Appendix

Hilbert Space

We present some basic ideas and definitions from Hilbert space theory. For a more detailed exposition see the work of Korevaar [1] or Naylor and Sell [1]. All vector spaces will be assumed complex, although the facts for real spaces are essentially the same.

Definition. A vector space $\mathcal{V}$ is an **inner product space (pre-Hilbert space)** with inner product $(-, -)$ if $(\mathbf{u}, \mathbf{v}) \in \mathfrak{C}$ for each $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ and

(a) $(\mathbf{u}, \mathbf{v}) = \overline{(\mathbf{v}, \mathbf{u})}$.
(b) $(a_1\mathbf{u}_1 + a_2\mathbf{u}_2, \mathbf{v}) = a_1(\mathbf{u}_1, \mathbf{v}) + a_2(\mathbf{u}_2, \mathbf{v})$, $a_j \in \mathfrak{C}$, $\mathbf{u}_j, \mathbf{v} \in \mathcal{V}$.
(c) $(\mathbf{u}, \mathbf{u}) \geq 0$ and $(\mathbf{u}, \mathbf{u}) = 0$ only if $\mathbf{u} = \boldsymbol{\theta}$.

We define the **length (norm)** of a vector by $\|\mathbf{u}\| = [(\mathbf{u}, \mathbf{u})]^{1/2}$. Clearly, $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \boldsymbol{\theta}$.

Lemma A1 (Schwarz inequality). If $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ then $|(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$. Equality is obtained if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

Lemma A2 (Triangle inequality). If $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ then $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Every finite-dimensional vector space with an inner product is a pre-Hilbert space. We examine some examples of pre-Hilbert spaces which are not finite-dimensional.

Example 1. By l_2 we mean the set of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$ of complex numbers x_i such that $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. Here l_2 is a vector space with operations

$$\text{(A.1)} \qquad a\mathbf{x} = (ax_1, ax_2, \ldots), \qquad \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots),$$

where $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$. Indeed, $\sum_{i=1}^{\infty} |x_i + y_i|^2 \leq \sum_{i=1}^{\infty} (|x_i|^2 + |y_i|^2 + 2|x_i y_i|) \leq 2 \sum_{i=1}^{\infty} (|x_i|^2 + |y_i|^2) < \infty$ for $\mathbf{x}, \mathbf{y} \in l_2$. Here we have used the property $(a - b)^2 = a^2 + b^2 - 2ab \geq 0$ for all real numbers a, b. The space l_2 is a pre-Hilbert space with inner product

$$\text{(A.2)} \qquad (\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

Indeed, the series for $(\mathbf{x}, \mathbf{y})$ converges for all $\mathbf{x}, \mathbf{y} \in l_2$.

Example 2. Let $[a, b]$, $a < b$, be a closed interval on the real line and let $C[a, b]$ be the set of all functions $f(x)$ which are defined and continuous on $[a, b]$. Then $C[a, b]$ is a vector space with operations

$$\text{(A.3)} \qquad \begin{aligned} (\alpha f)(x) &= \alpha f(x), & \alpha &\in \mathfrak{C} \\ (f + g)(x) &= f(x) + g(x), & f, g &\in C[a, b]. \end{aligned}$$

The expression

$$\text{(A.4)} \qquad (f, g) = \int_a^b f(x)\overline{g(x)}\, dx$$

defines an inner product on $C[a, b]$.

Example 3. Let $\mathfrak{M}$ be a closed bounded connected subset of R_m whose boundary is piecewise smooth and let $C(\mathfrak{M})$ be the set of all functions $f(x)$, $x \in R_m$, which are defined and continuous on $\mathfrak{M}$. Using the definitions (A.3) we can make $C(\mathfrak{M})$ into a vector space. Furthermore if $w \in C(\mathfrak{M})$ and $w(x) > 0$ for all $x \in \mathfrak{M}$ then the expression

$$\text{(A.5)} \qquad (f, g) = \int_{\mathfrak{M}} f(x)\overline{g(x)}w(x)\, dx, \qquad dx = dx_1 \cdots dx_m$$

defines an inner product on $C(\mathfrak{M})$. Here $w(x)$ is a **weight function**.

Example 4. Let G be a compact linear Lie group and let $C(G)$ be the vector space of all continuous functions on G. Then $C(G)$ is a pre-Hilbert space with respect to the inner product

$$\text{(A.6)} \qquad (f, g) = \int_G f(A)\overline{g(A)}\, \delta A, \qquad f, g \in C(G),$$

where δA is the normalized invariant measure on G.

Example 5. Let $C^2(R_m)$ be the set of all functions $f(x)$ defined and continuous in R_m and such that

$$\int_{R_m} |f(x)|^2\, dx < \infty, \qquad dx = dx_1 \cdots dx_m.$$

(Note that continuous functions on R_m need not be bounded.) Then $C^2(R_m)$ is a vector space under the usual operations (A.3). Indeed, $|2ab| \leq a^2 + b^2$, for a, b real, so

$$\text{(A.7)} \qquad 2\int_{R_m} |f(x)g(x)|\,dx \leq \int_{R_m} |f(x)|^2\,dx + \int_{R_m} |g(x)|^2\,dx < \infty$$

for $f, g \in C^2(R_m)$. Thus the integrals $\int_{R_m} f\bar{g}\,dx$ and $\int_{R_m} |fg|\,dx$ converge. This shows that

$$\int_{R_m} |f(x) + g(x)|^2\,dx \leq \int_{R_m} (|f(x)|^2 + 2|f(x)g(x)| + |g(x)|^2)\,dx < \infty$$

and $f + g \in C^2(R_m)$. Furthermore, by (A.7) the expression

$$\text{(A.8)} \qquad (f, g) = \int_{R_m} f(x)\overline{g(x)}\,dx$$

defines an inner product on $C^2(R_m)$.

Definition. The pre-Hilbert spaces $\mathcal{V}, \mathcal{W}$ are **isomorphic** (as pre-Hilbert spaces) if there is a vector space isomorphism $\mathbf{T}\colon \mathcal{V} \to \mathcal{W}$ such that $(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{T}\mathbf{v}_1, \mathbf{T}\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. Here the first inner product belongs to $\mathcal{V}$ and the second to $\mathcal{W}$.

Example. Every m-dimensional pre-Hilbert space $\mathcal{V}_m$ is isomorphic to $\mathfrak{C}_m$. (Choose an ON basis for $\mathcal{V}_m$.)

Let $\{\mathbf{v}_j\}$, $j = 1, 2, \ldots$, be a sequence of vectors in the pre-Hilbert space $\mathcal{V}$. The sequence $\{\mathbf{v}_j\}$ is said to be **Cauchy** if for every $\epsilon > 0$ there exists a positive integer N_ϵ with the property $\|\mathbf{v}_k - \mathbf{v}_j\| < \epsilon$ whenever $k, j > N_\epsilon$. A Cauchy sequence **converges** in case there is a $\mathbf{v} \in \mathcal{V}$ such that $\lim_{j\to\infty} \|\mathbf{v} - \mathbf{v}_n\| = 0$. If a Cauchy sequence converges to both $\mathbf{v}$ and $\mathbf{w}$ then $\|\mathbf{v} - \mathbf{w}\| = \|\mathbf{v} - \mathbf{v}_j + \mathbf{v}_j - \mathbf{w}\| \leq \|\mathbf{v} - \mathbf{v}_j\| + \|\mathbf{v}_j - \mathbf{w}\| \to 0$ as $j \to \infty$, so $\mathbf{v} = \mathbf{w}$. Thus the limit of a convergent Cauchy sequence is unique. If a sequence $\{\mathbf{v}_j\}$ converges to $\mathbf{v}$, $\lim_{j\to\infty} \|\mathbf{v}_j - \mathbf{v}\| = 0$, then $\{\mathbf{v}_j\}$ is Cauchy. Indeed, $\|\mathbf{v}_j - \mathbf{v}_k\| = \|\mathbf{v}_j - \mathbf{v} + \mathbf{v} - \mathbf{v}_k\| \leq \|\mathbf{v}_j - \mathbf{v}\| + \|\mathbf{v} - \mathbf{v}_k\| \to 0$ as $j, k \to \infty$.

Definition. A pre-Hilbert space $\mathcal{H}$ is a **Hilbert space** if every Cauchy sequence $\{\mathbf{v}_j\}$ in $\mathcal{H}$ converges to an element of $\mathcal{H}$.

Example 1. Every finite-dimensional pre-Hilbert space is a Hilbert space. (Prove it!)

Example 2. The space l_2 is a Hilbert space.

Example 3. The pre-Hilbert spaces $C(\mathfrak{M})$, $C(G)$ and $C^2(R_m)$ are not Hilbert spaces. In each case it is easy to construct a Cauchy sequence of continuous functions which do not converge to an element of the pre-Hilbert space.

The inner product in a pre-Hilbert space is continuous in its two arguments.

Lemma A3. Let $\{\mathbf{u}_j\}$, $\{\mathbf{v}_j\}$ be convergent Cauchy sequences in the pre-Hilbert space $\mathcal{V}$. If $\mathbf{v}_j \to \mathbf{v}$, $\mathbf{u}_j \to \mathbf{u}$ as $j \to \infty$ then $\lim_{j\to\infty} (\mathbf{u}_j, \mathbf{v}_j) = (\mathbf{u}, \mathbf{v})$.

In case $\mathbf{u}_j = \mathbf{v}_j$, Lemma A3 yields $\lim_{j\to\infty} \|\mathbf{v}_j\| = \|\mathbf{v}\|$, i.e., the norm is continuous with respect to convergence in $\mathcal{V}$.

Let $\mathcal{S}$ be a subset of the Hilbert space $\mathcal{H}$. The subset $\mathcal{S}$ is **dense** in $\mathcal{H}$ if for every $\mathbf{u} \in \mathcal{H}$ there exists a Cauchy sequence $\{\mathbf{u}_j\}$ in $\mathcal{S}$ such that $\mathbf{u}_j \to \mathbf{u}$.

Example. The set of all $\mathbf{x} = (x_1, x_2, \ldots)$ in l_2 with only finitely many nonzero components x_i is dense in l_2.

A subspace $\mathcal{W}$ of the Hilbert space $\mathcal{H}$ is **closed** in $\mathcal{H}$ if every Cauchy sequence in $\mathcal{W}$ converges to an element of $\mathcal{W}$. The **closure** of a possibly nonclosed subspace $\mathcal{W}$ is the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{W}$. (We order the subspaces by inclusion.)

Lemma A4. Let $\bar{\mathcal{W}}$ be the subset of $\mathcal{H}$ consisting of all $\mathbf{u} \in \mathcal{H}$ such that there exists a Cauchy sequence $\{\mathbf{u}_j\}$ in $\mathcal{W}$ with $\mathbf{u}_j \to \mathbf{u}$. Then $\bar{\mathcal{W}}$ is the closure of $\mathcal{W}$.

Theorem A1. Let $\mathcal{W}$ be a pre-Hilbert space. Then there exists a Hilbert space $\mathcal{H}$ (unique up to isomorphism) such that $\mathcal{W}$ is dense in $\mathcal{H}$. Indeed $\mathcal{H} = \bar{\mathcal{W}}$.

The proof of this theorem is not obvious since we do not know *a priori* that there exists a Hilbert space containing $\mathcal{W}$ as a subspace. Until $\mathcal{H}$ is constructed the meaning of $\bar{\mathcal{W}}$ is not clear. To construct $\mathcal{H}$ one considers the Cauchy sequences $\{\mathbf{u}_j\}$ in $\mathcal{W}$ which do not converge. Then one adds new elements $\mathbf{u}$ to $\mathcal{W}$ so that $\mathbf{u}_j \to \mathbf{u}$. If $\{\mathbf{v}_j\}$ is a Cauchy sequence in $\mathcal{W}$ such that $\|\mathbf{u}_j - \mathbf{v}_j\| \to 0$ then also $\mathbf{v}_j \to \mathbf{u}$. It can be shown that $\mathcal{W}$ together with the ideal elements $\{\mathbf{u}\}$ forms a Hilbert space $\mathcal{H}$. See books by Korevaar [1] or Helwig [1] for the details.

By Theorem A1 we can always assume we are dealing with a Hilbert space. (If $\mathcal{W}$ is not a Hilbert space we merely close it to obtain the Hilbert space $\bar{\mathcal{W}}$.) This is fortunate because Hilbert spaces have many nice features not shared by pre-Hilbert spaces.

As stated earlier, $C(\mathfrak{M})$ is not a Hilbert space. However, by the preceding theorem $C(\mathfrak{M})$ is dense in a Hilbert space denoted $L_2(\mathfrak{M})$. It can be shown that to each element f in the closure of $C(\mathfrak{M})$ we can associate a function $f(x)$ on $\mathfrak{M}$. Here $f(x)$ is in general not continuous. If $f(x)$, $g(x)$ are in $L_2(\mathfrak{M})$ then there exist Cauchy sequences $\{f_j\}, \{g_j\}$ in $C(\mathfrak{M})$ such that $f_j \to f$, $g_j \to g$ in the norm. We define the integral $\int_{\mathfrak{M}} f\bar{g}\,dx$ by

$$\text{(A.9)} \qquad \int_{\mathfrak{M}} f(x)g(\bar{x})\,dx = (f, g) = \lim_{j\to\infty}(f_j, g_j) = \lim_{j\to\infty}\int_{\mathfrak{M}} f_j(x)\overline{g_j(x)}\,dx.$$

The integral $\int_{\mathfrak{M}} f\bar{g}\,dx$ is called the **Lebesgue integral** and $L_2(\mathfrak{M})$ is the space of Lebesgue square-integrable functions on $\mathfrak{M}$. If $f(x)$ and $g(x)$ are functions on $\mathfrak{M}$ such that the Riemann integrals $\int_{\mathfrak{M}} |f|^2\,dx$ and $\int_{\mathfrak{M}} |g|^2\,dx$ converge, then $f, g \in L_2(\mathfrak{M})$ and (A.9) is just the ordinary Riemann integral. However, there exist functions in $L_2(\mathfrak{M})$ which are so discontinuous that they are not Riemann square-integrable. The spaces of continuous and of Riemann square-integrable functions on $\mathfrak{M}$ form pre-Hilbert but not Hilbert spaces. However, the closure of each of these pre-Hilbert spaces is the Hilbert space $L_2(\mathfrak{M})$.

Note: Actually the elements of $L_2(\mathfrak{M})$ are not functions but equivalence classes of functions. We say that two Lebesgue square-integrable functions f, g are **equivalent** if $\int_{\mathfrak{M}} |f(x) - g(x)|^2\,dx = 0$. Equivalent functions correspond to the same Hilbert space element. This distinction does not arise on the subspace $C(\mathfrak{M})$ since if f and g are equivalent continuous functions on $\mathfrak{M}$ then $f(x) = g(x)$ for all $x \in \mathfrak{M}$.

The reader unfamiliar with Lebesgue integration need not despair. Since $C(\mathfrak{M})$ is dense in $L_2(\mathfrak{M})$ we will ordinarily be able to restrict our computations to $C(\mathfrak{M})$.

In complete analogy with the above discussion, the closures of the pre-Hilbert spaces $C(G)$ and $C^2(R_m)$ are $L_2(G)$ and $L_2(R_m)$, the Hilbert spaces of Lebesgue square-integrable functions on G and R_m, respectively.

Definition. A Hilbert space $\mathcal{H}$ is **separable** if it contains a countable dense subset $\{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$.

Example. The space l_2 is separable. Consider the subset $\mathcal{S}$ of all $\mathbf{x} = \{x_1, x_2, \ldots\}$ such that each $x_j = a_j + ib_j$, where the real numbers a_j, b_j are rational, and only a finite number of the x_j are nonzero. The set $\mathcal{S}$ is countable and dense in l_2.

It can be shown that $L_2(\mathfrak{M})$, $L_2(G)$, and $L_2(R_m)$ are separable. In fact every Hilbert space studied in this book is separable. Therefore, from now on, "Hilbert space" means "separable Hilbert space."

Let $\mathfrak{M}$ be a subspace of $\mathcal{H}$. Then the set

$$\mathfrak{M}^\perp = \{\mathbf{u} \in \mathcal{H}: (\mathbf{u}, \mathbf{v}) = 0 \quad \text{for all} \quad \mathbf{v} \in \mathfrak{M}\}$$

is clearly a subspace of $\mathcal{H}$. Moreover, $\mathcal{M}^\perp$ is closed in $\mathcal{H}$ since if $\{\mathbf{u}_j\}$ is a Cauchy sequence in $\mathcal{M}^\perp$ with $\mathbf{u}_j \to \mathbf{u}$ and $\mathbf{v} \in \mathcal{M}$ we have $(\mathbf{u}, \mathbf{v}) = \lim_{j\to\infty} (\mathbf{u}_j, \mathbf{v}) = 0$, so $\mathbf{u} \in \mathcal{M}^\perp$.

Theorem A2. Let $\mathcal{M}$ be a **closed** subspace of the Hilbert space $\mathcal{H}$. Then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, i.e., every $\mathbf{u} \in \mathcal{H}$ can be written uniquely in the form $\mathbf{u} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in \mathcal{M}$, $\mathbf{w} \in \mathcal{M}^\perp$.

For $\mathcal{H}$ finite-dimensional this theorem can be proved easily by introducing an appropriate ON basis. In the infinite-dimensional case the proof is not so obvious. The theorem is not true unless $\mathcal{M}$ is closed. (A finite-dimensional subspace of $\mathcal{H}$ is always closed.)

Definition. A countable set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$ in $\mathcal{H}$ is **orthonormal** (ON) if $(\mathbf{u}_j, \mathbf{u}_k) = \delta_{jk}, j, k = 1, 2, \ldots$. For any $\mathbf{u} \in \mathcal{H}$ the numbers $a_j = (\mathbf{u}, \mathbf{u}_j)$ are the **Fourier coefficients** of $\mathbf{u}$ with respect to the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$.

Definition. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ be a countable set in $\mathcal{H}$. We say $\sum_{j=1}^\infty \mathbf{v}_j$ **converges** in $\mathcal{H}$ if the partial sums $\mathbf{s}_k = \sum_{j=1}^k \mathbf{v}_j$, $k = 1, 2, \ldots$ form a Cauchy sequence in $\mathcal{H}$. The **sum s** of the convergent series is the limit of the Cauchy sequence $\{\mathbf{s}_k\}$.

Theorem A3. Let $\{\mathbf{u}_j\}$ be an ON set in $\mathcal{H}$ and let $a_j \in \mathcal{C}$, $j = 1, 2, \ldots$. Then $\sum_{j=1}^\infty a_j\mathbf{u}_j$ converges in $\mathcal{H}$ if and only if $\sum_{j=1}^\infty |a_j|^2 < \infty$.

This theorem is not true for pre-Hilbert spaces because there the partial sums of $\sum_{j=1}^\infty a_j\mathbf{u}_j$ may form a Cauchy sequence which does not converge.

Definition. An ON sequence $\{\mathbf{u}_j\}$ in $\mathcal{H}$ is an **orthonormal basis** (**ON basis**) if every $\mathbf{u} \in \mathcal{H}$ can be expressed as $\mathbf{u} = \sum_{j=1}^\infty a_j\mathbf{u}_j$ for some constants $a_j \in \mathcal{C}$.

If $\{\mathbf{u}_j\}$ is an ON basis then the constants a_j in the expansion of $\mathbf{u} \in \mathcal{H}$ are uniquely determined. Indeed

$$(\mathbf{u}, \mathbf{u}_j) = \lim_{k\to\infty} \left(\sum_{l=1}^k a_l\mathbf{u}_l, \mathbf{u}_j\right) = \lim a_j = a_j,$$

so $a_j = (\mathbf{u}, \mathbf{u}_j)$, the Fourier coefficient of $\mathbf{u}$ with respect to $\mathbf{u}_j$. Thus,

$$\mathbf{u} = \sum_{j=1}^\infty (\mathbf{u}, \mathbf{u}_j)\mathbf{u}_j. \tag{A.10}$$

Furthermore,

$$||\mathbf{u}||^2 = \lim_{k\to\infty} \left|\left| \sum_{j=1}^k (\mathbf{u}, \mathbf{u}_j)\mathbf{u}_j \right|\right|^2 = \sum_{j=1}^\infty |(\mathbf{u}, \mathbf{u}_j)|^2. \tag{A.11}$$

This is the **Parseval equality**.

Theorem A4. An ON sequence $\{\mathbf{u}_j\}$ is an ON basis for $\mathcal{H}$ if and only if the only vector $\mathbf{v} \in \mathcal{H}$ such that $(\mathbf{v}, \mathbf{u}_j) = 0, j = 1, 2, \ldots$, is $\mathbf{v} = \boldsymbol{\theta}$.

Theorem A5. Every separable Hilbert space $\mathcal{H}$ has an ON basis.

Indeed $\mathcal{H}$ has an infinite number of ON bases if $\dim \mathcal{H} > 0$.

Definition. Let $\mathcal{H}_j$, $j = 1, 2, \ldots$, be Hilbert spaces, where j runs over a finite or countably infinite number of values. Let $\mathcal{H} = \sum_{j=1}^{\infty} \oplus \mathcal{H}_j$ be the set of all sequences

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j, \ldots), \qquad \mathbf{x}_j \in \mathcal{H}_j,$$

such that $\sum_{j=1}^{\infty} ||\mathbf{x}_j||^2 < \infty$, where $||\mathbf{x}_j||$ is the norm of $\mathbf{x}_j$ in $\mathcal{H}_j$. Then $\mathcal{H}$ is an inner product space with operations

$$a\mathbf{x} = (a\mathbf{x}_1, \ldots, a\mathbf{x}_j, \ldots), \qquad a_j \in \mathfrak{C},$$

$$\mathbf{x} + \mathbf{y} = (\mathbf{x}_1 + \mathbf{y}_1, \ldots, \mathbf{x}_j + \mathbf{y}_j, \ldots), \qquad \mathbf{x}, \mathbf{y} \in \mathcal{H},$$

$$(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{\infty} (\mathbf{x}_j, \mathbf{y}_j),$$

where $(\mathbf{x}_j, \mathbf{y}_j)$ is the inner product in $\mathcal{H}_j$. Here $\mathcal{H}$ is called the **direct sum** of the Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_j, \ldots$.

The verification that $\mathcal{H}$ is an inner product space under the above operations is similar to the corresponding proof for l_2. Moreover, by mimicking the completeness proof for l_2 one can show that $\sum \oplus \mathcal{H}_j$ is a Hilbert space.

Let $\mathcal{H}$ be a Hilbert space. A linear operator $\mathbf{T}: \mathcal{H} \longrightarrow \mathcal{H}$ is **bounded** if

$$||\mathbf{T}|| = \sup_{||\mathbf{u}||=1} ||\mathbf{T}\mathbf{u}|| < \infty, \tag{A.12}$$

i.e., if the least upper bound of the set $\{||\mathbf{T}\mathbf{u}||: ||\mathbf{u}|| = 1\}$ is finite. If $\mathbf{T}$ is bounded the number $||\mathbf{T}||$ is called the **norm** of $\mathbf{T}$. The sum, product, and scalar multiplication of bounded operators are defined exactly as in the finite-dimensional case.

Lemma A5. Let $\mathbf{S}, \mathbf{T}$ be bounded operators on $\mathcal{H}$. Then (1) $||\mathbf{T}\mathbf{u}|| \leq ||\mathbf{T}|| \cdot ||\mathbf{u}||$, $\mathbf{u} \in \mathcal{H}$; (2) $||\mathbf{S} + \mathbf{T}|| \leq ||\mathbf{S}|| + ||\mathbf{T}||$; (3) $||\mathbf{S}\mathbf{T}|| \leq ||\mathbf{S}|| \cdot ||\mathbf{T}||$; (4) $||a\mathbf{T}|| = |a| \cdot ||\mathbf{T}||$, $a \in \mathfrak{C}$. In particular the sum and product of two bounded operators are bounded operators.

The proof of these results is identical with the proof in Section 5.1 of the corresponding results for operators on finite-dimensional spaces.

Lemma A6. If $\mathbf{s} = \sum_{j=1}^{\infty} \mathbf{v}_j$, where $\sum \mathbf{v}_j$ is a convergent series in $\mathcal{H}$ and $\mathbf{T}$ is a bounded operator, then $\mathbf{T}\mathbf{s} = \sum \mathbf{T}\mathbf{v}_j$.

A bounded operator $\mathbf{T}$ from $\mathcal{H}$ onto $\mathcal{H}$ is **unitary** if $(\mathbf{Tu}, \mathbf{Tv}) = (\mathbf{u}, \mathbf{v})$, all $\mathbf{u}, \mathbf{v} \in \mathcal{H}$, i.e., $\mathbf{T}$ preserves inner product. The operator $\mathbf{T}$ is **symmetric** or **self-adjoint** if $(\mathbf{Tu}, \mathbf{v}) = (\mathbf{u}, \mathbf{Tv})$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}$.

The **matrix** $T = (T_{kj})$ of a bounded operator $\mathbf{T}$ with respect to the ON basis $\{\mathbf{u}_j\}$ is defined by

$$\text{(A.13)} \quad \mathbf{Tu}_j = \sum_{k=1}^{\infty} T_{kj}\mathbf{u}_k, \qquad T_{kj} = (\mathbf{Tu}_j, \mathbf{u}_k), \qquad j, k = 1, 2, \ldots .$$

To the sum of two operators $\mathbf{S} + \mathbf{T}$ corresponds the sum of their matrices and to the product $\mathbf{ST}$ corresponds the matrix product:

$$\text{(A.14)} \qquad (S + T)_{kj} = S_{kj} + T_{kj}, \qquad (ST)_{kj} = \sum_{l=1}^{\infty} S_{kl}T_{lj}.$$

The **adjoint** $\mathbf{T}^*$ of the bounded operator $\mathbf{T}$ is defined by the relation

$$\text{(A.15)} \qquad (\mathbf{Tu}, \mathbf{v}) = (\mathbf{u}, \mathbf{T}^*\mathbf{v}), \qquad \text{all} \quad \mathbf{u}, \mathbf{v} \in \mathcal{H}.$$

In particular, if T_{kj} are the matrix elements of $\mathbf{T}$ with respect to an ON basis $\{\mathbf{u}_j\}$ then the corresponding matrix elements of $\mathbf{T}^*$ are

$$(T^*)_{kj} = \bar{T}_{jk}: \quad \mathbf{T}^*\mathbf{u}_j = \sum_{k=1}^{\infty} \bar{T}_{jk}\mathbf{u}_k .$$

Thus $\mathbf{T}^*$ is a uniquely determined linear operator on $\mathcal{H}$. Moreover, $\mathbf{T}^*$ is bounded and $||\mathbf{T}|| = ||\mathbf{T}^*||$. Note that $\mathbf{T}$ is self-adjoint if and only if $\mathbf{T} = \mathbf{T}^*$.