

## On the signature matrix of semantic property

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### Notation

$L$  : the number of semantic dimensions

$K$  : the number of semantic dimensions in a property represented as  $K$ -polytope

$N$  : number of semantic aspects in a property

$\mathcal{P}$  : set of points forming the  $K$ -polytope of a semantic property

$A_i$  : denotes the  $i$ -th semantic aspect of a semantic property

$P_i$  : denotes semantic property

$V_i$  : denotes primitive semantic particle

$\vec{r}_c$  : in the context of a property: the center of mass of the property

In the context of an ensemble of properties: the center of mass of the ensemble

$\vec{r}_i$  : In the context of a property: semantic position of the aspect  $A_i$

In the context of an ensemble of properties: the center of mass of the property

$l_i$  : the type of the aspect  $A_i$

$\theta_j$  : angle between the current aspect and semantic axis  $x_j$

$\Theta$  : a vector with all angular coordinates of the current aspect to the semantic axes

### Matrix Representation of Semantic Property

$$\vec{r}_c = \frac{\sum_{i=1}^{|\mathcal{P}|} m_i \vec{r}_i}{\sum_{l=1} m_l} \quad (1)$$

If  $m_l = m = \text{const}$

$$\vec{r}_c = \frac{\sum_{i=1}^{|\mathcal{P}|} \vec{r}_i}{|\mathcal{P}|} \quad (2)$$

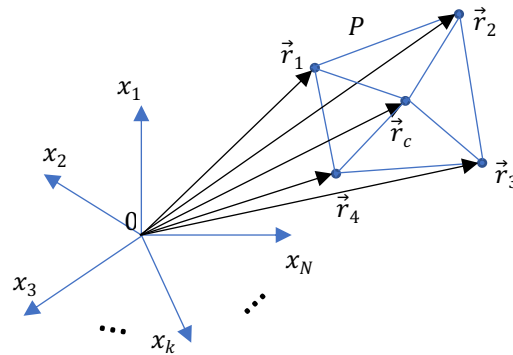


Figure 1

$$\vec{p}_i = \vec{r}_i - \vec{r}_c \quad (3)$$

$$\vec{p}_i = \left(1 - \frac{m_i}{\sum_{l=1} m_l}\right) \vec{r}_i - \sum_{j=1, j \neq i}^{|\mathcal{P}|} \frac{m_j}{\sum_{l=1} m_l} \vec{r}_j \quad (4)$$

With  $\hat{m}_i = \frac{m_i}{\sum_{l=1} m_l}$  we write:

$$\vec{p}_i = (1 - \hat{m}_i) \vec{r}_i - \sum_{j=1, j \neq i}^{|\mathcal{P}|} \hat{m}_j \vec{r}_j \quad (5)$$

In a matrix form:

$$P = \begin{bmatrix} 1 - \hat{m}_1 & -\hat{m}_2 & \cdots & -\hat{m}_N \\ -\hat{m}_1 & 1 - \hat{m}_2 & \cdots & -\hat{m}_N \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{m}_1 & -\hat{m}_2 & \cdots & 1 - \hat{m}_N \end{bmatrix} \begin{Bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_N \end{Bmatrix} \quad (6)$$

or succinctly

$$P = MX \quad (7)$$

$$\text{where } M = \begin{bmatrix} 1 - \hat{m}_1 & -\hat{m}_2 & \cdots & -\hat{m}_N \\ -\hat{m}_1 & 1 - \hat{m}_2 & \cdots & -\hat{m}_N \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{m}_1 & -\hat{m}_2 & \cdots & 1 - \hat{m}_N \end{bmatrix} \quad X = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,L} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,L} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N,1} & x_{N,2} & \cdots & x_{N,L} \end{bmatrix} \quad (8)$$

While  $P$  is traversing the semantic space each aspect  $A_i$  retains its properties – this means that each  $\vec{p}_i$  is invariant when  $P$  traverses semantic space. That is -  $l_i$  and  $\theta_i$  remain invariant.

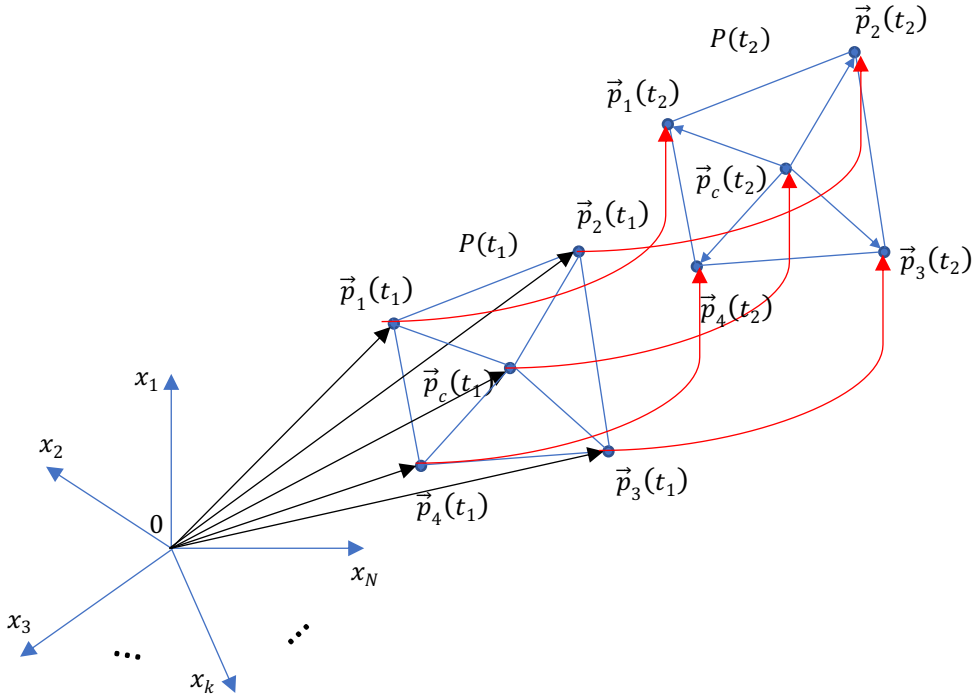


Figure 2

The last statement, obviously, implies that there does not exist inverse matrix  $M^{-1}$  as the set of  $N \times N$  matrices  $X$  which map to a given matrix  $P$  is a continuum.

Thus, we conclude that each semantic property is uniquely defined by the pair of two quantities: the semantic signature matrix  $P$  and the mass vector of the property  $\mathbf{m} = \{m_1, m_2, \dots, m_N\}$ .

## In Situ Position of Semantic Property

### **Definition:** *In-situ position of semantic property*

This is the initial position in Semantic Space from which each semantic property starts its travel to bound state.

Each semantic property will be mapped to a portion of semantic space which will contain its initial / in-situ position. Such partitioning of the semantic space into generally non-overlapping feasible sectors is necessary in order to assure the effectiveness of the future reinforcements based on past realized property assemblies.

In order to determine the portion of semantic space which will map to semantic property we will look into the singular value decomposition of the property signature matrix  $P$  given with

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,L} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,L} \\ \vdots & \vdots & \vdots & \vdots \\ p_{N,1} & p_{N,2} & \cdots & p_{N,L} \end{bmatrix} \quad (9)$$

Recall, in (9) each row corresponds to an aspect definition encoding the aspect type  $l_i$  and its angular coordinates  $\theta_i$ . Thus, we will be looking for factorization in the form:

$$P = U\Sigma V^T \quad (10)$$

where  $U$  is  $N \times N$  orthonormal matrix,  $V$  is  $L \times L$  orthonormal matrix and  $\Sigma$  is  $N \times L$  diagonal matrix with at most  $N$  non-zero values on the main diagonal. Let us denote those non-zero values on the main diagonal of  $\Sigma$  with  $\sigma_1, \sigma_2, \dots, \sigma_N$ . We can always normalize  $P$  such that  $\hat{\sigma}_1 + \hat{\sigma}_2 + \cdots + \hat{\sigma}_N = 1$ . Then the entropy of the property  $P$  is given with:

$$H(P) = -(\hat{\sigma}_1 \log \hat{\sigma}_1 + \hat{\sigma}_2 \log \hat{\sigma}_2 + \cdots + \hat{\sigma}_N \log \hat{\sigma}_N) \quad (11)$$

Here we use  $\lim_{\varepsilon \rightarrow 0} \varepsilon \log \varepsilon \rightarrow 0$ .

The higher the entropy  $H(P)$ , the higher the information content of the property  $P$ .

One can argue that the in-situ positions of semantic properties with higher information content should be farther from the semantic center compared to the properties with less information content. The argument is that population of properties with high information content is larger than the population of properties with low information content. In fact, we will assume that the population with information content  $H$  is proportional to the surface of  $L$  dimensional sphere with radius  $H$ . Thus, the population increases as the following ratio  $\left(\frac{H_1}{H_2}\right)^{L-1}$ .

Therefore, in order to have maximum utilization of semantic space we will restrict the space of feasible initial (in-situ) positions for a property  $P$  with information content  $H(P)$  to be the surface of  $L$  dimensional sphere with radius  $H(P)$ . Thus, all properties with the same information content value  $H$  share the same feasible space for their initial (in-situ) positions. Additionally to the information content  $H(P)$  we would like to account for the [effective rank](#) of  $P$  and this is done through additional

normalization coefficient  $\sum_{i=1}^N \sigma_i$ . Thus, for a radius of the *feasibility sphere* we will use the *normalized information content* given with:

$$H^* = -(\hat{\sigma}_1 \log \hat{\sigma}_1 + \hat{\sigma}_2 \log \hat{\sigma}_2 + \dots + \hat{\sigma}_N \log \hat{\sigma}_N) \sum_{i=1}^N \sigma_i = H \sum_{i=1}^N \sigma_i \quad (12)$$

Thus if  $pop(H^*)$  is the population with normalized information content  $H^*$

$$\frac{pop(H_1^*)}{pop(H_2^*)} \sim \left(\frac{H_1^*}{H_2^*}\right)^{L-1} \quad (13)$$

**Definition:** *feasible sphere for in-situ positions of semantic property*

Let the information content of the semantic property  $P$  is denoted with  $H^*$ . Then the feasible sphere for the in-situ positions of  $P$  is the region of semantic space given with the equation  $(H^*)^2 = x_1^2 + x_2^2 + \dots + x_N^2$ .

Further we will partition the feasible sphere for all properties with the same normalized information content  $H^*$  based on the set of non-zero singular values for each property. Specifically, if we have  $k$  non-zero singular values  $\sigma_1, \sigma_2, \dots, \sigma_k$  let us denote those axes which correspond to the  $k$  non-zero singular values with  $\varsigma_1, \varsigma_2, \dots, \varsigma_k$ . Let us denote with  $\mathfrak{S}$  the  $k$ -dimensional ellipsoid corresponding to  $\sigma_1, \sigma_2, \dots, \sigma_k$ . In other words,  $\mathfrak{S}$  is given with:

$$\mathfrak{S}: \left(\frac{\varsigma_1}{\sigma_1}\right)^2 + \left(\frac{\varsigma_2}{\sigma_2}\right)^2 + \dots + \left(\frac{\varsigma_k}{\sigma_k}\right)^2 \leq 1 \quad (14)$$

Let us denote with  $\mathfrak{S}^*$  the *scaled*  $k$ -dimensional ellipsoid corresponding to *scaled* singular values  $\sigma_1^*, \sigma_2^*, \dots, \sigma_k^*$  which are obtained from the normalized singular values  $\hat{\sigma}_i$  with the relation

$$\sigma_i^* = kH^* \hat{\sigma}_i \quad (15)$$

$$\mathfrak{S}^*: \left(\frac{\varsigma_1}{\sigma_1^*}\right)^2 + \left(\frac{\varsigma_2}{\sigma_2^*}\right)^2 + \dots + \left(\frac{\varsigma_k}{\sigma_k^*}\right)^2 \leq 1 \quad \text{s.t.} \quad \frac{\sigma_1^* + \sigma_2^* + \dots + \sigma_k^*}{k} = H^* \quad (16)$$

Let us denote with  $\mathfrak{R}$  the feasible sphere of property  $P$ , which is given with the equation  $(H^*)^2 = x_1^2 + x_2^2 + \dots + x_N^2$ .

**Definition:** *feasible in-situ positions for semantic property*

The feasible in-situ positions of a semantic property, denoted with  $\mathfrak{P}$ , is a region on the surface of the  $N$ -dimensional feasible sphere which is inter. Given with the sector on the feasible sphere  $\mathfrak{R}$  determined by the intersection of  $\mathfrak{R}$  with  $\mathfrak{S}^*$  where the latter is defined with (16).

Special cases for the region of feasible in-situ positions

//TODO: list special cases here for various  $k, N$  combinations and specific relationships between  
//  $\sigma_1^*, \sigma_2^*, \dots, \sigma_k^*$

Example: Let  $L = 3$  and  $P$  is a property with three aspects i.e.  $N = 3$ . Let the normalized information content of  $P$  is  $H^*(P)$ . Let  $P$  has two non-zero singular values -  $\sigma_1 > 0$  and  $\sigma_2 > 0$  while  $\sigma_3 = 0$ . Then the feasible in-situ positions for  $P$  will be any point on the sector  $AB$  shown on the Figure 3 below.

Figure 3:

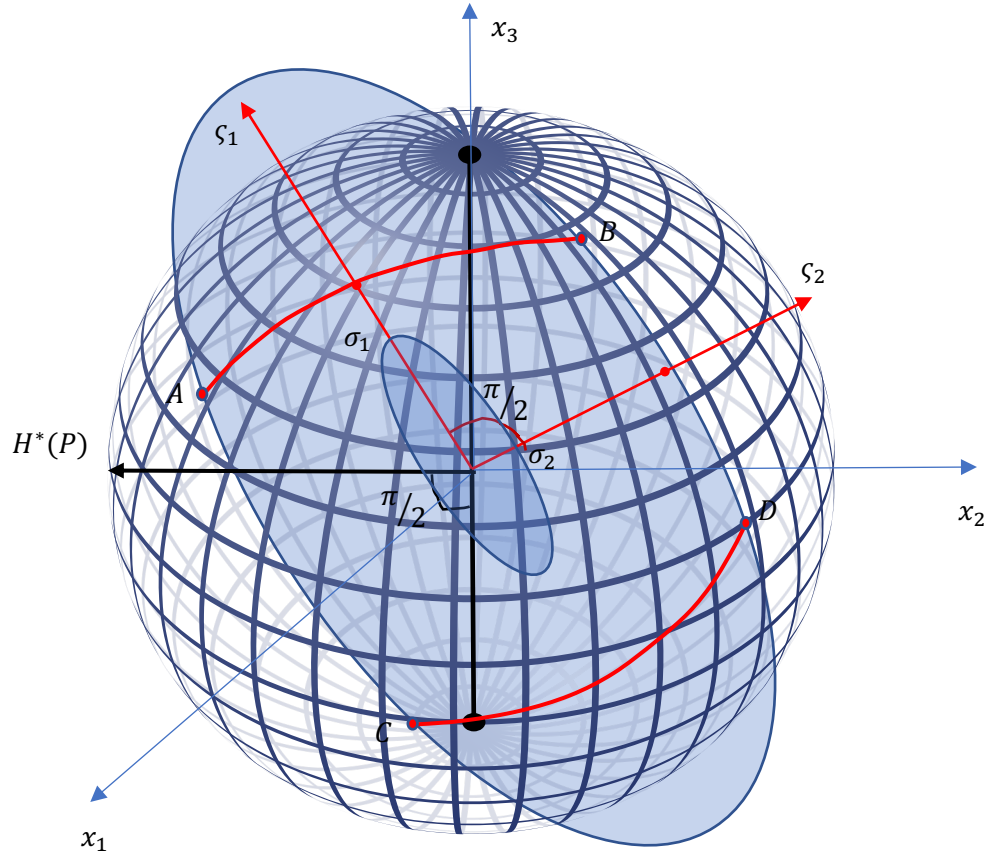


Figure 4:

