Chapter 10

The Harmonic Oscillator Group

10.1 The Harmonic Oscillator

The two most important nonrelativistic systems whose Schrödinger equations can be completely solved are the hydrogen atom and the harmonic oscillator. We have seen that the tractability of the hydrogen atom is related to its high degree of symmetry and we shall reach similar conclusions for the harmonic oscillator.

We start with a system in one-dimensional space. In suitable units the Hamiltonian for a spinless particle subject to a harmonic oscillator potential is

$$\mathbf{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}. \tag{1.1}$$

The Hilbert space $\mathcal{H}$ consists of functions $\Psi(x)$ square-integrable on the real line. The inner product is

$$(\Psi, \Phi) = \int_{-\infty}^{\infty} \Psi(x)\overline{\Phi(x)}\, dx. \tag{1.2}$$

Although the eigenvalue problem

$$\mathbf{H}\Psi = \lambda\Psi \tag{1.3}$$

can be solved with special function theory, we can achieve greater insight by adopting a formal Lie-algebraic approach. (Our approach can be made rigorous by careful attention to the domains of definition of $\mathbf{H}$ and other unbounded operators on $\mathcal{H}$ which we shall define shortly; Helwig [1].) Consider the operators

(1.4) $$\mathbf{J}^{\pm} = \pm \frac{1}{\sqrt{2}}\left(\frac{d}{dx} \mp x\right)$$

on $\mathcal{H}$. It is straightforward to verify the commutation relations

(1.5) $$[\mathbf{J}^3, \mathbf{J}^{\pm}] = \pm\mathbf{J}^{\pm}, \qquad [\mathbf{J}^+, \mathbf{J}^-] = -\mathbf{E},$$

where $\mathbf{E}$ is the identity operator and

(1.6) $$\mathbf{J}^3 = \mathbf{H}.$$

Furthermore, from the abstract relations (1.5) alone we can check that the operator

(1.7) $$\mathbf{C} = \mathbf{J}^+\mathbf{J}^- - \mathbf{E}\mathbf{J}^3 = \mathbf{J}^-\mathbf{J}^+ - \mathbf{E} - \mathbf{E}\mathbf{J}^3$$

commutes with $\mathbf{J}^{\pm}$ and $\mathbf{J}^3$. Thus $\mathbf{J}^{\pm}$, $\mathbf{J}^3$ and $\mathbf{E}$ form a basis for a four-dimensional complex Lie algebra $\mathcal{G}$ and $\mathbf{C}$ is an invariant operator for $\mathcal{G}$ (analogous to the Casimir operator for semisimple algebras). Corresponding to an irred rep of $\mathcal{G}$ we expect $\mathbf{C}$ to be a multiple of the identity operator. In fact for the model of $\mathcal{G}$ defined by (1.1) and (1.4) we find

(1.8) $$\mathbf{C} = -\tfrac{1}{2}\mathbf{E}.$$ two

Under the assumption that $\Psi(x)$, $\Phi(x)$, and their first derivatives vanish as $|x| \to \infty$, the formal relations

(1.9) $$(\Psi, \mathbf{J}^3\Phi) = (\mathbf{J}^3\Psi, \Phi), \qquad (\Psi, \mathbf{J}^+\Phi) = (\mathbf{J}^-\Psi, \Phi)$$

can easily be verified. (Integrate by parts.) Let Ψ be a normalized eigenvector of $\mathbf{J}^3 = \mathbf{H}$ with eigenvalue λ. The commutation relations (1.5) imply

(1.10) $$\mathbf{J}^3(\mathbf{J}^{\pm}\Psi) = (\lambda \pm 1)\mathbf{J}^{\pm}\Psi,$$

so $\mathbf{J}^{\pm}$ are raising and lowering operators in the usual sense. Given an eigenvector Ψ with eigenvalue λ we can obtain a ladder of eigenvectors with eigenvalues $\lambda + n$. We assume the vectors $\mathbf{J}^{\pm}\Psi$ still belong to $\mathcal{H}$. Then

(1.11) $$(\mathbf{J}^+\Psi, \mathbf{J}^+\Psi) = (\mathbf{J}^-\mathbf{J}^+\Psi, \Psi) = (\{\mathbf{J}^3 + \tfrac{1}{2}\mathbf{E}\}\Psi, \Psi) = \lambda + \tfrac{1}{2}.$$

Similarly,

(1.12) $$0 \le \|\mathbf{J}^-\Psi\|^2 = \lambda - \tfrac{1}{2}.$$

It follows that $\lambda \ge \frac{1}{2}$, so the eigenvalues of $\mathbf{J}^3$ are bounded below.

If λ_0 is the lowest eigenvalue of $\mathbf{J}^3$ and Ψ_0 is a corresponding eigenvector then $\mathbf{J}^-\Psi_0 = 0$, since $\lambda_0 - 1$ is not an eigenvalue. By (1.12), $\lambda_0 = \frac{1}{2}$. Using (1.4) we can solve this first-order differential equation for Ψ_0 and obtain

(1.13) $$\Psi_0(x) = \pi^{-1/4}\exp(-x^2/2),$$

where the factor $\pi^{-1/4}$ is chosen so $\|\Psi_0\| = 1$. From (1.7) and (1.8), $\mathbf{J}^3\Psi_0 = (\mathbf{J}^+\mathbf{J}^- + \frac{1}{2}\mathbf{E})\Psi_0 = \frac{1}{2}\Psi_0$, so $\mathbf{J}^3$ has a lowest eigenvalue $\frac{1}{2}$ with multiplicity one. From (1.10) and (1.11) we can define normalized eigenvectors Ψ_n with eigenvalues $n + \frac{1}{2}$ recursively by

$$\mathbf{J}^+\Psi_n = (n+1)^{1/2}\Psi_{n+1}, \qquad n = 0, 1, 2, \ldots. \tag{1.14}$$

Thus we use the raising operator $\mathbf{J}^+$ to move up the ladder of eigenvalues. According to (1.11), $\|\mathbf{J}^+\Psi_n\|^2 = n + 1 > 0$, so this process never ends. Furthermore, the commutation relations imply the formulas

$$\mathbf{J}^3\Psi_n = (n + \tfrac{1}{2})\Psi_n, \qquad \mathbf{J}^-\Psi_n = n^{1/2}\Psi_{n-1}. \tag{1.15}$$

We have shown that $\mathbf{H}$ has eigenvalues $n + \frac{1}{2}$, $n = 0, 1, \ldots$. It is left to the reader to verify that there are no other eigenvalues and each eigenspace is one-dimensional.

Substituting the operators (1.1) and (1.4) into (1.14) and (1.15), we obtain a second-order differential equation and two recurrence formulas for the special functions $\Psi_n(x)$. We can obtain a generating function from the first-order operator $\mathbf{J}^+$. A simple computation using Theorem 5.31 yields

$$[(\exp \alpha \mathbf{J}^+)\Psi](x) = \exp(-\tfrac{1}{4}\alpha^2 - 2^{-1/2}\alpha x)\Psi(x + 2^{-1/2}\alpha).$$

On the other hand, (1.14) implies

$$[(\exp \alpha \mathbf{J}^+)\Psi_n](x) = \sum_{k=0}^{\infty} \left[\frac{(n+k)!}{n!}\right]^{1/2} \frac{\alpha^k}{k!}\Psi_{n+k}(x).$$

Comparing these equations, we have the identity ($\beta = 2^{-1/2}\alpha$)

$$\left[\exp\left(-\frac{\beta^2}{2} - \beta x\right)\right]\Psi_n(x + \beta) = \sum_{k=0}^{\infty}\left[\frac{2^k(n+k)!}{n!}\right]^{1/2}\frac{\beta^k}{k!}\Psi_{n+k}(x). \tag{1.16}$$

In the special case $n = 0$, (1.13) yields

$$\pi^{-1/4}\exp(-\beta^2 - 2\beta x - \tfrac{1}{2}x^2) = \sum_{k=0}^{\infty} 2^{k/2}\beta^k(k!)^{-1/2}\Psi_k(x), \tag{1.17}$$

a simple generating function for the $\Psi_k(x)$. Comparing this with the well-known generating function

$$\exp(-\beta^2 + 2\beta x) = \sum_{k=0}^{\infty} \beta^k H_k(x)/k!$$

For the Hermite polynomials $H_k(x)$ (Erdélyi *et al.* [2, p. 194]) we obtain

$$\Psi_k(x) = \pi^{-1/4}(k!)^{-1/2}(-1)^k 2^{-k/2}[\exp(-x^2/2)]H_k(x). \tag{1.18}$$

The above series converge for all x and β. Since the $\{\Psi_n(x)\}$ form an ON set in $\mathcal{H}$ we easily obtain the formula

$$\int_{-\infty}^{\infty} H_n(x)H_k(x)\exp(-x^2)\,dx = \pi^{1/2}2^n n!\,\delta_{nk}. \tag{1.19}$$

Our Lie-algebraic analysis of the harmonic oscillator problem has not only determined the eigenvalues of $\mathbf{H}$ but also enabled us to derive the eigenfunctions and a number of their properties in a very simple manner.

The generalized Lie derivatives (1.1) and (1.4) determine the action of a connected four-parameter Lie group G on $\mathcal{H}$ called the **harmonic oscillator group**. Here G is *not* a symmetry group of the Hamiltonian since $\mathbf{J}^{\pm}$ do not commute with $\mathbf{H}$. However, a knowledge of the rep theory of G enables us to determine not only the multiplicities of the eigenvalues but also the eigenvalues themselves. Such a group G is called a **dynamical symmetry group** of the quantum mechanical system. [To be more precise, we are actually interested in the real Lie algebra $\mathcal{G}'$ generated by the skew-Hermitian operators

$$iJ^3, \quad \mathbf{E}, \quad \mathbf{J}^+ - \mathbf{J}^-, \quad i(\mathbf{J}^+ + \mathbf{J}^-). \tag{1.20}$$

These operators determine a unitary irred rep of the real dynamical symmetry group G' on $\mathcal{H}$, where $L(G') = \mathcal{G}'$. However, for Lie-algebraic purposes it is more convenient to work with the complexified algebra $\mathcal{G}$ determined by (1.5).]

The above analysis shows that the harmonic oscillator system in one dimension forms a model of an irred rep of $\mathcal{G}$. (The proof of irreducibility is left to the reader.) Another model of this same rep is provided by the annihilation and creation operators for bosons. In this model the annihilation operator $\mathbf{a}$ and the creation operator $\mathbf{a}^*$ act on a Hilbert space $\mathcal{H}$ and satisfy the commutation relations

$$[\mathbf{a}^*, \mathbf{a}] = -\mathbf{E}. \tag{1.21}$$

Furthermore the number-of-particles operator $\mathbf{N} = \mathbf{a}^*\mathbf{a}$ satisfies the commutation relations

$$[\mathbf{N}, \mathbf{a}^*] = \mathbf{a}^*, \qquad [\mathbf{N}, \mathbf{a}] = -\mathbf{a}. \tag{1.22}$$

There is an ON basis $\{|n\rangle, n = 0, 1, 2, \ldots\}$ for $\mathcal{H}$ such that

$$\begin{gathered} \mathbf{a}|n\rangle = n^{1/2}|n-1\rangle, \qquad \mathbf{a}^*|n\rangle = (n+1)^{1/2}|n+1\rangle, \\ \mathbf{N}|n\rangle = n|n\rangle. \end{gathered} \tag{1.23}$$

The eigenstates $|n\rangle$ of $\mathbf{N}$ are considered to be states of n bosons, which explains the names for $\mathbf{a}$, $\mathbf{a}^*$, and $\mathbf{N}$. Clearly, the operators $\mathbf{a}$, $\mathbf{a}^*$, $\mathbf{N}$, and $\mathbf{E}$ generate the Lie algebra $\mathcal{G}$ and expressions (1.23) determine an irred rep of $\mathcal{G}$ equivalent to that of the one-dimensional harmonic oscillator.

The harmonic oscillator in three-space has Hamiltonian

$$\mathbf{H} = -\frac{1}{2}\Delta + \frac{r^2}{2} = \frac{1}{2}\sum_{j=1}^{3}\left(-\frac{d^2}{dx_j{}^2} + x_j{}^2\right). \tag{1.24}$$

We can view this system as composed of three one-dimensional noninteracting harmonic oscillators. Thus the eigenfunctons of $\mathbf{H}$ are of the form

$$\Psi_{n_1n_2n_3}(x) = \Psi_{n_1}(x_1)\Psi_{n_2}(x_2)\Psi_{n_3}(x_3) \tag{1.25}$$

with eigenvalue $\lambda_N = n_1 + n_2 + n_3 + \frac{3}{2} = N + \frac{3}{2}$, $n_j = 0, 1, \ldots$. The multiplicity of λ_N is equal to the number of ways we can select nonnegative integers n_j such that $n_1 + n_2 + n_3 = N$. A simple combinatorial argument gives the multiplicity $(N+1)(N+2)/2$. Here $G \times G \times G$ is a dynamical symmetry group for this system, which enables us to compute the eigenvalues and their multiplicities. However, to get a better understanding of the multiplicities it is useful to compute the ordinary symmetry group of **H**. From (1.24) it is clear that the angular momentum operators commute with **H**, so $SO(3)$ [or $SU(2)$] is a symmetry group. However, the dimensions of the reps $D^{(l)}$ do not coincide with the multiplicities $(N+1)(N+2)/2$. This suggests the existence of a larger symmetry group.

To investigate this group we consider the annihilation and creation operators

$$(1.26)\quad \mathbf{a}_j = -2^{-1/2}\left(\frac{d}{dx_j} + x_j\right), \qquad \mathbf{a}_j^* = 2^{-1/2}\left(\frac{d}{dx_j} - x_j\right), \qquad 1 \le j \le 3,$$

with commutation relations

$$(1.27)\qquad [\mathbf{a}_j, \mathbf{a}_k^*] = \delta_{jk}\mathbf{E}, \qquad [\mathbf{a}_j, \mathbf{a}_k] = [\mathbf{a}_j^*, \mathbf{a}_k^*] = \mathbf{0}.$$

From (1.24)

$$(1.28)\qquad \mathbf{H} = \frac{1}{2}\sum_{j=1}^{3} (\mathbf{a}_j^*\mathbf{a}_j + \mathbf{a}_j\mathbf{a}_j^*) = \sum_{j=1}^{3} \mathbf{a}_j^*\mathbf{a}_j + \frac{3}{2}\mathbf{E}.$$

It is easy to verify that the nine operators $\mathbf{E}_{jk} = \mathbf{a}_j^*\mathbf{a}_k$ commute with **H** and satisfy the commutation relations

$$(1.29)\qquad [\mathbf{E}_{jk}, \mathbf{E}_{hl}] = \delta_{kh}\mathbf{E}_{jl} - \delta_{lj}\mathbf{E}_{hk}$$

[see (7.3), Section 9.7]. (Note that the $\mathbf{E}_{jk}$ preserve N, while operators such as $\mathbf{a}_j\mathbf{a}_k$ do not.) Clearly, the $\mathbf{E}_{jk}$ generate a complex Lie algebra isomorphic to $gl(3, \mathfrak{C})$. The skew-Hermitian operators in this algebra form a real Lie algebra isomorphic to $u(3)$. A basis for $u(3)$ is given by

$$(1.30)\quad i(\mathbf{E}_{jk} + \mathbf{E}_{kj}), \quad \mathbf{E}_{jk} - \mathbf{E}_{kj}, \quad k \neq j; \qquad i\mathbf{E}_{jj}, \quad j, k = 1, 2, 3.$$

The angular momentum operators, given by $\mathbf{E}_{jk} - \mathbf{E}_{kj}$, $k \neq j$, generate a subalgebra isomorphic to $so(3)$.

Under the action of the $\mathbf{E}_{jk}$ the eigenspace of **H** corresponding to eigenvalue λ_N is decomposed into a direct sum of $u(3)$-irred subspaces. Using the results of Section 9.1 we can explicitly carry out this decomposition. The highest weight vector is easily shown to be the eigenfunction with $n_1 = N$, $n_2 = n_3 = 0$. Thus, the rep $[N, 0, 0]$ of $u(3)$ occurs exactly once. Moreover, from (2.24), Section 9.2, $\dim[N, 0, 0] = (N+1)(N+2)/2$, which is the multiplicity of λ_N. Thus, the eigenspace of λ_N transforms as $[N, 0, 0]$. As we have seen, the angular momentum operators generate a subalgebra $so(3)$

of $u(3)$. To determine the branching rule for the subalgebra of angular momentum operators we could compute the weight vectors corresponding to the generator $\mathbf{L}^3 = \mathbf{E}_{12} - \mathbf{E}_{21}$ (this is not easy). The results are

$$(1.31) \qquad [N, 0, 0] \mid so(3) \cong \begin{cases} \displaystyle\sum_{j=0}^{N/2} \oplus D^{(N-2j)}, & N \quad \text{even} \\ \displaystyle\sum_{j=0}^{(N-1)/2} \oplus D^{(N-2j)}, & N \quad \text{odd.} \end{cases}$$

Thus, for N even the reps $D^{(l)}$ occur with l even. An alternate proof of (1.31) can be obtained from the character formula for $U(3)$. The subgroup $SO(3)$ is embedded in $U(3)$ in the natural way and it is straightforward to expand $\chi^{N00} \mid SO(3)$ as a sum of simple characters of $SO(3)$.

We say that $U(3)$ is the symmetry group of the three-dimensional harmonic oscillator. The global action of $U(3)$ on $\mathcal{H}$ is fairly difficult to determine in this case since the $\mathbf{E}_{jl}$ are second-order partial differential operators to which local Lie theory does not apply. The group action is expressable in terms of integral operators. For details see the work of Bargmann [1] or Miller [1]. These references also give the action of the harmonic oscillator group on $\mathcal{H}$.

10.2 Representations of the Harmonic Oscillator Group

The Lie algebra $\mathcal{G}$ of the complex harmonic oscillator group G is defined by the commutation relations

$$(2.1) \quad [\mathcal{J}^3, \mathcal{J}^\pm] = \pm\mathcal{J}^\pm, \qquad [\mathcal{J}^+, \mathcal{J}^-] = -\mathcal{E}, \qquad [\mathcal{E}, \mathcal{J}^\pm] = [\mathcal{E}, \mathcal{J}^3] = 0.$$

We present a brief survey of irred reps of $\mathcal{G}$ which occur in physical theories. Let ρ be a rep of $\mathcal{G}$ on a complex vector space V and set

$$(2.2) \qquad J^\pm = \rho(\mathcal{J}^\pm), \qquad J^3 = \rho(\mathcal{J}^3), \qquad I = \rho(\mathcal{E}).$$

These operators satisfy relations (2.1) again.

The faithful irred reps of $\mathcal{G}$ are all infinite-dimensional. Indeed, if ρ is irred and finite-dimensional then I must be a multiple μE of the identity operator on V, since I commutes with $\rho(\alpha)$ for all $\alpha \in \mathcal{G}$. Thus

$$(2.3) \qquad \mathrm{tr}([J^+, J^-]) = \mathrm{tr}(\mu E) = \mu \dim V$$

and $\mu = 0$ because the trace of a commutator is zero. Hence, $I = 0$ and ρ is not faithful. (A rep ρ of a Lie algebra $\mathcal{G}$ is **faithful** if $\rho(\alpha) \neq 0$ for every $\alpha \neq 0$ in $\mathcal{G}$.)

We make no attempt to classify all irred reps of $\mathcal{G}$ and simply examine a few reps of particular importance. An easy way to construct such reps of $\mathcal{G}$ is via realizations in terms of generalized Lie derivatives in one complex variable. [We tried this same approach for $sl(2)$ in Section 5.10.] Clearly, the generalized Lie derivatives

$$(2.4)\qquad J^3 = \lambda + z\frac{d}{dz}, \qquad J^+ = \mu z, \qquad J^- = \frac{\xi}{z} + \frac{d}{dz}, \qquad I = \mu,$$

satisfy the commutation relations (2.1) for all constants λ, μ, and ξ.

For an arbitrary rep ρ the operator

$$(2.5)\qquad C = J^+J^- - IJ^3$$

commutes with all $\rho(\alpha)$, $\alpha \in \mathcal{G}$, as the reader can check. If ρ is irred we expect that C is a multiple ωE of the identity operator on V. (However, we have not proved this since Theorem 3.5 applies only to finite-dimensional reps.) For our model (2.4) we find $C = \mu(\xi - \lambda)$.

The operators (2.4) determine a local multiplier rep of G. To compute this rep we need an explicit definition of G. Recall that G is only determined locally by $\mathcal{G}$. Among the linear Lie groups with Lie algebra $\mathcal{G}$ we select the one, unique up to isomorphism, which is simply connected.

Definition. The complex **harmonic oscillator group** G consists of all matrices

$$(2.6)\qquad g(a, b, c, \tau) = \begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad a, b, c, \tau \in \mathfrak{C}.$$

Here G is a four-parameter complex linear Lie group. In terms of the parameters,

(2.7)
$$g(a, b, c, \tau)g(a', b', c', \tau') = g(a + a' + cb'e^{\tau}, b + b'e^{\tau}, c + c'e^{-\tau}, \tau + \tau')$$
$$g^{-1}(a, b, c, \tau) = g(bc - a, -be^{-\tau}, -ce^{\tau}, -\tau).$$

The matrices $\mathcal{J}^{\pm}, \mathcal{J}^3, \mathcal{E}$ defined by

$$(2.8)\qquad g(a, b, c, \tau) = (\exp b\mathcal{J}^+)(\exp c\mathcal{J}^-)(\exp \tau\mathcal{J}^3)(\exp a\mathcal{E})$$

form a basis for $\mathcal{G}$ satisfying the commutation relations (2.1).

Let $\mathbf{T}$ be the local multiplier rep of G determined by the generalized Lie derivatives (2.4). The group identity (2.8) implies

$$(2.9)\qquad \mathbf{T}(g) = (\exp bJ^+)(\exp cJ^-)(\exp \tau J^3)(\exp aI)$$

for $|b|, |c|, |\tau|, |a|$ sufficiently small. For simplicity we choose $\lambda = -\omega$, $\xi = 0$ in (2.4). Applying local Lie theory to compute the factors of (2.9) and composing the result, we find

$$(2.10)\qquad [\mathbf{T}(g)f](z) = \exp[\mu(bz + a) - \omega\tau]f(e^{\tau}z + e^{\tau}c)$$

for $f(z)$ analytic in some neighborhood of $z = 0$. [Compare the analogous computation for $SL(2)$ in Section 5.10.] Since $\mathbf{T}$ is a local rep we have

$$\mathbf{T}(gg')f = \mathbf{T}(g)[\mathbf{T}(g')f] \tag{2.11}$$

for g and g' in a suitably small neighborhood of the identity. Moreover, if we restrict f to the space $\mathfrak{A}$ of entire functions then (2.10) is defined for all $g \in G$ and $\mathbf{T}(g)f \in \mathfrak{A}$. In this case the identity (2.11) holds for all $g, g' \in G$, as the reader can prove directly from (2.7) and (2.10).

Every $f \in \mathfrak{A}$ has a unique power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

which converges for all $z \in \mathfrak{C}$. Thus the functions $h_n(z) = z^n$, $n \geq 0$, form a basis for $\mathfrak{A}$. With respect to this basis we define matrix elements $T_{lk}(g)$ by

$$[\mathbf{T}(g)h_k](z) = \sum_{l=0}^{\infty} T_{lk}(g)h_l(z), \qquad k = 0, 1, \ldots; \tag{2.12}$$

$$\{\exp[\mu(bz + a) + (k - \omega)\tau]\}(z + c)^k = \sum_{l=0}^{\infty} T_{lk}(g)z^l. \tag{2.13}$$

The group property (2.11) yields the addition theorem

$$T_{lk}(gg') = \sum_{j=0}^{\infty} T_{lj}(g)T_{jk}(g'), \qquad g, g' \in G. \tag{2.14}$$

We can obtain the matrix elements explicitly by expanding the left-hand side of (2.13) in a power series and computing the coefficient of z^l:

$$T_{lk}(g) = \{\exp[\mu a + (k - \omega)\tau]\}(\mu b)^{l-k} \sum_s \frac{(\mu bc)^s k!}{(l - k + s)!\,(k - s)!\,s!}, \tag{2.15}$$

where the sum is taken over all integers s such that the summand is defined. From Erdélyi *et al.* [1, p. 268] we find

$$T_{lk}(g) = \{\exp[\mu a + (k - \omega)\tau]\}c^{k-l}L_l^{(k-l)}(-\mu bc), \tag{2.16}$$

where $L_m^{(n)}(x)$ is an associated Laguerre polynomial. Substituting this result into (2.13) we obtain the generating function

$$e^{-bz}(z + 1)^k = \sum_{l=0}^{\infty} L_l^{(k-l)}(b)z^l. \tag{2.17}$$

Furthermore, the addition theorem (2.14) yields

(2.18)

$$e^{-cb'}(c + c')^n L_l^{(n)}[(b + b')(c + c')] = \sum_{j=0}^{\infty} c^{j-l}L_l^{(j-l)}[bc'](c)^{l+n-j}L_j^{(l+n-j)}[b'c'],$$

where the integers $l, l + n$ are nonnegative and $b, b', c, c' \in \mathfrak{C}$.

To exhibit the rep $\uparrow_{\omega,\mu}$ of $\mathcal{G}$ induced by $\mathbf{T}$ we label the basis vectors in terms of their eigenvalues with respect to J^3:

$$f_m(z) = h_n(z) = z^n, \qquad m = n - \omega. \tag{2.19}$$

Then

(2.20) $\quad J^3 = -\omega + z(d/dz), \qquad J^+ = \mu z, \qquad J^- = d/dz, \qquad I = \mu,$

and direct computation yields

$$J^3 f_m = m f_m, \qquad J^+ f_m = \mu f_{m+1}, \qquad J^- f_m = (\omega + m) f_{m-1},$$
$$I f_m = \mu f_m, \qquad C f_m = \mu \omega f_m, \tag{2.21}$$

where $m = -\omega + n, n = 0, 1, \ldots$, and $\mu, \omega \in \mathfrak{C}$ with $\mu \neq 0$. As the reader can verify, the Lie algebra rep $\uparrow_{\omega,\mu}$ is irred on the infinite-dimensional vector space of all finite linear combinations of the basis vectors $\{f_m\}$.

The rep $\uparrow_{0,1}$ corresponds to the harmonic oscillator problem in one dimension. Indeed, setting $\omega = 0$, $\mu = 1$, and $|n\rangle = f_n(z)(n!)^{-1/2}$ we obtain

(2.22)
$$J^3|n\rangle = n|n\rangle, \qquad J^+|n\rangle = (n+1)^{1/2}|n+1\rangle, \qquad J^-|n\rangle = n^{1/2}|n-1\rangle,$$

in agreement with (1.23). [We have normalized our basis vectors $\{f_m\}$ so no square roots appear in (2.21).]

We have defined a class $\uparrow_{\omega,\mu}$ of irred reps of $\mathcal{G}$ and used a simple model to compute the matrix elements $T_{lk}(g)$ of this rep extended to G. These matrix elements are uniquely determined by expressions (2.21) and are model-independent. Thus, the model of $\uparrow_{0,1}$ provided by the operators (1.4), (1.6), and basis vectors

$$f_n(x) = (-1)^n 2^{-n/2}(\exp -\tfrac{1}{2}x^2) H_n(x) \tag{2.23}$$

must have matrix elements (2.16). From (1.4) we find

(2.24)
$$[\mathbf{T}(g)f](x) = \{\exp[\tfrac{1}{4}(c^2 - b^2 - 2bc) - 2^{-1/2}x(b + c)]\} f(x + (b - c)2^{-1/2})$$

for $g(0, b, c, 0) = (\exp b\mathcal{J}^+) \exp c\mathcal{J}^-$. Substituting (2.16), (2.23), and (2.24) into

$$\mathbf{T}(g) f_k = \sum_{l=0}^{\infty} T_{lk}(g) f_l \tag{2.25}$$

we obtain (after some simplification)

$$[\exp(-b^2 - 2bx)] H_k(x + b + c) = \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(bc) H_l(x). \tag{2.26}$$

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This expression converges for all b, c, and x.

Problems

10.1 Prove (1.31).

10.2 Show that the 4 × 4 matrices (2.6) define a rep of the harmonic oscillator group which cannot be expressed as a direct sum of irred reps.

10.3 Compute the recurrence relations and differential equation for Hermite polynomials which result on application of the operators $\mathbf{J}^{\pm}$, $\mathbf{J}^3$, $\mathbf{C}$ (Section 10.1) to Ψ_n.

10.4 Show that $E^+(2)$, the proper Euclidean group in the plane, is isomorphic to the group of matrices

$$g(x, y, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Check that these matrices define a rep of $E^+(2)$ which cannot be expressed as a direct sum of irred reps.

10.5 Show that the generalized Lie derivatives

$$J_1 = i\rho \cos\alpha, \qquad J_2 = i\rho \sin\alpha, \qquad J_3 = -\partial/\partial\alpha,$$

ρ a nonzero real constant, $i = \sqrt{-1}$, $0 < \alpha \leq 2\pi \pmod{2\pi}$, span a Lie algebra isomorphic to $L(E^+(2))$. Compute the operators $\mathbf{T}(g)$ of the multiplier rep of $E^+(2)$ determined by the J_k. [Use the coordinates $g[r, \varphi, \theta] = g(x, y, \theta)$, where $x + iy = re^{i\varphi}$, $r \geq 0$, φ real.]

10.6 Verify that the $\mathbf{T}(g)$ computed in the preceding problem define a unitary rep of $E^+(2)$ on the Hilbert space $L_2[0, 2\pi]$ (see Section 6.2). Compute the matrix elements $T_{n,m}(g) = \langle \mathbf{T}(g)f_m, f_n\rangle$ with respect to the ON basis $f_m(\alpha) = e^{im\alpha}$, $m = 0, \pm 1, \pm 2, \ldots$, and show that these elements can be expressed in terms of Bessel functions. What properties of Bessel functions follow from the unitarity of the $\mathbf{T}(g)$ and the group property $\mathbf{T}(g_1)\mathbf{T}(g_2) = \mathbf{T}(g_1 g_2)$?