ORDERING THE PRIMITIVE PYTHAGOREAN TRIPLES BY GENERALIZED PELLIAN SEQUENCES

While it is well-known that the Pell numbers: 0, 1, 2, 5, 12, 29, 70, . . . generate the Pythagorean triples with consecutive legs, it seems to be a great secret that the rest of the primitive triples can be generated, ordered and largely sorted by leg difference using similar sequences. But before getting to them, let's review. Recall that the primitive Pythagorean triples are the sets $\{a, b, c\}$ that each represent the lengths of the sides of a a right triangle that are all whole numbers with $\gcd(a,b,c) = 1$. Letting b be the even leg, each of these triples can be represented uniquely by a pair of positive integers u and v, where $v > u$, $\gcd(u,v) = 1$, u and v are of opposite parity and $\{a, b, c\} = \{v^2 - u^2, 2uv, v^2 + u^2\}$. Now, taking the Pell numbers in consecutive pairs starting at 1, 2 and substituting them for u and v so that $u = 1$, $v = 2$; then $u = 2$, $v = 5$; then $u = 5$, $v = 12$, and so on, gives us the triples {3, 4, 5}; {21, 20, 29}; {119, 120, 169}, . . . Note that the leg differences are 1 ($d = | b - a | = 1$).

For primitive triples, the next greater leg difference happens to be 7. There are actually two sequences, having the same recursion relation, $P_{n+2} = 2P_{n+1} + P_n$, as the Pell numbers, that generate all of the primitive triples with $d = 7$ without redundancy. And they are: 2, 3, 8, 19, 46, . . . and 1, 4, 9, 22, 53, . . . Substituting their consecutive pairs of terms for u and v yields the desired triples:

{5, 12, 13}, {55, 48, 73}, {297, 304, 425}, {1755, 1748, 2477}, . . .

{15, 8, 17}, {65, 72, 97}, {403, 396, 565}, {2325, 2332, 3293}, . . .

Again, note that the leg differences are 7, and observe that we can represent the initial two terms of the sequences as n, $n + m$ and $n - m$, $3n - 2m$, where $n = 2$ and $m = 1$. In this case, $d = |\, b - a \,| = 2n^2 - m^2 = 2(2)^2 - 1^2 = 7$.

After $d = 7$, the next greater possible difference corresponds to when $n = 3$ and $m = 1$. Thus, $d = 2n^2 - m^2 = 2(3)^2 - 1^2 = 17$. And the twin sequences that generate the primitive triples with this leg difference are

$$n,\ n + m,\ \ldots \quad = \quad 3, 4, 11, 26, 63, \ldots$$

$$n - m,\ 3n - 2m,\ \ldots = \quad 2, 7, 16, 39, 94, \ldots$$

But sometimes there are multiple permissible n and m values that give the same difference. For example, $n = 8,\ m = 3$ and $n = 10,\ m = 9$ both correspond to $d = 2(8)^2 - 3^2 = 2(10)^2 - 9^2 = 119$. Each pair of n and m values initiates twin sequences. So it requires four infinite sequences to generate the primitive triples with $d = 119$.

If we focus on the leg difference from a distance, everything falls into place. Fixing the leg differences of primitive triples results in Pell-type equations of the form $2n^2 - m^2 = d$. And the solutions to such equations are recursive sequences (generalized Pellian sequences). It is easy to show that our recursion relation preserves leg difference (i.e., all consecutive pairs of terms satisfy the Pell-type equation if the initial terms do).

PROOF: Observe that if P_n and P_{n+1} are any two consecutive terms of any of our sequences, including the Pell numbers themselves, the leg difference of the triple they yield would be $d = |\,(P_{n+1})^2 - (P_n)^2 - 2P_n\,P_{n+1}\,|$.

By the recursion, $P_{n+2} = 2P_{n+1} + P_n$, and the leg difference of the triple generated by P_{n+1} and P_{n+2} would be $d = |\,(P_{n+2})^2 - (P_{n+1})^2 - 2P_{n+1}\ P_{n+2}\,|$

$= |\,(2P_{n+1} + P_n)^2 - (P_{n+1})^2 - 2P_{n+1}(2P_{n+1} + P_n)\,|$

$= |\,4(P_{n+1})^2 + 4P_nP_{n+1} + (P_n)^2 - (P_{n+1})^2 - 4(P_{n+1})^2 - 2P_nP_{n+1}\,|$

$= |\,(P_n)^2 - (P_{n+1})^2 + 2P_n\ P_{n+1}\,|$

$= |\,(P_{n+1})^2 - (P_n)^2 - 2P_n\ Pn_{+1}\,|$.

QED

So, let me sum up the overall result and prove that it holds true.

THEOREM: *All of the primitive Pythagorean triples can be generated, ordered and largely sorted without redundancy by substitution into* $\{v^2 - u^2,\ 2uv,\ v^2 + u^2\}$ *of consecutive pairs of terms of the Pell numbers and similar sequences formed by the same recursion relation,* $P_{n+2} = 2Pn_{+1} + P_n$, *and initial values* $n,\ n + m$ *and* $n - m,\ 3n - 2m$, *where* n *and* m *are positive integers,* $n > m,\ gcd(m,n) = 1,$ *and* m *is odd.* (The leg difference of triples generated from both sequences is $d = 2n^2 - m^2$. And the Pell numbers can be considered the special case of a singleton sequence: $n, n + m, \ldots,$ where $n = m = 1$.)

Since I've shown that the recursion preserves leg differences and it is easy to verify that the $\gcd(n,n + m) = \gcd(n - m ,3n - 2m) = 1$ and that the recursion preserves this relative primality as well as the opposite parities, it remains for me to prove that the generation of the primitive triples is exhaustive and without redundancy.

PROOF: Let $\{a, b, c\}$ be an arbitrary primitive Pythagorean triple. There exists a pair of positive integers, u and v, such that $v > u$, $\gcd(u,v) = 1$, u and v are of opposite parity, and $\{a, b, c\} = \{v^2 - u^2, 2uv, v^2 + u^2\}$.

Case 1: If $v < 2u$, then $n = u, m = v - u$. u, v are the initial terms $n, n + m$.

Case 2: If $v = 2u$, then $u = 1, v = 2$, the second and third Pell numbers.

Case 3: If $v > 3u$, then $n = v - 2u, m = v - 3u$. u, v are the initial terms $n - m$, $3n - 2m$.

Case 4: If $2u < v < 3u$, then u, v are advanced terms in one of the sequences and can be determined by backtracking, via the reverse recursion relation $P_{n-2} = P_n - 2P_{n-1}$ for as long as the terms remain positive and strictly decreasing, until one of the following occurs.

Case 4a: If $P_{n-2} < 0$, then $P_n < 2P_{n-1}$ and u, v are advanced terms of the sequence $n, n + m, \ldots$ where $n = P_{n-1}$, $m = P_n - P_{n-1}$.

Case 4b: If $P_{n-2} = 0$, then $P_n = 2P_{n-2}$ and u, v are advanced terms of the Pell numbers.

Case 4c: If $P_{n-2} > P_{n-1}$, then $P_n > 3P_{n-1}$ and u, v are advanced terms of the sequence $n - m$, $3n - 2m, \ldots$ where $n = P_n - 2P_{n-1}$, $m = P_n - 3P_{n-1}$.

This covers all possibilities and proves that u and v must be in some specific position in one of the sequences.

QED

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