THE PROBABILISTIC ROOTS OF THE QUANTUM MECHANICAL PARADOXES

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Indice

O. The goal of any mathematical investigation of the foundations of a physical theory is to clarify to what extent the mathematical formalism of that theory is uniquely determined by some clearly and explicitely stated physical assumptions. The achievement of that goal is particularly relevant in the case of the quantum theory where the novelty of the formalism, its being far away from any immediate intuition, the substantial failure met, for many years, by any attempt to deduce the quantum formalism from plausible physical assumptions, intersected with the never solved problems concerning the interpretation of the theory. That with quantum theory a new kind of probability theory was involved, was clear since the very beginnings of quantum mechanics (cf. [28]), even if it was not so clear which of the axioms of classifical probability had to be substituted, which physically meaningful statement had to replace it, how and if a physically meaningful statement could justify the apparently strange quantum mechanical formalism. The lack of clear answers to these questions had a tremendous impact on the process of interpretation and misinterpretation of quantum theory. The attempts to answer these questions motivated the development of a new branch of probability theory -quantum probability- and led to definite mathematical answers to these questions. In the present paper we want to discuss how these mathematical results allow to solve in a rather natural way some old problems concerning the interpretation of quantum theory and its mathematical foundations.

1 Introduction

Assume that some astronomers of a civilization, which considers euclidean geometry not as one of the possible mathematical models of space but as the obvious expression of the properties of physical space and the only possible one, measure the angles among three stars which they are sure to be coplanar and find that the sum of the inner angles of the stellar triangle differs from π.

Calling "Euclid's equality" the theorem according to which the sum of the inner angles of a triangle is π, we would expect that in order to conciliate the belief that the euclidean theorems are self-evident truths with the fact that the experimental data violate "Euclid's equality", these astronorners might be led to strange statements. May be some of them would suggest that stars have only a virtual existence when we don't look at them and that only when we use our telescope to perturb this indistinct virtual being they acquire a definite position; some other might advocate the use of a "non-local astronomy" according to which we cannot describe three stars within a single classical geometrical model, but we need a different geometrical model for each pair of stars; and so on... .

Some people will probably look at the anecdote above as to a ridiculously absurd one; yet substituting physicists for astronomers, electrons and similar objects for stars, and probability for geometry, the fanciful anecdote becomes contemporary history: history of the interpretation of quantum theory.

The present paper, as well as several previous ones [1], [2],... [12] is devoted to the exposition of a new approach to that problem which can be synthetized in the following three statesments:

I) Bayes' axiom (cf. Section 3) plays for probability the role played by the parallel axiom for geometry.

II) All the paradoxes of quantum theory arise from the implicit or explicit application of Bayes' axiom (or of the theorem of composite probabi1ities which is an equivalent formulation of it, cf.Lemma **??**) to the statistical data of quantum theory. This application being unjustified both physical]y and mathematically.

III) The whole mathematical formalism of quanturn theory can be deduced, within the framework of Schwingers algebra of measurements, from a single axiom, which is a model independent formulation of a consequence of Heisenberg indeterminacy principle.

Of course, the main conceptual points of this thesis have a long history in quantum theory. The fact that Bayes' definition of conditional probability (cf. formula (**??**) in the following) might be meaningless in a quantum mechanical context goes back at least to a 1937 paper of Strauss and has been clearly stated by Suppes [15]. We have shown that this formula is mathematically incompatible with the experimental data of quantum theory (cf. [6], [8], and the following S. 2). The proof given in [6] is a generalization of Wigner's proof [51] of Bell's inequality [14]; while the more complete results in [8] use different techniques. The remark that Wigner's proof of Bell'S inequality concerns in fact transition probabilities (and not, as in Bell's origina1 version, correlations) is due to Bub [16]. The possibility of realizing a given set of transition probabilities within the complex Hilbert space framework has been studied by M. Roos (cf. [37], [38] and these proceedings) in view of a possible experimental verification of the superposition principle (cf. also [9] and S. 5). Roos' inequality [38] goes in the converse direction as Bell's one: the former gives conditions for the realization of a given set of transition probabilities within the Hilbert space model of quantum theory; the latter do the same for the usual probabilistic model (cf. S. [8] for a precise discussion). The first necessary and sufficient condition for the existence of a kolmogorovian model compatible with a given set of joint probabilities of pairs of two-valued observables is due to Corleo, Gutkowski, Masotto and Valdes [17], [25]; their condition is equivalent to the one formulated in [8] in terms of transition probabilitis. Recently Suppes and Zanotti([47][48], and these proceedings) have given an independent proof of this result using correlations as statistical data, like in the original paper by Bell. The use of transition probabilities instead of correlations has the advantage to be applicable to individual systems (and not only composite ones) and this helps in clearing the ground from confusions concerning the relevance of "locality arguments" for such questions. The suggestion to interpret Bell's inequalities as a mathematical and experimental evidence that there are in nature some statistical data which cannot be fit into the framework of the classical probabilistic model is also contained in an early paper of G. Lochak [31] and in a recently appeared paper of I. Pitowsky [33] (I am grateful to H.Van den Berg for pointing out to me the latter refer- ence).

The idea that the basic object of quantum theory are the transition (or conditional) probabilities goes back to N. Bohr (a good exposition of Bohr's thoughti is in[39]); while A. Landé [30] made the first attempt to deduce the Hilbert space structure from an analysis of the experimentally given transition probabilities. The remark that "... The circumstance that therefore not the probabilities themselves but their amplitudes follow the usual rules of probability calculus.." goes back to Jordan [28 1 who, immediateiy after, adds that this circum- stance "...can, in a convqnient way, be denoted as interference of probability.. Looking at the just mentioned paper by Jordan one can easily recognize that the "...usual rules of probability calculus..." he refers to are in fact the theorem of composite probabilities, which must be replaced by the principle of conposite amplitudes.

This analysis is pushed forward by Feynman who further recognizes [20], [21] that the principle of composite amplitudes alone is sufficient to deduce the whole quantum mechanical formalism. Both the above mentioned authors implicitely assume the vaiidity of Bayes' postuiate and this leads to a recurrent confusion in their papera between the theorem of composite probability and the principle of additivity of probability on disjoint events: as argued in [7] (cf. also S. 2, 3 in the following) this is not a technical point because the measurement apparatus needed to measure the two kind of probabiiitiea are different. The fact that the act of meaaurement without taking into account the reauit re-eatabiiahea the validity of the theorem of composite probability (cf. [7] for more details) was correctly stated in probabilistic terms by W. Heisenberg (cf. [26] , pg. 61) without even mentioning such notorious terminology such as "collapse", "conscience",... Finally we owe to J.Schwinger [41] the introduction of the useful formalism of the "algebra of measurements" in terms of which the deep connection between the algebra of quantum theory and its statistics is best understood: in fact the main result in the classification of those non-kolmogarovian models which we call of Heisenberg type" (cf. S. 9) is that the experimentally measured transition probabilities determine (in an essentially unique way) the usual quantum-mechanical algebra.

The monumental work of J. von Neumann gave rise to the theory of von Neumann aigebras and was originated as a generalization of the quantum probabilistic formalism. Since von Neumann the developement of the theory of operator algebras has always been deeply connected with the quantum mechanical formalism. We mill not mention here this development, as well as the several other approaches to the foundations of quantum theory (cf. G.W. Mackey [32], J.M. Jauch [27], C. Piron [34] I.E. Segal [42],...), because here we will not be concerned with the inner developement or generaiization of the quantum probabiiistic formaiism but, rather, with those resuits, both experimentai and mathematical, which prove without any possible ambiguity the necessity of introducing non-kolmogarovian models in the statistical description of nature, and the relevance of this point of view for the solution of the interpretative problems of quantum theory. Our emphasis in the present paper will be on interpretation; for the proofs of the nathematical resuitm and for a more thorough discussion of some conceptuai anaiyses which me shaii pniy sketch me miii refer to published papers (mith a few exceptions where we refer to preprints). In conclusions let us emphasize that the main differences between the point of view of non-kolmogorovian probabiiity and the known proposals of abandoning the ciassical aristotelic logic (when we don't look at objects they do not obey the usual logic) or the usual concept of reallty (mhen me don't look at observables they don't take their values) are the following:

I) The necessity of abandoning the kolmogorovian model of probability can be proved only in terms of experimentally measurable quantities (cf. S. 2); while the necessity of giving up aristotelean logic or the usual concept of realism can be motivated only postulating the validity of Bayes'axiom even in situations in which it cannot be experimentally checked.

II) If we accept the necessity of using non-kolmogarovian models in the description of quantum theory, then we don't need to abandon neither aristotelean logic nor the classical concept of realism. While if we abandon either of these concepts, nevertheless we must abandon the kolmogarovian model because the fact that the experimental data don't fit into this model is independent on any interpretation.

III) The traditional interpretations of quantun theory compel us to make definite statements on events which are in principle out of any experimental control (observables not looked at don't assume values or do not obey usual logic). Quantum probability doesn't need to get involved in the problem of how things behave when nobody is looking at them.

2 The oldest Bell's inequality: the two-slit experinent

As R. Feyrman says ([22] pg. 146) the two-slit experiment ... is formulated so to include all the misteries of quantum mechanics.." Confirming this statement, and in order to give an intuitive support to the more general considerations which will follow, we will illustrate the main contention of quantum probability theory starting from an analysis of the 2-slit experiment. In fact, for lack of space, we will analyze only "one-half" of the two slit experiment, i.e. the part concerning the non validity of the theorem of composite probabilities when no attespt is made to decide which of a set of alternatives is choosen. An exhaustive discussion of the "other half" of the experiment, i.e. the fact that the formal validity of that theorem is re-established by the physical possibiltty of distinguishing among the alternatives, is contained in the paper [7].

The scheme of the experiment is well known and we recall it just to fix the notations: a source S emits identically prepared particles (by this we mean that some observable of the particles - energy, frequency, ... - has a definite value) to wards a screen $\sum_1$ with two slits, denoted 1 and 2. The particles which pass the screen $\sum_1$ are collected on another screen $\sum_2$ parallel to $\sum_1$ and, denoting X a small portion of the screen $\sum_2$ one counts the relative frequencies (probabilities) of the particles hitting the region X - i.e. the quotient $N(X)/N$ of the number $N(X)$ of particles hitting the region X over the number N of particles hitting the screen $\sum_2$. This counting is carried out in three different experimental situations:

a) with both slits 1 and 2 open

b) with slit 1 closed and slit 2 open

c) with slit 2 ciosed and slit 1 open.

Let us denote $P(X)$, $P(X|2)$, $P(X|1)$ the probability of arrivai in X in case (a), (b), (c) respectively. $P(X|j)(= 1, 2)$ is the conditional probability that a particle arrives in X given that we know it passed through the slit j. The event the particle reached the region X of the screen E" will be simply denoted X. The theorem of composite probabilities in this case simply amounts to the statement:

$$P(X) = P(1) \cdot P(X|1) + (P(2) \cdot P(X|2) \tag{1}$$

where $P(j)(j = 1, 2)$ denotes the a priori probability that a particle passes through hole j; often one assumes that $P(1) = P(2) = 1/2$, but nothing in the following analysis will depend on these values or even on our knowledge of them therefore we leave them unspecified. It is well known however that, according to the experimental data

$$P(X) \neq P(1) \cdot (P(X|1) + P(2) \cdot P(X|2) \tag{2}$$

Since the theoretical deduction of equality (**??**) involves only elementary steps, anybody can realize that the experimental validity of inequaiity (**??**) mines the foundations of some fundamental principle, as it is always the case when something previously considered very elementary, turns out to be questionable or false. Let us therefore make explicit the elementary steps which lead from the experimental data (i.e. $P(X), P(X|1), p(x|2)$) to the theoretical equality (**??**) because we know from the experiments that at least one of these steps is founded on an empirically false prejudice. We will denote 1, 2 - the event that the particle passes through hole 1, 2 and with X the event that the particle reaches the screen $\sum_2$ at X.

Step I - If a particle reaches the screen E2 then it passed at least through one of the two slits. We will express this fact with the identity:

$$1 \vee 2 = I \text{event always realized} \tag{3}$$

A particle can pass through one and only one of the slits at a time:

$$1 \wedge 2 = \oslash \text{impossible event} \tag{4}$$

Step II- The rules of classical aristotelean logic allow us to deduce from (**??**) and (**??**) the identities:

$$X = (X \wedge 1) \vee (X \wedge 2) \tag{5}$$

$$\oslash = (X \wedge 1) \wedge (X \wedge 2) \tag{6}$$

which express the fact that a particle can arrive in X only if it has passed through one and only one of the slits 1 or 2.

Step III - Tge postulate of additivity of the probabilities of disjoint events and (**??**),(**??**) imply that the identity

$$P(X) = P(X \wedge 1) + P(X \wedge 2) \tag{7}$$

Step IV- The joint probabilities $P(X \wedge j)(j = 1, 2)$ are related to the conditional probabilities $P(X|j)$ through Bayes' identity

$$P(X|j) = \frac{P(X \wedge j)}{P(j)}; \quad j = 1, 2 \tag{8}$$

Step V- The identities (**??**) and (**??**) imply the theorem of composite probabilities, i.e.:

$$P(X) = P(1) \cdot P(X|1) + P(2) \cdot P(X|2) \tag{9}$$

The conclusion Step V follows, as a trivial mathematical consequence from the preceeding four steps. Since this conclusion is experimentally wrong, at least one of these steps must be wrong. Which one? All the different interpretations of quantum theory can be classified according to which of the above four steps they consider to be the first wrong one. The Copenhagen interpretation, the most widespread at present, confutes the first step. This point of view is clearly expressed by the words of Feynman ([20], pag. 536) ... we concluded on logical bases that, since

$$P(X) \neq P(1) \cdot P(X|1) + P(2) \cdot P(X|2) \tag{10}$$

[here we adapt Feynman's notations to ours] it is not true that the electron passes either through hole 1 or through hole 2..." And, shortly after ([20] Pg. 538), Feynman himseif explains that in order to be compatible with the experimental data, this sentence should be understood in the sense that "...when no attempt is made to determine through which hole the electron passes one cannot say that it must pass through one or the other hole..". Thus the Copenhagen school from one side rightly asserts that physics should not make statements which in principle cannot be subjected to any experimentai verification; from the other, to avoid contradiction with the inequality (**??**), is led to state that when no attempt is made to determine through which hole the electron passes then at least one of the events (**??**) or (**??**) must not happen and this is a typical such statement. Similarly the quantum logic point of view [35], even admitting Step I, confutes Step II which uses the distributivity axiom of the boolean propositional calculus. The arguments of Feynman can be applied also to this case and show (cf. [11] for detaiis) that the non validity of the distributive axiom should concern particles only when nobody looks at them: the experiments made to check (**??**) or (**??**) confirm their validity.

The non validity of Step III, i.e. of the additivity postulate of probability, has been considered by some authors; for exampie P. Jordan (cf. Postulate (C.)) of[28]) and Feynman ([23] , pg. 1.10) who propose to substitute for it the additivity postulate for amplitudes. However these authors tend to look at the non validity of (**??**) as a consequence of the non validity of Step I (this is particularly explicit in Feynman [20] ,pg. 536). Moreover, the additivity postulate for probabilities, whenever experimentally verified holds and when no experimental verification is possible, as it is the case for the right hand side of (**??**), its validity can be mantained at a purely existencial level in the following sense: accepting Steps I and II and the frequency interpretation of probability, then all particles arriving in X can be subdivided into two classes, those coming from hole 1 and those coming from hole 2, the cardinalities $N(X\wedge 1)$ and $N(X\wedge 2)$ of these classes might not be experimentally measurable but, whatever they are, the relative frequencies $P(X \wedge 1)$, $P(X \wedge 2)$ must satisfy (**??**).

The basic remark of the quantum probability point of view is that (**??**) is equivalent to (**??**) only if Bayes' identity (**??**) is postulated. Let us make more explicit what does this mean: first of all remark that, as Feynman shows, the right hand side of (**??**) is not measurable without altering the conditions of the experiment. Therefore what we are really doing when postulating the validity of (**??**) is to postulate the existence of four strictly positive numbers $P(1)$, $P(2)$, $P(X \wedge 1)$, $P(X \wedge 2)$ subjected to the relations:

$$P(1) + P(2) = 1 \tag{11}$$

$$P(Z \wedge 1) + P(X \wedge 2) = P(X) \tag{12}$$

$$\frac{P(X \wedge 1)}{P(1)} = P(X|1) \tag{13}$$

$$\frac{P(X \wedge 2)}{P(2)} = P(X|2) \tag{14}$$

But this is a system of four equations in four unknowns for which the existence of a solution is not a question of postulates but something which can be uniquely decided once given the "coefficients" $P(X)$, $P(X|1)$, $P(X|2)$. It is a simple computation to show that the system above admits a strictiy positive solution if and only if

$$0 < \frac{P(X) - P(X|2)}{P(X|1) - P(X|2)} < 1 \tag{15}$$

therefore to postulate the validity of Bayes' identity (**??**) is equivalent, in our case, to postulate the validity of the following, far from obvious, statement: the laws of nature must be such that, however choosen three events

X, 1, 2, the pro- babilities $P(X)$, $P(X|1)$, $P(X|2)$ satisfy the inequalities (**??**). But, since the probabilities $P(X)$, $P(X|1)$, $P(X|2)$ are experimentally measurable, whether or not the laws of nature satisfy (**??**) is not a question of postulates, but of experiments. Many experiments have been done (cf. in particular the papers by Gozzini, Rauch, Zeilinger, in these proceedings) and they have indisputably proved that there are natural phenomena which violate (**??**) or equivalently, that there are in nature sets of statistical data to which Bayes' formula for the conditional probabilities, and therefore the whole classical model of probability theory, cannot be applied for mathematical rather than interpretative reasons. Since the inequalities (**??**) are interpretation free, neither the Copenhagen nor the quantum logic interpretation can help preventing the use of a non-kolmogorovian probabilistic model; on the contrary once accepted (as we must in any case) the empirical groundiessness of the Bayes' postulate, one is not committed to any statement on the behaviour of things when nobody looks at them; in particular one can quietly believe, if he likes, that an observable assumes one and only one of its values even if nobody looks at it. Summing up, since the inequalities (**??**) are equivalent to the inequality (**??**), we can conclude that the whole debate on the interpretation of quantum mechanics began, many years ago, with a discussion on a Bell's type inequaiity.

The discussion above concerns a particular, even if important, case. We must now show that the non validity of Bayes' axiom is really universal for quantum theory and that all the interpretative problems of this theory arise from an injustified application of this axiom. This will conclude the critical part of our programme. The constructive part will consist in the explicit formulation of the postulates of the non- kolmogorovian probabilistic niodels and in the investigation of the structure and the applications of these models.

3 The hidden postulate of classical probability

Let A, B denote two events; $P(A|B)$ will denote the conditional probability of the event A given that one knows that the event B has happened. If A, B denote states of a system one also speaks of transition probability. Physically conditioning with the event B" means that one has prepared an ensemble of systems of the same type (say neutrons) so that on each of the systems the event 3 is verified (e.g. all the systems have spin +1 in the b-direction in appropriate units). The joint probability of the events A and B, i.e. the probabiiity that both A and B happen, is denoted $P(A \cap B)$. In the following we will call conditional probabiiity with condition B, any probabiiity measure $P(\bullet|B)$ satisfying the identity: $P(B|B) = 1$.

Bayes Axiom. For any two events A and B, the conditional probabiiity $P(A|B)$ and the joint probability $P(A \cap B)$ are related by the identity

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{16}$$

In most probability textbooks the identity (**??**) is not formulated as an axiom but as a definition (even an informai definition, cf. [29]). From the physical point of view, whenever the two sides of the identity (**??**) can be experinentally measured with different physical operations, the validity of the identity (**??**) cannot be taken for granted. If, moreover, the right hand side cannot be experimentally measured but the left hand side can, then the attempt to define the forner in terms of the latter is equivalent to postulate the existence of two numbers in [0,1] satisfying (**??**). For a single pair A, B one can always find such numbers; but for three or more events sone constraints anse and existence is no longer granted. More specifically: if $A_1, ..., A_n$ are events then one can trivially find $\binom{n}{2}$ probability measures $P_{jk}(j, k = 1, ..., n)$ satisfying

$$P(A_j|A_k) = P_{jk}(A_j \cap A_k)/P_{jk}(A_k) \tag{17}$$

however a single probability measure $P(\bullet)$ satisfying

$$P(A_j|A_k) = P(A_j \cap A_k)/P(A_k) \tag{18}$$

for each pair of indices $j, k = 1, ..., n$ might not exist. By a non-local" classical probabilistic model for the set of transition probabilities $\{P(A_j|A_k) : j, k = 1, ..., n\}$ we mean a family of $\binom{n}{2}$ - or a few less - probability measures satisfying (**??**) for each j and k. The term locality" in the above has historical reasons and has nothing to do with space-time. The use of a different probability measure for each pair (A_j, A_k) is justified by saying that the probability measure should take into account the experimental setup needed to verify the events A_j and A_k. Me won't discuss in the following the advantages and disadvantages of non-local models but rather the consequences of the erroneous implicit assumption that for any set of con- ditional probabilities $\{P(A_j|A_k) : j,k = 1,...,n\}$ there is a single probability measure $P(\bullet)$ satisfying (**??**) for any pair j, k. First of all renark that if $B_1,...,B_n$ are disioint events and $A \subset \cup_{j=1}^n B_j$. then the validity of Bayes' identity

$$P(A|B_j) = P(A \cap B_j)/P(B_j) \tag{19}$$

for each $j = 1,...,n$ implies the so-called theorem of composite probabilities:

$$P(A) = \sum_{j=1}^{n} P(B_j)P(A|B_j) \tag{20}$$

The converse implication also holds, more precisely one has:

Lemma 1 *Let (B_j) be a family of disjoint events and let for each B_j be given a conditional probability $P(\bullet|B_j)$ (in the sense specified at the beginning of this S.). The following statements are equivalent:*

*I) for each $A \subseteq \bigcup_{j=1}^n B_j$, Baye's identity (**??**) holds.*

*II) for each $A \subseteq \bigcup_{j=1}^n B_j$, the theorem of composite probabilities (**??**) holds.*

Proof 1 *The implication $(i) \Rightarrow (ii)$ is trivial. To proove the converse, remark that since $P(B|B) = 1$, then*

$$P(A|B_j) = P(A \cap B_j|B_j) \tag{21}$$

*hence it will be sufficient to prove (**??**) in the assumption that $A \subseteq B_k$ for some k. In this case, if (**??**) holds, using (**??**) one finds: $P(A) = P(B_k)P(A|B_k)$ which is the thesis.*

Remark-Usually one considers the case in which $\bigcup_{j=1}^n B_j = I$ (i.e. the event always realized). In this case the inclusion $A \subseteq I$ is true for arbitrary A.

Thus the relation between Bayes' axiom and the theorem of composite probabilities is the same as the one between the parallel axiom and the theorem that the sum of the inner angles of a triangle is π: they are equivalent. This establishes in general what we have shown in the preceeding S. in the case of the two-slit experiment. In particular this provides a tool to check the experimental validity of Bayes' axiom even in cases in which the right hand side of (**??**) Is not experimentally measurable.

4 Zeno's Paradox

Let us denote A a complete set of quantum mechanical observables, and $[A_t = a]$ the event that the value a is the result of a measurement, actually performed on a system, of the observable A at time t. In quantum mechanics, the event $[A_t = a]$ uniquely determines a pure state; hence the transition probabilities $P(B_u = b|A_t = a)$ where $t < u, B$ is any observable and b anv value of B, do not depend on the past history of the system. The mathematical formulation of this condition of lack of memory is called, in classical probability theory, a Markov property and is expressed by the validity of the equality:

$$\begin{gathered} P(B_u = b|[A_t = a] \cap [A^{(n)}_{t_n} = a_n] \cap ... \cap [A^{(0)}_{t_0} = a_0]) = \\ = P(B_u = b|A_t = a) \end{gathered} \tag{22}$$

for any choice of the observables $A, B, A^{(0)}, ..., A^{(n)}$ of their values $a, b, a_0, ..., a_n$; and of the times $t_0 < t_1 < ... < t_n < t < u$ (for a more detailed discussion of this point and its connec-tion with the quantum theory of measurement, cf'.[7]). In particular, choosing in (**??**) all the observablesequal to A and all the values equal to a, one finds:

$$\begin{gathered} P(A_t = a|[A_{t_n} = a] \cap [A_{t_{n-1}} = a] \cap ... \cap [A_{t_o} = a]) = \\ P(A_t = a|A_{t_n} = a) \end{gathered} \tag{23}$$

where $t_0 < t_1 < ... < t_n < t$ are arbitrary instants of time. Now let us postulate the validity of Baye's axiom for the left hand inside of (**??**), that is:

$$\begin{gathered} P([A_t = a] \cap [A_{t_n} = a] \cap ... \cap [A_{t_0} = a]) = \\ P(A_t = a|A_{t_n} = a) \bullet P([A_{t_n} = a] \cap ... \cap [A_{t_0} = a]) \end{gathered} \tag{24}$$

From (**??**), by iteration, we obtain:

$$\begin{gathered} P([A_t = a] \cap [A_{t_n} = a] \cap ... \cap [A_{t_0} = a]) = \\ P(A_y = a|A_{t_n} = a) \bullet P(A_{t_n} = a|A_{t_{n-1}} = a) \bullet ... \bullet P(A_{t_1} = a|)P(A_{t_0} = a) \end{gathered} \tag{25}$$

Conversely the validity of (**??**) for earch n implies (**??**).

Assuming the translation invariance of the transition probabilities $P(A_t = a|A_s = a')$ (surely true in quatum mechanics, in absence of exterior forces) and choosing $t_0 = 0; T_{k+1} - t_k = t/(n+1)$,(**??**) becomes

$$\begin{gathered} P([A_t = a] \cap [A_{t-\frac{t}{n+1}} = a] \cap ... \cap [A_0 = a]) = \\ P(A_{\frac{t}{n+1}} = a|A_0 = a)^{n+1} \bullet P(A_0 = a) \end{gathered} \tag{26}$$

It is a known fact that, substituting in the right hand side of (**??**) the quantum mechanical expression for the transition probability, i.e. (in obvious natations):

$$P(A_s = a|A_0 = a) = | < \psi_a, e^{\text{-isH}} \psi_a > |^2 \tag{27}$$

one reaches the paradoxical conclusion that

$$\lim_{n\to\infty} P([A_t = a] \cap [A_{t\left(\frac{n}{n+1}\right)} = a] \cap \ldots \cap [A_{\frac{t}{n+1}} = a]|A_0 = a) = 1 \tag{28}$$

Here again the formula which gives rise to the paradox follows (is even equivalent to) Bayes' axiom (**??**), and it is a simple exercise to verify that the quantum transition probabilities cannot fullfil that identity. To postulate the validity of (**??**) is erroneous also for conceptual reasons: the joint probabilities in the two sides of (**??**) are evaluated under different experimental conditions. The left hand side under the condition of making $n + 2$ measurements at times $t_0, ..., t_n, t$; the right hand side, under the condition of making $n+1$ measurements at time $t_0, ..., t_n$. Denoting $P_{t_0}, ..., t_n, t$ and $P_{t_0}, ..., t_n$ the corresponding probabilities, one knows that the family $(P_{t_0}, ..., t_n)$ is not an agreeing one: for example, restricting $P_{t_0}, ..., t_n, t$ to $(t_0, ..., t_n)$ means that you perform the measurement at time t and don't read the result, and since this is different from not doing the measurement, the resulting measure is not $P_{t_0,...,t_n}$.

This means that even if we want to postulate Baye's identity (**??**), that identity should connect $P_{t_0,...,t_n,t}$ with its restriction to $(t_0, ...t_n)$ and not, as implicitely assumed in the deduction of (**??**) from (**??**) with $P_{t_0,...,t_n}$.

Moreover, even postulating the validity of the identity (**??**) for this restriction, there will be no reason, nor physical neither mathematical, to suppose that the transition probabilities defined by it are given by the usual quantum mechanical formula (**??**) (calculations in simple examples show that in general they are quite different). Finally it should be remarked that the mathematical model for the quantum theory of measurement, deduced from quantum Markov theory (cf. 13 I), does not lead to Zeno's paradox.

5 Physical Superpositions

The general situation, illustrated by the two-slit experiment, is best explained again in a paper by Feynmann [21] . Let $A, B, C, ...$ denote some physical observables and a, b, c, their values. The identity:

$$P(A = a|C = c) = \sum_b P(A = a|B = b) \bullet P(B = b|C = c) \qquad (29)$$

(the sum being extended to all the values of B) follows from, and in fact is equivalent to, the theorem of composite probabiòities plus Dirac's jurnp assumption", which is nothing but the Markov Property as formulated in the preceeding S. (cf. 4.1.), i.e.:

$$P(A = a|[B = b] \wedge [C = c]) = P(A = a|B = b) \qquad (30)$$

where, in the left-hand side it is understood that the measurement of C has been performed before that of B. All the transition probabilities in (**??**) can be checked experimentally, and the results disprove the validity of (**??**). It is known that, in order to find agreement with the experiments, one must introduce a conditional probability ampiitude", i.e. a complex number $\psi(A = a|B = b)$ related to the conditional probability by the relation:

$$P(A = a|B = b) = |\psi(A = a|B = b)|^2 \qquad (31)$$

and satisfying the principle of composite amplitudes:

$$\psi(A = a|C = c) = \sum_b \psi(A = a|B = b) \bullet \psi(B = b|C = c) \qquad (32)$$

Feynmann comments this situation as follows [21]: ...If (**??**) is correct, ordinariiy (**??**) is incorrect [here again I have changed the numeration. NdA] . The logical error made in deducing (**??**) consisted, of course, in assuming that to get from a to c the system had to go through a condition such that B had to have some definite value b ." and further he addS that ...Looking at probability from a frequency point f view (**??**) simply results from the statement that in each experimemt giving a and c, B had some value. The only way (**??**) could be wrong is the statement B had sonie value" must sometimes be meaningless. Noting that (**??**) replaces (**??**) only under the circumstance that we make no attempt to measure B, we are led to say that the statement B had some value" may be meaningless whenever we make no attempt to measure B" But if B has no values when nobody looks at it, then what happens? The usually accepted answer is that the system is in a physicai superposition" of states in each of which B has a definite vaiue. Nobody has ever seen a physical superposition" of states. One needs such a concept oniy in case he has a strong emotionai affectiOn for Bayes' postulate. In fact it is only assuming that this postuiate is a truth of nature, that the non validity of (**??**) becomes a physical puzzle. As remarked by many authors, first of all Einstein, the concept of physical superposition" is not only unnecessary but harmfull. In fact, since the result of any act of ineasurement is always a definite value and never a superposition, the problem arises of when this physical transition: superposition → definite value, takes place; what causes this transition; how to describe it. The attempts to answer these problems gave rise to the quantum theory of measurement. A purely probabilistic approach to that theory has been described in [7] . A lucid exposition of the point of view, shared nowadays by most physicists, is described by R. Schlegel in [40]. We will come back to the existence of physical superpositions in the discussion of the EPR paradox (cf S. 7.); before this we need some more remarks on the respective regions of validity of the principle of composite amplitudes and the theorem of composite probabilities. The principle of composite amplitudes can be trivially deduced within the usual Hilbert space formulation of quantum theory. The converse deduction is also true, as specified by the following Proposition:

Proposition 1 *Let be given a family $F = \{A, B, C, ...\}$ of physical observables. Denote sp A the set of values of A, and assume that sp A has n (different) points for each $A \in F$. Let be given, for each pair $A, B \in F, a \in spA, b \in spB$, a complex nuntber $\psi(A = a|B = b)$ satisfying:*

$$\psi(A = a|A = a') = \delta_{a,a'} \tag{33}$$

$$\text{The map}(i, j) \to \psi(A = a_j|B = b_j); i, j = l, ..., n \tag{34}$$

is a positive definite Kernel on $(l, ..., n)$,

$$\psi(A = a|C = c) = \sum_b \psi(A = a|B = b) \bullet \psi(B = b|C = c) \tag{35}$$

$$\sum_a |\psi(A = a|B = b)|^2 = 1 \tag{36}$$

Then there exists a C-Hilbert space $\mathcal{H}$, and for every observable A an orthonormal basis $\{\varphi^A_a : a \in spA\}$ of $\mathcal{H}$ such that for any $A, B \in F, a \in spA, b \in spB$:

$$\langle \varphi^A_a, \varphi^B_b \rangle = \psi(A = a|B = b) \tag{37}$$

Conversely, for any C-Hilbert Space $\mathcal{H}$ and any family $(\psi\genfrac{}{}{0pt}{}{A}{a})$ of ortho-normal bases of $\mathcal{H}$, the function ψ defined by (??) satisfies (??), (??), (??), (??).

Proof 2 *Fin an observable $A \in \mathcal{F}$ and let $\langle$ be an Hilbert space with an ortho-normal basis $\{\varphi\binom{A}{a}\}$. Defining for any observable $B \in \mathcal{F}$ and $b \in spB$:*

$$\varphi\genfrac{}{}{0pt}{}{B}{b} = \sum_a \psi(A = a|B = b) \bullet \varphi\genfrac{}{}{0pt}{}{A}{a} \tag{38}$$

The family of o.n. bases $\{\varphi\genfrac{}{}{0pt}{}{A}{a} : A \in \mathcal{F}; a \in spA\}$ has the required properties. The converse is trivial.

Remark - The Hilbert space and the family of o.n. bases $(\varphi\genfrac{}{}{0pt}{}{A}{a})$ above are unique up to isomorphism, in the sense that if $\mathcal{H}$ is another Hilbert space and $(\chi\genfrac{}{}{0pt}{}{A}{a})$ another fainily of orthonormal bases, then there exists a unique unitary operator $V : \mathcal{H} \to \mathcal{H}'$ satisfying

$$V\varphi\genfrac{}{}{0pt}{}{A}{a} = \chi\genfrac{}{}{0pt}{}{A}{a}; \quad \forall A \in \mathcal{F}; \quad \forall_a \in spA \tag{39}$$

The vector $\varphi\genfrac{}{}{0pt}{}{A}{a}$ (better - the associated ray) is called the state of the physical system associated to the preparation $A = a$. The identity

$$\varphi\genfrac{}{}{0pt}{}{A}{a} = \sum_b \psi(B = b|A = a)\varphi\genfrac{}{}{0pt}{}{B}{b} \tag{40}$$

is the mathematical formulation of the fact that the state $\varphi\genfrac{}{}{0pt}{}{A}{a}$ is a superposition of states in wgich $B = b$

Summing up: the principle of composite amplitudesis equivalent to the superpositiOn principle. Therefore if we can deduce the principle of composite amplitudes from some physically plausible assumptions made on experimentally measurable quantities, we will have established quantum theory on a solid conceptual ground, independent]y on its empirica1 success. In [9] we have shown how this goal can be accomplished.

6 Mixtures, and the physical formulation of Baye's postulate

We will now rephrase Bayes' postulate in a language more familiar to the quantum physicists. We will deal on]y with a very special case, which is all we need for our discussion of the EPR paradox (of S. 7), and leave the obvious generalization to readers. Consider two events, denoted 1 and 2, defining two quantunt mechanical pure states, ψ_1, ψ_2. Assume that the events are disjoint and that at least one of them should happen:

$$wedge 2 = \phi; \quad 1 \vee 2 = I \tag{41}$$

Thus (ψ_1, ψ_2) is an orthonormal basis in a Hilbert space $calH$. As discussed at length in S. 2, 3, Baye's postulate in this case is reduced to the statement that for any event X one has:

$$P(X) = P(1) \bullet P(X|1) + P(2) \bullet P(X|2) \tag{42}$$

where $P(j)$ denotes the probability of the event $j (= 1, 2)$. Denoting Q_X the projection operator corresponding to X in the usual quantum formalism (**??**) can be written in the form:

$$P(X) = P(1) Tr(Q_X P_{\psi_1}) + P(2) Tr(Q_X P_{\psi_2}) \tag{43}$$

where P_ψ denote the rank one projection into the state ψ and Tr-the trace. The fact that (**??**) must hold for an arbitrary projection Q_X means that the statistical description of the system is given by the dendity matrix:

$$W = P(1) P_{\psi_1} + P(2) P_{\psi_2} \tag{44}$$

Summing up: when translated into quantum–mechanical formalism, Baye's postulate reduces to the following statement:

QMB. Let 1,2 be states of a physical system corresponding to the orthonormal basis (ψ_1, ψ_2) of the quantum mechanical state space of a system. If on an ensamble of systems we know with certainly that the fraction $P(1)$ is in the state 1 and the fraction $P(2)$ in the state 2, then the statistical description of this ensemble is given by the mixture (**??**). In the statement above, we purpously left unspecified how do we "know with certainty" that each system of the ensemble is either in state 1 or in state 2. In case our knowledge comes from an experiment, in the sense that all the systems have interacted with an apparatus which can in principle allow to decide which of the two alternatives is realized, then the above statement is very well verified by the experiment. In this case, using the mixture (**??**), we are not only expressing a physical property of the system, but also our knowledge that at a certain time to a measurement was performed on it. However, until a measurement is not made, we have not the right to describe the system with the mixture (**??**) because this would be equivalent to the application of Bayes' postulate to a situation in which the experiments have shown that it is usually not applicable.

The ostulate of realism has to do with the fact that each system of the ensemble is either in state 1 or in state 2, but in both, and this is quite different from the statement that the validity of the postulate of realism is equivalent to the validity of Bayes' axiom. Summiflg up: the phenomenon called collapse of the wave packet" when stated in probabil language sounds as follows: the act of an incomplete measurement re-establishes the validity of the theorem of composite probabilities. (cf.[7] for details).

7 The EPR paradox

Once accepted the idealization of instantaneous measurement" it is clear that the notion of ,physiCal superposition implies action at distance for earnple an instantaneous measurement of position would instantaneously change a particle whose positiOn is "superposed" throught the space into a localized particle-like object. Considering systems which are separated in spade-time but connected by a conservation law, the argument of Einstein, Podolsky and Rosen [19] shows that the same conclusion can be reached without using the idealization of instantaneous measurement. This is one aspect of the EPR paradox, the other one will be discussed in the following. From the point of view of quantum probability the superposition principle, i.e. the principle of composite amplitudes means only that we are describing a set of statistical data by means of a particular non kolmogorovian model and the empirical necessity to use such a model by no means implies the necessity to postulate the existence of something like a "physical superposition". Thus, from the point of view of quantum probability the argument, according to which quantum theory should imply the existence of superluminal signals, even if at an unobserable level, sterns from the fact that one has deduced a condlusion on the physical world (the existence of physicai superposition) from a demostrably wrong mathematical premise (the applicability of the theorem of composite probabilities to a given set of statistical data). A more sophisticated discussion of the EPR paradox is due to B. d'Espagnat [19] (cf. also the paper by F. Selleri- and Tarozzi [43] , which we will follow in our exposition). In it one doesn't make use of the noti-On of physicai superpOSi-tion but only needs the universally accepted difference between such mathematical objects as superpositions and mixtures. The argument of these authors starts from Bohm's version of the EPR paradox: one considers two particles, say 1 and 2, far away from each other, and the system (1,2) composed of these two particles. It is assumed that at time O, hence at any time before the measurement, the system (1,2) is in the singlet state

$$\psi_2 = \frac{1}{\sqrt{2}}(u_1^+ \otimes u_2^- - u_1^- \otimes u_2^+) \tag{45}$$

where $u_j^\pm (j = 1,2)$ are the eigenstates of the z-component of the spin of particle j. One comsiders an ensemble of N copies of the system (1,2) all prepared in the singlet state. With these notations the main steps of the argument are the following ([43], pag. 10):

1- At time t, one measures the z-component of the spin of particle 1 on each of the composed systems (1,2), while particle 2 is left isolated. Quantum theory says that approximatively for 1/2 of the systems the result obtained will be +1/2 (in appropriate units). This statement is confirmed by the experiments.

2- Denoting $\sigma_2^j(t)(j = 1,2)$ the z-component of the spin of particle j at time t, the singlet assumption and the conservation of spin implies that at each time t:

$$\sigma_z^1(t) + \sigma_z^2(t) = 0 \tag{46}$$

The prediction is confirmed by the experiments too. Therefore, since (**??**) is equivalent to:

$$\sigma_z^1(t) = +1/2 \Leftrightarrow \sigma_z^2(t) = -1/2 \tag{47}$$

whenever the result of the $\sigma_z^1(t_0)-$ measurement is $+1/2$, we know with certainty that $\sigma_z^2(t_0) = -1/2$. Thus (since we assume that our measurement is a first kind one) we know with certainty that for any $t \neq t_0$ one has

$$\sigma_z^1(t) = +1/2; \quad \sigma_z^2(t) = -1/2 \tag{48}$$

we will express that fact saying that the system (1,2) is the state $(+,-)$; similarly for the state $(-,+)$.

3- ...But, at time $t = t_0$, when particle 1 interacts with an instrument, nothing can happen to particle 2, which can be as far away as one whishes from 1. All what is true for 2 for $t \neq t_0$ must have been true before (namely for $t < t_0$)... [43] This means that if without acting on the z-component of the spin of particle 2 we know with certainty that $\sigma_z^2(t_0) = -1/2$ (resp. $+1/2$) then the same must be true before t_0.

4- Using the argument at step (2) we conclude that, before and after time t_0, the state of the system (1,2) was $(+,-)$ (resp. $(-,+)$).

5- But according to quantum theory, if at time t the system (1,2) is in state $(+,-)$ (resp. $(-,+)$) then, in the mathematical model, the system (1,2) is described by the density matrix $P_{u_1^+ \otimes u_2^-}$ (resp. $Pu_1^- \otimes u_2^+$) where, as usual, P_ψ denotes the rank one projection cnto the state ψ.

6- Because of step (1) we then know with certainty that in the ensemble of N system, approximatively 1/2 are in the state $(+,-)$ and the other half in state $(-,+)$.

7- From Step (6) and from the physical formulation of Bayes' postulate (cf. S. 6) we deduce that at any time before or later to the statistical description of the system (1,2) is given by the density matrix:

$$W = P(1)P_{u_1^+ \otimes u_2^-} + P(2)P_{u_1^- \otimes u_2^+} \tag{49}$$

8- But this contradicts, in an observable manner, the initial assumption that at any time before t_0, the system (1,2) was described by the singlet state $W_0 = P_{\psi_s}, \psi_s$ being the singlet state (**??**).

The conclusion of Selleri and Tarozzi is that the statement of step (3) above: . . .at time $t = t_0$ noting can happen to particie 2, which is very far away from 1..." is incompatibie with quantum mechanics. Since the above statement is a formulation of Einstein locality" (cf. [43], pag. 7), they conclude that quantum theory in incompatibie with Einstein locality.

However at step (7) above these authors bave (impliciteiy in I 13 I) ueed the physical fcrrnuiation of Bayes' postulate. Therefore a more correct restatement of their conciusion should be "...if we insist to appiy Bayes' postuiate to the statistie- al data of quantuni mechanics, then quantum theory is incompatibie with Einstein locality..." But why should one insist in applying Bayes' postulate to the statistical data of quantum theory, when both mathematics and the experiments provide evidence against this application?

8 Inequalities and statistical invariants

Having shown that there are in nature sets of statisticai data which cannot be described by the classical kolmogorovian probabilistic model, a number of questions naturally arise: given a set of statistical data, how to decide wether to describe it with the classical or with the quantum model? are there other models besides these two? etc... Until now the answer to the first question has been purely empirical: if the data come, say, from CERN, one uses the quantum model; if they come from more classical environments, one uses the kolmogorovian model. The notion of statistical invariant has been introduced to find a theoretical answer to this problem. This notion is best illustrated again by the example of the two slit experiment: in this case the statistical data are the probabilities $P(X), P(X|1), (P(X|2)$ and we have shown in S. 2 that a kolmogorovian model for these data exists if and oniy if

$$0 < \frac{P(X) - P(X|2)}{P(X|1) - P(X|2)} < 1 \tag{50}$$

The inequaiity (**??**) is an invariant of the set of statisticai data $P(X), P(X|1), (P(X|2)$ in the sense that changing these data in a way that (**??**) is still satisfied (or stili not satisfied) will not change the property of admitting (or non admitting) a kolmogorovian model. For the given set of statistical data, we can also ask the question of the existence of a compiex, a set of statistical data, we will cali a statistical invariant relative to this set, any famiiy of conditions which are ex- pressed oniy in terms of the initiai data and which are necessary and sufficient for the possibility of describing these data within the given probabilistic model. Let us formulate precisely this definition for the two best known probabilistic modeis: let $\mathcal{F} = \{A, B, C, ...\}$ be a family of n-valued observables ($n < \infty$, independent on the observabie, and all the values are distinct); let, for any pair of observables A, B and of their values a, b, be given a transition probability $P(A = a|B = b)$ considered as experimental data.

Definition 1 *A non kolmogorovia model for the family of transition probabilities* $\{P(A = a|B = b) : A, B \in \{; a \in spA, b \in spB\}$ *is defined by:*

I) A probability space (ω, δ, μ)

II) For any $A \in \mathcal{F}$*- a measurable partition* $\{A_a : a \in spA\}$ *of* ω *such that*

$$P(A = a|B = b) = \frac{\mu(A_a \cap B_b)}{\mu(B_b)} \tag{51}$$

for any $A, B \in \{; a \in spA, b \in spB$

Definition 2 *A C (resp R-orQ,...) Hilbert Space model for any above family of transition probabilities is defined by:*

I) A C (resp. R, $\underline{Q}-, ...$*) Hilbert space* $\mathcal{H}$

II) For any $A \in \mathcal{F}-$ *an orthonormal basis* $\{\varphi \begin{matrix} A \\ a \end{matrix} : a \in spA\}$ *such that*

$$P(A = a|B = b) = | < \varphi \begin{matrix} A \\ a \end{matrix}, \varphi \begin{matrix} B \\ b \end{matrix} > |^2 \tag{52}$$

for any $A, B \in \{; a \in spA, b \in spB$

The quatum mechanical transition probabilities must therefore satisfy the symmetry condition:

$$P(A = a|B = b) = P(B = b|A = a) \tag{53}$$

Under this condition for two ovservables a kolmogorovian model always exists if $n < +\infty$ but a C-Hilbert space might not exist in $n \geq 3$ (cf.[6]). For three two-valued observables the statistical invariants have been explicitely computed for the kolmogorovian, $C-$, $R-$, and quaternion models. They are fine enough to distinguish among the first three cases. For lack of space we refer to [8], [9] for a precise formulation and comments; however, what should be clear from the above few remarks, is that all the Bell's type inequalities [14], [17], [18], [24], [25], [9], [50], [51] have a simple probabilistic interpretation: they are necessary conditions on a set of statistical data for the existence of a kolmogorovian model.

9 Outline of the deduction of the quantum formalism

Let A be an observable relative to a given system, $a \in spA$ - a value of A. To the pair (A, a) we associate the elementary filter A_a corresponding to the ideal seiective measurement that from an ensemble of independent, identicaliy prepared systems accepts those for which the observable A takes the value a and rejects all the other ones (cf. [41]). Elementary filters can be applied in series, and the symbol $A_a \cdot B_b$ means that to the given ensemble one applies first the filter A_a and then B_b. In order to avoid possible confusions it is important to keep in mimd that the apparatus $A_a \cdot B_b$ should be considered as a whole, in the sense that one counts the particles coming out from the filter B_b but not those who pass the filter A_a and enter b_b. The analogous convenction will be stipulated for products of more than two fiiters. The above shows that a notion of time is implicit in the muitiplication of filter, however in first approximation we will not make it explicit, assuming that the effect due to time variation are negligible. In other words, what we are assuming is that, for very small $\epsilon > O$, the resuits of the apphication of the filters $A_a(t) \bullet B_b(t + \varepsilon)$ will be independent on ε and, by time stationariety, on t. This is what one means with the heuristic statement the filter B_b is applied ' immediately after" the filter A_a" Two elementary filters A_a, B_b are called compatible if the following conditions are satisfied:

I) it is possible to realize an apparatus, denoted $A_a + B_b$ such that for each single system emerging from it one knows with oertainty that either $A = a$ or $B = b$.

II)

$$A_a \bullet B_b = B_b \bullet A_a$$

As shown by J. Schwinger [41], the operations + and $\bullet$ enjoy the usual properties of addition and multiphication. The multiphication of an elementary filter A_a for a scalar p, between O and 1, can be defined as that apparatus - denoted $p\ A_a$- which from an ensemble of systems accepts at random only a fraction p of those for which $A = a$, and rejects all the other ones. These operations allow to embed the family of elementary filters in a larger mathematical structure, i.e. an algebra $\mathcal{A}$ with unity, over the reals the so-called Schwinger's algebra of mesurements.

With the goal of formulating in this context Heisenberg's indeterminacy principle, iet us distinguish in it two aspects: a purely existential statement, namely

HP1) There exist in nature pairs of incompatible observables, which is independent on the pair of observables involved; and a more specific statemerit concerning how much" the two observables are incompatibie - this one necessariiy depends on the pair of observables.

The statement (HP1) says only that the algebra of measurements cannot be commutative, but this is too little to give on idea of its structure. A more quantitative consequence of Heisenberg' principle is the so-called Dirac's jump assumption which, as seen in S. 4. can be formulated as a purely statistical property: the Markov property (cf. 4.1). This implies, in particular, that about any System emerging from the filter $A_a \bullet B_b \bullet A_a$, we can only say with certainty that for that system $A = a$ and not, as in the ciassical case, that $A = aandB = b$. Moreover, in general, not all the systems which passed the first filter A_a will pass also the filter B_b. Thus, if we consider two identic- ally prepared ensembies and apply to one of them the filter A_a and to the other one the filter $A_a \bullet B_b \bullet A_a$ then only a fraction of the systems which passed the former will pass the latter. It is an experimentaliy verifiable fact that this fraction is just the transition probabiiity $P(A = a|B = b)$. In symbols:

$$A_a \bullet B_b \bullet A_a = P(A = a|B = b)A_a \tag{54}$$

This statemerit, which establishes a definite connection between the structure of the algebra of measurements and the empirically given statistical data, will be assumed as the basic postulate of the quantum mechanical algebra of measurements. More precisely:

Definition 3 *Let T be any set and let be given, for any $x \in T$ an observable $A(x)$ with values $a_1(x), ..., a_n(x) (n < +\infty$, independent on x). Le tbe given, for any $x, y, \in T$, a transition probability matrix*

$$P(x, y) = \{P_{\alpha\beta}(x, y) : \alpha, \beta = 1, ..., n\}$$

$$P_{\alpha\beta}(x, y) = P(A(x) = a_\alpha(x) | A(y) = a_\beta(y))$$

(by definition $P(x, x) =$ the identity matrix).

We will say that the family of transition probability matrices $\{P(x, y) : x, y \in T\}$ admits an Heisenberg model if there exist:

I) a real associative algebra with unity, $\mathcal{A}$.

II) for each $x \in T$, a family $\{A_\alpha(x) : \alpha = 1, ..., n\}$ of elements of $\mathcal{A}$ satisfying:

$$A_a(x) \bullet A'_\alpha(x) = \delta_{\alpha\alpha'} A_\alpha(x); \forall \alpha, \alpha' \tag{55}$$

$$\sum_\alpha A_\alpha(x) = 1 \tag{56}$$

$$A_\alpha(x) \bullet A_\beta(y) \bullet A_\alpha(x) = p_{\alpha\beta}(x, y) A_\alpha(x) \tag{57}$$

The Heisenberg model above will be called standard if, denoting k the center of the algebra $\mathcal{A}$ one has:

III) For any $A \in \mathcal{A}$ and $\lambda \in k$

$$\lambda \bullet A = 0 \Leftrightarrow \lambda = 0 or A = 0$$

IV) For any pair $x, y \in T, x \neq y$, the products $\{A_\alpha(x) \bullet A_\beta(y) : \alpha, \beta = 1, ..., n\}$ are a linear basis of $\mathcal{A}$ over k

It will be called minimal if it is standard and k has the smallest possible dimension over R, i.e.

V) If $\{\mathcal{A}', k'\}$ is another standard Heisenberg model for the family $\{P(x, y)\}$ then k can be isomorphically embedded into k'.

The classification problem for Heisenberg models consists in describing which families of transition probabilities admit one projection onto Ka (x), *defined with respect to the an Heisenberg model and which Schwinger algebras anse as Heisenberg models of some set of transition probabilities. Our main result is the solution of the classification problem for standard Heisenberg models. To state the result we need one more definition:*

Definition 4 *Let $P = (p_{\alpha\beta})$ be an nxn transition probability matrix, and let k be a commutative, associative algebra the reals with unity. A k-valued matrix $U = (u_{\alpha\beta})$ satisfying:*

$$\sum_{\alpha}(u_{\alpha\beta})^* u_{\alpha\beta'} = \delta_{\beta\beta'} \tag{58}$$

$$\sum_{\beta} u_{\alpha\beta} \bullet (u_{\alpha'\beta})^* = \delta_{\alpha\alpha'} \tag{59}$$

where

$$(u_{\alpha\beta})^* = p_{\alpha\beta}/u_{\alpha\beta} \tag{60}$$

will be called a k-valued transition amplitude matrix for P (we assume $u_{\alpha\beta} \neq 0$)

Theorem 1 *Let T be a set; $\{P(x,y) : x, y \in T\}$ be a family of transition probability matrices. The following are equivalent:*

I) The family $\{P(x,y)\}$ admits a standard Heisenberg model with center k

II) For each pair , $x, y, \in T$, one has

$$P_{\alpha\beta}(x,y) = p_{\beta\alpha}(y,x) \tag{61}$$

and there exists a k-valued transition amplitude $U(x,y)$ for $P(x,y)$ such that the family $\{U(x,y)\}$ satisfies the conditions:

$$U(x,y) \bullet U(y,z) = U(x,z); \forall x, y, z \in T \tag{62}$$

$$U(x,x) = id$$

III) There exists a k-module $\mathcal{H}$ and, for each $x \in T$, a k-basis $(a_\alpha(x))$ of $\mathcal{H}$ such that, denoting $A_\alpha(x)$ the linear rankone projection onto $k \bullet a_\alpha(x)$, defined with respect to the $(a_\alpha(x))$-basis, one has:

$$A_\alpha(x) \bullet A_\beta(y) \bullet A_\alpha(x) = p_{\alpha\beta}(x,y) A_\alpha(x) \tag{63}$$

for any $x, y \in T; \alpha, \beta = 1, ..., n$.

*Writing equation (**??**) in terms of the coefficients $(u_{\alpha\beta}(x,y))$ of $U(x,y)$, one immediately recognizes that (**??**) is a generalized form of the principle of composite amplitudes. Moreover, using the techniques mentioned in S. 8 it is possible, at least in principle, to determine the minimal dimension of K over R as a function only of the trarisition probabilities $\{p_{\alpha\beta}(x,y)\}$. Finally, a simple generalization of Proposition (**??**) allows to characterize those Heisenberg models in which $\mathcal{H}$ can be choosen to be an Hilbert space.*

Summing up: we have shown that, within the framework of Schwinger's algebra of measurernents, and assuming as a postulate only a simple and experimentally verifiable consequence of the Heinsenberg indeterminacy principle, the whole mathematicai apparatus of quantum theory can be deduced only from a detailed mathematical analysis of the transition probabilities. Our analysis yields moreover, some examples of non-kolmogorovian models of a new type (i.e. those minimal Heisenberg models which are not of Hilbert space type) whose structure seems to be interesting to investigate, at least from a mathematical point of view.

Some more refined non-Kolmogorovian models can be constructed and lead to a generalization of gauge theories (cf. the author's paper in: Proceeding of the workshop: "Quantum probability and applications to the quantum theory of irreversible processes" Rome, Sept. 1982 (to appear in Springer Lecture Notes in Mathematics).

Riferimenti bibliografici

[1] Accardi, L. *Non-Commutative Markov chains.* in: Proceedings School of Math. Phys. Università di Camerino, Sept.-Oct. 1974.

[2] Accardi, L. *Non relativistic quantum mechanics as a non-commutative Markov process*. Advances in Math. **20** (1976), 329-366.

[3] Accardi, L. *Noncommutative Markov chains Associated to a preassigned evolution: an application to the quantum theory of measurement.* Advances in Math. **29** (1978), 226- 243.

[4] Accardi, L. *On the quantum Feynman-Kac formula*. Rendiconti del Seminario Matematico e Fisico di Milano, 48 (1980), 135-179.

[5] Accardi, L. *Quantum Markov Processes.* in: Dynamical Systems and Microphysics; **A**. Blaquiere, F. Fer, A. Marzolio eds., Springer 1980.

[6] Accardi, L. **Topics in quantum probability**. Physics Report **77** (1981), 169-192.

[7] Accardi, L. **Probabilità e teoria quantistica**. Physis **23** (1981), 485-524.

[8] Accardi, L., Fedullo A. *On the statistical meaning of complex numbers in quantum theory*. Lett. Nuovo Cimento *34* (1982), 161-172.

[9] Accardi, L. *Foundations of Quantum Probability*. Rend. Sem. Mat. Univers. Politecn. Torino (1982), 249-270.

[10] Accardi, L., Cecchini C. *Conditional expectations in von Neumann algebras and a theorem of Takesaki*, Journ. of Funct. anal. **45** (1982), 245-273

[11] Accardi, L. *Dio gioca ai dai? (Ciò che è vivo e ciò che è morto nel dibattito sull'interpretazione della teoria quantistica).* Il Saggiatore (to appear).

[12] Accardi, L. *History of the theorem of composite probabilities from 1925 to 1948*. Communication to the Congress "La storia della Matematica in Italia", Cagliari, Ottobre 1982.

[13] Bell, J.S. *On the Problem of Hidden variables in quantum Mechanics.* Rev. of Mod. Phys. **38** (1966), 447-452.

[14] Bell, J.S. Physics **1** (1965), 195.

[15] Beltrametti E., Cassinelli C. *The logic of Quantum Mechanics*. Encyclopedia of Mathematics, Addison-Wesley (1982).

[16] Bub J. *The statistics of non-boolean event structure*. Foundations of Phylosophy of Statistical Theories in the physical sciences. voi. III; W.L. Harper, C.A. Hooker (eds.) Reidel (1973).

[17] COrleo G., Gutkowski D., Masotto G., Valdes M.V. *Are Bell's type inequaiities sufficient conditions for local hidden variabie theories?* Nuovo Cimento **B** 25 (1975), 413-424.

[18] D'Espagnat, B. *Conceptions de la physique contemporaine*. Paris 1965.

[19] Einstein A., Podolsky B., Rosen N. Phys. Rev. 47 (1935), 777.

[20] Feynrnan R.P. *The concept of probability in quantum mechanics*. Proceedings II-d Berkeley Symp. on math. Stat. and Prob. Tiniversity of California Press, Berkeley, Los Angeles (1951), 533-541.

[21] Feynman R.P. *Space–time approach to non–relativistic quantum mechanics*. Rev. of Mod. Phys. **20** (1948), 367–385.

[22] Feynman R.P. *The character of physical law* (1965). Ed. Boringhieri (1971).

[23] Feynman R.P., Leighton R.B., Sands M.; Lectures on physics vol. III. Addison-Wesley (1966).

[24] Garuccio A., Selleri F., *Systematic derivation of all the Inequalities of Einstein Locality*. Foundations of Physics **10** (1980), 209-216.

[25] Gutkowski 0., JLasotto G. *An inequality stronger than Bell's Inequality*. Nuovo Cimento B **22** (1974), 121.

[26] Heisenberg W. *The physical principles of the quantum theory*. Dover (1949).

[27] Jauch J.M. *Foundations of Quantum Mechanics*. Addison–Wesley (1968).

[28] Jordan P. *Uber eine neue Begrundigung der Quantenmechanik*. Zeitschr. Fur Physik **40** (1927), 809.

[29] Kolmogorov A.N. *Foundations of the theory of Probability.* Chelsea, New York (1950).

[30] Landé A. *New foundations of Quantum Mechanics.* Cambridge University Press (1965).

[31] Lochak G. Has *Bell's inequality a general meaning for Hidden-variable theories?*, Foundations of Physics **6** (1976), 173-184.

[32] Mackey G.W. *Mathematical foundations of quantum mechanics.* W.A. Benjaniin Inc. (1963).

[33] Pitowsky I. *Resolution of the einstein-Podolsky–Rosen paradox.* Phys. Rev. Lett. **48** (1982), 1299-1302.

[34] Piron C. *Foundations of quantum physics.* W.A. Benjamin Inc' (1976).

[35] Putnam H. *Is logic empirical?* in: The Logico-algebraic approach to quantum mechanics. C.A. Hooker (ed.)., vol. Il, Reidel.

[36] Putnam H. *Three valued logic.* in: The Logico-algebraic approach to quantum mechanics. C.A. Hooker (ed.), voi. I, Reidel.

[37] Roos 14. *On the construction of a unitary matrix with elements of given moduli.* Journ. of Math. Phys. **5** (1964), 1609-1611.

[38] Roos 14. *Studies of the principle of superposition in quantum mechanics.* Commentationes Physico-Matematical **33** (1966) 14. i.

[39] Scheibe, E. *The logical analysis of quaritun mechanics.* Pergamon Press.

[40] Schiegel, 8. *Superposition and interaction.* The University of Chicago Press (1980).

[41] Schwinger, i. *Quantum kinematics and Dynamics* Academic Presa (1970).

[42] Segal, I.E. *Mathematical problems of relativistic physics.* Lectures in Appi. Math. Amer. Math. Soc. (1963).

[43] Selleri F., Tarozzi 6. *Quantum mechanics reality and separability.* La Rivista dei Nuovo Cimento **2** (1981).

[44] Strauss, 14. *Mathematica as Logical Syntax - A method to formalize the language of a physicai theory* (1937). in: The Logico-Algebraic approach to quantum mechanics. C.A. Hooker (ed.), Reidel (1975).

[45] Suppes, P. *The probabiliatic argument Vor a non-clasaical iogic of quantum mechanica* (i966). cf. ref. 44.

[46] Suppes P., Zanotti M. *Stochastic incompieteness of quantum mechanica.* in: Logic and Probabiiity in quantum mechancs. P. Suppes (ed.), Reidel.

[47] Suppes P., Zanotti M. *Bell's inequalities are necessary and sufficient conditions for extension of probabiiity on the partial boolean algebra of four observables to the fuli boolean algebra.* (preprint, March 1982).

[48] Suppes P., Zanotti M. *Bell's inequality is implied by existence of a joint probability distribution of the four observables.* (submitted to Nuovo Cimento).

[49] Santos, E. *Local hidden variable theories violating Bell inequalities* (preprint).

[50] Svetlichny, G. *A Bell-type inequality distinguishes Three–Body from two–body non separability.* (preprint).

[51] Wigner, E.P. *On hidden variables in quantum mechanics.* Amer. Journ. of Phys. **38** (1970), 1005-1009.