Chapter 9

Representations of the Classical Groups

9.1 Representations of the General Linear Groups

In Section 4.3 we computed all the tensor irred reps of the general linear groups $GL(m) = GL(m, \mathfrak{C})$. The reps were determined by Young frames. Here we use Lie-algebraic methods to determine all analytic irred finite-dimensional reps of $GL(m)$. A comparison of the Lie-algebraic method with the method based on Young symmetrizers will yield results which are not easily obtainable from either method alone. The Lie-algebraic approach to the rep theory of $GL(m)$ is patterned closely after the corresponding treatment of $SL(2)$ and $SU(2)$ in Section 7.3.

Recall that $GL(m)$ is an m^2-dimensional complex Lie group. Its Lie algebra $gl(m)$ consists of all $m \times m$ complex matrices. The unimodular group $SL(m) = SL(m, \mathfrak{C})$ is an $(m^2 - 1)$-dimensional subgroup of $GL(m)$ with Lie algebra $sl(m)$ consisting of all $m \times m$ complex matrices of trace zero.

As a basis for $gl(m)$ we choose the matrices $\mathcal{E}_{hj}$, $1 \leq h, j \leq m$, where $\mathcal{E}_{hj}$ is the matrix with a one for the entry in row h, column j, and zeros everywhere else. It is easy to verify the commutation relations

$$[\mathcal{E}_{hj}, \mathcal{E}_{kl}] = \delta_{jk}\mathcal{E}_{hl} - \delta_{lh}\mathcal{E}_{kj}. \tag{1.1}$$

Denote the diagonal elements of the basis by $\mathcal{H}_h = \mathcal{E}_{hh}$, $h = 1, \ldots, m$. The set $\mathscr{h}_m$ of all diagonal matrices

$$\mathcal{H} = \sum_{j=1}^{m} \lambda_j \mathcal{H}_j = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda_m \end{pmatrix} \tag{1.2}$$

forms an m-dimensional commutative subalgebra of $gl(m)$. From (1.1)

$$[\mathcal{H}, \mathcal{E}_{kl}] = (\lambda_k - \lambda_l)\mathcal{E}_{kl}, \qquad [\mathcal{H}, \mathcal{H}'] = Z, \qquad \mathcal{H}, \mathcal{H}' \in \mathscr{h}_m. \tag{1.3}$$

It is easy to show from (1.3) that $\mathscr{h}_m$ is a maximal commutative subalgebra of $gl(m)$, i.e., if $\mathscr{h}'$ is a commutative subalgebra of $gl(m)$ and $\mathscr{h}' \supseteq \mathscr{h}_m$ then $\mathscr{h}' = \mathscr{h}_m$.

From now on we use the notation $\mathcal{E}_{kl}$ only for $k \neq l$ and reserve $\mathcal{H}$, $\mathcal{H}_j$ to denote elements of $\mathscr{h}_m$. The mapping $\mathcal{H} \to \operatorname{ad} \mathcal{H}$, where

$$\operatorname{ad} \mathcal{H}(\mathcal{A}) = [\mathcal{H}, \mathcal{A}], \qquad \mathcal{A} \in gl(m), \tag{1.4}$$

defines a rep of $\mathscr{h}_m$ on $gl(m)$, the **adjoint representation,** as we saw in Section 5.6. According to (1.3) the element $\mathcal{E}_{kl}$ is a simultaneous eigenvector for all operators $\operatorname{ad} \mathcal{H}(\lambda_1, \ldots, \lambda_m)$ and corresponds to the eigenvalue $\lambda_k - \lambda_l$. The nonzero elements of $\mathscr{h}_m$ are eigenvectors corresponding to the eigenvalue zero.

Note that the eigenvalues $\lambda_k - \lambda_l = \alpha(\mathcal{H})$ are linear functionals on the elements $\mathcal{H} = \sum \lambda_j \mathcal{H}_j$ of $\mathscr{h}_m$. These $m(m-1)$ distinct functionals for $k \neq l$ are called **roots.** The eigenvector $\mathcal{E}_{kl}$ is called the **branch** belonging to the root $\lambda_k - \lambda_l = \alpha$. We will sometimes write $\mathcal{E}_{kl} = \mathcal{E}_\alpha$ to denote this branch. Furthermore, we define $\mathcal{H}_\alpha = \mathcal{H}_k - \mathcal{H}_l$ for $\alpha = \lambda_k - \lambda_l$.

Lemma 9.1.

(a) If α is a root then $-\alpha$ is a root.
(b) $[\mathcal{E}_\alpha, \mathcal{E}_{-\alpha}] = \mathcal{H}_\alpha \neq Z$.
(c) $[\mathcal{E}_\alpha, \mathcal{E}_\beta] = Z$ if $\alpha + \beta$ is not a root and $\alpha \neq -\beta$.
(d) $[\mathcal{E}_\alpha, \mathcal{E}_\beta] = \pm\mathcal{E}_{\alpha+\beta}$ if $\alpha + \beta$ is a root.
(e) $[\mathcal{H}, \mathcal{E}_\alpha] = \alpha(\mathcal{H})\mathcal{E}_\alpha$, $[\mathcal{H}_\alpha, \mathcal{E}_\alpha] = 2\mathcal{E}_\alpha$.

Proof. (a) If $\alpha = \lambda_k - \lambda_l$ is a root then $-\alpha = \lambda_l - \lambda_k$ is a root. (b) $[\mathcal{E}_\alpha, \mathcal{E}_{-\alpha}] = [\mathcal{E}_{kl}, \mathcal{E}_{lk}] = \mathcal{H}_k - \mathcal{H}_l = \mathcal{H}_\alpha$. (c) If $\alpha = \lambda_h - \lambda_j$ and $\beta = \lambda_k - \lambda_l$ and $\alpha + \beta = \lambda_h + \lambda_k - \lambda_j - \lambda_l$ is not a root or zero, then $j \neq k$, $h \neq l$, and $[\mathcal{E}_{hj}, \mathcal{E}_{kl}] = Z$ by (1.1). (d) If $\alpha + \beta$ is a root then from (c) either $j = k$, in which case $[\mathcal{E}_{hj}, \mathcal{E}_{kl}] = \mathcal{E}_{hl} = \mathcal{E}_{\alpha+\beta}$, or $l = h$, in which case $[\mathcal{E}_{hj}, \mathcal{E}_{kl}] = -\mathcal{E}_{kj} = -\mathcal{E}_{\alpha+\beta}$. (e) This follows directly from (1.3). Q.E.D.

Let ρ be a rep of $gl(m)$ by operators $\rho(\mathcal{A})$, $\mathcal{A} \in gl(m)$, on the complex vector space V. Setting $\rho(\mathcal{E}_\alpha) = E_\alpha$, $\rho(\mathcal{H}) = H = \sum \lambda_j H_j$, we obtain the relations

$$\begin{aligned} &[E_\alpha, E_\beta] = \begin{cases} H_\alpha & \text{if} \quad \alpha + \beta = 0, \\ 0 & \text{if} \quad \alpha + \beta \quad \text{is nonzero and not a root,} \\ \pm E_{\alpha+\beta} & \text{if} \quad \alpha + \beta \quad \text{is a root.} \end{cases} \\ &[H, E_\alpha] = \alpha(\mathcal{H})E_\alpha, \qquad [H, H_j] = 0, \qquad 1 \leq j \leq m. \end{aligned} \tag{1.5}$$

where $\alpha = \lambda_k - \lambda_l$ and $\beta = \lambda_h - \lambda_j$ are roots and $[A, B] = AB - BA$ for linear operators A, B on V. To determine the reps of $gl(m)$ it is enough to determine the possible operators $\{E_\alpha, H_j\}$ which satisfy the commutation relations (1.5).

If ρ is a rep of $gl(m)$ on the n-dimensional space V then V has a basis of simultaneous eigenvectors of the operators $H = \rho(\mathcal{H})$, $\mathcal{H} \in \hbar_m$. A vector $\mathbf{v} \neq \mathbf{\theta}$ is a simultaneous eigenvector if there exist constants $c_1, \ldots, c_m$ such that $H_j\mathbf{v} = c_j\mathbf{v}$, $1 \leq j \leq m$. Then

$$H\mathbf{v} = \Lambda(\mathcal{H})\mathbf{v}, \qquad \Lambda = \sum_{j=1}^{m} c_j\lambda_j \tag{1.6}$$

for $H = \sum \lambda_j H_j$. The linear functional $\Lambda(\mathcal{H})$ on $\hbar_m$ is called a **weight** and $\mathbf{v}$ is a **weight vector.** Before proving the existence of a basis of weight vectors we note that in the special case where $V = gl(m)$ and ρ is the adjoint rep of $gl(m)$ acting on itself, the weights are just the $m(m-1)$ roots $\alpha = \lambda_k - \lambda_l$ plus the zero weight. The weight vectors $\{\mathcal{E}_\alpha, \mathcal{H}_j\}$ form a basis for the rep space.

Lemma 9.2. If ρ is a rep of $gl(m)$ on V then it contains at least one weight.

Proof. Since V is complex the operator $H_1 = \rho(\mathcal{H}_1)$ has a least one eigenvalue c_1. Let W_1 be the nonzero eigenspace of V corresponding to eigenvalue c_1. If $\mathbf{v} \in W_1$ then $H_1(H_j\mathbf{v}) = H_j(H_1\mathbf{v}) = c_1 H_j\mathbf{v}$, so $H_j\mathbf{v} \in W_1$ for $2 \leq j \leq m$. Since W_1 is invariant under H_2, $H_2 \mid W_1$ has an eigenvalue c_2. Let $W_2 \subseteq W_1$ be the corresponding nonzero eigenspace. Then $H_k\mathbf{v} = c_k\mathbf{v}$, $k = 1, 2$, for $\mathbf{v} \in W_2$. Continuing in this manner, we finally obtain a nonzero vector $\mathbf{w} \in W_m$ such that $H_j\mathbf{w} = c_j\mathbf{w}$, $1 \leq j \leq m$. Clearly, $\Lambda = \sum \lambda_j c_j$ is a weight. Q.E.D.

Note. The above proof merely demonstrates that a set of commuting operators on a finite-dimensional vector space has a simultaneous eigenvector.

The next result shows that by applying the operator E_α to a weight vector we may be able to generate a new weight.

Lemma 9.3. Let $\mathbf{v}$ be a weight vector with weight Λ. If α is a root and $E_\alpha\mathbf{v} \neq \mathbf{\theta}$ then $\Lambda + \alpha$ is a weight with weight vector $E_\alpha\mathbf{v}$.

Proof. If $H\mathbf{v} = \Lambda\mathbf{v}$ then $H(E_\alpha\mathbf{v}) = E_\alpha H\mathbf{v} + \alpha E_\alpha\mathbf{v} = (\Lambda + \alpha)E_\alpha\mathbf{v}$, as follows from the second relation (1.5). Thus $\Lambda + \alpha$ is a weight if $E_\alpha\mathbf{v} \neq \mathbf{\theta}$. Q.E.D.

Theorem 9.1. If ρ is an irred rep of $gl(m)$ on V then V contains a basis of weight vectors.

Proof. By Lemma 9.2 there is a weight vector $\mathbf{v} \neq \boldsymbol{\theta}$ in V with weight Λ. Consider the set $\mathcal{W}$ of all vectors of the form

$$\mathbf{v}, \quad E_{\alpha_1}\mathbf{v}, \quad E_{\alpha_1}E_{\alpha_2}\mathbf{v}, \ldots, \quad E_{\alpha_1}\cdots E_{\alpha_k}\mathbf{v}, \ldots,$$

where the α_j run over all roots of $gl(m)$. By Lemma 9.3, each nonzero element $E_{\alpha_1}\cdots E_{\alpha_k}\mathbf{v}$ of $\mathcal{W}$ is a weight vector with weight $\Lambda + \alpha_1 + \cdots + \alpha_k$. Let W be the subspace of V spanned by the elements of $\mathcal{W}$. By construction W is invariant under all operators E_α and H. Hence the nonzero subspace W is invariant under ρ. Since ρ is irred, $W = V$. Now choose a maximal linearly independent set of vectors from $\mathcal{W}$. This set is clearly a basis of weight vectors for V. Q.E.D.

The proof of this theorem is valid only for irred reps. However, we will show later that every finite-dimensional rep of $gl(m)$ can be decomposed into a direct sum of irred reps. Thus, Theorem 9.1 is true for all reps.

Let ρ be an irred rep of $gl(m)$ and let $\{\mathbf{v}_j : j = 1, \ldots, n\}$ be a basis of weight vectors from ρ with weights Λ_j. Then every weight Λ of ρ is one of the Λ_j. Indeed if $\Lambda \neq \Lambda_j$ for any j then there exists an $\mathcal{H} \in \mathscr{h}_m$ such that $\Lambda(\mathcal{H}) \neq \Lambda_j(\mathcal{H})$, $1 \leq j \leq n$. This means that the nonzero eigenspace of $H = \rho(\mathcal{H})$ corresponding to eigenvalue $\Lambda(\mathcal{H})$ is linearly independent of the eigenspaces corresponding to the eigenvalues $\Lambda_j(\mathcal{H})$. However, the latter eigenspaces span V by Theorem 9.1. This is a contradiction, so no such weight Λ exists.

Corollary 9.1. If ρ is an n-dimensional rep of $gl(m)$ there are at most n distinct weights.

Let α be a root. Since the rep ρ has only a finite number of weights there must exist a weight Λ^* such that $\Lambda^* + \alpha$ is not a weight. Let $\mathbf{v}_0$ be a weight vector corresponding to Λ^*, so

$$H\mathbf{v}_0 = \Lambda^*(\mathcal{H})\mathbf{v}_0, \qquad E_\alpha\mathbf{v}_0 = \boldsymbol{\theta} \tag{1.7}$$

by Lemma 9.3. We define a sequence of weight vectors recursively by

$$E_{-\alpha}\mathbf{v}_j = \mathbf{v}_{j+1}, \qquad j = 0, 1, 2, \ldots. \tag{1.8}$$

By Lemma 9.3,

$$H\mathbf{v}_j = (\Lambda^* - j\alpha)(\mathcal{H})\mathbf{v}_j, \tag{1.9}$$

so either $\mathbf{v}_j = \boldsymbol{\theta}$ or $\mathbf{v}_j$ is a weight vector with weight $\Lambda^* - j\alpha$. Since ρ is finite-dimensional there must exist a positive integer q such that $\mathbf{v}_q \neq \boldsymbol{\theta}$ and $\mathbf{v}_{q+1} = \boldsymbol{\theta}$. The $q + 1$ weight vectors $\mathbf{v}_0, \ldots, \mathbf{v}_q$ are called an **α-ladder** of **ladder length** q. The corresponding weights $\Lambda^*, \Lambda^* - \alpha, \ldots, \Lambda^* - q\alpha$ also constitute an **α-ladder**. According to (1.8) we can move down the ladder by applying the operator $E_{-\alpha}$. On the other hand, using

(1.10) $$[E_\alpha, E_{-\alpha}] = H_\alpha,$$

we can show that application of E_α enables us to move up the α-ladder. For convenience we set $\Lambda^*(\mathcal{H}_\alpha) = \Lambda_\alpha^*$.

Lemma 9.4. $E_\alpha \mathbf{v}_j = r_j \mathbf{v}_{j-1}$, $j = 0, 1, \ldots, q+1$, where $r_j = j\Lambda_\alpha^* - \frac{1}{2}j(j-1)\alpha_\alpha$.

Proof. Induction on j. According to (1.7), $r_0 = 0$. Suppose the lemma is valid for $j \leq k \leq q$. We must verify the result for $j = k+1$. From (1.10) and the induction hypothesis,

(1.11) $$E_\alpha \mathbf{v}_{k+1} = E_\alpha E_{-\alpha} \mathbf{v}_k = (E_{-\alpha}E_\alpha + H_\alpha)\mathbf{v}_k = (r_k + \Lambda_\alpha^* - k\alpha_\alpha)\mathbf{v}_k = r_{k+1}\mathbf{v}_k.$$

Thus $r_{k+1} = r_k + \Lambda_\alpha^* - k\alpha_\alpha = (k+1)\Lambda_\alpha^* - \frac{1}{2}(k+1)k\alpha_\alpha$. Q.E.D.

From the first equality in (1.11) we have $r_{q+1} = 0$, since $E_{-\alpha}\mathbf{v}_q = \mathbf{\theta}$. Thus

$$(q+1)\Lambda_\alpha^* - \tfrac{1}{2}(q+1)q\alpha_\alpha = 0.$$

Lemma 9.5. $q = 2\Lambda_\alpha^*/\alpha_\alpha$.

Remark. From the commutation relations $[H_\alpha, E_{\pm\alpha}] = \pm\alpha_\alpha E_{\pm\alpha}$, $[E_\alpha, E_{-\alpha}] = H_\alpha$, it follows that the operators $E_{\pm\alpha}$, H_α form a basis for a subalgebra of $gl(m)$ isomorphic to $sl(2)$. Thus the construction of the α-ladder of weights containing Λ^* is essentially the same as the construction of the irred reps of $sl(2)$ in Section 7.6.

As we have shown earlier, $\alpha_\alpha = \alpha(\mathcal{H}_\alpha) = 2$ for $gl(m)$ since $\alpha = \lambda_k - \lambda_l$ and $\mathcal{H}_\alpha = \mathcal{H}_k - \mathcal{H}_l$. Thus $q = \Lambda_\alpha^*$ for $gl(m)$. However, it is convenient to use the notation α_α because with its use we can verify Lemma 9.5 for other classical groups.

Now let Λ be any weight and consider the linear functionals $\Lambda, \Lambda + \alpha, \Lambda + 2\alpha, \ldots$. There will be a smallest nonegative integer h such that $\Lambda + h\alpha$ is a weight but $\Lambda + (h+1)\alpha$ is not a weight. Then $\Lambda^* = \Lambda + h\alpha$ is a maximal weight in the sense of (1.7) and there exists an α-ladder

(1.12) $$\Lambda^*, \quad \Lambda^* - \alpha, \quad \Lambda^* - 2\alpha, \ldots, \quad \Lambda^* - (2\Lambda_\alpha^*/\alpha_\alpha)\alpha,$$

with ladder length $2\Lambda_\alpha^*/\alpha_\alpha = 2\Lambda_\alpha/\alpha_\alpha + 2h$. Since the ladder length is a nonnegative integer it follows that $2\Lambda_\alpha/\alpha_\alpha$ is an integer. Furthermore, in terms of Λ the α-ladder (1.12) is

(1.13) $$\Lambda + h\alpha, \Lambda + (h-1)\alpha, \ldots, \quad \Lambda - [2(\Lambda_\alpha/\alpha_\alpha) + h]\alpha.$$

The midpoint of this ladder is $\frac{1}{2}\{[\Lambda + h\alpha] + [\Lambda - 2(\Lambda_\alpha/\alpha_\alpha)\alpha - h\alpha]\} = \Lambda - (\Lambda_\alpha/\alpha_\alpha)\alpha$. (The midpoint is a weight on the ladder if and only if the ladder contains an odd number of weights.) Similarly we can find a smallest nonnegative integer k such that $\Lambda - k\alpha = \Lambda^{**}$ is a weight but $\Lambda^{**} - \alpha$ is not a weight. In analogy with Lemma 9.4 it is easy to show that Λ^{**} is the lowest rung on an α-ladder of length $q' = -2\Lambda_\alpha^{**}/\alpha_\alpha = -2\Lambda_\alpha/\alpha_\alpha + 2k$. The midpoint of this ladder is again $\Lambda - (\Lambda_\alpha/\alpha_\alpha)\alpha$. (Prove it!) Since both ladders have the same midpoints and no gaps. they must necessarily coincide. [For example, if the first ladder were longer than the second the weight $\Lambda^* - 2(\Lambda_\alpha^*/\alpha_\alpha)\alpha$ would lie lower than Λ^{**}, which is impossible.] Thus there is only one ladder (1.13) and Λ belongs to it. Note that Λ lies a distance $(\Lambda_\alpha/\alpha_\alpha)\alpha$ from the midpoint of the ladder. Hence if we reflect the α-ladder in its midpoint, Λ will be mapped into the functional $\Lambda - 2(\Lambda_\alpha/\alpha_\alpha)\alpha$ which is the same distance from the midpoint but on the opposite side. In particular $\Lambda - 2(\Lambda_\alpha/\alpha_\alpha)\alpha$ is a weight. We have proved the following result.

Theorem 9.2. If Λ is a weight and α is a root then $2\Lambda_\alpha/\alpha_\alpha$ is an integer and $\Lambda - 2(\Lambda_\alpha/\alpha_\alpha)\alpha$ is a weight.

Corollary 9.2. The weights of the form $\Lambda + j\alpha$ belonging to ρ are just those for which $j = -k, -k+1, \ldots, h-1, h$, where $\Lambda^* = \Lambda + h\alpha$ is maximal and $\Lambda^{**} = \Lambda - k\alpha$ is minimal, i.e., there are no gaps in the α-ladder. Here, $k - h = 2\Lambda_\alpha/\alpha_\alpha$.

Proof. Suppose $\Lambda^* = \Lambda + h\alpha$ is the maximal weight constructed in the proof of the theorem and let p be the largest integer such that $(\Lambda^*)' = \Lambda^* + p\alpha$ is a weight. Suppose $p > 0$. Then $(\Lambda^*)'$ is maximal and $S^\alpha(\Lambda^*)' = (\Lambda^*)' - 2[(\Lambda_\alpha^*)'/\alpha_\alpha]\alpha = \Lambda^* - 2(\Lambda_\alpha^*/\alpha_\alpha)\alpha - p\alpha$ is also a weight which lies lower on the α-ladder than Λ^*, since $\Lambda_\alpha^* \geq 0$. According to the proof of the theorem, all functionals on the α-ladder between $S^\alpha(\Lambda^*)'$ and $(\Lambda^*)'$ are weights, so Λ^* is not maximal. This contradiction shows that $p = 0$.

Similarly there is no weight on the α-ladder below $S^\alpha\Lambda^* = \Lambda - k\alpha$. By Lemma 9.5 the length of the α-ladder is $h + k = 2\Lambda_\alpha^*/\alpha_\alpha = 2(\Lambda_\alpha/\alpha_\alpha) + 2h$ so $2\Lambda_\alpha/\alpha_\alpha = k - h$. Q.E.D.

The map $S^\alpha\Lambda = \Lambda - 2(\Lambda_\alpha/\alpha_\alpha)\alpha$ of the weights of ρ onto themselves is called a **Weyl reflection.** Each Weyl reflection permutes the weights. Hence the totality of reflections S^α as α runs over the roots generates a group of permutations of the weights, called the **Weyl group.** As we have shown, $S^\alpha\Lambda$ is the reflection of Λ with respect to the midpoint of the α-ladder on which Λ lies. In particular if Λ is the highest weight on the ladder then $S^\alpha\Lambda$ is the lowest weight.

Theorem 9.2 greatly restricts the possible weights $\Lambda = \sum c_j\lambda_j$. The requirement that $2\Lambda_\alpha/\alpha_\alpha$ be an integer for all roots $\alpha = \lambda_k - \lambda_l$ implies $\Lambda_\alpha = c_k - c_l$ is an integer for all $k \neq l$.

If ρ is irred and $\mathbf{v} \in V$ is a weight vector with weight Λ then from Theorem 9.1 the possible weights of ρ are all of the form

$$\Lambda + \alpha_1 + \alpha_2 + \cdots + \alpha_j, \qquad \alpha_1, \ldots, \alpha_j \quad \text{roots.}$$

Thus the difference $\Lambda - \Lambda'$ of any two weights of ρ can be expressed as a sum of roots $\alpha_i = \lambda_k - \lambda_l$.

Definition. A linear functional $b_1\lambda_1 + b_2\lambda_2 + \cdots + b_m\lambda_m$ on $\mathscr{h}_m$ is **real** if all the constants b_j are real. A real functional is **positive** if the first nonzero b_j is positive, reading from left to right. A (possibly complex) linear functional Λ is **greater** than another functional Λ' if $\Lambda - \Lambda'$ is a real positive functional.

Since a sum of roots is always a real functional it follows that the difference of any two weights of ρ is real. Thus the above definition defines a lexicographic ordering of the weights of ρ. We say $\Lambda = \sum c_j\lambda_j$ is greater than $\Lambda' = \sum c_j'\lambda_j$ ($\Lambda > \Lambda'$) provided the first nonzero difference $c_l - c_l'$ is positive. With this total ordering it makes sense to speak of the **highest weight** of ρ. (Note that the roots $\alpha = \lambda_k - \lambda_l$ are positive provided $k < l$. If $\alpha > 0$ then $-\alpha$ is negative. Since the roots and zero are the weights of the adjoint rep, the highest weight of the adjoint rep is $\lambda_1 - \lambda_m$.)

If Λ is the highest weight of the irred rep ρ then $\Lambda + \alpha$ cannot be a weight for any positive root α, since $\Lambda + \alpha > \Lambda$. Thus, if $\mathbf{v}$ is a weight vector with weight Λ then $E_\alpha\mathbf{v} = \boldsymbol{\theta}$ for all $\alpha > 0$. A basis of weight vectors for V can be selected from the set of all vectors of the form

$$E_{\alpha_1}E_{\alpha_2}\cdots E_{\alpha_k}\mathbf{v}, \qquad k = 1, 2, \ldots. \tag{1.14}$$

If $k = 1$, the vectors $E_{\alpha_1}\mathbf{v}$ are zero unless $\alpha_1 < 0$. For $k = 2$ the vectors $E_{\alpha_1}E_{\alpha_2}\mathbf{v}$ are zero unless $\alpha_2 < 0$. If $\alpha_2 < 0$ and $\alpha_1 > 0$ then

(1) $E_{\alpha_1}E_{\alpha_2}\mathbf{v} = (E_{\alpha_2}E_{\alpha_1} \pm E_{\alpha_1+\alpha_2})\mathbf{v} = \pm E_{\alpha_1+\alpha_2}\mathbf{v}$ if $\alpha_1 + \alpha_2 \neq 0$ is a root.
(2) $E_{\alpha_1}E_{\alpha_2}\mathbf{v} = E_{\alpha_2}E_{\alpha_1}\mathbf{v} = \boldsymbol{\theta}$ if $\alpha_1 + \alpha_2 \neq 0$ is not a root.
(3) $E_{\alpha_1}E_{-\alpha_1}\mathbf{v} = (E_{-\alpha_1}E_{\alpha_1} + H_{\alpha_1})\mathbf{v} = \Lambda_{\alpha_1}\mathbf{v}$.

We have used the commutation relations (1.5) to derive these results. Proceeding in this way, we see that all weight vectors (1.14) can be written as linear combinations of weight vectors

$$E_{-\beta_1}E_{-\beta_2}\cdots E_{-\beta_l}\mathbf{v}, \qquad \beta_j > 0, \quad j = 1, \ldots, l. \tag{1.15}$$

Furthermore, we can express the vectors (1.14) as linear combinations of the vectors (1.15) by a procedure which depends only on the commutation relations (1.5), not on ρ or V.

Clearly we can choose a subset of the vectors (1.15) as a basis for V. Each such basis vector corresponds to the weight $\Lambda - \sum_{j=1}^{l} \beta_j$ with $\beta_j > 0$. The only possible basis vector (1.15) with highest weight Λ is $\mathbf{v}$ itself. This proves the first statement in the following theorem.

Theorem 9.3. The weight space belonging to the highest weight Λ in an irred rep ρ is one-dimensional. Two irred reps with the same highest weight are equivalent.

Proof. Suppose ρ and ρ' are irred reps of $gl(m)$ on V and V', respectively, with the same highest weight Λ. Let $\mathbf{v}$ and $\mathbf{v}'$ be weight vectors belonging to Λ in V and V'. Weight vectors of the form

$$\mathbf{w} = E_{-\beta_1}E_{-\beta_2}\cdots E_{-\beta_l}\mathbf{v}, \qquad \mathbf{w}' = E'_{-\beta_1}E'_{-\beta_2}\cdots E'_{-\beta_l}\mathbf{v}', \tag{1.16}$$

$\beta_j > 0$, span V and V'. We define a mapping $\mathbf{S}$ from V to V' by

$$\mathbf{S}\Big(\sum_{k=1}^{p} a_k\mathbf{w}_k\Big) = \sum_{k=1}^{p} a_k\mathbf{w}_k', \qquad p = 1, 2, \ldots, \quad a_k \in \mathcal{C}, \tag{1.17}$$

where corresponding vectors $\mathbf{w}_k \in V$, $\mathbf{w}_k' \in V'$ are of the form (1.16) and belong to the same weight. It is not clear that this mapping is well-defined. Assuming this for the present, it follows that $\mathbf{S}$ is a linear mapping of V onto V'. Furthermore, the vectors $E_\alpha\mathbf{w}_k$ and $E_\alpha'\mathbf{w}_k'$ for any root α can be written as linear combinations of corresponding weight vectors $\mathbf{w}_l$ and $\mathbf{w}_l'$ by a procedure based solely on the commutation relations. The expansion coefficients in the primed and unprimed spaces will be the same. Similarly $H_\alpha\mathbf{w}_k = (\Lambda - \sum \beta_j)_\alpha\mathbf{w}_k$, $H_\alpha'\mathbf{w}_k' = (\Lambda - \sum \beta_j)_\alpha\mathbf{w}_k'$ where $\Lambda - \sum \beta_j$ is the weight to which $\mathbf{w}_k$ and $\mathbf{w}_k'$ belong. As a consequence,

$$E_\alpha'\mathbf{S} = \mathbf{S}E_\alpha, \qquad H_\alpha'\mathbf{S} = \mathbf{S}H_\alpha, \tag{1.18}$$

for all roots α. Since $\mathbf{S}$ is nonzero it follows from (1.18) and the Schur lemmas that ρ and ρ' are equivalent reps.

To finish the proof we must verify that $\mathbf{S}$ is well-defined, i.e., that whenever $\sum_{k=1}^{p} a_k\mathbf{w}_k = \boldsymbol{\theta}$ in V, then $\mathbf{S}(\sum a_k\mathbf{w}_k) = \sum_{k=1}^{p} a_k\mathbf{w}_k' = \boldsymbol{\theta}'$ in V'. Consider the set W' of all vectors $\mathbf{z}' = \sum a_k\mathbf{w}_k'$ such that $\mathbf{z} = \sum a_k\mathbf{w}_k = \boldsymbol{\theta}$ in V. Clearly W' is a subspace of V'. Furthermore, by (1.18), $E_\alpha'\mathbf{z}' = E_\alpha'\mathbf{S}\mathbf{z} = \mathbf{S}E_\alpha\mathbf{z}$, where $E_\alpha\mathbf{z} = \sum a_k E_\alpha\mathbf{w}_k = \boldsymbol{\theta}$, so $E_\alpha'\mathbf{z}' \in W'$ for any root α. Similarly, $H_\alpha'\mathbf{z}' \in W'$. Thus W' is invariant under ρ'. Since ρ' is irred, either $W' = V'$ or $W' = \{\boldsymbol{\theta}\}$. But $\mathbf{v}' \notin W'$. For, if $\sum a_k\mathbf{w}_k = \boldsymbol{\theta}$ and $\sum a_k\mathbf{w}_k' = \mathbf{v}'$, each of the $\mathbf{w}_k'$ with nonzero coefficient a_k must be a multiple $b_k\mathbf{v}'$ of $\mathbf{v}'$ since $\mathbf{v}'$ is a highest weight vector. We can assume that each $\mathbf{w}_k$ is a multiple $b_k\mathbf{v}$ of $\mathbf{v}$ and $\sum a_k\mathbf{w}_k' = \sum a_k b_k\mathbf{v}' = \mathbf{v}'$. Thus $\sum a_k b_k = 1$ and $\sum a_k\mathbf{w}_k = \sum a_k b_k\mathbf{v} = \mathbf{v} \neq \boldsymbol{\theta}$. This contradiction shows that $W' = \{\boldsymbol{\theta}\}$ and $\mathbf{S}$ is well-defined. Q.E.D.

Corollary 9.3. If Λ is a weight such that $\Lambda + \alpha$ is not a weight for all roots $\alpha > 0$ then Λ is the highest weight.

According to this theorem each irred rep ρ is uniquely determined by its highest weight Λ. In particular the highest weights can be used to label the irred reps. Let us determine the possible highest weights

$$\Lambda = c_1\lambda_1 + c_2\lambda_2 + \cdots + c_m\lambda_m.$$

In order that Λ be a weight it is necessary that the differences $c_k - c_l$ be integers for all $k \neq l$. If Λ is a highest weight then $\Lambda + \alpha$ is not a weight for all $\alpha > 0$, so Λ is the maximal weight on each α-ladder containing it. From Lemma 9.5 we have $2\Lambda_\alpha/\alpha_\alpha = \Lambda_\alpha \geq 0$. The positive roots are $\alpha = \lambda_k - \lambda_l$, $k < l$, so $\Lambda_\alpha = c_k - c_l \geq 0$. These are the only restrictions on highest weights if we consider Lie algebra reps alone. However, if we restrict ourselves to reps of $gl(m)$ which extend to **global** reps of $GL(m)$ we get an additional requirement on Λ. Let $\mathcal{H} = \sum \lambda_j \mathcal{H}_j \in \mathscr{h}_m$ and suppose $\mathbf{v}$ is a highest weight vector. Setting $H = \rho(\mathcal{H})$, we have $H\mathbf{v} = \Lambda(\mathcal{H})\mathbf{v}$. Then

$$\exp \mathcal{H} = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & Z \\ & & \ddots & \\ Z & & & e^{\lambda_m} \end{pmatrix} \in GL(m), \tag{1.19}$$

$$(\exp H)\mathbf{v} = e^{\Lambda(\mathcal{H})}\mathbf{v} = \exp(c_1\lambda_1 + \cdots + c_m\lambda_m)\mathbf{v}. \tag{1.20}$$

It is clear that the addition of any integer multiple of $2\pi i$ to a λ_j leaves the group element $\exp \mathcal{H}$ unchanged. Thus, if the Lie algebra rep induces a global group rep, the addition of an integer multiple of $2\pi i$ to a λ_j must leave (1.20) unchanged. This is possible only if the c_j are integers.

We conclude that the possible highest weights are

$$\Lambda^* = p_1\lambda_1 + p_2\lambda_2 + \cdots + p_m\lambda_m, \qquad p_j \quad \text{integer}, \tag{1.21}$$

$$p_1 \geq p_2 \geq \cdots \geq p_m. \tag{1.22}$$

The corresponding irred reps ρ will be denoted $(p_1, \ldots, p_m)$. We shall show that there exists an irred rep of $GL(m)$ corresponding to each such set of integers p_j.

Since $GL(m)$ is not compact we cannot directly apply the theory of Chapter 6 to show that every rep of $GL(m)$ decomposes into a direct sum of irred reps. However, using a technique (the unitary trick) from Chapter 7 we can relate the reps of $GL(m)$ to those of the (compact) unitary group $U(m)$. The real Lie algebra $u(m)$ of $U(m)$ consists of all $m \times m$ matrices $i\mathcal{B}$, where $\mathcal{B}$ is self-adjoint. Now every $\mathcal{A} \in gl(m)$ can be expressed uniquely in the form

$$\mathcal{A} = \mathcal{B} + i\mathcal{C}, \qquad \mathcal{B} = \tfrac{1}{2}(\mathcal{A} + \bar{\mathcal{A}}^t), \qquad \mathcal{C} = -\tfrac{1}{2}i(\mathcal{A} - \bar{\mathcal{A}}^t),$$

where $\mathcal{B}$ and $\mathcal{C}$ are self-adjoint. Thus $u(m)$ is a real form of $gl(m)$ and there is a 1–1 relationship between complex reps ρ of $gl(m)$ and ρ' of $u(m)$. In particular, ρ' is the restriction of ρ to the subalgebra $u(m)$ and ρ is the extension of ρ' to the complexified algebra $u(m)^c = gl(m)$. The reps ρ and ρ' are simultaneously reducible or irred and a decomposition of one rep into irred components induces a decomposition of the other. From Section 5.8, this 1–1 relationship also holds between the finite-dimensional analytic reps of $GL(m)$ and their restrictions to $U(m)$. Since $U(m)$ is compact we deduce that every finite-dimensional analytic rep of $GL(m)$ [or $gl(m)$] can be decomposed into a direct sum of irred reps (This result is false if the rep is infinite-dimensional or if it is not analytic.)

Now we begin the construction of all analytic irred reps ρ of $GL(m)$. The one-dimensional reps

(1.23) $\quad A \longrightarrow (\det A)^p, \qquad A \in GL(m), \qquad p \quad$ an integer,

are clearly analytic and irred. It follows from Corollary 5.3 that the induced rep of $gl(m)$ is

$$\mathcal{A} \longrightarrow p\,\mathrm{tr}(\mathcal{A}), \qquad \mathcal{A} \in gl(m).$$

Choosing $\mathcal{H} \in \mathscr{h}_m$ as in (1.2) we see that each of these reps has a single weight

(1.24) $$\Lambda = p(\lambda_1 + \lambda_2 + \cdots + \lambda_m).$$

Thus, we have constructed the reps $(p, \ldots, p)$, $p = 0, \pm 1, \ldots$.

Let ρ, ρ' be reps of a Lie algebra $\mathcal{G}$ on the vector spaces V, V', respectively. We define the **tensor product representation** $\rho \otimes \rho'$ of $\mathcal{G}$ on $V \otimes V'$ by

(1.25) $$\rho \otimes \rho'(\alpha)(\mathbf{v} \otimes \mathbf{v}') = (\rho(\alpha)\mathbf{v}) \otimes \mathbf{v}' + \mathbf{v} \otimes (\rho'(\alpha)\mathbf{v}'),$$
$$\alpha \in \mathcal{G}, \quad \mathbf{v} \in V, \quad \mathbf{v}' \in V'.$$

It is straightforward to verify that $\rho \otimes \rho'$ is indeed a rep of $\mathcal{G}$. In fact it is just the Lie algebra rep induced by the corresponding tensor product of group reps.

Lemma 9.6. Let ρ, ρ' be reps of $gl(m)$ on V and V'. The weights of $\rho \otimes \rho'$ are all functionals of the form $\Lambda + \Lambda'$, where Λ is a weight of ρ and Λ' is a weight of ρ'.

Proof. Let $\{\mathbf{v}_j\}$ be a basis of weight vectors for V and $\{\mathbf{v}_k'\}$ a basis of weight vectors for V'. Then $H\mathbf{v}_j = \Lambda_j(\mathcal{H})\mathbf{v}_j$ and $H'\mathbf{v}_k' = \Lambda_k'(\mathcal{H})\mathbf{v}_k'$, where Λ_j, Λ_k' are the weights of ρ and ρ', respectively. Choose the vectors $\{\mathbf{v}_j \otimes \mathbf{v}_k'\}$ as a basis for $V \otimes V'$. The action of $\rho \otimes \rho'(\mathcal{H}) = H + H'$ on this basis is

(1.26) $$\rho \otimes \rho'(\mathcal{H})\mathbf{v}_j \otimes \mathbf{v}_k' = (H\mathbf{v}_j) \otimes \mathbf{v}_k' + \mathbf{v}_j \otimes (H'\mathbf{v}_k')$$
$$= (\Lambda_j(\mathcal{H}) + \Lambda_k'(\mathcal{H}))\mathbf{v}_j \otimes \mathbf{v}_k'.$$

Thus $\{\mathbf{v}_j \otimes \mathbf{v}_k'\}$ is a weight basis with weights $\Lambda_j + \Lambda_k'$. By the remarks preceding Corollary 9.1 these are the only weights of $\rho \otimes \rho'$. Q.E.D.

Now suppose ρ is the irred rep $(p_1, \ldots, p_m), p_1 \geq p_2 \geq \cdots \geq p_m$, and ρ' is the irred rep $(p_1', \ldots, p_m')$. Furthermore, suppose both ρ and ρ' induce global irred reps of $GL(m)$ on V and V'. Then $\rho \otimes \rho'$ determines a rep of $gl(m)$ which extends to $GL(m)$. It follows easily from the preceding lemma that the highest weight in $\rho \otimes \rho'$ is

$$\Lambda^* = (p_1 + p_1')\lambda_1 + (p_2 + p_2')\lambda_2 + \cdots + (p_m + p_m')\lambda_m, \tag{1.27}$$

i.e., the sum of the highest weights in ρ and ρ'. Furthermore, the weight space of Λ^* has dimension one. Now $\rho \otimes \rho'$ can be decomposed into irred reps and the weight Λ^* must belong to exactly one of the irred pieces. Since Λ^* is the maximal weight of $\rho \otimes \rho'$ this irred piece must be the rep

$$(p_1 + p_1', \ldots, p_m + p_m'). \tag{1.28}$$

Thus the existence of $(p_1, \ldots, p_m)$ and $(p_1', \ldots, p_m')$ implies the existence of the rep (1.28). We shall use this method to prove the existence of the irred reps $(p_1, \ldots, p_m)$ for all integers $p_1 \geq p_2 \geq \cdots \geq p_m$. Unfortunately our procedure only proves existence. To obtain explicit expressions for the reps we fall must back on Young symmetrizer methods developed in Section 4.3.

Consider the tensor product of $\rho = (p_1, \ldots, p_m)$ and the one-dimensional rep $\rho' = (p, p, \ldots, p)$. In this special case $\rho \otimes \rho' \cong (p_1 + p, \ldots, p_m + p)$ is irred.(Prove it!) The group operators of $\rho \otimes \rho'$ are

$$(\det A)^p \mathbf{T}(A), \tag{1.29}$$

where the $\mathbf{T}(A)$ are the operators of ρ. Thus we can limit ourselves to the construction of reps for which $p_1 \geq p_2 \geq \cdots \geq p_m \geq 0$. The remaining reps can be obtained from (1.29).

In Section 4.3 we determined all tensor irred reps of $GL(m)$. Each such rep was determined by the Young frame $[f_1, f_2, \ldots, f_m]$, $f_1 \geq f_2 \geq \cdots \geq f_m \geq 0$. We will now explore the relationship between these reps and the reps $(p_1, \ldots, p_m)$.

Let us first consider the reps

$$[1^s] = [1, 1, \ldots, 1, 0, \ldots, 0], \qquad s = 1, 2, \ldots, m, \tag{1.30}$$

of $GL(m)$. Here the rep space consists of completely skew-symmetric tensors

$$a^{i_1 i_2 \cdots i_s}, \qquad 1 \leq i_j \leq m. \tag{1.31}$$

This space is $\binom{m}{s}$-dimensional since the $\binom{m}{s}$ independent components for which $i_1 < i_2 < \cdots < i_s$ completely determine the tensors. The group action is

$$[T(A)a]^{i_1 \cdots i_s} = \sum_{j_1 \cdots j_s = 1}^{m} A_{i_1 j_1} \cdots A_{i_s j_s} a^{j_1 \cdots j_s}, \qquad A \in GL(m). \tag{1.32}$$

As a basis for the rep space we choose the tensors $\mathbf{a}(k_1 \cdots k_s)$, $1 \leq k_1 < k_2 < \cdots < k_s \leq m$. Here $\mathbf{a}(k_1 \cdots k_s)$ is the skew-symmetric tensor with component $a^{k_1 \cdots k_s} = 1$ and all linearly independent components zero. It follows immediately from (1.32) and (1.19) that the $\mathbf{a}(k_1 \cdots k_s)$ form a weight basis for the rep. Indeed

$$\mathbf{Ha}(k_1 \cdots k_s) = (\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_s})\mathbf{a}(k_1 \cdots k_s), \tag{1.33}$$

so the $\binom{m}{s}$ weights of $[1^s]$ are

$$\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_s}, \qquad 1 \leq k_1 < k_2 < \cdots < k_s \leq m. \tag{1.34}$$

Each weight has multiplicity one and the highest weight is

$$\lambda_1 + \lambda_2 + \cdots + \lambda_s, \tag{1.35}$$

so $[1^s] \cong (1, \ldots, 1, 0, \ldots, 0) = (1^s)$.

In the above discussion we have used facts about Young frames to conclude that $[1^s]$ is irred. However, we can give an independent proof of irreducibility based on Theorem 9.2. Let

$$\Lambda = n_1\lambda_1 + \cdots + n_k\lambda_k + \cdots + n_l\lambda_l + \cdots + n_m\lambda_m \tag{1.36}$$

be a weight belonging to an irred rep ρ and let $\alpha = \lambda_k - \lambda_l$ be a root. Then $S^\alpha\Lambda = \Lambda - \Lambda_\alpha\alpha$ is also a weight. A simple computation gives

$$\begin{aligned} S^\alpha\Lambda &= \Lambda - (n_k - n_l)(\lambda_k - \lambda_l) \\ &= n_1\lambda_1 + \cdots + n_k\lambda_l + \cdots + n_l\lambda_k + \cdots + n_m\lambda_m, \end{aligned} \tag{1.37}$$

i.e., $S^\alpha\Lambda$ is obtained from Λ by interchanging λ_l and λ_k. Thus the group generated by the Weyl reflections is the group S_m of all permutations of $\lambda_1, \ldots, \lambda_m$. If Λ is a weight belonging to ρ then every linear functional obtained from Λ by permuting $\lambda_1, \ldots, \lambda_m$ is also a weight belonging to ρ.

Let us apply this result to show that $[1^s]$ is irred. By (1.35), the highest weight of $[1^s]$ is $\Lambda^* = \lambda_1 + \lambda_2 + \cdots + \lambda_s + 0\lambda_{s+1} + \cdots + 0\lambda_m$, so the rep space contains the irred rep (1^s). Now (1^s) must contain the $\binom{m}{s}$ distinct weights (1.35) obtained by applying all permutations of $\lambda_1, \ldots, \lambda_m$ to Λ^*. However, $\dim[1^s] = \binom{m}{s}$ so $[1^s] \cong (1^s)$.

It is now simple to prove the existence of irred reps $(p_1, \ldots, p_m)$ for all integers $p_1 \geq p_2 \geq \cdots \geq p_m$. Consider the rep

$$[1^1]^{\otimes k_1} \otimes [1^2]^{\otimes k_2} \otimes \cdots \otimes [1^m]^{\otimes k_m} = \rho, \tag{1.38}$$

where $k_1, \ldots, k_m$ are nonnegative integers. Using an obvious generalization of Lemma 9.6 we see that the highest weight of ρ is $\Lambda^* = p_1\lambda_1 + p_2\lambda_2 + \cdots + p_m\lambda_m$, where

$$
\begin{aligned}
p_1 &= k_1 + k_2 + \cdots + k_m \\
p_2 &= \phantom{k_1 + {}} k_2 + \cdots + k_m \\
&\ \ \vdots \\
p_{m-1} &= \qquad\qquad k_{m-1} + k_m \\
p_m &= \qquad\qquad\qquad\ \ k_m .
\end{aligned}
\tag{1.39}
$$

Clearly $p_1 \geq p_2 \geq \cdots \geq p_m \geq 0$. Since Λ^* occurs in ρ with multiplicity one it follows that the irred rep $(p_1, \ldots, p_m)$ of $GL(m)$ is contained in ρ with multiplicity one. We can obtain all reps $(p_1, \ldots, p_m)$ with $p_m \geq 0$ by choosing the integers $k_m = p_m$, $k_{m-1} = p_{m-1} - p_m, \ldots, k_2 = p_2 - p_3$, $k_1 = p_1 - p_2$. Then using (1.29) we can relax the requirement $p_m \geq 0$.

Let us now determine which rep $(p_1, \ldots, p_m)$ is equivalent to the irred tensor rep with Young frame $[f_1, \ldots, f_m]$. The elements of the rep space of $[f_1, \ldots, f_m]$ are tensors

$$
F^B, \qquad B = \begin{matrix} i_1 & \cdots & i_{f_1} \\ j_1 & \cdots & j_{f_2} \\ \vdots & & \\ z_1 & \cdots & z_{f_m} \end{matrix},
\tag{1.40}
$$

with $f_1 + f_2 + \cdots + f_m$ indices, each index taking the values one to m. These tensors are defined in terms of a Young symmetrizer by (3.47), Section 4.3. The action of the induced Lie algebra rep on each tensor component F^B is easily shown to be

$$
(\mathbf{H}F)^B = (\lambda_{i_1} + \cdots + \lambda_{i_f} + \lambda_{j_1} + \cdots + \lambda_{z_{f_m}})F^B.
\tag{1.41}
$$

Since the tensors F^B are skew-symmetric with respect to interchange of indices in the same column of B, the highest possible weight vector is the tensor with component $F^{B_0} = 1$ and all linearly independent components zero. where

$$
B_0 = \begin{matrix} 1 & 1 \cdots & 1 \\ 2 & 2 \cdots & 2 \\ \vdots & & \\ m & m \cdots & m \end{matrix}
\tag{1.42}
$$

(see the discussion of F^{B_0} at the end of Section 4.3). Clearly, this weight vector corresponds to the highest weight $\Lambda^* = f_1\lambda_1 + f_2\lambda_2 + \cdots + f_m\lambda_m$. Thus, $[f_1, \ldots, f_m] \cong (f_1, \ldots, f_m)$. This completes the construction of irred reps of the complex Lie group $GL(m)$.

Since $u(m)$ is a real form of $gl(m)$ and $U(m)$ is a connected subgroup of the connected group $GL(m)$ it follows that there is a 1–1 relationship between finite-dimensional analytic reps of $U(m)$ and $GL(m)$. In particular every rep of $GL(m)$ restricts to a rep of $U(m)$ and every rep of $U(m)$ extends to a unique rep of $GL(m)$. One of these reps is irred if and only if the other is. Hence, the irred reps of $U(m)$ are also denoted $[f_1, \ldots, f_m]$, $f_1 \geq f_2 \geq \cdots \geq f_m$, where we allow the f_j to be negative.

The real Lie group $GL(m) = GL(m, \mathfrak{C})$ is $2m^2$-dimensional. As a basis for the real Lie algebra we choose the elements $\mathcal{E}_{kl}$ and $i\mathcal{E}_{kl}$, where $\mathcal{E}_{kl}$ is the $m \times m$ matrix with a one in row k, column l and zeros everywhere else. Let ρ be a complex rep of $gl(m)$ and set $\rho(\mathcal{E}_{kl}) = E_{kl}$, $\rho(i\mathcal{E}_{kl}) = F_{kl}$. If we introduce a new basis $C_{kl} = \frac{1}{2}(E_{kl} - iF_{kl})$, $D_{kl} = \frac{1}{2}(E_{kl} + iF_{kl})$ for the complexified Lie algebra the commutation relations become

$$(1.43)\qquad \begin{aligned} &[C_{kl}, C_{k'l'}] = \delta_{lk'}C_{kl'} - \delta_{l'k}C_{k'l} \\ &[D_{kl}, D_{k'l'}] = \delta_{lk'}D_{kl'} - \delta_{l'k}D_{k'l}, \qquad [C_{kl}, D_{k'l'}] = 0. \end{aligned}$$

Thus, if we denote the real Lie algebra $gl(m)$ by $gl_r(m)$, relations (1.43) show $[gl_r(m)]^c \cong gl(m) \oplus gl(m)$. Hence the irred reps of $[gl_r(m)]^c$ can be expressed as products $\rho' \otimes \rho''$ of irred reps of $gl(m)$.

Suppose the reps ρ', ρ'' of $gl(m)$ induce matrix reps $\rho'(A)$, $\rho''(A)$ of $GL(m)$. Then reasoning exactly as in (3.3), Section 8.3, we see that $\rho' \otimes \rho''$ induces the matrix rep $\rho'(A)\rho''(\bar{A}) = (\rho'(A)_{kl}\rho''(\bar{A})_{k'l'})$ of $GL_r(m)$. Even if ρ' and ρ'' do not induce global reps of $GL(m)$, $\rho' \otimes \rho''$ may induce a global rep. Indeed the reps $A \longrightarrow (\det A)^\alpha$ and $A \longrightarrow (\det \bar{A})^\alpha$ are only local for arbitrary $\alpha \in \mathfrak{C}$. but their product $A \longrightarrow (\det A)^\alpha(\det \bar{A})^\alpha = |\det A|^{2\alpha}$ is global. We conclude that the analytic irred reps of the real Lie group $GL_r(m)$ are

$$(1.44)\qquad |\det A|^c \cdot [f_1, f_2, \ldots, f_m] \otimes [f_1', f_2', \ldots, f_m'],$$

where $f_1 \geq f_2 \geq \cdots \geq f_m$, $f_1' \geq f_2' \geq \cdots \geq f_m'$, $c \in \mathfrak{C}$.

9.2 Character Formulas

Since $U(m)$ is a compact group we can use the techniques of Chapter 6 to deduce its simple characters and their orthogonality relations. The following results are due essentially to Weyl [3].

The matrices $\mathcal{H}_k$, $k = 1, \ldots, m$, (1.2), form a basis for the abelian subalgebra $\mathfrak{h}_m$ of $gl(m)$. Similarly the matrices $i\mathcal{H}_k$ form a basis for the real abelian subalgebra $\tilde{h}_m$ of $u(m)$:

$$(2.1) \qquad \tilde{h}_m = \left\{ \begin{pmatrix} i\varphi_1 & & Z \\ & \ddots & \\ Z & & i\varphi_m \end{pmatrix} = \sum_{k=1}^{m} i\varphi_k \mathcal{H}_k : \varphi_k \quad \text{real} \right\}.$$

Here $\tilde{h}_m$ is the Lie algebra of the abelian subgroup Λ_m of $U(m)$:

$$(2.2) \qquad \Lambda_m = \left\{ \begin{pmatrix} e^{i\varphi_1} & & Z \\ & \ddots & \\ Z & & e^{i\varphi_m} \end{pmatrix} = \Phi \right\}.$$

Since a unitary matrix can be diagonalized by a unitary similarity transformation, every $A \in U(m)$ is unitary similar to a $\Phi \in \Lambda_m$, where the $e^{i\varphi_k}$ are the eigenvalues of A:

$$(2.3) \qquad A = U\Phi U^{-1}, \qquad U \in U(m).$$

Thus A is conjugate to an element of Λ_m. Furthermore, distinct $\Phi, \Phi' \in \Lambda_m$ are conjugate if and only if they have the same diagonal elements (in a different order). Thus we can use the parameters $\varphi_1, \ldots, \varphi_m$, $0 \le \varphi_j < 2\pi$ of Φ to denote the conjugacy classes of $U(m)$. Since a character χ is constant on conjugacy classes we can write $\chi = \chi(\varphi_1, \ldots, \varphi_m)$, where χ is a symmetric function of its m arguments.

To compute the simple characters we choose parameters on $U(m)$ as follows: The first m parameters $\varphi_1, \ldots, \varphi_m$ pick out the conjugacy class in which an element lies and the remaining $m^2 - m$ parametrize the elements in a fixed conjugacy class.

Given $A \in U(m)$ we assign the parameters $\varphi_1, \ldots, \varphi_m$ from (2.3). For definiteness we assume $\varphi_1 \le \varphi_2 \le \cdots \le \varphi_m$. We can obtain all elements in the conjugacy class of A by forming $U\Phi U^{-1}$ and letting U run over $U(m)$. This suggests that the elements in the conjugacy class can be parametrized by the matrices U. However, U is described by m^2 real parameters, so we would obtain $m^2 + m$ parameters for $A \in U(m)$. Since $U(m)$ is m^2-dimensional, some of these parameters must be redundant.

The redundancy occurs in the choice of U. Suppose the eigenvalues of Φ are distinct. Then $U\Phi U^{-1} = V\Phi V^{-1}$ for $U, V \in U(m)$ if and only if $U = V\Phi'$ for $\Phi' \in \Lambda_m$. The m parameters of Φ' are redundant so the elements of a conjugacy class are uniquely determined by $m^2 - m$ local parameters whose exact choice need not concern us. If two eigenvalues of Φ are identical then $m + 2$ parameters of U are redundant. (Prove it!) Since Φ has $m - 1$ independent eigenvalues in this case, a matrix A with two eigenvalues identical is determined by $(m^2 - m - 2) + (m - 1) = m^2 - 3$ parameters.

Thus the submanifold of such matrices has three dimensions less than the manifold; it does not affect the invariant measure.

Let us assume that in some neighborhood $\mathcal{W}$ of $A_0 \in U(m)$ we have

$$A = U\Phi U^{-1}, \qquad A \in \mathcal{W}, \tag{2.4}$$

where $U = U(b_1, \ldots, b_{m^2-m})$, $\Phi = \Phi(\varphi_1, \ldots, \varphi_m)$, and the φ_j are distinct. The φ_j and b_j define local coordinates for A. To determine the invariant measure on $U(m)$ with respect to this coordinate system we choose a real analytic curve $A(t)$ in $\mathcal{W}$ and compute $A^{-1}\dot{A} \in u(m)$. The parameters $b_j(t)$, $\varphi_j(t)$ are analytic functions of t, so from (2.4) we obtain

$$A^{-1}\dot{A} = U\Phi^{-1}U^{-1}(\dot{U}\Phi U^{-1} + U\dot{\Phi}U^{-1} - U\Phi U^{-1}\dot{U}U^{-1}), \tag{2.5}$$

or

$$U^{-1}(A^{-1}\dot{A})U = \Phi^{-1}(U^{-1}\dot{U})\Phi - U^{-1}\dot{U} + \Phi^{-1}\dot{\Phi}. \tag{2.6}$$

Here $A^{-1}\dot{A} = \mathcal{A}$, $U^{-1}\dot{U} = \mathcal{U}$, and $\Phi^{-1}\dot{\Phi} = \mathcal{E}$ are elements of $u(m)$, i.e., these matrices are skew-Hermitian. Furthermore, $U^{-1}(A^{-1}\dot{A})U = B^{-1}\dot{B} = \mathcal{B} \in u(m)$, where $B(t) = U^{-1}A(t)U \in U(m)$. Writing (2.6) in terms of matrix elements, we obtain

$$\mathcal{B}_{jk} = [(\epsilon_j/\epsilon_k) - 1]\mathcal{U}_{jk} + i\dot{\varphi}_j\delta_{jk}, \tag{2.7}$$

where $\epsilon_j = e^{i\varphi_j}$. Thus,

$$\mathcal{B}_{jk} = [(\epsilon_j/\epsilon_k) - 1]\mathcal{U}_{jk}, \qquad j \neq k, \qquad \mathcal{B}_{jj} = i\dot{\varphi}_j. \tag{2.8}$$

A convenient basis for $u(m)$ is given by $i\mathcal{H}_k$, $k = 1, \ldots, m$, and $\mathcal{E}_{jk} - \mathcal{E}_{kj}$, $i(\mathcal{E}_{jk} + \mathcal{E}_{kj})$, $k \neq j$, $k, j = 1, \ldots, m$. From Section 6.1, if $U(m)$ has local coordinates $t_1, \ldots, t_{m^2}$ then the invariant measure on $U(m)$ is given (up to a multiplicative constant) by

$$dA = V_A(t_1 \ldots, t_{m^2})\, dt_1 \ldots dt_{m^2},$$

where

$$V_A = |\det(\alpha_j{}^k)|, \qquad A^{-1}\,\partial A/\partial t_j = \sum_k \alpha_j{}^k \mathcal{A}_k \tag{2.9}$$

and $\{\mathcal{A}_k\}$ is the basis for $u(m)$ given above. Furthermore, since U is unitary and $B^{-1}\dot{B} = U^{-1}(A^{-1}\dot{A})U$ it follows from the proof of Theorem 6.1 that $V_A = |\det(\beta_j{}^k)|$, where

$$\mathcal{B} = B^{-1}\,\partial B/\partial t_j = \sum \beta_j{}^k \mathcal{A}_k. \tag{2.10}$$

Using the parameters $\varphi_1, \ldots, \varphi_m, b_1, \ldots, b_{m^2-m}$ as well as expressions (2.8) and (2.10) it is straightforward to verify the formula

$$dA = \prod_{j \neq k} |(\epsilon_j/\epsilon_k) - 1|\, d\omega_u\, d\varphi_1 \cdots d\varphi_m, \tag{2.11}$$

where

$$d\omega_u = W(b_1, \ldots, b_{m^2-m})\, db_1 \cdots db_{m^2-m}$$

depends only on the b-parameters.

Since $|(1-\epsilon_j/\epsilon_k)(1-\epsilon_k/\epsilon_j)| = (\epsilon_j - \epsilon_k)\overline{(\epsilon_j - \epsilon_k)}$, we can write

$$|\prod_{j \neq k} [(\epsilon_j/\epsilon_k) - 1]| = \Delta\bar{\Delta}$$

where

$$\Delta(\epsilon_1, \ldots, \epsilon_m) = \prod_{j<k} (\epsilon_j - \epsilon_k) \tag{2.12}$$

is skew-symmetric in its m arguments.

Consider a continuous function $f(\varphi_1, \ldots, \varphi_m)$ on $U(m)$ which is constant on conjugacy classes. Performing the b-integration, we find

$$\begin{aligned} \int_{U(m)} f\, dA &= \int d\omega_u \int f\Delta\bar{\Delta}\, d\varphi_1 \cdots d\varphi_m \\ &= \int_0^{2\pi} \cdots \int_0^{2\pi} f(\varphi_1, \ldots, \varphi_m)\, \Delta\bar{\Delta}\, d\varphi_1 \cdots d\varphi_m, \end{aligned} \tag{2.13}$$

where the measure $d\omega_u$ has been normalized so that $\int d\omega_u = 1$. Thus, the inner product on $L_2(U(m))$ of two functions f, g constant on conjugacy classes is

$$(f, g) = V^{-1} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\varphi_1, \ldots, \varphi_m)\bar{g}(\varphi_1, \ldots, \varphi_m)\, \Delta\bar{\Delta}\, d\varphi_1 \cdots d\varphi_m, \tag{2.14}$$

where $V = \int_0^{2\pi} \cdots \int_0^{2\pi} \Delta\bar{\Delta}\, d\varphi_1 \cdots d\varphi_m$. [Note: the integral in (2.14) gives an answer $m!$ times too big since a conjugacy class is determined by $\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_m$ and f and g are symmetric. However, V is also $m!$ times too large, so the factor $m!$ cancels.]

Since the characters of $U(m)$ are constant on conjugacy classes we can use (2.14) to compute the inner product of two characters. Let χ be a simple character and let $\mathbf{v}_1, \ldots, \mathbf{v}_s$ be a basis of weight vectors for the corresponding rep, with weights $\Lambda_1, \ldots, \Lambda_s$, respectively. In terms of this basis the matrix $\rho(\Phi)$ of $\Phi(\varphi_1, \ldots, \varphi_m)$, (2.2), is

$$\rho(\Phi) = \begin{pmatrix} e^{i\Lambda_1(\varphi)} & & & Z \\ & \cdot & & \\ & & \cdot & \\ Z & & & e^{i\Lambda_s(\varphi)} \end{pmatrix}. \tag{2.15}$$

Here $\exp[i\Lambda_j(\varphi_1, \ldots, \varphi_m)] = \epsilon_1^{p_1} \cdots \epsilon_m^{p_m}$, where $\Lambda_j = p_1\varphi_1 + \cdots + p_m\varphi_m$. By definition,

$$\chi(\varphi_1, \ldots, \varphi_m) = \operatorname{tr} \rho(\Phi) = \sum_{p_j} c_{p_1 \cdots p_m} \epsilon_1^{p_1} \cdots \epsilon_m^{p_m}, \tag{2.16}$$

where the integer $c_{p_1 \cdots p_m}$ is the multiplicity of the corresponding weight in the rep. Since χ is a symmetric function of $\varphi_1, \ldots, \varphi_m$ the integers $c_{p_1 \cdots p_m}$ are symmetric in $p_1, \ldots, p_m$. Thus a weight Λ' obtained from Λ by permuting the coefficients $p_1, \ldots, p_m$ has the same multiplicity as Λ. Recalling the definition of the Weyl reflection S^α we find the following result.

Lemma 9.7. If Λ is a weight and $\Lambda' = S^\alpha \Lambda$ for some root α then Λ and Λ' have the same multiplicity.

If χ is the character of the rep $[f_1, \ldots, f_m]$, $f_1 \geq f_2 \geq \cdots \geq f_m$, then the term of highest weight on the right-hand side of (2.16) is $\epsilon_1^{f_1} \cdots \epsilon_m^{f_m}$ and the coefficient of this term is $c_{f_1 \cdots f_m} = 1$.

Now consider the product $\xi = \chi\Delta$. Since χ is symmetric and Δ is skew-symmetric, ξ is a skew-symmetric function of $\epsilon_1, \ldots, \epsilon_m$. Furthermore, ξ is a finite sum

$$\xi = \sum d_{q_1 \cdots q_m} \epsilon_1^{q_1} \cdots \epsilon_m^{q_m}, \qquad \epsilon_j = e^{i\varphi_j}, \tag{2.17}$$

where the $d_{q_1 \cdots q_m}$ are integers. The highest-order term in (2.17) is clearly $1 \cdot \epsilon_1^{f_1+m-1} \epsilon_2^{f_2+m-2} \cdots \epsilon_{m-1}^{f_{m-1}+1} \epsilon_m^{f_m} = \epsilon_1^{l_1} \epsilon_2^{l_2} \cdots \epsilon_m^{l_m}$, where $l_j = f_j + m - j$ and $l_1 > l_2 > \cdots > l_m$. Since ξ is skew-symmetric and contains $\epsilon_1^{l_1} \cdots \epsilon_m^{l_m}$ it must also contain the terms

$$\xi(l_1 \ldots, l_m) = \sum_{s \in S_m} \delta_s \, \epsilon_{s(1)}^{l_1} \cdots \epsilon_{s(m)}^{l_m} = |\epsilon^{l_1}, \ldots, \epsilon^{l_m}|, \tag{2.18}$$

where δ_s is the parity of the permutation $s \in S_m$ and $|\epsilon^{l_1}, \ldots, \epsilon^{l_m}|$ is the determinant of

$$\begin{pmatrix} \epsilon_1^{l_1} & \epsilon_1^{l_2} & \cdots & \epsilon_1^{l_m} \\ \epsilon_2^{l_1} & \epsilon_2^{l_2} & \cdots & \epsilon_2^{l_m} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \epsilon_m^{l_1} & \epsilon_m^{l_2} & & \epsilon_m^{l_m} \end{pmatrix}. \tag{2.19}$$

Note that $\Delta(\epsilon_1, \ldots, \epsilon_m) = \xi(m-1, m-2, \ldots, 1, 0)$. If $\xi \neq \xi(l_1, \ldots, l_m)$, then $\xi - \xi(l_1, \ldots, l_m)$ is skew-symmetric and contains a highest-order term $c' \epsilon_1^{l_1'} \cdots \epsilon_m^{l_m'}$, $l_1' > l_2' > \cdots > l_m'$. (Note that the skew-symmetry of ξ guarantees that any term $\epsilon_1^{p_1} \cdots \epsilon_j^{p_j} \cdots \epsilon_k^{p_k} \cdots \epsilon_m^{p_m}$ with $p_j = p_k$ has coefficient zero.) Hence, $\xi - \xi(l_1, \ldots, l_m) = c'\xi(l_1', \ldots, l_m') + \cdots$. Since ξ is a finite sum of terms, this process must end eventually and we obtain

$$\xi = \xi(l_1, \ldots, l_m) + c'\xi(l_1', \ldots, l_m') + c''\xi(l_1'', \ldots, l_m'') + \cdots. \tag{2.20}$$

Now

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \epsilon_1^{p_1} \cdots \epsilon_m^{p_m} \overline{\epsilon_1^{q_1} \cdots \epsilon_m^{q_m}} \, d\varphi_1 \cdots d\varphi_m = (2\pi)^m \delta_{p_1 q_1} \cdots \delta_{p_m q_m}$$

and it follows easily from (2.18) that

$$\begin{aligned} \int_0^{2\pi} \cdots \int_0^{2\pi} \xi(l_1, \ldots, l_m) \bar{\xi}(l_1', \ldots, l_m') \, d\varphi_1 \cdots d\varphi_m \\ = (2\pi)^m m! \, \delta_{l_1 l_1'} \cdots \delta_{l_m l_m'}. \end{aligned} \tag{2.21}$$

In particular

$$V = \int_0^{2\pi} \cdots \int_0^{2\pi} |\xi(m-1, m-2, \ldots, 1, 0)|^2 \, d\varphi_1 \cdots d\varphi_m = (2\pi)^m m!$$

and

$$(2.22) \qquad (\chi, \chi) = V^{-1} \int_0^{2\pi} \cdots \int_0^{2\pi} |\chi\Delta|^2 \, d\varphi_1 \cdots d\varphi_m = 1 + (c')^2 + (c'')^2 + \cdots,$$

where we have used (2.20) and the relation $\xi = \chi\Delta$. Since χ is a simple character, $(\chi, \chi) = 1$ and $c' = c'' = \cdots = 0$.

Theorem 9.4. The character of the irred rep $[f_1, \ldots, f_m]$ of $U(m)$ is

$$(2.23) \qquad \chi^{f_1 \cdots f_m}(\epsilon_1, \ldots, \epsilon_m) = |\epsilon^{l_1}, \ldots, \epsilon^{l_m}| / |\epsilon^{m-1}, \ldots, \epsilon, 1| = \xi/\Delta,$$

where $l_j = f_j + m - j$.

Expression (2.23) makes sense only if no two of the ϵ_j are equal, since otherwise both the numerator and denominator are zero. However, from (2.16) we see that χ is defined and continuous for all $\epsilon_1, \ldots, \epsilon_m$ on the unit circle. Thus, we can compute the character for equal ϵ_j by taking an appropriate limit in expression (2.23).

For example, the dimension $N(f_1, \ldots, f_m)$ of the rep $[f_1, \ldots, f_m]$ is just $\chi^{f_1 \cdots f_m}_{(1, \ldots, 1)}$. To obtain this value of the character we set $\epsilon_j = \epsilon^{m-j}$ in (2.23) and pass to the limit as $\epsilon \to 1$. Then

$$(2.24) \qquad N(f_1, \ldots, f_m) = \lim_{\epsilon \to 1} \left\{ \left[\prod_{j<k} (\epsilon^{l_j} - \epsilon^{l_k}) \right] \Big/ \left[\prod_{j<k} (\epsilon^{m-j} - \epsilon^{m-k}) \right] \right\} = \Delta(l_1, \ldots, l_m)/\Delta(m-1, m-2, \ldots, 1, 0)$$

since $\epsilon^{l_j} - \epsilon^{l_k} = e^{i\varphi l_j} - e^{i\varphi l_k} \to i(l_j - l_k)\varphi$ as $\varphi \to 0$. Note that $N(f_1 + p, \ldots, f_m + p) = N(f_1, \ldots, f_m)$ for any integer p. This is in agreement with the observation

$$[f_1 + p, \ldots, f_m + p] \cong [p^m] \otimes [f_1, \ldots, f_m],$$

where $[p^m]$ is one-dimensional.

If the requirement that $|\epsilon_j| = 1$ is relaxed, expression (2.23) also defines the character of the irred rep $[f_1, \ldots, f_m]$ of $GL(m)$. Indeed, if $A \in GL(m)$ has m distinct eigenvalues $\epsilon_1, \ldots, \epsilon_m$ then there exists $B \in GL(m)$ such that

$$(2.25) \qquad A = B \begin{pmatrix} \epsilon_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \epsilon_m \end{pmatrix} B^{-1} = BD(\epsilon)B^{-1}.$$

Thus the conjugacy class to which A belongs can be described by the parameters ϵ_j. If ρ is a matrix realization of $[f_1, \ldots, f_m]$ then

$$\chi^{f_1\cdots f_m}(\epsilon_1, \ldots, \epsilon_m) = \operatorname{tr} \rho(A) = \operatorname{tr} \rho(D) = \sum_{p_j} c_{p_1\cdots p_m} \epsilon_1^{p_1} \cdots \epsilon_m^{p_m}, \tag{2.26}$$

where $c_{p_1\cdots p_m}$ is the multiplicity of the weight $\Lambda = \sum p_j \lambda_j$. This expression is exactly the same as (2.16) except that the ϵ_j need not have modulus one. Thus, formula (2.23) must be valid for $\chi^{f_1\cdots f_m}(\epsilon_1, \ldots, \epsilon_m)$. If the eigenvalues of A are not all distinct it may not be possible to diagonalize A by similarity transformations. However, we can always find a sequence of matrices $\{A^{(n)} \in GL(m), n = 1, 2, \ldots\}$ such that each $A^{(n)}$ has m distinct eigenvalues $\epsilon_j^{(n)}$, $1 \le j \le m$, and $A^{(n)} \longrightarrow A$, $\epsilon_j^{(n)} \longrightarrow \epsilon_j$ as $n \longrightarrow \infty$. Since $\chi^{f_1\cdots f_m}$ is continuous we have $\chi^{f_1\cdots f_m}(A) = \lim_{n\to\infty} \chi^{f_1\cdots f_m}(A^{(n)}) = \lim_{n\to\infty} \chi^{f_1\cdots f_m}(\epsilon_1^{(n)}, \ldots, \epsilon_m^{(n)})$ and the character is given by an appropriate limiting form of (2.23) even if the eigenvalues of A are not distinct.

9.3 The Irreducible Representations of $GL(m, R)$, $SL(m, \mathfrak{C})$, and $SU(m)$

Since $gl(m, R)$ is a real form of $gl(m, \mathfrak{C})$ there is a 1–1 relationship between reps of these two Lie algebras. Every analytic rep of the complex Lie group $GL(m)$ restricts to an analytic rep of the real Lie group $GL(m, R)$. Conversely, an analytic rep of $GL(m, R)$ uniquely extends to an analytic rep of the **complex** Lie group $GL(m)$. Thus, the irred reps of $GL(m, R)$ can be denoted

$$[f_1, \ldots, f_m], \qquad f_1 \ge f_2 \ge \cdots \ge f_m. \tag{3.1}$$

These are the restrictions of the corresponding reps of $GL(m)$ to $GL(m, R)$. Every finite-dimensional analytic rep of $GL(m, R)$ can be decomposed into a direct sum of irred reps.

The rep theory of the group

$$GL(m, R)^+ = \{A \in GL(m, R)\colon \det A > 0\} \tag{3.2}$$

is slightly different because some Lie algebra reps of $gl(m, R)$ induce global reps of $GL(m, R)^+$ which do not extend to $GL(m, R)$. Indeed, the irred reps of $GL(m, R)^+$ are of the form

$$(\det A)^c \otimes [f_1, \ldots, f_m], \qquad c \in \mathfrak{C}, \quad 0 \le \operatorname{Re} c < 1. \tag{3.3}$$

Next we consider $SL(m) = SL(m, \mathfrak{C})$, where $m \ge 2$. The restriction of the tensor rep $T \cong [f_1, \ldots, f_m], f_1 \ge f_2 \ge \cdots \ge f_m \ge 0$, of $GL(m)$ to $SL(m)$ yields a matrix rep of $SL(m)$ whose matrix elements are homogeneous polynomials of order $f_1 + \cdots + f_m = s$ in the matrix elements of $A \in SL(m)$. If $T' = T \mid SL(m)$ then for any $B \in GL(m)$ we have $T(B) = (\det B)^{s/m} T'(A)$, where $B = (\det B)^{1/m} A$. Note that $\det A = 1$, so $A \in SL(m)$. Since T is irred, it follows from the Schur lemmas that T' is irred. Thus, the tensor reps $[f_1, \ldots, f_m]$ of $SL(m)$ are irred.

However, these reps are no longer pairwise nonequivalent. Indeed we have shown earlier that

$$[f_1 + p, \ldots, f_m + p] \cong [p^m] \otimes [f_1, \ldots, f_m] \tag{3.4}$$

for $GL(m)$, where $[p^m]$ is the rep

$$[p^m]: B \longrightarrow (\det B)^p, \qquad B \in GL(m), \tag{3.5}$$

and p is an integer. Clearly, on restriction to $B \in SL(m)$ we have $[p^m] \cong [0]$. Thus, the reps with signatures $[f_1 + p, \ldots, f_m + p]$ and $[f_1, \ldots, f_m]$ are equivalent for all integers p. By choosing $p = -f_m$ we can always assume that the irred reps take the form $[g_1, \ldots, g_{m-1}, 0]$, $g_1 \geq g_2 \geq \cdots \geq g_{m-1} \geq 0$. We shall adopt a Lie-algebraic approach to verify that these are the only analytic irred reps of the complex Lie group $SL(m)$ and that they are pairwise nonequivalent.

The rep theory of the Lie algebra $sl(m)$ is almost identical to the theory for $gl(m)$ developed in Section 9.1. As we mentioned earlier, $sl(m)$ is the $(m^2 - 1)$-dimensional Lie algebra of all $m \times m$ complex matrices with trace zero. The set of all diagonal matrices

$$\mathcal{H} = \sum_{j=1}^{m} \lambda_j \mathcal{H}_j = \begin{pmatrix} \lambda_1 & & Z \\ & \ddots & \\ Z & & \lambda_m \end{pmatrix}, \qquad \lambda_1 + \cdots + \lambda_m = 0, \tag{3.6}$$

forms an $(m-1)$-dimensional maximal commutative subalgebra $\mathscr{h}_{m-1}$ of $sl(m)$. Let $\mathcal{E}_{hj}$ be the $m \times m$ matrix with a one in row h, column j and zeros everywhere else. The $m^2 - 1$ matrices $\mathcal{E}_{hj}$, $1 \leq h, j \leq m$, $h \neq j$, and $\mathcal{H}_j - \mathcal{H}_m$, $j = 1, 2, \ldots, m-1$, form a basis for $sl(m)$. We easily obtain the commutation relations

$$\begin{aligned} &[\mathcal{E}_{hj}, \mathcal{E}_{kl}] = \delta_{jk}\mathcal{E}_{hl} - \delta_{lh}\mathcal{E}_{kj}, \qquad &&[\mathcal{H}, \mathcal{H}'] = Z, \\ &[\mathcal{H}, \mathcal{E}_{kl}] = (\lambda_k - \lambda_l)\mathcal{E}_{kl}, &&\mathcal{H}, \mathcal{H}' \in \mathscr{h}_{m-1}. \end{aligned} \tag{3.7}$$

Just as in Section 9.1, we consider the adjoint rep $\mathcal{H} \longrightarrow \text{ad}\,\mathcal{H}$ of $\mathscr{h}_{m-1}$ on $sl(m)$. According to the second of expressions (3.7) the element $\mathcal{E}_{kl}$ is a simultaneous eigenvector for all operators ad $\mathcal{H}(\lambda_1, \ldots, \lambda_m)$ with eigenvalue $\alpha(\mathcal{H}) = \lambda_k - \lambda_l$. As before we call the linear functional $\alpha(\mathcal{H})$ a **root** and the corresponding eigenvector $\mathcal{E}_{kl} \equiv \mathcal{E}_\alpha$ the **branch** belonging to root α. Set $\mathcal{H}_\alpha = \mathcal{H}_k - \mathcal{H}_l \in \mathscr{h}_{m-1}$ for $\alpha = \lambda_k - \lambda_l$. Then the $\mathcal{E}_\alpha$ and $\mathcal{H}_\alpha$ span $sl(m)$ as α runs over the $m(m-1)$ distinct roots.

Lemma 9.8.

(a) If α is a root then $-\alpha$ is a root.
(b) $[\mathcal{E}_\alpha, \mathcal{E}_{-\alpha}] = \mathcal{H}_\alpha \neq Z$.

(c) $[\mathcal{E}_\alpha, \mathcal{E}_\beta] = Z$ if $\alpha + \beta$ is not a root and $\alpha \neq -\beta$.
(d) $[\mathcal{E}_\alpha, \mathcal{E}_\beta] = \pm\mathcal{E}_{\alpha+\beta}$ if $\alpha + \beta$ is a root.
(e) $[\mathcal{H}, \mathcal{E}_\alpha] = \alpha(\mathcal{H})\mathcal{E}_\alpha$, $[\mathcal{H}_\alpha, \mathcal{E}_\alpha] = 2\mathcal{E}_\alpha$.

This lemma and its proof are identical with Lemma 9.1. The only difference between these results and the corresponding results for $gl(m)$ is that $\lambda_1 + \cdots + \lambda_m = 0$ in (3.6). We can consider $\lambda_1, \ldots, \lambda_{m-1}$ as independent variables while $\lambda_m = -\lambda_1 - \cdots - \lambda_{m-1}$.

Let ρ be a finite-dimensional rep of $sl(m)$ on the vector space V and set $\rho(\mathcal{H}) = H$, $\rho(\mathcal{E}_\alpha) = E_\alpha$. Then the operators $\{H, E_\alpha\}$ satisfy the commutation relations given above. Furthermore, the proof of Theorem 9.1 shows that V has a basis of weight vectors. The construction of α-ladders of weights and Theorems 9.2 and 9.3 carry over immediately to reps of $sl(m)$. In particular each irred rep ρ is uniquely determined by its highest weight $\Lambda^*(\lambda_1, \ldots, \lambda_{m-1})$ which has multiplicity one. (We write each weight in the unique form $\Lambda = q_1\lambda_1 + \cdots + q_{m-1}\lambda_{m-1}$ and adopt the usual lexicographic ordering.)

Let us determine the possible weights

$$(3.8)\qquad \begin{aligned} \Lambda(\mathcal{H}) &= \sum_{j=1}^{m} p_j\lambda_j = \sum_{j=1}^{m-1} q_j\lambda_j, \qquad q_j = p_j - p_m, \\ \mathcal{H} &= \sum_{j=1}^{m} \lambda_j\mathcal{H}_j = \sum_{j=1}^{m-1} \lambda_j(\mathcal{H}_j - \mathcal{H}_m) \in \mathscr{h}_{m-1}. \end{aligned}$$

If Λ is a weight then $2\Lambda_\alpha/\alpha_\alpha$ is an integer for each root $\alpha = \lambda_k - \lambda_l$, where $\Lambda_\alpha = \Lambda(\mathcal{H}_\alpha) = p_k - p_l$, $\alpha_\alpha = 2$. For $1 \leq k, l \leq m - 1$ we find $p_k - p_l = q_k - q_l$ is an integer. However, if $k = 1, \ldots, m - 1$, $l = m$ then $p_k - p_m = q_k$ is an integer. Thus, the possible weights take the form $\Lambda = \sum q_j\lambda_j$, where the q_j are integers.

Now supppose $\Lambda^* = \sum q_j\lambda_j$ is a highest weight vector. Then $\Lambda_\alpha^* \geq 0$ for all positive roots α. The positive roots are $\alpha = \lambda_k - \lambda_l$, $1 \leq k < l \leq m - 1$, and $\alpha = \lambda_k - \lambda_m = \lambda_k + \lambda_1 + \cdots + \lambda_{m-1}$. Thus, if Λ^* is maximal then $q_k - q_l \geq 0$ for $k < l$ and $q_k \geq 0$, i.e., $q_1 \geq q_2 \geq \cdots \geq q_{m-1} \geq 0$. We shall show that each such linear form actually *is* the maximal weight of an irred rep ρ of $sl(m)$ which defines a global rep of $SL(m)$.

Consider the irred rep $[f_1, \ldots, f_m]$ of $GL(m)$. Its highest weight is $\Lambda^* = f_1\lambda_1 + \cdots + f_m\lambda_m$. It is characterized by the fact that $\Lambda^* + \alpha$ is not a weight for any root $\alpha > 0$. If we restrict the rep to $SL(m)$ the weights $\Lambda = p_1\lambda_1 + \cdots + p_m\lambda_m$ on $\mathscr{h}_m$ will restrict to weights $\Lambda' = (p_1 - p_m)\lambda_1 + \cdots + (p_{m-1} - p_m)\lambda_{m-1}$ on $\mathscr{h}_{m-1}$. Furthermore, the positive roots α of $gl(m)$ restrict to the positive roots of $sl(m)$. Thus, $(\Lambda')^* + \alpha$ is not a weight for any root $\alpha > 0$ of $sl(m)$ and

$$(3.9)\qquad (\Lambda')^* = (f_1 - f_m)\lambda_1 + \cdots + (f_{m-1} - f_m)\lambda_{m-1}$$

is the maximal weight belonging to $[f_1, \ldots, f_m] \mid SL(m)$. Since this rep is irred we see again that

$$[f_1, \ldots, f_m] \cong [f_1 - f_m, \ldots, f_{m-1} - f_m, 0]$$

and $q_1\lambda_1 + \cdots + q_{m-1}\lambda_{m-1}$ for integers $q_1 \geq q_2 \geq \cdots \geq q_{m-1} \geq 0$ is the highest weight of the rep $[q_1, \ldots, q_{m-1}, 0]$.

The Lie algebra $su(m)$ is a real form of $sl(m)$ and there is a 1–1 relationship between analytic reps of the complex group $SL(m)$ and those of $SU(m)$. Since $SU(m)$ is compact we can use the unitary trick to conclude that every rep of $SL(m)$ can be decomposed into a direct sum of irred reps. Furthermore, the irred reps of $SU(m)$ are just the restrictions of the reps $[f_1, \ldots, f_{m-1}, 0]$ to $SU(m)$.

The simple characters $\psi^{f_1\cdots f_m}$ of $SU(m)$ are the restrictions of the characters (2.23) to $SU(m)$, i.e., $\psi^{f_1\cdots f_m} = \chi^{f_1\cdots f_m}$ with $\epsilon_1\epsilon_2\cdots\epsilon_m = 1$. It follows from remarks made above that $\psi^{f_1\cdots f_m} = \psi^{f_1+p,\ldots,f_m+p}$. For example, we compute the simple characters $\psi^{f_1f_2}$ of $SU(2)$. Here $\epsilon = \epsilon_1 = \epsilon_2^{-1}$ and

$$\begin{aligned} \psi^{f_1f_2} &= \begin{vmatrix} \epsilon^{f_1+1} & \epsilon^{f_2} \\ \epsilon^{-f_1-1} & \epsilon^{-f_2} \end{vmatrix} \Bigg/ \begin{vmatrix} \epsilon & 1 \\ \epsilon^{-1} & 1 \end{vmatrix} \\ &= (\epsilon^{(f_1-f_2+1)} - \epsilon^{-(f_1-f_2+1)})/(\epsilon - \epsilon^{-1}) \\ &= \{\sin[(f_1 - f_2 + 1)\tau/2]\}/\sin(\tau/2), \end{aligned} \tag{3.10}$$

where $\epsilon = e^{i\tau/2}$. This is in agreement with (2.24), Section 7.2, and shows that

$$[f_1, f_2] \cong [f_1 - f_2, 0] \cong \mathbf{D}^{((f_1-f_2)/2)}. \tag{3.11}$$

We can also consider $SL(m)$ as a real $2(m^2 - 1)$-dimensional Lie group. In analogy with the theory of the real group $GL(m)$ one can easily show that the analytic irred reps of the real group $SL(m)$ are

$$[f_1, \ldots, f_{m-1}, 0] \otimes \overline{[g_1, \ldots, g_{m-1}, 0]}. \tag{3.12}$$

Finally we note a general method for decomposing reps of the complex group $SL(m)$ into irred parts. Let ρ be an analytic rep of $SL(m)$ such that

$$\rho \cong \alpha_1[f_1, \ldots, f_{m-1}, 0] \oplus \alpha_2[g_1, \ldots, g_{m-1}, 0] \oplus \cdots, \tag{3.13}$$

where the reps on the right-hand side are listed in lexicographic order. Then ρ contains the highest weight $f_1\lambda_1 + \cdots + f_{m-1}\lambda_{m-1}$ with multiplicity α_1. Suppose we have a list of the weights of ρ, each weight listed as many times as its multiplicity in ρ. Remove all the weights corresponding to α_1 copies of $[f_1, \ldots, f_{m-1}, 0]$ from the list. Then the highest weight remaining will be $g_1\lambda_1 + \cdots + g_{m-1}\lambda_{m-1}$ with multiplicity α_2. Next remove all weights corresponding to α_2 copies of $[g_1, \ldots, g_{m-1}, 0]$. We can continue in this manner until all the weights of ρ have been removed. The process is useful when we know the weights of ρ and want to derive the decomposition (3.13).

As an example we rederive the Clebsch–Gordan series for reps of $SL(2)$. The weights of $[f_1, 0]$ are $n\lambda_1 + (f_1 - n)\lambda_2 = (2n - f_1)\lambda_1$, $n = 0, 1, \ldots, f_1$, as we see from (1.41) and $\lambda_1 + \lambda_2 = 0$. By Lemma 9.6, the weights of $\rho \cong [f_1, 0] \otimes [g_1, 0]$ are $(2n + 2p - f_1 - g_1)\lambda_1$, $n = 0, 1, \ldots, f_1$, $p = 0, 1, \ldots, g_1$. Assuming $f_1 \geq g_1$ we see that the weight $2s - f_1 - g_1$ has multiplicity $s + 1$ if $0 \leq s \leq g_1$, $g_1 + 1$ if $g_1 + 1 \leq s \leq f_1$, and $f_1 + g_1 + 1 - s$ if $f_1 + 1 \leq s \leq f_1 + g_1$. The highest weight of this rep is $(f_1 + g_1)\lambda_1$ with multiplicity one. Thus, ρ contains $[f_1 + g_1, 0]$. If we remove the weights of $[f_1 + g_1, 0]$ the highest remaining weight is $(f_1 + g_1 - 2)\lambda_1$ with multiplicity one. Thus ρ contains $[f_1 + g_1 - 2, 0]$. Continuing in this manner we obtain the reps $[f_1 + g_1 - 2k, 0]$, $0 \leq k \leq g_1$, each with multiplicity one. At this point all weights of ρ are used and the process ends. Thus

$$(3.14) \qquad [f_1, 0] \otimes [g_1, 0] \cong \sum_{k=1}^{g_1} \oplus [f_1 + g_1 - 2k, 0],$$

which is the Clebsch–Gordan series (2.26), Chapter 7.

9.4 The Symplectic Groups and Their Representations

Recall that the symplectic group $Sp(m)$ consists of all $2m \times 2m$ complex matrices A such that

$$(4.1) \qquad A^{\mathrm{t}}JA = J,$$

where J is the skew-symmetric matrix (4.1), Section 5.4. (More generally, we can consider $Sp(m)$ as the set of all linear operators $\mathbf{A}$ on a $2m$-dimensional complex vector space V such that $\langle \mathbf{Au}, \mathbf{Av} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$, where $\langle -, - \rangle$ is a nonsingular skew-symmetric bilinear form on V: see Weyl [3].)

From Theorem 5.15, the Lie algebra $sp(m)$ of $Sp(m)$ is the space of all $2m \times 2m$ complex matrices $\mathcal{A}$ such that

$$(4.2) \qquad \mathcal{A}^{\mathrm{t}} = J\mathcal{A}J.$$

Setting

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{pmatrix}, \qquad J = \begin{pmatrix} Z & E_m \\ -E_m & Z \end{pmatrix},$$

where the $\mathcal{A}_j$ are $m \times m$ matrices, we find

$$\mathcal{A}_1{}^{\mathrm{t}} = -\mathcal{A}_4, \qquad \mathcal{A}_2{}^{\mathrm{t}} = \mathcal{A}_2, \qquad \mathcal{A}_3{}^{\mathrm{t}} = \mathcal{A}_3.$$

Thus, $sp(m)$ is $(2m^2 + m)$-dimensional. Denoting by $\mathcal{E}_{jk}$ the matrix with a one in row j, column k and zeros everywhere else we obtain the basis

$$(4.3) \qquad \mathcal{E}_{jk} - \mathcal{E}_{k+m,j+m}, \quad \mathcal{E}_{j,k+m} + \mathcal{E}_{k,j+m}, \quad \mathcal{E}_{j+m,k} + \mathcal{E}_{k+m,j}, \qquad 1 \leq j, k \leq m.$$

The set $\mathscr{h}_m$ of all diagonal matrices

$$\mathcal{H}(\lambda_1, \ldots, \lambda_m) = \sum_{j=1}^{m} \lambda_j \mathcal{H}_j \tag{4.4}$$

forms a maximal abelian subalgebra of $sp(m)$. Here $\mathcal{H}_j = \mathcal{E}_{jj} - \mathcal{E}_{j+m,j+m}$.

The adjoint rep of $\mathscr{h}_m$ on $sp(m)$ is

$$\begin{aligned} [\mathcal{H}, \mathcal{E}_{jk} - \mathcal{E}_{k+m,j+m}] &= (\lambda_j - \lambda_k)(\mathcal{E}_{jk} - \mathcal{E}_{k+m,j+m}) \\ [\mathcal{H}, \mathcal{E}_{j,k+m} + \mathcal{E}_{k,j+m}] &= (\lambda_j + \lambda_k)(\mathcal{E}_{j,k+m} + \mathcal{E}_{k,j+m}) \\ [\mathcal{H}, \mathcal{E}_{j+m,k} + \mathcal{E}_{k+m,j}] &= (-\lambda_j - \lambda_k)(\mathcal{E}_{j+m,k} + \mathcal{E}_{k+m,j}). \end{aligned} \tag{4.5}$$

Thus the roots are $\alpha = \lambda_j - \lambda_k$ $(j \neq k)$ with branches $\mathcal{E}_\alpha = \mathcal{E}_{jk} - \mathcal{E}_{k+m,j+m}$, $\alpha = \lambda_j + \lambda_k$ $(j \leq k)$ with branches $\mathcal{E}_\alpha = \mathcal{E}_{j,k+m} + \mathcal{E}_{k,j+m}$, and $\alpha = -\lambda_j - \lambda_k$ $(j \leq k)$ with branches $\mathcal{E}_\alpha = \mathcal{E}_{j+m,k} + \mathcal{E}_{k+m,j}$. There are $2m^2$ distinct roots

$$\alpha = \pm\lambda_j \pm \lambda_k, \quad j < k, \qquad \alpha = \pm 2\lambda_j.$$

By tedious computations we can verify the relations

$$[\mathcal{H}, \mathcal{H}'] = Z, \qquad [\mathcal{H}, \mathcal{E}_\alpha] = \alpha(\mathcal{H})\mathcal{E}_\alpha,$$

$$[\mathcal{E}_\alpha, \mathcal{E}_\beta] = \begin{cases} Z & \text{if } \alpha + \beta \text{ is nonzero and not a root,} \\ N_{\alpha\beta}\mathcal{E}_{\alpha+\beta} & \text{if } \alpha + \beta \neq 0 \text{ is a root,} \\ \mathcal{H}_\alpha & \text{if } \alpha = -\beta. \end{cases} \tag{4.6}$$

Here $N_{\alpha\beta}$ is a nonzero constant depending on α and β, and

$$\mathcal{H}_\alpha = \begin{cases} \pm\mathcal{H}_j \pm \mathcal{H}_k & \text{if } \alpha = \pm\lambda_j \pm \lambda_k, \ j < k, \\ \pm\mathcal{H}_j & \text{if } \alpha = \pm 2\lambda_j. \end{cases} \tag{4.7}$$

It follows from (4.6) that the proofs of Theorems 9.1–9.3 go through virtually unchanged for reps ρ of $sp(m)$. If $\Lambda = \sum p_j \lambda_j$ is a weight then

$$\Lambda_\alpha = \Lambda(\mathcal{H}_\alpha) = \begin{cases} \pm p_j \pm p_k & \text{if } \alpha = \pm\lambda_j \pm \lambda_k, \ j < k, \\ \pm p_j & \text{if } \alpha = \pm 2\lambda_j. \end{cases} \tag{4.8}$$

In particular,

$$\alpha_\alpha = \alpha(\mathcal{H}_\alpha) = 2 \tag{4.9}$$

for all roots α.

If Λ is a weight then $2\Lambda_\alpha/\alpha_\alpha$ is an integer and $S^\alpha\Lambda = \Lambda - (2\Lambda_\alpha/\alpha_\alpha)\alpha$ is a weight. Writing $\Lambda = \sum p_j\lambda_j$, we see $2\Lambda_\alpha/\alpha_\alpha = \Lambda_\alpha$ and from (4.8) the p_j must be integers. Furthermore, $S^\alpha\Lambda = \sum p_l\lambda_l - (\pm p_j \pm p_k)(\pm\lambda_j \pm \lambda_k)$, so $S^\alpha\Lambda$ is obtained from Λ by interchanging λ_j and λ_k if $\alpha = \lambda_j - \lambda_k$ or by replacing λ_j with $-\lambda_k$ and λ_k with $-\lambda_j$ if $\alpha = \pm(\lambda_j + \lambda_k)$, $j < k$. If $\alpha = \pm 2\lambda_j$ then $S^\alpha\Lambda = \sum p_l\lambda_l - 2p_j\lambda_j$, so $S^\alpha\Lambda$ is obtained from Λ by replacing λ_j with $-\lambda_j$. The Weyl group has order $m!2^m$.

From Theorem 9.3 an irred rep ρ of $sp(m)$ is uniquely determined by its highest weight Λ^* which occurs with multiplicity one. Furthermore, $\Lambda_\alpha^* \geq 0$

for all positive roots α. With the usual lexicographic ordering, the positive roots are $\lambda_j - \lambda_k, \lambda_j + \lambda_k, j < k$, and $2\lambda_j$. From (4.8), Λ^* is a highest weight only if $p_1 \geq p_2 \geq \cdots \geq p_m \geq 0$. Thus, the possible irred reps of $sp(m)$ can be denoted

$$(4.10) \qquad (p_1, \ldots, p_m), \qquad p_1 \geq \cdots \geq p_m \geq 0.$$

We will show that there exists an irred rep corresponding to each signature (4.10). Furthermore, we will show that each Lie algebra rep induces a global group rep of $Sp(m)$.

Since $Sp(m)$ is a subgroup $GL(2m)$, we can construct the rep $[f_1, \ldots, f_{2m}]$ of $Sp(m)$ by restriction from the corresponding rep of $GL(2m)$. However, the restricted reps may not be irred.

First we consider the natural action of $Sp(m)$ on the $2m$-dimensional vector space V, i.e., the rep [1]. The weights of this rep are easily see to be $\pm\lambda_j$, $1 \leq j \leq m$, each with multiplicity one, and the highest weight is λ_1. This rep is irred since, applying the Weyl reflections S^α to λ_1, we get all of the weights $\pm\lambda_j$. Thus, $(1, 0, \ldots, 0) = (1) \cong [1]$. Similarly, it follows from (4.4), (1.19), and (1.33) that the $\binom{2m}{l}$-dimensional rep $[1^l]$, $l = 1, 2, \ldots, m$, has weights

$$(4.11) \qquad \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_l}, \qquad 1 \leq i_1 < i_2 < \cdots < i_l \leq 2m,$$

each with multiplicity one. Here $\lambda_{i_j} = -\lambda_k$ if $i_j = m + k$, $k > 0$. The highest weight is clearly

$$(4.12) \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_l.$$

At this point we assume that every finite-dimensional analytic rep of $Sp(m)$ can be decomposed into a direct sum of irred reps. (We will prove this later.) Then from (4.12) there exist irred reps of $Sp(m)$ with signature (1^l), $1 \leq l \leq m$. (Note that we use the reps $[1^l]$ of $GL(2m)$ only for $1 \leq l \leq m$. The reps for $m + 1 \leq l \leq 2m$ have been omitted.)

Now consider the rep

$$(4.13) \qquad [1]^{\otimes k_1} \otimes [1^2]^{\otimes k_2} \otimes \cdots \otimes [1^m]^{\otimes k_m} \cong \rho,$$

where $k_1, \ldots, k_m$ are nonnegative integers. The highest weight of ρ is $p_1\lambda_1 + \cdots + p_m\lambda_m$, where

$$(4.14) \qquad \begin{aligned} p_1 &= k_1 + k_2 + \cdots + k_m \\ p_2 &= \phantom{k_1 +{}} k_2 + \cdots + k_m \\ &\;\;\vdots \\ p_{m-1} &= \phantom{k_1 + k_2 + \cdots +{}} k_{m-1} + k_m \\ p_m &= \phantom{k_1 + k_2 + \cdots + k_{m-1} +{}} k_m, \end{aligned}$$

and this weight occurs with multiplicity one. We can construct the irred rep with signature $(p_1, \ldots, p_m)$ by choosing $k_1 = p_1 - p_2, k_2 = p_2 - p_3, \ldots, k_{m-1} = p_{m-1} - p_m, k_m = p_m$. Thus all our candidates for highest weights actually occur in the irred reps of $Sp(m)$.

Our construction has several gaps. We have not shown that every rep of $Sp(m)$ can be decomposed into a direct sum of irred reps. Second, using Lie-algebraic methods we have computed only reps of the connected component of the identity in $Sp(m)$. If $Sp(m)$ has more than one component [like $O(m)$ or $L(4)$] then there are more irred reps than those we have listed.

If $A \in Sp(m)$ then $A^tJA = J$. Taking the determinant of both sides of this equation, we have $(\det A)^2 = 1$, or $\det A = \pm 1$. If there exist group elements for which $\det A = -1$ then such elements are not in the connected component of the identity. We will show that $Sp(m)$ is connected and $\det A = +1$ always.

The group $USp(m) = Sp(m) \cap U(2m)$ is a subgroup of $Sp(m)$. Furthermore, as the reader can verify, its Lie algebra $usp(m)$ is a real $(2m^2 + m)$-dimensional subalgebra of $sp(m)$. Thus, $usp(m)$ is a real form of $sp(m)$ and there is a 1–1 correspondence between complex reps of these Lie algebras. It follows that the irred reps $(p_1, \ldots, p_m)$ of $Sp(m)$ constructed above, restrict to irred reps of $USp(m)$. We will soon show that $USp(m)$ is connected, so its irred reps are uniquely determined by the irred reps if $usp(m)$. Since $USp(m)$ is compact, every analytic rep of this group or every finite-dimensional complex rep of its Lie algebra decomposes into a direct sum of irred reps. This proves that every rep of $sp(m)$ decomposes into irred reps.

We now examine the structure of $USp(m)$. The elements A of this group are both symplectic and unitary. Thus A preserves the forms

$$y^tJx, \qquad y^t\bar{x} \tag{4.15}$$

simultaneously, where x is the column vector $(x_1, \ldots, x_m, x_1', \ldots, x_m')$. Indeed $(Ay)^tJ(Ax) = y^t(A^tJA)x = y^tJx$ with a similar proof for $y^t\bar{x}$. Since A is unitary, it has $2m$ eigenvalues $\epsilon_1, \ldots, \epsilon_{2m}$ each of modulus one. Let x be an eigenvector of A with eigenvalue ϵ: $Ax = \epsilon x$. Now $(A^{-1})^tJ = JA$ since A is symplectic and $(A^{-1})^t = \bar{A}$ since A is unitary. Thus, $\bar{A}J = JA$ and $JAx = \epsilon Jx = \bar{A}Jx$. Taking the complex conjugate, we obtain

$$A(J\bar{x}) = \bar{\epsilon}(J\bar{x}). \tag{4.16}$$

Thus, if x is an eigenvector of A with eigenvalue ϵ then $J\bar{x}$ is an eigenvector with eigenvalue $\bar{\epsilon}$.

Let $x^{(1)}, \ldots, x^{(l)}$ be an ON basis for the eigenspace $\mathcal{C}_\epsilon$ of A (usual scalar product $[x^{(j)}]^t\bar{x}^{(k)} = \delta_{jk}$). Then the $J\bar{x}^{(j)}$, $1 \le j \le l$, form an ON set in $\mathcal{C}_{\bar{\epsilon}}$ since $(J\bar{x}^{(j)})^t\overline{(J\bar{x}^{(k)})} = -[\bar{x}^{(j)}]^tJ^2x^{(k)} = [\bar{x}^{(j)}]^tx^{(k)} = \delta_{jk}$. Similarly, an ON basis $\{y^{(j)}\}$ of $\mathcal{C}_{\bar{\epsilon}}$ is mapped into an ON set $\{J\bar{y}^{(j)}\}$ in $\mathcal{C}_\epsilon$. It follows that $\dim \mathcal{C}_\epsilon = \dim \mathcal{C}_{\bar{\epsilon}} = l$ and $\{J\bar{x}^{(j)}\}$ is an ON basis for $\mathcal{C}_{\bar{\epsilon}}$.

If ϵ is a complex eigenvalue then $\bar{\epsilon} \neq \epsilon$, $\{x^{(j)}\}$, $1 \leq j \leq l$, is an ON basis of $\mathcal{C}_\epsilon$ and $\{J\bar{x}^{(j)}\}$ is an ON basis of $\mathcal{C}_{\bar{\epsilon}}$. Now suppose ϵ is real, i.e., $\epsilon = \pm 1$. Let $x^{(1)}$ be an element of $\mathcal{C}_\epsilon$ with length one. Then $J\bar{x}^{(1)}$ is a unit vector in $\mathcal{C}_\epsilon$ which is perpendicular to $x^{(1)}$. Indeed $(x^{(1)})^t\overline{(J\bar{x}^{(1)})} = (x^{(1)})^t J x^{(1)} = 0$ since J is skew-symmetric. Thus $\{x^{(1)}, J\bar{x}^{(1)}\}$ is an ON set in $\mathcal{C}_\epsilon$. If this set is not a basis we can find a unit vector $x^{(2)} \in \mathcal{C}_\epsilon$ orthogonal to the above set. A simple computation shows that $J\bar{x}^{(2)}$ is also a unit vector in $\mathcal{C}_\epsilon$ and $\{x^{(j)}, J\bar{x}^{(j)}: j = 1, 2\}$ is an ON set. Continuing in this fashion we eventually obtain an ON basis $\{x^{(j)}, J\bar{x}^{(j)}: j = 1, \ldots, l\}$ for $\mathcal{C}_\epsilon$. In particular, the dimension of $\mathcal{C}_\epsilon$ is even for ϵ real.

Lemma 9.9. The $2m$ eigenvalues of $A \in USp(m)$ occur in pairs: $\epsilon_1, \ldots, \epsilon_m, \bar{\epsilon}_1, \ldots, \bar{\epsilon}_m$. There exists a corresponding ON basis $x^{(1)}, \ldots, x^{(m)}, J\bar{x}^{(1)}, \ldots, J\bar{x}^{(m)}$ of eigenvectors.

Let $\mathbf{f}_j, \mathbf{f}_j'$ be the vectors in V with components $x^{(j)}, -J\bar{x}^{(j)}$, respectively, for $1 \leq j \leq m$. Then the $\{\mathbf{f}_j, \mathbf{f}_j'\}$ form an ON basis for V with respect to the usual inner product, and

$$
\begin{aligned}
& (x^{(j)})^t J x^{(k)} = (x^{(j)})^t\overline{(J\bar{x}^{(k)})} = 0 \\
(4.17)\qquad & (-J\bar{x}^{(j)})^t J(-J\bar{x}^{(k)}) = \overline{(x^{(j)})^t J x^{(k)}} = 0 \\
& (x^{(j)})^t J(-J\bar{x}^{(k)}) = (x^{(j)})^t \bar{x}^{(k)} = \delta_{jk}.
\end{aligned}
$$

It follows from this that the $2m \times 2m$ matrix U with columns $(x^{(1)}, \ldots, x^{(m)}, -J\bar{x}^{(1)}, \ldots, -J\bar{x}^{(m)})$ satisfies $U^tJU = J$ and $U^t\bar{U} = E_{2m}$, i.e., $U \in USp(m)$. With this matrix and relations (4.17) it is straightforward to verify the following result.

Theorem 9.5. If $A \in USp(m)$ then there exists $U \in USp(m)$ such that

$$
U^{-1}AU = D(\epsilon) = \begin{pmatrix} \epsilon_1 & & & & & \\ & \ddots & & & & \\ & & \epsilon_m & & & \\ & & & \bar{\epsilon}_1 & & \\ & & & & \ddots & \\ & & & & & \bar{\epsilon}_m \end{pmatrix} \in USp(m).
$$

This shows that every $A \in USp(m)$ is conjugate in the group to a diagonal matrix $D(\epsilon)$. Since $A = UD(\epsilon)U^{-1}$ it is clear that A can be connected to the identity element by an analytic curve $A(t)$ in $USp(m)$ with $A(1) = A$, $A(0) = E_{2m}$. [Let the eigenvalues $\epsilon(t)$ approach $+1$ as $t \to 0$.] Thus $USp(m)$ is connected.

We now sketch a proof of the fact that $Sp(m)$ is connected, leaving many of the details to the reader. As a byproduct we obtain a parametrization of $Sp(m)$.

Any nonsingular matrix A can be written uniquely in the form $A = HU$ where H is a positive-definite Hermitian matrix and U is unitary. (This is the **polar decomposition** of A; see Lancaster [1]). Here $H^2 = AA^*$ and H is the unique positive-definite square root of AA^*.

If $A \in Sp(m)$ then $A^* = \bar{A}^t \in Sp(m)$ and $AA^* \in Sp(m)$. Since the Hermitian matrix $AA^* = H^2$ is positive-definite it can be diagonalized by a unitary similarity transformation. Furthermore, the eigenvalues of H^2 are positive. Let ϵ be an eigenvalue of H^2 with eigenvector x. Now $H^2x = \epsilon x$ and $\bar{H}^{-2}Jx = (H^{-2})^tJx = JH^2x = \epsilon Jx$ since $H^2 \in Sp(m)$ and $(\bar{H}^2)^t = H^2$. Thus.

$$H^2(J\bar{x}) = \epsilon^{-1}(J\bar{x}) \tag{4.18}$$

and $J\bar{x}$ is an eigenvector of H^2 with eigenvalue ϵ^{-1}. Proceeding almost exactly as in the proof of Theorem 9.5, we can show that the eigenvalues of H^2 take the form $\epsilon_1, \ldots, \epsilon_m, \epsilon_1^{-1}, \ldots, \epsilon_m^{-1}, \epsilon_j > 0$. Furthermore, there exists $W \in USp(m)$ such that

$$H^2 = AA^* = W\begin{pmatrix} \epsilon_1 & & & & & \\ & \ddots & & & Z & \\ & & \epsilon_m & & & \\ & & & \epsilon_1^{-1} & & \\ & Z & & & \ddots & \\ & & & & & \epsilon_m^{-1} \end{pmatrix}W^{-1} \in Sp(m). \tag{4.19}$$

It is clear that the matrix H is given by

$$H = W\begin{pmatrix} \epsilon_1^{1/2} & & & & & \\ & \ddots & & & Z & \\ & & \epsilon_m^{1/2} & & & \\ & & & \epsilon_1^{-1/2} & & \\ & Z & & & \ddots & \\ & & & & & \epsilon_m^{-1/2} \end{pmatrix}W^{-1} = WD(\epsilon)W^{-1}. \tag{4.20}$$

Since $D(\epsilon)$ belongs to $Sp(m)$ we have $H \in Sp(m)$. Hence $U = H^{-1}A \in Sp(m) \cap U(2m) = USp(m)$. so every $A \in Sp(m)$ can be written uniquely in the form $A = HU$, where H is given by (4.20) and $U \in USp(m)$. Since $USp(m)$ is connected, so is $Sp(m)$. In particular det $A = +1$, as follows directly from det A = det H det U and Theorem 9.5.

We have shown that the irred reps $(p_1, \ldots, p_m)$ of $Sp(m)$ are all tensor reps. For an explicit description of the irred tensor spaces see Weyl [2, 3].

The construction of simple characters for $USp(m)$ is analogous to that for $U(2m)$ so we present only the results. From Theorem 9.5 the conjugacy class in which $A \in USp(m)$ lies is determined by the eigenvalues $\epsilon_1, \ldots, \epsilon_m$, $\bar{\epsilon}_1, \ldots, \bar{\epsilon}_m$ $(\bar{\epsilon}_j = \epsilon_j^{-1})$ of A. Writing $\epsilon_j = \exp(i\varphi_j), -\pi \leq \varphi_j < \pi$, we see that a character χ of $USp(m)$ can be written $\chi(\varphi_1, \ldots, \varphi_m)$. Here χ is invariant under any permutation of the φ_j and the transformations $\varphi_k \longrightarrow -\varphi_k$ for any k, i.e., under the Weyl group. In analogy with (2.14), the simple characters satisfy the orthogonality relations

$$(4.21) \qquad (\chi^{p_1\cdots p_m}, \chi^{q_1\cdots q_m}) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \chi^{p_1\cdots p_m}(\varphi_1, \ldots, \varphi_m) \times \bar{\chi}^{q_1\cdots q_m}(\varphi_1, \ldots, \varphi_m)\, \Delta\bar{\Delta}\, d\varphi_1 \cdots d\varphi_m = \delta_{p_1q_1} \cdots \delta_{p_mq_m},$$

where

$$(4.22) \qquad \Delta(\varphi_1, \ldots, \varphi_m) = \prod_{j=1}^{m} (\epsilon^j - \epsilon^{-j}) \prod_{1 \leq j < k \leq m} (\epsilon^j + \epsilon^{-j} - \epsilon^k - \epsilon^{-k}).$$

Now Δ is skew-symmetric with respect to the Weyl group, so $\xi = \chi\Delta$ is skew-symmetric, i.e., ξ changes sign under a transposition $\epsilon_j \leftrightarrow \epsilon_k$ and under the exchange $\epsilon_j \leftrightarrow \epsilon_j^{-1}$ for fixed j. Proceeding as in (2.17), we find the possible choices for ξ which give simple χ are

$$(4.23) \qquad \xi(l_1, \ldots, l_m) = |\epsilon^{l_1} - \epsilon^{-l_1}, \ldots, \epsilon^{l_m} - \epsilon^{-l_m}|$$

where the integers l_j satisfy $l_1 > l_2 > \cdots > l_m > 0$, and the determinant $|\cdot|$ is defined by (2.19). In particular

$$\xi(m, m-1, \ldots, 2, 1) = \Delta.$$

The ratio $\chi = \xi/\Delta$ is a finite sum of terms $\epsilon_1^{j_1} \cdots \epsilon_m^{j_m}$. The term with highest weight is $\epsilon_1^{l_1-m}\epsilon_2^{l_2-m+1} \cdots \epsilon_m^{l_m-1}$ and it occurs with multiplicity one. Thus $p_j = l_j - m + j - 1$.

Theorem 9.6.

$$\chi^{p_1,\ldots,p_m} = \frac{|\epsilon^{l_1} - \epsilon^{-l_1}, \ldots, \epsilon^{l_m} - \epsilon^{-l_m}|}{|\epsilon^m - \epsilon^{-m}, \ldots, \epsilon - \epsilon^{-1}|}, \qquad l_j = p_j + m - j + 1,$$
$$1 \leq j \leq m.$$

Corollary 9.4. The dimension of $(p_1, \ldots, p_m)$ is

$$N(p_1, \ldots, p_m) = P(l_1, \ldots, l_m)/P(m, m-1, \ldots, 1),$$
$$P(l_1, \ldots, l_m) = \prod_{1 \leq j \leq m} l_j \prod_{1 \leq j < k \leq m} (l_j^2 - l_k^2).$$

We can use the characters of $USp(m)$ to determine how a symmetry class of tensors $[f_1, \ldots, f_{2m}]$ of $GL(2m)$ decomposes into irred reps $(p_1, \ldots, p_m)$ when $GL(2m)$ is restricted to $Sp(m)$ or $USp(m)$. One need only restrict the character $\chi^{f_1,\ldots,f_{2m}}$ of $U(2m)$, (2.23), to $USp(m)$ and then write it as a sum of simple characters of $USp(m)$. See Weyl [3] for more details.

The reader can check that the analytic irred reps of $Sp(m, R)$ are just the restrictions of the reps $(p_1, \ldots, p_m)$ of $Sp(m)$ to $Sp(m, R)$. Furthermore, the analytic irred reps of the real $2(2m^2 + m)$-dimensional Lie group $Sp(m, \mathfrak{C})$ are $(p_1, \ldots, p_m) \otimes \overline{(p_1', \ldots, p_m')}$.

9.5 The Orthogonal Groups and Their Representations

The usual realization of the complex orthogonal group $O(m, \mathfrak{C})$ is the set of all $m \times m$ complex matrices A such that

$$A^tA = E_m. \tag{5.1}$$

However, we can also consider $O(m, \mathfrak{C})$ as the set of all linear operators $\mathbf{A}$ on an m-dimensional complex vector space V such that $(\mathbf{Au}, \mathbf{Av}) = (\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$, where $(-, -)$ is a nondegenerate symmetric bilinear form on V. There always exists a basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$ for V such that

$$(\mathbf{f}_j, \mathbf{f}_k) = (\mathbf{f}_k, \mathbf{f}_j) = \delta_{jk}, \qquad 1 \leq j, k \leq m, \tag{5.2}$$

(see the book of Cullen [1]). Writing $\mathbf{u} = \sum u_j\mathbf{f}_j$, $\mathbf{v} = \sum v_k\mathbf{f}_k$ we find

$$(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{m} u_jv_j.$$

Furthermore, if A is the matrix of $\mathbf{A}$ in the $\mathbf{f}$-basis a simple computation yields $\mathbf{A} \in O(m, \mathfrak{C})$ if and only if $A^tA = E_m$.

The realization (5.1) is not very convenient for a study of the irred reps of $O(m, \mathfrak{C})$ via the Lie algebra route, for the Lie algebra $o(m)$ in this realization consists of skew-symmetric matrices. Such matrices have all zeros on the diagonal and this is inconvenient since we have become accustomed to the use of a maximal abelian subalgebra $\mathscr{h}_k$ of diagonal matrices. We get around this difficulty by choosing a new basis for V.

If $m = 2n$, $n = 1, 2, \ldots$, we set

$$\mathbf{e}_j = 2^{-1/2}(\mathbf{f}_j + i\mathbf{f}_{n+j}), \qquad \mathbf{e}_j' = 2^{-1/2}(\mathbf{f}_j - i\mathbf{f}_{n+j}), \qquad 1 \leq j \leq n. \tag{5.3}$$

Then

$$(\mathbf{e}_j, \mathbf{e}_k) = (\mathbf{e}_j', \mathbf{e}_k') = 0, \qquad (\mathbf{e}_j, \mathbf{e}_k') = \delta_{jk} \tag{5.4}$$

and if $\mathbf{u} = \sum (u_j\mathbf{e}_j + u_j'\mathbf{e}_j')$ and $\mathbf{v} = \sum (v_j\mathbf{e}_j + v_j'\mathbf{e}_j')$ we find

$$(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} (u_jv_j' + u_j'v_j). \tag{5.5}$$

If $m = 2n + 1, n = 1, 2, \ldots$, we define $\mathbf{e}_j, \mathbf{e}_j', 1 \le j \le n$, by (5.3) and set $\mathbf{e}_0 = \mathbf{f}_{2n+1}$. Then if $\mathbf{v} = \sum (v_j\mathbf{e}_j + v_j'\mathbf{e}_j') + v_0\mathbf{e}_0$ we find

$$(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} (u_j v_j' + u_j' v_j) + u_0 v_0. \tag{5.6}$$

From (5.5) the matrices A of orthogonal transformations $\mathbf{A} \in O(2n, \mathfrak{C})$ with respect to the $\{\mathbf{e}_j, \mathbf{e}_j'\}$ basis are those which satisfy

$$A^t K A = K, \qquad K = \begin{pmatrix} Z_n & E_n \\ E_n & Z_n \end{pmatrix}, \tag{5.7}$$

where E_n is the $n \times n$ identity matrix and Z_n is the $n \times n$ zero matrix.

Similarly the matrices A of operators $\mathbf{A} \in O(2n + 1, \mathfrak{C})$ must satisfy

$$A^t K A = K, \qquad K = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \cdot & & Z_n & E_n \\ \cdot & & & \\ \cdot & & E_n & Z_n \\ 0 & & & \end{pmatrix}, \tag{5.8}$$

where K is a $(2n + 1) \times (2n + 1)$ matrix. Clearly our new realizations of $O(2n, \mathfrak{C})$ and $O(2n + 1, \mathfrak{C})$ are isomorphic (even unitary equivalent) to the old ones. The justification for handling odd m and even m differently is that the rep theory for these two classes is distinctly different.

In our new realization the Lie algebra $o(m)$ is the space of $m \times m$ complex matrices $\mathfrak{A}$ such that

$$\mathfrak{A}^t K + K\mathfrak{A} = Z_m. \tag{5.9}$$

We consider the case $m = 2n$ first. Writing

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{A}_1 & \mathfrak{A}_2 \\ \mathfrak{A}_3 & \mathfrak{A}_4 \end{pmatrix},$$

where the $\mathfrak{A}_j$ are $n \times n$ matrices, we obtain

$$\mathfrak{A}_1{}^t = -\mathfrak{A}_4, \qquad \mathfrak{A}_2{}^t = -\mathfrak{A}_2, \qquad \mathfrak{A}_3{}^t = -\mathfrak{A}_3. \tag{5.10}$$

It follows that the dimension of $o(2n)$ is $2n^2 - n$. Denoting by $\mathcal{E}_{jk}$ the matrix with a one in row j, column k and zeros elsewhere, we find the basis

$$\begin{aligned} &\mathcal{E}_{jk} - \mathcal{E}_{k+n,j+n}, \qquad j, k = 1, \ldots, n, \\ &\mathcal{E}_{j+n,k} - \mathcal{E}_{k+n,j}, \qquad \mathcal{E}_{j,k+n} - \mathcal{E}_{k,j+n}, \qquad j \neq k. \end{aligned} \tag{5.11}$$

The set $\mathfrak{h}_n$ of all diagonal matrices

$$\mathcal{H}(\lambda_1, \ldots, \lambda_n) = \sum_{j=1}^{n} \lambda_j \mathcal{H}_j \tag{5.12}$$

is a maximal abelian subalgebra of $o(2n)$, where $\mathcal{H}_j = \mathcal{E}_{jj} - \mathcal{E}_{j+n,j+n}$.

A straightforward computation shows that the adjoint rep of $\mathfrak{h}_n$ on $o(2n)$ is given by

$$(5.13)\qquad \begin{aligned} [\mathcal{H}, \mathcal{E}_{jk} - \mathcal{E}_{k+n,j+n}] &= (\lambda_j - \lambda_k)(\mathcal{E}_{jk} - \mathcal{E}_{k+n,j+n}) \\ [\mathcal{H}, \mathcal{E}_{j+n,k} - \mathcal{E}_{k+n,j}] &= (-\lambda_j - \lambda_k)(\mathcal{E}_{j+n,k} - \mathcal{E}_{k+n,j}) \\ [\mathcal{H}, \mathcal{E}_{j,k+n} - \mathcal{E}_{k,j+n}] &= (\lambda_j + \lambda_k)(\mathcal{E}_{j,k+n} - \mathcal{E}_{k,j+n}), \end{aligned}$$

where $j \neq k$ and $1 \leq j, k \leq n$. Note the close relationship between (4.5) and (5.13). Clearly the roots are $\alpha = \lambda_j - \lambda_k, j \neq k, \alpha = -\lambda_j - \lambda_k, j < k$, and $\alpha = \lambda_j + \lambda_k, j < k$. There are $2n(n-1)$ distinct roots:

$$\alpha = \pm\lambda_j \pm \lambda_k, \qquad j < k.$$

By straightforward computation one can verify the formulas (4.6) and (4.7).

If $m = 2n + 1$ then $\mathcal{A} \in o(2n + 1)$ takes the form

$$\mathcal{A} = \begin{pmatrix} 0 & -a_2{}^t & -a_1{}^t \\ a_1 & \mathcal{A}_1 & \mathcal{A}_2 \\ a_2 & \mathcal{A}_3 & \mathcal{A}_4 \end{pmatrix},$$

where a_1 and a_2 are $n \times 1$ matrices and $\mathcal{A}_1, \ldots, \mathcal{A}_4$ satisfy (5.10). Thus, the dimension of $o(2n + 1)$ is $2n^2 + n$. If we consider the top row of $\mathcal{A}$ as row zero and the left-hand column as column zero then a basis for $o(2n + 1)$ is given by the matrices (5.11) plus the matrices

$$(5.14)\qquad \mathcal{E}_{k0} - \mathcal{E}_{0,k+n}, \quad \mathcal{E}_{0k} - \mathcal{E}_{k+n,0}, \qquad k = 1, \ldots, n.$$

The set $\mathfrak{h}_n'$ of diagonal matrices

$$(5.15)\qquad \mathcal{H}(\lambda_1, \ldots, \lambda_n) = \sum_{j=1}^{n} \lambda_j \mathcal{H}_j$$

is a maximal abelian subalgebra of $o(2n + 1)$.

The adjoint rep of $\mathfrak{h}_n'$ on $o(2n + 1)$ is given by expressions (5.13) plus

$$(5.16)\qquad \begin{aligned} [\mathcal{H}, \mathcal{E}_{k0} - \mathcal{E}_{0,k+n}] &= \lambda_k(\mathcal{E}_{k0} - \mathcal{E}_{0,k+n}) \\ [\mathcal{H}, \mathcal{E}_{0k} - \mathcal{E}_{k+n,0}] &= -\lambda_k(\mathcal{E}_{0k} - \mathcal{E}_{k+n,0}). \end{aligned}$$

Thus $o(2n + 1)$ has the same roots and branches as $o(2n)$ plus the simple roots $\pm\lambda_k$ with corresponding branches. In summary, the roots of $o(2n + 1)$ are

$$(5.17)\qquad \pm\lambda_j \pm \lambda_k, \quad j < k, \qquad \pm\lambda_k.$$

Again formulas (4.6) and (4.7) can be verified by direct computation.

It follows that the proofs of Theorems 9.1–9.3 apply to reps ρ of $o(2n)$ and $o(2n + 1)$. Here

$$(5.18)\qquad \mathcal{H}_\alpha = \begin{cases} \pm\mathcal{H}_j \pm \mathcal{H}_k & \text{if} \quad \alpha = \pm\lambda_j \pm \lambda_k, \quad j < k, \\ \pm\mathcal{H}_j & \text{if} \quad \alpha = \pm\lambda_j. \end{cases}$$

For example, if $\alpha = \lambda_j$ then $-\alpha = -\lambda_j$ and

$$[\mathcal{E}_\alpha, \mathcal{E}_{-\alpha}] = \mathcal{E}_{jj} - \mathcal{E}_{j+n,j+n} = \mathcal{K}_j = \mathcal{K}_\alpha.$$

If $\Lambda = \sum p_j\lambda_j$ is a weight of ρ then

$$\Lambda_\alpha = \Lambda(\mathcal{K}_\alpha) = \begin{cases} \pm p_j \pm p_k & \text{if} \quad \alpha = \pm\lambda_j \pm \lambda_k, \quad j < k, \\ \pm p_j & \text{if} \quad \alpha = \pm\lambda_j. \end{cases} \tag{5.19}$$

In particular $\alpha_\alpha = 2$ if $\alpha = \pm\lambda_j \pm \lambda_k$ and $\alpha_\alpha = 1$ if $\alpha = \pm\lambda_j$. Note that the latter case occurs only for $m = 2n + 1$.

By Theorem 9.2, $2\Lambda_\alpha/\alpha_\alpha$ is an integer for every root α of $o(m)$ and $S^\alpha\Lambda = \Lambda - (2\Lambda_\alpha/\alpha_\alpha)\alpha$ is a weight of ρ. Thus $\pm p_j \pm p_k, j < k$, are integers for all $o(m)$ and $\pm 2p_j$ are integers for $m = 2n + 1$. If Λ is a weight then the p_j are either all integers of all half integers. If ρ is irred then the weights of ρ either have all integer coefficients or all half-integer coefficients, since any weight Λ' of ρ can be obtained from a single weight Λ by adding suitable sums of roots, $\Lambda' = \Lambda + \alpha_1 + \cdots + \alpha_s$. The roots are weights with integer coefficients.

Let $\Lambda^* = \sum p_j\lambda_j$ be the highest weight of ρ. Then $\Lambda_\alpha^* \geq 0$ for all roots $\alpha > 0$. The positive roots of $o(m)$ are $\lambda_j + \lambda_k, \lambda_j - \lambda_k, 1 \leq j < k \leq n$, and λ_j (if m is odd). Thus $p_j \pm p_k \geq 0$ for $j < k$, and $p_j \geq 0$ (if m is odd). The possible highest weights satisfy

$$\begin{aligned} &p_1 \geq p_2 \geq \cdots \geq p_{n-1} \geq |p_n|, && m = 2n \\ &p_1 \geq p_2 \geq \cdots \geq p_{n-1} \geq p_n \geq 0, && m = 2n + 1, \end{aligned} \tag{5.20}$$

where the p_j are either all integral or all half-integral. In the case $m = 2n$ we have implicitly assumed $n \geq 2$ and omitted the abelian algebra $o(2)$. Note that p_n need not be positive for even m.

We will show that each signature (5.20) does correspond to the highest weight of an irred rep of $o(m)$. However, we will have some difficulty in using these results to determine the irred reps of $O(m, \mathfrak{C})$. First of all, $O(m, \mathfrak{C})$ is not connected. Indeed one can easily prove from the defining relations (5.7) and (5.8) that if $A \in O(m, \mathfrak{C})$ then $(\det A)^2 = 1$ or $\det A = \pm 1$. Furthermore, both signs occur. Thus $O(m, \mathfrak{C})$ has at least two connected components. (We will prove that there are only two.) The Lie algebra $o(m)$ only furnishes us with information about the connected component of the identity $SO(m, \mathfrak{C})$. Therefore, we have to look at more than the Lie algebra to determine the rep theory of $O(m, \mathfrak{C})$. The examples $O(3)$ and $SO(3)$ which we have treated earlier illustrate the problem to be solved here.

A more serious difficulty is that the half-integral Lie algebra reps do not induce global group reps of $SO(m, \mathfrak{C})$. Indeed, the matrix $\mathcal{K} \in \mathscr{h}_n$, (5.12) or (5.15), exponentiates to (in case $m = 2n + 1$, say)

$$(5.21) \qquad e^{\mathcal{K}} = \begin{pmatrix} 1 & 0 & \cdot & \cdots & & & 0 \\ 0 & e^{\lambda_1} & & & & Z & \\ \cdot & & \cdot & & & & \\ \cdot & & & e^{\lambda_n} & & & \\ \cdot & & & & e^{-\lambda_1} & & \\ & Z & & & & \cdot & \\ 0 & & & & & & e^{-\lambda_n} \end{pmatrix} \in SO(m, \mathbb{C}).$$

If ρ is irred, then in a weight basis the matrices $\rho(e^{\mathcal{K}})$ of the induced local group rep take the form

$$(5.22) \qquad \rho(e^{\mathcal{K}}) = \begin{pmatrix} e^{\Lambda_1(\mathcal{K})} & & Z \\ & \cdot & \\ Z & & e^{\Lambda_q(\mathcal{K})} \end{pmatrix},$$

where $\Lambda_1, \ldots, \Lambda_q$ are the weights of ρ. If we replace λ_j by $\lambda_j + 2\pi i$ in (5.21) then $e^{\mathcal{K}}$ remains unchanged. However, if we make these replacements in (5.22) for each j, $\rho(e^{\mathcal{K}})$ will remain unchanged only if the weights Λ_k all have integer coefficients. If the weights have half-integer coefficients then ρ does not induce a single-valued rep of $SO(m, \mathbb{C})$. If we replace λ_j by $\lambda_j + 4\pi i$, however, then $\rho(e^{\mathcal{K}})$ remains unchanged even for half-integral ρ. This suggests that such ρ define double-valued reps of $SO(m, \mathbb{C})$. We will see that there is a group Spin(m), locally isomorphic to $SO(m, \mathbb{C})$ and a homomorphism ν: Spin(m) $\to SO(m, \mathbb{C})$ which covers $SO(m, \mathbb{C})$ exactly twice. The double-valued reps of $SO(m, \mathbb{C})$ are single-valued reps of Spin(m).

Keeping these difficulties in mind, we return to the construction of the irred reps of $o(m)$. Since $O(m, \mathbb{C})$ is a subgroup of $GL(m)$ we obtain reps $[f_1, \ldots, f_m]$ of $O(m, \mathbb{C})$ by restriction of the corresponding irred reps of $GL(m)$. Most of these reps will no longer be irred, however.

The rep belonging to the symmetry class [1] is just the natural action of $O(m, \mathbb{C})$ on an m-dimensional vector space. We see immediately from (5.12) and (5.15) that the weights of the induced Lie algebra rep are $\pm\lambda_j$, $1 \leq j \leq n$, each with multiplicity one, plus the simple weight zero if m is odd. The highest weight is clearly λ_1, so $p_1 = 1, p_k = 0, 2 \leq k \leq n$. This rep is irred, as the reader can easily check. Thus, $[1] \cong (1)$.

The weights of the Lie algebra reps induced by the symmetry classes $[1^l]$, $1 \leq l \leq m - 1$, are

$$(5.23) \qquad \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_l}, \qquad 0 \leq i_1 < i_2 < \cdots < i_l \leq m,$$

where $\lambda_0 = 0$ ($m = 2n + 1$) and $\lambda_{i_j} = -\lambda_k$ if $i_j = n + k, k > 0$. The highest weight of this rep is

$$\lambda_1 + \lambda_2 + \cdots + \lambda_l. \tag{5.24}$$

Assuming that every rep of $o(m)$ can be decomposed into a direct sum of irred reps (which will be proved later) we have shown the existence of irred reps of $o(m)$ with signature (1^l), i.e., $p_1 = p_2 = \cdots = p_l = 1$, $p_{l+1} = \cdots = p_n = 0$.

Now suppose $m = 2n + 1$. In Section 9.6 we will construct a spinor rep of $o(2n + 1)$ with signature $(\frac{1}{2}^n)$. Then the highest weight of the rep

$$(1)^{\otimes k_1} \otimes (1^2)^{\otimes k_2} \otimes \cdots \otimes (1^{n-1})^{\otimes k_{n-1}} \otimes (\tfrac{1}{2}^n)^{\otimes k_n} \tag{5.25}$$

is $p_1\lambda_1 + \cdots + p_n\lambda_n$ with multiplicity one, where

$$p_j = \tfrac{1}{2}k_n + \sum_{h=j}^{n-1} k_j, \qquad 1 \leq j \leq n-1, \quad p_n = \tfrac{1}{2}k_n. \tag{5.26}$$

Here the p_j are integral or half-integral depending on whether k_n is even or odd. We can obtain any highest weight (5.20) by choosing the integers k_j such that $k_n = 2p_n, k_{n-1} = p_{n-1} - p_n, \ldots, k_2 = p_2 - p_3, k_1 = p_1 - p_2$. Thus we can find an irred rep of $o(2n + 1)$ for each set of integers or half-integers p_j satisfying (5.20).

In the next section we will show that $o(2n)$ has irred spinor reps with signatures $(\frac{1}{2}^n)$ and $(\frac{1}{2}^{n-1}, -\frac{1}{2})$. The highest weight of the rep

$$(1)^{\otimes k_1} \otimes (1^2)^{\otimes k_2} \otimes \cdots \otimes (1^{n-2})^{\otimes k_{n-2}} \otimes (\tfrac{1}{2}^n)^{\otimes k_{n-1}} \otimes (\tfrac{1}{2}^{n-1}, -\tfrac{1}{2})^{\otimes k_n} \tag{5.27}$$

is $p_1\lambda_1 + \cdots + p_n\lambda_n$ with multiplicity one, where

$$\begin{gathered} p_j = k_j + k_{j+1} + \cdots + k_{n-2} + \tfrac{1}{2}(k_{n-1} + k_n), \\ l = 1, 2, \ldots, n-2, \\ p_{n-1} = \tfrac{1}{2}(k_{n-1} + k_n), \qquad p_n = \tfrac{1}{2}(k_{n-1} - k_n). \end{gathered} \tag{5.28}$$

The p_j are all integral or half-integral depending on whether $k_{n-1} + k_n$ is even or odd. We can obtain any highest weight (5.20) with the choice

$$\begin{gathered} k_n = p_{n-1} - p_n, \quad k_{n-1} = p_{n-1} + p_n, \quad k_j = p_j - p_{j+1}, \\ j = 1, \ldots, n-2. \end{gathered} \tag{5.29}$$

The group $SO(m, R) = SO(m, \mathfrak{C}) \cap U(m)$ is a compact subgroup of $SO(m, \mathfrak{C})$. In the realization (5.1) of $O(m, \mathfrak{C})$ this is the group of real orthogonal matrices with determinant $+1$. In the realization (5.7) or (5.8) the matrices of $SO(m, R)$ are unitary equivalent to real orthogonal matrices with determinant $+1$ where the unitary equivalence is determined by (5.3). In the following we use the second realization of $SO(m, R)$. It is easy to verify that the real Lie algebra $o(m, R)$ is a real form of $o(m, \mathfrak{C})$.

Theorem 9.7. If $A \in SO(m, R)$ then there exists $U \in SO(m, R)$ such that

$$U^{-1}AU = D(\epsilon) = \begin{pmatrix} 1 & 0 & \cdots & & & & 0 \\ 0 & \epsilon_1 & & & & & \\ 0 & & \ddots & & & Z & \\ \vdots & & & \epsilon_n & & & \\ \vdots & & & & \bar{\epsilon}_1 & & \\ \vdots & & Z & & & \ddots & \\ 0 & & & & & & \bar{\epsilon}_n \end{pmatrix} \in SO(m, R), \quad |\epsilon_j| = 1.$$

The first row and column occur only for $m = 2n + 1$.

The proof of this theorem is analogous to that of Lemma 9.9 and theorem 9.5, so we omit it. Since $A = UD(\epsilon)U^{-1}$ it is clear, by the same argument as used for $USp(m)$, that A can be connected to E_m by an analytic curve in $SO(m, R)$. Thus, $SO(m, R)$ is connected.

To show that $SO(m, \mathfrak{C})$ is connected we mimic the corresponding proof for $Sp(m)$, leaving the details to the reader. If $A \in SO(m, \mathfrak{C})$ then by the polar decomposition, $A = HU$, Where H is Hermitian and positive-definite, and U is unitary. Now H is the unique positive-definite solution of $H^2 = AA^*$ $\in SO(m, \mathfrak{C})$. Since H^2 is Hermitian it can be diagonalized within $SO(m, \mathfrak{C})$:

$$H^2 = V \begin{pmatrix} 1 & 0 & \cdots & & & & 0 \\ 0 & \epsilon_1 & & & & & \\ 0 & & \ddots & & & Z & \\ \vdots & & & \epsilon_n & & & \\ \vdots & & & & \epsilon_1^{-1} & & \\ \vdots & & Z & & & \ddots & \\ 0 & & & & & & \epsilon_n^{-1} \end{pmatrix} V^{-1} = VC(\epsilon)V^{-1}, \qquad V \in SO(m, \mathfrak{C}).$$

The eigenvalues ϵ_j are positive, and the first row and column occur only if $m = 2n + 1$. It follows that H is given by

(5.30) $$H = VC(\epsilon^{1/2})V^{-1} \in SO(m, \mathfrak{C}).$$

Thus, $U = H^{-1}A \in SO(m, \mathfrak{C}) \cap U(m) = SO(m, R)$. Since $SO(m, R)$ is connected and H is given by (5.30) we see that $SO(m, \mathfrak{C})$ is connected. Thus any group rep of $SO(m, \mathfrak{C})$ or $SO(m, R)$ is uniquely determined by its induced Lie algebra rep of $o(m)$. Furthermore, since $SO(m, R)$ is compact and $o(m, R)$ is a real form of $o(m)$ every finite-dimensional analytic rep of $SO(m, \mathfrak{C})$ [and of $O(m, \mathfrak{C})$] can be decomposed into a direct sum of irred reps.

We have seen earlier that some of the irred reps $(p_1, \ldots, p_n)$ of $SO(m, \mathfrak{C})$

and $o(m)$ can be obtained from tensor reps. We clearly cannot obtain the (double-valued) half-integral reps in this way, but we can obtain all of the integral reps. In the case $m = 2n + 1$ the rep $[1^n]$ has highest weight $\lambda_1 + \cdots + \lambda_n$ [see (5.24)]. It is easy to verify that the rep $(p_1, \ldots, p_n)$, integral p_j, is contained in the tensor rep

$$(5.31) \qquad (1)^{\otimes k_1} \otimes (1^2)^{\otimes k_2} \otimes \cdots \otimes (1^{n-1})^{\otimes k_{n-1}} \otimes (1^n)^{\otimes k_n}$$

exactly once, where $k_n = p_n, k_{n-1} = p_{n-1} - p_n, \ldots, k_1 = p_1 - p_2$. A similar argument shows $(p_1, \ldots, p_n)$ is contained in the symmetry class $[p_1, \ldots, p_n]$ exactly once.

For $m = 2n$ the highest weight of $[1^n]$ is still $\Lambda = \lambda_1 + \cdots + \lambda_n$. From (5.23), $\Lambda' = \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} - \lambda_n$ is also a weight of $[1^n]$ and occurs with multiplicity one. Furthermore, if α is a positive root $\lambda_j + \lambda_k$ or $\lambda_j - \lambda_k$, $1 \le j < k \le n$, then $\Lambda' + \alpha$ is not a weight of $[1^n]$. (This is false for $m = 2n + 1$.) Thus Λ' must be the highest weight of the irred rep $(1^{n-1}, -1)$ contained in $[1^n]$ with multiplicity one. This shows that (1^n) and $(1^{n-1}, -1)$ are tensor reps. Finally the integral rep $(p_1, \ldots, p_n)$ is contained in the tensor rep

$$(5.32) \quad (1)^{\otimes k_1} \otimes (1^2)^{\otimes k_2} \otimes \cdots \otimes (1^{n-1})^{\otimes k_{n-1}} \otimes (1^{n-1}, -1)^{\otimes k_n} \otimes (1^n)^{\otimes k_{n+1}}$$

with multiplicity one, where $k_j = p_j - p_{j+1}$, $1 \le j \le n - 2$, and

$$(5.33) \quad \begin{array}{llll} k_{n-1} = p_{n-1} - p_n, & k_n = 0, & k_{n+1} = p_n & \text{if } \ p_n \ge 0 \\ k_{n-1} = p_{n-1} + p_n, & k_n = -p_n, & k_{n+1} = 0 & \text{if } \ p_n < 0. \end{array}$$

By counting dimensions one can show

$$(5.34) \quad \begin{aligned} &[1^l] \cong (1^l), \quad 1 \le l \le n - 1, \\ &[1^n] \cong \begin{cases} (1^n) & \text{if } \ m = 2n + 1 \\ (1^n) \oplus (1^{n-1}, -1) & \text{if } \ m = 2n. \end{cases} \end{aligned}$$

We have now determined all single-valued irred reps of $SO(m, \mathfrak{C})$. For m odd it is easy to extend these results to compute the irred reps of $O(m, \mathfrak{C})$. Indeed, if $m = 2n + 1$ then $-E_m \in O(m, \mathfrak{C})$ and $\det(-E_m) = -1$. Thus, $O(m, \mathfrak{C})$ has the coset decomposition $\{SO(m, \mathfrak{C}), -E_m \cdot SO(m, \mathfrak{C})\}$.

Let $\mathbf{T}$ be an irred rep of $O(m, \mathfrak{C})$. Since $-E_m$ commutes with all elements of $O(m, \mathfrak{C})$ it follows from the Schur lemmas that $\mathbf{T}(-E_m) = \lambda \mathbf{E}$. The property $(-E_m)^2 = E_m$ implies $\lambda = \pm 1$. Thus $\mathbf{T} \mid SO(m, \mathfrak{C})$ must still be irred. The irred reps of $O(2n + 1, \mathfrak{C})$ are

$$(5.35) \quad \begin{array}{ll} (p_1, \ldots, p_n)^+, & \mathbf{T}(-E_m) = \mathbf{E}; \\ (p_1, \ldots, p_n)^-, & \mathbf{T}(-E_m) = -\mathbf{E}. \end{array}$$

For reps of $O(2n, \mathfrak{C})$ the situation is somewhat more complicated. In this case $\det(-E_m) = +1$, so $-E_m$ belongs to the connected component of the identity. We choose the element

$$(5.36)\qquad S = \begin{pmatrix} & & & 0 & & & 0 \\ & & & \cdot & & & \cdot \\ & E_{n-1} & & \cdot & & Z_{n-1} & \cdot \\ & & & \cdot & & & \cdot \\ 0 & \cdot & \cdot & 0 & \cdot & \cdot & 1 \\ & & & \cdot & & & \cdot \\ & Z_{n-1} & & \cdot & & E_{n-1} & \cdot \\ & & & \cdot & & & \cdot \\ 0 & \cdot & \cdot & 1 & \cdot & \cdot & 0 \end{pmatrix}, \qquad \det S = -1, \quad S^2 = E_m.$$

Clearly, S and $SO(2n, \mathfrak{C})$ generate $O(2n, \mathfrak{C})$. Let $\mathbf{T}$ be an irred rep of $O(2n, \mathfrak{C})$ and set $\mathbf{S} = \mathbf{T}(S)$. Then $\mathbf{T}' = \mathbf{T} \mid SO(2n, \mathfrak{C})$ decomposes into a direct sum of irred reps. Let W be a subspace of the rep space V transforming irreducibly under $\mathbf{T}'$. Then the subspace $\mathbf{S}W$ also transforms irreducibly under $\mathbf{T}'$. Indeed if $\mathbf{w}^* = \mathbf{S}\mathbf{w} \in \mathbf{S}W$ and $A \in SO(2n, \mathfrak{C})$ then $S^{-1}AS \in SO(2n, \mathfrak{C})$ [since $\det(S^{-1}AS) = +1$] and

$$\mathbf{T}(A)\mathbf{w}^* = \mathbf{S}\mathbf{S}^{-1}\mathbf{T}(A)\mathbf{S}\mathbf{w} = \mathbf{S}(\mathbf{T}(S^{-1}AS)\mathbf{w}) \in \mathbf{S}W,$$

so $\mathbf{S}W$ is invariant under $\mathbf{T}'$. If W' is a nonzero invariant subspace of $\mathbf{S}W$ then $\mathbf{S}W'$ is a nonzero invariant subspace of $\mathbf{S}(\mathbf{S}W) = W$. Since W is irred, $\mathbf{S}W' = W$, so $W' = \mathbf{S}W$ and $\mathbf{S}W$ is irred.

The space $W + \mathbf{S}W$ is invariant under $\mathbf{T}$ and nonzero. Thus $W + \mathbf{S}W$ is the entire rep space V.

From (5.12) and (5.36) we have

$$\begin{aligned} &\mathfrak{K}(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n)S = S\mathfrak{K}(\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n), \qquad E_\alpha S = SE_{\tilde{\alpha}} \\ (5.37)\qquad &\alpha = p_1\lambda_1 + \cdots + p_{n-1}\lambda_{n-1} + p_n\lambda_n, \\ &\tilde{\alpha} = p_1\lambda_1 + \cdots + p_{n-1}\lambda_{n-1} - p_n\lambda_n. \end{aligned}$$

If $\mathbf{w}$ is a weight vector in W with weight $\Lambda = p_1\lambda_1 + \cdots + p_n\lambda_n$ then $\mathbf{S}\mathbf{w}$ is a weight vector in $\mathbf{S}W$ with weight $\Lambda' = p_1\lambda_1 + \cdots + p_{n-1}\lambda_{n-1} - p_n\lambda_n$. It follows that the weights of W and $\mathbf{S}W$ are related by a change of sign in λ_n. Let $p_1\lambda_1 + \cdots + p_{n-1}\lambda_{n-1} + p_n\lambda_n$ be the highest weight of W, $p_1 \geq \cdots \geq p_{n-1} \geq |p_n|$. It is straightforward to show that $p_1\lambda_1 + \cdots + p_{n-1}\lambda_{n-1} - p_n\lambda_n$ is the highest weight of $\mathbf{S}W$. If $p_n \neq 0$ then W and $\mathbf{S}W$ are linearly independent and

$$(5.38)\qquad \mathbf{T} \cong (p_1, \ldots, p_n) \oplus (p_1, \ldots, p_{n-1}, -p_n).$$

Note that W and $\mathbf{S}W$ have the same dimension. If $p_n = 0$ we define $\mathbf{h}^+ = \mathbf{w} + \mathbf{S}\mathbf{w}$, $\mathbf{h}^- = \mathbf{w} - \mathbf{S}\mathbf{w}$, where $\mathbf{w}$ is the highest weight vector of W. Note that $\mathbf{S}\mathbf{w}$ is the highest weight vector of $\mathbf{S}W$ and $\mathbf{w}$, $\mathbf{S}\mathbf{w}$ correspond to the same weight $\Lambda^* = p_1\lambda_1 + \cdots + p_{n-1}\lambda_{n-1}$. Now $\mathbf{S}\mathbf{h}^\pm = \pm\mathbf{h}^\pm$. It follows from (5.37) and (1.14) that the spaces $H^\pm$ spanned by all vectors of the form $E_{\alpha_1}E_{\alpha_2}\cdots$

$E_{\alpha_k}\mathbf{h}^{\pm}$ are invariant under $\mathbf{T}$. Since $\mathbf{T}$ is irred, either $H^+ = V$, $H^- = \{\theta\}$ or $H^+ = \{\theta\}$, $H^- = V$. In the first case we denote the rep by

(5.39)

$$(p_1, \ldots, p_{n-1}, 0)^+, \qquad \mathbf{S}\mathbf{w} = \mathbf{w}, \qquad \mathbf{T} \,|\, SO(2n, \mathfrak{C}) \cong (p_1, \ldots, p_{n-1}, 0)$$

and in the second case by

(5.40)

$$(p_1, \ldots, p_{n-1}, 0)^-, \qquad \mathbf{S}\mathbf{w} = -\mathbf{w}, \qquad \mathbf{T} \,|\, SO(2n, \mathfrak{C}) \cong (p_1, \ldots, p_{n-1}, 0).$$

This completes our catalog of irred reps of $O(2n, \mathfrak{C})$.

We have mentioned earlier that the Lie algebra of the compact group $SO(m, R)$ is a real form of the complex Lie algebra $o(m)$. Thus there is a 1–1 relationship between reps of $SO(m, \mathfrak{C})$ and $SO(m, R)$. Using the same techniques as in Section 9.2 we construct the characters for the irred reps $(p_1, \ldots, p_n)$ of $SO(m, R)$. We assume that the p_j are integral, although the results are virtually unchanged for half-integral p_j. Most results will be given without their straight-forward proofs.

From Theorem 9.7 a character χ of $SO(2n + 1, R)$ can be considered as a function $\chi(\varphi_1, \ldots, \varphi_n)$, $\epsilon_j = e^{i\varphi_j}$, $-\pi \leq \varphi_j < \pi$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is a weight basis for the corresponding rep and the weights are $\Lambda_1, \ldots, \Lambda_s$ then

$$(5.41) \qquad \chi = \operatorname{tr}[e^{iH(\varphi_1, \ldots, \varphi_n)}] = \sum_{j=1}^{s} e^{i\Lambda_j(\varphi)} = \sum_{q_1 \cdots q_n} c_{q_1 \cdots q_n} \epsilon_1^{q_1} \cdots \epsilon_n^{q_n},$$

where $\Lambda_j = q_1\varphi_1 + \cdots + q_n\varphi_n$ and $c_{q_1 \cdots q_n}$ is the multiplicity of this weight. Since χ is defined on conjugacy classes it follows from Theorem 9.7 that we can permute the φ_j or replace any subset of the φ_j by $-\varphi_j$ without changing the value of the character. Thus the character admits a symmetry group of order $2^n n!$. This is just the Weyl group. From (5.41) we see that the weights Λ and $S^\alpha\Lambda$ must have the same multiplicity for each root α.

The inner product on $L_2(O(2n + 1, R))$ for functions constant on conjugacy classes is

$$(5.42) \quad (f, g) = V^{-1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(\varphi_1, \ldots, \varphi_n) \bar{g}(\varphi_1, \ldots, \varphi_n)\, \Delta\bar{\Delta}\, d\varphi_1 \cdots d\varphi_n$$

where

$$V = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \Delta\bar{\Delta}\, d\varphi_1 \cdots d\varphi_n$$

and

$$(5.43) \quad \Delta(\varphi_1, \ldots, \varphi_n) = \prod_{j=1}^{n} (\epsilon_j^{1/2} - \epsilon_j^{-1/2}) \prod_{1 \leq j < k \leq n} (\epsilon_j + \epsilon_j^{-1} - \epsilon_k - \epsilon_k^{-1}).$$

Here Δ is skew-symmetric in its arguments and double-valued (due to the occurrence of $\epsilon_j^{1/2}$). Thus if χ is the character of $(p_1, \ldots, p_n)$, the function $\xi = \chi\Delta$ is skew-symmetric and double-valued. Furthermore, the substitution

$\varphi_j \longrightarrow -\varphi_j$ for a single j causes ξ to change sign. Note that ξ is a finite sum of the form (2.17) whose highest-order term is

$$1 \cdot \epsilon_1^{p_1+n-(1/2)} \epsilon_2^{p_2+n-(3/2)} \cdots \epsilon_n^{p_n+(1/2)}. \tag{5.44}$$

Because of the symmetry properties, since the sum for ξ contains (5.44) it must also contain

$$\xi(l_1, \ldots, l_n) = |\epsilon^{l_1} - \epsilon^{-l_1}, \ldots, \epsilon^{l_n} - \epsilon^{-l_n}| \tag{5.45}$$

where $|\cdot|$ is defined similarly to (2.19) and $l_j = p_j + n - j + \frac{1}{2}$. Here, $\Delta(\epsilon_1, \ldots, \epsilon_n) = \xi(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2})$.

Continuing in this fashion we obtain the expansion $\xi = \xi(l_1, \ldots, l_n) + c'\xi(l_1', \ldots, l_n') + \cdots$. However, the requirement $(\chi, \chi) = 1$ implies $1 = 1 + |c'|^2 + \cdots$, so $c' = \cdots = 0$ and $\xi = \xi(l_1, \ldots, l_n)$.

Theorem 9.8. The character of the rep $(p_1, \ldots, p_n)$ of $SO(2n+1, R)$ is

$$\chi^{p_1 \cdots p_n}(\epsilon_1, \ldots, \epsilon_n) = \frac{|\epsilon^{l_1} - \epsilon^{-l_1}, \ldots, \epsilon^{l_n} - \epsilon^{-l_n}|}{|\epsilon^{l_1^0} - \epsilon^{-l_1^0}, \ldots, \epsilon^{l_n^0} - \epsilon^{-l_n^0}|},$$

where $l_j = p_j + n - j + \frac{1}{2}$, $l_j^0 = n - j + \frac{1}{2}$. The dimension of this rep is

$$N(p_1, \ldots, p_n) = Q(l_1, \ldots, l_n)/Q(l_1^0, \ldots, l_n^0),$$

where

$$Q(l_1, \ldots, l_n) = \prod_{1 \le j < k \le n} (l_j^2 - l_k^2) \prod_{j=1}^{n} l_j.$$

If $m = 2n$ then (5.42) holds with

$$\Delta = \prod_{1 \le j < k \le n} (\epsilon_j + \epsilon_j^{-1} - \epsilon_k - \epsilon_k^{-1}). \tag{5.46}$$

Here Δ is skew-symmetric and single-valued. If χ is the character of $(p_1, \ldots, p_n)$ then $\xi = \chi\Delta$ is skew-symmetric and invariant under an *even* number of sign changes $\varphi_j \longrightarrow -\varphi_j$. Indeed from Theorem 9.7 it follows that we can perform an arbitrary permutation of $\varphi_1, \ldots, \varphi_n$ in $D(e^{i\varphi})$ and still get a diagonal matrix in the same conjugacy class of $SO(2n, R)$. Furthermore, if we make the replacement $\varphi_j \longrightarrow -\varphi_j$ for an even number of angles we stay in the same conjugacy class. However, as the reader can check, the diagonal matrix obtained from an odd number of replacements $\varphi_j \longrightarrow -\varphi_j$ is not conjugate to $D(e^{i\varphi})$ in $SO(2n, R)$, although it is conjugate in $O(2n, R)$. We see that Lemma 9.7 holds also for $o(2n)$ since from (5.19) the above symmetries generate the Weyl group of order $2^{n-1}n!$.

The term of highest weight belonging to ξ is

$$1 \cdot \epsilon_1^{p_1+n-1} \epsilon_2^{p_2+n-2} \cdots \epsilon_n^{p_n}.$$

The symmetry requirements and the condition $(\chi, \chi) = 1$ yield

$$2\xi = |\epsilon^{l_1} + \epsilon^{-l_1}, \ldots, \epsilon^{l_n} + \epsilon^{-l_n}| + |\epsilon^{l_1} - \epsilon^{-l_1}, \ldots, \epsilon^{l_n} - \epsilon^{-l_n}|.$$

Theorem 9.9. The character of the rep $(p_1, \ldots, p_n)$ of $SO(2n, R)$ is

$$\chi^{p_1\cdots p_n}(\epsilon_1, \ldots, \epsilon_n) = \frac{|\epsilon^{l_1} + \epsilon^{-l_1}, \ldots, \epsilon^{l_n} + \epsilon^{-l_n}| + |\epsilon^{l_1} - \epsilon^{-l_1}, \ldots, \epsilon^{l_n} - \epsilon^{-l_n}|}{|\epsilon^{l_1^0} + \epsilon^{-l_1^0}, \ldots, \epsilon^{l_n^0} + \epsilon^{-l_n^0}|}$$

where $l_j = p_j + n - j, l_j^0 = n - j$. The dimension is

$$N(p_1, \ldots, p_n) = \frac{R(l_1, \ldots, l_n)}{R(l_1^0, \ldots, l_n^0)}, \qquad R(l_1, \ldots, l_n) = \prod_{1 \leq j < k \leq n} (l_j^2 - l_k^2).$$

Note that the reps $(p_1, \ldots, p_{n-1}, \pm p_n)$ have the same dimension. This is also a consequence of the remark following (5.38).

For more detailed proofs of the above theorems see the work of Boerner [1] or Weyl [3].

9.6 Dirac Matrices and the Spin Representations of the Orthogonal Groups

In an attempt to formulate a relativistic theory describing electrons Dirac considered systems of equations of the form

$$\left(L_1\frac{\partial}{\partial x} + L_2\frac{\partial}{\partial y} + L_3\frac{\partial}{\partial z} + L_4\frac{\partial}{\partial t}\right)\Psi = \kappa\Psi, \tag{6.1}$$

where the L_j are square matrices, κ is a constant, and Ψ is a spinor field. He required that this system be compatible with the Klein–Gordon equation (5.37), Section 8.5. Using the same notation as in Section 8.5, we see from (6.1) that

$$\left(\sum_{j=1}^{4} L_j\frac{\partial}{\partial x_j}\right)^2\Psi = \kappa^2\Psi. \tag{6.2}$$

If we choose $\kappa = m_0$, the mass of the electron, then (6.2) becomes

$$\sum_{j,k=1}^{4} L_jL_k\frac{\partial^2\Psi}{\partial x_j\,\partial x_k} = m_0^{\,2}\Psi. \tag{6.3}$$

This is equivalent to the Klein–Gordon equation

$$\left(\frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}} + \frac{\partial^2}{\partial x_3^{\,2}} - \frac{\partial^2}{\partial x_4^{\,2}}\right)\Psi = m_0^{\,2}\Psi$$

provided

$$L_jL_k + L_kL_j = 2G_{jk}, \qquad 1 \leq j, k \leq 4, \tag{6.4}$$

where G is given by (1.1), Section 8.1. We have already seen that the 4×4 matrices (5.33) Section 8.5, are solutions of (6.4). There are many other solutions and we shall compute all of them.

For convenience we modify (6.4) by defining matrices α_j such that

$$\alpha_j = L_j, \quad j = 1, 2, 3, \qquad \alpha_4 = iL_4, \qquad i = \sqrt{-1}.$$

Then the relations become

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \tag{6.5}$$

i.e.,

$$\alpha_j^2 = 1, \quad \alpha_j\alpha_k = -\alpha_k\alpha_j \quad \text{if} \quad j \neq k. \tag{6.6}$$

Finally we generalize our problem by allowing the indices j and k to run from 1 to m. The associative algebra generated by $\alpha_1, \ldots, \alpha_m$ is called the **Clifford algebra** C_m. We shall not study C_m but rather the multiplicative group G_m generated by ± 1 and $\alpha_1, \ldots, \alpha_m$. A general element of G_m has the form

$$\pm\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_q}. \tag{6.7}$$

Making use of (6.6), we can always write this element in the standard form

$$\pm\alpha_{l_1}\alpha_{l_2}\cdots\alpha_{l_s}, \qquad l_1 < l_2 < \cdots < l_s, \quad s \leq m. \tag{6.8}$$

Indeed the relation $\alpha_j\alpha_k = -\alpha_k\alpha_j$, $k \neq j$, permits a reordering of the factors of (6.7) in the normal form (6.8). If two factors in (6.7) are equal one simply reorders the terms such that the two factors are adjacent and then uses $\alpha_j^2 = 1$ to reduce the length of the group element by two factors. Note that no element in standard form can have more than m factors since then two factors would have to be equal. Furthermore, the only elements with m factors in standard form are $\pm\alpha_1\alpha_2\cdots\alpha_m$. Thus the distinct elements in G_m are

$$1, \quad \alpha_{j_1}, \quad \alpha_{j_1}\alpha_{j_2}, \ldots, \quad \alpha_1\alpha_2\cdots\alpha_m; \qquad 1 \leq j_1 < j_2 < \cdots < j_{m-1}. \tag{6.9}$$

and their negatives. The order of the group is

$$2\left[\binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m-1} + \binom{m}{m}\right] = 2(1+1)^m = 2^{m+1}.$$

It is obvious that every matrix rep of (6.5) determines a matrix rep of G_m. Furthermore every matrix rep T of G_m such that $T(1) = E$ and $T(-1) = -E$, where E is the identity matrix, determines a solution of (6.5). Thus our problem reduces to the determination of the irred reps $\mathbf{T}$ of G_m such that $\mathbf{T}(\pm 1) = \pm\mathbf{E}$. Every rep with this property will be a direct sum of such irred reps.

The number of irred reps is equal to the number of conjugacy classes in G_m. It is straightforward to check that for m even, the classes $\{+1\}$, $\{-1\}$ contain one element each, while the remaining classes are of the form $\{\pm\alpha_{j_1}\cdots\alpha_{j_s}\}$ and contain two elements each. Thus the total number of conjugacy classes is $[(2^{m+1} - 2)/2] + 2 = 2^m + 1$. For odd m the results are the same except that the two elements with m factors each determine a conjugacy class with one element:

$$\{\alpha_1\alpha_2\cdots\alpha_m\}, \quad \{-\alpha_1\alpha_2\cdots\alpha_m\}.$$

Thus the total number of conjugacy classes is $2^m + 2$.

The elements ± 1 form a normal subgroup of G_m. Furthermore the factor group $G_m' \cong G_m/\{\pm 1\}$ of order 2^m is abelian since $\alpha_j\alpha_k = -\alpha_k\alpha_j$, and ± 1 correspond to the identity element in G_m'. Thus G_m' has 2^m irred reps $\mathbf{T}_j'$ all one-dimensional. The composed mappings

$$G_m \longrightarrow G_m' \xrightarrow{T_j'} \mathfrak{C}$$

determine 2^m equivalence classes of one-dimensional irred reps of G_m.

Consider the case where $m = 2n$ is even. Here G_m has $2^m + 1$ irred reps of which 2^m are one-dimensional. Since the sum of the squares of the dimensions of the irred reps equals $2^{m+1} = N_G$ we have

$$2^{2n} + q^2 = 2^{2n+1} \qquad \text{or} \qquad q = 2^n = 2^{m/2},$$

where q is the dimension of the remaining irred rep $\mathbf{T}$. The one-dimensional reps map ± 1 to the identity operator, so they are not acceptable as solutions of (6.5).

We will construct an explicit matrix realization of $\mathbf{T}$. Suppose first that $m = 2$, so $\mathbf{T}$ is two-dimensional. The matrices

$$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

satisfy the relations $\alpha_j^2 = 1$, $\alpha_1\alpha_2 = -\alpha_2\alpha_1 \neq 0$. Thus, these matrices necessarily determine a two-dimensional irred rep of G_2 equivalent to $\mathbf{T}$.

In the general case for $m = 2n$ we form the $2^n \times 2^n$ matrices

(6.10)

$$\alpha_j = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{j-1} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{n-j}$$

$$\alpha_{n+j} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{j-1} \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{n-j}$$

$$1 \leq j \leq n.$$

It is straightforward to check that these matrices satisfy relations (6.5). Since the matrices do not commute they cannot be obtained as a direct sum of one-dimensional reps of G_m. Thus, (6.10) defines a matrix rep equivalent to $\mathbf{T}$. It is easy to verify that $\mathbf{T}$ is a 1–1 rep of G_m.

Remark. Recall that if A, B are $m \times m$, $n \times n$ matrices, respectively, then $C = A \otimes B$ is the $mn \times mn$ matrix with matrix elements

$$C_{jk,ls} = A_{jl}B_{ks}.$$

It is easy to verify the relations

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$$

for $m \times m$ matrices A_j and $n \times n$ matrices B_j. Furthermore,

$$c(A \otimes B) = (cA) \otimes B = A \otimes (cB).$$

for any scalar c. These relations are easily extended to $C_1 \otimes \cdots \otimes C_k$.

Using the basic theorems on the rep theory of finite groups we can say more about the matrix realizations of (6.5) for even m. First, every irred matrix realization has dimension $2^{m/2}$ and any two such realizations α_j, α_j' are equivalent, i.e., there exists a nonsingular matrix S such that $\alpha_j' = S^{-1}\alpha_j S$, $1 \leq j \leq m$. In particular, every realization is equivalent to (6.10). Furthermore, if the α_j form a realization and S is nonsingular then the $S^{-1}\alpha_j S$ form an equivalent realization. All reducible realizations are direct sums of copies of the single irred rep.

Taking $m = 4$ we obtain all solutions of (6.4) which lead to the Dirac equation. The only irred realizations are in terms of 4×4 matrices and all such realizations are equivalent. All reducible realizations are given by $4k \times 4k$ matrices, where $k \geq 2$.

The case $m = 2n + 1$ is a little more complicated. There are $2^{2n+1} + 2$ equivalence classes of irred reps. Furthermore, since $G_m/\{1, -1\}$ is abelian of order 2^{2n+1} there are 2^{2n+1} one-dimensional reps. This leaves two irred reps of dimensions q_1 and q_2. Since the sum of the squares of the dimensions equals 2^{2n+2} we have

$$2^{2n+1} + q_1^2 + q_2^2 = 2^{2n+2} \qquad \text{or} \qquad q_1^2 + q_2^2 = 2^{2n+1}.$$

A solution of this diophantine equation is $q_1 = q_2 = 2^n$. Indeed, we can exhibit two nonequivalent irred reps of dimension 2^n. For $n = 1$ the matrices

$$(6.11) \qquad \alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy (6.5). (These are the **Pauli spin matrices.**) One can use the Schur lemmas to check the irreducibility. A second irred rep is $\alpha_j' = -\alpha_j$, $j = 1, 2, 3$, where the α_j are defined by (6.11). These reps cannot be equivalent because $i\alpha_1\alpha_2\alpha_3 = E_2 = 1$ for (6.11), while $i\alpha_1'\alpha_2'\alpha_3' = -E_2 = -1$. This also shows that neither of these reps is faithful (1–1), although their direct sum is faithful.

For general $m = 2n + 1$ the $2^n \times 2^n$ matrices α_j, α_{n+j}, $1 \leq j \leq n$, defined by (6.10) and the matrix

$$(6.12) \qquad \alpha_m = \alpha_{2n+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy relations (6.5) and, by Schur's lemmas, define an irred rep of G_{2n+1}.

Furthermore, the matrices $\alpha_k' = -\alpha_k$, $1 \le k \le m$, where the α_k are given by (6.10) and (6.12), also define an irred rep of G_{2n+1}. These reps are not equivalent because $i^n\alpha_1 \cdots \alpha_n = E_{2^n}$ for the first representation, while $i^n\alpha_1' \cdots \alpha_n' = -E_{2^n}$ for the second. Neither rep is faithful, but their direct sum is faithful. These results suffice to describe all realizations of relations (6.5) by matrices.

We have related the Clifford algebra C_m to the reps of a finite group G_m. We shall now show that C_m is also related to the reps of a Lie group. Suppose the quantities α_j satisfy the relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \qquad 1 \le j, k \le m.$$

We define elements $\alpha_{jk}, j \ne k, 1 \le j, k \le m+1$, by

$$\alpha_{jk} = \begin{cases} 2\alpha_j\alpha_k & \text{if } j \ne k, \quad 1 \le j, k \le m, \\ \alpha_k & \text{if } j = m+1, \quad 1 \le k \le m, \\ -\alpha_j & \text{if } k = m+1, \quad 1 \le j \le m. \end{cases} \tag{6.13}$$

Furthermore, we set $\alpha_{jj} = 0$. It follows that $\alpha_{jk} = -\alpha_{kj}$ and there are $m(m+1)/2$ independent quantities α_{jk}, $j < k$. The $\{\alpha_{jk}\}$ form a Lie algebra under the commutator bracket

$$[\alpha_{jk}, \alpha_{hl}] = \alpha_{jk}\alpha_{hl} - \alpha_{hl}\alpha_{jk}.$$

Indeed a straightforward computation yields

$$\begin{gathered} [\alpha_{m+1,k}, \alpha_{m+1,l}] = \alpha_{kl}, \qquad [\alpha_{m+1,k}, \alpha_{hl}] = 4\delta_{kh}\alpha_{m+1,l} - 4\delta_{kl}\alpha_{m+1,h}, \\ [\alpha_{jk}, \alpha_{hl}] = 4(\delta_{kh}\alpha_{jl} + \delta_{jl}\alpha_{kh} - \delta_{kl}\alpha_{jh} - \delta_{jh}\alpha_{kl}), \\ 1 \le j, k, h, l \le m. \end{gathered} \tag{6.14}$$

Relations (6.14) are the commutation relations of an $m(m+1)/2$-dimensional Lie algebra.

Let us compare these results with the commutation relations of the Lie algebra $so(m+1)$. It will be convenient to consider $so(m+1)$ as the space of all $(m+1) \times (m+1)$ skew-symmetric matrices. Then a basis for $so(m+1)$ is provided by the matrices $\mathcal{a}_{jk} = \mathcal{E}_{jk} - \mathcal{E}_{kj}$, $1 \le j < k \le m+1$. Taking account of the rules $\mathcal{a}_{jk} = -\mathcal{a}_{kj}$ for all j, k we easily derive the commutation relations

(6.15)

$$[\mathcal{a}_{jk}, \mathcal{a}_{hl}] = \delta_{kh}\mathcal{a}_{jl} + \delta_{jl}\mathcal{a}_{kh} - \delta_{kl}\mathcal{a}_{jh} - \delta_{jk}\mathcal{a}_{kl}, \qquad 1 \le j, k, h, l \le m+1.$$

Setting $\mathcal{a}_{jk} = \frac{1}{4}\alpha_{jk}$ and $\mathcal{a}_{j,m+1} = -\frac{1}{2}i\alpha_{j,m+1} = \frac{1}{2}i\alpha_j$ we see that relations (6.14) and (6.15) coincide. Thus, the $\{\alpha_{jk}\}$ span a Lie algebra isomorphic to $so(m+1)$, so any matrix realization of (6.5) determines a rep of $so(m+1)$ via the relations (6.14). Furthermore, the nontrivial irred reps of G_m computed above determine irred reps of $so(m+1)$. We can determine which irred reps of $so(m+1)$ we have obtained by computing the highest weights.

The maximal abelian subalgebra $\mathscr{h}_s$ of $so(m+1)$ obtained from (5.12) by the change of basis (5.3) can be chosen to take the form

$$\text{(6.16)}\qquad \begin{pmatrix} & & i\lambda_1 & & 0 \\ & Z_s & & \ddots & \\ & & & & i\lambda_s \\ -i\lambda_1 & & & & \vdots \\ & \ddots & & Z_s & \\ & & -i\lambda_s & & \\ 0 & \cdots & \cdot & & 0 \end{pmatrix}, \qquad m+1 = 2s+1,$$

in the space of skew-symmetric matrices, where the last row and column are missing if $m+1 = 2s$. A basis for $\mathscr{h}_s$ is given by $\mathcal{H}_j = i\mathcal{A}_{j,s+j} = i\mathcal{E}_{j,s+j} - i\mathcal{E}_{s+j,j}$, $1 \le j \le s$.

Now consider the rep of G_{2n} or $so(2n+1)$ given by (6,10). The corresponding operators H_j are

$$H_j = \tfrac{1}{4} i\alpha_{j,n+j} = \tfrac{1}{2} i\alpha_j\alpha_{n+j}, \qquad 1 \le j \le n,$$

or

$$\text{(6.17)}\qquad H_j = \tfrac{1}{2}\underbrace{E_2 \otimes \cdots \otimes E_2}_{j-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \underbrace{E_2 \otimes \cdots \otimes E_2}_{n-j}.$$

The eigenvalues of the H_j are $\pm\frac{1}{2}$. In particular, a weight basis for the rep is given by

$$\text{(6.18)}\qquad e(k_1 \cdots k_n) = e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_n}, \qquad k_l = \pm\tfrac{1}{2},$$

where

$$e_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Clearly,

$$H_j e(k_1 \cdots k_j \cdots k_n) = k_j e(k_1 \cdots k_j \cdots k_n),$$

and $e(k_1 \cdots k_n)$ is a weight vector with weight $\sum k_j\lambda_j$ since

$$H(\lambda_1, \ldots, \lambda_n) = \sum_{j=1}^{n} \lambda_j H_j.$$

The highest weight is $\frac{1}{2}\sum \lambda_j$ so we have constructed a realization of the fundamental spin rep $(\frac{1}{2}^n)$ of $so(2n+1)$.

Next we consider the rep of $G_{2n-1} = G_{2(n-1)+1}$ or $so(2n)$ determined by (6.10) and (6.12). The operators H_j are

$$H_j = \tfrac{1}{4} i\alpha_{j,(n-1)+j+1}, \quad 1 \le j \le n-1, \qquad H_n = i\mathcal{A}_{n,2n} = -\tfrac{1}{2}\alpha_{(n-1)+1}$$

or

$$(6.19)\qquad \begin{aligned} H_j &= \tfrac{1}{2}\underbrace{E_2 \otimes \cdots \otimes E_2}_{j-1} \otimes S_2 \otimes S_2 \otimes \underbrace{E_2 \otimes \cdots \otimes E_2}_{n-j-2} \\ H_n &= \tfrac{1}{2}S_2 \otimes \underbrace{E_2 \otimes \cdots \otimes E_2}_{n-2}, \qquad S_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \end{aligned}$$

Since the matrix S_2 has eigenvalues ± 1 we can perform a unitary similarity transformation on the H_j to obtain

$$H_j = \tfrac{1}{2}E_2 \otimes \cdots \otimes E_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes E_2 \otimes \cdots \otimes E_2$$

$$H_n = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes E_2 \otimes \cdots \otimes E_2.$$

Again these matrices have eigenvalues $\pm\frac{1}{2}$ and the $e(k_1 \cdots k_{n-1})$, $k_l = \pm\frac{1}{2}$, form a weight basis. We have

$$(6.20)\qquad \begin{aligned} H_j e(k_1 \cdots k_{n-1}) &= 2k_j k_{j+1} e(k_1 \cdots k_{n-1}), \\ 1 \le j &\le n-1, \quad k_n = \tfrac{1}{2}, \\ H_n e(k_1 \cdots k_{n-1}) &= k_1 e(k_1 \cdots k_{n-1}). \end{aligned}$$

We get the highest weight vector for $k_1 = \cdots = k_{n-1} = \frac{1}{2}$. Thus the above construction yields a model of the spin rep $(\frac{1}{2}^n)$ of $so(2n)$.

A second model is obtained by making the replacements $\alpha_j \to -\alpha_j$, $1 \le j \le n-1$. This leaves the operators H_j, $1 \le j \le n-1$, unchanged but causes H_n to be replaced by $-H_n$. Again we get the highest weight vector for $k_1 = \cdots = k_{n-1} = \frac{1}{2}$, but this time the highest weight is $\frac{1}{2}(\lambda_1 + \cdots + \lambda_{n-1} - \lambda_n)$. Thus we have constructed a model of the spin rep $(\frac{1}{2}^{n-1}, -\frac{1}{2})$.

As we demonstrated in the preceding section, the spin reps of $so(m)$ do not extend to single-valued reps of $SO(m, \mathfrak{C})$. It is shown explicitly by Boerner [1] and Freudenthal and De Vries [1] that these reps exponentiate to single-valued reps of a compact Lie group Spin(m). Here, Spin(m) is locally isomorphic to $SO(m, \mathfrak{C})$ and there is a 2–1 analytic homomorphism of Spin(m) onto $SO(m, \mathfrak{C})$.

9.7 Examples and Applications

We present several examples showing how the classical groups appear in physical theories.

Consider a family of linear operators $\mathbf{a}_j, \mathbf{a}_j^*$, $1 \le j \le m$, on an inner product space V, satisfying the commutation relations

$$(7.1)\qquad [\mathbf{a}_j, \mathbf{a}_k] = [\mathbf{a}_j^*, \mathbf{a}_k^*] = \mathbf{0}, \qquad [\mathbf{a}_j, \mathbf{a}_k^*] = \delta_{jk}\mathbf{E},$$

where $\mathbf{E}$ is the identity operator. [Such operators appear in the method of second quantization in quantum mechanics where they are defined on a Hilbert space $\mathcal{H}$. There $\mathbf{a}_j^*$ is called a **creation operator** for bosons and its adjoint $\mathbf{a}_j$ is called an **annihilation operator.** These operators are closely related to the harmonic oscillator problem in quantum mechanics and will be treated in detail in Chapter 10. Here, we consider only the abstract commutation relations (7.1).]

Now V must be infinite-dimensional for (7.1) to hold. For, if A and B are matrices such that $[A, B] = \lambda E$ then $\mathrm{tr}([A, B]) = 0$ implies $\lambda = 0$. If V consists of analytic functions in m variables $z_1, \ldots, z_m$ then a realization of (7.1) is provided by the assignment

$$\mathbf{a}_j = \partial/\partial z_j, \qquad \mathbf{a}_j^* = z_j. \tag{7.2}$$

It follows directly from the commutation relations (7.1) that the operators $\mathbf{E}_{jk} = \mathbf{a}_j^*\mathbf{a}_k$ satisfy relations

$$[\mathbf{E}_{jk}, \mathbf{E}_{hl}] = \delta_{kh}\mathbf{E}_{jl} - \delta_{jl}\mathbf{E}_{hk}, \qquad 1 \leq j, k, h, l \leq m, \tag{7.3}$$

in agreement with (1.1). Thus, making the identification $\mathbf{E}_{jk} \leftrightarrow \mathcal{E}_{jk}$ one can easily construct reps of each of the classical Lie algebras in terms of annihilation and creation operators for bosons. Furthermore one can use the models to decompose V into subspaces transforming irreducibly under these reps.

For our next example we consider a family of operators $\mathbf{a}_j, \mathbf{a}_j^*, 1 \leq j \leq n$, on a finite-dimensional vector space V satisfying the **anticommutation** relations

$$[\mathbf{a}_j, \mathbf{a}_k]_+ = [\mathbf{a}_j^*, \mathbf{a}_k^*]_+ = \mathbf{0}, \qquad [\mathbf{a}_j, \mathbf{a}_k^*]_+ = \delta_{jk}\mathbf{E}, \qquad 1 \leq j, k \leq n, \tag{7.4}$$

where $[\mathbf{a}, \mathbf{b}]_+ = \mathbf{ab} + \mathbf{ba}$. Setting

$$\mathbf{a}_j = \tfrac{1}{2}(\alpha_j - i\alpha_{n+j}), \qquad \mathbf{a}_j^* = \tfrac{1}{2}(\alpha_j + i\alpha_{n+j}), \tag{7.5}$$

we obtain the relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}\mathbf{E}, \qquad 1 \leq j, k \leq 2n. \tag{7.6}$$

Conversely, if the α_k satisfy (7.6) and $\mathbf{a}_j, \mathbf{a}_j^*$ are defined by (7.5) then the anticommutation relations (7.4) hold. Operators satisfying (7.4) are called **annihilation** and **creation operators** for fermions.

It follows from the preceding section that V can be decomposed into a direct sum of irred subspaces under the $\mathbf{a}_j$ and $\mathbf{a}_k^*$, each subspace transforming as the 2^n-dimensional rep $(\frac{1}{2}^n)$ of $so(2n + 1)$. A basis is given by (6.18) which we rewrite as

$$|p_1 \cdots p_n\rangle = f_{p_1} \otimes \cdots \otimes f_{p_n}, \qquad p_j = 0, 1, \tag{7.7}$$

where

$$f_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then from (6.10) and (7.5) we obtain

$$\begin{aligned}(7.8)\qquad &\mathbf{a}_j\,|\,p_1\cdots p_j\cdots p_n\rangle=(-1)^{p_1+\cdots+p_{j-1}}p_j\,|\,p_1\cdots p_j-1\cdots p_n\rangle\\ &\mathbf{a}_j{}^*\,|\,p_1\cdots p_j\cdots p_n\rangle=(-1)^{p_1+\cdots+p_{j-1}}(1-p_j)\,|\,p_1\cdots p_j+1\cdots p_n\rangle.\end{aligned}$$

The physical interpretation of these relations will be given in the next section.

It has been found experimentally that for many nuclear interactions (the **strong** interactions) one can consider the proton p and the neutron n as two states of the same particle. In particular the masses of p and n are approximately equal. Of course there is a small mass difference and p has charge $+1$ while n has charge zero. However, the mass and charge differences can be ascribed to so-called electromagnetic and weak interactions, which are considered as perturbations of the charge-independent strong interactions.

In the theory of strong interactions one considers p and n as two states of the nucleon N described by the state space $\mathcal{K}=\mathcal{K}_{1/2}\otimes \mathcal{I}_2$. Here $\mathcal{K}_{1/2}$ is the state space for a particle with spin $\frac{1}{2}$ as described in Sections 7.8 or 8.4 (for a relativistic theory) and $\mathcal{I}_2$ is a two-dimensional space with basis

$$(7.9)\qquad \mathbf{e}_{1/2}=\mathbf{p}=\begin{pmatrix}1\\0\end{pmatrix},\qquad \mathbf{e}_{-1/2}=\mathbf{n}=\begin{pmatrix}0\\1\end{pmatrix},$$

called **isobaric spin space.** A general element of $\mathcal{K}$ can be written uniquely in the form

$$\mathbf{\Psi}=\mathbf{\Psi}_p\otimes\mathbf{p}+\mathbf{\Psi}_n\otimes\mathbf{n},$$

where $\mathbf{\Psi}_p$, $\mathbf{\Psi}_n$ are themselves two-component spinors. Here $\mathbf{\Psi}_p\otimes\mathbf{p}$ is a pure proton state and $\mathbf{\Psi}_n\otimes\mathbf{n}$ is a pure neutron state. In general, $\mathbf{\Psi}$ is a superposition of proton and neutron states. The inner product in $\mathcal{K}$ is given by

$$(7.10)\qquad \langle\mathbf{\Psi}^{(1)},\mathbf{\Psi}^{(2)}\rangle_{\mathcal{K}}=\langle\mathbf{\Psi}_p^{(1)},\mathbf{\Psi}_p^{(2)}\rangle_{\mathcal{K}_{1/2}}+\langle\mathbf{\Psi}_n^{(1)},\mathbf{\Psi}_n^{(2)}\rangle_{\mathcal{K}_{1/2}}.$$

The state space for k nucleons is

$$\mathcal{K}^{\otimes k}=(\mathcal{K}_{1/2}\otimes\mathcal{I}_2)^{\otimes k}\cong(\mathcal{K}_{1/2})^{\otimes k}\otimes(\mathcal{I}_2)^{\otimes k}.$$

An element $\mathbf{\Psi}$ of $\mathcal{K}^{\otimes k}$ is determined by the spinor

$$(7.11)\qquad \mathbf{\Psi}_{s_1\cdots s_k,t_1\cdots t_k}(\mathbf{x}_1,\ldots,\mathbf{x}_k),\qquad s_j=\pm\tfrac{1}{2},\quad t_j=\pm\tfrac{1}{2}.$$

Here $\mathbf{x}_1,\ldots,\mathbf{x}_k$ are the position coordinates of the k particles, the s_j are the ordinary spin indices, and the t_j are isobaric spin indices. For example, if all components of $\mathbf{\Psi}$ are zero except that component for which $s_1=\cdots=s_k=-\frac{1}{2}$ and $t_1=\cdots=t_k=\frac{1}{2}$, then $\mathbf{\Psi}$ is a state of k protons each with 3-component of ordinary spin equal to $-\frac{1}{2}$.

If this k-particle system interacts with itself we demand as usual that the interaction admit the symmetry $\mathcal{E}_3$, (8.21), Section 7.8, where the group acts on the position vectors $\mathbf{x}_j$ and the spin indices s_k. [In a relativistic theory we demand that the interaction admit $\mathcal{P}$ as a symmetry group, (4.22), Section

8.4.] Furthermore, for strong interactions we also require that the operators $\mathbf{I}(A)$, $A \in SU(2)$, commute with the interaction Hamiltonian, where

(7.12)
$$[\mathbf{I}(A)\Psi]_{s_1\cdots s_k,t_1\cdots t_k}(\mathbf{x}_1,\ldots,\mathbf{x}_k) = \sum_{t_1'\cdots t_k'} A_{t_1t_1'}\cdots A_{t_kt_k'}\Psi_{s_1\cdots s_k,t_1'\cdots t_k'}(\mathbf{x}_1,\ldots,\mathbf{x}_k).$$

Note that $\mathbf{I}(A)$ acts only on the isobaric spin space $\mathcal{I}_2$ and does not affect the position vectors or ordinary spin indices. Thus for a relativistic theory of strong interactions the symmetry group is assumed to be $\mathcal{P} \times SU(2)$. (Actually one also requires invariance under space and time inversion, but we will not discuss this here.) Since we have already examined the implications of invariance with respect to symmetry groups acting on the position vectors and ordinary spin indices, we restrict the following discussion to the rep (7.12) of $SU(2)$ acting on the isobaric spin indices. Under this action $(\mathcal{I}_2)^{\otimes k}$ transforms according to the rep $(\mathbf{D}^{(1/2)})^{\otimes k}$. We can use the CG series, Section 7.7, to decompose this rep into irred components. If the system is in a state transforming as the canonical basis vector $\mathbf{f}_m^{(u)}$ of $\mathbf{D}^{(u)}$ in isobaric spin space at some time t_0, then the isobaric spin invariance of the system implies that at any later time t the system still transforms as $\mathbf{f}_m^{(u)}$. The mathematical analysis which exploits this invariance is similar to that leading up to expression (8.10) in Section 7.8, but the physical interpretation is different. As an example we consider an interacting two-nucleon system such that the result of the interaction is again a two-nucleon system. The isobaric spin space $\mathcal{I}_2 \otimes \mathcal{I}_2$ for this problem transforms as $\mathbf{D}^{(1/2)} \otimes \mathbf{D}^{(1/2)} \cong \mathbf{D}^{(1)} \otimes \mathbf{D}^{(0)}$ under $SU(2)$. The canonical basis vectors are

(7.13)
$$\mathbf{f}_1^{(1)} = \mathbf{p}\otimes\mathbf{p},\quad \mathbf{f}_0^{(1)} = 2^{-1/2}(\mathbf{p}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{p}),\quad \mathbf{f}_{-1}^{(1)} = \mathbf{n}\otimes\mathbf{n},$$
$$\mathbf{f}_0^{(0)} = 2^{-1/2}(\mathbf{p}\otimes\mathbf{n} - \mathbf{n}\otimes\mathbf{p}).$$

Since isobaric spin is conserved by the interaction a p–p system ends up as a p–p system and, similarly, an n–n system ends up as an n–n system. Furthermore, the transition probabilities between corresponding eigenstates of orbital and spin angular momentum are exactly the same for p–p and n–n systems. Both of these systems belong to $\mathbf{D}^{(1)}$, while a mixed system belongs to both $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(0)}$.

The Hamiltonian describing strong interactions between the two nucleons commutes with the action of the isospin group. However, if one takes into account the electromagnetic interaction between nucleons the perturbed Hamiltonian no longer commutes with the isospin operators and it becomes possible to distinguish between p and n. Thus the electromagnetic interaction breaks isospin symmetry.

The concept of isospin symmetry can be applied to other elementary particles. As an example we consider a family of eight baryons, all with (ordinary) spin $\frac{1}{2}$ and approximately the same mass. Two of these particles are p and n which transform as a canonical basis for the rep $\mathbf{D}^{(1/2)}$ in isospace. The sigma hyperons $\sum^+, \sum^0, \sum^-$ have charges $+1, 0, -1$, respectively, and transform as a canonical basis $\mathbf{f}_1^{(1)}, \mathbf{f}_0^{(1)}, \mathbf{f}_{-1}^{(1)}$ for $\mathbf{D}^{(1)}$ in isospace. (Thus the Hilbert space describing a single sigma hyperon takes the form $\mathcal{H}_{1/2} \otimes \mathcal{I}_3$, where $\mathcal{I}_3$ is the rep space for $\mathbf{D}^{(1)}$.) The Λ^0 hyperon has charge zero and transforms as a scalar $\mathbf{D}^{(0)}$ under isospin. Finally, the cascade particles Ξ^0, Ξ^- have charges $0, -1$, respectively, and transform as the canonical basis $\mathbf{f}_{1/2}^{(1/2)}, \mathbf{f}_{-1/2}^{(1/2)}$ of $\mathbf{D}^{(1/2)}$.

It is required that the isospin group $SU(2)$ be an invariance group for the strong interactions of these particles. Thus for nucleon–sigma hyperon scattering the isospin space transforms as $\mathbf{D}^{(1/2)} \otimes \mathbf{D}^{(1)} \cong \mathbf{D}^{(3/2)} \oplus \mathbf{D}^{(1/2)}$, and the isospin space has a basis consisting of a canonical basis for $\mathbf{D}^{(3/2)}$ and a canonical basis for $\mathbf{D}^{(1/2)}$. The basis vectors can be determined from the CG series, e.g., $\mathbf{f}_{3/2}^{(3/2)} = \mathbf{p} \otimes \mathbf{\Sigma}^+$. Isospin invariance implies that a system transforming as $\mathbf{f}_m^{(u)}$ before the scattering must transform the same way after the scattering even if particles are created or destroyed by the interaction. This leads to selection rules and conservation laws in analogy with those derived in Section 7.8. Again, electromagnetic effects break the isospin symmetry, so the conservation laws are only approximately correct. For a detailed examination of these rules and their physical significance or validity see the standard textbooks in elementary particle physics.

We now introduce one more concept important in the theory of strong interactions: the **hypercharge** Y. Let $I^{\pm}, I^3$ be the generators of isospin satisfying relations (1.23), Section 7.1. (We use the notation I rather than J to distinguish isospin from angular momentum.) The eight elementary particles listed above correspond to eigenvectors of I^3 and the eigenvalues of I^3 are conserved by strong interactions. We know that the charge of a quantum mechanical system is conserved under strong interactions, and each of the canonical basis vectors for isospin space listed above has definite charge. Thus we can uniquely define a charge operator Q by

$$Q\mathbf{r} = q\mathbf{r}, \tag{7.14}$$

where q is the charge of the particle $\mathbf{r}$ and $\mathbf{r}$ runs over a canonical basis for isospin space. For example,

$$Q\mathbf{n} = \mathbf{\theta}, \qquad Q\mathbf{p} = \mathbf{p} \tag{7.15}$$

for the nucleon pair n and p. We define the **hypercharge** operator Y by

$$Q = I^3 + \tfrac{1}{2}Y \qquad \text{or} \qquad Y = 2(Q - I^3). \tag{7.16}$$

it follows easily from (7.9) and (7.15) that $Y = 1$ for the pair n, p. A similar computation shows that the hypercharge of the $\mathbf{\Sigma}$-triplet and Λ^0-singlet is

$Y = 0$ and the hypercharge of the cascade doublet is $Y = -1$. Thus there is a close relation between the charge and the third component of isotopic spin, and this exact relation is determined by the hypercharge. Since charge and the eigenvalues of I^3 are additive quantum numbers conserved by strong interactions, hypercharge must also be an additive conserved quantity. (By additive we mean that a two-particle system consisting of particle a with hypercharge Y_a and particle b with hypercharge Y_b has total hypercharge $Y_a + Y_b$. The sum $Y_a + Y_b$ is conserved under strong interactions, though not Y_a or Y_b separately.)

With the success of the concept of isospin in providing some order in elementary particle physics, attempts were made to find a larger symmetry group G for strong interactions which included the isospin group $SU(2)$ as a proper subgroup. The most successful attempt to date is the "eightfold way" of Gell-Man and Ne'eman [1], in which the symmetry group is $SU(3)$.

Before describing this theory we discuss the irred reps of $SU(3)$. These reps can be denoted $[f_1, f_2] = [f_1, f_2, 0]$, where $f_1 \geq f_2 \geq 0$ are integers. The simple characters are given by Theorem 9.4 and the dimensions by

$$N(f_1, f_2) = \tfrac{1}{2}(f_1 - f_2 + 1)(f_1 + 2)(f_2 + 1). \tag{7.17}$$

We list some of the irred reps of low dimension.

(7.18)

	$[f_1, f_2]$	$N(f_1, f_2)$		$[f_1, f_2]$	$N(f_1, f_2)$
1	[0, 0]	1	**8**	[2, 1]	8
3	[1, 0]	3	**10**	[3, 0]	10
$\overline{\mathbf{3}}$	[1, 1]	3	$\overline{\mathbf{10}}$	[3, 3]	10
6	[2, 0]	6	**27**	[4, 2]	27
$\overline{\mathbf{6}}$	[2, 2]	6			

Here, we have adopted the notation in elementary particle physics where a rep is denoted by its dimension. (Using Young symmetrizers the reader should be able to show that $\overline{\mathbf{3}}, \overline{\mathbf{6}}$, and $\overline{\mathbf{10}}$ are the complex conjugate reps of **3**, **6**, and **10**, respectively.)

Clearly, **3** is the usual matrix realization of $SU(3)$. The weights are $\lambda_1, \lambda_2, \lambda_3 = -\lambda_1 - \lambda_2$. The rep $\overline{\mathbf{3}}$ is defined by the matrices $\bar{A}$, $A \in SU(3)$. The weights are $-\lambda_1, -\lambda_2, -\lambda_3 = \lambda_1 + \lambda_2$. The rep $\mathbf{8} \cong [2, 1]$ is just the adjoint rep of $SU(3)$ acting on $su(3)$. Indeed, $su(3)$ is eight-dimensional and the weights of the adjoint rep are just the roots of $su(3)$ plus the weight zero with multiplicity two. Thus the nonzero weights are $\pm(\lambda_1 - \lambda_2), \pm(\lambda_1 - \lambda_3) = \pm(2\lambda_1 + \lambda_2), \pm(\lambda_2 - \lambda_3) = \pm(\lambda_1 + 2\lambda_2)$. The highest weight is $2\lambda_1 + \lambda_2$, so the adjoint rep contains [2, 1]. Since dim [2, 1] = 8 it follows that the adjoint rep is irred.

We can identify $SU(2)$ with the subgroup

$$\begin{pmatrix} SU(2) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of $SU(3)$. Clearly $[f_1, f_2] \mid SU(2)$ decomposes into a direct sum of irred reps. We determine this decomposition for [2, 1]. On restriction to $SU(2)$, we find $\lambda_1 + \lambda_2 = 0 = \lambda_3$. Thus, the weights of $\rho \cong \mathbf{8} \mid SU(2)$ are $\pm 2\lambda_1, \pm\lambda_1$ (multiplicity two), and 0 (multiplicity two). The highest weight is $2\lambda_1$, so ρ contains $[2] \cong \mathbf{D}^{(1)}$. (Recall $[2u] \cong \mathbf{D}^{(u)}$.) Removing the weights $\pm 2\lambda_1, 0$ of [2] we see that λ_1 (multiplicity two) is the highest remaining weight. Therefore, ρ contains $[1] \oplus [1]$. Removing the four weights $\pm\lambda_1, \pm\lambda_1$ corresponding to these reps we are left with the single weight 0. Thus,

$$\mathbf{8} \mid SU(2) \cong \mathbf{D}^{(1)} \oplus 2\mathbf{D}^{(1/2)} \oplus \mathbf{D}^{(0)} \cong [2] \oplus 2[1] \oplus [0]. \tag{7.19}$$

In the eightfold way the eight baryons introduced above are identified with a basis for **8** which is canonical with respect to the decomposition (7.19). The Σ-triplet with hypercharge $Y = 0$ forms a canonical basis for [2], the two doublets with hypercharge $Y = 1$ and $Y = -1$ form canonical bases for the two occurrences of [1], and the Λ^0 with hypercharge $Y = 0$ belongs to [0]. The assignment

$$H(\lambda_1, \lambda_2) = \lambda_1 Q + \lambda_2(Y - Q) \tag{7.20}$$

defines the eight baryons as a weight basis for **8**. We shall not be concerned with the exact normalization of this basis.

Now consider these eight baryons as distinct states of a single particle with spin $\frac{1}{2}$. The Hilbert space describing such a single-particle system takes the form $\mathcal{H}_{1/2} \otimes \mathcal{V}_8$, where $\mathcal{V}_8$ is the eight-dimensional rep space for **8** constructed above. For very strong interactions involving this particle it is required that the interaction Hamiltonian commute with the action of $SU(3)$. Since the isospin group $SU(2)$ is identified as a subgroup of $SU(3)$, this requirement implies conservation of isospin, However, $SU(3)$ invariance clearly leads to additional selection rules and conservation laws. These rules can be obtained from the Wigner–Eckart theorem.

Other particle multiplets can be fitted with irred reps of $SU(3)$. In addition to the baryon octet there is an octet of pseudoscalar mesons which also transforms according to **8**. There is also a baryon decouplet transforming according to **10**. For very strong interactions involving a particle from the baryon octet and a particle from the pseudoscalar meson octet the space on which $SU(3)$ acts transforms according to $\mathbf{8} \otimes \mathbf{8}$. To make full use of the $SU(3)$ symmetry it is necessary to decompose $\mathbf{8} \otimes \mathbf{8}$ into irred reps. The result is

$$\mathbf{8} \otimes \mathbf{8} \cong \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus 2\mathbf{8} \oplus \mathbf{1}. \tag{7.21}$$

If the system is in one of the irred subspaces on the right-hand side of (7.21) before the interaction, it will lie in the same subspace after the interaction. The analysis is very similar to that given in Section 7.6. The weight vectors corresponding to irred reps on the right-hand side of (7.21) can themselves be considered as particles or "resonances." If such a resonance decays into a baryon and a meson the possible decay modes and their relative probabilities are given by expressing the resonance as a linear combination of weight vectors from the "natural" baryon–meson basis for $\mathbf{8} \otimes \mathbf{8}$ via the Clebsch–Gordan coefficients. The Clebsch–Gordan coefficients for $SU(3)$ have not been computed in the general case. The principal difficulty is that the tensor products may contain an irred rep with multiplicity greater than one, as illustrated by (7.21). However, the CG coefficients have been tabulated for all tensor products of importance in the eightfold way model, such as (7.21) (Dyson [1]).

The eightfold way model is presumed to be exact only for extremely strong interactions. The other possible interactions between particles are considered as perturbations which break the $SU(3)$ symmetry. We can imagine turning on these perturbations in sequence. First we reduce the symmetry from $SU(3)$ to $SU(2)$, the isospin group. This causes the baryon octet to split into a triplet, two doublets, and a singlet via (7.19). At this point we still have isospin symmetry. Now we turn on the electromagnetic and weak interactions to break the $SU(2)$ symmetry. The electromagnetic interactions conserve I^3 and Q, but the weak interactions conserve only Q. If the perturbing interactions are "small" with respect to the very strong interactions then we expect this model to yield experimental predictions which are at least qualitatively correct. Furthermore, we expect to explain the observed mass differences of the particles in the baryon octet in terms of the perturbing interactions. In fact, there is a great deal of experimental evidence validating the predictions of this model and the observed mass differences can be explained rather well by the Gell-Mann–Okubo mass formula (Dyson [1]). The model has no firm theoretical basis and may be disgarded in time, but it is certainly a useful means of classifying elementary particles and a beautiful application of symmetry groups.

The classical groups have been used extensively in atomic spectroscopy and nuclear physics. For a detailed study of these applications see the work of Hamermesh [1], Loebl [1, 2], or Tinkham [1]. In many cases the mathematical content of the application is the determination of multiplicities of irred reps $\mathbf{T}'$ belonging to a subgroup K of G in the restricted rep $\mathbf{T} \mid K$. Here $\mathbf{T}$ is an irred rep of G. The formulas for the multiplicities are called **branching laws.** Throughout this book we have computed branching laws for various groups of physical interest. Some of the most important laws for the classical groups are given by Boerner [1].

The group $SO(4, R)$ is of special importance in physics because of its relationship to the hydrogen atom. For a (spinless) particle in a spherically symmetric field the Hamiltonian takes the form

$$H = -(1/2m)\Delta + V(r), \qquad r = [x^2 + y^2 + z^2]^{1/2}.$$

For most choices of the potential V the (connected) symmetry group of $\mathbf{H}$ is just $SO(3)$. However, outside of the trivial case V a constant, there are two cases where the connected symmetry group of $\mathbf{H}$ is larger than $SO(3)$:

$$V(r) = c/r, \qquad V(r) = cr^2, \qquad c \quad \text{a constant.} \tag{7.22}$$

The hydrogen atom (a single particle in an attractive Coulomb field) corresponds to such a potential. Using appropriate units we can choose

$$\mathbf{H} = -\tfrac{1}{2}\Delta - (1/r). \tag{7.23}$$

It can be shown that the eigenvalues of $\mathbf{H}$ are all negative (the boundstate energy levels) (Helwig [1]). Let λ be an eigenvalue of $\mathbf{H}$ and $\mathfrak{C}_\lambda$ the corresponding eigenspace. We look for symmetric operators on $\mathfrak{C}_\lambda$ which commute with $\mathbf{H}$. Since r^{-1} is spherically symmetric the angular momentum operators $\mathbf{L} = (L_1, L_2, L_3)$ [(6.24), Section 7.6] commute with $\mathbf{H}$. Furthermore, a tedious computation shows that the operators $\mathbf{A} = (A_1, A_2, A_3)$,

$$\mathbf{A} = \frac{1}{2(-2\lambda)^{1/2}}\left(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L} + \frac{2\mathbf{r}}{r}\right), \tag{7.24}$$

are also symmetric and commute with $\mathbf{H}$. Here $\mathbf{p} = (-i\partial_x, -i\partial_y, -i\partial_z)$, $\mathbf{r} = (x, y, z)$, and $\mathbf{L} \times \mathbf{P}$ is the cross-product. For the physical significance of $\mathbf{A}$ (the Runge–Lenz vector) see the book by Pollard [1]. Note that $\mathbf{A}$ depends on λ. The six operators $\mathbf{L}$ and $\mathbf{A}$ satisfy the commutation relations

$$\begin{aligned} [L_j, L_k] &= i \sum_l \epsilon_{jkl} L_l, \qquad [L_j, A_k] = i \sum_l \epsilon_{jkl} A_l, \\ [A_j, A_k] &= i \sum_l \epsilon_{jkl} L_l, \qquad 1 \le j, k, l \le 3, \end{aligned} \tag{7.25}$$

where ϵ_{jkl} is the completely skew-symmetric tensor such that $\epsilon_{123} = +1$. These commutation relations are valid only on the domain $\mathfrak{C}_\lambda$. Setting

$$C_j = -\tfrac{1}{2}i(L_j + A_j), \qquad D_j = -\tfrac{1}{2}i(L_j - A_j), \qquad j = 1, 2, 3,$$

we obtain

$$[C_j, C_k] = \sum \epsilon_{jkl} C_l, \qquad [D_j, D_k] = \sum \epsilon_{jkl} D_l, \qquad [C_j, D_k] = 0. \tag{7.26}$$

Clearly the complexification of the Lie algebra generated by the operators (7.25) is isomorphic to $sl(2) \times sl(2)$. In particular, the commutation relations (7.26) and (3.1), Section 8.3, are identical. Thus the irred reps of the Lie algebra (7.25) can be denoted $\mathbf{D}^{(u,v)}$. The reps are defined by (3.2) and (3.3), Section 8.3. In particular, $\mathbf{C} \cdot \mathbf{C} = u(u + 1)$, $\mathbf{D} \cdot \mathbf{D} = v(v + 1)$, and dim $\mathbf{D}^{(u,v)} = (2u + 1)(2v + 1)$. It follows directly from (7.24) that $\mathbf{A} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{A}$

$= 0$ and $\mathbf{L}\cdot\mathbf{L} + \mathbf{A}\cdot\mathbf{A} = -[1 + (1/2\lambda)]$. Thus

(7.27) $$\mathbf{C}\cdot\mathbf{C} = \mathbf{D}\cdot\mathbf{D} = \tfrac{1}{4}(\mathbf{L}\cdot\mathbf{L} + \mathbf{A}\cdot\mathbf{A}) = -\tfrac{1}{4}[1 + (1/2\lambda)] = u(u+1),$$

so the possible irred reps on $\mathcal{C}_\lambda$ obtainable from our model (7.24) are $\mathbf{D}^{(u,u)}$, where

(7.28) $$\lambda = -1/2(2u+1)^2 = -1/2n^2, \qquad n = 2u + 1 = 1, 2, \ldots.$$

The possible energy eigenvalues are given by the Balmer series $\lambda_n = -1/2n^2$ and the degeneracy of the eigenvalues λ_n is n^2. Just as in Section 8.3, the restriction of this Lie algebra to the subalgebra $su(2)$ generated by the angular momentum operators L_1, L_2, L_3 yields

(7.29) $$\mathbf{D}^{(u,u)} \mid su(2) \cong \mathbf{D}^{(u)} \otimes \mathbf{D}^{(u)} \cong \mathbf{D}^{(2u)} \oplus \cdots \oplus \mathbf{D}^{(1)} \oplus \mathbf{D}^{(0)}.$$

The eigenspace corresponding to λ_n is not irred under the angular momentum operators but decomposes into a direct sum of the irred reps $\mathbf{D}^{(l)}$, $0 \le l \le n-1$, each rep with multiplicity one. It follows from (7.29) that we can choose n^2 vectors Ψ_{nlm} as a basis for $\mathcal{C}_{\lambda_n}$, where $l = 0, 1, \ldots, n-1$, $-l \le m \le l$.

The skew-Hermitian operators iL_j, iA_j, $1 \le j \le 3$, generate a real Lie algebra isomorphic to $so(4, R)$. Furthermore, the Lie algebra rep $\mathbf{D}^{(u,u)}$ on $\mathcal{C}_{\lambda_n}$ exponentiates to the irred rep $(2u, 0)$ of $SO(4, R)$. Note that $so(4, R)$ and $so(3, 1)$ are both real forms of $sl(2) \oplus sl(2) \cong so(4)$. The action of $SO(4, R)$ on the coordinate space is rather difficult to compute because the A_j are second-order differential operators (Kursunoğlu [1]).

The harmonic oscillator in three dimensions corresponds to the potential $-kr^2$, $k > 0$. The symmetry group of the Hamiltonian is $SU(3)$. We will study this example in Chapter 10. Outside of these two cases no examples of rotationally symmetric Hamiltonians with connected symmetry group larger than $SO(3)$ are known. The high degree of symmetry of these two examples enables one to find the bound-state energy levels and their multiplicities from group theory alone.

In conclusion, we note that the noncompact classical groups also have infinite-dimensional irred reps. These reps can be constructed using Lie algebras and weight vectors (Sherman [1]) or using the method of induced reps (Gel'fand and Naimark [1]).

9.8 The Pauli Exclusion Principle and the Periodic Table

Consider a Hilbert space $\mathcal{H}_s^{\otimes N}$ corresponding to a physical system of N indistinguishable particles (say electrons) with spin s and mass m. A typical Hamiltonian for this system is

(8.1) $$\mathbf{H} = -\sum_{j=1}^{N} (1/2m)\Delta_j + V(\mathbf{x}_1, \ldots, \mathbf{x}_N) + \mathbf{W},$$

where $\mathbf{x}_j$ is the position coordinate of particle j, V is a potential function, and

$$\mathbf{W} = (W^{\nu_1 \cdots \nu_N}_{\mu_1 \cdots \mu_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N))$$

is a spin-dependent interaction term. The Hilbert space consists of wave functions $\mathbf{\Psi} = \sum \Psi_{\mu_1 \cdots \mu_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N)\mathbf{e}^{\mu_1} \otimes \cdots \otimes \mathbf{e}^{\mu_N}$, where μ_j is the spin index of the jth particle (see Section 7.8) and the $\mathbf{e}^{\mu_j}$ form a canonical basis in spin space. Here

(8.2)
$$(\mathbf{W\Psi})_{\mu_1 \cdots \mu_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{\nu_j = -s}^{s} W^{\nu_1 \cdots \nu_N}_{\mu_1 \cdots \mu_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N)\Psi_{\nu_1 \cdots \nu_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N).$$

Let σ be a permutation of the integers $1, \ldots, N$ and define the permutation operator $\boldsymbol{\sigma}$ on $\mathcal{H}_s^{\otimes N}$ by

(8.3) $$(\boldsymbol{\sigma}\Psi)_{\mu_1 \cdots \mu_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \Psi_{\mu_{\sigma^{-1}(1)}, \ldots, \mu_{\sigma^{-1}(N)}}(\mathbf{x}_{\sigma^{-1}(1)}, \ldots, \mathbf{x}_{\sigma^{-1}(N)}).$$

Here $\boldsymbol{\sigma}$ is a self-adjoint operator on $\mathcal{H}_s^{\otimes N}$, as can be seen from the inner product (8.20), Section 7.8. The indistinguishability requirement for the N particles means that $\mathbf{H}$ is symmetric in the coordinates of these particles, i.e.,

(8.4) $$\boldsymbol{\sigma}\mathbf{H} = \mathbf{H}\boldsymbol{\sigma}$$

for all $\sigma \in S_N$. Thus the permutation group S_N is a symmetry group of $\mathbf{H}$. To see the implications of this fact let λ be an eigenvalue of $\mathbf{H}$ and W_λ the corresponding finite-dimensional eigenspace. If $\mathbf{\Psi} \in W_\lambda$ we have $\boldsymbol{\sigma}\mathbf{\Psi} \in W_\lambda$ for each $\sigma \in S_N$. Thus we can decompose W_λ into a direct sum of subspaces, each subspace transforming irreducibly under S_N. This decomposition was studied in Sections 3.7 and 4.2, and the irred reps were labeled by Young frames $\{f_1, \ldots, f_N\}$.

It has been found experimentally that not all eigenvectors $\mathbf{\Psi}$ in W_λ are physically meaningful. In particular, if the spin s is half-integral, the eigenvectors of $\mathbf{H}$ with eigenvalue λ (corresponding to a physical system) can occupy only that subspace of W_λ belonging to the rep $\{1^N\}$ of S_N. Indeed, the only allowed states $\mathbf{\Psi}$ permitted to such a system are completely skew-symmetric:

(8.5) $$\boldsymbol{\sigma}\mathbf{\Psi} = \delta_\sigma \mathbf{\Psi}.$$

(Here δ_σ is the parity of $\sigma \in S_N$.) Particles with half-integral spin $s = \frac{1}{2}, \frac{3}{2}, \ldots$ are called **fermions.**

If s is an integer, the eigenvectors corresponding to a physical system can occupy only that subspace of W_λ belonging to the completely symmetric rep $\{N\}$. Thus, the allowed eigenfunctions $\mathbf{\Psi}$ must satisfy

(8.6) $$\boldsymbol{\sigma}\mathbf{\Psi} = \mathbf{\Psi}$$

for all $\sigma \in S_N$. Particles with integral spin are called **bosons.**

All known elementary particles are either fermions or bosons. In the following discussion we consider only fermions.

If the Hamiltonian is a sum of single-particle Hamiltonians

$$\mathbf{H} = \sum_{j=1}^{N} \mathbf{H}_j, \qquad \mathbf{H}_j = -(1/2m)\,\Delta_j + V(\mathbf{x}_j) \tag{8.7}$$

then the eigenspace W_λ is spanned by tensor products of single-particle eigenfunctions,

$$\mathbf{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \mathbf{\Psi}^{(1)}(\mathbf{x}_1) \otimes \cdots \otimes \mathbf{\Psi}^{(N)}(\mathbf{x}_N), \qquad \mathbf{H}_j\mathbf{\Psi}^{(j)} = \lambda_j\mathbf{\Psi}^{(j)}, \tag{8.8}$$

where $\lambda = \lambda_1 + \cdots + \lambda_N$. If $\mathbf{\Psi} \in W_\lambda$ then $\boldsymbol{\sigma}\mathbf{\Psi} \in W_\lambda$ for each $\sigma \in S_N$. The vectors $\boldsymbol{\sigma}\mathbf{\Psi}$ span a subspace with dimension $N!$ if the $\{\mathbf{\Psi}^{(j)}\}$ are linearly independent. However, only the one-dimensional subspace consisting of skew-symmetric tensors

$$\tilde{\mathbf{\Psi}} = \sum_\sigma \delta_\sigma \mathbf{\Psi}^{(1)}(\mathbf{x}_{\sigma(1)}) \otimes \cdots \otimes \mathbf{\Psi}^{(N)}(\mathbf{x}_{\sigma(N)}) \tag{8.9}$$

is physically meaningful. Thus, for a system of N identical noninteracting fermions the skew-symmetry requirement allows us to discard all but one of our $N!$ linearly independent mathematical eigenfunctions. Furthermore, if two of the single-particle eigenfunctions $\mathbf{\Psi}^{(j)}, \mathbf{\Psi}^{(k)}, j \neq k$, are linearly dependent then $\tilde{\mathbf{\Psi}} \equiv \boldsymbol{\theta}$ in (8.9) and there are no permissible eigenfunctions. This yields the **Pauli principle:** A system of identical fermions cannot exist in a state in which two of the fermions are in the same single-particle state. The Pauli principle is a special case of (8.5) since it applies only to noninteracting systems.

To show the significance of the Pauli principle we discuss the N-electron atom. (Electrons are fermions since they have spin $\frac{1}{2}$.) By neglecting the motion of the (relatively) heavy nucleus we can consider an atom as a system of N electrons in a Coulomb field centered at $\mathbf{x} = \mathbf{0}$. The Hamiltonian is

$$\mathbf{H} = \mathbf{H}^{(0)} + \mathbf{H}^{(1)} + \mathbf{H}^{(2)}, \tag{8.10}$$

where

$$\mathbf{H}^{(0)} = \sum_{j=1} \mathbf{H}_j, \qquad \mathbf{H}^{(1)} = \sum_{j<k} V(|\mathbf{x}_j - \mathbf{x}_k|). \tag{8.11}$$

Here the $\mathbf{H}_j$ are single-particle Hamiltonians

$$\mathbf{H}_j = -(1/2m)\,\Delta_j - Nq^2/|\mathbf{x}_j| \tag{8.12}$$

and m, q are the mass and charge of the electron, respectively. The $V(|\mathbf{x}_j - \mathbf{x}_k|)$ is the potential due to the Coulomb interaction between a pair of electrons and $\mathbf{H}^{(2)}$ denotes the interaction between the spin and orbital angular momentum of the electrons. Only $\mathbf{H}^{(2)}$ acts on the spin indices of the state functions.

We present a group-theoretic discussion of this system under the assumption that $\mathbf{H}^{(1)}$ is "small" in comparison to $\mathbf{H}^{(0)}$ and $\mathbf{H}^{(2)}$ is "small" in comparison to $\mathbf{H}^{(1)}$. The perturbation theory built on this assumption is called the **Russell–Saunders** approximation. It leads to useful results for most atoms, particularly the lighter ones.

To a first approximation the system is described by the Hamiltonian $\mathbf{H}^{(0)}$, i.e., the electrons do not interact with each other, but only with the nucleus. Since $\mathbf{H}^{(0)}$ is a sum of single-particle Hamiltonians its eigenfunctions are tensor products of single-particle eigenfunctions. Furthermore, the single-particle Hamiltonian $\mathbf{H}_j$ is exactly that of the hydrogen atom. Omitting spin considerations for a moment, we see from (7.29) that the single-particle eigenfunctions can be labeled $\Psi_{nlm}(\mathbf{x})$, where the **principal** quantum number takes the value $n = 1, 2, \ldots$, the **orbital** quantum number takes the values $l = 0, 1, \ldots, n-1$, and $m = -l, -l+1, \ldots, l$. For fixed n and l the $2l+1$ vectors $\{\Psi_{nlm}\}$ correspond to an energy level nl of the hydrogen atom. The energy levels increase with increasing n. In the nonrelativistic idealized hydrogen atom, levels nl and nl' have the same energy for $l \neq l'$. However, with a more realistic model of the atom it can be shown that this degeneracy is partially removed and the energy levels increase slightly with increasing l. Thus we can label the distinct energy levels by nl and each such level has multiplicity $2l+1$. In atomic spectroscopy the orbital quantum numbers $l = 0, 1, 2, 3, \ldots$ are denoted $s, p, d, f, g, \ldots$. Hence, the levels of hydrogen in order of increasing energy are $1s, 2s, 2p, 3s, 3p, 4s, \ldots$.

We have not yet taken the electron spin into account. The spin space of the electron is two-dimensional and the single-particle Hamiltonian does not act on the spin indices. Thus, to each $\Psi_{nlm}(\mathbf{x})$ there correspond two eigenvectors $\Psi_{nlm}(\mathbf{x})\mathbf{e}^{1/2}$ and $\Psi_{nlm}(\mathbf{x})\mathbf{e}^{-1/2}$. We conclude that the multiplicity of the level nl is $2(2l+1)$.

Now we show that even in the rough first-order approximation where the electrons are noninteracting we can get useful qualitative information. In nature most atoms are found in the ground state (lowest energy state) rather than in some excited state. Using our model of the atom with Hamiltonian $\mathbf{H}^{(0)}$ we will explicitly compute the ground state for each N.

For hydrogen, $N = 1$, the single electron must be in a $1s$ state. For the helium atom, $N = 2$, the lowest energy level is obtained by choosing both electrons in a $1s$ state. In the case of lithium, $N = 3$, it is tempting to choose all three electrons in a $1s$ state, but this is forbidden by the Pauli principle since there are only two $1s$ states. Thus, the lowest energy level of lithium is obtained by choosing two electrons in $1s$ states and the third electron in a $2s$ state. For $N = 4$ the ground state is formed by two electrons in $1s$ states and two electrons in $2s$ states. Since there are only two $2s$ states the next electron must be in a $2p$ state to obtain the lowest energy level for $N = 5$. We can continue adding electrons one at a time until the six $2p$ states are filled. This occurs for $N = 10$ (neon). For $N = 11$ (sodium) the added electron must lie in a $3s$ state. We can continue in this manner adding electrons one at a time in the lowest possible eigenstates consistent with the Pauli principle.

It follows from our construction that electrons in an atom fall into **electron shells** labeled by the principal quantum number n. Electrons in the same shell have approximately the same energy, while the energy difference of electrons in different shells is relatively large.

The first five experimentally observed shells ordered in terms of increasing energy are listed in Table 9.1.

TABLE 9.1

Shell number	Electron states	Number of states in filled shell
1	$1s$	2
2	$2s, 2p$	8
3	$3s, 3p$	8
4	$4s, 3d, 4p$	18
5	$5s, 4d, 5p$	18

The observed composition of the first two shells is just as our simple model predicts. We would expect that the third shell would contain the ten $3d$ state as well as the eight $3s$ and $3p$ states. However, it has been found experimentally that in a complex atom the $3d$ states have higher energy than the $4s$ states and fall in the fourth shell. Similarly the $4d$ states lie in the fifth shell. (Our theoretical model ignores the mutual interaction between electrons. However, as the number of atomic electrons increases so does the electron interaction, so we would expect the model to be less accurate for many-electron atoms.)

As we have seen, for helium the first electron shell is filled, and neon contains exactly two filled shells. Similarly, exactly three, four, and five shells are filled for argon, krypton, and xenon, respectively. These atoms with filled shells all correspond to inert gases. On the other hand, the atoms of the alkali metals, lithium, sodium, potassium, and rubidium, consist of filled shells together with one electron in an s state of the next higher shell. Again these elements are observed to have similar chemical properties. Using the above ideas it is possible to divide the known elements into families with similar chemical properties, based on the structure of the electron shells. This theory provides a quantum mechanical derivation of Mendeleev's periodic table of the elements. Indeed the nth row of the periodic table corresponds exactly to the nth electron shell. For more details the reader should consult standard texts in atomic physics.

We now turn to the problem of analyzing the multiplicity structure of the eigenvalues of the Hamiltonian $\mathbf{H}$, (8.10)–(8.12), corresponding to an N-electron atom. Following the usual perturbation theory technique we first consider the Hamiltonian $\mathbf{H}^{(0)}$, a sum of single-particle Hamiltonians. It is clear that S_N [whose action is described by (8.3)] is a symmetry group of $\mathbf{H}^{(0)}$. Furthermore, the group $[SO(3)]^N = SO(3) \times \cdots \times SO(3)$ (N times) defined by (8.3), Section 7.8, is a symmetry group since each single-particle Hamiltonian $\mathbf{H}_j$ separately commutes with the action of $SO(3)$. [Actually $\mathbf{H}_j$ commutes with $O(3)$ but parity conservation contributes little to the following analysis.] Finally, since $\mathbf{H}^{(0)}$ does not act on the spin indices any unitary transformation in spin space commutes with $\mathbf{H}^{(0)}$. The spin space of an N-electron system is 2^N-dimensional, so the spin transformations form a symmetry group isomorphic to $U(2^N)$. By forming all possible products of symmetries corresponding to S_N, $[SO(3)]^N$, and $U(2^N)$ we can generate a larger symmetry group G_0. Unfortunately the structure of G_0 is so complicated that it is not very useful for perturbation theory. (Note that symmetries from S_N and $[SO(3)]^N$ may not commute.) Therefore, we temporarily restrict our attention to the direct product $K = [SO(3)]^N \times U(2^N)$, a proper subgroup of the maximal unitary symmetry group.

Let λ be an eigenvalue of $\mathbf{H}^{(0)}$ and W_λ the corresponding finite-dimensional eigenspace. Since K is compact, W_λ can be decomposed into a direct sum of K-irred subspaces. Each such subspace transforms according to an irred rep of the form

$$D^{(l_1)} \times D^{(l_2)} \times \cdots \times D^{(l_N)} \times [1], \tag{8.13}$$

where $D^{(l)}$ is the $(2l+1)$-dimensional irred rep of $SO(3)$ and [1] is the 2^N-dimensional rep of $U(2^N)$ equivalent to the usual $2^N \times 2^N$ matrix realization of this group. Note that such an irred subspace is of the form

$$V^{(l_1 \cdots l_N)} \otimes Z^{\otimes N}, \tag{8.14}$$

where $Z^{\otimes N}$ is the 2^N-dimensional spin space with basis $\{\mathbf{e}^{\mu_1} \otimes \cdots \otimes \mathbf{e}^{\mu_N}\}$, $V^{(l_1 \cdots l_N)}$ is the $(2l_1+1) \cdots (2l_N+1)$-dimensional space of scalar functions with basis

$$\Psi_{n_1 l_1 m_1}(\mathbf{x}_1) \cdots \Psi_{n_N l_N m_N}(\mathbf{x}_N), \qquad -l_j \le m_j \le l_j, \tag{8.15}$$

and Ψ_{nlm} is the hydrogen atom wave function with quantum numbers n, l, m. The group $U(2^N)$ acts on $Z^{\otimes N}$ irreducibly and $[SO(3)]^N$ acts on $V^{(l_1 \cdots l_N)}$ irreducibly. The dimension of this rep is

$$(2l_1+1) \cdots (2l_N+1)2^N. \tag{8.16}$$

The numbers (8.16) give information concerning the degeneracies of eigenvalues of $\mathbf{H}^{(0)}$, although, since K is not maximal, several irred subspaces correspond to the same eigenvalue.

Next we consider the effect of the perturbing potential $\mathbf{H}^{(1)}$, (8.11). Clearly the action of $A_1 \times \cdots \times A_N \in [SO(3)]^N$ on configuration space commutes with $\mathbf{H}^{(1)}$ if and only if $A_1 = \cdots = A_N$. These "diagonal" elements generate a subgroup of $[SO(3)]^N$ isomorphic to $SO(3)$. The permutation group S_N and $U(2^N)$ acting on the spin indices are still symmetry groups of $\mathbf{H}^{(1)}$. We can generate a much larger symmetry group G_1 by forming all possible finite products of symmetries associated with $SO(3)$, S_N, and $U(2^N)$. However, the symmetries from S_N and $U(2^N)$ do not commute in general so the structure of this group is very complicated. To get more useful results we consider a subgroup of G_1. The set of all matrices

$$\begin{pmatrix} B & & & Z \\ & B & & \\ & & \ddots & \\ Z & & & B \end{pmatrix} \in U(2^N),$$

where $B \in U(2)$, forms a group isomorphic to $U(2)$. Under the action of this group each of the spin indices of $\mathbf{\Psi}$ is transformed identically under the usual two-dimensional realization of $U(2)$. [If we restrict further to the subgroup $SU(2)$ we get the usual action of $SU(2)$ in spin space.] It is easy to verify that the actions of S_N and $U(2)$ mutually commute. Thus, the symmetry group of $\mathbf{H}^{(1)}$ generated by $SO(3)$, S_N, and $U(2)$ is the direct product group

(8.17) $$K_1 = SO(3) \times S_N \times U(2).$$

The irred reps of K_1 are of the form

(8.18) $$D^{(L)} \times \{f_1, \ldots, f_N\} \times [g_1, g_2],$$

where $D^{(L)}$ is the $(2L + 1)$-dimensional rep of $SO(3)$, the Young frame $\{f_1, \ldots, f_N\}, f_1 + \cdots + f_N = N$, denotes an irred rep of S_N as determined in Section 4.2, and the Young frame $[g_1, g_2]$ denotes an irred rep of $U(2)$. Thus each eigenspace of the Hamiltonian $\mathbf{H}^{(0)} + \mathbf{H}^{(1)}$ decomposes into a direct sum of subspaces, each transforming according to reps (8.18). However, the only subspaces of physical interest are those for which $\{f_1, \ldots, f_N\} \cong \{1^N\}$. Furthermore, the rep $[g_1, g_2]$ on restriction to $SU(2)$ is equivalent to $[g_1 - g_2, 0] \cong D^{((g_1 - g_2)/2)}$. We conclude that each irred subspace of physical interest is of the form

(8.19) $$D^{(L)} \times \{1^N\} \times [M + S, M - S],$$

where M and S are both integral or both half-integral. The dimension of this space is $(2L + 1)(2S + 1)$ and the action of the subgroup $SO(3) \times SU(2)$ on it yields the irred rep $D^{(L)} \times D^{(S)}$. Such a space is called a **term** and designated by the symbol ${}^{2S+1}L$ in spectroscopic notation. Each wave function in the space has orbital angular momentum L and spin angular momentum

S. (Recall that spin and orbital angular momentum are not coupled at this state of our perturbation procedure.)

Our derivation of (8.19) is not constructive since it does not show how to actually compute such reps for a given physical system. We analyze this problem in more detail by considering a configuration in which the N electrons are in states $n_1 l_1, \ldots, n_N l_N$. Ignoring spin for the moment, we see that the space $Y^{(l_1 \cdots l_N)}$ of all possible coordinate wave functions describing this system is invariant under the group $SO(3) \times S_N^{(1)}$, where $SO(3)$ is related to orbital angular momentum and $S_N^{(1)}$ denotes the action of the permutation group on the spatial coordinates $\mathbf{x}_j$ of the wave functions $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$. Similarly the 2^N-dimensional spin space $Z^{\otimes N}$ is invariant under $U(2) \times S_N^{(2)}$, where $U(2)$ [and its subgroup $SU(2)$] are related to spin angular momentum, and $S_N^{(2)}$ refers to the action of the permutation group on spin indices. Clearly, the elements of $W_\lambda' = Y^{(l_1 \cdots l_N)} \otimes Z^{\otimes N}$ form an eigenspace of the unperturbed Hamiltonian $\mathbf{H}^{(0)}$ corresponding to some eigenvalue λ. (Here, W_λ' may only be a proper subspace of the total eigenspace W_λ.)

We will decompose W_λ' into terms by first decomposing $Y^{(l_1 \cdots l_N)}$ into irred subspaces under $SO(3) \times S_N^{(1)}$ and $Z^{\otimes N}$ into irred subspaces under $U(2) \times S_N^{(2)}$. For $Y^{(l_1 \cdots l_N)}$ the results depend strongly on $l_1, \ldots, l_N$. All we can say in general is that the irred subspaces are of the form

$$D^{(L)} \times \{f_1, \ldots, f_N\}, \qquad f_1 + \cdots + f_N = N. \tag{8.20}$$

Important special cases of this decomposition are treated by Hamermesh [1] and Lomont [1]. For $Z^{\otimes N}$, on the other hand, we can proceed with complete generality. It follows from the results of Section 4.3 and Section 9.1 that $Z^{\otimes N}$ decomposes under $U(2) \times S_N^{(2)}$ into a direct sum of subspaces of the form

$$[g_1, g_2] \times \{g_1, g_2\}, \qquad g_1 + g_2 = N. \tag{8.21}$$

Indeed the $U(2)$-irred subspaces of $Z^{\otimes N}$ consist of symmetry classes of tensors, each class belonging to a two-rowed Young tableau with N boxes. The space (8.21) is that spanned by all symmetry classes of tensors corresponding to a single frame $[g_1, g_2]$. The dimension of this space is $(g_1 - g_2 + 1) \cdot \dim \{g_1, g_2\}$, where $\dim \{g_1, g_2\}$ is given by Theorem 4.2. Note that the multiplicity of the rep $\{g_1, g_2\}$ of $S_N^{(2)}$ in $Z^{\otimes N}$ is $g_1 - g_2 + 1$. Since $[g_1, g_2] \mid SU(2) \cong [g_1 - g_2, 0]$ the action of $SU(2) \times S_N^{(2)}$ on $Z^{\otimes N}$ decomposes into irred reps

$$D^{(S)} \times \{\tfrac{1}{2}N + S, \tfrac{1}{2}N - S\}, \qquad D^{(S)} \cong [2S, 0], \tag{8.22}$$

where $S = 0, 1, \ldots, N/2$ if N is even and $S = \frac{1}{2}, \frac{3}{2}, \ldots, N/2$ if N is odd. Each irred rep occurs with multiplicity one.

Thus, we can decompose $Y^{(l_1 \cdots l_N)} \otimes Z^{\otimes N}$ into a direct sum of irred reps

$$D^{(L)} \times D^{(S)} \times \{f_1, \ldots, f_N\} \times \{\tfrac{1}{2}N + S, \tfrac{1}{2}N - S\} \tag{8.23}$$

under the action of the group

$$SO(3) \times SU(2) \times S_N^{(1)} \times S_N^{(2)}. \tag{8.24}$$

At this point we couple the actions of S_N on the spatial coordinates and the spin indices, i.e., we restrict $S_N^{(1)} \times S_N^{(2)}$ to the diagonal subgroup $\{\sigma \times \sigma : \sigma \in S_N\}$, isomorphic to S_N. Then the action of the symmetry group

(8.25) $$SO(3) \times SU(2) \times S_N = K_1$$

on (8.23) is

(8.26) $$D^{(L)} \times D^{(S)} \times (\{f_1, \ldots, f_N\} \otimes \{\tfrac{1}{2}N + S, \tfrac{1}{2}N - S\}).$$

The only possible term in (8.26) is ${}^{2S+1}L$ and the multiplicity of this term is equal to the multiplicity of the alternating rep $\{1^N\}$ in the tensor product

(8.27) $$\{f_1, \ldots, f_N\} \otimes \{\tfrac{1}{2}N + S, \tfrac{1}{2}N - S\}.$$

Let $\{g_1, \ldots, g_p\}^t$ be the Young frame obtained by interchanging rows and columns in the Young frame $\{g_1, \ldots, g_p\}$. In the next section we will show that the multiplicity of $\{1^N\}$ in (8.27) is zero unless

(8.28) $$\{f_1, \ldots, f_N\} = \{\tfrac{1}{2}N + S, \tfrac{1}{2}N - S\}^t = \{2^{(N/2)-S}, 1^S\},$$

in which case the multiplicity is one.

Thus the only symmetry classes (8.20) that satisfy the Pauli principle are those whose Young frames consist of two columns. The space transforming according to (8.26) contains one or zero terms depending on whether or not (8.28) is satisfied. An explicit construction of the states in ${}^{2S+1}L$ requires a knowledge of the Clebsch–Gordan coefficients for $\{1^N\}$ in the tensor product (8.27), but this explicit construction is seldom necessary in physical problems.

To recapitulate, our method of term analysis is to decompose the orbital and spin wave functions separately into symmetry classes. The terms are formed from tensor products of these two classes that satisfy the Pauli principle.

We are now ready to consider the effect of the spin–orbit interaction $\mathbf{H}^{(2)}$. We assume that $\mathbf{H}^{(2)}$ does not commute with the orbital angular momentum operators $\mathbf{L}_k$ or the spin angular momentum operators $\mathbf{S}_k$ but that it does commute with the operators $\mathbf{J}_k = \mathbf{L}_k + \mathbf{S}_k$ of total angular momentum. Thus, $\mathbf{H}^{(2)}$ couples the actions of $SO(3)$ and $SU(2)$ on the orbital and spin indices. (Compare the analogous discussion in Section 7.8.)

Let ${}^{2S+1}L$ be a term corresponding to a given energy level of $\mathbf{H}^{(0)} + \mathbf{H}^{(1)}$. This space has dimension $(2S + 1)(2L + 1)$ and transforms according to the irred rep $D^{(S)} \times D^{(L)} \times \{1^N\}$ of the symmetry group $K_1 = SO(3) \times SU(2) \times S_N$. The symmetry group of $\mathbf{H}^{(2)}$ is the subgroup

(8.29) $$K_2 = SU(2) \times S_N,$$

where the action of $SU(2)$ is defined as in (8.21), Chapter 7. Decomposing the term ${}^{2S+1}L$ into subspaces irred under K_2, we find

(8.30) $$D^{(S)} \times D^{(L)} \times \{1^N\} \,|\, K_2 \cong \sum_{J=|S-L|}^{S+L} \oplus (D^{(J)} \times \{1^N\}).$$

The $(2J+1)$-dimensional subspace transforming according to $D^{(J)} \times \{1^N\}$ is called a **multiplet** and denoted ${}^{2S+1}L_J$. Thus, under a perturbing potential $\mathbf{H}^{(2)}$ the term ${}^{2S+1}L$ splits into $2K+1$ multiplet levels where $K = \min(S, L)$. This completes our analysis of the Hamiltonian $\mathbf{H} = \mathbf{H}^{(0)} + \mathbf{H}^{(1)} + \mathbf{H}^{(2)}$ on the basis of Russell–Saunders perturbation theory. This method is based on the chain of groups $G_0 \supset K_1 \supset K_2$. Although K_2 is uniquely determined as the symmetry group of $\mathbf{H}$ and G_0 as the symmetry group of a noninteracting system, the intermediate group K_1 is a function of our perturbation assumptions. Different perturbation schemes lead to different chains of symmetry groups $G_1 \supset \cdots \supset G_k$. For some examples see the work of Hamermesh [1] or Loebl [1].

As the simplest nontrivial example of the Russell–Saunders scheme we consider a configuration of two electrons in an np state. Here $N = 2$, $l_1 = l_2 = 1$, and the space $Y^{(l_1 l_2)}$ is nine-dimensional. Since the spin space $Z^{\otimes 2}$ is four-dimensional there are 36 possible states belonging to this configuration. However, applying the Pauli principle we see that only

$$\binom{6}{2} = 15$$

of these states can be occupied by a physical system. (Note that a single np electron can occupy $3 \times 2 = 6$ states.) We will decompose this configuration into terms. Under the action of $SO(3)$, $Y^{(11)}$ transforms according to $D^{(2)} \otimes D^{(1)} \otimes D^{(0)}$. We want to reduce $Y^{(11)}$ into irred subspaces under $SO(3) \times S_2^{(1)}$. To do this we first split up the space into symmetry classes of tensors, i.e., irred subspaces under the action T of $U(3) \times S_2^{(1)}$ [or $GL(3) \times S_2^{(1)}$]. According to the results of Section 4.3,

$$T \cong [2, 0] \times \{2, 0\} \oplus [1^2] \times \{1^2\}. \tag{8.31}$$

Thus, $Y^{(11)}$ is decomposed into spaces of symmetric tensors (dimension six) and skew-symmetric tensors which carry the reps $[2, 0]$ and $[1^2]$ of $U(3)$, respectively. It is easy to check that $[1^2] \mid SO(3) \cong D^{(1)}$, so we must have $[2, 0] \mid SO(3) \cong D^{(0)} \oplus D^{(2)}$. Thus

$$T \mid SO(3) \times S_2^{(1)} \cong (D^{(0)} \times \{2, 0\}) \oplus (D^{(2)} \times \{2, 0\}) \oplus (D^{(1)} \times \{1^2\}). \tag{8.32}$$

According to our general analysis, the action of $SU(2) \times S_2^{(2)}$ on $Z^{\otimes 2}$ leads to

$$(D^{(0)} \times \{1^2\}) \oplus (D^{(1)} \times \{2, 0\}). \tag{8.33}$$

Now $\{1^2\}^t = \{2, 0\}$ and $\{2, 0\}^t = \{1^2\}$, so the possible terms ${}^{2S+1}L$ transforming according to $D^{(L)} \times D^{(S)} \times \{1^2\}$ are ${}^1S$, ${}^1D$, ${}^3P$, each term occurring once (we use spectroscopic notation for the value of L). The total number of states in the three terms is $1 + 5 + 9 = 15$, in agreement with our earlier result. The decomposition of each term into multiplets follows directly from (8.30). For more complicated examples of term analysis see the work of Hamermesh [1], Lomont [1], or Wybourne [1].

9.9 The Group Ring Revisited

Here we derive some theoretical results of great utility in atomic and nuclear physics. Most of these results are related to the group ring of the symmetric group S_n.

We start by proving a theorem used in the previous section. Let $\{\lambda_j\}$, $\{\lambda_j'\}$ be Young frames with n boxes and let $\chi(\sigma)$, $\chi'(\sigma)$ be the corresponding simple characters of S_n. The rep $\{1^n\} \otimes \{\lambda_j\}$ of S_n has character $\psi\chi(\sigma) = \psi(\sigma)\chi(\sigma)$, where $\psi(\sigma) = \delta_\sigma$ and δ_σ is the parity of $\sigma \in S_n$.

Lemma 9.10. The rep $\{1^n\} \otimes \{\lambda_j\}$ is irred.

Proof. We use the results of Section 3.4 on group characters. By the Corollary to Theorem 3.7 a character ρ of S_n is simple if and only if $\langle\rho, \rho\rangle = 1$, where

(9.1)
$$\langle\chi_1, \chi_2\rangle = (1/n!) \sum_{\sigma\in S_n} \chi_1(\sigma)\overline{\chi_2(\sigma)}$$

for characters χ_1, χ_2. An elementary computation yields $\langle\psi\chi, \psi\chi\rangle = \langle\chi, \chi\rangle = 1$ since χ is simple. Therefore, $\psi\chi$ is a simple character. Q.E.D.

Denote the rep $\{1^n\} \otimes \{\lambda_j\}$ by $\{\tilde{\lambda}_j\}$.

Lemma 9.11. The multiplicity m of the alternating rep $\{1^n\}$ in $\{\lambda_j\} \otimes \{\lambda_j'\}$ is one if $\{\lambda_j'\} \cong \{\tilde{\lambda}_j\}$; otherwise $m = 0$.

Proof. We know $m = \langle\chi\chi', \psi\rangle$. Since the characters of S_n are all real, (9.1) implies $\langle\chi\chi', \psi\rangle = \langle\chi', \chi\psi\rangle$. The characters χ' and $\chi\psi$ are simple, so by the orthogonality relations for characters, $\langle\chi', \chi\psi\rangle = 1$ if and only if $\chi' \equiv \chi\psi$. Otherwise $\langle\chi', \chi\psi\rangle = 0$. Q.E.D.

Since the rep $\{\tilde{\lambda}_j\}$ is irred, it must correspond to some Young frame $\{\mu_j\}$ of S_n. To identify this frame we need a few facts relating characters to the structure of the group ring of S_n. These facts turn out to be valid for all finite groups. Thus we consider an arbitrary finite group G with group ring R_G. The following discussion is based on results derived in Section 3.7.

Let W be a subspace of R_G, invariant under the left regular rep $\mathbf{L}$: $\mathbf{L}(g)x = gx$, $g \in G$, $x \in R_G$. W is a left ideal of R_G and there exists an idempotent c in R_G such that $W = R_G c$, $c^2 = c$. Let $\chi(g)$ be the character of $\mathbf{L} \mid W$. Clearly, $\chi(g)$ is uniquely determined once c is known. We shall derive an expression defining $\chi(g)$ in terms of c. [Recall that every simple character $\chi(g)$ can be obtained in this way from some primitive left ideal W.]

Lemma 9.12. $\chi(g) = \sum_{h\in G} c(h^{-1}g^{-1}h)$, where $c = \sum_{h\in G} c(h)\cdot h$.

Proof. Let $c' = e - c$, where e is the identity element of R_G. Then c' is idempotent and $R_G = W \oplus W'$, where $W' = R_G c'$ is a left ideal. By definition, $\chi(g) = \operatorname{tr}(\mathbf{L}(g) \mid W)$, where $\mathbf{L}(g) \mid W$ is the restriction of $\mathbf{L}(g)$ to W. Let $\mathbf{P}$ be the projection operator $\mathbf{P}x = xc$, $x \in R_G$. Then $\mathbf{P}w = w$ for $w \in W$ and $\mathbf{P}w' = \theta$ for $w' \in W'$. Clearly $\mathbf{L}(g)\mathbf{P}x = \mathbf{L}(g)w$ for $x = w + w' \in R_G$. Thus $\chi(g) = \operatorname{tr}(\mathbf{L}(g)\mathbf{P})$, where $\mathbf{L}(g)\mathbf{P}$ is defined on R_G. We compute this trace using the natural basis $\{k : k \in G\}$ for R_G. We have

$$\mathbf{L}(g)\mathbf{P}k = gkc = \sum_{h \in G} c(h) \cdot gkh = \sum_{h \in G} c(k^{-1}g^{-1}h) \cdot h, \tag{9.2}$$

so the $c(k^{-1}g^{-1}h)$ are the matrix elements of $\mathbf{L}(g)\mathbf{P}$ in the natural basis. Summing the diagonal elements we obtain the lemma. Q.E.D.

We apply this result to S_n. Let T be a Young tableau corresponding to the Young frame $\{\lambda_j\}$. Then T defines a primitive idempotent

$$c = \sum_{p,q} \delta_q pq, \tag{9.3}$$

where p runs over all row permutations of T and q runs over all column permutations. Thus, $c = \sum c(s) \cdot s$, where $c(pq) = \delta_q$ and $c(s) = 0$ if s is not a pq. Let $\tilde{T}$ be the tableau obtained from T by interchanging rows and columns (see Fig. 9.1).

$$\underset{T}{\begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 7 & 8 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}} \qquad \underset{\tilde{T}}{\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 7 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}}$$

FIGURE 9.1

Clearly, the row permutations p of T are the column permutations $\tilde{q}$ of $\tilde{T}$ and the column permutations q of T are the row permutations $\tilde{p}$ of $\tilde{T}$. Thus the essential idempotent $\tilde{c}$ corresponding to $\tilde{T}$ is

$$\tilde{c} = \sum_{\tilde{p},\tilde{q}} \delta_{\tilde{q}} \tilde{p}\tilde{q} = \sum_{p,q} \delta_p qp = \sum_s \tilde{c}(s) \cdot s. \tag{9.4}$$

If $s = pq$ then $s^{-1} = q^{-1}p^{-1}$, $\delta_{p^{-1}} = \delta_p$, and $\tilde{c}(s^{-1}) = \delta_p$. If s is not a pq then $\tilde{c}(s^{-1}) = 0$. This proves the following:

$$c(s) = \delta_p \delta_q \tilde{c}(s^{-1}) = \delta_s \tilde{c}(s^{-1}) \tag{9.5}$$

for all $s \in S_n$. (Note that $\delta_p \delta_q = \delta_s$ if $s = pq$.)

If T is a standard tableau with frame $\{\lambda_j\}$ then $\tilde{T}$ is a standard tableau with frame $\{\tilde{\lambda}_j\}$ and this is a 1–1 relationship between standard tableaux corresponding to these two frames. Hence, by Theorem 4.2 the reps $\{\lambda_j\}$ and $\{\tilde{\lambda}_j\}$ have the same dimension f. By Lemma 4.6 the Young elements $fc/n!$ and $f\tilde{c}/n!$ are generating idempotents for the reps $\{\lambda_j\}$ and $\{\tilde{\lambda}_j\}$, respectively. Applying Lemma 9.12 we obtain

$$\chi(s) = \frac{f}{n!} \sum_{t \in S_n} c(t^{-1}s^{-1}t), \qquad \tilde{\chi}(s) = \chi(s^{-1}) = \frac{f}{n!} \sum_t \tilde{c}(t^{-1}st), \tag{9.6}$$

$$\tilde{c}(t^{-1}st) = \delta_{t^{-1}st}c(t^{-1}s^{-1}t)$$

for the characters of these reps. But $\delta_{t^{-1}st} = \delta_s$, so

$$\tilde{\chi}(s) = \delta_s \chi(s). \tag{9.7}$$

This proves $\{\tilde{\lambda}_j\} \cong \{1^n\} \otimes \{\lambda_j\}$.

Theorem 9.10. The multiplicity of the alternating rep $\{1^n\}$ in $\{\lambda_j\} \otimes \{\mu_j\}$ is zero unless $\{\mu_j\} \cong \{\tilde{\lambda}_j\}$, in which case the multiplicity is one. Here $\{\tilde{\lambda}_j\}$ is the frame obtained from $\{\lambda_j\}$ by interchanging rows and columns.

The frames $\{\lambda_j\}$, $\{\tilde{\lambda}_j\}$ are said to be **conjugate** and their corresponding reps are **conjugate reps.** If $\{\lambda_j\} \cong \{\tilde{\lambda}_j\}$ then $\{\lambda_j\}$ is **self-conjugate.**

Lemma 9.12, relating an idempotent in the group ring to the character of the group rep it generates, is very useful in applied problems. To illustrate this we reexamine the meaning of an induced rep of a finite group G as defined in Section 3.5. If H is a proper subgroup of G we can regard R_H as a subspace (not a subalgebra) of the group ring R_G. Let c be a primitive idempotent in R_H. Then under the action of the left regular rep of H the subspace $R_H c$ of R_H determines an irred rep $\mathbf{T}$ of H with character

$$\chi(h) = \sum_{k \in H} c(k^{-1}h^{-1}k), \qquad c = \sum_{h \in H} c(h) \cdot h. \tag{9.8}$$

All simple characters of H can be so obtained. Now c is also an idempotent in R_G, though not necessarily primitive. Thus, under the action of the left regular rep of G the left ideal $R_G c$ determines a rep $\mathbf{T}'$ of G with character

$$\chi'(g) = \sum_{t \in G} \dot{c}(t^{-1}g^{-1}t), \tag{9.9}$$

where $\dot{c}(t) = c(t)$ if $t \in H$ and $\dot{c}(t) = 0$ if $t \notin H$. Let

$$g_1 H, \quad g_2 H, \ldots, \quad g_m H, \quad n(G) = m \cdot n(H),$$

be the distinct left cosets of H in G, where $g_1 = e$. Then any $t \in G$ can be written uniquely in the form $t = g_j h$ for some $h \in H$. Thus from (9.8),

$$\chi'(g) = \sum_{j=1}^{m} \sum_{h \in H} \dot{c}(h^{-1}g_j^{-1}g^{-1}g_j h) = \sum_{j=1}^{m} \dot{\chi}(g_j^{-1}gg_j)$$

$$\dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases} \tag{9.10}$$

Now $\dot{\chi}(h^{-1}gh) = \dot{\chi}(g)$ for all $h \in H$, so

$$\sum_h \dot{\chi}(h^{-1}g_j^{-1}gg_j h) = n(H)\dot{\chi}(g_j^{-1}gg_j)$$

and

$$\chi'(g) = \frac{1}{n(H)} \sum_{t \in G} \dot{\chi}(tgt^{-1}), \tag{9.11}$$

where $t^{-1} = g_j h$. This expression for $\chi'(g)$ is identical with expression (5.27), Section 3.5, for the induced character χ^G. Thus the rep of G defined by the primitive idempotent c in R_H is equivalent to the induced rep $\mathbf{T}^G$.

Certain induced reps of the symmetric groups are of great importance in atomic physics. We can consider the direct product group $S_n \times S_m$ as a subgroup of S_{n+m}. (If we think of S_{n+m} as the permutation group on $n + m$ letters, then S_n permutes the first n letters alone and S_m permutes the last m letters). The irred reps of $S_n \times S_m$ are of the form $\{\lambda_j\} \times \{\mu_k\}$, where $\{\lambda_j\}$, $\{\mu_k\}$ are irred reps of S_n, S_m, respectively. The rep $\{\lambda_j\} \boxtimes \{\mu_k\}$ of S_{n+m} induced in the sense of Frobenius from $\{\lambda_j\} \times \{\mu_k\}$ is called the **outer product** of $\{\lambda_j\}$ and $\{\mu_k\}$ (Hamermesh [1]). Theorem 3.9 yields the following result.

Lemma 9.13. The multiplicity of the rep $\{\rho_l\}$ of S_{n+m} in the outer product $\{\lambda_j\} \boxtimes \{\mu_k\}$ equals the multiplicity of $\{\lambda_j\} \times \{\mu_k\}$ in the restriction of $\{\rho_l\}$ to the subgroup $S_n \times S_m$.

Simple algorithms have been developed which enable one to decompose any outer product $\{\lambda_j\} \boxtimes \{\mu_k\}$ into a direct sum of irred reps of S_{n+m}. For a discussion of these procedures see the work of Littlewood [1] or Hamermesh [1]. Here we merely show the importance of outer products for quantum mechanics.

Outer products allow us to decompose a tensor product of irred reps of $U(k)$ [or $GL(k)$ or $SU(k)$] into a direct sum of irred reps, i.e., they enable us to determine the Clebsch–Gordan series for $U(k)$. Consider the irred reps $[\lambda_1, \ldots, \lambda_k]$ and $[\mu_1, \ldots, \mu_k]$ of $U(k)$, where $\lambda_1 + \cdots + \lambda_k = n$ and $\mu_1 + \cdots + \mu_k = m$. We can regard the tensor product rep $[\lambda_j] \otimes [\mu_j]$ of $U(k)$ as defined on a subspace W of $V^{\otimes(n+m)} \simeq V^{\otimes n} \otimes V^{\otimes m}$, where V is a k-dimensional vector space. Here $W = W_1 \otimes W_2$, W_1 is a symmetry class of tensors in $V^{\otimes n}$ corresponding to a tableau with frame $\{\lambda_j\}$, and W_2 is a symmetry class of tensors in $V^{\otimes m}$ corresponding to a tableau with frame $\{\mu_j\}$. It is a consequence of Theorem 4.11 that $W_1 = \hat{c}_1 V^{\otimes n}$, $W_2 = \hat{c}_2 V^{\otimes m}$, where c_1, c_2 are the Young symmetrizers corresponding to $\{\lambda_j\}$ and $\{\mu_j\}$, respectively. (We can assume that the c_i are primitive idempotents.) Now c_1 belongs to the group ring R_n of S_n and c_2 belongs to the group ring R_m of S_m. Since $S_n \times S_m$ is a subgroup of S_{n+m} we can also regard c_1, c_2 as commuting idempotents in R_{n+m}. Thus, $(c_1c_2)^2 = c_1{}^2c_2{}^2 = c_1c_2$ and c_1c_2 is an idempotent in R_{n+m}. This proves $W = \widehat{c_1c_2}V^{\otimes(n+m)}$ and associates the $U(k)$-invariant space of tensors W uniquely with the right ideal $\mathcal{I} = c_1c_2R_{n+m}$. By Theorem 4.11 again, the decomposition of W into $U(k)$-irred subspaces is equivalent to the decomposition of $\mathscr{I}$ into a direct sum of primitive right ideals in R_{n+m}. We have observed above that $\mathscr{I}$ transforms under the right regular rep of S_{n+m} as the outer product $\{\lambda_j\} \boxtimes \{\mu_j\}$. (Our switch from left regular to right regular rep in no way changes this result.) Thus there is a 1–1 correspondence between irred reps $\{\rho_l\}$ of S_{n+m} occurring in the outer product and $U(k)$-irred subspaces of W transforming according to $[\rho_l]$. (To this assertion we must add the proviso that $\{\rho_l\}$ contain at most k rows since otherwise the tensors in the symmetry class $[\rho_l]$ will all be zero.)

Theorem 9.11. The $U(k)$-irred reps occurring in the tensor product $[\lambda_j] \otimes [\mu_j]$, $\sum \lambda_j = n$, $\sum \mu_j = m$, are of the form $[\rho_j]$ where $\sum \rho_j = n + m$. The multiplicity of $[\rho_j]$ in $[\lambda_j] \otimes [\mu_j]$ equals the multiplicity of the rep $\{\rho_j\}$ of S_{n+m} in the outer product $\{\lambda_j\} \boxtimes \{\mu_j\}$.

Corollary 9.5. The multiplicity of $[\rho_j]$ in $[\lambda_j] \otimes [\mu_j]$ equals the multiplicity of the rep $\{\lambda_j\} \times \{\mu_j\}$ of $S_n \times S_m$ in $\{\rho_j\} \mid S_n \times S_m$.

Proof. Immediate from Lemma 9.13.

The above theorem and its corollary demonstrate the importance of the outer product in atomic and elementary particle physics. Use of the outer product enables one to work out the Clebsch–Gordan series for $GL(k)$, $U(k)$, and $SU(k)$ in a straightforward manner. For details concerning the applications see the work of Hamermesh [1] or Wybourne [1].

9.10 Semisimple Lie Algebras

In this chapter we have investigated the rep theory of the classical groups and their Lie algebras. The classical Lie algebras belong to a larger family of Lie algebras called semisimple. Semisimple Lie algebras have been widely studied and there is a vast mathematical literature on their structure and reps. Here we present a number of definitions and results, mostly without proof, to show the relationship between the theory of the classical Lie algebras as presented in this book and the more general theory of semisimple Lie algebras. All of these results are proved in detail in the textbooks by Freudenthal and De Vries [1], Hausner and Schwartz [1], and Jacobson [1]. With the orientation provided here the reader should have no trouble understanding these texts.

Let $\mathcal{G}$ be a Lie algebra. We define a sequence $\{\mathcal{G}^{(n)}\}$ of ideals in $\mathcal{G}$ inductively by $\mathcal{G}^{(1)} = \mathcal{G}$, $\mathcal{G}^{(n+1)} = [\mathcal{G}^{(n)}, \mathcal{G}^{(n)}]$, $n = 1, 2, \ldots$. Clearly, $\mathcal{G}^{(n+1)} \subseteq \mathcal{G}^{(n)}$. We say $\mathcal{G}$ is **solvable** if $\mathcal{G}^{(n)} = \{\theta\}$ for n sufficiently large. We say $\mathcal{G}$ is **semisimple** if it contains no proper solvable ideals and dim $\mathcal{G} > 1$. (An ideal is itself a Lie algebra.) $\mathcal{G}$ is **simple** (dim $\mathcal{G} > 1$) if it contains no proper ideals. Clearly, if $\mathcal{G}$ is simple then it is semisimple. On the other hand, we have the following result.

Theorem 9.12. A Lie algebra $\mathcal{G}$ is semisimple if and only if it can be expressed as a direct sum of simple Lie algebras:

$$\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_k, \qquad \mathcal{G}_j \quad \text{simple.}$$

Simple Lie algebras are the building blocks out of which the semisimple Lie algebras are constructed. Another characterization of semisimple algebras is as follows.

Theorem 9.13 (Cartan's criterion). Let $\gamma_1, \ldots, \gamma_n$ be a basis for $\mathcal{G}$ and define the structure constants c_{ij}^l by

$$[\gamma_i, \gamma_j] = \sum_{l=1}^{n} c_{ij}^l \gamma_l. \tag{10.1}$$

Then $\mathcal{G}$ is semisimple if and only if $\det \mathfrak{F} \neq 0$, where $\mathfrak{F}$ is the matrix with components $\mathfrak{F}_{ik} = \sum_{j,l=1}^{n} c_{ij}^l c_{kl}^j = \mathfrak{F}_{ki}$.

It is straightforward to determine which of the Lie algebras of the complex classical groups are simple or semisimple. The algebra $gl(m, \mathbb{C})$, $m > 1$, is not semisimple, because the set of all multiples of the identity matrix forms a proper solvable ideal. However, the algebras $sl(m, \mathbb{C})$, $m \geq 2$, are simple, as are $sp(m, \mathbb{C})$, $m \geq 1$. The one-dimensional algebra $so(2, \mathbb{C})$ is abelian, but $so(m, \mathbb{C})$ is simple for $m \geq 3$ with the exception of $so(4, \mathbb{C}) \cong sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$, which is semisimple. In the mathematical literature the simple classical Lie algebras are denoted A_m, B_m, C_m, D_m according to the following list

$sl(m+1, \mathbb{C})$	A_m	$m \geq 1$	$m(m+2)$
$so(2m+1, \mathbb{C})$	B_m	$m \geq 2$	$m(2m+1)$
$sp(m, \mathbb{C})$	C_m	$m \geq 3$	$m(2m+1)$
$so(2m, \mathbb{C})$	D_m	$m \geq 4$	$m(2m-1)$.

The last number in each row is the dimension of the corresponding Lie algebra. The algebras B_1, C_1, C_2, D_3 are also simple, but due to the isomorphisms $B_1 \cong C_1 \cong A_1$, $C_2 \cong B_2$, $D_3 \cong A_3$, they are already included on our list. No two algebras in the above list are isomorphic.

The (complex) simple Lie algebras can be classified up to isomorphism. In addition to the four infinite families $A_m - D_m$ there are exactly five simple algebras E_6, E_7, E_8 of dimensions 52, 78, and 133, respectively, F_4 of dimension 52, and G_2 of dimension 14. Thus, with the exception of these five algebras, the **exceptional** algebras, we have already studied all complex simple Lie algebras. The construction of the exceptional Lie algebras is by no means a trivial matter, but these algebras are rarely used in theoretical physics, so we omit their definition and rep theory. (The exceptional algebra G_2 has been applied in atomic spectroscopy, see the work of Racah [1]).

It is an elementary consequence of the Cartan criterion that any real form of a complex semisimple Lie algebra is itself semisimple. Furthermore, the complexification of any real semisimple algebra is also semisimple. By Theorem 9.12 any real semisimple Lie algebra can be expressed as a direct sum of real simple Lie algebras. The real simple algebras have been classified up to isomorphism. They are of two types. Each real algebra of the first type is obtained by considering a complex simple algebra with dimension n as a real algebra with dimension $2n$. Thus every complex simple algebra A_m – $D_m, E_6, E_7, E_8, F_4, G_2$ is a real simple algebra of twice the dimension. The real simple algebras of the second type are real forms of the complex simple algebras. The Lie algebras of the groups given in Table 9.2 constitute all algebras of type two which are real forms of the classical complex algebras.

TABLE 9.2[a]

Complex form	Dimension	Real forms
A_m	$m(m+2)$	$SU(m+1)$
	$m \geq 1$	$SU(m+1-q, q), q = 1, \ldots, [(m+1)/2]$
		$SL(m+1, R), m > 1$
		$SU^*(m+1), m+1$ even
B_m	$m(2m+1)$	$SO(2m+1)$
	$m \geq 2$	$SO(2m+1-q, q), q = 1, \ldots, m$
C_m	$m(2m+1)$	$USp(m)$
	$m \geq 3$	$Sp(m-q, q), q = 1, \ldots, [m/2]$
		$Sp(m, R)$
D_m	$m(2m-1)$	$SO(2m)$
	$m \geq 4$	$SO(2m-q, q), q = 1, \ldots, m$
		$SO^*(2m)$

[a]See Eqs. (10.2)–(10.6).

In Table 9.2

$$SU(p, q) = \{A \in SL(p+q, \mathfrak{C}) : \bar{A}^t G^p A = G^p\},$$

$$G^p_{ij} = \begin{cases} \delta_{ij}, & 1 \leq i \leq p \\ -\delta_{ij}, & p+1 \leq i \leq p+q; \end{cases} \tag{10.2}$$

$$SU^*(2n) = \{A \in SL(2n, \mathfrak{C}) : AJ = J\bar{A}\}, \qquad J = \begin{pmatrix} Z & E_n \\ -E_n & Z \end{pmatrix}; \tag{10.3}$$

$$SO(p, q) = \{A \in SO(p+q, R) : A^t G^p A = G^p\}; \tag{10.4}$$

$$Sp(p, q) = \{A \in Sp(p+q, \mathfrak{C}) : \bar{A}^t H^p A = H^p\}; \tag{10.5}$$

$$H^p = \begin{pmatrix} -E_p & & Z & \\ & E_q & & \\ & & -E_p & \\ Z & & & E_q \end{pmatrix};$$

(10.6) $SO^*(2n) = \{A \in SO(2n, \mathfrak{C}) : \bar{A}^t J A = J\}$.

The symbol $[k]$ is the largest integer $\leq k$.

No two algebras in the above list are isomorphic. In addition to the classical algebras of type two there are 17 more algebras which are real forms of the exceptional Lie algebras.

A Lie group G is **locally simple** if it contains no proper normal local Lie subgroups. The commutator subgroup $G^{(1)}$ of G is the group generated by all elements of the form $ghg^{-1}h^{-1}$, $g, h \in G$. Here $G^{(1)}$ is a Lie subgroup of G with Lie algebra $\mathcal{G}^{(1)} = [\mathcal{G}, \mathcal{G}]$. We define groups $G^{(n)}$ inductively by $G^{(n+1)} = (G^{(n)})^{(1)}$. Then $L(G^{(n+1)}) = \mathcal{G}^{(n+1)} = [\mathcal{G}^{(n)}, \mathcal{G}^{(n)}]$. The group G is **solvable** if $G^{(n)} = \{e\}$ for n sufficiently large. A Lie group is **semisimple** if it has no proper solvable normal Lie subgroup.

Theorem 9.14. A Lie group G is locally simple (semisimple, solvable) if and only if $\mathcal{G} = L(G)$ is simple (semisimple, solvable).

According to this result, Theorem 9.12, and our list of simple Lie algebras, the semisimple Lie groups can be classified completely, at least in a neighborhood of the identity. Moreover, use of topological methods enables one to list all global connected semisimple Lie groups.

The real simple Lie algebras $su(m+1)$ $(m \geq 1)$, $so(2m+1)$ $(m \geq 2)$, $usp(m)$ $(m \geq 3)$, and $so(2m)$ $(m \geq 4)$ are called **compact** since the global connected Lie groups associated with these algebras are all compact. The groups associated with all other real classical simple Lie algebras are noncompact. Each simple algebra $A_m - D_m$, E_6, E_7, E_8, F_4, G_2 has exactly one compact real form.

As a final comment on the theory we mention the **Casimir operator** C. Let $\mathcal{G}$ be a simisimple Lie algebra of matrices with basis $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and let T be a finite-dimensional rep of $\mathcal{G}$. If the $\{\mathcal{A}_j\}$ satisfy the commutation relations (10.1) then so do the operators $T_j = T(\mathcal{A}_j)$. The Casimir operator associated with T is

$$C = \sum_{j,k=1}^{n} (\mathfrak{F}^{-1})_{jk} T_j T_k, \tag{10.7}$$

where $\mathfrak{F}$ is the symmetric nonsingular matrix defined in Theorem 9.13. To demonstrate the significance of C we introduce the **Killing form** for $\mathcal{G}$,

$$(\mathcal{A}, \mathcal{B}) = \mathrm{tr}(\mathrm{Ad}\,\mathcal{A}\,\mathrm{Ad}\,\mathcal{B}), \qquad \mathcal{A}, \mathcal{B} \in \mathcal{G}, \tag{10.8}$$

where $(\mathrm{Ad}\,\mathcal{A})(\mathcal{C}) = [\mathcal{A}, \mathcal{C}]$ is a linear operator on $\mathcal{G}$ (see Section 5.6). It is easy to show that $(\mathcal{A}_j, \mathcal{A}_k) = \mathfrak{F}_{jk}$, so by Theorem 9.13 the Killing form of $\mathcal{G}$ is nondegenerate. Furthermore,

$$([\mathcal{A}, \mathcal{B}], \mathcal{C}) + (\mathcal{B}, [\mathcal{A}, \mathcal{C}]) = 0,$$

or, exponentiating,

$$(10.9) \qquad (e^{\mathcal{A}}\mathcal{B}e^{-\mathcal{A}}, e^{\mathcal{A}}\mathcal{C}e^{-\mathcal{A}}) = (\exp(\mathrm{Ad}\,\mathcal{A})\mathcal{B}, \exp(\mathrm{Ad}\,\mathcal{A})\mathcal{C}) = (\mathcal{B}, \mathcal{C})$$

for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{G}$.

We leave it to the reader to show that C is defined independent of the basis in $\mathcal{G}$, i.e., if we introduce a new basis $\{\mathcal{B}_j\}$, $\mathcal{B}_j = \sum h_{lj}\mathcal{A}_l$, $\mathcal{F}'_{jk} = (\mathcal{B}_j, \mathcal{B}_k)$, and compute $\sum (\mathcal{F}'^{-1})_{jk}T(\mathcal{B}_j)T(\mathcal{B}_k)$ we get C again. Now set $\mathcal{B}_j = \exp(\mathrm{Ad}\, t\mathcal{C})\mathcal{A}_j$, where $\mathcal{C} \in \mathcal{G}$. The elements $\{\mathcal{B}_j\}$ form a basis for $\mathcal{G}$ and

$$(10.10) \qquad \mathcal{F}'_{jk} = (\mathcal{B}_j, \mathcal{B}_k) = (\mathcal{A}_j, \mathcal{A}_k) = \mathcal{F}_{jk}, \qquad T(\mathcal{B}_j) = e^{tT(\mathcal{C})}T_j e^{-tT(\mathcal{C})}$$

by (10.9). Thus,

$$C = \sum_{j,k} (\mathcal{F}^{-1})_{jk} e^{tT(\mathcal{C})}T_jT_k e^{-tT(\mathcal{C})} = e^{tT(\mathcal{C})}Ce^{-tT(\mathcal{C})}.$$

Differentiating with respect to t and setting $t = 0$ we find

$$(10.11) \qquad T(\mathcal{C})C = CT(\mathcal{C})$$

for all $\mathcal{C} \in \mathcal{G}$. Thus if T is irred, C must be a multiple of the identity operator, $C = aE$. The value of a is a function of T and can be used to label the rep. We have already observed the utility of Casimir operators for the semisimple algebras $sl(2)$ [(3.2), Section 7.3] and $so(3, 1)$ [(3.2), Section 8.3].

The algebras $A_m - D_m$ are said to be of **rank** m and the irred reps of these algebras are designated by m integers $[\lambda_1, \ldots, \lambda_m]$. It can be shown that corresponding to each simple algebra of rank m one can find m independent invariant operators $C_1 = C, C_2, \ldots, C_m$ such that $T(\mathcal{A})C_j = C_jT(\mathcal{A})$ for each rep T of $\mathcal{G}$ and such that the values of the C_j for each irred T completely determine $\lambda_1, \ldots, \lambda_m$ (Racah [2]).

Problems

9.1 Fill in the details of the proof in the text that $Sp(m)$ is connected.

9.2 Prove Theorem 9.7.

9.3 Prove Theorem 9.8 in detail.

9.4 Using weights, decompose the reps $\mathbf{3} \otimes \mathbf{3}$, $\mathbf{3} \otimes \overline{\mathbf{3}}$, $\overline{\mathbf{3}} \otimes \overline{\mathbf{3}}$, and $\mathbf{8} \otimes \mathbf{8}$ of $SU(3)$ into irred reps.

9.5 Consider the electromagnetic interaction as an operator proportional to I^3 which perturbs the strong interactions. Derive the selection rules $I \longrightarrow I, I \pm 1$ for the matrix elements of this operator relating states of different isobaric spin. This theory predicts that to a first approximation, electromagnetic interactions permit transitions only between states whose isobaric spins differ by zero or one.

9.6 Compute the possible terms for the following configurations: (1) two electrons in an *nd* state, (2) three electrons in an *np* state, (3) one electron in an *ns* state and one electron in an *n′p* state.

9.7 Prove: The adjoint rep of a semisimple Lie algebra is faithful. The adjoint rep of a simple Lie algebra is irred.

9.8 Prove: A real compact semisimple Lie algebra has a negative-definite Killing form. (Conversely, a real Lie algebra with negative-definite Killing form is compact, but the verification is more difficult; see the work of Helgason [1, p. 122].)

9.9 Show that the matrix rep

$$g(c) = \begin{pmatrix} 1 & \ln|c| \\ 0 & 1 \end{pmatrix}, \qquad c \neq 0, \quad c \in \mathfrak{C},$$

of $GL(1)$ cannot be expressed as a direct sum of irred reps. Does this example contradict the results of Section 9.1?

9.10 Let $[f_1, \ldots, f_m]$ be an irred rep of $U(m)$ with character $\chi(\epsilon_1, \ldots, \epsilon_m)$ given by Theorem 9.4. We can consider $U(m-1)$ as the subgroup of $U(m)$ such that the character of $[f_1, \ldots, f_m] \mid U(m-1)$ is $\chi(\epsilon_1, \ldots, \epsilon_{m-1}, 1)$. Derive the branching law $[f_1, \ldots, f_m] \mid U(m-1) \cong \sum \oplus [h_1, h_2, \ldots, h_{m-1}]$, where the direct sum is taken over all integers h_j such that $f_1 \geq h_1 \geq f_2 \geq h_2 \geq f_3 \geq \cdots \geq f_{m-1} \geq h_{m-1} \geq f_m$ (see the work of Boerner [1]).