Chapter 8

The Lorentz Group and Its Representations

8.1 The Homogeneous Lorentz Group

The homogeneous Lorentz group in four-space $L(4)$ is the set of all 4×4 real matrices Λ such that $\Lambda^t G \Lambda = G$, where

$$G = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}. \tag{1.1}$$

It is straightforward to verify that $L(4)$ satisfies the group axioms. In particular, if $\Lambda \in L(4)$ then $\Lambda^{-1} = G\Lambda^t G \in L(4)$. Also, E and G belong to $L(4)$. If $\Lambda \in L(4)$ then so is $-\Lambda$ and $\Lambda^t = G\Lambda^{-1}G$.

If $x = (x_1, \ldots, x_4)$ and $y = (y_1, \ldots, y_4)$ are column four-vectors such that $y = \Lambda x$, $\Lambda \in L(4)$, then

$$y_1^2 + y_2^2 + y_3^2 - y_4^2 = y^t G y = (\Lambda x)^t G(\Lambda x) = x^t(\Lambda^t G \Lambda)x = x^t G x.$$

Thus the form $x^t G x$ is invariant under the action of Λ. Conversely, if Λ is a 4×4 real matrix such that $(\Lambda x)^t G(\Lambda x) = x^t G x$ for all real four-vectors x, then $\Lambda \in L(4)$. By the methods of Section 5.4 it is easy to show that $L(4)$ is a linear Lie group with Lie algebra

$$so(3, 1) = \{\mathcal{A} : \mathcal{A}^t = -G\mathcal{A}G\}, \tag{1.2}$$

[see (10.4), Section 9.10]. Note that $G^2 = E$. Any element of $so(3, 1)$ can be written in the form

$$(1.3) \qquad \mathfrak{A} = \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 & \beta_1 \\ \alpha_3 & 0 & -\alpha_1 & \beta_2 \\ -\alpha_2 & \alpha_1 & 0 & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{pmatrix},$$

where the real parameters α_j, β_j are arbitrary. Thus $so(3, 1)$ is six-dimensional and $L(4)$ is a six-parameter Lie group. The exponential mapping $\mathfrak{A} \to \exp \mathfrak{A}$ maps $so(3, 1)$ homeomorphically onto a neighborhood of the identity in $L(4)$.

As a basis for $so(3, 1)$ we choose the matrices $\mathfrak{L}_j$, $j = 1, 2, 3$, defined by setting $\alpha_j = 1$ and all other parameters zero, and the matrices $\mathfrak{B}_j$, $j = 1, 2, 3$, defined by setting $\beta_j = 1$ and all other parameters zero. The commutation relations are

$$(1.4) \qquad \begin{aligned} &[\mathfrak{L}_i, \mathfrak{L}_j] = \sum_k \epsilon_{ijk}\mathfrak{L}_k, \qquad [\mathfrak{B}_i, \mathfrak{L}_j] = \sum_k \epsilon_{ijk}\mathfrak{B}_k \\ &[\mathfrak{B}_i, \mathfrak{B}_j] = -\sum_k \epsilon_{ijk}\mathfrak{L}_k, \qquad 1 \le i, j \le 3, \end{aligned}$$

where ϵ_{ijk} is the completely skew-symmetric tensor such that $\epsilon_{123} = +1$.

Note that $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3$ form a basis for a subalgebra of $so(3, 1)$ isomorphic to $so(3)$. Furthermore, the matrices

$$(1.5) \qquad \begin{pmatrix} & & & 0 \\ & R & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad R \in O(3),$$

form a Lie subgroup of $L(4)$ isomorphic to $O(3)$. For convenience we identify this subgroup with $O(3)$. The corresponding subalgebra is spanned by the matrices $\mathfrak{L}_j$.

The one-parameter subgroups $\exp \varphi\mathfrak{L}_j$ all belong to $SO(3)$ [see (1.4), Chapter 7]. On the other hand, a simple computation yields

$$(1.6) \qquad \exp b\mathfrak{B}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh b & \sinh b \\ 0 & 0 & \sinh b & \cosh b \end{pmatrix} \in L(4),$$

with similar results for $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Since the matrix elements of (1.6) are not bounded it follows that $L(4)$ is not a compact group.

We will find it convenient to complexify the Lie algebra. A useful basis for the complexified algebra is

$$(1.7)\qquad \begin{aligned} \mathcal{L}^{\pm} &= \pm\mathcal{L}_2 + i\mathcal{L}_1, & \mathcal{L}^3 &= -i\mathcal{L}_3,\\ \mathcal{B}^{\pm} &= \pm\mathcal{B}_2 + i\mathcal{B}_1, & \mathcal{B}^3 &= -i\mathcal{B}_3. \end{aligned}$$

The commutation relations are

(1.8)

$$\begin{aligned} &[\mathcal{L}^+, \mathcal{L}^-] = 2\mathcal{L}^3, \quad [\mathcal{L}^3, \mathcal{L}^{\pm}] = \pm\mathcal{L}^{\pm}, \quad [\mathcal{L}^3, \mathcal{B}^{\pm}] = \pm\mathcal{B}^{\pm},\\ &[\mathcal{L}^+, \mathcal{B}^3] = -\mathcal{B}^+, \quad [\mathcal{L}^-, \mathcal{B}^3] = \mathcal{B}^-, \quad [\mathcal{L}^+, \mathcal{B}^-] = [\mathcal{B}^+, \mathcal{L}^-] = 2\mathcal{B}^3,\\ &[\mathcal{L}^+, \mathcal{B}^+] = [\mathcal{L}^-, \mathcal{B}^-] = [\mathcal{L}^3, \mathcal{B}^3] = Z,\\ &[\mathcal{B}^3, \mathcal{B}^{\pm}] = \mp\mathcal{L}^{\pm}, \quad [\mathcal{B}^+, \mathcal{B}^-] = -2\mathcal{L}^3. \end{aligned}$$

Note that $\mathcal{L}^{\pm}, \mathcal{L}^3$ form a basis for the subalgebra $sl(2)$ of the complexified Lie algebra.

A third useful basis is obtained by choosing

$$(1.9)\qquad \begin{aligned} \mathcal{C}^{\pm} &= \tfrac{1}{2}(\mathcal{L}^{\pm} + i\mathcal{B}^{\pm}), & \mathcal{D}^{\pm} &= \tfrac{1}{2}(\mathcal{L}^{\pm} - i\mathcal{B}^{\pm}),\\ \mathcal{C}^3 &= \tfrac{1}{2}(\mathcal{L}^3 + i\mathcal{B}^3), & \mathcal{D}^3 &= \tfrac{1}{2}(\mathcal{L}^3 - i\mathcal{B}^3). \end{aligned}$$

Then the commutation relations become

$$(1.10)\qquad \begin{aligned} &[\mathcal{C}^3, \mathcal{C}^{\pm}] = \pm\mathcal{C}^{\pm}, \quad [\mathcal{C}^+, \mathcal{C}^-] = 2\mathcal{C}^3,\\ &[\mathcal{D}^3, \mathcal{D}^{\pm}] = \pm\mathcal{D}^{\pm}, \quad [\mathcal{D}^+, \mathcal{D}^-] = 2\mathcal{D}^3, \quad [\mathcal{C}, \mathcal{D}] = Z, \end{aligned}$$

i.e., any $\mathcal{C}$ matrix commutes with any $\mathcal{D}$ matrix. It follows from (1.10) that $so(3, 1)^c \cong sl(2) \oplus sl(2)$. This result holds only for the complexified Lie algebra. It is *not* true that $so(3, 1)$ is the direct sum of two nontrivial real Lie algebras.

Let us return to an examination of the group $L(4)$. If $\Lambda \in L(4)$ then $\Lambda^t G\Lambda = G$. Taking the determinant of this expression we find $(\det \Lambda)^2 = 1$, or $\det \Lambda = \pm 1$. Both signs are possible since $E, G \in L(4)$, with $\det E = -\det G = 1$.

In terms of components, $\Lambda = (\Lambda_{ik}) \in L(4)$ provided

$$(1.11)\qquad \sum_{h=1}^{4} \Lambda_{hj} G_{hh} \Lambda_{hl} = G_{jl}, \qquad 1 \le j, l \le 4.$$

For $j = l = 4$ this reads

$$(1.12)\qquad \sum_{h=1}^{3} \Lambda_{h4}^2 - \Lambda_{44}^2 = -1.$$

(Also $\sum \Lambda_{4h}^2 - \Lambda_{44}^2 = -1$ since $\Lambda^t \in L(4)$. Thus $|\Lambda_{44}| \ge 1$, so $\Lambda_{44} \ge 1$ or $\Lambda_{44} \le -1$. If $\Lambda_{44} \ge 1$, then Λ is **forward-timelike,** otherwise Λ is **backward-timelike.** Since E is forward-timelike and G is backward-timelike it is clear that both cases occur. The forward-timelike matrices form a subgroup of $L(4)$. Indeed, if Λ and Λ' are forward-timelike then $(\Lambda\Lambda')_{44} = \sum_{j=1}^{3} \Lambda_{4j}\Lambda'_{j4} + \Lambda_{44}\Lambda'_{44} > 0$ since $|\sum \Lambda_{4j}\Lambda'_{j4}| \leq [\sum \Lambda_{4j}^2 \sum \Lambda'^2_{j4}]^{1/2} \leq [(\Lambda_{44}^2 - 1)(\Lambda'^2_{44} - 1)]^{1/2} < \Lambda_{44}\Lambda'_{44}$. Similarly it is easy to check that the inverse $\Lambda^{-1} = G\Lambda^t G$ of a forward-timelike transformation is forward-timelike.

Using these results we can separate $L(4)$ into four components:

(1.13)

$$L^{\uparrow +}: \Lambda_{44} \geq 1, \quad \det \Lambda = +1 \qquad L^{\uparrow -}: \Lambda_{44} \geq 1, \quad \det \Lambda = -1,$$
$$L^{\downarrow +}: \Lambda_{44} \leq -1, \quad \det \Lambda = +1 \qquad L^{\downarrow -}: \Lambda_{44} \leq -1, \quad \det \Lambda = -1.$$

Every element of $L(4)$ lies in a unique component. It is easy to show that the components are disconnected in the sense that no analytic curve in $L(4)$ can connect two distinct components. The component $L^{\uparrow +}$ is itself a group, the **proper Lorentz group.** It is clear that $L^{\uparrow +}$ contains the connected component of the identity in $L(4)$.

Lemma 8.1.
(a) $L^{\uparrow -} = SL^{\uparrow +} = L^{\uparrow +}S$, where $S = -G$.
(b) $L^{\downarrow +} = (-E)L^{\uparrow +}$.
(c) $L^{\downarrow -} = GL^{\uparrow +} = L^{\uparrow +}G$.

Proof. (a) Clearly $S = -G \in L^{\uparrow -}$. If $\Lambda \in L^{\uparrow +}$ then $\det(S\Lambda) = \det(\Lambda S) = \det S = -1$ and $(S\Lambda)_{44} = (\Lambda S)_{44} = \Lambda_{44} \geq 1$, so $S\Lambda$ and ΛS belong to $L^{\uparrow -}$. Thus $L^{\uparrow -} \supseteq SL^{\uparrow +}$, $L^{\uparrow -} \supseteq L^{\uparrow +}S$. Conversely if $\Lambda \in L^{\uparrow -}$ then $S\Lambda$ and ΛS belong to $L^{\uparrow +}$. Setting $S\Lambda = \Lambda_1$, $\Lambda S = \Lambda_2$ and using the relation $S^2 = E$, we obtain $\Lambda = S\Lambda_1 = \Lambda_2 S$. Therefore, $L^{\uparrow -} = SL^{\uparrow +} = L^{\uparrow +}S$. Parts (b) and (c) are proved in the same manner. Q.E.D.

The matrices S, $-E$ and G are of special importance in the theory of $L(4)$. Here S is called **space inversion,** G is **time inversion,** and $-E = SG = GS$ is **total inversion.** We will discuss the physical significance of these names in the next section.

It follows from the lemma that a parametrization of the whole group can be obtained directly from a parametrization of the proper Lorentz group $L^{\uparrow +}$. We can choose local coordinates for $L^{\uparrow +}$ by merely selecting six independent matrix elements. However, the following construction yields a more useful coordinate system.

Lemma 8.2. Let $\Lambda \in L^{\uparrow +}$. Then $\Lambda \in SO(3)$ if and only if $\Lambda_{44} = +1$.

Proof. From (1.12), $\sum_{h=1}^{3} \Lambda_{h4}^2 = \sum_{h=1}^{3} \Lambda_{4h}^2 = \Lambda_{44}^2 - 1$. Since $\Lambda \in L^{\uparrow +}$ we have $\Lambda_{44} \geq 1$. Thus $\Lambda_{h4} = \Lambda_{4h} \equiv 0$, $1 \leq h \leq 3$, if and only if $\Lambda_{44} = 1$. By (1.11), $\Lambda_{44} = 1$ if and only if Λ takes the form (1.5). Q.E.D.

Lemma 8.3. Let $\Lambda, \Lambda' \in L^{\uparrow +}$ and suppose $\Lambda e = \Lambda' e$, where

$$e = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then there exists a unique $R \in SO(3)$ such that $\Lambda = \Lambda' R$. Conversely, if $\Lambda' \in L^{\uparrow +}$ and $R \in SO(3)$ such that $\Lambda = \Lambda' R$ then $\Lambda e = \Lambda' e$. We are considering $SO(3)$ as the subgroup of matrices (1.5).

Proof. If $\Lambda e = \Lambda' e$ for $\Lambda, \Lambda' \in L^{\uparrow +}$ then $R = (\Lambda')^{-1}\Lambda \in L^{\uparrow +}$ and $Re = e$. Thus $R_{h4} = 0$ for $1 \le h \le 3$ and $R_{44} = 1$. By the preceding lemma, $R \in SO(3)$.

Conversely, if $\Lambda' \in L^{\uparrow +}$ and $R \in SO(3)$ then $Re = e$ and $\Lambda' Re = \Lambda' e$. Q.E.D.

Theorem 8.1. Every $\Lambda \in L^{\uparrow +}$ can be represented in the form

$$\Lambda = R_1(\exp b\mathcal{B}_3)R_2, \qquad R_1, R_2 \in SO(3).$$

Proof. It is obvious that all elements of the form $R_1(\exp b\mathcal{B}_3)R_2$ lie in $L^{\uparrow +}$. We will show that such elements exhaust $L^{\uparrow +}$. Suppose $\Lambda \in L^{\uparrow +}$. Then

$$\Lambda e = \Lambda \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda_{14} \\ \Lambda_{24} \\ \Lambda_{34} \\ \Lambda_{44} \end{pmatrix}, \qquad \Lambda_{44} \ge 1.$$

If $\Lambda_{44} = 1$ then $\Lambda \in SO(3)$ by Lemma 8.2 and the theorem follows with $b = 0$. If $\Lambda_{44} > 1$ then

$$\Lambda_{14}^2 + \Lambda_{24}^2 + \Lambda_{34}^2 = \Lambda_{44}^2 - 1 = r^2 > 0, \tag{1.14}$$

where we assume $r > 0$. Since $\Lambda_{44}^2 - r^2 = 1$ there exists a unique number $b > 0$ such that $r = \sinh b$, $\Lambda_{44} = \cosh b$. Indeed, $b = \ln[\Lambda_{44} + (\Lambda_{44}^2 - 1)^{1/2}]$. Then (1.6) implies

$$(\exp b\mathcal{B}_3)e = \begin{pmatrix} 0 \\ 0 \\ r \\ \Lambda_{44} \end{pmatrix}.$$

According to (1.14) there exist spherical coordinates r, θ_1, φ_1, such that

$$\Lambda_{14} = r \sin\theta_1 \cos\varphi_1, \qquad \Lambda_{24} = r \sin\theta_1 \sin\varphi_1, \qquad \Lambda_{34} = r\cos\theta_1.$$

It follows from (1.20), Chapter 7, that the matrix $R_1 = R(\varphi_1 + \pi/2, \theta_1, 0)$ (Euler parameters) satisfies

$$R_1 \begin{pmatrix} 0 \\ 0 \\ r \\ \Lambda_{44} \end{pmatrix} = \begin{pmatrix} \Lambda_{14} \\ \Lambda_{24} \\ \Lambda_{34} \\ \Lambda_{44} \end{pmatrix}.$$

Clearly, $R_1 \exp b\mathfrak{B}_3 \in L^{\uparrow +}$ and $R_1(\exp b\mathfrak{B}_3)e = \Lambda e$. By Lemma 8.3 there exists a unique $R_2 \in SO(3)$ such that $\Lambda = R_1(\exp b\mathfrak{B}_3)R_2$. Q.E.D.

If Λ is not an element of $SO(3)$ then the preceding factorization is unique. If the Euler parameters of R_2 are $\varphi_2, \theta_2, \psi_2$ we have

(1.15) $$\Lambda = R_1(\varphi_1 + \tfrac{1}{2}\pi, \theta_1, 0)(\exp b\mathfrak{B}_3)R_2(\varphi_2, \theta_2, \psi_2)$$

and the six parameters $\varphi_1, \theta_1, b, \varphi_2, \theta_2, \psi_2$ serve as coordinates for Λ. If $\theta_1, \theta_2 = 0, \pi$ these coordinates are not 1–1. Similarly, if $\Lambda \in SO(3)$ then $b = 0$ and only the product R_1R_2 is prescribed, not the individual factors. However, those points at which the coordinates are not 1–1 form a lower-dimensional manifold on the group and do not affect the invariant measure.

It follows from (1.15) that any $\Lambda \in L^{\uparrow +}$ can be connected to the identity element by an analytic curve lying entirely in $L^{\uparrow +}$. Indeed we can choose the curve $(t\varphi_1, \ldots, t\psi_2)$, $0 \le t \le 1$. Thus $L^{\uparrow +}$ coincides with the connected component containing the identity in $L(4)$. This proves that $L(4)$ consists of four connected components. The Lie algebra yields information only about $L^{\uparrow +}$. To study the other three connected components we make use of Lemma 8.1.

In Section 7.1 we showed that $SU(2)$ was a double covering group of $SO(3)$. There is a similar relationship between $SL(2) = SL(2, \mathfrak{C})$ and $L^{\uparrow +}$. Indeed, $sl(2)$ considered as a six-dimensional *real* Lie algebra is isomorphic to $so(3, 1)$. Thus, the real Lie groups $SL(2)$ and $L^{\uparrow +}$ are locally isomorphic. To show this we recall that $sl(2)$ consists of all 2×2 complex matrices $\mathfrak{a}$ with trace zero:

(1.16) $$\mathfrak{a} = \begin{pmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{pmatrix}, \qquad z_j \in \mathfrak{C}.$$

Writing $z_j = x_j + iy_j$, we see that $sl(2)$ is a six-dimensional Lie algebra over the reals. As a basis for $sl(2)$ we shoose the matrices $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ [(1.8), Chapter 7] and

(1.17) $$\mathfrak{F}_1 = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \qquad \mathfrak{F}_2 = \begin{pmatrix} 0 & -\frac{1}{2}i \\ \frac{1}{2}i & 0 \end{pmatrix}, \qquad \mathfrak{F}_3 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

These matrices satisfy the commutation relations (1.4), with $\mathfrak{L}_j$ replaced by $\mathcal{J}_j$ and $\mathfrak{B}_j$ by $\mathfrak{F}_j$.

To explicitly exhibit the global relationship between $SL(2)$ and $L^{\uparrow +}$ we consider the four-dimensional space S of all 2×2 skew-Hermitian matrices $\mathcal{S} = -\bar{\mathcal{S}}^t$. Each such matrix can be uniquely written as

$$\mathcal{S} = \begin{pmatrix} i(x_4 - x_3) & -x_2 + ix_1 \\ x_2 + ix_1 & i(x_4 + x_3) \end{pmatrix}, \quad x_j \text{ real.} \tag{1.18}$$

[These matrices form the Lie algebra of $U(2)$.] The mapping

$$\mathcal{S} \longrightarrow \mathcal{K} = A\mathcal{S}\bar{A}^t, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \tag{1.19}$$

is a rep of $SL(2)$ on S. Indeed $\bar{\mathcal{K}}^t = A\bar{\mathcal{S}}^t\bar{A}^t = -A\mathcal{S}\bar{A}^t = -\mathcal{K}$, so $\mathcal{K} \in S$. The homomorphism property is just as obvious. Now $\det \mathcal{K} = \det(A\mathcal{S}\bar{A}^t) = \det \mathcal{S}$, so, writing

$$\mathcal{K} = \begin{pmatrix} i(y_4 - y_3) & -y_2 + iy_1 \\ y_2 + iy_1 & i(y_4 + y_3) \end{pmatrix} \tag{1.20}$$

we obtain

(1.21)
$$y_1^2 + y_2^2 + y_3^2 - y_4^2 = \det \mathcal{K} = \det \mathcal{S} = x_1^2 + x_2^2 + x_3^2 - x_4^2.$$

From (1.19), the y_j are linear combinations of the x_k:

$$y_i = \sum_{k=1}^{4} L(A)_{jk} x_k, \qquad 1 \le j \le 4. \tag{1.22}$$

From (1.21) and the remarks following (1.1) we conclude that $L(A) \in L(4)$. Furthermore, since (1.19) defines a rep of $SL(2)$ we have the group property $L(AB) = L(A)L(B)$, $A, B \in SL(2)$.

The map $A \longrightarrow L(A)$ is continuous in the parameters of A and $SL(2)$ is connected. Therefore, $L(A)$ must lie in $L^{\uparrow +}$, the connected component of the identity in $L(4)$. We have established the existence of a real analytic homomorphism $A \longrightarrow L(A)$ of $SL(2)$ into $L^{\uparrow +}$. Clearly the kernel of this homomorphism is $\{\pm E_2\}$. Thus, $L(A) = L(-A)$ and exactly two elements of $SL(2)$ map onto each element in the range of the homomorphism.

Suppose $A \in SU(2)$, a real subgroup of $SL(2)$. Then $\bar{A}^t = A^{-1}$ and a comparison of (1.9), Chapter 7, with (1.19) shows that $L(A) = R(A) \in SO(3)$, where $R(A)$ is defined by (1.12), Chapter 7 ($y_4 = x_4$). Thus the homomorphism maps the subgroup $SU(2)$ 2–1 onto the subgroup $SO(3)$ of $L^{\uparrow +}$. We will use this result to show that $A \longrightarrow L(A)$ is a homomorphism of $SL(2)$ **onto** $L^{\uparrow +}$.

Let us compute $L(\exp b\mathcal{F}_3)$ where $\mathcal{F}_3 \in sl(2)$ is given by (1.17). Clearly

$$\exp b\mathcal{F}_3 = \begin{pmatrix} e^{-b/2} & 0 \\ 0 & e^{b/2} \end{pmatrix} \tag{1.23}$$

and

$$\begin{pmatrix} e^{-b/2} & 0 \\ 0 & e^{b/2} \end{pmatrix}\begin{pmatrix} i(x_4 - x_3) & -x_2 + ix_1 \\ x_2 + ix_1 & i(x_4 + x_3) \end{pmatrix}\begin{pmatrix} e^{-b/2} & 0 \\ 0 & e^{b/2} \end{pmatrix}$$
$$= \begin{pmatrix} i(y_4 - y_3) & -y_2 + iy_1 \\ y_2 + iy_1 & i(y_4 + y_3) \end{pmatrix},$$

where $y_1 = x_1, y_2 = x_2, y_3 = x_3 \cosh b + x_4 \sinh b$, and $y_4 = x_3 \sinh b + x_4 \cosh b$, so $L(\exp b\mathfrak{F}_3) = \exp b\mathfrak{B}_3$ [expression (1.6)]. Now suppose $\Lambda \in L^{\uparrow +}$ is given by (1.15). If $A_1 = A(\varphi_1 + \pi/2, \theta_1, 0)$ and $A_2 = A(\varphi_2, \theta_2, \psi_2)$ are elements of $SU(2)$ expressed in Euler coordinates, we have

$$L(A_1(\exp b\mathfrak{F}_3)A_2) = L(A_1)L(\exp b\mathfrak{F}_3)L(A_2) = R_1(\exp b\mathfrak{B}_3)R_2 = \Lambda,$$

so the map $A \longrightarrow L(A)$ covers $L^{\uparrow +}$. We can also use the parameters $\varphi_1, \theta_1, b, \varphi_2, \theta_2, \psi_2$ as coordinates on $SL(2)$, where

(1.24)
$$0 \le \varphi_1, \varphi_2 < 2\pi, \qquad 0 \le \theta_1, \theta_2 \le \pi, \qquad 0 \le b, \qquad -2\pi \le \psi_2 < 2\pi.$$

The parameters of $-A$ are the same as those of A except that ψ_2 is replaced by $\psi_2 \pm 2\pi$. On $L^{\uparrow +}$ the parameters range over the same values except that ψ_2 is restricted to $0 \le \psi_2 < 2\pi$.

Since our group homomorphism is locally 1–1 it induces a Lie algebra isomorphism $\mathfrak{a} \longrightarrow L(\mathfrak{a})$ of $sl(2)$ onto $so(3, 1)$. It is straightforward to check that $L(\mathfrak{J}_j) = \mathfrak{L}_j, L(\mathfrak{F}_j) = \mathfrak{B}_j, 1 \le j \le 3$.

If $\mathbf{T}$ is a rep of the proper Lorentz group by operators $\mathbf{T}(\Lambda)$ then the operators $\mathbf{T}'(A) = \mathbf{T}(L(A))$ define a rep of $SL(2)$ such that $\mathbf{T}'(-A) = \mathbf{T}'(A)$. On the other hand, if $\mathbf{S}$ is a rep of $SL(2)$ such that $\mathbf{S}(A) = \mathbf{S}(-A)$ then the operators $\mathbf{S}'(L(A)) = \mathbf{S}(A)$ define a rep of $L^{\uparrow +}$. Thus, there is a 1–1 correspondence between single-valued reps of $L^{\uparrow +}$ and reps $\mathbf{S}$ of $SL(2)$ such that $\mathbf{S}(-E_2)$ is the identity operator.

Since $SL(2)$ and $L(4)$ are not compact, the results of Chapter 6 do not hold for these groups. In particular a finite-dimensional rep of $SL(2)$ is not necessarily equivalent to a unitary rep. For example the matrices $L(A)$, $A \in SL(2)$, define a four-dimensional irred rep of $SL(2)$. Since the matrix elements of $L(A)$ are unbounded this rep cannot be equivalent to a unitary matrix rep.

Furthermore, we shall see that $SL(2)$ has infinite-dimensional unitary irred reps, which is not possible for compact groups. An arbitrary rep of $SL(2)$ cannot necessarily be decomposed into a direct sum of irred reps.

Suppose $\mathbf{S}$ is a finite-dimensional irred rep of $SL(2)$. Since $-E_2$ commutes with all elements of $SL(2)$, the operator $\mathbf{S}(-E_2)$ commutes with all $\mathbf{S}(A)$. By the Schur lemmas, $\mathbf{S}(-E_2) = \alpha\mathbf{E}$, where $\mathbf{E}$ is the identity operator. Furthermore, $[\mathbf{S}(-E_2)]^2 = \mathbf{S}(E_2) = \mathbf{E}$, so $\alpha^2 = 1$ and $\alpha = \pm 1$. Thus,

$\mathbf{S}(-E_2) = \pm\mathbf{E}$. If $\alpha = +1$ then $\mathbf{S}$ defines a single-valued irred rep of $L^{\uparrow+}$. However, if $\alpha = -1$ then $\mathbf{S}(-A) = -\mathbf{S}(A)$ and $\mathbf{S}$ determines a double-valued rep of $L^{\uparrow+}$. These are the only possibilities.

In quantum mechanics the double-valued reps appear naturally for the same reasons that double-valued reps of $SO(3)$ appear. Thus $SL(2)$ is the group to study for quantum mechanical Lorentz invariance.

8.2 The Physical Significance of Lorentz Invariance

We briefly discuss a realization of the Lorentz group which appears in Einstein's special theory of relativity. In this theory space–time is viewed as a four-dimensional real manifold called **Minkowski space.** The elements or points of this space are **events.** In Minkowski space we distinguish a family of coordinate systems called **inertial frames** or **observers.** With respect to an inertial frame the coordinates of an event are denoted $x = (x_1, x_2, x_3, x_4) = (\mathbf{x}, x_4)$, where the cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$ are the spatial coordinates of the event and $x_4 = ct$, where t is the time coordinate of the event. Here c is the velocity of light in a vacuum. The points of Minkowski space are swept out as the x_j range over all real numbers.

Let $\mathscr{I}$ be an inertial frame and let p, q be events with coordinates x, y in $\mathscr{I}$. Here x and y are column 4-vectors. We define the squared space–time distance between these two events by

$$(2.1)\qquad \|x - y\|^2 = \sum_{j=1}^{3} (x_j - y_j)^2 - (x_4 - y_4)^2 = (x - y)^t G(x - y),$$

where G is given by (1.1). Now suppose $\mathscr{I}'$ is another coordinate system with respect to which the events p, q have coordinates x', y', respectively. We postulate that $\mathscr{I}'$ is an inertial frame (with respect to $\mathscr{I}$) provided

$$(2.2)\qquad \|x - y\|^2 = \|x' - y'\|^2$$

for all pairs of events p, q, i.e., provided the space–time distance between events is preserved. By a computation analogous to that carried out in Section 2.2 one can show that if $\mathscr{I}'$ is inertial then the relationship between the coordinates of the event p in $\mathscr{I}$ and $\mathscr{I}'$ is

$$(2.3)\qquad x_j' = \sum_{k=1}^{4} \Lambda_{jk} x_k + a_j, \qquad j = 1, \ldots, 4,$$

where $\Lambda \in L(4)$ and $a = (a_1, \ldots, a_4)$ is a real four-tuple. Conversely, if $\mathscr{I}$ is inertial and $\mathscr{I}'$ is a coordinate system related to $\mathscr{I}$ by (2.3) then $\mathscr{I}'$ is inertial. (For a proof that the coordinate transformation must be linear see the work of Rätz [1].)

It is clear from definition (2.2) that the inertial frames form an equivalence class. That is, (a) $\mathscr{I}$ is inertial with respect to $\mathscr{I}$, (b) if $\mathscr{I}'$ is inertial with respect to $\mathscr{I}$ then $\mathscr{I}$ is inertial with respect to $\mathscr{I}'$, and (c) if $\mathscr{I}'$ is inertial with respect to $\mathscr{I}$ and $\mathscr{I}''$ is inertial with respect to $\mathscr{I}'$ then $\mathscr{I}''$ is inertial with respect to $\mathscr{I}$. Once one inertial frame is chosen it is easy to obtain the rest.

Let p be an event with coordinates x, x', x'' in the inertial frames $\mathscr{I}, \mathscr{I}'$ $\mathscr{I}''$. Then the relations between these coordinates are given by

(1) $x_s' = \sum_k \Lambda_{sk} x_k + a_s$.

(2) $x_l'' = \sum_s \Lambda'_{ls} x_s' + a_l'$.

(3) $x_l'' = \sum_k \Lambda''_{lk} x_k + a_k'', \quad \Lambda, \Lambda', \Lambda'' \in L(4)$.

From (1) and (2) we have

$$x_l'' = \sum_k \left(\sum_s \Lambda'_{ls} \Lambda_{sk}\right) x_k + \sum_s \Lambda'_{ls} a_s + a_l'.$$

A comparison of this expression with (3) yields

$$\Lambda'' = \Lambda'\Lambda, \qquad a'' = \Lambda' a + a'. \tag{2.4}$$

It follows that the set of all pairs $\{a, \Lambda\}$ forms a group with product

$$\{a', \Lambda'\}\{a, \Lambda\} = \{\Lambda' a + a', \Lambda'\Lambda\}, \qquad \Lambda, \Lambda' \in L(4), \quad a, a' \in R_4. \tag{2.5}$$

This is a ten-parameter Lie group called the **Poincaré** or **inhomogeneous Lorentz** group P. There is a 1–1 relationship between inertial frames and elements of P.

In the theory of special relativity it is postulated that the laws of physics must take the same form in any inertial frame. Since the elements of P determine the coordinate changes from one inertial frame to another, this means the dynamical equations of physics must be invariant under the Poincaré group. For differential equations we mean this invariance in the same sense as Euclidean invariance in Section 7.11. From (2.5) the set of all elements $\{\mathbf{b}, R\}$, $R \in O(3)$, $\mathbf{b} = (a_1, a_2, a_3, 0)$, forms a subgroup of P isomorphic to $E(3)$. Thus, Poincaré-invariant equations are automatically Euclidean-invariant. We shall determine the possible Poincaré-invariant equations in Section 8.5.

Let p be an event and consider the set i_p of all inertial frames in which the coordinates of p are $(0, 0, 0, 0)$, i.e., the inertial frames whose origin of coordinates is p. Let us fix a system $\mathscr{I} \in i_p$. Then if $\mathscr{I}' \in i_p$ there is a $\{a, \Lambda\} \in P$ such that the coordinates x in $\mathscr{I}$ and x' in $\mathscr{I}'$ are related by $x_s' = \sum \Lambda_{sk} x_k + a_s$. This equation must hold for $x = x' = (0, 0, 0, 0)$, so a is the zero vector. Similarly if $\mathscr{I}'$ is a coordinate system related to $\mathscr{I}$ by $x_s' = \sum \Lambda_{sk} x_k$, $\Lambda \in L(4)$, then $\mathscr{I}' \in i_p$. Thus there is a 1–1 correspondence between elements of i_p and elements of $L(4)$ a subgroup of P. In the following we restrict ourselves to inertial frames in i_p.

We now investigate the physical significance of Lorentz transformations.

Let x be a column four-vector with spatial components $\mathbf{x} = (x_1, x_2, x_3)$ and time component $x_4 = ct$. Under space inversion $Sx = (-\mathbf{x}, x_4)$, under time inversion $Gx = (\mathbf{x}, -x_4)$, and under total inversion $SGx = -x$, so the meaning of these coordinate transformations is clear. According to Lemma 8.1 we need only determine the physical significance of transformations in $L^{\uparrow +}$. To do this we prove a variant of Theorem 8.1.

Theorem 8.2. Every $\Lambda \in L^{\uparrow +}$ can be represented uniquely in the form $\Lambda = V(\mathbf{b})R$, where $R \in SO(3)$ and

$$V(\mathbf{b}) = \exp(b_1\mathcal{B}_1 + b_2\mathcal{B}_2 + b_3\mathcal{B}_3).$$

The group elements $V(\mathbf{b})$ are called **velocity transformations.**

Proof. By Theorem 8.1.

$$\Lambda = R_1(\varphi_1 + \tfrac{1}{2}\pi, \theta_1, 0)(\exp b\mathcal{B}_3)R_2(\varphi_2, \theta_2, \psi_2),$$

where

$$(2.6) \qquad \Lambda_{14} = r\sin\theta_1\cos\varphi_1, \qquad \Lambda_{24} = r\sin\theta_1\sin\varphi_1, \qquad \Lambda_{34} = r\cos\theta_1$$

and $r = \sinh b, b \geq 0$. Suppose $r > 0$, in which case this factorization is unique. Now $\Lambda = R_1(\exp b\mathcal{B}_3)R_1^{-1}(R_1R_2)$. The matrices $B(t) = R_1 \times (\exp tb\mathcal{B}_3)R_1^{-1}$ form a one-parameter subgroup of $L^{\uparrow +}$ as t runs over all real numbers, and the tangent matrix at the identity is $bR_1\mathcal{B}_3R_1^{-1}$. A direct computation gives

$$R_1\mathcal{B}_3R_1^{-1} = (\cos\varphi_1\sin\theta_1)\mathcal{B}_1 + (\sin\varphi_1\sin\theta_1)\mathcal{B}_2 + (\cos\theta_1)\mathcal{B}_3.$$

Since the tangent matrix completely determines the one-parameter subgroup we have

$$(2.7) \qquad R_1(\exp tb\mathcal{B}_3)R_1^{-1} = \exp(t[b_1\mathcal{B}_1 + b_2\mathcal{B}_2 + b_3\mathcal{B}_3]) = V(t\mathbf{b}),$$

where

$$(2.8) \qquad \mathbf{b} = (b\cos\varphi_1\sin\theta_1, b\sin\varphi_1\sin\theta_1, b\cos\theta_1), \qquad r = \sinh b.$$

Setting $t = 1$, we obtain $\Lambda = V(\mathbf{b})R$, where $R = R_1R_2 \in SO(3)$. By construction this factorization is unique if $r > 0$, i.e., if $\Lambda \notin SO(3)$. However, if $r = 0$ then $b = 0$ and $\Lambda \in SO(3)$. In this case $V(\mathbf{b}) = E$ and $\Lambda = R$, so again the factorization is unique. Q.E.D.

Since the $\mathcal{B}_j$ are symmetric matrices it follows that $V(\mathbf{b})$ is a positive-definite symmetric matrix. Thus the product $\Lambda = V(\mathbf{b})R$ is just the well-known **polar decomposition** of a real nonsingular matrix into the product of a positive-definite symmetric matrix and an orthogonal matrix.

Let $\mathcal{I}'$ be the inertial frame related to $\mathcal{I}$ by the velocity transformation $x' = V(\mathbf{b})x$. In frame $\mathcal{I}$ the origin of spatial coordinates at time t has coordinates $x = (0, 0, 0, ct)$. In frame $\mathscr{I}'$ this event has coordinates

$$
\begin{aligned}
(2.9) \quad x' &= V(\mathbf{b})x \\
&= ct(\sinh b \sin\theta_1 \cos\varphi_1, \sinh b \sin\theta_1 \sin\varphi_1, \sinh b \cos\theta_1, \cosh b) \\
&= (\mathbf{x}', ct').
\end{aligned}
$$

Thus, in $\mathscr{I}'$ the coordinates of the event are related by the equations

$$
(2.10) \qquad \mathbf{x}' = \mathbf{v}t', \qquad \mathbf{v} = \frac{c}{b}(\tanh b)\mathbf{b} = \frac{rc}{(1+r^2)^{1/2}}\hat{\mathbf{b}},
$$

where $\hat{\mathbf{b}}$ is a unit three-vector in the direction of $\mathbf{b}$. The spatial origin of coordinates in system $\mathscr{I}$ is moving with uniform velocity $\mathbf{v}$ with respect to the spatial origin of coordinates in $\mathscr{I}'$. Note that

$$
\sinh b = \frac{v/c}{(1 - v^2/c^2)^{1/2}}, \qquad \cosh b = \frac{1}{(1 - v^2/c^2)^{1/2}}, \qquad v = \|\mathbf{v}\|.
$$

From the definition of $V(\mathbf{b})$ it is easy to show that this velocity transformation leaves invariant any vector $x = (\mathbf{x}, 0)$ such that $\mathbf{x}\cdot\mathbf{v} = 0$, i.e., $\mathbf{x}\cdot\mathbf{b} = 0$. Indeed

$$
(b_1\mathscr{B}_1 + b_2\mathscr{B}_2 + b_3\mathscr{B}_3)\begin{pmatrix}\mathbf{x}\\0\end{pmatrix} = \begin{pmatrix}\boldsymbol{\theta}\\ \mathbf{b}\cdot\mathbf{x}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\theta}\\0\end{pmatrix}.
$$

In the special case where the velocity $\mathbf{v}$ is in the direction of the positive z axis then

$$
V(\mathbf{b}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\gamma & v/\gamma c \\ 0 & 0 & v/\gamma c & 1/\gamma \end{pmatrix}, \qquad \gamma = (1 - v^2/c^2)^{1/2},
$$

and the coordinate transformation becomes

$$
(2.11) \qquad
\begin{aligned}
&x' = x, \qquad y' = y, \qquad z' = \frac{z + vt}{(1 - v^2/c^2)^{1/2}}, \\
&t' = \frac{t + zv/c^2}{(1 - v^2/c^2)^{1/2}}, \qquad \mathbf{x} = (x, y, z).
\end{aligned}
$$

Equations (2.11) are the usual Lorentz transformations discussed in textbooks on special relativity. The physical significance of $R \in SO(3)$ is obvious, so a Lorentz transformation $\Lambda = V(\mathbf{b})R$ can be interpreted as a rotation of spatial coordinates followed by a velocity transformation.

Warning. The velocity transformations do not form a subgroup of $L^{\uparrow +}$ because the product of two velocity transformations is not necessary a velocity transformation.

In the above discussion we have given a passive interpretation of Lorentz transformations: The space remains fixed and the observers (inertial frames) transform under $L(4)$. Alternatively, we could adopt the active interpretation: There is one fixed coordinate system and the Lorentz group transforms the points of Minkowski space. In the active interpretation a velocity transformation maps a state in which a particle is at rest into a state where the particle has velocity $\mathbf{v}$.

8.3 Representations of the Lorentz Group

To find the analytic irred reps of $L^{\uparrow +}$ we compute the analytic irred reps $\mathbf{T}$ of $SL(2)$ considered as a real Lie group and determine which of these reps satisfy $\mathbf{T}(-E_2) = \mathbf{E}$. We have already computed the irred reps $\mathbf{D}^{(u)}$ of $SL(2)$ which are analytic functions of the complex group parameters. If $D^{(u)}(A)$ is a matrix realization of $\mathbf{D}^{(u)}$ then the complex conjugate matrices $\overline{D^{(u)}}(A)$ also define an irred rep of $SL(2)$ which is analytic in the real group parameters but not in the complex group parameters (Prove it!) Since any rep equivalent to a complex analytic rep is complex analytic it follows that $\mathbf{D}^{(u)}$ and $\overline{\mathbf{D}^{(u)}}$ are nonequivalent irred reps.

As a convenient basis for the real six-dimensional Lie algebra $sl(2) \cong so(3, 1)$ we choose the matrices $\mathcal{J}_j, \mathcal{F}_j = i\mathcal{J}_j, 1 \leq j \leq 3$, where the $\mathcal{J}_j$ are defined by (1.8), Chapter 7. These matrices satisfy the commutation relations (1.4) with $\mathcal{L}_j$ replaced by $\mathcal{J}_j$ and $\mathcal{B}_j$ by $\mathcal{F}_j$. Now we forget the origin of our basis as a set of matrices and merely consider the abstract Lie algebra $sl(2)$ spanned by linearly independent basis elements $\mathcal{J}_j, \mathcal{F}_j$ with commutation relations (1.4). From (1.9) we see that $sl(2)^c$, the complexification of our real Lie algebra, has a basis $\mathcal{C}_j, \mathcal{D}_k$ with commutation relations

$$(3.1) \qquad [\mathcal{C}_j, \mathcal{D}_k] = 0, \qquad [\mathcal{C}_j, \mathcal{C}_k] = \sum_l \epsilon_{jkl}\mathcal{C}_l, \qquad [\mathcal{D}_j, \mathcal{D}_k] = \sum_l \epsilon_{jkl}\mathcal{D}_l,$$

where $\mathcal{C}_j = (\mathcal{J}_j - i\mathcal{F}_j)/2, \mathcal{D}_j = (\mathcal{J}_j + i\mathcal{F}_i)/2$. The Lie algebra of the group $SU(2) \times SU(2) = G$ is another real form of the complex algebra (3.1). Since G is compact we know that its global irred reps are just $\mathbf{D}^{(u,v)} = \mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)}$, $2u, 2v = 0, 1, 2, \ldots$. Therefore, the possible irred finite-dimensional reps of $L(G)$ are just the Lie algebra reps induced by $\mathbf{D}^{(u,v)}$. Since there is a 1–1 correspondence between reps of a complex Lie algebra and reps of any of its real forms we conclude the the irred reps of $sl(2)$ and $sl(2)^c$ are $\mathbf{D}^{(u,v)}$. Indeed if we denote the operators corresponding to such a rep by $C_j = \mathbf{T}(\mathcal{C}_j)$, $D_j = \mathbf{T}(\mathcal{D}_j)$ and set

$$C^{\pm} = \pm C_2 + iC_1, \qquad C^3 = -iC_3, \qquad D^{\pm} = \pm D_2 + iD_1, \qquad D^3 = -iD_3,$$

then there exists a basis $\{f^{(u,v)}_{mn}\}$ for the rep space $\mathcal{V}^{(u,v)}$ corresponding to the

$(2u+1)(2v+1)$-dimensional rep $\mathbf{D}^{(u,v)}$ such that

$$\begin{aligned}
&C^3 f_{mn}^{(u,v)} = m f_{mn}^{(u,v)}, \qquad C^{\pm} f_{mn}^{(u,v)} = [(u \pm m + 1)(u \mp m)]^{1/2} f_{m\pm 1,n}^{(u,v)} \\
(3.2)\qquad &D^3 f_{mn}^{(u,v)} = n f_{mn}^{(u,v)}, \qquad D^{\pm} f_{mn}^{(u,v)} = [(v \pm n + 1)(v \mp n)]^{1/2} f_{m,n\pm 1}^{(u,v)} \\
&-C\cdot C f_{mn}^{(u,v)} = u(u+1) f_{mn}^{(u,v)}, \qquad -D\cdot D f_{mn}^{(u,v)} = v(v+1) f_{mn}^{(u,v)}.
\end{aligned}$$

[We call a basis satisfying (3.2) **canonical.**] The operators $C\cdot C = C_1C_1 + C_2C_2 + C_3C_3$ and $D\cdot D$ commute with the C_j and D_j, so they must be multiples of the identity operator for any irred rep of $sl(2)$.

Now we will show that the Lie algebra reps $\mathbf{D}^{(u,v)}$ induce global reps of $SL(2)$. To begin we consider the complex analytic rep $\mathbf{D}^{(u)}$ of $SL(2)$ determined in Chapter 7. Clearly, the induced Lie algebra rep has the property $F_j = iJ_j, 1 \le j \le 3$. Thus $C^{\pm} = J^{\pm}$, $C^3 = J^3$, and the D-operators are zero. We conclude that $\mathbf{D}^{(u)}$ is equivalent to the rep $\mathbf{D}^{(u,0)}$. On the other hand the Lie algebra rep induced by $\bar{\mathbf{D}}^{(v)}$ has the property $F_j = -iJ_j$. Hence $D^{\pm} = J^{\pm}$, $D^3 = J^3$, and the C-operators are zero. This shows that $\bar{\mathbf{D}}^{(v)}$ is equivalent to $\mathbf{D}^{(0,v)}$. Similarly, if we compute the Lie algebra rep induced by the group rep $\mathbf{D}^{(u)} \otimes \bar{\mathbf{D}}^{(v)}$ of $SL(2)$ on $\mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$ we get exactly the results (3.2), by making the identification $f_{mn}^{(u,v)} = f_m^{(u)} \otimes g_n^{(v)}$, where $\{f_m^{(u)}\}$ and $\{g_n^{(v)}\}$ are canonical bases for $\mathcal{V}^{(u)}$ and $\mathcal{V}^{(v)}$, respectively.

To sum up, we have shown that a complete set of finite-dimensional analytic irred reps of the real Lie group $SL(2)$ is given by $\mathbf{D}^{(u,v)}$, $2u, 2v = 0, 1, 2, \ldots$. The matrix elements of these reps with respect to a suitable (not canonical) basis are

$$(3.3)\qquad \mathbf{T}(A)h_{mn}^{(u,v)} = \sum_{m'=-u}^{u} \sum_{n'=-v}^{v} T_{m'm}^{(u)}(A)\overline{T_{n'n}^{(v)}}(A)h_{m'n'}^{(u,v)}.$$

Note: If the vectors $f_m^{(u)}$ form a canonical basis for $\mathbf{D}^{(u)}$ it is *not* true that the complex conjugate vectors $\bar{f}_m^{(u)}$ form a canonical basis for $\bar{\mathbf{D}}^{(u)}$. To see this, choose a matrix realization of $\mathbf{D}^{(u)}$ so that the $f_m^{(u)}$ are $(2u+1)$-component column vectors. This group rep induces a matrix Lie algebra rep of $sl(2)$. The matrices F_j, J_j satisfy the properties

$$\begin{aligned}
(3.4)\qquad &C^3 f_m^{(u)} = m f_m^{(u)}, \qquad C^{\pm} f_m^{(u)} = [(u \pm m + 1)(u \mp m)]^{1/2} f_{m\pm 1}^{(u)}, \\
&D^{\pm} = D^3 = Z,
\end{aligned}$$

where the C and D matrices are defined by the expression following (3.1). Now denote the corresponding matrices induced from the complex conjugate matrix rep $\bar{D}^{(u)}$ with stars. Then $J_j^* = \bar{J}_j$, $F_j^* = \bar{F}_j$, so

$$C_j^* = (J_j^* - iF_j^*)/2 = \bar{D}_j, \qquad D_j^* = (J_j^* + iF_j^*)/2 = \bar{C}_j.$$

Thus

$$C^{*\pm} = C^{*3} = Z, \qquad D^{*\pm} = -\bar{C}^{\mp}, \qquad D^{*3} = -\bar{C}^3.$$

Substituting these results in (3.4), we find

$$C^{*\pm} = C^{*3} = Z, \quad D^{*3}\bar{f}_m^{(u)} = -\overline{C^3 f_m^{(u)}} = -m\bar{f}_m^{(u)},$$
$$D^{*\pm}\bar{f}_m^{(u)} = -\overline{C^{\mp} f_m^{(u)}} = -[(u \mp m + 1)(u \pm m)]^{1/2}\bar{f}_{m\mp 1}^{(u)}.$$

This shows that the vectors $g_m^{(u)} = (-1)^{u-m}\bar{f}_{-m}^{(u)}$ form a canonical basis for $\bar{\mathbf{D}}^{(u)}$.

With respect to a canonical basis the matrix elements of $\bar{\mathbf{D}}^{(u)}$ are

$$\mathbf{T}(A)g_m^{(u)} = \sum_{n=-u}^{u} (-1)^{m-n}\overline{T_{-n,-m}^{(u)}(A)}g_n^{(u)},$$

where the $T_{nm}^{(u)}(A)$ are the matrix elements of $\mathbf{D}^{(u)}$ in a canonical basis. It follows immediately from (3.4) that $\mathbf{T}(-E_2) = (-1)^{2(u+v)}\mathbf{E}$, so $\mathbf{D}^{(u,v)}$ determines a single-valued rep of the Lorentz group if and only if $u + v$ is an integer.

By construction $\mathbf{D}^{(u,v)} \cong \mathbf{D}^{(u)} \otimes \bar{\mathbf{D}}^{(v)}$. Taking the complex conjugate of

$$\mathbf{D}^{(u)} \otimes \mathbf{D}^{(u')} \cong \sum_{w=|u-u'|}^{u+u'} \oplus \mathbf{D}^{(w)}$$

we obtain an analogous relation for $\bar{\mathbf{D}}^{(u)} \otimes \bar{\mathbf{D}}^{(u')}$. [Note: Even though we have defined $\bar{\mathbf{D}}^{(u)}$ by taking the complex conjugate of a matrix realization of $\mathbf{D}^{(u)}$ with respect to a fixed basis, it is easy to show that $\bar{\mathbf{D}}^{(u)}$ is basis-independent. Indeed, one merely verifies that two matrix reps $T(A)$, $T'(A)$ are equivalent if and only if $\overline{T(A)}$ and $\overline{T'(A)}$ are equivalent.] Thus

$$\begin{aligned}\mathbf{D}^{(u,v)} \otimes \mathbf{D}^{(u',v')} &\cong (\mathbf{D}^{(u)} \otimes \bar{\mathbf{D}}^{(v)}) \otimes (\mathbf{D}^{(u')} \otimes \bar{\mathbf{D}}^{(v')}) \\ &\cong (\mathbf{D}^{(u)} \otimes \mathbf{D}^{(u')}) \otimes (\bar{\mathbf{D}}^{(v)} \otimes \bar{\mathbf{D}}^{(v')}) \\ &\cong \sum_{w=|u-u'|}^{u+u'} \sum_{z=|v-v'|}^{v+v'} \oplus \mathbf{D}^{(w,z)}\end{aligned} \tag{3.5}$$

is the CG series for irred reps of $SL(2)$. Note that each irred rep $\mathbf{D}^{(w,z)}$ occurring in the decomposition of $\mathbf{D}^{(u,v)} \otimes \mathbf{D}^{(u',v')}$ has multiplicity one. Therefore, it is easy to project out the subspace $\mathcal{W}^{(w,z)}$ of $\mathcal{V}^{(u,v)} \otimes \mathcal{V}^{(u',v')}$ which transforms irreducibly under $\mathbf{D}^{(w,z)}$. Indeed from (3.3) and the results of Section 7.7 a canonical basis for $\mathcal{W}^{(w,z)}$ is given by the vectors

$$h_{kl}^{(w,z)} = \sum_{mnm'n'} C(u, m; u', m' \mid w, k)C(v, n; v', n' \mid z, l) f_{mn}^{(u,v)} \otimes f_{m'n'}^{(u'v')}, \tag{3.6}$$
$$-w \le k \le w, \quad -z \le l \le z,$$

where the $C(-|-)$ are the CG coefficients (7.21), Chapter 7. The coefficients for the real Lie group $SL(2)$ are products of the coefficients for the complex group $SL(2)$.

If we restrict the rep $\mathbf{D}^{(u,v)}$ of $SL(2)$ to the subgroup $SU(2)$ it decomposes into a direct sum of irred reps of $SU(2)$. To determine the decomposition we note the rep $\bar{\mathbf{D}}^{(u)}$ of $SU(2)$ is equivalent to $\mathbf{D}^{(u)}$. Indeed $\bar{\mathbf{D}}^{(u)}$ is irred and every irred rep of $SU(2)$ is equivalent to some $\mathbf{D}^{(v)}$. Since dim $\bar{\mathbf{D}}^{(u)} = 2u + 1$ we must have $\bar{\mathbf{D}}^{(u)} \cong \mathbf{D}^{(u)}$. Alternatively, the character $\chi^{(u)}$ of $\mathbf{D}^{(u)}$ is real, so $\mathbf{D}^{(u)}$ and $\bar{\mathbf{D}}^{(u)}$ have the same character. Using this result we obtain

$$\mathbf{D}^{(u,v)} \mid SU(2) \cong \mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)} \cong \sum_{w=|u-v|}^{u+v} \oplus \mathbf{D}^{(w)}. \tag{3.7}$$

Note the special cases $\mathbf{D}^{(u,0)} \mid SU(2) \cong \mathbf{D}^{(0,u)} \mid SU(2) \cong \mathbf{D}^{(u)}$.

Although the canonical basis $\{f_{mn}^{(u,v)}\}$ is very convenient for computational purposes, it does not clearly exhibit the decomposition (3.7) when the Lie algebra $sl(2)$ is restricted to $su(2)$. According to (3.7) there exists an ON basis $\{f_k^{(w)}: |u - v| \le w \le u + v, -w \le k \le w\}$ for $\mathfrak{V}^{(u,v)}$ such that

$$J^3 f_k^{(w)} = k f_k^{(w)}, \qquad J^{\pm} f_k^{(w)} = [(w \pm k + 1)(w \mp k)]^{1/2} f_{k\pm 1}^{(w)}. \tag{3.8}$$

Recall that the J-operators satisfy the commutation relations of $su(2)$. We could use CG coefficients to express the $\{f_k^{(w)}\}$ basis in terms of the $\{f_{mn}^{(u,v)}\}$ basis. However, it is more instructive to compute the action of the Lie algebra on the $\{f_k^{(w)}\}$ basis directly.

For this purpose we choose the operators $J^{\pm}, J^3, F^{\pm}, F^3$ with commutation relations (1.8), $(J \equiv \mathfrak{L}, F \equiv \mathfrak{B})$, as the generators of our rep. The action of the J-operators on $f_k^{(w)}$ is given by (3.8). To determine the action of the F-operators we note from (1.8) that the operators $\mathbf{Q}_1 = -F^+$, $\mathbf{Q}_0 = \sqrt{2}\, F^3$, and $\mathbf{Q}_{-1} = F^-$ transform as a spherical tensor of rank one under the action of $SU(2)$. According to the Wigner–Eckart theorem (10.14), Chapter 7,

$$\begin{aligned} (F^{\pm} f_k^{(w)}, f_{k'}^{(w')}) &= \mp N(w, w') C(1, \pm 1; w, k \mid w', k'), \\ (F^3 f_k^{(w)}, f_{k'}^{(w')}) &= 2^{-1/2} N(w, w') C(1, 0; w, k \mid w', k'). \end{aligned} \tag{3.9}$$

In particular, these matrix elements are zero unless $w' = w \pm 1, w$. Explicit expressions for the CG coefficients are given in (7.28), Chapter 7, so we need only compute the constants N. These constants can be obtained from the remaining commutation relations

$$[F^3, F^{\pm}] = \mp J^{\pm}, \qquad [F^+, F^-] = -2J^3. \tag{3.10}$$

It follows from (3.9) that

$$\begin{aligned} F^{\pm} f_k^{(w)} = &\pm [(w \mp k)(w \mp k - 1)]^{1/2} A_w f_{k\pm 1}^{(w-1)} \\ &- [(w \pm k + 1)(w \mp k)]^{1/2} B_w f_{k\pm 1}^{(w)} \\ &\pm [(w \pm k + 1)(w \pm k + 2)]^{1/2} C_{w+1} f_{k\pm 1}^{(w+1)}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} F^3 f_k^{(w)} = &[(w - k)(w + k)]^{1/2} A_w f_k^{(w-1)} \\ &- k B_w f_k^{(w)} - [(w + k + 1)(w - k + 1)]^{1/2} C_{w+1} f_k^{(w+1)}, \end{aligned} \tag{3.12}$$

where the constants A_w, B_w, C_w depend only on w. We can simplify the above formulas by renormalizing the vectors $f_k^{(w)}$. If we introduce new basis vectors

$f_k'^{(w)} = \alpha_w f_k^{(w)}$, where the α_w are nonzero complex numbers, then Eq. (3.8) will remain unchanged in the primed basis while (3.11) and (3.12) will maintain the same form with A_w, B_w, C_w replaced by

$$A_w' = (\alpha_w/\alpha_{w-1})A_w, \qquad B_w' = B_w, \qquad C_w' = (\alpha_{w-1}/\alpha_w)C_w. \tag{3.13}$$

The new basis vectors $f_k'^{(w)}$ will be orthogonal but not necessarily of length one. Note the product $A_w'C_w' = A_w C_w$ is invariant under renormalization and must be nonzero for $|u - v| + 1 \leq w \leq u + v$ since $\mathbf{D}^{(u,v)}$ is irred. Thus we can choose the constants α_w so $A_w = C_w$. We will suppose that this is the case in expressions (3.11) and (3.12).

Now we use the commutation relations (3.10) to compute A_w and B_w. Substituting (3.8), (3.11), and (3.12) in $[F^+, F^3]f_k^{(w)} = J^+ f_k^{(w)}$ and equating coefficients of $f_{k+1}^{(w')}$ on both sides of the resulting relations, we find

$$[(w+1)B_w - (w-1)B_{w-1}]A_w = [(w+2)B_{w+1} - wB_w]A_{w+1} = 0, \tag{3.14}$$

$$(2w-1)A_w{}^2 - (2w+3)A_{w+1}^2 - B_w{}^2 = 1. \tag{3.15}$$

The other two equations (3.10) lead to the same results. Since $A_w \neq 0$ it follows from (3.14) that

$$B_{w+1} = wB_w/(w+2), \qquad w = |u-v|, \ldots, u+v.$$

The solution is

$$B_w = B_{w_0} w_0(w_0+1)/[w(w+1)] = iw_0 w_1/[w(w+1)], \tag{3.16}$$

where $iw_1 = B_{w_0}(w_0+1)$, $w_0 = |u-v|$. We will determine the constant w_1 later. Substituting (3.16) into (3.15), we get a recurrence relation for $A_w{}^2$:

$$(2w-1)A_w{}^2 - (2w+3)A_{w+1}^2 = 1 + \frac{w_0{}^2 w_1{}^2}{w^2(w+1)^2}, \tag{3.17}$$
$$w = w_0, \ldots, u+v-1.$$

Since $f_k^{(w_0-1)}$ does not belong to $\mathcal{V}^{(u,v)}$ we must require $A_{w_0} = 0$. With this restriction Eq. (3.17) determine $A_w{}^2$. The solution is

$$A_w = \frac{i}{w}\left[\frac{(w^2 - w_0{}^2)(w^2 - w_1{}^2)}{4w^2 - 1}\right]^{1/2}. \tag{3.18}$$

(By choosing the normalization factors α_w appropriately we can always assume $|\arg A_w| \leq \pi/2$.)

To determine w_1 we note from (3.11) that $A_{u+v+1} = 0$ since $f_k^{(u+v+1)}$ does not belong to $\mathcal{V}^{(u,v)}$. Therefore, (3.18) implies $w_1{}^2 = (u+v+1)^2$, or $w_1 = \pm(u+v+1)$. To determine the proper sign we must distinguish between $\mathbf{D}^{(u,v)}$ and $\mathbf{D}^{(v,u)}$.

It follows from (3.2) that $-C \cdot C = u(u+1)\mathrm{E}$ and $-D \cdot D = v(v+1)\mathrm{E}$ for the rep $D^{(u,v)}$. If we express the C_j and D_j in terms of the operators (3.11) and (3.12) and use (3.16)and (3.18) we find

$$(3.19)\qquad \begin{aligned} -C\cdot C &= \tfrac{1}{4}(w_0 + w_1 + 1)(w_0 + w_1 + 3)\mathbf{E}, \\ -D\cdot D &= \tfrac{1}{4}(w_1 - w_0 + 1)(w_1 - w_0 + 3)\mathbf{E}. \end{aligned}$$

Thus, if we allow w_0 to be negative we can make the unique assignment

$$(3.20)\qquad w_0 = u - v, \qquad w_1 = u + v + 1.$$

This is permissible since expressions (3.11), (3.12), (3.16), and (3.18) depend only on $w_0{}^2$, $w_1{}^2$, and $w_0 w_1$. In particular the rep defined by the pair (w_0, w_1) is equivalent to the rep $(-w_0, -w_1)$. Here $2w_0$, $2w_1$, and $w_0 + w_1$ are integers with $|w_0| \le |w_1|$.

Summing up, there is a basis $\{f_k^{(w)}\}$ for the rep space of $\mathbf{D}^{(u,v)}$ such that

$$(3.21)\qquad J^{\pm} f_k^{(w)} = [(w \pm k + 1)(w \mp k)]^{1/2} f_{k\pm 1}^{(w)}, \qquad J^3 f_k^{(w)} = k f_k^{(w)},$$

$$(3.22)\qquad \begin{aligned} F^{\pm} f_k^{(w)} = {} & \pm[(w \mp k)(w \mp k - 1)]^{1/2} A_w f_{k\pm 1}^{(w-1)} \\ & - [(w \pm k + 1)(w \mp k)]^{1/2} B_w f_{k\pm 1}^{(w)} \\ & \pm [(w \pm k + 1)(w \pm k + 2)]^{1/2} A_{w+1} f_{k\pm 1}^{(w+1)}, \end{aligned}$$

$$(3.23)\qquad \begin{aligned} F^3 f_k^{(w)} = {} & [w^2 - k^2]^{1/2} A_w f_k^{(w-1)} - k B_w f_k^{(w)} \\ & - [(w+1)^2 - k^2]^{1/2} A_{w+1} f_k^{(w+1)} \end{aligned}$$

$$w = |w_0|, |w_0| + 1, \ldots, |w_1|, \qquad -w \le k \le w,$$

where

$$(3.24)\qquad B_w = \frac{i w_0 w_1}{w(w+1)}, \qquad A_w = \frac{i}{w}\left[\frac{(w^2 - w_0{}^2)(w^2 - w_1{}^2)}{4w^2 - 1}\right]^{1/2}$$

and $w_0 = u - v$, $w_1 = u + v + 1$.

In many respects the basis $\{f_k^{(w)}\}$ is more convenient than the basis $\{f_{mn}^{(u,v)}\}$. This is particularly true in problems where one is interested in the restriction of a rep of $SL(2)$ to the subgroup $SU(2)$.

The noncompact group $SL(2)$ also has bounded infinite-dimensional irred reps. If $\mathbf{T}$ is such a rep, a slight extension of the results of Section 6.3 shows that $\mathbf{T}\,|\,SU(2)$ decomposes into a direct sum of irred reps of the compact group $SU(2)$:

$$(3.25)\qquad \mathbf{T}\,|\,SU(2) \cong \sum_{2w=0}^{\infty} \oplus\, a_w \mathbf{D}^{(w)}.$$

For the present we assume that the multiplicity a_w of $\mathbf{D}^{(w)}$ is either zero or one, i.e., each rep $\mathbf{D}^{(w)}$ appears at most once in the decomposition. Furthermore, we assume that the usual relationships between the bounded operators $\mathbf{T}(A)$, $A \in SU(2)$, and $J^{\pm}$, J^3, $F^{\pm}$, F^3 hold for these infinite-dimensional reps.

Let I be the set of all w such that $\mathbf{D}^{(w)}$ is contained in the decomposition of $\mathbf{T}\,|\,SU(2)$. There exists a basis $\{f_k^{(w)}: w \in I, -w \le k \le w\}$ for the rep space such that Eq. (3.8) are satisfied. Indeed for fixed w, $\{f_k^{(w)}\}$ is a canonical basis for $\mathbf{D}^{(w)}$. Applying the Wigner–Eckart theorem, we see that the operators $F^{\pm}, F^3$ satisfy Eq. (3.11) and (3.12) exactly as in the finite-dimensional case. Let w_0 be the smallest number in the index set I. It follows from (3.11), (3.12), and the irreducibility of $\mathbf{T}$ that $I = \{w_0 + n : n = 0, 1, 2, \ldots\}$. Thus

$$\mathbf{T} \mid SU(2) \cong \sum_{n=0}^{\infty} \oplus \mathbf{D}^{(w_0+n)}.$$

(There can be no gaps in the sequence of $\mathbf{D}^{(w)}$ since $\mathbf{T}$ is irred. If the sequence is finite then $\mathbf{T}$ is isomorphic to one of the finite-dimensional reps $\mathbf{D}^{(u,v)}$ which we have already classified.)

The computation of relations (3.21)–(3.24) is exactly the same for $\mathbf{T}$ as for finite-dimensional reps. The only difference is that w_1 is no longer an integer or half-integer such that $w_1 = w_0 + k$ for some integer $k \geq 0$; otherwise $\mathbf{T}$ would be finite-dimensional. Thus, w_1 is an arbitrary complex number not satisfying the above requirement. We conclude that the infinite-dimensional irred reps of $SL(2)$ can be labeled by the parameters (w_0, w_1) where $2w_0$ is a nonnegative integer and w_1 is a complex number such that $w_1 \neq w_0 + k$, $k = 0, 1, 2, \ldots$. However, it is not clear that each of these Lie algebra reps can be exponentiated to a global irred rep of $SL(2)$. Naimark [2] proves that there is in fact a global group rep corresponding to each of our Lie algebra reps. Furthermore, Naimark shows that any irred rep $\mathbf{T}$ of $SL(2)$ when restricted to $SU(2)$ contains each $\mathbf{D}^{(u)}$ at most once.

Let us check to see which of our reps are unitary. If $\mathbf{T}$ is unitary a simple computation (which should be familiar to the reader by now) shows that

$$(J^3)^* = J^3, \qquad (J^+)^* = J^-, \qquad (F^3)^* = F^3, \qquad (F^+)^* = F^-. \tag{3.26}$$

Just as in Section 7.7 we can use the requirements on the J-operators to prove $(f_k^{(w)}, f_{k'}^{(w')}) = 0$ unless $w = w'$ and $k = k'$. Furthermore $\| f_k^{(w)} \| = \| f_j^{(w)} \|$, $-w \leq k, j \leq w$. Since F^3 is symmetric we have

$$(F^3 f_k^{(w)}, f_k^{(w')}) = (f_k^{(w)}, F^3 f_k^{(w')}).$$

Substituting (3.23) into this expression, we find

$$B_w = \bar{B}_w, \qquad A_w \| f_k^{(w-1)} \|^2 = -\bar{A}_w \| f_k^{(w)} \|^2. \tag{3.27}$$

The relation $(F^+)^* = F^-$ yields no additional constraints. By (3.24), B_w is real if and only if (a) $w_1 = ic$, c real, or (b) $w_0 = 0$.

Writing $A_w = R_w + iI_w$ in terms of real and imaginary parts, we see that the second relation (3.27) implies $R_w = 0$, $\| f_k^{(w)} \| = \| f_k^{(w-1)} \|$. Since all of the basis vectors have the same length we can normalize them so $\| f_k^{(w)} \| = 1$. By (3.24) the requirement $R_w = 0$ is identically satisfied in case (a) since $-w_1{}^2 = c^2 \geq 0$. In case (b) this requirement will be satisfied provided

$(w^2 - w_1{}^2)/(4w^2 - 1) \geq 0$ for $w = 0, 1, 2, \ldots$. This is possible if and only if $0 \leq w_1{}^2 \leq 1$. Thus $-1 \leq w_1 \leq 1$.

This discussion shows that the unitary irred reps (w_0, w_1) of $SL(2)$ fall into two classes:

(3.28) The principal series: w_1 pure imaginary.

(3.29) The complementary series: $w_0 = 0$, w_1 real, $|w_1| \leq 1$.

The only finite-dimensional unitary rep is the identity rep $(0, 1) \cong \mathbf{D}^{(0,0)}$.

We now construct models of the corresponding Lie group reps, starting with the finite-dimensional reps $\mathbf{D}^{(u,v)}$. From (2.1), Section 7.2, we know $\mathbf{D}^{(u,0)} \cong \mathbf{D}^{(u)}$, $2u = 0, 1, 2, \ldots$, has a model in terms of operators

$$[\mathbf{T}(A)f](z) = (bz + d)^{2u} f\left(\frac{az + c}{bz + d}\right), \qquad A \in SL(2), \tag{3.30}$$

acting on the $(2u + 1)$-dimensional space $\mathcal{V}^{(u)}$ of polynomials with order $2u$ in z. The vectors $f_m^{(u)} = (-z)^{u+m}/[(u + m)!(u - m)!]^{1/2}$ form a canonical basis.

It follows that $\mathbf{D}^{(0,v)} \cong \bar{\mathbf{D}}^{(v)}$ has the model

$$[\mathbf{T}(A)f](\bar{z}) = \overline{(bz + d)}^{2v} f\left(\frac{\bar{a}\bar{z} + \bar{c}}{\bar{b}\bar{z} + \bar{d}}\right) \tag{3.31}$$

on the $(2v + 1)$-dimensional space $\bar{\mathcal{V}}^{(v)}$ of polynomials with order $2v$ in $\bar{z}$. The vectors $g_n^{(v)} = (-1)^{v-n}\bar{f}_n^{(v)} = (\bar{z})^{v-n}/[(v + n)!(v - n)!]^{1/2}$ form a canonical basis.

According to (3.5), $\mathbf{D}^{(u,v)} \cong \mathbf{D}^{(u)} \otimes \bar{\mathbf{D}}^{(v)}$. Thus $\mathbf{D}^{(u,v)}$ has a model defined by operators

$$[\mathbf{T}(A)f](z, \bar{z}) = (bz + d)^{2u}\overline{(bz + d)}^{2v} f\left(\frac{az + c}{bz + d}, \frac{\bar{a}\bar{z} + \bar{c}}{\bar{b}\bar{z} + \bar{d}}\right) \tag{3.32}$$

acting on the $(2u + 1)(2v + 1)$-dimensional space of polynomials with order $2u$ in z and $2v$ in $\bar{z}$.

It is clear from (3.32) that $\bar{\mathbf{D}}^{(u,v)} \cong \mathbf{D}^{(v,u)}$. Only the diagonal reps $\mathbf{D}^{(u,v)}$ are equivalent to their own complex conjugates. Such reps are called **real.**

It is easy to find a model of the unitary reps in the principal series (w_0, ic), (3.28). Indeed by comparing the eigenvalues of the invariant operators $C \cdot C$ and $D \cdot D$ in the $\{f_{mn}^{(u,v)}\}$ and $\{f_k^{(w)}\}$ bases we have concluded that $w_0 = u - v$, $w_1 = u + v + 1$. This suggests that the action of (w_0, ic) can be obtained from (3.32) by setting $2u = w_0 + ic - 1$ and $2v = -w_0 + ic - 1$:

$$[\mathbf{T}(A)f](z) = |bz + d|^{-2w_0 + 2ic - 2}(bz + d)^{2w_0} f\left(\frac{az + c}{bz + d}\right). \tag{3.33}$$

Here we regard $f(z) = f(x, y)$ as a function of the two real variables x, y, where $z = x + iy$, and we suppress the argument z. If the operators $\mathbf{T}(A)$

act on the Hilbert space $L_2(R_2)$,

$$(f_1, f_2) = \int_{R_2} f_1(x,y)\overline{f_2(x,y)}\,dx\,dy,$$

then one can show that they define a global irred unitary rep of $SL(2)$ whose induced Lie algebra rep is equivalent to (w_0, ic), $2w_0 = 0, 1, 2, \ldots$, c real (Naimark [2]).

It is just as easy to formally compute the action of the unitary reps from the complementary series. However, in this case there is some difficulty in determining the proper Hilbert space on which the rep acts. Naimark works out the details.

The computation of the infinitesimal generators (generalized Lie derivatives) for all of the above models is straightforward but will not be carried out here. Furthermore, we will now limit ourselves to finite-dimensional reps. The infinite-dimensional unitary reps of the homogeneous Lorentz group seem to be of less importance for physical applications (see, however, Ruhl [1]).

Because of the isomorphism between the Lie algebras $sl(2)$ and $su(2) \oplus su(2)$ we can conclude that any finite-dimensional rep of $SL(2)$ or $L^{\uparrow +}$ can be decomposed into a direct sum of irred reps $\mathbf{D}^{(u,v)}$. (This is false for infinite-dimensional reps.) We shall use this fact to compute the (finite-dimensional) irred reps of the general Lorentz group $L(4)$. We shall also compute the irred reps of the **complete** Lorentz group $L^{\uparrow} = \{SL^{\uparrow +}, L^{\uparrow +}\}$ obtained by adding the space reflection S to the proper Lorentz group.

Let $\mathbf{T}$ be an irred rep of $L^{\uparrow}$ and let $\mathbf{S} = \mathbf{T}(S)$. Since S commutes with all rotations [see (1.5)] it follows that

$$\mathbf{S}J^{\pm}\mathbf{S}^{-1} = J^{\pm}, \qquad \mathbf{S}J^{3}\mathbf{S}^{-1} = J^{3}, \qquad \mathbf{S} = \mathbf{S}^{-1}.$$

On the other hand, by (1.3)

$$\mathbf{S}B^{\pm}\mathbf{S}^{-1} = -B^{\pm}, \qquad \mathbf{S}B^{3}\mathbf{S}^{-1} = -B^{3}.$$

In terms of the C- and D-operators [(1.9)] these results become

$$\mathbf{S}C^{\pm}\mathbf{S}^{-1} = D^{\pm}, \qquad \mathbf{S}C^{3}\mathbf{S}^{-1} = D^{3}. \tag{3.34}$$

Suppose the rep $\mathbf{D}^{(u,v)}$ is contained in $\mathbf{T}\,|\,L^{\uparrow +}$. Then there exist vectors $\{f_{mn}^{(u,v)}: -u \le m \le u, -v \le n \le v\}$ spanning a subspace $\mathcal{V}^{(u,v)}$ of the rep space $\mathcal{V}$ which transform under the C- and D-operators according to (3.2). Define vectors $g_{nm}^{(v,u)} \in \mathcal{V}$ by $g_{nm}^{(v,u)} = \mathbf{S}f_{mn}^{(u,v)}$. Then by (3.34) and (3.2)

$$C^{3}g_{nm}^{(v,u)} = ng_{nm}^{(v,u)}, \qquad C^{\pm}g_{nm}^{(v,u)} = [(v \pm n + 1)(v \mp n)]^{1/2}g_{n\pm 1,m}^{(v,u)}, \tag{3.35}$$

with similar results for the D-operators. Thus the vectors $\{g_{nm}^{(v,u)}\}$ span a subspace $\mathcal{V}^{(v,u)}$ of $\mathcal{V}$ which transforms under $\mathbf{D}^{(v,u)}$. Since $\mathbf{S}^2 = \mathbf{E}$ we have $\mathbf{S}g_{nm}^{(v,u)} = \mathbf{S}^2 f_{mn}^{(u,v)} = f_{mn}^{(u,v)}$. Furthermore, the space $\mathcal{V}^{(u,v)} + \mathcal{V}^{(v,u)}$ is invariant under **T**. Since **T** is irred this space must coincide with $\mathcal{V}$ itself. There are two possibilities depending on whether or not $u = v$. If $u \neq v$ the set $\{f_{mn}^{(u,v)}, g_{kl}^{(v,u)}\}$ is linearly independent and $\mathcal{V} = \mathcal{V}^{(u,v)} \oplus \mathcal{V}^{(v,u)}$. We designate the $2(2u + 1) \times (2v + 1)$-dimensional rep by

$$\mathbf{D}^{(u,v)} \oplus \mathbf{D}^{(v,u)}, \qquad u \neq v. \tag{3.36}$$

Conversely, it is easy to show that each pair of reps of $L^{\uparrow+}$ taking the form (3.36) does define an irred rep of $L^{\uparrow}$.

Now suppose $u = v$ and define new vectors $h_{mn}^{\pm} = f_{mn}^{(u,u)} \pm g_{mn}^{(u,u)}$. The $\{h_{mn}^{+}\}$ span a subspace $\mathcal{V}^{(+)}$, while the $\{h_{mn}^{-}\}$ span $\mathcal{V}^{(-)}$. Here

$$\begin{aligned} C^3 h_{mn}^{\pm} &= m(f_{mn}^{(u,u)} \pm g_{mn}^{(u,u)}) = m h_{mn}^{\pm}, \qquad D^3 h_{mn}^{\pm} = n h_{mn}^{\pm} \\ C^+ h_{mn}^{\pm} &= [(u + m + 1)(u - m)]^{1/2} h_{m+1,n}^{\pm}, \\ D^+ h_{mn}^{\pm} &= [(u + n + 1)(u - n)]^{1/2} h_{mn+1}^{\pm}, \end{aligned} \tag{3.37}$$

with similar results for C^- and D^-. Also,

$$\mathbf{S} h_{mn}^{\pm} = \mathbf{S} f_{mn}^{(u,u)} \pm \mathbf{S} g_{mn}^{(u,u)} = g_{nm}^{(u,u)} \pm f_{mn}^{(u,u)} = \pm h_{nm}^{\pm}. \tag{3.38}$$

As a consequence, both $\mathcal{V}^{(+)}$ and $\mathcal{V}^{(-)}$ are invariant under **T**. Since **T** is irred, either $\mathcal{V} = \mathcal{V}^{(+)}$ and $\mathcal{V}^{(-)} = \{\theta\}$ or $\mathcal{V} = \mathcal{V}^{(-)}$ and $\mathcal{V}^{(+)} = \{\theta\}$. If the first case holds then $h_{mn}^{-} = \theta$, so $f_{mn}^{(u,u)} = g_{mn}^{(u,u)} = \mathbf{S} f_{nm}^{(u,u)}$ and **S** transposes the lower indices of the basis vectors $f_{mn}^{(u,u)}$ for $\mathcal{V}$. We denote the corresponding irred rep by $\mathbf{D}_{+}^{(u,u)}$. If the second case holds then $h_{mn}^{+} = \theta$ and $\mathbf{S} f_{nm}^{(u,u)} = -f_{mn}^{(u,u)}$. We denote this rep by $\mathbf{D}_{-}^{(u,u)}$. Here $\dim \mathbf{D}_{+}^{(u,u)} = \dim \mathbf{D}_{-}^{(u,u)} = (2u + 1)^2$.

Thus the possible irred reps of $L^{\uparrow}$ are $\mathbf{D}^{(u,v)} \oplus \mathbf{D}^{(v,u)}$, $u > v$, and $\mathbf{D}_{+}^{(u,u)}$, $\mathbf{D}_{-}^{(u,u)}$. Only those reps such that $u + v$ is an integer are single-valued on $L^{\uparrow}$. The remaining reps are double-valued on $L^{\uparrow}$ but they are single-valued reps of the group generated by S and $SL(2)$.

We can obtain the irred reps of $L(4)$ by noting that $L(4) = \{L^{\uparrow}, I \cdot L^{\uparrow}\}$, where $I = -E$ is the total inversion operation. Since I commutes with all elements of $L(4)$ and $I^2 = E$ it follows that $\mathbf{T}(I) = \pm \mathbf{E}$ for each irred rep **T**. Thus, to each irred rep of $L^{\uparrow}$ there correspond exactly two reps of $L(4)$. In one rep $\mathbf{T}(I) = \mathbf{E}$ and in the other $\mathbf{T}(I) = -\mathbf{E}$. If $u + v$ is not an integer then these reps are double-valued on $L(4)$ but single-valued on the group generated by S, I, and $SL(2)$.

We mention some of the simplest examples of our reps. The usual 4×4 matrix realization (1.1) of $L^{\uparrow+}$ is equivalent to the real rep $\mathbf{D}^{(1/2,1/2)}$ of $SL(2)$. The usual 2×2 matrix realization is equivalent to $\mathbf{D}^{(1/2,0)}$. The matrices $\bar{A}, A \in SL(2)$, define the rep $\mathbf{D}^{(0,1/2)}$. The usual 4×4 realization of $L^{\uparrow}$ is equivalent to $\mathbf{D}_{-}^{(1/2,1/2)}$. A quantity transforming under $L^{\uparrow}$ according to $\mathbf{D}_{+}^{(0,0)}$ is called a **scalar;** one transforming according to $\mathbf{D}_{-}^{(0,0)}$ is a **pseudoscalar. Vectors** and **pseudovectors** transform according to $\mathbf{D}_{-}^{(1/2,1/2)}$ and

$\mathbf{D}_+^{(1/2,1/2)}$, respectively. Finally the usual 4×4 realization of $L(4)$ is equivalent to $\mathbf{D}_-^{(1/2,1/2)}$ with $\mathbf{T}(I) = -\mathbf{E}$.

8.4 Models of the Representations

Let V be a complex two-dimensional vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2\}$. We define a model of the rep $\mathbf{D}^{(1/2,0)}$ of $SL(2)$ on V by

$$\mathbf{A}\mathbf{v}_j = \sum_{l=1}^{2} A_{lj}\mathbf{v}_l, \quad A = (A_{lj}) \in SL(2). \tag{4.1}$$

The vectors $\mathbf{f}_{1/2}^{(1/2)} = \mathbf{v}_1$ and $\mathbf{f}_{-1/2}^{(1/2)} = \mathbf{v}_2$ form a canonical basis.

Similarly we can define a model of the rep $\mathbf{D}^{(0,1/2)} \cong \bar{\mathbf{D}}^{(1/2,0)}$ on the two-dimensional vector space W with basis $\{\mathbf{w}_1, \mathbf{w}_2\}$:

$$\bar{\mathbf{A}}\mathbf{w}_j = \sum_{l=1}^{2} \bar{A}_{lj}\mathbf{w}_l. \tag{4.2}$$

The vectors $\mathbf{g}_{1/2}^{(1/2)} = \mathbf{w}_2$ and $\mathbf{g}_{-1/2}^{(1/2)} = -\mathbf{w}_1$ form a canonical basis.

Now consider a rep of $SL(2)$ on the $2^{(p+q)}$-dimensional space $V^{\otimes p} \otimes W^{\otimes q}$ defined by

$$\begin{aligned} \mathbf{A}(\mathbf{v}_{\alpha_1} \otimes \cdots \otimes \mathbf{v}_{\alpha_p} \otimes \mathbf{w}_{\beta_1} \otimes \cdots \otimes \mathbf{w}_{\beta_q}) \\ = \sum_{\alpha_j', \beta_k' = 1}^{2} A_{\alpha_1'\alpha_1} \cdots A_{\alpha_p'\alpha_p}\bar{A}_{\beta_1'\beta_1} \cdots \bar{A}_{\beta_q'\beta_q}\mathbf{v}_{\alpha_1'} \otimes \cdots \otimes \mathbf{w}_{\beta_q'}. \end{aligned} \tag{4.3}$$

The elements $\mathbf{a}$ of this space are called **spinors** of rank $p + q$. In terms of the components $a^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}$ of $\mathbf{a}$ with respect to the basis $\mathbf{v}_{\alpha_1}, \otimes \cdots \otimes \mathbf{w}_{\beta_q}$ the group action (4.3) reads

$$\mathbf{A}a^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} = \sum_{\alpha_j', \beta_k' = 1}^{2} A_{\alpha_1\alpha_1'} \cdots \bar{A}_{\beta_q\beta_q'}a^{\alpha_1'\cdots\alpha_p'\beta_1'\cdots\beta_q'}. \tag{4.4}$$

It is evident that the spinors of rank $p + q$ transform according to the rep $(\mathbf{D}^{(1/2,0)})^{\otimes p} \otimes (\mathbf{D}^{(0,1/2)})^{\otimes q}$. We can use the Clebsch–Gordan series (3.5) repeatedly to decompose this rep into irred reps $\mathbf{D}^{(u,v)}$ but the resulting expression is complicated. However, it is easy to verify that $\mathbf{D}^{(p/2,q/2)}$ is the irred rep of highest weight contained in the reducible rep and its multiplicity is exactly one. We show how to determine the subspace transforming under $\mathbf{D}^{(p/2,q/2)}$.

Let $\mathcal{S}^p$ be the subspace of completely symmetric spinors in $V^{\otimes p}$. The elements of $\mathcal{S}^p$ are symmetric in the spinor indices $a^{\alpha_1\cdots\alpha_p}$. As shown in Section 4.3, $\dim \mathcal{S}^p = p + 1$ and $\mathbf{a} \in \mathcal{S}^p$ is uniquely determined by the independent components $a^{11\cdots 1,22\cdots 2} = a^{(s)}$, where s is the number of twos and $p - s$ the number of ones, $s = 0, 1, \ldots, p$. Furthermore, $\mathcal{S}^p$ is invariant under the induced action of $SL(2)$ on $V^{\otimes p}$. We have shown earlier that $\mathcal{S}^p$

transforms irreducibly under the rep $[p, 0] = [p]$ of $GL(2)$ [Section 4.3]. We will now show that $\mathcal{S}^p$ remains irred when $GL(2)$ is restricted to the subgroup $SL(2)$. Let $\mathbf{a}^{s_0}$ be the element of $\mathcal{S}^p$ such that $a^{(s_0)} = 1$ and all independent components of $\mathbf{a}^{s_0}$ are zero. Clearly, the tensors $\mathbf{a}^0, \mathbf{a}^1, \ldots, \mathbf{a}^p$ form a basis for $\mathcal{S}^p$. Furthermore a direct computation from (4.4) with $q = 0$ shows

$$(\exp tC^3)\mathbf{a}^{s_0} = (\exp tJ^3)\mathbf{a}^{s_0} = \exp[t(\tfrac{1}{2}p - s_0)]\mathbf{a}^{s_0}.$$

Recall that the D-operators are zero for this rep. It follows that the highest weight vector in $\mathcal{S}^p$ with respect to C^3 has eigenvalue $p/2$. Thus $\mathcal{S}^p$ must contain a subspace transforming according to $\mathbf{D}^{(p/2,0)}$. Since $\dim \mathbf{D}^{(p/2,0)} = p + 1 = \dim \mathcal{S}^p$, $\mathcal{S}^p$ is irred.

An exactly similar argument shows that the subspace $\bar{\mathcal{S}}^q$ of completely symmetric spinors in $W^{\otimes q}$ transforms according to $\mathbf{D}^{(0,q/2)}$. Now the subspace $\mathcal{S}^p \otimes \bar{\mathcal{S}}^q$ of $V^{\otimes p} \otimes W^{\otimes q}$ consists of spinors $a^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}$ symmetric in the indices $\alpha_1, \ldots, \alpha_p$ and in the indices $\beta_1, \ldots, \beta_q$ simultaneously. Furthermore, $\mathcal{S}^p \otimes \bar{\mathcal{S}}^q$ transforms under $\mathbf{D}^{(p/2,0)} \otimes \mathbf{D}^{(0,q/2)} \cong \mathbf{D}^{(p/2,q/2)}$. This shows that $\mathcal{S}^p \otimes \bar{\mathcal{S}}^q$ is the subspace of $V^{\otimes p} \otimes W^{\otimes q}$ which carries the rep $\mathbf{D}^{(p/2,q/2)}$. Letting p and q range over all nonnegative integers we can obtain models of all reps $\mathbf{D}^{(u,v)}$ of $SL(2)$.

The use of spinors to provide models of $SL(2)$ reps is very popular in mathematical physics. An extensive spinor calculus has been evolved which enables one to perform operations on spinors to yield new spinors. For example, if $a^{\alpha_1\cdots\beta_q}$ is a spinor of rank $p + q$ and $b^{\alpha_1'\cdots\beta_{q'}'}$ is a spinor of rank $p' + q'$, then the quantity with components $a^{\alpha_1\cdots\beta_q}b^{\alpha_1'\cdots\beta_{q'}'}$ transforms as a spinor of rank $(p + p') + (q + q')$. For more details on the spinor calculus see the work of Gel'fand *et al.* [1].

Let V be a four-dimensional real vector space with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ and define a rep of $L^{\uparrow +}$ on V by

$$\Lambda\mathbf{v}_j = \sum_{l=1}^{4} \Lambda_{lj}\mathbf{v}_l, \qquad \Lambda \in L^{\uparrow +}. \tag{4.5}$$

This rep is clearly irred; in fact it is equivalent to $\mathbf{D}^{(1/2,1/2)}$. We will verify this explicitly.

If we restrict the rep (4.5) to the subgroup $SO(3)$ then $\mathbf{v}_4$ remains fixed and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ transform under the vector rep $\mathbf{D}^{(1)}$. The only four-dimensional irred reps of $L^{\uparrow +}$ are $\mathbf{D}^{(3/2,0)}$, $\mathbf{D}^{(0,3/2)}$, $\mathbf{D}^{(1/2,1/2)}$ and the first two of these reps remain irred when restricted to $SO(3)$. However,

$$\mathbf{D}^{(1/2,1/2)} \mid SO(3) \cong \mathbf{D}^{(1)} \oplus \mathbf{D}^{(0)}, \tag{4.6}$$

in agreement with our comments above, so (4.5) defines a rep equivalent to $\mathbf{D}^{(1/2,1/2)}$. One can verify from (1.3) that the vectors

$$\mathbf{f}_{\pm 1}^{(1)} = (1/\sqrt{2})(\pm\mathbf{v}_1 - i\mathbf{v}_2), \qquad \mathbf{f}_0^{(1)} = \mathbf{v}_3, \qquad \mathbf{f}_0^{(0)} = -i\mathbf{v}_4 \tag{4.7}$$

form a canonical basis which exhibits the decomposition (4.6). Furthermore from (1.9) the vectors

(4.8) $$\mathbf{f}^{(1/2,1/2)}_{\pm 1/2,\mp 1/2} = \mathbf{v}_3 \pm \mathbf{v}_4, \qquad \mathbf{f}^{(1/2,1/2)}_{\pm 1/2,\pm 1/2} = \pm\mathbf{v}_1 - i\mathbf{v}_2$$

form a canonical basis satisfying relations (3.2).

We can extend the action (4.5) to the space $V^{\otimes n}$. If $\mathbf{a} \in V^{\otimes n}$ with tensor components $a^{i_1 \cdots i_n}$, $1 \le i_j \le 4$, then the action of $L^{\uparrow +}$ on $V^{\otimes n}$ is given by

(4.9) $$\Lambda a^{i_1 \cdots i_n} = \sum_{j_1 \cdots j_n = 1}^{4} \Lambda_{i_1 j_1} \cdots \Lambda_{i_n j_n} a^{j_1 \cdots j_n}.$$

Clearly this rep is equivalent to $(\mathbf{D}^{(1/2,1/2)})^{\otimes n}$ and the Clebsch–Gordan series (3.5) can be used to decompose it into irred reps. Note that every irred part of $(\mathbf{D}^{(1/2,1/2)})^{\otimes n}$ is a single-valued rep of $L^{\uparrow +}$ and every single-valued irred rep can be so obtained. The elements of $V^{\otimes n}$ are called **tensors** in distinction to the spinors (4.4) which lead to double-valued reps.

The Lorentz group acts as a natural transformation group on Minkowski space according to the formula

(4.10) $$x \longrightarrow \Lambda^{-1}x, \qquad \Lambda \in L^{\uparrow +},$$

where $x = (x, y, z, ct)$ is a column four-vector. The Lie derivatives corresponding to this action are

(4.11)
$$L_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad L_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad L_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$B_1 = -t\frac{\partial}{\partial x} - x\frac{\partial}{\partial t}, \qquad B_2 = -t\frac{\partial}{\partial y} - y\frac{\partial}{\partial t}, \qquad B_3 = -t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t}.$$

In these equations and for the computations to follow, we choose units in which $c = 1$.

The components x, y, z, t form a basis for a realization of $\mathbf{D}^{(1/2,1/2)}$ under the action (4.10). Indeed, comparing (4.11) with (3.21)–(3.24) we see that a canonical basis exhibiting the decomposition (4.6) is given by

(4.12) $$f^{(1)}_{\pm 1} = (1/\sqrt{2})(\pm x - iy), \qquad f^{(1)}_0 = z, \qquad f^{(0)}_0 = it.$$

Furthermore, the vectors

(4.13) $$f^{(1/2,1/2)}_{\pm 1/2,\mp 1/2} = z \mp t, \qquad f^{(1/2,1/2)}_{\pm 1/2,\pm 1/2} = \pm x - iy$$

form a canonical basis satisfying relations (3.2).

Another model of $\mathbf{D}^{(1/2,1/2)}$ which will prove useful is obtained by using (4.10) to induce a group rep on the four-dimensional space $\mathfrak{D}$ spanned by the derivatives $\partial/\partial x_j$, $j = 1, \ldots, 4$, where $x = (x_1, \ldots, x_4) = (x, y, z, t)$. Indeed if $x_j' = \sum (\Lambda^{-1})_{jl} x_l$ then $x_l = \sum \Lambda_{lj} x_j'$ and

$$\frac{\partial}{\partial x_j'} = \sum_{l=1}^{4} \frac{\partial x_l}{\partial x_j'} \frac{\partial}{\partial x_l} = \sum_{l=1}^{4} \Lambda_{lj} \frac{\partial}{\partial x_l}.$$

The derivatives

$$\partial_{\pm 1/2, \mp 1/2} = \frac{\partial}{\partial z} \pm \frac{\partial}{\partial t}, \qquad \partial_{\pm 1/2, \pm 1/2} = \pm \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \tag{4.14}$$

form a canonical basis satisfying relations (3.2) for $u = v = \frac{1}{2}$.

At this point it is convenient to describe the relationship between energy and momentum of a particle in the theory of special relativity. Let $\mathcal{I}$ be an inertial frame with coordinates $x = (x_1, \ldots, x_4) = (x, y, z, t)$. We describe the path of a particle with mass m in this frame using the parametric equations $x_j = h_j(s)$, $1 \le j \le 4$, where the parameter s is determined by

$$\begin{aligned} ds &= [1 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2]^{1/2} dt \\ &= \pm[dt^2 - dx^2 - dy^2 - dz^2]^{1/2} = (1 - \mathbf{v}\cdot\mathbf{v})^{1/2}\, dt \end{aligned} \tag{4.15}$$

and $\mathbf{v}$ is the velocity of the particle at time t. [The sign on the right-hand side of (4.15) is the sign of dt.] Since no massive particle can have a velocity as great as the velocity of light ($c = 1$ in this case), ds is always real. The **world time** between two events q_1 and q_2 with coordinates $x^{(1)} = (\mathbf{x}^{(1)}, t^{(1)})$, $x^{(2)} = (\mathbf{x}^{(2)}, t^{(2)})$, $t^{(1)} \neq t^{(2)}$ which lie on the path of the particle is

$$s_2 - s_1 = \int_{t^{(1)}}^{t^{(2)}} ds, \tag{4.16}$$

where the integral is taken along the particle path from $x^{(1)}$ to $x^{(2)}$. Note that s_1 and s_2 are not uniquely determined by (4.16) but only their difference $s_2 - s_1$. The expression $dt^2 - dx^2 - dy^2 - dz^2$ is obviously invariant under the Lorentz group, so the world time between two events q_1, q_2 is the same for *all* inertial frames $\mathcal{I}'$ related to $\mathcal{I}$ by an element of $L^{\uparrow}$. However, if $\mathcal{I}'$ is related to $\mathcal{I}$ by an element of $L^{\downarrow +}$ or $L^{\downarrow -}$ then dt and dt' have opposite signs and $ds = -ds'$. In particular, under time inversion $(dx, dy, dz, dt) \to (dx, dy, dz, -dt)$. In this case the magnitude of the world time between two events is conserved but the sign is reversed.

If the particle is moving with uniform velocity $\mathbf{v}$ (with resepct to $\mathcal{I}$) then the frame $\mathcal{I}'$ with spatial axes parallel to the spatial axes of $\mathcal{I}$ and spatial origin of coordinates embedded in the particle is also an inertial frame. In $\mathcal{I}'$ the world time difference between q_1 and q_2 is just the ordinary time interval between the two events as determined by a clock fixed in the particle.

The **momentum** $\boldsymbol{p}$ of the particle is defined as

$$\boldsymbol{p} = (m\, dx/ds, m\, dy/ds, m\, dz/ds), \tag{4.17}$$

where $x(s), y(s), z(s)$ are the spatial coordinates of the particle with respect to $\mathcal{I}$. The **total energy** is given by

$$E = (\boldsymbol{p}\cdot\boldsymbol{p} + m^2)^{1/2} \tag{4.18}$$

and the **four-vector momentum** by

$$p = (\boldsymbol{p}, E) = (p_1, p_2, p_3, p_4). \tag{4.19}$$

Note that

$$p_4^{\,2} - p_3^{\,2} - p_2^{\,2} - p_1^{\,2} = m^2. \tag{4.20}$$

Now ds is Lorentz-invariant and (dx, dy, dz) transforms under the Lorentz group exactly as (x, y, z). Let $x = (x, y, z, t) = (\mathbf{x}, t)$ be the coordinates in $\mathcal{g}$ of an event q such that $t > 0$ and $t^2 - x^2 - y^2 - z^2 = m^2 > 0$. Then we can write

$$x = (\mathbf{x}, t) = (\mathbf{x}, (\mathbf{x}\cdot\mathbf{x} + m^2)^{1/2}). \tag{4.21}$$

Comparing (4.19) and (4.21), we see that both of these vectors must transform in exactly the same manner under $L^{\uparrow}$. (Here we are assuming that the mass of a particle is the same in all inertial frames.) Since x transforms according to $\mathbf{D}^{(1/2,1/2)}$, so does $\boldsymbol{p}$. In particular the expression for the four-momentum of a particle takes the same form in all inertial systems, as it must in order to be physically meaningful. This shows that expression (4.20) is also Lorentz-invariant. Note, however, that x and $\boldsymbol{p}$ do not transform in the same way under time inversion G. Under G, x goes to $(\mathbf{x}, -t)$ and $\boldsymbol{p}$ goes to $(-\boldsymbol{p}, E)$.

In relativistic quantum physics the states of a one-particle system at time t are given by spinor-valued functions $\boldsymbol{\Psi} = \{\Psi_\mu(x)\}$, $\mu = 1, \ldots, q$, where $x = (x, y, z, t)$. The action of the Poincaré group $\mathcal{P}$ on these state functions is given by

$$[\mathbf{T}(a, A)\Psi]_\mu(x) = \sum_{\nu=1}^{q} T_{\mu\nu}(A)\Psi_\nu(L(A^{-1})(x - a)), \tag{4.22}$$
$$a \in R_4, \qquad A \in SL(2),$$

where $L(A) \in L^{\uparrow +}$ is given by (1.18)–(1.22) and $T(A)$ is a $q \times q$ matrix rep of $SL(2)$. Here $\mathcal{P}$ is the set of all pairs $\{a, A\}$ with group product

$$\{a_1, A_1\}\{a_2, A_2\} = \{a_1 + L(A_1)a_2, A_1A_2\}. \tag{4.23}$$

The map

$$\{a, A\} \longrightarrow \{a, L(A)\}$$

is a homomorphism of $\mathcal{P}$ onto the ordinary Poincaré group P, (2.5), which covers each element of P exactly twice.

The construction of state functions for relativistic k-particle systems is analogous to that discussed in (11.6), Chapter 7, and is left to the reader. Furthermore, we shall be concerned only with the group-theoretic properties of the transformation (4.22) and shall omit any discussion of Hilbert spaces containing the state vectors $\boldsymbol{\Psi}$. For such a discussion see the work of Schweber [1].

In general, functions $\boldsymbol{\Psi}$ which transform under $\mathcal{P}$ by (4.22) are called **spinor fields.** If $T(A) = T(-A)$ for all $A \in SL(2)$ then (4.22) defines a single-valued rep of P and the functions are called **tensor fields.** Among the important tensor and spinor fields of relativistic physics are the four-momentum and the vector four-potential, $T \cong D^{(1/2,1/2)}$, and the Dirac electron field, $T \cong D^{(1/2,0)} \oplus D^{(0,1/2)}$ (Roman [1], Landau and Lifshitz [3]).

If we set $a = \theta$ in expression (4.22) we obtain a rep of $SL(2)$. The Lie algebra rep of $sl(2)$ induced by this group action takes the form

$$\text{(4.24)} \qquad \mathcal{J}_j = \mathcal{S}_j + L_j, \qquad \mathcal{B}_j = \mathcal{K}_j + B_j \qquad j = 1, 2, 3,$$

where the Lie derivatives L_j, B_j are given by (4.11) and the matrices

$$\text{(4.25)} \qquad \mathcal{S}_j = \frac{d}{dt} T(\exp t\mathcal{J}_j)|_{t=0}, \qquad \mathcal{K}_j = \frac{d}{dt} T(\exp t\mathcal{B}_j)|_{t=0},$$
$$\mathcal{J}_j, \mathcal{B}_j \in sl(2),$$

act on the spinor indices of $\mathbf{\Psi}$. Suppose we restrict the group rep $\mathbf{T}(A) = \mathbf{T}(\theta, A)$ to the subgroup $SU(2)$. Then the matrix rep $T(A)$ will decompose into a direct sum of irred reps $D^{(u)}$ of $SU(2)$. The spinor components of $\mathbf{\Psi}$ can always be chosen so that $T(A) \mid SU(2)$ explicitly exhibits this direct sum decomposition:

$$T(A) = \begin{pmatrix} T^{(u_1)}(A) & & Z \\ & \ddots & \\ Z & & T^{(u_k)}(A) \end{pmatrix}, \qquad A \in SU(2).$$

Thus, on restriction to $SU(2)$ the field $\mathbf{\Psi}$ transforms as a sum of spinor fields of weights $s = u_1, \ldots, u_k$ with respect to $SU(2)$. This last statement is meant in the sense of (8.19), Chapter 7. The above remarks constitute the relativistic interpretation of spin. If a particle state function transforms according to (4.22) with $T \cong D^{(u,v)}$ in a relativistic theory then the formula

$$\text{(4.26)} \qquad \mathbf{D}^{(u,v)} \mid SU(2) \cong \mathbf{D}^{(u+v)} \oplus \mathbf{D}^{(u+v-1)} \oplus \cdots \oplus \mathbf{D}^{(|u-v|)}$$

shows that this particle can have spins $s = u + v, u + v - 1, \ldots, |u - v|$. However, there is no known particle with more than one spin. For particles transforming according to $\mathbf{D}^{(u,0)}$ or $\mathbf{D}^{(0,u)}$ this restriction to one spin is achieved automatically: $s = u$. However, for particles which transform according to $\mathbf{D}^{(u,v)}$ with $u, v > 0$ it is necessary to subject the spinor function $\mathbf{\Psi}$ to certain additional constraints which, in a fixed inertial coordinate system, require that all components Ψ_μ of $\mathbf{\Psi}$ are zero except those transforming according to a single rep $\mathbf{D}^{(s)}$ of $SU(2)$.

For example, the photon transforms according to the four-dimensional representation $\mathbf{D}^{(1/2,1/2)}$. Since $\mathbf{D}^{(1/2,1/2)} \mid SU(2) \cong \mathbf{D}^{(1)} \oplus \mathbf{D}^{(0)}$ we would expect the photon to have spins one and zero. However, the system of equations obeyed by the photon includes a supplementary condition which suppresses the component transforming according to $\mathbf{D}^{(0)}$ and we say that the photon has spin one (see the work of Jauch and Rohrlich [1]). The Dirac electron field transforms according to $\mathbf{D}^{(1/2,0)} \oplus \mathbf{D}^{(0,1/2)}$. On restriction to $SU(2)$ we obtain $\mathbf{D}^{(1/2)} \oplus \mathbf{D}^{(1/2)}$, so the electron has the single spin $\frac{1}{2}$, even though $\mathbf{D}^{(1/2)}$ occurs with multiplicity two.

8.5 Lorentz-Invariant Equations

In Section 8.2 we enunciated the basic principle of relativistic physics: The equations and laws of a physical theory must have the same form in any inertial coordinate system. Stated another way, the equations of a physical theory must maintain their form under the action of the Poincaré group. We shall use this principle to classify, under suitable conditions, the possible linear differential equations which can appear in a relativistic theory. Our analysis will be analogous to that for the Euclidean-invariant equations in Section 7.11.

Let $\Psi_\mu(x)$ be a q-component spinor field transforming according to the rule (4.22) under the Poincaré group. We suppose that the components Ψ_μ satisfy a system of q linear partial differential equations in the independent variables $x = (x, y, z, t)$. By introducing new components if necessary we can assume the system takes the form

$$\left(C_1\frac{\partial}{\partial x} + C_2\frac{\partial}{\partial y} + C_3\frac{\partial}{\partial z} + C_4\frac{\partial}{\partial t} + D\right)\mathbf{\Psi}(x) = 0, \tag{5.1}$$

where C_j and D are $q \times q$ matrix functions of x and $\mathbf{\Psi}(x) = (\Psi_\mu(x))$ is a q-component column vector. We will investigate the conditions under which the system (5.1) maintains its form under the action (4.22) of $\mathcal{P}$. First, it is clear that (5.1) is invariant under all translations of coordinates if and only if the matrices C_j and D are constant. Assuming these matrices constant we reduce the problem to one of invariance under $SL(2)$ (the homogeneous Lorentz group):

$$[\mathbf{T}(A)\mathbf{\Psi}]_\mu(x) = \Psi_\mu'(x) = \sum_{\nu=1}^{q} T_{\mu\nu}(A)\Psi_\nu(L(A^{-1})x), \tag{5.2}$$

or

$$\Psi_\mu'(x') = \sum_\nu T_{\mu\nu}(A)\Psi_\nu(x), \qquad x' = L(A)x, \tag{5.3}$$

where $T(A)$ is a $q \times q$ matrix rep of $SL(2)$. Lorentz invariance of (5.1) means exactly that if we replace x by x' and $\Psi_\mu(x)$ by $\Psi_\mu'(x')$ then the resulting system of equations is equivalent to the original system, i.e., the primed equations are linear combinations of the unprimed equations and conversely.

To simplify the discussion we assume D is nonsingular. Then multiplying (5.1) on the left by D^{-1} we obtain the equivalent system

$$\left(L_1\frac{\partial}{\partial x} + L_2\frac{\partial}{\partial y} + L_3\frac{\partial}{\partial z} + L_4\frac{\partial}{\partial t}\right)\mathbf{\Psi} = \kappa\mathbf{\Psi}, \tag{5.4}$$

where κ is a nonzero constant. (We could choose $\kappa = -1$ but it is preferable to leave it arbitrary.)

We may assume without loss of generality that the matrix rep $T(A)$ has already been decomposed into a direct sum of irred reps:

(5.5)
$$T(A) = \begin{pmatrix} D^{(0,0)}(A) & & & \alpha_{00} & & & \\ & \ddots & & & & Z & \\ & & D^{(0,0)}(A) & & & & \\ & & & \ddots & & & \\ & & & & D^{(u,v)}(A) & & \alpha_{uv} \\ & Z & & & & \ddots & \\ & & & & & & D^{(u,v)}(A) \\ & & & & & & & \ddots \end{pmatrix},$$

where $D^{(u,v)}(A)$ is a matrix realization of $\mathbf{D}^{(u,v)}$ and α_{uv} is the multiplicity of $\mathbf{D}^{(u,v)}$ in $T(A)$. We can label the components of $\mathbf{\Psi}$ as $\Psi^{uv}_{mn,k}$, the spin component corresponding to the canonical basis vector $f^{(u,v)}_{mn}$ in the kth occurrence of $\mathbf{D}^{(u,v)}$ in (5.5). Here we are using the basis (3.2). Thus,

$$[\mathbf{T}(A)\mathbf{\Psi}]^{uv}_{mn,k}(x) = \sum_{m'=-u}^{u} \sum_{n'=-v}^{v} D^{(u,v)}_{mn,m'n'}(A)\Psi^{uv}_{m'n',k}(L(A^{-1})x). \tag{5.6}$$

The partial derivatives on the left-hand side of (5.4) can be expressed as linear combinations of the derivatives $\partial_{\pm 1/2, \pm 1/2}$, (4.14), which form a canonical basis for a realization of $\mathbf{D}^{(1/2,1/2)}$. Thus the left-hand side of (5.4) is a linear combination of terms $\partial_{\pm 1/2, \pm 1/2}\Psi^{uv}_{m'n',k}$. For fixed u, v, and k, and m', n' ranging over $-u \le m' \le u$, $-v \le n' \le v$, these $4(2u+1)(2v+1)$ quantities form a basis for the rep

$$\mathbf{D}^{(1/2,1/2)} \otimes \mathbf{D}^{(u,v)} \cong \mathbf{D}^{(u+1/2,v+1/2)} \oplus \mathbf{D}^{(u+1/2,v-1/2)} \oplus \mathbf{D}^{(u-1/2,v+1/2)} \oplus \mathbf{D}^{(u-1/2,v-1/2)}. \tag{5.7}$$

If either u or v is zero, this expression has an obvious modification. By (5.7) and (3.6), the new basis functions

$$h^{u'v'}_{m'n'}(uv, k) = \sum_{mnjl} C(\tfrac{1}{2}, j; u, m \mid u', m')C(\tfrac{1}{2}, l; v, n \mid v', n')\, \partial_{j,l}\Psi^{uv}_{mn,k} \tag{5.8}$$

transform irreducibly according to $\mathbf{D}^{(u',v')}$. Here $u' = u \pm \frac{1}{2}$ and $v' = v \pm \frac{1}{2}$ for $u, v > 0$. Again the results must be slightly modified if either $u = 0$ or $v = 0$.

Due to the unitarity of the CG coefficients we can uniquely express each of the terms $\partial_{j,l}\Psi^{uv}_{mn,k}$ on the left-hand side of (5.4) as linear combinations of the $h^{u'v'}_{m'n'}(uv, k)$. The resulting system takes the form

$$\sum_{m'n'k_1u_1v_1} A^{uv,u'v',k}_{mn,m'n'}(u_1v_1, k_1)h^{u'v'}_{m'n'}(u_1v_1, k_1) = \kappa\Psi^{uv}_{mn,k}. \tag{5.9}$$

We consider a subsystem of $(2u+1)(2v+1)$ equations (5.9) for which

u, v, k are fixed and $-u \le m \le u, -v \le n \le v$. From (5.3) and (5.6) we obtain

$$\Psi'^{uv}_{mn,k}(x') = \sum_{jl} D^{(u,v)}_{mn,jl}(A)\Psi^{uv}_{jl,k}(x), \tag{5.10}$$

$$h'^{u'v'}_{m'n'}(u_1v_1, k_1)(x') = \sum_{j'l'} D^{(u',v')}_{m'n',j'l'}(A)h^{u'v'}_{j'l'}(u_1v_1, k_1)(x). \tag{5.11}$$

If follows that our subsystem will maintain its form under the action of $SL(2)$ if and only if the left-hand side of the subsystem also transforms according to $\mathbf{D}^{(u,v)}$. The necessary and sufficient condition for invariance is that all constants $A^{uv,u'v'}$ are zero except those for which $u = u'$, $v = v'$, $m = m'$, and $n = n'$. Furthermore, the nonzero constants must be independent of the spin indices m and n. Thus, any invariant system takes the form

$$\sum_{u_1v_1k_1} A^{uv,k}_{u_1v_1,k_1} h^{uv}_{mn}(u_1v_1, k_1) = \kappa\Psi^{uv}_{mn,k}, \qquad k = 1, \ldots, \alpha_{uv} \tag{5.12}$$

where h^{uv}_{mn} is given by (5.8) and the pair (u, v) ranges over all irred reps in $T(A)$. The constants $A^{uv,k}_{u_1v_1,k_1}$ are arbitrary and there is one equation for each component of $\mathbf{\Psi}$. We see from this analysis that the component $\Psi^{uv}_{mn,k}$ on the right is coupled with those components $\Psi^{u_1v_1}_{m_1n_1,k_1}$ on the left such that

(5.13)

$$u_1 = \begin{cases} u + \frac{1}{2}, u - \frac{1}{2} & \text{if} \quad u > 0 \\ \frac{1}{2} & \text{if} \quad u = 0, \end{cases} \qquad v_1 = \begin{cases} v + \frac{1}{2}, v - \frac{1}{2} & \text{if} \quad v > 0 \\ \frac{1}{2} & \text{if} \quad v = 0. \end{cases}$$

Note that there are no nontrivial invariant equations in which the spinor indices transform according to a single irred rep $\mathbf{D}^{(u,v)}$. With a single $\mathbf{D}^{(u,v)}$ we could not achieve a coupling (5.13).

In case the matrix D in (5.1) is singular or not square the analogous discussion in Section 7.11 is applicable. If $D = Z$ we can construct invariant equations of the form (5.12) with $\kappa = 0$ although the number of such equations need not be equal to the number of components of $\mathbf{\Psi}$. In this case it is possible to construct invariant equations in which the spinor indices transform according to a single irred rep $\mathbf{D}^{(u,v)}$. For an arbitrary singular matrix D one can construct systems of the form (5.12) in which κ is zero for some equations and nonzero for others in the system.

Naimark [2] presents a complicated derivation of results equivalent to Eq. (5.12) based on computations using the Lie algebra of the Lorentz group. His derivation has the useful feature that it generalizes to the case where the matrices L_j in (5.4) are infinite. In this case the infinite-dimensional irred reps of the Lorentz group may appear.

It is worth mentioning that all Lorentz-invariant equations are automatically Euclidean-invariant since $\mathcal{E}^+(3)$ is a subgroup of $\mathcal{P}$. Thus the Lorentz-invariant equations are already contained in the analysis of Section 7.11.

Our results must be modified if we demand invariance under the complete Lorentz group $L^{\uparrow}$ obtained by adding space reflection S to the proper Lorentz group. In Section 4.3 we showed that the irred reps of $L^{\uparrow}$ were

$$\text{(5.14)} \qquad \mathbf{D}^{(u,v)} \oplus \mathbf{D}^{(v,u)}, \qquad u > v, \qquad \mathbf{S}f_{mn}^{(u,v)} = f_{nm}^{(v,u)},$$

$$\text{(5.15)} \qquad \mathbf{D}_{\pm}^{(u,u)}, \qquad \mathbf{S}f_{mn}^{(u,u)} = \pm f_{nm}^{(u,u)},$$

where $\mathbf{S}$ is the operator corresponding to space reflection in each rep space. It follows from (4.14) that $\partial_{j,l} \to -\partial_{l,j}$ under space reflection, so the $\partial_{j,l}$ form a canonical basis for the rep $\mathbf{D}_{-}^{(1/2,1/2)}$ of $L^{\uparrow}$.

Suppose $\mathbf{\Psi} = \{\Psi_\mu(x)\}$ is a spinor field transforming under $L^{\uparrow}$. In addition to the transformation equations (5.2) we have

$$\text{(5.16)} \qquad [\mathbf{T}(S)\Psi]_\mu(x) = \Psi_\mu{}'(x) = \sum_{\nu=1}^{q} T_{\mu\nu}(S)\Psi_\nu(S^{-1}x),$$

where the matrices $T(A)$, $T(S)$ generate a rep of $L^{\uparrow}$ (possibly double-valued). Here $T(S)^2 = E$. The matrix rep can be decomposed into a direct sum of irred reps of $L^{\uparrow}$. (Prove it!) Thus, each component $\Psi_{mn,k}^{uv}$, $u \neq v$, is associated with a component $\Psi_{nm,k}^{vu}$ such that

$$\text{(5.17)} \qquad \mathbf{T}(S)\Psi_{mn,k}^{uv}(x') = \Psi_{nm,k}^{vu}(x), \qquad x' = Sx,$$
$$-u \le m \le u, \quad -v \le n \le v.$$

For $u = v$ there are possible components $\Psi_{mn,k}^{uu+}$ and $\Psi_{mn,k}^{uu-}$ such that

$$\text{(5.18)} \qquad \mathbf{T}(S)\Psi_{mn,k}^{uu\pm}(x') = \pm\Psi_{nm,k}^{uu\pm}(x).$$

We assume $\mathbf{\Psi}$ satisfies the equations (5.4) with $\kappa \neq 0$ and require that this system is $L^{\uparrow}$-invariant. Clearly, the system is $L^{\uparrow +}$ invariant so it can be expressed in the form (5.12). To guarantee $L^{\uparrow}$-invariance we need only determine the requirements on the constants $A_{u_1v_1,k_1}^{uv,k}$ in order that the system of equations remains invariant under space inversion.

Choose one of the equations (5.12) and replace x by $x' = Sx$ and $\Psi_{m'n',k'}^{u'v'}(x)$ by $\mathbf{T}(S)\Psi_{m'n',k'}^{u'v'}(x')$ on both sides of the equation. If $u \neq v$ then the right-hand side becomes $\kappa\Psi_{nm,k}^{uv}$, while the vectors $h_{mn}^{uv}(u_1v_1, k_1)$ become

$$\text{(5.19)} \qquad \mathbf{T}(S)h_{mn}^{uv}(u_1v_1, k_1) = \begin{cases} -h_{nm}^{vu}(v_1u_1, k_1) & \text{if } u_1 \neq v_1 \\ \mp h_{mn}^{vu}(u_1\pm, k_1) & \text{if } u_1 = v_1 \text{ and } \Psi_{m_1n_1,k_1}^{u_1v_1} = \Psi_{m_1n_1,k_1}^{u_1u_1\pm}. \end{cases}$$

Here we have used (5.8), (5.17), (5.18), and the fact that $\partial'_{j,l} = -\partial_{l,j}$. If the system is $L^{\uparrow}$-invariant then this transformed equation must be identical with the original equation for the component $\kappa\Psi_{nm,k}^{uv}$. But from (5.19) this is possible if and only if

$$\text{(5.20)} \qquad A_{u_1v_1,k_1}^{uv,k} = \begin{cases} -A_{v_1u_1,k_1}^{vu,k} & \text{for } u_1 \neq v_1 \\ \mp A_{u_1\pm,k_1}^{vu,k} & \text{for } u_1 = v_1, \quad \text{parity } \pm. \end{cases}$$

If the term on the right-hand side is $\kappa\Psi_{mn,k}^{u,u\pm}$ then under space inversion it is mapped to $\pm\kappa\Psi^{u,u\pm}_{nm,k}$. Again, for $L^{\uparrow}$-invariance the transformed equation must be the same as the original equation for the component $\kappa\Psi^{uu\pm}_{nm,k}$. This is possible if and only if

$$A^{u\pm,k}_{u_1v_1,k_1} = \begin{cases} \mp A^{u\pm,k}_{v_1u_1,k_1} & \text{for} \quad u_1 \neq v_1, \\ \mp A^{u\pm,k}_{u_1+,k_1} & \text{for} \quad u_1 = v_1, \quad \text{positive parity}, \\ \pm A^{u\pm,k}_{u_1-,k_1} & \text{for} \quad u_1 = v_1, \quad \text{negative parity}, \end{cases} \tag{5.21}$$

as can be shown by a proof similar to that of (5.20). Expressions (5.20) and (5.21) are necessary and sufficient for $L^{\uparrow}$-invariance of the system (5.12).

One of the simplest examples of an $L^{\uparrow +}$-invariant equation is the **Klein–Gordon Equation**

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\varphi(x) = m_0{}^2\varphi(x). \tag{5.22}$$

Here $\varphi(x)$ transforms as a scalar field under $\mathcal{P}$:

$$[\mathbf{T}(a, \Lambda)\varphi](x) = \varphi(\Lambda^{-1}(x - a)). \tag{5.23}$$

In relativistic physics this equation describes fields corresponding to particles with mass m_0. Since the spin index of φ transforms according to $\mathbf{D}^{(0,0)}$ and $\mathbf{D}^{(0,0)} \mid SO(3) \cong \mathbf{D}^{(0)}$, these particles must have spin zero.

To see the connection between the Klein–Gordon equation and elementary particles recall that in classical relativistic physics the relation between momentum and energy of a particle with mass m_0 is

$$E^2 - p_1{}^2 - p_2{}^2 - p_3{}^2 = m_0{}^2, \tag{5.24}$$

[Eq. (4.20)]. In quantum physics we associate the classical momenta and energy with differential operators according to the rule

$$\begin{aligned} p_1 &\longleftrightarrow i\,\partial/\partial x = \mathbf{P}_1, & p_2 &\longleftrightarrow i\,\partial/\partial y = \mathbf{P}_2, \\ p_3 &\longleftrightarrow i\,\partial/\partial z = \mathbf{P}_3, & E &\longleftrightarrow i\,\partial/\partial t = \mathbf{H}. \end{aligned} \tag{5.25}$$

From (5.24) and the usual correspondence principle between classical and quantum physics we see that the state function $\varphi(\mathbf{x}, t)$ describing a particle of mass m_0 satisfies the equation

$$(\mathbf{H}^2 - \mathbf{P}_1{}^2 - \mathbf{P}_2{}^2 - \mathbf{P}_3{}^2)\varphi(\mathbf{x}, t) = m_0{}^2\varphi(\mathbf{x}, t). \tag{5.26}$$

Making the substitutions (5.25), we obtain the Klein–Gordon equation.

Let us write (5.22) in the canonical form (5.12). We introduce four new components $\varphi_{m,n}(x) = \partial_{m,n}\varphi(x)$, $m, n = \pm\frac{1}{2}$, which form a canonical basis for $\mathbf{D}^{(1/2,1/2)} \otimes \mathbf{D}^{(0,0)} \cong \mathbf{D}^{(1/2,1/2)}$. From (4.14), the Klein–Gordon equation is equivalent to the system

$$\begin{gathered} \partial_{m,n}\varphi = \varphi_{m,n}, \qquad m, n = \pm\tfrac{1}{2} \\ \tfrac{1}{2}(-\partial_{1/2,1/2}\varphi_{-1/2,-1/2} + \partial_{1/2,-1/2}\varphi_{-1/2,1/2} + \partial_{-1/2,1/2}\varphi_{1/2,-1/2} \\ - \partial_{-1/2,-1/2}\varphi_{1/2,1/2}) = m_0{}^2\varphi. \end{gathered} \tag{5.27}$$

The indices of the spinor field $\mathbf{\Phi} = (\varphi_{m,n}, \varphi)$ transform according to $\mathbf{D}^{(1/2,1/2)} \oplus \mathbf{D}^{(0,0)}$. Now the most general $L^{\uparrow +}$-invariant system of equations for such a field takes the form

$$
(5.28) \qquad \begin{aligned}
&a \sum_{jl=-1/2}^{1/2} C(\tfrac{1}{2}, j; 0, 0 \,|\, \tfrac{1}{2}, m) C(\tfrac{1}{2}, l; 0, 0 \,|\, \tfrac{1}{2}, n)\, \partial_{j,l}\varphi = \kappa \varphi_{m,n},\\
&b \sum_{jlm_1n_1=-1/2}^{1/2} C(\tfrac{1}{2}, j; \tfrac{1}{2}, m_1 \,|\, 0, 0) C(\tfrac{1}{2}, l; \tfrac{1}{2}, n_1 \,|\, 0, 0)\, \partial_{j,l}\varphi_{m,n} = \kappa\varphi.
\end{aligned}
$$

According to table (7.27), Section 7.7 these two systems are identical provided $\kappa = 1, a = 1, b = -1/m_0^2$. [Recall that$C(\frac{1}{2}, j; 0, 0 \,|\, \frac{1}{2}, m) = \delta_{jm}$.]

Now consider the behavior of (5.22) under space inversion. Under the group $L^\uparrow$, φ transforms as $\mathbf{D}_+^{(0,0)}$ or $\mathbf{D}_-^{(0,0)}$, i.e., as a **scalar** or a **pseudoscalar.** Thus $[\mathbf{T}(S)\varphi](x) = \pm\varphi(Sx)$. In either case it is obvious that the Klein–Gordon equation remains invariant under space inversion. However, it is instructive to verify this result for the system (5.28). If φ is a scalar then $\mathbf{\Phi} = (\varphi_{m,n}, \varphi)$ transforms as $\mathbf{D}_-^{(1/2,1/2)} \oplus \mathbf{D}_+^{(0,0)}$. It then follows from (5.21) that the system is $L^\uparrow$-invariant. Similarly, if φ is pseudoscalar then $\mathbf{\Phi}$ transforms as $\mathbf{D}_+^{(1/2,1/2)} \oplus \mathbf{D}_-^{(0,0)}$ and (5.28) is $L^\uparrow$-invariant.

Note that (5.27) is Lorentz-invariant even if $m_0 = 0$, in which case it corresponds to the system (5.1) with D singular.

We cannot write a nontrivial first-order system of equations for a spinor field transforming as $\mathbf{D}^{(0,0)}$. The next simplest possibility is $\mathbf{D}^{(1/2,0)}$. A particle described by such a spinor field would have spin $\frac{1}{2}$. As we have already remarked, this field cannot satisfy a system of the form (5.12) with $\kappa \neq 0$ since $\mathbf{D}^{(1/2,0)}$ cannot couple with itself. However, for $\kappa = 0$ the relation $\mathbf{D}^{(1/2,1/2)} \otimes \mathbf{D}^{(1/2,0)} \simeq \mathbf{D}^{(1,1/2)} \oplus \mathbf{D}^{(0,1/2)}$ suggests the system

$$
\sum_{j=-1/2}^{1/2} C(\tfrac{1}{2}, j; \tfrac{1}{2}, -j \,|\, 0, 0)\, \partial_{j,l}\Psi_{-j} = 0, \qquad l = \pm\tfrac{1}{2},
$$

or

$$
(5.29) \qquad \partial_{1/2,l}\Psi_{-1/2} - \partial_{-1/2,l}\Psi_{1/2} = 0, \qquad l = \pm\tfrac{1}{2}.
$$

The left-hand side of this system transforms as $\mathbf{D}^{(0,1/2)}$. (We reject a system whose left-hand side transforms according to $\mathbf{D}^{(1,1/2)}$ since it would subject the two spinor components to six conditions). Expression (5.29) is the equation of the **two-component neutrino.** This equation cannot possibly be invariant under space reflection because $\mathbf{\Psi}$ transforms as $\mathbf{D}^{(1/2,0)}$. Thus $\mathbf{T}(S)\mathbf{\Psi}$ transforms as $\mathbf{D}^{(0,1/2)}$ and the system does not admit S as a symmetry. It is easy to verify the formulas

$$
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\Psi_{\pm 1/2}(x) = 0
$$

from (5.29), so each component of the field is a solution of the Klein–Gordon equation with $m_0 = 0$. We conclude that the neutrino has spin $\frac{1}{2}$, mass zero, and does not conserve parity, i.e., does not transform according to a rep of the complete Lorentz group.

From experimental results the electron is known to have spin $\frac{1}{2}$, nonzero mass, and to conserve parity in those reactions in which it takes part. Thus we would expect a spinor field $\mathbf{\Psi}$ corresponding to an electron to have spin $\frac{1}{2}$ and to satisfy an $L^{\uparrow}$-invariant first-order system. The simplest possibility is that $\mathbf{\Psi}$ transforms as $\mathbf{D}^{(1/2,0)} \oplus \mathbf{D}^{(0,1/2)}$ under $L^{\uparrow}$. Then Eq. (5.12) take the form

$$
\begin{aligned}
&a \sum_{l=-1/2}^{1/2} C(\tfrac{1}{2}, l; \tfrac{1}{2}, -l \,|\, 0, 0)\, \partial_{m,l} \Psi_{-l}^{-} = \kappa \Psi_m^{+} \\
(5.30)\qquad & \\
&b \sum_{j=-1/2}^{1/2} C(\tfrac{1}{2}, j; \tfrac{1}{2}, -j \,|\, 0, 0)\, d_{j,m} \Psi_{-j}^{+} = \kappa \Psi_m^{-}, \qquad m = \pm\tfrac{1}{2},
\end{aligned}
$$

where $\mathbf{\Psi} = \{\Psi_m^{+}, \Psi_m^{-}\}$ is a four-component spinor, $\{\Psi_m^{+}\}$ forms a canonical basis for $\mathbf{D}^{(1/2,0)}$, and $\{\Psi_m^{-}\}$ forms a canonical basis for $\mathbf{D}^{(0,1/2)}$. Under space inversion Ψ_m^{+} goes to Ψ_m^{-} and Ψ_m^{-} goes to Ψ_m^{+}. Thus the system (5.30) is invariant under space inversion if and only if $a = -b$. If we choose $a = -b = \sqrt{2}$, (5.30) becomes

$$
(5.31)\qquad
\begin{aligned}
\partial_{m,1/2} \Psi_{-1/2}^{-} - \partial_{m,-1/2} \Psi_{1/2}^{-} &= \kappa \Psi_m^{+}, \\
-\partial_{1/2,m} \Psi_{-1/2}^{+} + \partial_{-1/2,m} \Psi_{1/2}^{+} &= \kappa \Psi_m^{-},
\end{aligned}
$$

or in matrix form

$$
(5.32)\qquad \left(L_1 \frac{\partial}{\partial x} + L_2 \frac{\partial}{\partial y} + L_3 \frac{\partial}{\partial z} + L_4 \frac{\partial}{\partial t}\right) \mathbf{\Psi} = m_0 \mathbf{\Psi}, \qquad i\kappa = m_0,
$$

where

$$
L_1 = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad
L_2 = i \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},
$$

$$
(5.33)\qquad L_3 = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad
L_4 = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
$$

$$
\mathbf{\Psi} = \begin{pmatrix} \Psi_{1/2}^{+} \\ \Psi_{-1/2}^{+} \\ \Psi_{1/2}^{-} \\ \Psi_{-1/2}^{-} \end{pmatrix}.
$$

Note that the matrices L_j satisfy the relations

$$(5.34) \qquad L_j L_k + L_k L_j = 2G_{jk},$$

Where $G = (G_{jk})$ is the matrix (1.1).

From (5.32)

$$(5.35) \qquad \left(\sum_{j=1}^{4} L_j \,\partial/\partial x_j\right)^2 \Psi = m_0{}^2 \Psi.$$

On the other hand, we can use (5.34) to evaluate the left side of this expression:

$$(5.36) \qquad \left(\sum_{j=1}^{4} L_j \frac{\partial}{\partial x_j}\right)^2 = \sum_{j,k=1}^{4} L_j L_k \frac{\partial^2}{\partial x_j\, \partial x_k} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}.$$

Thus, (5.35) becomes

$$(5.37) \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\Psi = m_0{}^2 \Psi,$$

so each component of Ψ satisfies the Klein–Gordon equation. The system (5.32) is one form of the **Dirac electron equation.** A solution of this equation corresponds to a particle of mass m_0 and spin $\frac{1}{2}$ which conserves parity. We will investigate other possible forms of the Dirac equation in Section 9.6.

Maxwell's equations for an electromagnetic field in a vacuum provide another important example of a Lorentz-invariant system. See the work of Gel'fand *et al.* [1] for the details.

Problems

8.1 Let y be a four-vector such that $y^t G y = -m^2 < 0$ and $y_4 > 0$. (We say y is **forward-timelike.**) Show that there exists a $\Lambda \in L^{\uparrow +}$ such that $x = \Lambda y$ where $x_1 = x_2 = x_3 = 0$, $x_4 = m$.

8.2 Use the polar decomposition to obtain an alternate proof of Theorem 8.2.

8.3 Let $\mathbf{D}^{(u,v)}$ be a finite-dimensional irred rep of the real Lie group $SL(2, \mathbb{C})$. Express the $\{f_{mn}^{(u,v)}\}$ basis in terms of the $\{f_k^{(w)}\}$ basis (Section 8.3).

8.4 Verify directly that the operators (3.33) define a global irred unitary rep of $SL(2)$ on $L_2(R_2)$ whose induced Lie algebra rep is equivalent to (w_0, ic).

8.5 Decompose the reps $(\mathbf{D}_+^{(1/2,1/2)})^{\otimes n}$ and $(\mathbf{D}_-^{(1/2,1/2)})^{\otimes n}$ of $L^\uparrow$ into irred reps for $n = 2, 3, 4, 5$.

8.6 Discuss the Lorentz invariance of Maxwell's equations using the methods of Section 8.5. Include a discussion of invariance under space inversion. (See Landau and Lifshitz [3] for the relativistic transformation properties of Maxwell's equations.)

8.7 Discuss the simplest relativistic equations suitable for describing a particle with spin $\frac{3}{2}$. Which equations are invariant under space inversion?

8.8 Answer Problem 8.7 for particles with spin one.