.

The expressions (4.12) can be used to compute the characters $\chi^{\{\lambda_j\}}$ directly. Indeed, as $g$ ranges over $G_m$ the $\varepsilon_1, \ldots, \varepsilon_m$ independently range over all nonzero complex numbers, so the coefficients $\chi_\nu^{\{\lambda_j\}}$ on the right-hand side of (4.12) are uniquely determined. It is evident from an examination of this expression that the characters of $S_\alpha$ are real. In fact the characters take on only integer values. Although $\chi_\nu^{\{\lambda_j\}}$ can be computed directly from (4.12) by expanding in powers of the $\varepsilon_j$, this process is difficult even for low values of $m$ and $\alpha$. In practice the characters are usually determined by graphical procedures or recursion relations which are derived from (4.12).

As an example we show how to compute the dimension of the irred rep $\{\lambda_j\}$ of $S_\alpha$. This number is equal to $\chi^{\{\lambda_j\}}(s)$ for $s = e$ in the conjugacy class $(\alpha, 0, \ldots, 0)$, i.e., $\nu_1 = \alpha$, $\nu_2 = \cdots = \nu_\alpha = 0$. Therefore, the dimension $N^{\{\lambda_j\}}$ is determined by

$$(4.14) \qquad (\varepsilon_1 + \cdots + \varepsilon_\alpha)^\alpha \,|\varepsilon^{m-1}, \ldots, \varepsilon, 1| = \sum_{\{\lambda_j\}} N^{\{\lambda_j\}} \,|\varepsilon^{l_1}, \ldots, \varepsilon^{l_m}|.$$

[In order that the coefficient $N^{\{\lambda_j\}}$ appear in (4.14) it is necessary that all but the first $m$ terms $\lambda_1, \ldots, \lambda_m$ of $\{\lambda_1, \ldots, \lambda_\alpha\}$ be zero. Otherwise, the value of $m$ is immaterial. To be definite we can choose $m = \alpha$.] The determinant $|\varepsilon^{m-1}, \ldots, \varepsilon, 1|$ changes sign under the interchange of two rows. Thus, the left-hand side of (4.14) is a skew-symmetric function of the $\varepsilon_j$, i.e., it changes sign under the interchange of any two of these variables. If we expand $\sigma_1{}^\alpha \,|\varepsilon^{m-1}, \ldots, \varepsilon, 1|$ as a sum of monomials

$$(4.15) \qquad \varepsilon_1^{\beta_1}\varepsilon_2^{\beta_2} \cdots \varepsilon_m^{\beta_m}$$

no terms with $\beta_i = \beta_j$ for $i \neq j$ can occur with nonzero coefficient in the expansion. Indeed if the variables $\varepsilon_i, \varepsilon_j$ are raised to equal powers then the term is invariant under an interchange of $\varepsilon_i$ and $\varepsilon_j$ while $\sigma_1{}^\alpha \,|\varepsilon^{m-1}, \ldots, 1|$ changes sign, so the offending term must have zero coefficient. Furthermore, by skew-symmetry the occurrence of the monomial (4.15) on the right-hand side with coefficient $c$ implies the occurrence of each of the monomials

$$\delta_t \varepsilon_{t(1)}^{\beta_1} \varepsilon_{t(2)}^{\beta_2} \cdots \varepsilon_{t(m)}^{\beta_m}$$

with coefficient $c$, where $t$ is any permutation of the integers $1, \ldots, m$ and $\delta_t$ is the parity of $t$.

The right-hand side of (4.14) reads

$$\sum_{\{\lambda_j\}} N^{\{\lambda_j\}} \sum_{t \in S_m} \delta_t \varepsilon_{t(1)}^{\lambda_1 + m - 1} \varepsilon_{t(2)}^{\lambda_2 + m - 2} \cdots \varepsilon_{t(m)}^{\lambda_m},$$

so $N^{\{\lambda_j\}}$ is the coefficient of the term

$$(4.16) \qquad \varepsilon_1^{\lambda_1 + m - 1} \varepsilon_2^{\lambda_2 + m - 2} \cdots \varepsilon_m^{\lambda_m}$$

in which the highest power comes first, the next highest power comes second, and so on. The coefficients of the terms which are not in the ordered form (4.16) are obtained from the skew-symmetry.

To obtain $N^{\{\lambda_j\}}$ we multiply

$$|\varepsilon^{m-1}, \ldots, \varepsilon, 1| = \sum_{t \in S_m} \delta_t \varepsilon_{t(1)}^{m-1} \varepsilon_{t(2)}^{m-2} \cdots \varepsilon_{t(m-1)} \cdot 1 \tag{4.17}$$

$\alpha$ times by $\sigma_1 = \varepsilon_1 + \cdots + \varepsilon_m$ and compute the coefficient of (4.16) in the resulting expression. The only term in (4.17) which contributes to the result is the ordered term

$$\varepsilon_1^{m-1} \varepsilon_2^{m-2} \cdots \varepsilon_{m-1} \cdot 1 \tag{4.18}$$

since at some stage of the multiplication process each of the other monomials would have two variables raised to the same power. Thus, $N^{\{\lambda_j\}}$ is equal to the total number of ways we can obtain (4.16) from (4.18) by means of $\alpha$ successive multiplications by the variables $\varepsilon_i$, making sure that at each step no two variables are raised to the same power. Clearly, the number of times we multiply by $\varepsilon_1$ is greater than or equal to the number of times we multiply by $\varepsilon_2$, etc.

An example should clarify the situation. Let us compute the dimension of the rep $\{2, 1^3\}$ of $S_5$. For $m = \alpha = 5$, (4.18) becomes $\varepsilon_1{}^4\varepsilon_2{}^3\varepsilon_3{}^2\varepsilon_4$. We must end up with $\varepsilon_1{}^6\varepsilon_2{}^4\varepsilon_3{}^3\varepsilon_4{}^2$, Eq. (4.16), by multiplying by one variable $\varepsilon_i$ at a time, making sure that at each step the exponents of no two variables are equal. The possibilities are as follows:

$$\begin{array}{llll} (1) & \varepsilon_4\varepsilon_3\varepsilon_2\varepsilon_1\varepsilon_1(\varepsilon_1{}^4\varepsilon_2{}^3\varepsilon_3{}^2\varepsilon_4), & (2) & \varepsilon_4\varepsilon_3\varepsilon_1\varepsilon_2\varepsilon_1(\varepsilon_1{}^4\varepsilon_2{}^3\varepsilon_3{}^2\varepsilon_4) \\ (3) & \varepsilon_4\varepsilon_1\varepsilon_3\varepsilon_2\varepsilon_1(\varepsilon_1{}^4\varepsilon_2{}^3\varepsilon_3{}^2\varepsilon_4), & (4) & \varepsilon_1\varepsilon_4\varepsilon_3\varepsilon_2\varepsilon_1(\varepsilon_1{}^4\varepsilon_2{}^3\varepsilon_3{}^2\varepsilon_4). \end{array} \tag{4.19}$$

We conclude that the dimension of $\{2, 1^3\}$ is four.

Our method has a graphical interpretation. From our rule for obtaining $N^{\{\lambda_j\}}$, this integer is just the number of distinct ways we can fill in the frame of $\{\lambda_1, \ldots, \lambda_\alpha\}$ successively with $\alpha$ dots, making sure that at each step every row in the frame has at least as many dots as the rows below it. The dots are filled in from left to right in any given row. Multiplication by $\varepsilon_i$ corresponds to the application of a dot in row $i$. As an example we again consider the frame $\{2, 1^3\}$. If we number the dots from 1 to 5 in the order of their application we obtain the results

$$(1)\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \qquad (2)\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \qquad (3)\ \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \qquad (4)\ \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \tag{4.20}$$

which correspond exactly to the expressions (1)–(4) of (4.19). Note that the tableaux (4.20) are just the standard tableaux associated with the frame $\{2, 1^3\}$.

The reader should now recognize that the number of ways we can fill in a frame $\{\lambda_j\}$ with dots satisfying the above rules is exactly the number of standard tableaux associated with $\{\lambda_j\}$. This proves Theorem 4.2.

Similar but more complicated graphical methods can be used to compute the characters $\chi_\nu^{[\lambda_j]}$ (see Hamermesh [1], Boerner [1], Murnaghan [1]).

## Problems

**4.1** Compute the conjugacy classes of $S_6$ and the number of elements in each class.

**4.2** Apply the methods of Chapter 3 to deduce the character tables of $S_3$, $S_4$, and $S_5$. Hint: Use formula (5.27), Section 3.5, to derive characters of $S_n$ from the simple characters of $S_{n-1} \subset S_n$.

**4.3** List the possible frames of $S_5$ and the standard tableaux corresponding to each frame.

**4.4** Let $c$ be an essential idempotent corresponding to the frame $\{2, 1\}$. Determine the invariant subspace $R_3c$ directly and use it to compute a matrix rep of $S_3$ equivalent to $\{2, 1\}$.

**4.5** Construct the ring element $c = PQ$ corresponding to the tableau of Fig. 4.3 and verify directly that $c$ is an essential idempotent. What is the dimension of the irred rep of $S_6$ determined by $c$?

**4.6** Compute the dimension of the rep $\{4, 3, 1, 1\}$ of $S_9$.

**4.7** Decompose the space $V^{\otimes 4}$ into subspaces irred under $GL(m, \mathfrak{C})$, dim $V = m \geq 2$.

**4.8** Use formula (4.12) to obtain the character tables for $S_3$ and $S_4$.

**4.9** Let $G$ be a group, not necessarily finite, and let $T^{(1)}, \ldots, T^{(r)}$ be irred nonequivalent matrix reps of $G$. Show that the matrix elements $\{T_{ij}^{(k)}(g)\}$ $(1 \leq i, j \leq n_k;\ 1 \leq k \leq r)$ are linearly independent functions on $G$. (This is not easy. See Curtis and Reiner [1, Chapter IV].)