Chapter 3

Group Representation Theory

3.1 A Group Representation

Let V be a vector space, real or complex, and denote by $GL(V)$ the group of all nonsingular linear transformations of V onto itself.

Definition. A **representation (rep)** of a group G with **representation space** V is a homomorphism $\mathbf{T}: g \longrightarrow \mathbf{T}(g)$ of G into $GL(V)$. The **dimension** of the representation is the dimension of V.

As a consequence of this definition we have the following:

(1.1)
$$\mathbf{T}(g_1)\mathbf{T}(g_2) = \mathbf{T}(g_1 g_2), \qquad \mathbf{T}(g)^{-1} = \mathbf{T}(g^{-1}), \qquad \mathbf{T}(e) = \mathbf{E}, \qquad g_1, g_2, g \in G,$$

where $\mathbf{E}$ is the identity operator on V. Unless otherwise specified, only finite-dimensional reps of finite groups will be studied in the present chapter. This finiteness restriction will be lifted later. It will also be assumed unless otherwise mentioned that V is defined over the complex field $\mathfrak{C}$.

Definition. An ***n*-dimensional matrix rep** of G is a homomorphism T: $g \longrightarrow T(g)$ of G into $GL(n, \mathfrak{C})$ [or $GL(n, R)$].

The $n \times n$ matrices $T(g)$, $g \in G$, satisfy multiplication properties analogous to (1.1). Any group rep $\mathbf{T}$ of G with rep space V defines many matrix reps. For, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V, the matrices $T(g) = (T(g)_{kj})$ defined by

$$\mathbf{T}(g)\mathbf{v}_k = \sum_{j=1}^{n} T(g)_{jk}\mathbf{v}_j, \qquad 1 \le k \le n. \tag{1.2}$$

form an n-dimensional matrix rep of G. Every choice of a basis for V yields a new matrix rep of G defined by $\mathbf{T}$. However, any two such matrix reps T, T' are equivalent in the sense that there exists a matrix $S \in GL(n, \mathfrak{C})$ such that

$$T'(g) = ST(g)S^{-1} \tag{1.3}$$

for all $g \in G$. In fact if T, T' correspond to the bases $\{\mathbf{v}_i\}$, $\{\mathbf{v}_i'\}$ respectively, then for S we can take the matrix (S_{ji}) defined by

$$\mathbf{v}_i = \sum_{j=1}^{n} S_{ji}\mathbf{v}_j', \qquad i = 1, \ldots, n. \tag{1.4}$$

Definition. Two complex n-dimensional matrix reps T and T' are **equivalent** ($T \simeq T'$) if there exists an $S \in GL(n, \mathfrak{C})$ such that (1.3) holds.

Equivalent matrix reps can be viewed as arising from the same operator rep.

Conversely, given an n-dimensional matrix rep $T(g)$ we can define many n-dimensional operator reps of G. If V is an n-dimensional vector space with basis $\{\mathbf{v}_i\}$ we can define the group rep $\mathbf{T}$ by expression (1.2), i.e., we define the operator $\mathbf{T}(g)$ by the right-hand side of (1.2). Every choice of a vector space V and a basis $\{\mathbf{v}_i\}$ for V yields a new operator rep defined by T. However, if V, V' are two such n-dimensional vector spaces with bases $\{\mathbf{v}_i\}$, $\{\mathbf{v}_i'\}$ respectively, then the reps $\mathbf{T}$ and $\mathbf{T}'$ are related by

$$\mathbf{T}'(g) = \mathbf{S}\mathbf{T}(g)\mathbf{S}^{-1}, \tag{1.5}$$

where $\mathbf{S}$ is an invertible operator from V onto V' defined by

$$\mathbf{S}\mathbf{v}_i = \mathbf{v}_i', \qquad 1 \le i \le n.$$

Definition. Two n-dimensional group reps $\mathbf{T}, \mathbf{T}'$ of G on the spaces V, V' are **equivalent** ($\mathbf{T} \simeq \mathbf{T}'$) if there exists an invertible linear transformation $\mathbf{S}$ of V onto V' such that expression (1.5) holds.

The reader can easily check that equivalent operator reps correspond to equivalent matrix reps, i.e., there is a 1–1 correspondence between classes of equivalent operator reps and classes of equivalent matrix reps. (Note: The above definitions can be modified in an obvious manner to yield definitions of equivalence classes of **real** operator and matrix reps and to establish their 1–1 correspondence.)

In order to determine all possible reps of a group G it is enough to find one rep $\mathbf{T}$ in each equivalence class. The remaining reps $\mathbf{T}'$ in each class are given by (1.5), where $\mathbf{S}$ runs over all invertible operators from V to V' and V' runs over all n-dimensional vector spaces. It is a matter of choice whether we study operator reps or matrix reps. For theoretical purposes the operator reps are usually more convenient, while matrix reps are more useful for computations.

Most applications of groups to the physical sciences occur via representation theory. Group reps appear naturally in the study of physical problems with inherent symmetry and analysis of the reps aids the solution of these problems. We present some examples of group reps.

Example 1. The matrix groups $GL(n, \mathfrak{C})$, $SL(n, \mathfrak{C})$, $O(n)$, etc. are n-dimensional matrix reps of themselves.

Example 2. Any group of operators on a vector space is a rep of itself. In particular, the point groups considered as linear operators on the vector space R_3 define three-dimensional reps of themselves.

Example 3. Let G be a group of order n. We formally define an n-dimentional vector space R_G consisting of all elements of the form

$$\sum_{g \in G} x(g)\cdot g, \qquad x(g) \in \mathfrak{C}. \tag{1.6}$$

Two vectors $\sum x(g)\cdot g$ and $\sum y(g)\cdot g$ are equal if and only if $x(g) = y(g)$ for all $g \in G$. The sum of two vectors and the scalar multiple of a vector are defined by

$$\begin{aligned} \sum x(g)\cdot g + \sum y(g)\cdot g &= \sum [x(g) + y(g)]\cdot g \\ \alpha \sum x(g)\cdot g &= \sum \alpha x(g)\cdot g \end{aligned} \tag{1.7}$$

The zero vector of R_G is $\theta = \sum 0\cdot g$. Furthermore, the vectors $1\cdot g$, $g \in G$, form a natural basis for R_G. (From now on we write $1\cdot g = g \in R_G$.) We define the **product** of two elements $x = \sum x(g)\cdot g$, $y = \sum y(h)\cdot h$ in a natural manner:

$$\begin{aligned} xy = (\sum x(g)\cdot g) \sum y(h)\cdot h &= \sum_{g, h \in G} x(g)y(h)\cdot gh \\ &= \sum_{k \in G} xy(k)\cdot k, \end{aligned} \tag{1.8}$$

where

$$xy(g) = \sum_{h \in G} x(h)y(h^{-1}g).$$

It is easy to verify the following relations:

$$\begin{gathered} (x + y)z = xz + yz, \qquad x(y + z) = xy + xz, \qquad x, y, z \in R_G, \\ (xy)z = x(yz), \quad \alpha(xy) = (\alpha x)y = x(\alpha y), \quad ex = xe = x, \quad \alpha \in \mathfrak{C}, \end{gathered} \tag{1.9}$$

where e is the identity element of G. Thus, R_G is an algebra, called the **group algebra** or **group ring** of G. The mapping $\mathbf{L}$ of G into $GL(R_G)$ given by

$$\mathbf{L}(g)x = gx, \qquad x \in R_G, \tag{1.10}$$

defines an n-dimensional rep of G, the **(left) regular** rep. In fact,

$$\mathbf{L}(g_1g_2)x = g_1g_2x = \mathbf{L}(g_1)g_2x = \mathbf{L}(g_1)\mathbf{L}(g_2)x$$

and the $\mathbf{L}(g)$ are linear operators.

This example provides us with a rep of any finite group, and is of great importance for theoretical purposes. Another natural rep of G on R_G is the **(right) regular** rep defined by

$$\mathbf{R}(g)x = xg^{-1}, \qquad x \in R_G, \qquad g \in G. \tag{1.11}$$

[Check that g^{-1} is needed on the right-hand side of (1.11) to make $\mathbf{R}$ a rep.]

Example 4. Consider the Helmholtz equation

$$\Delta u(\mathbf{x}) + k^2u(\mathbf{x}) = 0, \tag{1.12}$$

where $\mathbf{x} = (x_1, x_2, x_3) \in R_3$, $k \geq 0$, and

$$\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^{\,2}}.$$

The set of all solutions $u(\mathbf{x})$ of (1.12) (defined for all $\mathbf{x} \in R_3$) forms an infinite-dimensional vector space V_k. In particular, any finite linear combination of solutions of (1.12) is a solution. We show that the operators $\mathbf{T}(g)$, $g \in E(3)$, given by

$$[\mathbf{T}(g)u](\mathbf{x}) = u(\mathbf{g}^{-1}\mathbf{x}), \tag{1.13}$$

where

$$\mathbf{gx} = \{\mathbf{a}, \mathbf{O}\}\mathbf{x} = \mathbf{Ox} + \mathbf{a}, \qquad \mathbf{O} \in O(3), \qquad \mathbf{a} \in T(3),$$

define an (infinite-dimensional) rep of $E(3)$ on V_k. The homomorphism property follows from

$$[\mathbf{T}(g_1g_2)u](\mathbf{x}) = u(\mathbf{g}_2^{-1}\mathbf{g}_1^{-1}\mathbf{x}) = [\mathbf{T}(g_2)u](\mathbf{g}_1^{-1}\mathbf{x}) = [\mathbf{T}(g_1)\mathbf{T}(g_2)u](\mathbf{x}).$$

Note that $\mathbf{T}(g)\mathbf{u}$ is a function whose value at $\mathbf{x}$ is the value of u at $\mathbf{g}^{-1}\mathbf{x}$. The reader should check that use of $\mathbf{gx}$ on the right-hand side of (1.13) would *not* lead to a homomorphism.

In order to prove our assertion we must show that V_k is invariant under the operators $\mathbf{T}(g)$. Write $\mathbf{x}' = \mathbf{g}^{-1}\mathbf{x}$. Since

$$\mathbf{g}^{-1} = \{-\mathbf{O}^{-1}\mathbf{a}, \mathbf{O}^{-1}\}, \qquad \mathbf{O} \in O(3)$$

a simple computation gives

$$\frac{\partial}{\partial x_l} = \sum_{i=1}^{3} O_{il}\frac{\partial}{\partial x_i'}.$$

Therefore,

$$\Delta = \sum_{l,i,j=1}^{3} O_{li}O_{lj} \frac{\partial^2}{\partial x_i' \, \partial x_j'} = \sum_{i=1}^{3} \frac{\partial^2}{\partial (x_i')^2} = \Delta' \tag{1.14}$$

since O is an orthogonal matrix. If $u \in V_k$ then

$$\Delta[\mathbf{T}(g)u](\mathbf{x}) = \Delta[u(\mathbf{x}')] = \Delta' u(\mathbf{x}') = -k^2 u(x') = -k^2[\mathbf{T}(g)u](\mathbf{x}),$$

so $\mathbf{T}(g)u \in V_k$. The existence of this group rep has important consequences in the study of the solutions of the Helmholtz equation. These consequences will be explored in Chapter 8.

Example 5. The square integrable solutions $\Psi(\mathbf{x})$ of the Schrödinger equation

$$-\frac{\hbar^2}{2m} \Delta\Psi(\mathbf{x}) + V(\mathbf{x})\Psi(\mathbf{x}) = E\Psi(\mathbf{x}) \tag{1.15}$$

describing a particle of mass m and energy E subject to the potential field $V(\mathbf{x})$, form a vector space W_E. Suppose $V(\mathbf{x})$ is invariant under the action of some subgroup G of $O(3)$:

$$V(\mathbf{gx}) = V(\mathbf{x}), \qquad g \in G.$$

[For example, if $V(\mathbf{x})$ has rotational symmetry, G may be $O(3)$. Another possibility is a point group.] Then the operators

$$[\mathbf{T}(g)\Psi](\mathbf{x}) = \Psi(\mathbf{g}^{-1}\mathbf{x}), \qquad g \in G,$$

satisfy the homomorphism property and map solutions of (1.15) into other solutions. Furthermore, for $\Psi \in W_E$

$$\int_{R_3} |\Psi(\mathbf{g}^{-1}\mathbf{x})|^2 \, d^3\mathbf{x} = \int_{R_3} |\Psi(\mathbf{x})|^2 \, d^3\mathbf{x} < \infty, \qquad d^3\mathbf{x} = dx_1 \, dx_2 \, dx_3, \tag{1.16}$$

since the Jacobian of the coordinate transformation is $+1$. Therefore, $\mathbf{T}(g)\Psi \in W_E$ and the operators $\mathbf{T}(g)$ define a length-preserving rep of G on W_E, where the inner product $\langle -, - \rangle$ is given by

$$\langle \Psi_1, \Psi_2 \rangle = \int_{R_3} \Psi_1(\mathbf{x})\overline{\Psi_2(\mathbf{x})} \, d^3\mathbf{x}.$$

It is easy to verify that

$$\langle \mathbf{T}(g)\Psi_1, \mathbf{T}(g)\Psi_2 \rangle = \langle \Psi_1, \Psi_2 \rangle, \qquad g \in G.$$

Thus, the operators $\mathbf{T}(g)$ are **unitary** with respect to $\langle -, - \rangle$ and they define a **unitary rep** of G on W_E. In most quantum mechanical problems the eigenspaces W_E are zero-dimensional except for a countable number of values E_n (the bound-state energy levels) where they have finite nonzero dimension. We shall show later that a knowledge of the symmetry group of Schrödinger's equation furnishes us with important information about the eigenspaces W_{E_n} even in cases where (1.15) cannot be explicitly solved.

Let $\mathbf{T}$ be a rep of the finite group G on a finite-dimensional inner product space V. The rep $\mathbf{T}$ is said to be **unitary** if for all $g \in G$

$$\langle \mathbf{T}(g)\mathbf{v}, \mathbf{T}(g)\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle, \qquad \mathbf{v}, \mathbf{w} \in V, \tag{1.17}$$

i.e., if the operators $\mathbf{T}(g)$ are unitary. Recall that an **orthonormal (ON) basis** for the n-dimensional space V is a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$, where $\langle -, - \rangle$ is the inner product on V. The matrices $T(g)$ of the operators $\mathbf{T}(g)$ with respect to an ON basis $\{\mathbf{v}_i\}$ are unitary matrices

$$\overline{T(g)_{ji}} = T(g^{-1})_{ij} = [T(g)^{-1}]_{ij}.$$

Hence, they form a unitary matrix rep of G. Unitary operator and matrix reps have useful properties which make them desirable in both theoretical and computational problems. The following theorem shows that for finite groups at least, we can always restrict ourselves to unitary reps.

Theorem 3.1. Let $\mathbf{T}$ be a rep of G on the inner product space V. Then $\mathbf{T}$ is equivalent to a unitary rep on V.

Proof. First we define a new inner product $(-, -)$ on V with respect to which $\mathbf{T}$ is unitary. For $\mathbf{u}, \mathbf{v} \in V$ let

$$(\mathbf{u}, \mathbf{v}) = \frac{1}{n(G)} \sum_{g \in G} \langle \mathbf{T}(g)\mathbf{u}, \mathbf{T}(g)\mathbf{v} \rangle. \tag{1.18}$$

[Note that $(\mathbf{u}, \mathbf{v})$ is an average of the numbers $\langle \mathbf{T}(g)\mathbf{u}, \mathbf{T}(g)\mathbf{v} \rangle$ taken over the group.] It is easy to check that $(-, -)$ is an inner product on V. Furthermore,

$$\begin{aligned}(\mathbf{T}(h)\mathbf{u}, \mathbf{T}(h)\mathbf{v}) &= \frac{1}{n(G)} \sum_{g \in G} \langle \mathbf{T}(gh)\mathbf{u}, \mathbf{T}(gh)\mathbf{v} \rangle \\ &= \frac{1}{n(G)} \sum_{g' \in G} \langle \mathbf{T}(g')\mathbf{u}, \mathbf{T}(g')\mathbf{v} \rangle = (\mathbf{u}, \mathbf{v}),\end{aligned}$$

where the next to last equality follows from the fact that if g runs through the elements of G exactly once, then so does gh. Now $\mathbf{T}$ is unitary with respect to the new inner product, but not the old one. Let $\{\mathbf{u}_i\}$ be an ON basis of V with respect to $(-, -)$ and let $\{\mathbf{v}_i\}$ be an ON basis with respect to $\langle -, - \rangle$. Define the nonsingular linear operator $\mathbf{S}: V \to V$ by $\mathbf{S}\mathbf{u}_i = \mathbf{v}_i$. Then for $\mathbf{w} = \sum \alpha_i \mathbf{u}_i$ and $\mathbf{x} = \sum \beta_j \mathbf{u}_j$ we find

$$\langle \mathbf{S}\mathbf{w}, \mathbf{S}\mathbf{x} \rangle = \sum_{i,j} \alpha_i \bar{\beta}_j \langle \mathbf{S}\mathbf{u}_i, \mathbf{S}\mathbf{u}_j \rangle = \sum_i \alpha_i \bar{\beta}_i = (\mathbf{w}, \mathbf{x}),$$

so

$$\begin{aligned}\langle \mathbf{S}\mathbf{T}(g)\mathbf{S}^{-1}\mathbf{w}, \mathbf{S}\mathbf{T}(g)\mathbf{S}^{-1}\mathbf{x} \rangle &= (\mathbf{T}(g)\mathbf{S}^{-1}\mathbf{w}, \mathbf{T}(g)\mathbf{S}^{-1}\mathbf{x}) \\ &= (\mathbf{S}^{-1}\mathbf{w}, \mathbf{S}^{-1}\mathbf{x}) = \langle \mathbf{w}, \mathbf{x} \rangle.\end{aligned}$$

Thus, the rep $\mathbf{T}'(g) = \mathbf{S}\mathbf{T}(g)\mathbf{S}^{-1}$ is unitary on V. Q.E.D.

We can always assume that a rep **T** on V is unitary. Indeed, we can always define an inner product on V with respect to which **T** is unitary. Moreover, if V is already equipped with a given inner product $\langle -, - \rangle$ then we can find a unitary rep (with respect to $\langle -, - \rangle$) which is equivalent to **T**.

Theorem 3.1 and its proof are also valid for reps on a real vector space V. In this case we can find an inner product on V with respect to which the operators $\mathbf{T}(g)$ are **orthogonal**.

3.2 Reducible Representations

In this section **T** will be a finite-dimensional rep of a finite group G acting on the (real or complex) vector space V.

Definition. A subspace W of V is **invariant** under **T** if $\mathbf{T}(g)\mathbf{w} \in W$ for every $g \in G$, $\mathbf{w} \in W$.

If W is invariant under **T** we can define a rep $\mathbf{T}' = \mathbf{T} \mid W$ of G on W by

$$\mathbf{T}'(g)\mathbf{w} = \mathbf{T}(g)\mathbf{w}, \qquad \mathbf{w} \in W. \tag{2.1}$$

This rep is called the **restriction** of **T** to W. If **T** is unitary so is $\mathbf{T}'$.

Definition. The rep **T** is **reducible** if there is a proper subspace W of V which is invariant under **T**. Otherwise, **T** is **irreducible (irred)**.

A rep is irred if the only invariant subspaces of V are $\{\boldsymbol{\theta}\}$ and V itself. One-dimensional and zero-dimensional reps are necessarily irred. However, the trivial zero-dimensional rep will be ignored in all the material to follow.

We now give a matrix interpretation of reducibility. Suppose **T** is reducible and W is a proper invariant subspace of V. If $\dim W = k$ and $\dim V = n$ we can find a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for V such that $\mathbf{v}_1, \ldots, \mathbf{v}_k$, $1 \leq k \leq n$, form a basis for W. Then the matrices of the operators $\mathbf{T}(g)$ with respect to this basis take the form

$$\begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \begin{pmatrix} T'(g) & *** \\ Z & T''(g) \end{pmatrix} \end{array}$$

The $k \times k$ matrices $T'(g)$ and the $(n-k) \times (n-k)$ matrices $T''(g)$ separately define matrix reps of G. In particular $T'(g)$ is the matrix of the rep $\mathbf{T}'(g)$, (2.1), with respect to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of W. Here Z is the zero matrix.

Every reducible rep can be decomposed into irred reps in an almost unique manner. Thus the problem of constructing all reps of G simplifies to the problem of constructing all irred reps. The irred reps emerge as fundamental building blocks for the theory of reps of finite groups. To prove these statements in the simplest fashion we assume, as we can, that **T** is unitary.

If W is a proper subspace of the inner product space V and

$$W^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0, \quad \text{all} \quad \mathbf{w} \in W\} \tag{2.2}$$

is the subspace of all vectors perpendicular to W, it is an easy exercise in linear algebra to prove that $V = W \oplus W^{\perp}$ (V is the **direct sum** of W and $W^{\perp}$). That is, every $\mathbf{v} \in V$ can be written uniquely in the form

$$\mathbf{v} = \mathbf{w} + \mathbf{w}', \qquad \mathbf{w} \in W, \qquad \mathbf{w}' \in W^{\perp}.$$

Theorem 3.2. If **T** is a reducible unitary rep of G on V and W is a proper invariant subspace of V, then $W^{\perp}$ is also a proper invariant subspace of V. In this case we write $\mathbf{T} = \mathbf{T}' \oplus \mathbf{T}''$ and say that **T** is the **direct sum** of $\mathbf{T}'$ and $\mathbf{T}''$, where $\mathbf{T}', \mathbf{T}''$ are the (unitary) restrictions of **T** to $W, W^{\perp}$, respectively.

Proof. We must show $\mathbf{T}(g)\mathbf{u} \in W^{\perp}$ for every $g \in G$, $\mathbf{u} \in W^{\perp}$. Now for every $\mathbf{w} \in W$,

$$\langle \mathbf{T}(g)\mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{T}(g^{-1})\mathbf{w} \rangle = 0$$

since $\mathbf{T}(g^{-1})\mathbf{w} \in W$. The first equality follows from (1.17) and unitarity. Thus, $\mathbf{T}(g)\mathbf{u} \in W^{\perp}$. Q.E.D.

Suppose **T** is reducible and V_1 is a proper invariant subspace of V of smallest dimension. Then, necessarily, the restriction $\mathbf{T}_1$ of **T** to V_1 is irred and we have the direct sum decomposition $V = V_1 \oplus V_1^{\perp}$, where $V_1^{\perp}$ is invariant under **T**. If $V_1^{\perp}$ is not irred we can find a proper irred subspace V_2 of smallest dimension such that $V_1^{\perp} = V_2 \oplus V_2^{\perp}$ by repeating the above argument. We continue in this fashion until eventually we obtain the direct sum decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_l \quad \text{or} \quad \mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \cdots \oplus \mathbf{T}_l \tag{2.3}$$

where the V_i are mutually orthogonal proper invariant subspaces of V which transform irreducibly under the restrictions $\mathbf{T}_i$ of **T** to V_i. The decomposition process comes to an end after a finite number of steps because V is finite-dimensional. Some of the $\mathbf{T}_i$ may be equivalent. If a_1 of the reps $\mathbf{T}_i$ are equivalent to $\mathbf{T}_1$, a_2 to $\mathbf{T}_2, \ldots, a_k$ to $\mathbf{T}_k$ and $\mathbf{T}_1, \ldots, \mathbf{T}_k$ are pairwise nonequivalent, we write

$$\mathbf{T} = \sum_{j=1}^{k} \oplus\, a_j \mathbf{T}_j. \tag{2.4}$$

With this notation we are identifying equivalent reps. It is a straightforward exercise to show that, given a_j copies of $\mathbf{T}_j$, $1 \le j \le k$, one can construct a rep of G equivalent to **T**.

Theorem 3.3. Every finite-dimensional unitary rep of a finite group can be decomposed into a direct sum of irred unitary reps.

The above decomposition is not unique since the irred subspaces $V_1, \ldots, V_l$ are not uniquely determined. However, it will be shown in the next section that the integers a_j in (2.4) are uniquely determined. Thus, up to equivalence we can determine uniquely how many times a particular irred rep of G occurs in the decomposition of $\mathbf{T}$. The integer a_j is called the **multiplicity** of $\mathbf{T}_j$ in $\mathbf{T}$.

It follows from Theorem 3.1 that the statement of Theorem 3.3 still holds when the word "unitary" is deleted. To get a matrix interpretation of Theorem 3.3 choose a basis for V by combining bases $\{\mathbf{v}_i^{(j)}\}$, $j = 1, \ldots, l$, for $V_1, V_2, \ldots$, and V_l. In terms of this basis the matrix $T(g)$ of $\mathbf{T}(g)$ is given by

$$\begin{pmatrix} T_1(g) & & & Z \\ & T_2(g) & & \\ & & \ddots & \\ Z & & & T_l(g) \end{pmatrix} \tag{2.5}$$

where $n_j = \dim V_j$ and $T_j(g)$ is the matrix of $\mathbf{T}_j(g)$ with respect to the basis $\{\mathbf{v}_i^{(j)}\}$, $1 \leq i \leq n_j$.

3.3 Irreducible Representations

The fundamental problem in the representation theory of a finite group is the construction of a complete set of nonequivalent irred reps. A secondary problem is the determination of a practical method for decomposing a reducible rep into irred reps. The following two theorems (Shur's lemmas) are crucial.

Theorem 3.4. Let $\mathbf{T}$, $\mathbf{T}'$ be irred reps of the group G on the finite-dimensional vector spaces V, V', respectively and let $\mathbf{A}$ be a nonzero linear transformation mapping V into V' such that

$$\mathbf{T}'(g)\mathbf{A} = \mathbf{A}\mathbf{T}(g) \tag{3.1}$$

for all $g \in G$. Then $\mathbf{A}$ is a nonsingular linear transformation of V onto V', so $\mathbf{T}$ and $\mathbf{T}'$ are equivalent.

Proof. Let $N_\mathbf{A}$ be the **null space** and $R_\mathbf{A}$ the **range** of $\mathbf{A}$:

$$N_\mathbf{A} = \{\mathbf{v} \in V : \mathbf{A}\mathbf{v} = \boldsymbol{\theta}\} \qquad R_\mathbf{A} = \{\mathbf{v}' \in V' : \mathbf{v}' = \mathbf{A}\mathbf{v} \quad \text{for some} \quad \mathbf{v} \in V\}.$$

The subspace $N_\mathbf{A}$ of V is invariant under $\mathbf{T}$ since $\mathbf{A}\mathbf{T}(g)\mathbf{v} = \mathbf{T}'(g)\mathbf{A}\mathbf{v} = \boldsymbol{\theta}$ for all $g \in G$, $\mathbf{v} \in V$. Since $\mathbf{T}$ is irred, $N_\mathbf{A}$ is either V or $\{\boldsymbol{\theta}\}$. The first possibility implies $\mathbf{A} = \mathbf{Z}$, the zero operator, which is impossible. Therefore, $N_\mathbf{A} = \{\boldsymbol{\theta}\}$. The subspace $R_\mathbf{A}$ of V' is invariant under $\mathbf{T}'$ because $\mathbf{T}'(g)\mathbf{A}\mathbf{v} =$

$\mathbf{A}\mathbf{T}(g)\mathbf{v} \in R_{\mathbf{A}}$ for all $\mathbf{v} \in V$. But $\mathbf{T}'$ is irred so $R_{\mathbf{A}}$ is either V' or $\{\boldsymbol{\theta}\}$. If $R_{\mathbf{A}} = \{\boldsymbol{\theta}\}$ then $\mathbf{A} = \mathbf{Z}$, which is impossible. Therefore $R_{\mathbf{A}} = V'$ which implies that $\mathbf{T}$ and $\mathbf{T}'$ are equivalent. Q.E.D.

Corollary 3.1. Let $\mathbf{T}, \mathbf{T}'$ be nonequivalent finite-dimensional irred reps of G. If $\mathbf{A}$ is a linear transformation from V to V' which satisfies (3.1) for all $g \in G$ then $\mathbf{A} = \mathbf{Z}$.

The results in the remainder of this section apply only to complex reps.

Theorem 3.5. Let $\mathbf{T}$ be a rep of the group G on the finite-dimensional complex vector space V. Then $\mathbf{T}$ is irred if and only if the only transformations $\mathbf{A}: V \to V$ such that

$$\mathbf{T}(g)\mathbf{A} = \mathbf{A}\mathbf{T}(g) \tag{3.2}$$

for all $g \in G$ are $\mathbf{A} = \lambda\mathbf{E}$, where $\lambda \in \mathbb{C}$ and $\mathbf{E}$ is the identity operator on V.

Proof. It is well known that a linear operator on a finite-dimensional complex vector space always has at least one eigenvalue. (This statement is false for a real vector space.) Let λ be an eigenvalue of an operator $\mathbf{A}$ which satisfies (3.2) and define the eigenspace C_λ by

$$C_\lambda = \{\mathbf{v} \in V: \mathbf{A}\mathbf{v} = \lambda\mathbf{v}\}.$$

Clearly C_λ is a subspace of V and $\dim C_\lambda > 0$. Furthermore, C_λ is invariant under $\mathbf{T}$ because

$$\mathbf{A}\mathbf{T}(g)\mathbf{v} = \mathbf{T}(g)\mathbf{A}\mathbf{v} = \lambda\mathbf{T}(g)\mathbf{v}$$

for $\mathbf{v} \in C_\lambda$, $g \in G$, so $\mathbf{T}(g)\mathbf{v} \in C_\lambda$. If $\mathbf{T}$ is irred then $C_\lambda = V$ and $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for all $\mathbf{v} \in V$.

Conversely, suppose $\mathbf{T}$ is reducible. Then there exists a proper invariant subspace V_1 of V and by Theorem 3.2, a proper invariant subspace V_2 such that $V = V_1 \oplus V_2$. Any $\mathbf{v} \in V$ can be written uniquely as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_j \in V_j$. We define the projection operator $\mathbf{P}$ on V by $\mathbf{P}\mathbf{v} = \mathbf{v}_1 \in V_1$. Then $\mathbf{P}\mathbf{T}(g)\mathbf{v} = \mathbf{T}(g)\mathbf{P}\mathbf{v} = \mathbf{T}(g)\mathbf{v}_1$ (verify this), and $\mathbf{P}$ is clearly not a multiple of $\mathbf{E}$. Q.E.D.

Choosing a basis for V and a basis for V' we can immediately translate Shur's lemmas into statements about irred matrix reps.

Corollary 3.2. Let T and T' be $n \times n$ and $m \times m$ complex irred matrix reps of the group G, and let A be an $m \times n$ matrix such that

$$T'(g)A = AT(g) \tag{3.3}$$

for all $g \in G$. If T and T' are nonequivalent then $A = Z$, the zero matrix.

(In particular, this is true if $n \neq m$.) If $T = T'$ then $A = \lambda E_n$, where $\lambda \in \mathfrak{C}$ and E_n is the $n \times n$ identity matrix.

Note that the proofs of Shur's lemmas use only the concept of irreducibility and the fact that the rep spaces are finite-dimensional. The homomorphism property of reps and the fact that G is finite are not needed.

Theorem 3.5 is extremely useful because it yields a practical method for determining if a group rep is irred. The original definition of irreducibility, while useful for theoretical purposes, is too complicated to verify directly in most practical problems. Theorem 3.5 can easily be translated to a theorem about matrix reps, whose obvious statement and proof are left to the reader.

Let G be a finite group and select one irred rep $\mathbf{T}^{(\mu)}$ of G in each equivalence class of irred reps. Then every irred rep is equivalent to some $\mathbf{T}^{(\mu)}$ and the reps $\mathbf{T}^{(\mu_1)}, \mathbf{T}^{(\mu_2)}$ are nonequivalent if $\mu_1 \neq \mu_2$. The parameter μ indexes the equivalence classes of irred reps. (We will soon show that there are only a finite number of these classes.) Introduction of a basis in each rep space $V^{(\mu)}$ leads to a matrix rep $T^{(\mu)}$. The $T^{(\mu)}$ form a complete set of irred $n_\mu \times n_\mu$ matrix reps of G, one from each equivalence class. Here $n_\mu = \dim V^{(\mu)}$. If we wish, we can choose the $T^{(\mu)}$ to be unitary.

The following trick leads to an extremely useful set of relations in rep theory, the orthogonality relations. Given two irred matrix reps $T^{(\mu)}, T^{(\nu)}$ of G, choose an arbitrary $n_\mu \times n_\nu$ matrix B and form the $n_\mu \times n_\nu$ matrix

$$A = N^{-1} \sum_{g \in G} T^{(\mu)}(g) B T^{(\nu)}(g^{-1}) \tag{3.4}$$

where $N = n(G)$. Here, A is just the average of the matrices $T^{(\mu)}(g)BT^{(\nu)}(g^{-1})$ over the group G. We will show that A satisfies

$$T^{(\mu)}(h)A = AT^{(\nu)}(h) \tag{3.5}$$

for all $h \in G$. This result and Corollary 3.2 imply that if $\mu \neq \nu$ then $A = Z$, whereas if $\mu = \nu$ then $A = \lambda E_{n_\mu}$ for some $\lambda \in \mathfrak{C}$. The verification of (3.5) follows from

$$\begin{aligned} T^{(\mu)}(h)A &= N^{-1} \sum_{g \in G} T^{(\mu)}(h) T^{(\mu)}(g) B T^{(\nu)}(g^{-1}) \\ &= N^{-1} \sum_{g \in G} T^{(\mu)}(hg) B T^{(\nu)}((hg)^{-1}) T^{(\nu)}(h) = AT^{(\nu)}(h). \end{aligned}$$

We have used the fact that as g runs over each of the elements of G exactly once, so does $g' = hg$. Applying Corollary 3.2, we obtain the result $A = \lambda(\mu, B)\delta_{\mu\nu}E_{n_\mu}$ where $\delta_{\mu\nu}$ is the Kronecker delta, and the constant $\lambda \in \mathfrak{C}$ depends on μ and B. To derive all possible consequences of this identity it is enough to let B run through the $n_\mu \times n_\nu$ matrices $B^{(l,m)} = (B_{jk}^{(l,m)})$, where

$$B_{jk}^{(l,m)} = \begin{cases} 1 & \text{if} \quad j = l, \quad k = m, \qquad 1 \le j \le n_\mu, \quad 1 \le k \le n_\nu, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Making these substitutions, we obtain

$$\sum_{g\in G} T^{(\mu)}_{il}(g)T^{(\nu)}_{ms}(g^{-1}) = N\lambda\delta_{\mu\nu}\delta_{is}, \quad 1\le i,l\le n_\mu,\ 1\le m,s\le n_\nu. \tag{3.7}$$

Here, λ may depend on μ, l, and m, but not on i or s. To evaluate λ, set $\nu=\mu$, $s=i$, and sum on i to obtain

$$n_\mu N\lambda = \sum_{g\in G}\sum_{i=1}^{n_\mu} T^{(\mu)}_{mi}(g^{-1})T^{(\mu)}_{il}(g) = \sum_{g\in G} T^{(\mu)}_{ml}(e) = N\delta_{ml}$$

since $N = n(G)$. Therefore, $\lambda = \delta_{ml}n_\mu^{-1}$. We can simplify (3.7) slightly if we assume (as we can) that all of the matrix reps $T^{(\nu)}(g)$ are unitary. Then

$$T^{(\nu)}_{ms}(g^{-1}) = \overline{T^{(\nu)}_{sm}}(g)$$

and (3.7) reduces to

$$\sum_{g\in G} T^{(\mu)}_{il}(g)\overline{T^{(\nu)}_{sm}}(g) = (N/n_\mu)\delta_{is}\,\delta_{lm}\,\delta_{\mu\nu}. \tag{3.8}$$

Equations (3.7) and (3.8) are called the **orthogonality relations** for the matrix elements of irred reps of G. We have derived these remarkable relations without any detailed knowledge of the structure of G.

To better understand the orthogonality relations it is convenient to consider the elements x of the group ring R_G as complex-valued functions $x(g)$ on the group G. The relation between this approach and the definition of R_G as given in Example 3, Section 3.1, is provided by the correspondence

$$x = \sum_{g\in G} x(g)\cdot g \longleftrightarrow x(g). \tag{3.9}$$

The elements of the N-tuple $(x(g_1), \ldots, x(g_N))$, where g_i ranges over G, can be regarded as the components of $x \in R_G$ in the natural basis provided by the elements of G. Furthermore the 1–1 mapping (3.9) leads to the relations

$$\begin{aligned} &x+y \longleftrightarrow x(g)+y(g), \qquad \alpha x \longleftrightarrow \alpha x(g),\\ &xy \longleftrightarrow xy(g) = \sum_{h\in G} x(h)y(h^{-1}g)\end{aligned} \tag{3.10}$$

where the expression defining $xy(g)$ is called the **convolution product** of $x(g)$ and $y(g)$. Thus, we can consider R_G as the ring of all complex-valued functions $x(g)$ on G where addition, scalar multiplication, and convolution product are defined by (3.10). Indeed, the ring of functions just constructed is algebraically isomorphic to R_G with the isomorphism given by (3.9). Under this isomorphism the element $h = 1\cdot h \in R_G$ is mapped into the function

$$h(g) = \begin{cases} 1 & \text{if } g = h\\ 0 & \text{otherwise.}\end{cases}$$

Now consider the right regular rep on R_G. Writing

$$\mathbf{R}(h)x = \sum_{g\in G} [\mathbf{R}(h)x](g)\cdot g = xh^{-1} = \sum_g x(gh)\cdot g \qquad x\in R_G, \tag{3.11}$$

we obtain

$$[\mathbf{R}(h)x](g) = x(gh), \qquad h \in G, \tag{3.12}$$

as the action of $\mathbf{R}(h)$ on our new model of R_G. From Theorem 3.1, there exists an inner product on the N-dimensional vector space R_G with respect to which the right regular rep $\mathbf{R}$ is unitary. In particular, the following inner product works:

$$\langle x, y \rangle = N^{-1} \sum_{g \in G} x(g)\overline{y(g)}, \qquad x, y \in R_G. \tag{3.13}$$

The reader can easily verify that this is an inner product and that the $\mathbf{R}(h)$ operators are unitary with respect to it. Now note that for fixed μ, i, j with $1 \le i, j \le n_\mu$ the matrix element $T_{ij}^{(\mu)}(g)$ defines a function on G, hence an element of R_G. Furthermore, comparing (3.13) with (3.8), we see that the functions

$$\varphi_{ij}^{(\mu)}(g) = n_\mu^{1/2} T_{ij}^{(\mu)}(g), \qquad 1 \le i, j \le n_\mu, \tag{3.14}$$

where μ ranges over all equivalence classes of irred reps of G, form an ON set in R_G. Since R_G is N-dimensional the ON set can contain at most N elements. Thus there are only a finite number, say α, of nonequivalent irred reps of G. Each irred matrix rep μ yields $n_\mu{}^2$ vectors of the form (3.14). The full ON set $\{\varphi_{ij}^{(\mu)}\}$ spans a subspace of R_G of dimension

$$n_1{}^2 + n_2{}^2 + \cdots + n_\alpha{}^2 \le N. \tag{3.15}$$

The inequality (3.15) is a strong restriction on the possible number and dimensions of irred reps of G. This result can be strengthened even more by showing that the ON set $\{\varphi_{ij}^{(\mu)}\}$ is actually a **basis** for R_G. Since the dimension N of R_G is equal to the number of basis vectors, we obtain the equality

$$n_1{}^2 + n_2{}^2 + \cdots + n_\alpha{}^2 = N. \tag{3.16}$$

To prove this result, let V be the subspace of R_G spanned by the ON set $\{\varphi_{ij}^{(\mu)}\}$. From (3.14) and the homomorphism property of the matrices $T^{(\mu)}(g)$ there follows

$$[\mathbf{R}(h)\varphi_{ij}^{(\mu)}](g) = \varphi_{ij}^{(\mu)}(gh) = \sum_{k=1}^{n_\mu} T_{kj}^{(\mu)}(h)\varphi_{ik}^{(\mu)}(g) \in V. \tag{3.17}$$

Thus, V is invariant under $\mathbf{R}$. According to Theorem 3.2, $V^\perp$ is also invariant under $\mathbf{R}$ and $R_G = V \oplus V^\perp$. Here, $V^\perp$ is defined with respect to the inner product (3.13). If $V^\perp \neq \{\boldsymbol{\theta}\}$ then it contains a subspace W transforming under some irred rep $\mathbf{T}^{(\nu)}$ of G. Thus, there exists an ON basis $x_1, \ldots, x_{n_\nu}$ for W such that

$$[\mathbf{R}(g)x_i](h) = x_i(hg) = \sum_{j=1}^{n_\mu} T_{ji}^{(\nu)}(g)x_j(h), \qquad 1 \le i \le n_\nu. \tag{3.18}$$

Setting $h = e$ in (3.18) and letting g run over G, we find

$$x_i(g) = \sum_j x_j(e)T_{ji}^{(\nu)}(g) = \sum_j x_j(e)\varphi_{ji}^{(\nu)}(g)/n_\nu^{1/2},$$

so $x_i \in V$. Thus, $W \subseteq V \cap V^\perp$. This is possible only if $W = \{\boldsymbol{\theta}\}$. Therefore, $V^\perp = \{\boldsymbol{\theta}\}$ and $V = R_G$.

Theorem 3.6. The functions

$$\{\varphi_{ij}^{(\mu)}(g)\}, \qquad \mu = 1, \ldots, \alpha, \quad 1 \leq i, j \leq n_\mu,$$

form an ON basis for R_G. Every function $x \in R_G$ can be written uniquely in the form

$$x(g) = \sum_{i,j,\mu} a_{ij}^\mu \varphi_{ij}^{(\mu)}(g), \qquad a_{ij}^\mu = \langle x, \varphi_{ij}^{(\mu)} \rangle.$$

Relation (3.17) also yields the interesting fact that for fixed μ and i, $1 \leq i \leq n_\mu$, the n_μ vectors $\{\varphi_{ij}^{(\mu)} : 1 \leq j \leq n_\mu\}$ form an ON basis for a subspace $V_i^{(\mu)}$ of R_G which transforms under the irred rep $\mathbf{T}^{(\mu)}$ of G. Thus,

$$R_G = \sum_{\mu, i} \oplus V_i^{(\mu)}, \qquad 1 \leq \mu \leq \alpha, \quad 1 \leq i \leq n_\mu,$$

and the rep $\mathbf{T}^{(\mu)}$ occurs with multiplicity n_μ in the right regular rep $\mathbf{R}$.

$$\mathbf{R} = \sum_{\mu=1}^{\alpha} \oplus n_\mu \mathbf{T}^{(\mu)}. \tag{3.19}$$

3.4 Group Characters

The orthogonality relations and decomposition theorems of the preceding section suffer from the defect that they are basis-dependent. To determine in what sense our results are unique we free them from a dependence on the choice of basis vectors for V.

Let $\mathbf{T}$ be a rep of the finite group G on the n-dimensional vector space V. With respect to some fixed basis in V the operators $\mathbf{T}(g)$ define a matrix rep in terms of $n \times n$ matrices $T(g)$. We define the **character** of $\mathbf{T}$ as the function

$$\chi(g) = \operatorname{tr} T(g), \qquad g \in G. \tag{4.1}$$

Since the trace satisfies

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

for any two $n \times n$ matrices, we find

$$\operatorname{tr}(ST(g)S^{-1}) = \operatorname{tr}(T(g)S^{-1}S) = \operatorname{tr}(T(g)) \tag{4.2}$$

for all nonsingular $n \times n$ matrices S. Thus, equivalent matrix reps have the same character and $\chi(g)$ is independent of basis. Furthermore, we will soon show that two reps with equal characters are equivalent. Thus, there is a 1–1 relationship between equivalence classes of reps of G and group characters on G. If χ is the character of an irred rep it is called **simple**; if the rep is reducible, χ is a **compound** character.

The orthogonality relations for matrix elements immediately lead to orthogonality relations for characters. Let $\chi^{(\mu)}$ be the character of the irred rep $T^{(\mu)}$, $\mu = 1, \ldots, \alpha$. Setting $i = l$ and $m = s$ in (3.7) and summing i from 1 to n_μ, and m from 1 to n_ν, we obtain

$$\sum_{g \in G} \chi^{(\mu)}(g)\chi^{(\nu)}(g^{-1}) = N\delta_{\mu\nu}. \tag{4.3}$$

If we assume, as we can, that the matrix rep $T(g)$ is unitary then

$$\chi(g^{-1}) = \operatorname{tr} T(g^{-1}) = \operatorname{tr} \overline{T(g)}^t = \overline{\operatorname{tr} T(g)} = \overline{\chi(g)}, \tag{4.4}$$

a result which is now seen to be valid independent of basis. Substituting this result into (4.3), we obtain the orthogonality relations

$$\langle \chi^{(\mu)}, \chi^{(\nu)} \rangle = \delta_{\mu\nu}, \qquad 1 \leq \mu, \nu \leq \alpha, \tag{4.5}$$

where the inner product is defined by (3.17). Thus, simple characters of G form an ON set in R_G.

Now let $\mathbf{T}(g)$ be an arbitrary rep with character $\chi(g)$. It follows from (2.4) and (2.5) that with respect to one basis at least, we can write

$$\chi(g) = \sum_{\mu=1}^{\alpha} a_\mu \chi^{(\mu)}(g) \tag{4.6}$$

where a^μ is the multiplicity of $\mathbf{T}^{(\mu)}$ in $\mathbf{T}$. However, the orthogonality relations (4.5) imply

$$\langle \chi, \chi^{(\mu)} \rangle = a_\mu, \qquad 1 \leq \mu \leq \alpha. \tag{4.7}$$

Since the left-hand side of (4.7) is basis-independent, so is the right-hand side.

Theorem 3.7. The multiplicity a_μ of the irred rep $\mathbf{T}^{(\mu)}$ in $\mathbf{T}$ is given by (4.7). Since reps with the same multiplicities are equivalent, reps with equal characters are equivalent.

Thus, the multiplicities a_μ are unique even though the exact decomposition of the rep space into irred subspaces may be nonunique.

Corollary 3.3. Let $\chi(g)$ be a group character of G. Then $\langle \chi, \chi \rangle$ is a nonnegative integer and $\chi(g)$ corresponds to an irred rep if and only if $\langle \chi, \chi \rangle = 1$.

Proof. We can write χ as a unique sum of simple characters:

$$\chi(g) = \sum_{\mu=1}^{\alpha} a_\mu \chi^{(\mu)}(g).$$

Since the $\chi^{(\mu)}$ form an ON set there follows

$$\langle \chi, \chi \rangle = \sum_{\mu=1}^{\alpha} a_\mu{}^2. \tag{4.8}$$

The right-hand side equals one if and only if one of the a_μ is one and the rest are zero. Q.E.D.

We list a few additional properties of characters. If $\mathbf{T}$ is an n-dimensional rep with character χ then

$$\chi(e) = \operatorname{tr} E_n = n. \tag{4.9}$$

Thus $\chi(e)$ is always equal to the dimension of the rep. Furthermore,

$$\chi(hgh^{-1}) = \operatorname{tr}[T(h)T(g)T(h)^{-1}] = \operatorname{tr} T(g) = \chi(g), \qquad g, h \in G, \tag{4.10}$$

so χ is constant on each conjugacy class of G. Suppose G has k conjugacy classes containing $m_1, \ldots, m_k$ elements, respectively, with $m_1 + \cdots + m_k = N$. Then the orthogonality relations (4.5) read

$$N^{-1} \sum_{i=1}^{k} \chi_i^{(\mu)} \overline{\chi_i^{(\nu)}} m_i = \delta_{\mu\nu} \tag{4.11}$$

where $\chi_i^{(\mu)}$ is the value of $\chi^{(\mu)}(g)$ with g in the ith conjugacy class. Relations of the form (4.11) are not as esthetically pleasing as (4.5) but they are useful for practical computations.

Let us examine the relationship between group characters and the subspace F of R_G consisting of all functions $\psi(g)$ such that

$$\psi(hgh^{-1}) = \psi(g), \qquad g, h \in G,$$

i.e., all functions which are constant on conjugacy classes. Each $\psi \in F$ is uniquely determined by k complex numbers, the value assumed by ψ on the k conjugacy classes of G. Thus F is k-dimensional. Clearly, the α simple characters of G form an ON set in F with respect to the inner product $\langle -, - \rangle$. In fact, $\alpha = k$ and these characters form an ON basis for F.

Theorem 3.8. The number α of nonequivalent irred reps of G is equal to the number of conjugacy classes in G.

Proof. Let $\psi \in F$. Since $F \subseteq R_G$ we can expand ψ in the form

$$\psi(g) = \sum_{\mu,i,j} a_{ij}^\mu T_{ij}^{(\mu)}(g)$$

where the $T_{ij}^{(\mu)}(g)$ are the matrix elements of a complete set of nonequivalent unitary irred reps of G. Since (summing over repeated indices)

$$\begin{aligned}\psi(g) &= N^{-1} \sum_{h \in G} \psi(hgh^{-1}) = N^{-1} \sum_{h} a_{ij}^\mu T_{il}^{(\mu)}(h) T_{lm}^{(\mu)}(g) T_{mj}^{(\mu)}(h^{-1}) \\ &= a_{ij}^\mu T_{lm}^{(\mu)}(g) \langle T_{il}^{(\mu)}, T_{jm}^{(\mu)} \rangle = \sum_{i,\mu} (a_{ii}^\mu / n_\mu) \chi^{(\mu)}(g),\end{aligned}$$

$\psi(g)$ is a linear combination of simple characters. Therefore, the simple characters form an ON basis for F. Q.E.D.

In terms of the $\alpha \times \alpha$ matrix A with elements

$$A_{\mu i} = (m_i/N)^{1/2}\chi_i^{(\mu)}, \qquad 1 \le i, \mu \le \alpha,$$

the first orthogonality relation (4.11) reads $A\bar{A}^t = E_\alpha$. Thus, $\bar{A}^t = A^{-1}$ and $\bar{A}^t A = E_\alpha$, or

$$\sum_{\mu=1}^{\alpha} \bar{\chi}_i^{(\mu)}\chi_j^{(\mu)} = (N/m_i)\delta_{ij}, \qquad 1 \le i, j \le \alpha. \tag{4.12}$$

This is known as the **second orthogonality relation** for characters.

As an example of character methods we verify expression (3.19) for the decomposition of the right regular rep **R** into irred reps. We begin by computing the character χ of **R** in the natural basis for R_G provided by the group elements. Now,

$$\mathbf{R}(h)g = gh^{-1}, \qquad h, g \in G,$$

so $\mathbf{R}(h)$ acts on the natural basis by permuting the basis vectors. If $h \ne e$ then no basis vector is left fixed under $\mathbf{R}(h)$. Thus, the matrix of $\mathbf{R}(h)$ in the natural basis has matrix elements which are zeros and ones, and if $h \ne e$ the diagonal matrix elements are all zero:

$$\chi(h) = \begin{cases} N & \text{if} \quad h = e \\ 0 & \text{if} \quad h \ne e. \end{cases}$$

Writing

$$\chi = \sum_{\mu=1}^{\alpha} a_\mu \chi^{(\mu)}$$

we obtain

$$a_\mu = \langle \chi, \chi^{(\mu)} \rangle = \overline{\chi^{(\mu)}(e)} = n_\mu.$$

Therefore, the multiplicity of $\mathbf{T}^{(\mu)}$ in **R** equals the dimension of $\mathbf{T}^{(\mu)}$. (The results for the decomposition of the left regular rep **L** are the same, so **R** and **L** are equivalent reps.)

Later we shall present a detailed derivation of the simple characters for the crystallographic point groups and the symmetric groups. Here we consider only the simple case where G is an abelian group of order N. Then G contains N conjugacy classes with one element each. Thus $\alpha = N$ and the relation

$$n_1^2 + n_2^2 + \cdots + n_N^2 = N$$

implies $n_1 = \cdots = n_N = 1$. The N nonequivalent irred reps of G are one-dimensional. In this special case the simple characters $\chi^{(\mu)}(g)$ coincide with the irreducible 1×1 matrix reps. Thus, the characters satisfy the homomorphism property

$$\chi^{(\mu)}(g_1)\chi^{(\mu)}(g_2) = \chi^{(\mu)}(g_1 g_2). \tag{4.13}$$

Since $g^N = e$ for every $g \in G$ there follows

$$[\chi^{(\mu)}(g)]^N = \chi^{(\mu)}(g^N) = \chi^{(\mu)}(e) = 1,$$

so $\chi^{(\mu)}(g)$ is an Nth root of unity. In order to explicitly list the simple characters for any abelian group it would be necessary to study the structure theory of such groups. However, if G is cyclic it is easy to give complete results. Let g_0 be an element of order N which generates G. Then

$$[\chi^{(\mu)}(g_0)]^N = 1 \tag{4.14}$$

for each of the N simple characters of G. Furthermore, the numbers $\chi^{(\mu)}(g_0)$ uniquely determine $\chi^{(\mu)}$, so these N numbers must be distinct. The equation $\omega^N = 1$ has exactly N solutions,

$$\omega_\mu = \exp(2\pi i\mu/N), \qquad \mu = 0, 1, \ldots, N-1.$$

Thus, the simple characters can be uniquely defined by

$$\chi^{(\mu)}(g_0{}^n) = \exp(2\pi i n\mu/N), \qquad \mu, n = 0, 1, \ldots, N-1. \tag{4.15}$$

The reader should understand that the above discussion applies only to complex reps of a group G. Character arguments can be applied to real reps only with special care. To understand the difficulties involved here, consider a real irred matrix rep T of G. We can also consider T as a complex matrix rep T^c of G. However, T^c may not be irred. For example, in an appropriate basis the generator $C(\pi/2)$ of the cyclic group C_4, considered as a transformation group in the plane, corresponds to the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The two dimensional real rep generated by this matrix is irred since the matrix has complex eigenvalues $\pm i$ and cannot be diagonalized by a real similarity transformation. However, considered as a complex matrix rep it is reducible.

3.5 New Representations from Old Ones

Let G be a group of order N. We discuss some methods for using known reps of G to construct new reps. The right and left regular reps and the identity rep are already familiar. (The **identity representation** of G is the irred one-dimensional rep defined by mapping each $g \in G$ into 1.) Furthermore, if G is defined as a matrix group, this matrix realization automatically yields a rep. To construct more reps we will probably have to rely on one of the methods presented below. Ultimately, we want to explicitly construct all irred reps of G (or at least their characters) and to explicitly decompose an arbitrary rep of G into irred reps.

First we review two techniques which have been studied earlier. If $\mathbf{T}_1$, $\mathbf{T}_2$ are reps of G on the vector spaces V_1, V_2, respectively, the **direct sum** $\mathbf{T}_1 \oplus \mathbf{T}_2$ is a rep acting on $V_1 \oplus V_2$ (vector space direct sum) and defined by

$$(5.1) \qquad [\mathbf{T}_1 \oplus \mathbf{T}_2(g)]\mathbf{v}_1 \oplus \mathbf{v}_2 = \mathbf{T}_1(g)\mathbf{v}_1 \oplus \mathbf{T}_2(g)\mathbf{v}_2, \qquad \mathbf{v}_i \in V_i.$$

It is easy to show that the character χ of this rep is $\chi(g) = \chi_1(g) + \chi_2(g)$, where χ_1, χ_2 are the characters of $\mathbf{T}_1, \mathbf{T}_2$, respectively. This procedure can easily be extended to define the direct sum of any finite number of reps. We know already that every rep $\mathbf{T}$ is equivalent to a rep

$$\sum_{\mu=1}^{\alpha} \oplus a_\mu \mathbf{T}^{(\mu)}$$

where the $\mathbf{T}^{(\mu)}$ are a complete set of nonequivalent irred reps of G and the multiplicities a_μ are uniquely determined.

If $\mathbf{T}$ has rep space V and W is a proper invariant subspace of V then the rep $\mathbf{T}' = \mathbf{T} \mid W$ on W defined by

$$(5.2) \qquad \mathbf{T}'(g)\mathbf{w} = \mathbf{T}(g)\mathbf{w}, \qquad g \in G, \quad \mathbf{w} \in W$$

is called the **restriction** of $\mathbf{T}$ to W. We have seen that every reducible rep of G can be written as a direct sum of certain of its irred restrictions.

Let V and V' be vector spaces of dimensions n, n', respectively and let $\{\mathbf{v}_i\}, \{\mathbf{v}_j'\}$ be bases for these spaces. We define $V \otimes V'$, the **tensor product** of V and V', as the nn'-dimensional space with basis $\{\mathbf{v}_i \otimes \mathbf{v}_j'\}$, $1 \leq i \leq n$, $1 \leq j \leq n'$. Thus, any $\mathbf{w} \in V \otimes V'$ can be written uniquely in the form

$$(5.3) \qquad \mathbf{w} = \sum_{ij} \alpha_{ij} \mathbf{v}_i \otimes \mathbf{v}_j'.$$

If $\mathbf{v} = \sum \alpha_i \mathbf{v}_i$ and $\mathbf{v}' = \sum \beta_j \mathbf{v}_j'$ we define the vector $\mathbf{v} \otimes \mathbf{v}' \in V \otimes V'$ by

$$(5.4) \qquad \mathbf{v} \otimes \mathbf{v}' = \sum_{i=1}^{n} \sum_{j=1}^{n'} \alpha_i \beta_j \mathbf{v}_i \otimes \mathbf{v}_j'.$$

If $\mathbf{w} \in V \otimes V'$ can be written in the form $\mathbf{w} = \mathbf{v} \otimes \mathbf{v}'$ then $\mathbf{w}$ is said to be **indecomposable.** The example $\mathbf{w} = \mathbf{v}_1 \otimes \mathbf{v}_2' + \mathbf{v}_2 \otimes \mathbf{v}_1'$ shows that if n, $n' \geq 2$ not every $\mathbf{w}$ is indecomposable. As a consequence of definition (5.4) it is easy to verify the following properties:

$$(5.5a) \qquad \alpha(\mathbf{v} \otimes \mathbf{v}') = (\alpha\mathbf{v}) \otimes \mathbf{v}' = \mathbf{v} \otimes (\alpha\mathbf{v}'), \qquad \alpha \in \mathfrak{C},$$

$$(5.5b) \qquad (\mathbf{u} + \mathbf{v}) \otimes \mathbf{v}' = \mathbf{u} \otimes \mathbf{v}' + \mathbf{v} \otimes \mathbf{v}', \qquad \mathbf{u}, \mathbf{v} \in V,$$

$$(5.5c) \qquad \mathbf{v} \otimes (\mathbf{u}' + \mathbf{v}') = \mathbf{v} \otimes \mathbf{u}' + \mathbf{v} \otimes \mathbf{v}', \qquad \mathbf{u}', \mathbf{v}' \in V'.$$

Although our definition of tensor product appears to depend on the choice of bases $\{\mathbf{v}_i\}$ and $\{\mathbf{v}_j'\}$ it is actually independent of this choice. For, let $\{\mathbf{u}_i\}$,

$\{\mathbf{u}_j'\}$ be new bases related to the old bases by

$$\mathbf{v}_l = \sum_{i=1}^{n} A_{il}\mathbf{u}_i, \qquad \mathbf{v}_k' = \sum_{j=1} A'_{jk}\mathbf{u}_j'$$

where the matrices A and A' are nonsingular. From relations (5.5a)–(5.5c) the basis vectors $\mathbf{v}_l \otimes \mathbf{v}_k'$ can all be expressed as linear combinations of the nn' vectors $\mathbf{u}_i \otimes \mathbf{u}_j'$. Since $V \otimes V'$ is nn'-dimensional it follows that the set $\{\mathbf{u}_i \otimes \mathbf{u}_j'\}$ is also a basis for $V \otimes V'$. This shows that the definition of $V \otimes V'$ is independent of basis. The definition of an indecomposable vector is also independent of basis.

Suppose $\mathbf{T}, \mathbf{T}'$ are reps of G on the spaces V, V', respectively. The **tensor product** $\mathbf{T} \otimes \mathbf{T}'$ is the rep of G on $V \otimes V'$ defined by

(5.6) $$[\mathbf{T} \otimes \mathbf{T}'(g)]\mathbf{v} \otimes \mathbf{v}' = \mathbf{T}(g)\mathbf{v} \otimes \mathbf{T}'(g)\mathbf{v}', \qquad g \in G,$$

and linearity of the operator $\mathbf{T} \otimes \mathbf{T}'(g)$. It is straightforward to verify the rep property of these operators. Let $\{\mathbf{v}_i\}$, $\{\mathbf{v}_j'\}$ be bases of V, V', and let $T(g)$, $T'(g)$ be the corresponding matrix reps of $\mathbf{T}, \mathbf{T}'$. Then the matrix rep of $\mathbf{T} \otimes \mathbf{T}'$ with respect to $\{\mathbf{v}_i \otimes \mathbf{v}_j'\}$ is defined by

(5.7) $$[\mathbf{T} \otimes \mathbf{T}'(g)]\mathbf{v}_i \otimes \mathbf{v}_j' = \sum_{l=1}^{n} \sum_{k=1}^{n'} T_{li}(g)T'_{kj}(g)\mathbf{v}_l \otimes \mathbf{v}_k'$$

or

$$[T \otimes T'(g)]_{lk,ij} = T_{li}(g)T'_{kj}(g).$$

(Note the double-suffix notation.) The character $\chi \otimes \chi'(g)$ is

(5.8) $$\chi \otimes \chi'(g) = \sum_{l=1}^{n} \sum_{k=1}^{n'} T_{ll}(g)T'_{kk}(g) = \chi(g)\chi'(g).$$

Thus, the character of the tensor product is the product of the characters of the factors. As an immediate consequence of this result we see that the reps $\mathbf{T} \otimes \mathbf{T}'$ and $\mathbf{T}' \otimes \mathbf{T}$ are equivalent.

The above definitions have obvious generalizations to define n-fold tensor products $V^{(1)} \otimes \cdots \otimes V^{(n)}$ and tensor product reps $\mathbf{T}^{(1)} \otimes \cdots \otimes \mathbf{T}^{(n)}$ of the reps $\mathbf{T}^{(j)}$ on $V^{(j)}$. The dimension of the tensor product space is the product of the dimensions of the factor spaces $V^{(j)}$.

Let $\{\mathbf{T}^{(\mu)}\}$, $1 \leq \mu \leq \alpha$, be a complete set of nonequivalent irred reps of G. We can form tensor product reps $\mathbf{T}^{(\mu)} \otimes \mathbf{T}^{(\nu)}$, $1 \leq \mu, \nu \leq \alpha$, with characters $\chi^{(\mu)} \otimes \chi^{(\nu)}(g) = \chi^{(\mu)}(g)\chi^{(\nu)}(g)$. These reps can then be decomposed into irred reps

(5.9) $$\mathbf{T}^{(\mu)} \otimes \mathbf{T}^{(\nu)} \simeq \sum_{\xi=1}^{\alpha} \oplus\, a_\xi \mathbf{T}^{(\xi)}$$

where

$$a_\xi = \langle \chi^{(\mu)}\chi^{(\nu)}, \chi^{(\xi)} \rangle.$$

The expansion (5.9) is called a **Clebsch–Gordan series**. Many important problems in mathematical physics reduce to the computation of the multiplicities a_ξ. Suppose we have agreed on a complete set of nonequivalent irred matrix reps $T^{(\xi)}(g)$ of G. Let $\{\mathbf{v}_i^{(\mu)}\}$, $\{\mathbf{v}_j^{(\nu)}\}$ be bases of $V^{(\mu)}$, $V^{(\nu)}$ whose associated matrix reps of $\mathbf{T}^{(\mu)}, \mathbf{T}^{(\nu)}$ are $T^{(\mu)}, T^{(\nu)}$, respectively. The set $\{\mathbf{v}_i^{(\mu)} \otimes \mathbf{v}_j^{(\nu)}\}$ clearly defines a basis of $V^{(\mu)} \otimes V^{(\nu)}$. On the other hand, by (5.9) there exists another basis $\{\mathbf{w}_l^{\xi,s}\}$, $1 \leq \xi \leq \alpha$, $1 \leq s \leq a_\xi$, for $V^{(\mu)} \otimes V^{(\nu)}$ such that for fixed ξ and s, the vectors $\{\mathbf{w}_l^{\xi,s}, 1 \leq l \leq n_\xi\}$ form a basis for an irred subspace transforming under $\mathbf{T}^{(\xi)}$ and inducing the matrix rep $T^{(\xi)}$. (We have by no means uniquely defined the basis $\{\mathbf{w}_l^{\xi,s}\}$. For practical computations it is necessary to be explicit as to how each basis vector is chosen. This matter will be taken up later.) In terms of the "natural basis" $\{\mathbf{v}_i^{(\mu)} \otimes \mathbf{v}_j^{(\nu)}\}$ the tensor product induces the $n_\mu n_\nu$-dimensional matrix rep

$$[T^{(\mu)} \otimes T^{(\nu)}(g)]_{lk,ij} = T_{li}^{(\mu)}(g) T_{kj}^{(\nu)}(g) \tag{5.10}$$

while in terms of the $\{\mathbf{w}_l^{\xi,s}\}$ basis the matrix rep is

$$\begin{pmatrix} \overbrace{T^{(1)}(g) \quad \cdots \quad T^{(1)}(g)}^{a_1} & & & & & Z \\ & \overbrace{T^{(2)}(g) \quad \cdots}^{a_2} & & & & \\ & & \ddots & & & \\ Z & & & \overbrace{T^{(\alpha)}(g) \quad \cdots \quad T^{(\alpha)}(g)}^{a_\alpha} & & \end{pmatrix} \tag{5.11}$$

These two bases are related by expressions of the form

$$\mathbf{w}_l^{\xi,s} = \sum_{ij} (\mu i, \nu j \mid \xi s l) \mathbf{v}_i^{(\mu)} \otimes \mathbf{v}_j^{(\nu)}. \tag{5.12}$$

The expansion coefficients $(\mu i, \nu j \mid \xi s l)$ are called **Clebsch–Gordan (CG) coefficients**. These coefficients form an $n_\mu n_\nu \times n_\mu n_\nu$ matrix. This matrix is clearly invertible with inverse matrix elements defined by

$$\mathbf{v}_i^{(\mu)} \otimes \mathbf{v}_j^{(\nu)} = \sum_{\xi k l} (\xi s l \mid \mu i, \nu j) \mathbf{w}_l^{\xi,s}, \qquad 1 \leq i \leq n_\mu, \quad 1 \leq j \leq n_\nu. \tag{5.13}$$

As an immediate consequence we have the relations

$$\begin{gathered} \sum_{\xi s l} (\mu i, \nu j \mid \xi s l)(\xi s l \mid \mu i', \nu j') = \delta_{ii'} \delta_{jj'} \\ \sum_{ij} (\xi s l \mid \mu i, \nu j)(\mu i, \nu j \mid \xi' s' l') = \delta_{\xi\xi'} \delta_{ss'} \delta_{ll'}. \end{gathered} \tag{5.14}$$

Furthermore, if we assume, as we can, that the $T^{(\xi)}(g)$ are unitary matrices and the above bases are ON with respect to an inner product $\langle -, - \rangle$ on $V^{(\mu)} \otimes V^{(\nu)}$ then the matrix formed by the CG coefficients is unitary, i.e.,

$$(\xi s l \mid \mu i, \nu j) = \overline{(\mu i, \nu j \mid \xi s l)}. \tag{5.15}$$

Although the "natural basis" is the easiest to compute it is the **w**-basis which is the most useful in applications, since this basis explicitly exhibits the decomposition of $\mathbf{T}^{(\mu)} \otimes \mathbf{T}^{(\nu)}$ into irred reps. By (5.12), to obtain this new basis from $\{\mathbf{v}_i^{(\mu)} \otimes \mathbf{v}_j^{(\nu)}\}$ it is sufficient to know the CG coefficients. For this reason much effort has been expended in the compilation of CG coefficients. We will return to this problem later.

If $\mathbf{T}_1$ and $\mathbf{T}_2$ are reps of the groups G_1 and G_2, respectively, we can define a rep $\mathbf{T}$ of the direct product group $G_1 \times G_2$ on $V_1 \otimes V_2$ by

$$(5.16) \quad \mathbf{T}(g_1 g_2)\mathbf{v}_1 \otimes \mathbf{v}_2 = \mathbf{T}_1(g_1)\mathbf{v}_1 \otimes \mathbf{T}_2(g_2)\mathbf{v}_2 \qquad g_i \in G_i, \quad \mathbf{v}_i \in V_i.$$

If $\mathbf{T}_1$ is n_1-dimensional and $\mathbf{T}_2$ is n_2-dimensional then $\mathbf{T}$ is $n_1 n_2$-dimensional. Furthermore, an elementary computation similar to (5.8) shows that the character χ of $\mathbf{T}$ is

$$(5.17) \qquad \chi(g_1 g_2) = \chi_1(g_1)\chi_2(g_2)$$

where χ_i is the character of $\mathbf{T}_i$. If $\mathbf{T}_1$ and $\mathbf{T}_2$ are irred then

$$\langle \chi, \chi \rangle_{G_1 \times G_2} = \langle \chi_1, \chi_1 \rangle_{G_1} \langle \chi_2, \chi_2 \rangle_{G_2} = 1$$

so χ is irred. Let $\chi_1^{(\mu)}$, $1 \le \mu \le \alpha_1$, be the simple characters of G_1 corresponding to reps of dimension $n_\mu^{(1)}$. Let $\chi_2^{(\nu)}$, $1 \le \nu \le \alpha_2$, and $n_\nu^{(2)}$ be similar quantities for G_2. Then the characters $\chi^{(\mu,\nu)}(g_1 g_2) = \chi_1^{(\mu)}(g_1)\chi_2^{(\nu)}(g_2)$ belong to $\alpha_1\alpha_2$ nonequivalent irred reps $\mathbf{T}^{(\mu,\nu)}$ of $G_1 \times G_2$, since

$$\langle \chi^{(\mu,\nu)}, \chi^{(\mu',\nu')} \rangle_{G_1 \times G_2} = \delta_{\mu\mu'}\delta_{\nu\nu'}.$$

Now G_1 has α_1 conjugacy classes and G_2 has α_2 conjugacy classes, so $G_1 \times G_2$ must have exactly $\alpha_1\alpha_2$ conjugacy classes. Thus, every irred rep of $G_1 \times G_2$ is equivalent to exactly one of the irred reps $\mathbf{T}^{(\mu,\nu)}$. We have shown that a knowledge of the irred characters and reps of the factors G_1, G_2 immediately yields the irred characters and reps of $G_1 \times G_2$.

If $\mathbf{T}$ is a rep of G on V we can obtain a rep $\mathbf{T}_H$ of any subgroup H of G by restricting $\mathbf{T}$ to H,

$$(5.18) \qquad \mathbf{T}_H(h) = \mathbf{T}(h), \qquad h \in H.$$

We sometimes write $\mathbf{T}_H = \mathbf{T} \mid H$. The character $\chi_H = \chi \mid H$ of this rep is given by $\chi_H(h) = \chi(h)$.

On the other hand, there is a method due to Frobenius for constructing a rep of G from a rep of the subgroup H. Let $\mathbf{T}$ be a rep of H on the space V. Denote by $\mathcal{V}^G$ the vector space of all functions $\mathbf{f}(g)$ with domain G and range contained in V where addition and scalar multiplication of functions are the vector operations. Here, for a fixed $g \in G$, $\mathbf{f}(g)$ is a vector in V. Let V^G be the subspace of $\mathcal{V}^G$ defined by

$$(5.19) \quad V^G = \{\mathbf{f} \in \mathcal{V}^G : \mathbf{f}(hg) = \mathbf{T}(h)\mathbf{f}(g) \quad \text{for all} \quad h \in H, \quad g \in G\}$$

We define a rep $\mathbf{T}^G$ of G on V^G by

$$[\mathbf{T}^G(g)\mathbf{f}](g') = \mathbf{f}(g'g), \qquad g, g' \in G, \quad \mathbf{f} \in V^G. \tag{5.20}$$

It is clear that V^G is invariant under G and the operators $\mathbf{T}^G(g)$ satisfy the homomorphism property. Here, $\mathbf{T}^G$ is called an **induced representation**. Let

$$Hg_1, \ldots, Hg_m, \qquad n(G) = m \cdot n(H)$$

be the distinct right cosets of H, where $g_1 = e$. Any $\mathbf{f} \in V^G$ is uniquely determined by the m vectors $\mathbf{f}(g_1), \ldots, \mathbf{f}(g_m)$ since for $g = hg_i$ in the right coset Hg_i we have

$$\mathbf{f}(g) = \mathbf{f}(hg_i) = \mathbf{T}(h)\mathbf{f}(g_i).$$

Let $\{\mathbf{v}_j\}$, $1 \le j \le d$, be a basis for V and define elements $\mathbf{e}_j{}^k(g)$ of V^G by

$$\mathbf{e}_j{}^k(g_i) = \delta_{ik}\mathbf{v}_j, \qquad 1 \le i, k \le m, \quad 1 \le j \le d. \tag{5.21}$$

The functions $\{\mathbf{e}_j{}^k\}$ form a basis for V^G, so the induced rep is md-dimensional. Let $T(h)$ be the matrix of $\mathbf{T}(h)$ relative to the $\{\mathbf{v}_j\}$ basis. We will use $T(h)$ to compute the matrix rep of G defined by $\mathbf{T}^G$ relative to the $\{\mathbf{e}_j{}^k\}$ basis:

$$\begin{aligned}[\mathbf{T}^G(g)\mathbf{e}_j{}^k](g_s) &= \mathbf{e}_j{}^k(g_s g) = \mathbf{e}_j{}^k(hg_r) = \mathbf{T}(h)\mathbf{e}_j{}^k(g_r)\\ &= \sum_{i=1}^{d} T_{ij}(h)\mathbf{e}_i{}^r(g_k) = \sum_{i=1}^{d} T_{ij}(h)\mathbf{e}_i{}^l(g_s)\end{aligned}$$

where g_r and g_l are the representatives of the right cosets containing $g_s g$ and $g_k g^{-1}$, respectively. From (5.21) we have $h = g_l g g_k^{-1}$ for $s = l$, i.e., $r = k$. We conclude that

$$\mathbf{T}^G(g)\mathbf{e}_j{}^k = \sum_{i=1}^{d} T_{ij}(h)\mathbf{e}_i{}^l = \sum_{i,l} \dot{T}_{ij}(g_l g g_k^{-1})\mathbf{e}_i{}^l \tag{5.22}$$

where

$$\dot{T}_{ij}(g) = \begin{cases} T_{ij}(g) & \text{if} \quad g \in H \\ 0 & \text{if} \quad g \notin H. \end{cases} \tag{5.23}$$

If we order the basis $\{\mathbf{e}_j{}^k\}$ in the sequence

$$\mathbf{e}_1{}^1, \ldots, \mathbf{e}_d{}^1, \mathbf{e}_1{}^2, \ldots, \mathbf{e}_d{}^2, \ldots, \mathbf{e}_1{}^m, \ldots, \mathbf{e}_d{}^m$$

then the matrix of $\mathbf{T}^G(g)$ with respect to this basis is

$$T^G(g) = \begin{pmatrix} \dot{T}(g_1 g g_1^{-1}) & \cdots & \dot{T}(g_1 g g_m^{-1}) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \dot{T}(g_m g g_1^{-1}) & \cdots & \dot{T}(g_m g g_m^{-1}) \end{pmatrix}. \tag{5.24}$$

That is, the $md \times md$ matrix $T^G(g)$ is partitioned into an $m \times m$ array of $d \times d$ matrix blocks. The block in the jth row and kth column of the array is $\dot{T}(g_j g g_k^{-1})$. The character is clearly

(5.25) $$\chi^G(g) = \sum_{k=1}^{m} \sum_{l=1}^{d} \dot{T}_{ll}(g_k g g_k^{-1}) = \sum_{k=1}^{m} \dot{\chi}(g_k g g_k^{-1})$$

where $\chi(h)$ is the character of $\mathbf{T}$ and

(5.26) $$\dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H. \end{cases}$$

We can write (5.25) in a more convenient form by noting that

$$\dot{\chi}(hg_k g(hg_k)^{-1}) = \dot{\chi}(g_k g g_k^{-1}), \qquad h \in H.$$

Therefore,

(5.27) $$\chi^G(g) = [n(H)]^{-1} \sum_{t\in G} \dot{\chi}(tgt^{-1}).$$

To recapitulate, given the character χ corresponding to a rep $\mathbf{T}$ of H, we can define the character χ^G of the induced rep $\mathbf{T}^G$ by expression (5.27). One of the most useful induced reps is that obtained from the one-dimensional identity rep of H. Then $\chi(h) = 1$ for all $h \in H$ and

(5.28) $$\chi^G(g) = \frac{n(G)}{n(H)} \frac{m_g}{n_g}$$

where n_g is the number of elements in G conjugate to g and m_g is the number of elements in $H \cap G$ conjugate to g. (Prove it!)

An important result on induced reps is the **Frobenius reciprocity theorem.** Let H be a subgroup of G and let $\mathbf{T}$, $\mathbf{Q}$ be irred reps of H and G with characters χ, ψ, respectively.

Theorem 3.9. The multiplicity of the irred rep $\mathbf{Q}$ in $\mathbf{T}^G$ is equal to the multiplicity of the irred rep $\mathbf{T}$ in $\mathbf{Q} \mid H = \mathbf{Q}_H$.

Proof. It is enough to show that

(5.29) $$\langle \chi^G, \psi \rangle_G = \langle \chi, \psi_H \rangle_H$$

since the left-hand side is the multiplicity of $\mathbf{Q}$ in $\mathbf{T}^G$ and the right-hand side is the multiplicity of $\mathbf{T}$ in $\mathbf{Q}_H$. Using (5.26) and (5.27) we have

$$\begin{aligned} \langle \chi^G, \psi \rangle_G &= [n(G)]^{-1} \sum_{g\in G} \chi^G(g)\bar{\psi}(g) \\ &= [n(G)n(H)]^{-1} \sum_{g,s\in G} \dot{\chi}(sgs^{-1})\bar{\psi}(g). \end{aligned}$$

Since $\psi(sgs^{-1}) = \psi(g)$ and sgs^{-1} ranges over G as g does for fixed $s \in G$, there follows

$$\begin{aligned} \langle \chi^G, \psi \rangle_G &= [n(H)]^{-1} \sum_{t\in G} \dot{\chi}(t)\bar{\psi}(t) \\ &= [n(H)]^{-1} \sum_{t\in H} \chi(t)\bar{\psi}(t) = \langle \chi, \psi_H \rangle_H. \quad \text{Q.E.D.} \end{aligned}$$

The Frobenius reciprocity theorem is important because it enables one to decompose any induced rep into a direct sum of irred reps.

3.6 Character Tables

We now apply the results of the preceding sections to compute the simple characters and reps of the crystallographic point groups. For many applications only the simple characters are needed, not the rep matrices themselves. Furthermore, a simple character yields much information about its corresponding group rep and it is often possible to construct the group rep rather easily once the character is known. Although we study primarily the crystallographic point groups, the techniques used in the construction are applicable to any finite group.

Let G be a finite group of order N and $\mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(\alpha)}$ a complete set of nonequivalent irred reps with dimensions $n_1, \ldots, n_\alpha$. The group G has α conjugacy classes and

(6.1) $$n_1{}^2 + n_2{}^2 + \cdots + n_\alpha{}^2 = N.$$

Furthermore, the characters $\chi^{(\mu)}$ obey the orthogonality relations

(6.2) $$\langle \chi^{(\mu)}, \chi^{(\nu)} \rangle = N^{-1} \sum_{i=1}^{\alpha} m_i \chi_i^{(\mu)} \overline{\chi_i^{(\nu)}} = \delta_{\mu\nu}, \qquad 1 \leq \mu, \nu \leq \alpha,$$

and

(6.3) $$\sum_{\mu=1}^{\alpha} \chi_i^{(\mu)} \overline{\chi_j^{(\mu)}} = \delta_{ij} N/m_i, \qquad 1 \leq i, j \leq \alpha,$$

where $\chi_i^{(\mu)}$ is the value of $\chi^{(\mu)}(g)$ for g an element of the ith conjugacy class $\mathcal{K}_i$ and m_i is the number of elements in $\mathcal{K}_i$. We assume $\mathcal{K}_1 = \{e\}$ and $m_1 = 1$. Thus, $\chi_1^{(\mu)} = \chi^{(\mu)}(e) = n_\mu$. A **character table** for G is a table of the form

(6.4)

	$\mathcal{K}_1$	$m_2\mathcal{K}_2$	$\cdots$	$m_\alpha\mathcal{K}_\alpha$
$\chi^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$	$\cdots$	$\chi_\alpha^{(1)}$
$\chi^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_\alpha^{(2)}$
$\cdot$				
$\cdot$				
$\cdot$				
$\chi^{(\alpha)}$	$\chi_1^{(\alpha)}$	$\chi_2^{(\alpha)}$	$\cdots$	$\chi_\alpha^{(\alpha)}$

listing all simple characters of G. We already know $\chi^{(1)}(g) = 1$ for all $g \in G$, the character of the one-dimensional rep in which $\mathbf{T}(g) = \mathbf{E}$. To obtain the rest of the table we use the orthogonality relations and various devices for constructing reps which were discussed in the previous section. We always assume the characters are ordered so that $1 = n_1 \leq n_2 \leq \cdots \leq n_\alpha$.

It is worth noting that isomorphic groups have the same reps. The rep theory of a group is determined by its abstract group structure alone. Thus, the isomorphic groups D_n and C_{nv} have the same character tables even though these groups are not conjugate subgroups of $E(3)$. Similarly, the following pairs of isomorphic groups have the same character tables:

$$S_{2n} \cong C_{2n}, \qquad C_{2h} \cong D_2, \qquad T_d \cong O, \qquad D_{2nd} \cong D_{4n}. \tag{6.5}$$

A number of the crystallographic point groups can be expressed as direct products of groups of lower order:

$$\begin{aligned} &C_6 \cong C_3 \times C_2, \qquad C_{nh} \cong C_n \times C_2, \qquad D_{nh} \cong D_n \times C_2 \\ &D_{2n+1,d} \cong D_{2n+1} \times C_2, \qquad T_h \cong T \times C_2, \qquad O_h \cong O \times C_2 \\ &D_2 \cong C_2 \times C_2, \qquad D_6 \cong D_3 \times C_2. \end{aligned} \tag{6.6}$$

According to the discussion following expression (5.16) the simple characters of each of the direct product groups can be obtained by forming all possible products of simple characters belonging to the factors. Thus, to derive character tables for each of the 32 crystallographic point groups it is enough to study the groups $C_2, C_3, C_4, D_3, D_4, T, O$.

The character tables of the cyclic groups C_n follow from Eq. (4.15). Let g be a generator of C_2, $g^2 = e$. Then the conjugacy classes of C_2 are

$$\mathcal{E} = \{e\}, \qquad \mathcal{C}_2 = \{g\} \tag{6.7}$$

and the character table reads

(6.8)

C_2	$\mathcal{E}$	$\mathcal{C}_2$
$\chi^{(1)}$	1	1
$\chi^{(2)}$	1	-1

Let g be a generator of C_3, $g^3 = e$. The conjugacy classes are

$$\mathcal{E} = \{e\}, \qquad \mathcal{C}_3 = \{g\}, \qquad \mathcal{C}_3{}^2 = \{g^2\} \tag{6.9}$$

and we obtain

(6.10)

C_3	$\mathcal{E}$	$\mathcal{C}_3$	$\mathcal{C}_3{}^2$
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	ε	ε^2
$\chi^{(3)}$	1	ε^2	ε

$\varepsilon = \exp(2\pi i/3)$.

Finally, let g be a generator of C_4, $g^4 = e$. The conjugacy classes are

$$\mathcal{E} = \{e\}, \qquad \mathcal{C}_4 = \{g\}, \qquad \mathcal{C}_4{}^2 = \{g^2\}, \qquad \mathcal{C}_4{}^3 = \{g^3\} \tag{6.11}$$

and the character table is

(6.12)

C_4	$\mathcal{E}$	$\mathcal{C}_4$	$\mathcal{C}_4^2$	$\mathcal{C}_4^3$
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	i	-1	$-i$
$\chi^{(3)}$	1	-1	1	-1
$\chi^{(4)}$	1	$-i$	-1	i

The group D_3 (of order six) is not quite so easy to handle. The elements g, h with $g^3 = h^2 = e$ and $hgh = g^{-1}$ generate D_3. The conjugacy classes are

(6.13) $$\mathcal{E} = \{e\}, \qquad \mathcal{C}_3 = \{g, g^2\}, \qquad \mathcal{C}_2 = \{h, gh, g^2h\}.$$

Thus, there are three irred reps of dimensions n_1, n_2, n_3 with $n_1 = 1$ (the identity rep) and

$$n_1{}^2 + n_2{}^2 + n_3{}^2 = 6.$$

The only possible solution is $n_1 = n_2 = 1$, $n_3 = 2$. There is another one-dimensional rep in addition to the identity rep. This can easily be found by inspection: $\chi^{(2)}(g) = 1$, $\chi^{(2)}(h) = -1$.

To obtain the third character we use the orthogonality relations

$$0 = 6\langle\chi^{(3)}, \chi^{(1)}\rangle = 2 + 2\chi_2^{(3)} + 3\chi_3^{(3)}$$
$$0 = 6\langle\chi^{(3)}, \chi^{(2)}\rangle = 2 + 2\chi_2^{(3)} - 3\chi_3^{(3)}.$$

Solving these equations simultaneously, we find $\chi_2^{(3)} = -1$, $\chi_3^{(3)} = 0$. The complete table is

(6.14)

D_3	$\mathcal{E}$	$2\mathcal{C}_3$	$3\mathcal{C}_2$
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	1	-1
$\chi^{(3)}$	2	-1	0

The two-dimensional rep $\mathbf{T}^{(3)}$ is equivalent to the 2×2 matrix rep one obtains by considering D_3 as a transformation group in the plane, i.e., as the symmetry group of an equilateral triangle. Indeed, with respect to the basis pictured in Fig. 3.1, we can associate the matrices

(6.15) $$g \sim \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}, \qquad h \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the character χ satisfies $\chi(e) = \chi_1 = 2$, $\chi(g) = \chi_2 = -1$, $\chi(h) = \chi_3 = 0$, so $\chi = \chi^{(3)}$.

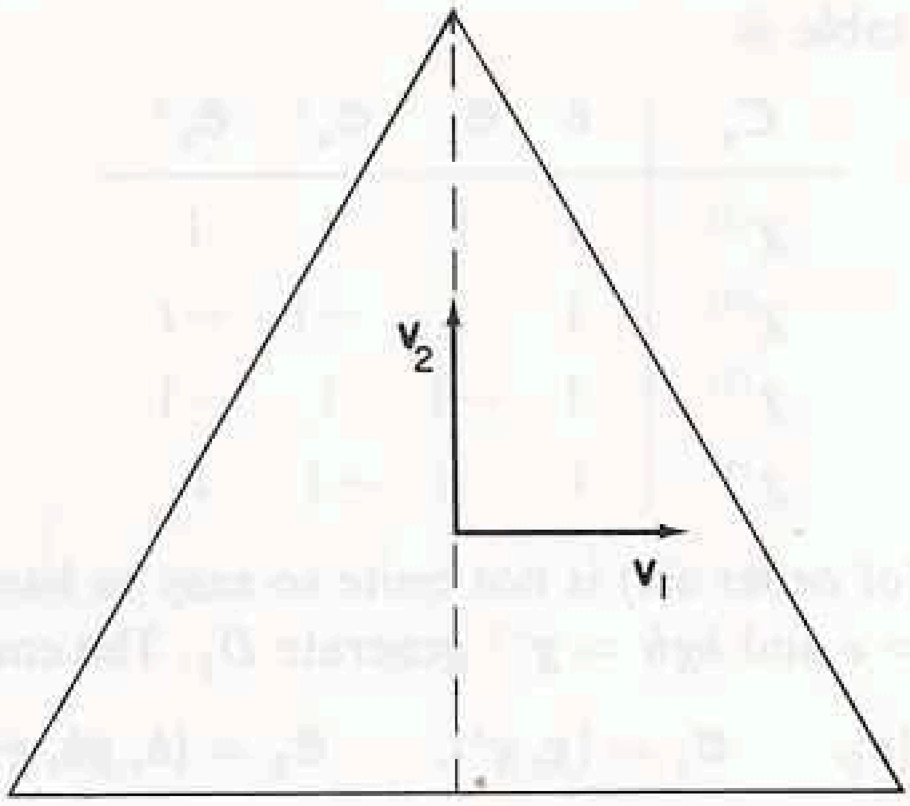

FIGURE 3.1

The group D_4 of order eight is generated by elements g, h such that $g^4 = h^2 = e$ and $(gh)^2 = e$. This group has five conjugacy classes:

(6.16)
$$\mathcal{E} = \{e\}, \qquad \mathcal{C}_4{}^2 = \{g^2\}, \qquad \mathcal{C}_4 = \{g, g^3\},$$
$$\mathcal{C}_2 = \{h, g^2h\}, \qquad \mathcal{C}_2' = \{gh, g^3h\}.$$

Thus, there are five irred reps of dimensions $n_1 = 1 \leq n_2 \leq \cdots \leq n_5$ such that

$$n_1{}^2 + n_2{}^2 + n_3{}^2 + n_4{}^2 + n_5{}^2 = 8.$$

The only possibility is $n_1 = n_2 = n_3 = n_4 = 1$, $n_5 = 2$. The one-dimensional reps can be determined by inspection. Thus,

(6.17)

D_4	$\mathcal{E}$	$\mathcal{C}_4{}^2$	$2\mathcal{C}_4$	$2\mathcal{C}_2$	$2\mathcal{C}_2'$
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	1	1	-1	1	-1
$\chi^{(4)}$	1	1	-1	-1	1
$\chi^{(5)}$	2	$\chi_2^{(5)}$	$\chi_3^{(5)}$	$\chi_4^{(5)}$	$\chi_5^{(5)}$

The orthogonality relations $\langle \chi^{(5)}, \chi^{(j)} \rangle = 0$, $j = 1, 2, 3, 4$, imply $\chi_2^{(5)} = -2$, $\chi_3^{(5)} = \chi_4^{(5)} = \chi_5^{(5)} = 0$. The reader can explicitly construct a 2×2 irred matrix rep $T^{(5)}$ in a manner similar to (6.15). One realization of $T^{(5)}$ is generated by

(6.18)
$$T^{(5)}(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T^{(5)}(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The tetrahedral group T contains 12 elements. Realizing the elements of T as permutations of the four vertices of a tetrahedron, we obtain the four conjugacy classes

(6.19)
$$\mathcal{K}_1 = \{1\}, \quad \mathcal{K}_2 = \{(12)(34), (13)(24), (14)(23)\}$$
$$\mathcal{K}_3 = \{(123), (142), (134), (243)\}, \quad \mathcal{K}_4 = \{(132), (124), (143), (234)\}.$$

There are four irred reps of dimensions $n_1 = 1 \leq n_2 \leq \cdots \leq n_4$ such that
$$n_1^2 + n_2^2 + n_3^2 + n_4^2 = 12.$$
The only possible solution is $n_1 = n_2 = n_3 = 1$, $n_4 = 3$. Note that the subgroup
$$D = \{1, (12)(34), (13)(24), (14)(23)\}$$
(the identity element and the three rotations of 180°) is normal in T. Therefore, the factor group T/D is cyclic of order three and has three one-dimensional nonequivalent irred reps given by (6.10). These reps $\mathbf{T}^{(i)\prime}$, $i = 1, 2, 3$, are defined by mapping a generator g of C_3 into 1, ε, or ε^2, respectively, where $\varepsilon = \exp(2\pi i/3)$. The combined homomorphisms
$$T \longrightarrow T/D \xrightarrow{\mathbf{T}^{(i)\prime}} \mathfrak{C}$$
yield three one-dimensional nonequivalent reps of T such that D is mapped into the identity operator. These reps are necessarily irred and they exhaust the possible one-dimensional reps of T. A simple computation gives the character table

(6.20)

T	$\mathcal{K}_1$	$3\mathcal{K}_2$	$4\mathcal{K}_3$	$4\mathcal{K}_4$
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	1	ε	ε^2
$\chi^{(3)}$	1	1	ε^2	ε
$\chi^{(4)}$	3	$\chi_2^{(4)}$	$\chi_3^{(4)}$	$\chi_4^{(4)}$

The fourth line of the table can be obtained from the orthogonality relations $\langle\chi^{(4)}, \chi^{(j)}\rangle = 0$, $j = 1, 2, 3$, which have the solution $\chi_2^{(4)} = -1$, $\chi_3^{(4)} = \chi_4^{(4)} = 0$. Note that a rep of the form
$$\mathbf{T} \cong a_1\mathbf{T}^{(1)} + a_2\mathbf{T}^{(2)} + a_3\mathbf{T}^{(3)}$$
has the property that $\mathbf{T}(g_1)$ and $\mathbf{T}(g_2)$ commute for all g_1, g_2 in the tetrahedral group. Indeed, the reps $\mathbf{T}^{(i)}$, $i = 1, 2, 3$, are one-dimensional so that with respect to a suitable basis the matrices of the operators can be simultaneously diagonalized. The natural three-dimensional rep of the tetrahedral group as

a transformation group on R_3 is not commutative. Therefore, this natural rep must be $\mathbf{T}^{(4)}$.

The octahedral group O, the direct symmetry group of the cube, contains 24 elements in five conjugacy classes. They are

$$\begin{aligned}
\mathcal{E} = \{e\}, \quad \mathcal{C}_4{}^2 &= \{\text{three rotations of } 180^\circ \text{ about fourfold axes}\}\\
\mathcal{C}_2 &= \{\text{six rotations of } 180^\circ \text{ about twofold axes}\}\\
\mathcal{C}_4 &= \{\text{three rotations of } 90^\circ \text{ and three rotations of } 270^\circ \text{ about fourfold axes}\}\\
\mathcal{C}_3 &= \{\text{four rotations of } 120^\circ \text{ and four rotations of } 240^\circ \text{ about threefold axes}\}.
\end{aligned} \tag{6.21}$$

There are five irred reps of dimensions $1 \le n_1 \le n_2 \le \cdots \le n_5$ such that

$$n_1{}^2 + \cdots + n_5{}^2 = 24.$$

The only possibility is $n_1 = n_2 = 1, n_3 = 2, n_4 = n_5 = 3$. It is clear from the drawing of a cube in Fig. 3.2 that points $ABCD$ are the vertices of a tetra-

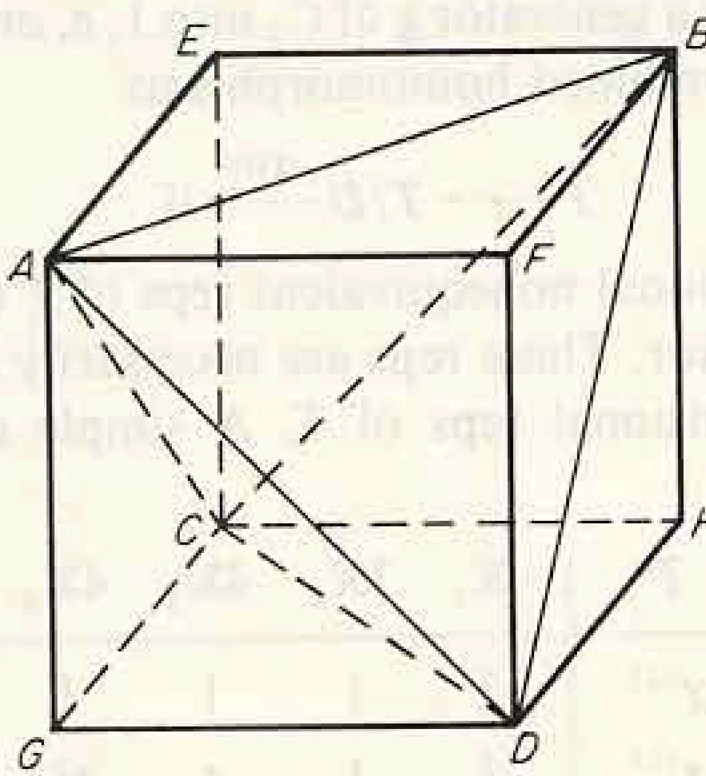

FIGURE 3.2

hedron. Every direct symmetry of the tetrahedron is also a direct symmetry of the cube. Thus O must contain the tetrahedral group T as a subgroup of index $24/12 = 2$. By Theorem 1.3, T is a normal subgroup of O and O/T is cyclic of order two. Now O/T has two one-dimensional reps given by (6.8). Just as in the preceding example, we can use these reps to obtain two one-dimensional irred reps of O such that T is mapped into the identity operator:

O	$\mathcal{E}$	$3\mathcal{C}_4{}^2$	$6\mathcal{C}_2$	$6\mathcal{C}_4$	$8\mathcal{C}_3$
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	-1	1

Next we construct the two-dimensional irred rep $\mathbf{T}^{(3)}$. Consider the simple character $\chi^{(2)}$ of T, table (6.20). The corresponding induced character χ^{O} is associated with a two-dimensional rep of O and is easily shown to be given by

O	ε	$3\mathcal{C}_4{}^2$	$6\mathcal{C}_2$	$6\mathcal{C}_4$	$8\mathcal{C}_3$
χ^{O}	2	2	0	0	-1

[See expression (5.27) defining an induced character.] Now

$$\langle\chi^{O}, \chi^{O}\rangle = (1/24)(2\cdot 2 + 3\cdot 2\cdot 2 + 6\cdot 0 + 6\cdot 0 + 8\cdot(-1)\cdot(-1)) = 1$$

so χ^{O} is simple. Since there is only one irred rep of dimension two we must have $\chi^{O} = \chi^{(3)}$. By means of (5.24) we could explicitly construct the irred rep $\mathbf{T}^{(3)}$.

The natural rep of O as a transformation group on R_3 must be irred since its restriction to T is the irred rep whose character is given by the bottom row of table (6.20). It follows that the character $\chi^{(4)}$ of this rep is

O	ε	$3\mathcal{C}_4{}^2$	$6\mathcal{C}_2$	$6\mathcal{C}_4$	$8\mathcal{C}_3$
$\chi^{(4)}$	3	-1	$\chi_3^{(4)}$	$\chi_4^{(4)}$	0

The values $\chi_3^{(4)}, \chi_4^{(4)}$ are not immediately determined since no elements of T lie in $\mathcal{C}_2$ or $\mathcal{C}_4$. To obtain these elements note that in suitable coordinate systems, a rotation of 180° about a twofold axis and a rotation of 90° about a fourfold axis can be represented by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. Taking traces we obtain $\chi_3^{(4)} = -1$, $\chi_4^{(4)} = 1$.

There remains only a single irred rep of dimension three. We could obtain the character of this rep by using the orthogonality relations. However, it is more informative to consider the tensor product $\mathbf{T}^{(5)} \simeq \mathbf{T}^{(2)} \otimes \mathbf{T}^{(4)}$. This is a three-dimensional rep with character $\chi^{(5)}(g) = \chi^{(2)}(g)\chi^{(4)}(g)$. Thus,

O	ε	$3\mathcal{C}_4{}^2$	$6\mathcal{C}_2$	$6\mathcal{C}_4$	$8\mathcal{C}_3$
$\chi^{(5)}$	3	-1	1	-1	0

Since

$$\langle\chi^{(5)}, \chi^{(5)}\rangle = (1/24)(9 + 3 + 6 + 6 + 0) = 1$$

it follows that $\chi^{(5)}$ is a simple character distinct from $\chi^{(4)}$. The complete character table is therefore

(6.22)

O	$\mathcal{E}$	$3\mathcal{C}_4^{\,2}$	$6\mathcal{C}_2$	$6\mathcal{C}_4$	$8\mathcal{C}_3$
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	-1	1
$\chi^{(3)}$	2	2	0	0	-1
$\chi^{(4)}$	3	-1	-1	1	0
$\chi^{(5)}$	3	-1	1	-1	0

The above results enable us to compute the simple characters and reps for each of the crystallographic point groups. Similar techniques yield the irred reps of any finite group, although in practice the required computations may be extremely difficult.

3.7 The Method of Projection Operators

Let G be a finite group and $\mathbf{T}$ a reducible unitary rep of G on the vector space V. Suppose $\mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(\alpha)}$ are a complete set of nonequivalent irred unitary reps of G. Then V can be decomposed into a direct sum of invariant subspaces

(7.1)
$$V = \sum_{\mu=1}^{\alpha} \sum_{i=1}^{a_\mu} \oplus V_i^{(\mu)}$$

where the restriction of $\mathbf{T}$ to $V_i^{(\mu)}$ is equivalent to the irred rep $\mathbf{T}^{(\mu)}$. Here a_μ is the multiplicity of $\mathbf{T}^{(\mu)}$ in $\mathbf{T}$. In this section we study a method which allows us to explicitly perform the decomposition (7.1). Furthermore, we examine the subspaces $V_i^{(\mu)}$ and determine to what extent they are unique.

Let $\mathbf{A}$ be a linear operator on the finite-dimensional inner product space V. Recall that $\mathbf{A}^*$, the **adjoint** of A, is the linear operator on V uniquely defined by

(7.2)
$$\langle \mathbf{Au}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{A}^*\mathbf{v} \rangle$$

for all $\mathbf{u}, \mathbf{v} \in V$, where $\langle -, - \rangle$ is the inner product. With respect to an ON basis for V the matrix for $\mathbf{A}^*$ is the conjugate transpose of the matrix for $\mathbf{A}$. The operator $\mathbf{A}$ is **self-adjoint** if $\mathbf{A} = \mathbf{A}^*$. If $\mathbf{A}^2 = \mathbf{A}$ then $\mathbf{A}$ is a **projection operator.** The **range** $R_\mathbf{A}$ and the **null space** $N_\mathbf{A}$ of a linear transformation are the V subspaces

(7.3)
$$R_\mathbf{A} = \{\mathbf{w} \in V: \mathbf{w} = \mathbf{Av} \quad \text{for some} \quad \mathbf{v} \in V\}, \qquad N_\mathbf{A} = \{\mathbf{v} \in V: \mathbf{Av} = \boldsymbol{\theta}\}.$$

Let $\mathbf{P}$ be a projection operator on V and let $W = R_\mathbf{P}$. Any $\mathbf{v} \in V$ can be written uniquely in the form

$$\mathbf{v} = \mathbf{Pv} + (\mathbf{E} - \mathbf{P})\mathbf{v} = \mathbf{w} + \mathbf{w}'$$

where $\mathbf{w} \in W$, $\mathbf{w}' \in W' = R_{(\mathbf{E}-\mathbf{P})}$. In particular, $\mathbf{Pw} = \mathbf{P}^2\mathbf{v} = \mathbf{Pv} = \mathbf{w}$ for $\mathbf{w} \in W$ and $\mathbf{Pw}' = \mathbf{P}(\mathbf{E} - \mathbf{P})\mathbf{v} = \boldsymbol{\theta}$ for $\mathbf{w}' \in W'$. If $\mathbf{u} \in W \cap W'$, then $\mathbf{Pu} = \mathbf{u}$ since $\mathbf{u} \in W$ and $\mathbf{Pu} = \boldsymbol{\theta}$ since $\mathbf{u} \in W'$. Thus, $W \cap W' = \{\boldsymbol{\theta}\}$ and

$$V = W \oplus W'. \tag{7.4}$$

We have shown that the projection $\mathbf{P}$ induces a direct sum decomposition of V. Conversely, if W and W' are subspaces of V such that (7.4) is valid, then the assignment $\mathbf{Pv} = \mathbf{w}$ defined by the decomposition

$$\mathbf{v} = \mathbf{w} + \mathbf{w}', \qquad \mathbf{v} \in V, \qquad \mathbf{w} \in W, \qquad \mathbf{w}' \in W'$$

determines a projection operator on V. Indeed, $\mathbf{Pw} = \mathbf{w}$ so $\mathbf{P}^2\mathbf{v} = \mathbf{Pw} = \mathbf{w} = \mathbf{Pv}$ and $\mathbf{P}^2 = \mathbf{P}$. Different choices of the supplementary space W' lead to different projection operators $\mathbf{P}$. If $W' = W^\perp$, then the corresponding projection operator is self-adjoint.

Theorem 3.10. There is a 1–1 relationship between subspaces W of V and self-adjoint projection operators $\mathbf{P}$ on V, given by $W = R_\mathbf{P}$, $W^\perp = N_\mathbf{P}$.

Proof. The decomposition $V = W \oplus W^\perp$ defines a projection $\mathbf{P}$ with $W = R_\mathbf{P}$, $W^\perp = N_\mathbf{P}$. If $\mathbf{v}_1, \mathbf{v}_2 \in V$ with

$$\mathbf{v}_i = \mathbf{w}_i + \mathbf{w}_i', \qquad \mathbf{w}_i \in W, \qquad \mathbf{w}_i' \in W^\perp, \qquad i = 1, 2,$$

then

$$\langle \mathbf{Pv}_1, \mathbf{v}_2 \rangle = \langle \mathbf{w}_1, \mathbf{w}_2 + \mathbf{w}_2' \rangle = \langle \mathbf{w}_1 + \mathbf{w}_1', \mathbf{w}_2 \rangle = \langle \mathbf{v}_1, \mathbf{Pv}_2 \rangle$$

so $\mathbf{P}^* = \mathbf{P}$.

Conversely, suppose $\mathbf{P}$ is a self-adjoint projection operator on V and set $W = R_\mathbf{P}$. Since $\mathbf{E} = \mathbf{P} + (\mathbf{E} - \mathbf{P})$ we can write

$$\mathbf{v} = \mathbf{Pv} + (\mathbf{E} - \mathbf{P})\mathbf{v} = \mathbf{w} + \mathbf{w}', \qquad \mathbf{v} \in V. \tag{7.5}$$

By definition $\mathbf{Pv} \in W$, while for any vector $\mathbf{Pu}, \mathbf{u} \in V$, in W we obtain

$$\langle \mathbf{Pu}, (\mathbf{E} - \mathbf{P})\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{P}(\mathbf{E} - \mathbf{P})\mathbf{v} \rangle = \langle \mathbf{u}, (\mathbf{P} - \mathbf{P})\mathbf{v} \rangle = 0 \tag{7.6}$$

so $(\mathbf{E} - \mathbf{P})\mathbf{v} \in W^\perp$. Thus $N_\mathbf{P} = R_{(\mathbf{E}-\mathbf{P})} = W^\perp$. Q.E.D.

Theorem 3.11. Let $\mathbf{T}$ be a finite-dimensional rep of G on the inner product space V. If V can be decomposed in the form $V = W_1 \oplus W_2$, where W_1 and W_2 are invariant under $\mathbf{T}$ then the projection operator $\mathbf{P}$ on V defined by

$$\mathbf{Pv} = \mathbf{w}_1 \quad \text{for} \quad \mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2, \quad \mathbf{w}_i \in W_i \tag{7.7}$$

satisfies

$$\mathbf{T}(g)\mathbf{P} = \mathbf{PT}(g) \qquad \text{for all} \quad g \in G. \tag{7.8}$$

Conversely, if $\mathbf{P}$ is a projection operator on V satisfying (7.8) then $V = W_1 \oplus W_2$, where $W_1 = R_\mathbf{P}$ and $W_2 = N_\mathbf{P}$ are invariant under $\mathbf{T}$.

Proof. Suppose W_1, W_2 are invariant under $\mathbf{T}$ and $V = W_1 \oplus W_2$. Then any $\mathbf{v} \in V$ can be written uniquely in the form $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_i \in W_i$, and

$$\mathbf{PT}(g)\mathbf{v} = \mathbf{P}(\mathbf{T}(g)\mathbf{w}_1 + \mathbf{T}(g)\mathbf{w}_2) = \mathbf{T}(g)\mathbf{w}_1 = \mathbf{T}(g)\mathbf{P}\mathbf{v},$$

where $\mathbf{P}$ is defined by (7.7). Conversely, if $\mathbf{P}$ is a projection operator satisfying (7.8) and $\mathbf{w}_1 \in W_1 = R_\mathbf{P}$, then

$$\mathbf{T}(g)\mathbf{w}_1 = \mathbf{T}(g)\mathbf{P}\mathbf{w}_1 = \mathbf{PT}(g)\mathbf{w}_1 \in R_\mathbf{P}$$

so W_1 is invariant under $\mathbf{T}$. Similarly, if $\mathbf{w}_2 \in W_2 = N_\mathbf{P}$, then

$$\mathbf{PT}(g)\mathbf{w}_2 = \mathbf{T}(g)\mathbf{P}\mathbf{w}_2 = \boldsymbol{\theta}$$

so $N_\mathbf{P}$ is invariant under $\mathbf{T}(g)$. Q.E.D.

This theorem establishes a 1–1 relationship between decompositions of V into a direct sum of two invariant subspaces and projection operators on V which commute with the operators $\mathbf{T}(g)$. We now determine which operators correspond to subspaces which transform irreducibly under $\mathbf{T}$. Let $P(\mathbf{T})$ be the set of all projection operators on V which commute with $\mathbf{T}(g)$ for all $g \in G$ and let $IP(\mathbf{T})$ be the set of all $\mathbf{P} \in P(\mathbf{T})$ which **cannot** be written in the form

(7.9)
$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2, \qquad \mathbf{P}_i \in P(\mathbf{T}), \qquad \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{Z}, \qquad \mathbf{P}_1, \mathbf{P}_2 \neq \mathbf{Z},$$

where $\mathbf{Z}$ is the zero operator on V.

Theorem 3.12. Let W be a proper invariant subspace of V and $\mathbf{P} \in P(\mathbf{T})$ a projection operator on W. Then W is irred under $\mathbf{T}$ if and only if $\mathbf{P} \in IP(\mathbf{T})$.

Proof. Suppose W is reducible under $\mathbf{T}$. Then W contains proper invariant subspaces W_1 and W_2 such that $W = W_1 \oplus W_2$. Also, the invariant subspace $W' = N_\mathbf{P}$ satisfies

$$V = W \oplus W' = W_1 \oplus W_2 \oplus W'.$$

Defining the projection operators $\mathbf{P}_1, \mathbf{P}_2 \in P(\mathbf{T})$ by

$$\mathbf{P}_1\mathbf{v} = \mathbf{w}_1, \qquad \mathbf{P}_2\mathbf{v} = \mathbf{w}_2,$$

where $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}'$, with $\mathbf{w}_i \in W_i$, $\mathbf{w}' \in W'$, we obtain $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. Furthermore, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{Z}$, since $W_1 \cap W_2 = \{\boldsymbol{\theta}\}$.

Conversely, suppose $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$, with $\mathbf{P}_1, \mathbf{P}_2 \in P(\mathbf{T})$, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{Z}$, and $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2 \neq \mathbf{Z}$. Then

$$\mathbf{P}^2 = (\mathbf{P}_1 + \mathbf{P}_2)^2 = \mathbf{P}_1{}^2 + \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_2{}^2 = \mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}$$

so $\mathbf{P} \in P(\mathbf{T})$. Set $W = R_{\mathbf{P}}, W_1 = R_{\mathbf{P}_1}, W_2 = R_{\mathbf{P}_2}$. For any $\mathbf{w} \in W$,

$$\mathbf{w} = \mathbf{P}\mathbf{w} = \mathbf{P}_1\mathbf{w} + \mathbf{P}_2\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$$

with $\mathbf{w}_i = \mathbf{P}_i\mathbf{w} \in W_i$. If $\mathbf{w} \in W_1 \cap W_2$ then $\mathbf{w} = \mathbf{P}_1\mathbf{P}_2\mathbf{w} = \boldsymbol{\theta}$ since $\mathbf{P}_1\mathbf{P}_2 = \mathbf{Z}$. Thus $W = W_1 \oplus W_2$, where W, W_1, W_2 are nonzero invariant subspaces of V. Q.E.D.

We now return to the decomposition (7.1) of V into irred subspaces under the action of the operators $\mathbf{T}(g)$. Let $T^{(\mu)}(g)$, $1 \leq \mu \leq \alpha$, be a complete set of nonequivalent unitary matrix reps of G corresponding to the operator reps $\mathbf{T}^{(\mu)}$. We can find an ON basis $\{\mathbf{v}_{ij}^{(\mu)}, 1 \leq j \leq n_\mu\}$ for each irred subspace $V_i^{(\mu)}$ such that

$$\mathbf{T}(g)\mathbf{v}_{ij}^{(\mu)} = \sum_{k=1}^{n_\mu} T_{kj}^{(\mu)}(g)\mathbf{v}_{ik}^{(\mu)}, \qquad 1 \leq j \leq n_\mu. \tag{7.10}$$

Corresponding to each simple character $\chi^{(\mu)}$ of G we define the linear operator $\mathbf{P}_\mu$ on V by

$$\mathbf{P}_\mu = \frac{n_\mu}{n(G)} \sum_{g \in G} \overline{\chi^{(\mu)}(g)}\mathbf{T}(g). \tag{7.11}$$

Since $hg = h(gh)h^{-1}$, for $g, h \in G$, i.e., the elements gh and hg lie in the same conjugacy class of G, it follows easily that $\mathbf{P}_\mu$ commutes with the operators $\mathbf{T}(h)$. Now

$$\mathbf{P}_\mu\mathbf{v}_{ij}^{(\nu)} = \frac{n_\mu}{n(G)} \sum_g \sum_{l=1}^{n_\mu} \sum_{k=1}^{n_\nu} T_{kj}^{(\nu)}(g)\overline{T_{ll}^{(\mu)}(g)}\mathbf{v}_{ik}^{(\nu)} = \delta_{\mu\nu}\mathbf{v}_{ij}^{(\nu)}, \tag{7.12}$$

where we have used (7.10), (7.11), and the orthogonality relations for matrix elements. It follows from this result and Theorems 3.10 and 3.11, that $\mathbf{P}_\mu \in P(\mathbf{T})$ is the self-adjoint projection operator on the invariant subspace

$$V^{(\mu)} = \sum_{i=1}^{a_\mu} \oplus V_i^{(\mu)}. \tag{7.13}$$

In general the spaces $V_i^{(\mu)}$ occurring in the decomposition (7.1) are not uniquely determined. However, since the definition of the projection operators $\mathbf{P}_\mu$ is basis-independent, the spaces $V^{(\mu)}$ *are* uniquely determined. The ambiguity occurs in the decomposition of each $V^{(\mu)}$ into irreducible subspaces. If $a_\mu > 1$ there is no unique way to perform the decomposition (7.13).

To carry out this nonunique decomposition we define operators

$$\mathbf{P}_\mu^{lk} = \frac{n_\mu}{n(G)} \sum_{g \in G} \overline{T_{lk}^{(\mu)}(g)}\mathbf{T}(g), \qquad l, k = 1, \ldots, n_\mu. \tag{7.14}$$

Since

$$\mathbf{P}_\mu^{lk}\mathbf{v}_{ij}^{(\nu)} = \frac{n_\mu}{n(G)} \sum_{g \in G} \sum_{m=1}^{n_\mu} T_{mj}^{(\nu)}(g)\overline{T_{lk}^{(\mu)}(g)}\mathbf{v}_{im}^{(\nu)} = \delta_{\mu\nu}\delta_{jk}\mathbf{v}_{il}^{(\nu)}, \tag{7.15}$$

it follows that $\mathbf{P}_\mu^{lk}$ is the self-adjoint projection operator on the a_μ-dimensional space $W_k^{(\mu)}$ spanned by the ON basis vectors $\{\mathbf{v}_{lk}^{(\mu)}, i = 1, \ldots, a_\mu\}$. Furthermore, the relations

$$\mathbf{P}_\mu^{lk}\mathbf{P}_{\mu'}^{l'k'} = \delta_{\mu\mu'}\delta_{kl'}\mathbf{P}_\mu^{lk'}, (\mathbf{P}_\mu^{lk})^* = \mathbf{P}_\mu^{kl}, \tag{7.16}$$

$$\mathbf{P}_\mu = \sum_{k=1}^{n_\mu} \mathbf{P}_\mu^{kk} \tag{7.17}$$

follow easily from (7.15).

If $\mathbf{v} \in V$ such that for some l, k, μ we have $\mathbf{P}_\mu^{lk}\mathbf{v} = \mathbf{w}_l \neq \boldsymbol{\theta}$, then the n_μ vectors $\mathbf{w}_j = \mathbf{P}_\mu^{jk}\mathbf{v} = \mathbf{P}_\mu^{jl}\mathbf{w}_l$, $1 \leq j \leq n_\mu$, are all nonzero and span an invariant subspace of V which transforms irreducibly according to the rep $\mathbf{T}^{(\mu)}$. Indeed, if

$$\mathbf{v} = \sum_{\nu lm} \alpha_{lm}^{(\nu)}\mathbf{v}_{lm}^{(\nu)}$$

then

$$\mathbf{w}_j = \mathbf{P}_\mu^{jk}\mathbf{v} = \sum_{l=1}^{a_\mu} \alpha_{lk}^{(\mu)}\mathbf{v}_{lj}^{(\mu)}$$

and

$$\mathbf{T}(g)\mathbf{w}_j = \sum_{l=1}^{a_\mu} \alpha_{lk}^{(\mu)}\mathbf{T}(g)\mathbf{v}_{lj}^{(\mu)} = \sum_{l=1}^{n_\mu} T_{lj}^{(\mu)}(g)\mathbf{w}_l.$$

Furthermore, if $\mathbf{v}, \mathbf{v}' \in V$ are such that the vectors $\mathbf{w}_l = \mathbf{P}_\mu^{lk}\mathbf{v}$ and $\mathbf{w}_l' = \mathbf{P}_\mu^{lk}\mathbf{v}'$ are orthogonal for some fixed l, k, μ then

$$\begin{aligned}\langle \mathbf{P}_\mu^{jk}\mathbf{v}, \mathbf{P}_\mu^{ik}\mathbf{v}'\rangle &= \langle \mathbf{P}_\mu^{jl}\mathbf{P}_\mu^{lk}\mathbf{v}, \mathbf{P}_\mu^{il}\mathbf{P}_\mu^{lk}\mathbf{v}'\rangle \\ &= \langle \mathbf{P}_\mu^{jl}\mathbf{w}_l, \mathbf{P}_\mu^{il}\mathbf{w}_l'\rangle = \delta_{ij}\langle \mathbf{w}_l, \mathbf{w}_l'\rangle = 0\end{aligned} \tag{7.18}$$

so $\mathbf{w}_j \perp \mathbf{w}_i'$ for $1 \leq i, j \leq n_\mu$.

From the above considerations we can decompose V into irred subspaces as follows: For each $\mu = 1, \ldots, \alpha$ apply the projection operator $\mathbf{P}_\mu^{11}$ to V and let $W_1^{(\mu)}$ be the range of this operator. Choose an ON basis $\{\mathbf{w}_{l1}^{(\mu)}, 1 \leq l \leq a_\mu\}$ for the a_μ-dimensional space $W_1^{(\mu)}$. Then the vectors $\{\mathbf{w}_{lj}^{(\mu)}, 1 \leq j \leq n_\mu\}$, where $\mathbf{w}_{lj}^{(\mu)} = \mathbf{P}_\mu^{j1}\mathbf{w}_{l1}^{(\mu)}$, form an ON basis for an invariant subspace $V_l^{(\mu)}$ of V such that the restriction of $\mathbf{T}$ to $V_l^{(\mu)}$ is equivalent to $\mathbf{T}^{(\mu)}$. In fact,

$$\mathbf{T}(g)\mathbf{w}_{lj}^{(\mu)} = \sum_k T_{kj}^{(\mu)}(g)\mathbf{w}_{lk}^{(\mu)}, \qquad 1 \leq j \leq n_\mu.$$

Furthermore,

$$V = \sum_{\mu=1}^{\alpha}\sum_{l=1}^{a_\mu} \oplus V_l^{(\mu)} \tag{7.19}$$

and the $V_l^{(\mu)}$ are mutually orthogonal. The totality of ON vectors $\{\mathbf{w}_{lj}^{(\mu)}\}$ form a basis for V since the number of elements in this set is equal to the dimension of V. Since this decomposition depends on the choice of basis vectors $\{\mathbf{w}_{l1}^{(\mu)}\}$ for $W_1^{(\mu)}$ and matrix reps $T^{(\mu)}(g)$, it is not unique.

An interesting special case occurs when $\mathbf{T}$ is the left regular rep $\mathbf{L}$ of G on the group ring R_G. We can use the convolution structure on R_G to derive additional information about the projection operators and irred subspaces. Recall that the action of $\mathbf{L}$ on R_G is given by

$$\mathbf{L}(g)x = gx, \qquad x \in R_G, \quad g \in G.$$

We can consider $\mathbf{L}$ as a unitary rep on R_G with inner product defined by (3.13). The multiplicity of the irred rep $\mathbf{T}^{(\mu)}$ in $\mathbf{L}$ is n_μ. All of our results concerning projection operators can immediately be specialized to R_G. For example the projection operators $\mathbf{P}_\mu$, (7.11), become

$$\mathbf{P}_\mu x = \frac{n_\mu}{n(G)} \sum_{g \in G} \overline{\chi^{(\mu)}(g)} \mathbf{L}(g)x = p_\mu x, \qquad x \in R_G, \tag{7.20}$$

where

$$p_\mu = \frac{n_\mu}{n(G)} \sum_{g \in G} \overline{\chi^{(\mu)}(g)} \cdot g \in R_G.$$

However, we can analyze such operators in another manner.

Let W be an invariant subspace of R_G and let $\mathbf{P}$ be a projection operator determined by the decomposition $R_G = W \oplus W'$ with W' also invariant under $\mathbf{L}$, i.e., $W = R_\mathbf{P}$, $W' = N_\mathbf{P}$. Then $\mathbf{P}$ commutes with $\mathbf{L}(g)$ and

$$g(\mathbf{P}x) = \mathbf{P}(gx), \qquad x \in R_G, \quad g \in G.$$

Let $e' = \mathbf{P}e$, where e is the identity element of R_G. Then for any $x = \sum x(g) \cdot g$ we have

$$\mathbf{P}x = \sum x(g) \cdot \mathbf{P}g = \sum x(g) \cdot g\mathbf{P}e = \sum x(g) \cdot ge' = xe'. \tag{7.21}$$

Furthermore, $\mathbf{P}^2 x = \mathbf{P}(xe') = x(e')^2 = \mathbf{P}x = xe'$ and setting $x = e$ we obtain

$$(e')^2 = e' \tag{7.22}$$

so e' is an idempotent. (An element y of R_G is called an **idempotent** if $y^2 = yy = y$.) Note that $W = \{xe' : x \in R_G\}$. Conversely, if $e' \in R_G$ satisfies (7.22) it is easy to verify that $\mathbf{P}$ defined by

$$\mathbf{P}x = xe', \qquad x \in R_G$$

is a projection operator which commutes with left multiplication by elements of G. It follows that there is a 1–1 correspondence between idempotents e' and projections $\mathbf{P}$ commuting with $\mathbf{L}$. The idempotent e is associated with the operator $\mathbf{E}$. If the idempotents e_1, e_2 are associated with projection operators $\mathbf{P}_1$ and $\mathbf{P}_2 = \mathbf{E} - \mathbf{P}_1$, respectively, then the relation $\mathbf{E} = \mathbf{P}_1 + \mathbf{P}_2$ implies $e = e_1 + e_2$. Furthermore, the relation $\mathbf{P}_1\mathbf{P}_2 = \mathbf{Z}$ implies $e_2 e_1 = 0$ since

$$0 = \mathbf{P}_1\mathbf{P}_2 e = (\mathbf{P}_2 e)e_1 = ee_2e_1 = e_2e_1.$$

Proceeding in this manner we see that properties of projection operators $\mathbf{P}$ on R_G which commute with left multiplication can be translated into properties of the corresponding generating idempotents e'. For example, Theorem 3.12 yields the following theorem.

Theorem 3.13. Let W be a subspace of R_G invariant under $\mathbf{L}$ and let $\mathbf{P}$ be a projection operator on R_G such that $W = R_\mathbf{P}$, and $\mathbf{P}x = xe'$, all $x \in R_G$. Then W is irred if and only if there do not exist elements e_1, e_2 of R_G such that

(7.23)
$$e' = e_1 + e_2, \qquad e_1{}^2 = e_1 \neq 0, \qquad e_2{}^2 = e_2 \neq 0, \qquad e_1e_2 = e_2e_1 = 0.$$

An idempotent e' is called **primitive** if there exists no decomposition of the form (7.23). Thus, if e' is primitive then the set

$$W = \{xe' : x \in R_G\}$$

is an irred subspace of R_G under $\mathbf{L}$. Conversely, if W is irred then every idempotent that generates W is primitive.

We shall now give another criterion for a primitive idempotent which is frequently simpler to verify than (7.23). If e' is an idempotent and $\mathbf{P}x = xe'$ is the corresponding projection operator then we can write $R_G = W_1 \oplus W_2$, where the invariant subspaces W_1 and W_2 can be characterized by

$$W_1 = R_\mathbf{P} = \{xe' : x \in R_G\}, \qquad W_2 = N_\mathbf{P} = \{x(e - e') : x \in R_G\}.$$

Furthermore $xe' = x$ for all $x \in W_1$, and $ye' = 0$ for all $y \in W_2$.

Theorem 3.14. If e' is a primitive idempotent then $e'xe' = \lambda_x e'$, $\lambda_x \in \mathfrak{C}$, for each $x \in R_G$. Conversely, if e' is idempotent and $e'xe' = \lambda_x e'$ for each $x \in R_G$ then e' is primitive.

Proof. Suppose e' is a primitive idempotent. Then for any $x \in R_G$ the operator $\mathbf{A}$ defined by

$$\mathbf{A}y = ye'xe', \qquad y \in R_G$$

commutes with the $\mathbf{L}(g)$. Furthermore, $\mathbf{A}y \in W_1$ for $y \in W_1$ and $\mathbf{A}y = 0$ for $y \in W_2$. Thus $\mathbf{A}_1$, the restriction of $\mathbf{A}$ to W_1, commutes with the $\mathbf{L}(g)$ and maps the irred space W_1 into itself. Theorem 3.5 implies $\mathbf{A}_1 = \lambda_x \mathbf{E}_{W_1}$ for some $\lambda_x \in \mathfrak{C}$. Thus $\mathbf{A} = \lambda_x \mathbf{P}$ or $e'xe' = \lambda_x e'$.

Conversely, suppose e' is idempotent and $e'xe' = \lambda_x e'$ for each $x \in R_G$. Let $e' = e_1 + e_2$, with $e_1{}^2 = e_1$, $e_2{}^2 = e_2$, $e_1e_2 = e_2e_1 = 0$. Then $e'e_1e' = (e_1 + e_2)e_1(e_1 + e_2) = e_1$, so $e_1 = \lambda e'$. Since e_1 and e' are idempotent it follows that $\lambda^2 = \lambda$ or $\lambda = 0, 1$. Thus, there exists no decomposition of e' for which e_1, e_2 are both nonzero and e' is primitive. Q.E.D.

Theorem 3.14 will prove very useful in Chapter 4 when we discuss the rep theory of the symmetric groups.

Suppose W_1 and W_2 are invariant subspaces of R_G which define equivalent reps of G. Then there exists an invertible transformation $\mathbf{S}$ from W_1 onto W_2 such that $g\mathbf{S}w = \mathbf{S}gw$ for all $g \in G$, $w \in W_1$. Furthermore $x\mathbf{S}w = \mathbf{S}(xw)$ for all $x \in R_G$. Let e_1, e_2 be generating idempotents for W_1 and W_2, respectively, and set $c = \mathbf{S}e_1 \in W_2$. Then for any $w \in W_1$ we have $\mathbf{S}w = \mathbf{S}we_1 = w\mathbf{S}e_1 = wc$. Therefore, any equivalence mapping $\mathbf{S}$ from W_1 to W_2 is given in terms of right multiplication by a ring element c. Since $c \in W_2$ we have $c = ce_2$. Furthermore $c = \mathbf{S}e_1 = \mathbf{S}e_1e_1 = e_1\mathbf{S}e_1 = e_1c$, so $c = e_1ce_2$. Thus we can assume that the equivalence is given by a nonzero element of the form e_1xe_2, $x \in R_G$. If W_1 and W_2 are irred we can say more.

Theorem 3.15. Two irred subspaces W_1, W_2 with primitive idempotents e_1, e_2 define equivalent reps if and only if there exist nonzero elements e_1xe_2, $x \in R_G$. Each such element defines an equivalence mapping $\mathbf{S}$ from W_1 to W_2.

Proof. If W_1 and W_2 define equivalent reps, then by the argument in the preceding paragraph, $e_1ce_2 \neq 0$, where $c = \mathbf{S}e_1$. Conversely, if $e_1xe_2 \neq 0$ then $\mathbf{S}w = we_1xe_2$, $w \in W_1$, is a nonzero mapping from W_1 into W_2 which commutes with the operators $\mathbf{L}(g)$. By Theorem 3.4 and the hypothesis of irreducibility, W_1 and W_2 define equivalent reps. Q.E.D.

The subspaces W of R_G which are invariant under the left regular rep are called left ideals.

Definition. A **left ideal** W is a subspace of R_G such that $xw \in W$ for all $x \in R_G$, $w \in W$.

If $c \in R_G$ the set

$$R_Gc = \{xc : x \in R_G\} \tag{7.24}$$

is clearly a left ideal. Moreover, we have shown that every left ideal can be obtained in this form. A left ideal W is said to be **minimal** if it contains no proper left ideal, i.e., if W is irred under the left regular rep. There is a similar definition of **right ideals**, which are just the subspaces of R_G invariant under the right regular rep.

A two-sided ideal is a subspace of R_G which is invariant under both the left and the right regular reps.

Definition. A **two-sided ideal** U is a subspace of R_G such that $xuy \in U$ for all $x, y \in R_G$, $u \in U$.

Let n be the multiplicity of the irred rep $\mathbf{T}$ in $\mathbf{L}$. Then there exist n linearly independent irred subspaces W_j, $1 \leq j \leq n$, each transforming under $\mathbf{T}$ so that the space

$$U = W_1 \oplus \cdots \oplus W_n$$

contains all irred subspaces W of R_G such that $\mathbf{L}|W$ is equivalent to $\mathbf{T}$. We have shown above that U is independent of the choice of the W_i. We will now show that U is a two-sided ideal. Let $u \in U$. Then

$$u = u_1 + \cdots + u_n, \qquad u_i \in W_i.$$

If $y \in R_G$ it follows from Theorem 3.15 that $u_i y$ is either zero or an element of a left ideal equivalent to W_i. In either case, $u_i y \in U$ for $1 \leq i \leq n$. Thus $uy \in U$ for all $u \in U$, $y \in R_G$. This proves that U is a right ideal. On the other hand, U is a left ideal since each of the W_i is a left ideal.

Finally, we will show that U is a **minimal** two-sided ideal. That is, U contains no proper two-sided ideal U'. For, if $U' \subseteq U$ and $U' \neq \{0\}$ then U' contains a minimal left ideal W. By Theorem 3.15 there exist ring elements $c_1, \ldots, c_n$ such that $W_i = Wc_i$, $1 \leq i \leq n$. Since U' is a right ideal, $W_i \subseteq U'$. Therefore, $U = U'$.

Let U_μ be the minimal two-sided ideal corresponding to the irred rep $\mathbf{T}^{(\mu)}$. [Note that $U_\mu = V^{(\mu)}$, (7.13), in the case $V = R_G$.] Then

$$R_G = U_1 \oplus \cdots \oplus U_\alpha \tag{7.25}$$

and $U_\mu U_\nu = \{0\}$ for $\mu \neq \nu$. Indeed, the first expression follows from (7.13) and (7.19). To prove the second formula note that $U_\mu U_\nu \subseteq U_\mu \cap U_\nu = \{0\}$ since U_μ and U_ν are disjoint two-sided ideals. A proof of the relation $U_\mu U_\mu = U_\mu$ is left to the reader.

3.8 Applications

We now study several applications of the rep theory of finite groups to problems in theoretical physics. These examples have been selected so that they can be understood without an extensive knowledge of physics. Some of the most important applications which require a knowledge of the rep theory of certain Lie groups, particularly of the group $SO(3)$, will be discussed in later chapters.

Our first application concerns the use of symmetry groups to determine the structure of tensors occurring in physical theories. We start by defining a tensor.

Let V be an m-dimensional vector space, real or complex, and consider the n-fold tensor product

$$V^{\otimes n} = V \otimes V \otimes \cdots \otimes V \quad (n\text{-times}). \tag{8.1}$$

If $\{\mathbf{v}_j, 1 \leq j \leq m\}$ is a basis for V then the mn vectors $\{\mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_n}, 1 \leq j_1, \ldots, j_n \leq m\}$ form a basis for $V^{\otimes n}$. The elements of $V^{\otimes n}$ are called **(contravariant) tensors** of **rank** n. Every tensor $\mathbf{a}$ can be written uniquely in the form

$$\mathbf{a} = \sum_{j_1 \cdots j_n} a_{j_1 \cdots j_n} \mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_n}. \tag{8.2}$$

In terms of a new basis $\{\mathbf{v}_k'\}$ for V related to $\{\mathbf{v}_j\}$ by

$$\mathbf{v}_j = \sum_{j=1}^{m} g_{kj} \mathbf{v}_k', \quad 1 \leq j \leq m, \tag{8.3}$$

we find

$$\mathbf{a} = \sum_{k_1 \cdots k_n} a'_{k_1 \cdots k_n} \mathbf{v}'_{k_1} \otimes \cdots \otimes \mathbf{v}'_{k_n} \tag{8.4}$$

where the tensor components a and a' are related by

$$a'_{k_1 \cdots k_n} = \sum_{j_1 \cdots j_n} a_{j_1 \cdots j_n} g_{k_1 j_1} \cdots g_{k_n j_n}. \tag{8.5}$$

The matrices $g = (g_{kj})$ are nonsingular and any nonsingular matrix defines a change of basis. One should carefully distinguish between the tensor $\mathbf{a}$ and the tensor components $a_{j_1 \cdots j_n}$. The components of a fixed tensor depend on the basis chosen in $V^{\otimes n}$.

Let G be a group of linear transformations $\mathbf{g}$ on V. (We do not require that G be finite.) Then, as discussed in Section 3.5, we can define a rep $\mathbf{T}^{\otimes n}$ of G on $V^{\otimes n}$ by

$$\mathbf{T}^{\otimes n}(g)\mathbf{w}_1 \otimes \mathbf{w}_2 \otimes \cdots \otimes \mathbf{w}_n = \mathbf{g}\mathbf{w}_1 \otimes \mathbf{g}\mathbf{w}_2 \otimes \cdots \otimes \mathbf{g}\mathbf{w}_n, \quad g \in G, \tag{8.6}$$

for all $\mathbf{w}_1, \ldots, \mathbf{w}_n \in V$. As usual, we choose a basis $\{\mathbf{v}_j\}$ for V and define the matrix (g_{kj}) corresponding to each $g \in G$ by

$$\mathbf{g}\mathbf{v}_j = \sum_{k=1}^{m} g_{kj} \mathbf{v}_k, \quad 1 \leq j \leq m. \tag{8.7}$$

Then the tensor $\mathbf{a}$, Eq. (8.2) is transformed into the tensor $\mathbf{T}^{\otimes n}(g)\mathbf{a}$, where

$$\mathbf{T}^{\otimes n}(g)\mathbf{a} = \sum_{k_1 \cdots k_n} a'_{k_1 \cdots k_n} \mathbf{v}_{k_1} \otimes \cdots \otimes \mathbf{v}_{k_n}, \tag{8.8}$$

$$a'_{k_1 \cdots k_n} = \sum_{j_1 \cdots j_n} a_{j_1 \cdots j_n} g_{k_1 j_1} \cdots g_{k_n j_n}. \tag{8.9}$$

Expressions (8.5) and (8.9) are identical, but their interpretations are different. In the first case (**passive**) the tensor is fixed and the basis is changed. In the second case (**active**) the basis remains fixed while the tensor $\mathbf{a}$ is mapped into a new tensor by the operator $\mathbf{T}^{\otimes n}(g)$. For the present we consider only the active case and fix the basis $\{\mathbf{v}_j\}$. Then every tensor $\mathbf{a}$ is uniquely determined by its components $a_{j_1 \cdots j_n}$ with respect to the basis and we can consider the rep $\mathbf{T}^{\otimes n}$ to be defined by (8.9). Another useful rep of G is obtained by forming the tensor product $\mathbf{Q} \otimes \mathbf{T}^{\otimes n}$, where $\mathbf{Q}$ is the one-dimensional rep $\mathbf{Q}(g) = \det g$.

(Recall that the value of the determinant is independent of basis in V.) The basis space for this rep is $V^{\otimes n}$ again and $g \in G$ acts on the tensor $\mathbf{a}$ with components $a_{j_1 \cdots j_n}$ to transform it into a tensor with components

$$a'_{k_1 \cdots k_n} = \det(g) \sum_{j_1 \cdots j_n} a_{j_1 \cdots j_n} g_{k_1 j_1} \cdots g_{k_n j_n}. \tag{8.10}$$

Frequently the above reps occur in physical theories where V is a real three-dimensional inner product space and $G = O(3)$, the group of all length-preserving linear transformations on V. Then with respect to an ON basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for V, the matrix O of each $\mathbf{O} \in O(3)$ is orthogonal: $O^t = O^{-1}$. The tensors $\mathbf{a} \in V^{\otimes n}$ which transform according to the rep $\mathbf{T}^{\otimes n}$,

$$a'_{k_1 \cdots k_n} = \sum_{j_1 \cdots j_n = 1}^{3} a_{j_1 \cdots j_n} O_{k_1 j_1} \cdots O_{k_n j_n} \tag{8.11}$$

are called **polar tensors** of rank n. Those which transform according to

$$a'_{k_1 \cdots k_n} = \det O \sum_{j_1 \cdots j_n = 1}^{3} a_{j_1 \cdots j_n} O_{k_1 j_1} \cdots O_{k_n j_n} \tag{8.12}$$

are called **axial tensors** of rank n. Note that $\det O = \pm 1$.

We give some familiar examples of polar and axial tensors. The action of $O(3)$ as a transformation group on R_3, considered in Chapter 2, defines a rep of $O(3)$ in which each $\mathbf{v} \in R_3$ transforms as a polar vector (polar tensor of rank 1). The well-known vector product or cross product $\mathbf{u} \times \mathbf{v}$ of two polar vectors transforms as an axial vector. In particular, under the inversion operator $\mathbf{I} \in O(3)$, $\mathbf{u} \longrightarrow -\mathbf{u}$, $\mathbf{v} \longrightarrow -\mathbf{v}$, and $\mathbf{u} \times \mathbf{v} \longrightarrow (-\mathbf{u}) \times (-\mathbf{v}) = \mathbf{u} \times \mathbf{v}$. The scalar product of two polar vectors transforms as a **scalar** (polar tensor of rank zero), while the scalar product of a polar vector and an axial vector transforms as a **pseudoscalar** (axial tensor of rank zero).

Let S be some physical system (molecule, crystal, garbage truck, etc.) in three-dimensional space R_3. We choose an arbitrary point, say $\boldsymbol{\theta}$, as the origin in R_3 and construct an orthogonal coordinate system at $\boldsymbol{\theta}$ with ON basis vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ pointing along the coordinate axes. We position ourselves at the point $\boldsymbol{\theta}$ and measure various physical properties of the system S. These measurements are performed very rapidly; so fast that they all take place in a single instant of time t_0. (We will be concerned with determining the properties of the system at a given instant of time and not with the time evolution of the system.) Once we have measured some physical property ρ of S with respect to the ON basis $\{\mathbf{v}_i\}$ we can perform an orthogonal transformation $\mathbf{O}$ on R_3 and then measure the same physical property of the system $\mathbf{O}S$ with respect to $\{\mathbf{v}_i\}$. Hopefully, there will be some functional relationship $\mathbf{O}\rho = f(\mathbf{O}, \rho)$ between the new measurement $\mathbf{O}\rho$ and the old one $\mathbf{x}$. Indeed, the measurable quantities ρ frequently transform as components of an axial or polar tensor. We mention some examples, at least a few of which should be familiar to the reader. The temperature at $\boldsymbol{\theta}$ is a scalar, while the rotary power of an optically active crystal is a pseudoscalar. The electric field and the current density are polar vectors, while the magnetic field is an axial vector. The resistivity and moment of inertia tensors are polar tensors of rank two, as are the stress and strain tensors. The optical gyration tensor is axial of rank two.

Let G be the point symmetry group of the physical system S. (This means that S and $\mathbf{g}S$ are physically indistinguishable for all $\mathbf{g}$ in the point group G.) if $\mathbf{a} \in V^{\otimes n}$ is any tensor describing a physical property of S it necessarily follows that $\mathbf{T}^{\otimes n}(g)\mathbf{a} = \mathbf{a}$. In terms of tensor components this relation becomes

$$a_{k_1\cdots k_n} = \sum_{j_1\cdots j_n} a_{j_1\cdots j_n} g_{k_1 j_1} \cdots g_{k_n j_n}, \qquad k_i = 1, 2, 3, \tag{8.13}$$

valid for all $g \in G$, where g_{kj} are the matrix elements of g with respect to the $\mathbf{v}_i$ basis of R_3. [Equation (8.13) is valid for polar tensors of rank n. The results are modified in an obvious manner if $\mathbf{a}$ is an axial tensor.] This relation places a restriction on the tensor components of $\mathbf{a}$. If $G = \{e\}$ and the tensor components are subject to no symmetry requirements then the nth-rank tensors have 3^n independent components. However, if G is a nontrivial symmetry group then by (8.13) the 3^n components are not all independent. We can use the symmetry group to determine the maximal number of linearly independent components of an nth-rank tensor, hence the number of parameters needed to uniquely determine a physical property of S associated with the tensor.

As an example of the restrictions provided by the symmetries of S, suppose G contains the inversion I. Then (8.13) with $g = I$ yields

$$a_{k_1\cdots k_n} = (-1)^n a_{k_1\cdots k_n}$$

which shows that all polar tensors of odd rank are identically zero. Thus if some physical property of S is described by a polar tensor of odd rank, it follows from symmetry conditions alone that this tensor is identically zero. Similarly, if $I \in G$ then all axial tensors of even rank are zero.

The most common method used for computing the possible tensors invariant under G is the brute force method. One chooses a set $g_1, \ldots, g_l$ whose elements generate G and then substitutes each of these elements into (8.13) to obtain a system of identities relating the tensor components of $\mathbf{a}$. These identities must then be solved to determine the number of linearly independent components of $\mathbf{a}$ and the dependence of all components on a suitably chosen set of independent ones. If the components of $\mathbf{a}$ satisfy Eq. (8.13) for the generators of G then the components will automatically satisfy these equations for any $g \in G$.

We can develop more sophisticated methods to solve this problem by noting that the solutions $\mathbf{a} \in V^{\otimes n}$ of the equation

$$\mathbf{T}^{\otimes n}(g)\mathbf{a} = \mathbf{a}, \qquad \text{all} \quad g \in G,$$

form a subspace $V^{(1)}$ which is invariant under $\mathbf{T}^{\otimes n}$. Let dim $V^{(1)} = q$ and let $\mathbf{a}_1, \ldots, \mathbf{a}_q$ be a basis for $V^{(1)}$. Then each of the $\mathbf{a}_i$ generates a one-dimensional invariant subspace W_i of $V^{(1)}$ such that the action of $\mathbf{T}^{\otimes n}$ on W_i is equivalent to the irred identity rep $\mathbf{T}^{(1)}$:

$$V^{(1)} = W_1 \oplus \cdots \oplus W_q.$$

Thus, q is the multiplicity of the identity rep of G in $\mathbf{T}^{\otimes n}$ and the number of linearly independent tensor components for solutions $\mathbf{a}$ of (8.13). To find $V^{(1)}$ we can make use of the projection operator $\mathbf{P}_1$ of Section 3.7:

$$\mathbf{P}_1 = \frac{1}{n(G)} \sum_{g \in G} \overline{\chi^{(1)}(g)} \mathbf{T}^{\otimes n}(g) = \frac{1}{n(G)} \sum_{g \in G} \mathbf{T}^{\otimes n}(g). \tag{8.14}$$

Then

$$V^{(1)} = \{\mathbf{P}_1 \mathbf{b} : \mathbf{b} \in V^{\otimes n}\} \tag{8.15}$$

or $V^{(1)}$ is the space of all solutions $\mathbf{a}$ of the equation

$$\mathbf{P}_1 \mathbf{a} = \mathbf{a}. \tag{8.16}$$

We can use the orthogonality relations for characters to obtain a simple expression for q. Let χ be the character of the natural three-dimensional rep $\mathbf{T}$ of G as a transformation group on R_3,

$$\chi(g) = \operatorname{tr} \mathbf{g}, \qquad g \in G.$$

It follows from (1.13), Section 2.1, that

$$\chi(g) = 1 + 2 \cos \varphi \tag{8.17}$$

for $g = C_k(\varphi) \in G$ and

$$\chi(g) = -1 + 2 \cos \varphi \tag{8.18}$$

for $g = S_k(\varphi) \in G$, so the character χ is immediately determined from a description of the action of each g. Since $\mathbf{T}^{\otimes n}$ is the tensor product of n copies of $\mathbf{T}$, the character χ^n of this rep is

$$\chi^n(g) = [\chi(g)]^n. \tag{8.19}$$

[This result is correct for polar tensors of rank n. For axial tensors of rank n the character is

$$\chi'^n(g) = \varepsilon_g \chi^n(g) \tag{8.20}$$

where $\varepsilon_g = 1$ if $g \in G \cap SO(3)$ and $\varepsilon_g = -1$ if g is an improper rotation.] Then we have

$$q = \langle \chi^n, \chi^{(1)} \rangle = \frac{1}{n(G)} \sum_{g \in G} [\chi(g)]^n. \tag{8.21}$$

The reader may be wondering why we have applied character theory to the **real** rep $\mathbf{T}^{\otimes n}$ since we emphasized earlier that this theory applies only to

complex reps. To verify that Eq. (8.21) is valid we formally define the complex m-dimensional vector space V_c by

$$V_c = \{\mathbf{v}_1 + i\mathbf{v}_2 : \mathbf{v}_1, \mathbf{v}_2 \in V\}.$$

Then every element $\mathbf{a}$ of the 3^n-dimensional tensor product space $V_c^{\otimes n}$ can be written uniquely in the form

$$\mathbf{a} = \mathbf{b}_1 + i\mathbf{b}_2, \qquad \mathbf{b}_1, \mathbf{b}_2 \in V^{\otimes n}. \tag{8.22}$$

We define the complex rep $\mathbf{T}_c^{\otimes n}$ of G on $V_c^{\otimes n}$ by

$$\mathbf{T}_c^{\otimes n}(g)\mathbf{a} = \mathbf{T}^{\otimes n}(g)\mathbf{b}_1 + i\mathbf{T}^{\otimes n}(g)\mathbf{b}_2. \tag{8.23}$$

If $\{\mathbf{v}_j\}$ is an ON basis for V then it is also an ON basis for V_c and the set $\{\mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_n}\}$ is an ON basis for $V_c^{\otimes n}$. In terms of tensor components the action of $\mathbf{T}_c^{\otimes n}(g)$ is given by relations (8.7)–(8.9) where now the components $a_{j_1 \cdots j_n}$ take complex values. The multiplicity of the identity rep $\mathbf{T}^{(1)}$ in $\mathbf{T}^{\otimes n}$ is now given by the right-hand side of (8.21). Let this multiplicity be r. Then we can find a basis $\mathbf{a}_1, \ldots, \mathbf{a}_r$ for the subspace $V_c^{(1)}$ consisting of all elements of $V_c^{\otimes n}$ which are fixed under $\mathbf{T}_c^{\otimes n}$. By definition

$$\overline{\mathbf{T}_c^{\otimes n}(g)\mathbf{a}} = \mathbf{T}_c^{\otimes n}(g)\bar{\mathbf{a}} \tag{8.24}$$

where $\bar{\mathbf{a}} = \mathbf{b}_1 - i\mathbf{b}_2$ and $\mathbf{a} = \mathbf{b}_1 + i\mathbf{b}_2$. Thus, if $\mathbf{a} \in V_c^{(1)}$ then $\bar{\mathbf{a}} \in V_c^{(1)}$. Clearly, the $2r$ real tensors

$$\tfrac{1}{2}(\mathbf{a}_j + \bar{\mathbf{a}}_j), \qquad \tfrac{1}{2}i(\mathbf{a}_j - \bar{\mathbf{a}}_j), \qquad 1 \le j \le r, \tag{8.25}$$

span $V_c^{(1)}$ and also lie in $V^{(1)}$. Among these $2r$ real tensors there must be r which form a basis for $V_c^{(1)}$. Thus, $r \le q$. However $V^{(1)} \subseteq V_c^{(1)}$ so $r = q$. This justifies our use of characters to compute q.

As an example we compute the dimension q of the space $V^{(1)}$ of polar tensors of rank two, invariant under C_{4v}. Since $C_{4v} \cong D_4$ we can use the characters and conjugacy classes (6.16) of D_4 to perform the computation. From (8.17) and (8.18) the character χ of the natural rep of C_{4v} as a transformation group on R_3 is

(8.26)

	$\mathcal{E}$	$\mathcal{C}_4^2$	$2\mathcal{C}_4$	$2\mathcal{C}_2$	$2\mathcal{C}_{2'}$
χ	3	−1	1	1	1

Thus

$$q = \langle \chi^2, \chi^{(1)} \rangle = \tfrac{1}{8} \sum_{g \in D_4} \chi^2(g) = 2.$$

Tensors describing physical phenomena frequently possess internal symmetry properties which are independent of their external point symmetry properties. For example, the moment of inertia and stress and strain tensors are all **symmetric** polar tensors of rank two, i.e., $a_{jk} = a_{kj}$ for $1 \le j, k \le 3$. Other physically interesting tensors of rank two are **skew-symmetric**, $a_{jk} =$

$-a_{kj}$ for $1 \leq j, k \leq 3$. Here we consider only polar tensors of rank two. The procedure needed to extend our results to tensors of higher order should be clear after an examination of this simple case.

It follows easily from the transformation law (8.11) that every symmetric tensor $\mathbf{a} \in V^{\otimes 2}$ is mapped into a symmetric tensor $\mathbf{T}^{\otimes 2}(O)\mathbf{a}$ by any $O \in O(3)$. Similarly, a skew-symmetric tensor is mapped into a skew-symmetric tensor. If the tensor

$$\mathbf{a} = \sum_{j,k=1}^{3} a_{jk}\mathbf{v}_j \otimes \mathbf{v}_k$$

is both symmetric and skew symmetric then

$$a_{jk} = -a_{kj} = a_{kj} = 0$$

so $\mathbf{a} = \boldsymbol{\theta}$. Given any $\mathbf{a} \in V^{\otimes 2}$ we can construct a symmetric tensor $\mathbf{a}^S$ with components

$$a_{jk}^S = \tfrac{1}{2}(a_{jk} + a_{kj})$$

and a skew-symmetric tensor $\mathbf{a}^A$ with components

$$a_{jk}^A = \tfrac{1}{2}(a_{jk} - a_{kj}),$$

so $\mathbf{a} = \mathbf{a}^S + \mathbf{a}^A$. This shows that

$$V^{\otimes 2} = W^S \oplus W^A$$

where W^S and W^A are the invariant subspaces of all symmetric and skew-symmetric tensors, respectively. Here $\dim W^S = 6$, $\dim W^A = 3$. (Prove it!) The character χ^2 of the rep $\mathbf{T}^{\otimes 2}$ acting on $V^{\otimes 2}$ can be written

$$\chi^2(g) = \chi^S(g) + \chi^A(g), \tag{8.27}$$

where χ^S is the character of $\mathbf{T}^{\otimes 2} \mid W^S$ and χ^A is the character of $\mathbf{T}^{\otimes 2} \mid W^A$. The symmetric tensors which are fixed under G form a subspace $V_S^{(1)}$ of W^S, while the skew-symmetric tensors fixed by G form a subspace $V_A^{(1)}$ of W^A. If $q_S = \dim V_S^{(1)}$, $q_A = \dim V_A^{(1)}$, then $q_S + q_A = q$ and

$$q_S = \langle \chi^S, \chi^{(1)} \rangle, \qquad q_A = \langle \chi^A, \chi^{(1)} \rangle.$$

We will compute the character χ^S directly. It is easy to show that the set of six symmetric tensors

$$\{\mathbf{v}_j \otimes \mathbf{v}_k + \mathbf{v}_k \otimes \mathbf{v}_j, \quad 1 \leq j \leq k \leq 3\}$$

is a basis for W^S. Then

$$\mathbf{T}^{\otimes 2}(g)(\mathbf{v}_j \otimes \mathbf{v}_k + \mathbf{v}_k \otimes \mathbf{v}_j) = \tfrac{1}{2} \sum_{lh} (T_{lj}(g)T_{hk}(g) + T_{lk}(g)T_{hj}(g)) \times (\mathbf{v}_l \otimes \mathbf{v}_h + \mathbf{v}_h \otimes \mathbf{v}_l).$$

Taking the trace of this transformation, we find

$$\chi^S(g) = \tfrac{1}{2} \sum_{jk} (T_{jj}(g)T_{kk}(g) + T_{jk}(g)T_{kj}(g)) = \tfrac{1}{2}(\chi^2(g) + \chi(g^2)). \tag{8.28}$$

Furthermore,

(8.29) $$\chi^A(g) = \chi^2(g) - \chi^S(g) = \tfrac{1}{2}(\chi^2(g) - \chi(g^2)).$$

We apply these results to compute the dimension q_S of the space $V_S^{(1)}$ of symmetric polar tensors of rank two which are invariant under C_{4v}. From (8.26) and (6.16) we find

(8.30)

	$\mathcal{E}$	$\mathcal{C}_4{}^2$	$2\mathcal{C}_4$	$2\mathcal{C}_2$	$2\mathcal{C}_{2'}$
χ^2	9	1	1	1	1
χ^S	6	2	0	2	2
χ^A	3	-1	1	-1	-1

Thus,

$$q_S = \langle \chi^S, \chi^{(1)} \rangle = 2, \qquad q_A = 0.$$

Since $q_S = q = 2$ it follows that all second-rank polar tensors fixed under C_{4v} are symmetric. A skew-symmetric tensor of this type describing a physical property of a system with C_{4v} symmetry is zero. One physical consequence of this computation is that all solids with C_{4v} symmetry have moment of inertia tensors which are determined by two parameters. The homogeneous four-pyramid is such a solid. (In this special case we found $q = q_S$. However, this equality is the exception rather than the rule.)

Our next application of group rep theory pertains to perturbation theory in quantum mechanics. So as not to interrupt the continuity of our presentation we assume that the reader understands a few basic facts about Hilbert space, Lebesgue integration, and the Hamiltonian operator in quantum mechanics. The relevant definitions are presented in the appendix. We concentrate on algebraic and group-theoretic questions and ignore certain analytic difficulties pertaining to unbounded operators in Hilbert space.

Consider a nonrelativistic quantum mechanical system consisting of k particles with masses $m_1, \ldots, m_k$, respectively. We suppose that the interaction between the particles is described by a real-valued potential function $V(\mathbf{x}_1, \ldots, \mathbf{x}_k)$, where $\mathbf{x}_j \in R_3$ refers to the coordinates of the jth particle. The possible (pure) states of this sytem are elements of the Hilbert space $\mathcal{H}$ consisting of all complex valued functions

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k),$$

such that

$$\int_{(R_3)^k} |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)|^2 \, d^3\mathbf{x}_1 \cdots d^3\mathbf{x}_k < \infty.$$

The inner product $(-, -)$ on $\mathcal{H}$ is defined by

(8.31) $$(\Psi, \Phi) = \int_{(R_3)^k} \Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)\overline{\Phi(\mathbf{x}_1, \ldots, \mathbf{x}_k)} \, d^3\mathbf{x}_1 \cdots d^3\mathbf{x}_k.$$

The Hamiltonian operator $\mathbf{H}$ of this system is defined by

$$\text{(8.32)} \qquad \mathbf{H}\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k) = -\sum_{j=1}^{k} \frac{h^2}{2m_j} \Delta_j \Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k) + V(\mathbf{x}_1, \ldots, \mathbf{x}_k)\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

where

$$\Delta_j = \sum_{i=1}^{3} \frac{\partial}{\partial x_{ij}^2}, \qquad \mathbf{x}_j = (x_{1j}, x_{2j}, x_{3j}),$$

and $h = 2\pi\hbar$ is Planck's constant. In the remainder of this book we choose units such that $\hbar = 1$.

Note. The Hamiltonian operator is not defined for all $\Psi \in \mathcal{H}$. Expression (8.32) defines an element of $\mathcal{H}$ only if the function $\mathbf{H}\Psi$ is square integrable. A precise definition of the domain of $\mathbf{H}$ is difficult and we refer the interested reader to Helwig [1]. For most potential functions $V(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ which occur in quantum mechanics it is possible to define the Hamiltonian in a satisfactory manner such that the domain of $\mathbf{H}$ is dense in $\mathcal{H}$. Furthermore, it can be shown that $\mathbf{H}$ is a **symmetric operator**, i.e.,

$$\text{(8.33)} \qquad (\Psi, \mathbf{H}\Phi) = (\mathbf{H}\Psi, \Phi)$$

for all Ψ, Φ in the domain of $\mathbf{H}$. Equation (8.33) is easy to obtain formally but difficult to prove by a rigorous computation. In some of the following arguments we shall also proceed formally, as do almost all textbooks on applications of group theory to quantum mechanics. The needed rigor can be supplied by Helwig [1] and Kato [1].

In analogy with Example 5 of Section 3.1 we define a unitary rep of $E(3)$ on $\mathcal{H}$ by

$$\text{(8.34)} \qquad \mathbf{T}(g)\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \Psi(g^{-1}\mathbf{x}_1, \ldots, g^{-1}\mathbf{x}_k), \qquad g = \{\mathbf{a}, \mathbf{O}\} \in E(3).$$

The relation

$$\text{(8.35)} \qquad (\mathbf{T}(g)\Psi, \mathbf{T}(g)\Phi) = (\Psi, \Phi), \qquad \Psi, \Phi \in \mathcal{H}$$

follows from (8.31) and a simple change of variable. Let G be any subgroup of $E(3)$ consisting of transformations g such that

$$V(\mathbf{g}\mathbf{x}_1, \ldots, \mathbf{g}\mathbf{x}_k) = V(\mathbf{x}_1, \ldots, \mathbf{x}_k), \qquad \text{all} \quad \mathbf{x}_j \in R_3.$$

Then by an elementary computation similar to that carried out in Examples 4 and 5 of Section 3.1 we can show

$$\mathbf{T}(g)\mathbf{H}\Psi = \mathbf{H}\mathbf{T}(g)\Psi, \qquad g \in G,$$

for all vectors Ψ in the domain of $\mathbf{H}$. Thus, the operators $\mathbf{T}(g)$ define a unitary rep of G on $\mathcal{H}$ and these operators commute with $\mathbf{H}$. The group G is called a symmetry group of the Hamiltonian.

The fundamental problem for this quantum mechanical system is the determination of the eigenvalues and eigenvectors of $\mathbf{H}$, i.e., the solutions of the eigenvalue problem

$$\mathbf{H}\Psi = \lambda\Psi, \qquad \Psi \in \mathcal{H}. \tag{8.36}$$

(We study only the point spectrum of $\mathbf{H}$. Group-theoretic methods also apply to the continuous spectrum but such a treatment is beyond the scope of this book.) Equation (8.36) is called the (time-independent) **Schrödinger equation.** Since $\mathbf{H}$ is symmetric the eigenvalues are real. Indeed, suppose λ is an eigenvalue of $\mathbf{H}$ with eigenvector Ψ. We can normalize Ψ so that $(\Psi, \Psi) = 1$. Then

$$\lambda = (\mathbf{H}\Psi, \Psi) = (\Psi, \mathbf{H}\Psi) = \bar{\lambda}$$

so λ is real. Furthermore, if λ, μ are eigenvalues of $\mathbf{H}$ with corresponding eigenvectors Ψ, Φ then

$$\lambda(\Psi, \Phi) = (\mathbf{H}\Psi, \Phi) = (\Psi, \mathbf{H}\Phi) = \mu(\Psi, \Phi)$$

so $(\Psi, \Phi) = 0$, if $\lambda \neq \mu$.

Let λ be an eigenvalue of $\mathbf{H}$ and define the eigenspace $W_\lambda \subset \mathcal{H}$ by

$$W_\lambda = \{\Psi \in \mathcal{H} : \mathbf{H}\Psi = \lambda\Psi\}.$$

If $g \in G$ we have

$$\mathbf{H}\mathbf{T}(g)\Psi = \mathbf{T}(g)\mathbf{H}\Psi = \lambda\mathbf{T}(g)\Psi \tag{8.37}$$

for all $\Psi \in W_\lambda$. Therefore $\mathbf{T}(g)\Psi \in W_\lambda$ and W_λ is invariant under the unitary rep $\mathbf{T}$ of the symmetry group G. Suppose W_λ is finite-dimensional and G is a point group. Then we can decompose W_λ into a direct sum of subspaces which transform irreducibly under G:

$$W_\lambda = \sum_{\mu=1}^{\alpha} \sum_{i=1}^{a_\mu} \oplus W_i^{(\mu)}$$

Here, the restriction of $\mathbf{T}$ to $W_i^{(\mu)}$ is equivalent to the irred rep $\mathbf{T}^{(\mu)}$ of G and a_μ is the multiplicity of $\mathbf{T}^{(\mu)}$ in $\mathbf{T}$. The reps $\mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(\alpha)}$ form a complete set of nonequivalent irred unitary reps of G. If $T^{(1)}, \ldots, T^{(\alpha)}$ are a corresponding complete set of unitary matrix reps we can find an ON basis $\{\mathbf{w}_{ij}^{(\mu)}, 1 \leq j \leq n_\mu\}$ for each space $W_i^{(\mu)}$ such that

$$\mathbf{T}(g)\mathbf{w}_{ij}^{(\mu)} = \sum_{l=1}^{n_\mu} T_{lj}^{(\mu)}(g)\mathbf{w}_{il}^{(\mu)}.$$

Then the complete set of symmetry-adapted basis vectors $\{\mathbf{w}_{ij}^{(\mu)}\}$ forms an ON basis for W_λ. In this way we use the irred reps of G to label the eigenvectors of $\mathbf{H}$.

The **complete symmetry group** of the Hamiltonian $\mathbf{H}$ is the group K of all unitary operators $\mathbf{U}$ on $\mathcal{H}$ such that

$$\mathbf{U}\mathbf{H} = \mathbf{H}\mathbf{U}. \tag{8.38}$$

Since the elements of K are unitary operators, K defines a unitary rep of itself. Just as in (8.37), we can show that W_λ is invariant (even irred) under K, so the reps of K can be used to label the elements of W_λ. However, it may be very difficult to determine all elements of K. Thus we usually restrict ourselves to consideration of the subgroup G' consisting of all symmetries of $\mathbf{H}$ taking the form (8.34). The symmetry group G' can be determined by inspection. (Later we shall include in G' spin transformations and permutations of indistinguishable particles when this is appropriate.)

The eigenspace W_λ is usually irred under G'. The degeneracy of the eigenvalue λ, i.e., the dimension of W_λ, is then equal to the dimension of some irred rep of G'. For certain specially chosen potential functions $V(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ it is possible to find eigenvalues λ of $\mathbf{H}$ for which W_λ is not irred, but this is rare. In such a case the eigenvalue λ has an **accidental degeneracy**, i.e., a degeneracy which does not follow from the symmetry of the Hamiltonian. (See the discussion of the hydrogen atom, Section 9.7.) Accidental degeneracy can be removed by a slight alteration of the potential function which does not change the symmetry group of the Hamiltonian.

If a point symmetry group G is a proper subgroup of G' then W_λ need not transform irreducibly under G and in general W_λ will break up into a direct sum of irred reps of G. In practice, if a physicist finds that W_λ is not irred under the action of G he has strong reason for suspecting the existence of a larger symmetry group. Thus, he is likely to search for additional symmetries of $\mathbf{H}$.

The eigenvalue equation

$$\mathbf{H}\Psi = \lambda\Psi \tag{8.39}$$

has been solved exactly for only a few simple Hamiltonians. The Hamiltonians $\mathbf{H}$ for which (8.39) can be solved usually correspond to physical systems which exhibit a high degree of symmetry. The two most important examples are the hydrogen atom and the harmonic oscillator, which will be discussed in later chapters. For systems with lower symmetry, Eq. (8.39) usually cannot be solved explicitly and some sort of approximation has to be employed. Group-theoretic methods are of the utmost importance here because they yield information about the multiplicities of the eigenvalues even in those cases where (8.39) cannot be solved exactly.

As an example, suppose $\mathbf{H}$ admits the point group G as a symmetry group and suppose we can find an eigenvector Ψ of $\mathbf{H}$ with eigenvalue λ. Furthermore, suppose Ψ is an element of a subspace $V^{(\mu)}$ of $\mathcal{H}$ such that the action of G on $V^{(\mu)}$ is equivalent to the irred rep $T^{(\mu)}$. Then the nonzero subspace $V^{(\mu)} \cap W_\lambda$ is invariant under G. Since $V^{(\mu)}$ is irred it follows that $V^{(\mu)} = V^{(\mu)} \cap W_\lambda$, so $V^{(\mu)} \subseteq W_\lambda$ and $\dim W_\lambda \geq n_\mu = \dim T^{(\mu)}$. Thus, the eigenvalue λ has multiplicity at least n_μ. The reader should be able to construct an ON set of n_μ eigenvectors by a judicious application of the projection operators $\mathbf{P}_\mu^{jk}$, (7.14), to Ψ.

Consider a physical system with Hamiltonian

(8.40) $$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

where

(8.41) $$\mathbf{H}_1 = -\sum_{j=1}^{k} \frac{1}{2m_j} \Delta_j + V_1(\mathbf{x}_1, \ldots, \mathbf{x}_k), \qquad \mathbf{H}_2 = V_2(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

and suppose the eigenvalue equation

(8.42) $$\mathbf{H}_1\Psi = \lambda\Psi$$

can be solved explicitly. We think of the physical system with Hamiltonian $\mathbf{H}$ as obtained from the system with Hamiltonian $\mathbf{H}_1$ by the addition of a "small" perturbing potential V_2. If the perturbation is not too large we would expect the eigenvalues and eigenfunctions of the Hamiltonian $\mathbf{H}$ to be "close" to those of $\mathbf{H}_1$. Proceeding formally, let us consider a family of Hamiltonians

(8.43) $$\mathbf{H}(t) = \mathbf{H}_1 + t\mathbf{H}_2$$

where the real parameter t runs from 0 to 1. Then $\mathbf{H}(0) = \mathbf{H}_1$, $\mathbf{H}(1) = \mathbf{H}$. If the perturbing potential is not too big it is reasonable to suppose that the eigenvalues and eigenfunctions of $\mathbf{H}(t)$ will be continuous functions of t. To be more precise, let λ_0 be any isolated eigenvalue of $\mathbf{H}_1$ with finite multiplicity m. Then we suppose there exist m continuous functions $\lambda_1(t), \ldots, \lambda_m(t)$ and m eigenvectors $\Psi_{1t}, \ldots, \Psi_{mt}$ in $\mathcal{H}$ which are continuous functions of t (in the norm $\|\cdot\|$) and satisfy

(8.44) $$\mathbf{H}(t)\Psi_{lt} = \lambda_l(t)\Psi_{lt}, \qquad 0 \le t \le 1, \quad 1 \le l \le m.$$

It is assumed that the set $\{\Psi_{lt}\}$ is ON for each t and $\lambda_l(0) = \lambda_0$. It can be shown (Kato [1]) that for a wide variety of perturbing potentials V_2 the above suppositions are correct and in fact, the $\lambda_l(t)$ can be expanded in power series in t. Physicists commonly employ a perturbation theory to compute the first few terms in the power series and get an approximation for the desired eigenvalues $\lambda_l(1)$. One of the most important problems is to determine the multiplicities of the eigenvalues $\lambda_l(1)$ of $\mathbf{H}$, that is, to determine how the m-fold degenerate eigenvalue λ_0 of $\mathbf{H}_1$ splits into eigenvalues $\lambda_l(1)$ as the perturbing potential V_2 is turned on. Group theory yields exact information about this splitting.

Suppose the point group G is a symmetry group of both Hamiltonians $\mathbf{H}_1$ and $\mathbf{H}$. Then the operators $\mathbf{T}(g)$ will commute with both $\mathbf{H}_1$ and $\mathbf{H}_2$, so G will be a symmetry of $\mathbf{H}(t)$ for all t. Let W_0 be the eigenspace of $\mathbf{H}_1$ corresponding to eigenvalue λ_0. Then the restriction of $\mathbf{T}$ to W_0 can be decomposed into irred reps,

$$\mathbf{T}\,|\,W_0 \cong \sum_{\mu=1}^{\alpha} \oplus\, a_\mu{}^0 \mathbf{T}^{(\mu)}, \tag{8.45}$$

where $a_\mu{}^0$ is the multiplicity of $\mathbf{T}^{(\mu)}$ in $\mathbf{T}\,|\,W_0$. For each t between 0 and 1 the eigenvectors $\{\Psi_{lt}\}$ form an ON basis for the direct sum W_t of the eigenspaces of $\mathrm{H}(t)$ corresponding to the eigenvalues $\lambda_1(t), \ldots, \lambda_m(t)$. Thus W_t is invariant under $\mathbf{T}$ and we have the decomposition

$$\mathbf{T}\,|\,W_t \cong \sum_{\mu=1}^{\alpha} \oplus\, a_\mu{}^t \mathbf{T}^{(\mu)}. \tag{8.46}$$

The integers $a_\mu{}^t$ must remain fixed as t varies from 0 to 1. To see this we compute the character $\chi_t(g)$ of $\mathbf{T}\,|\,W_t$. Since the ON basis vectors $\{\Psi_{lt}\}$ are continuous in t the matrix elements

$$\langle \mathbf{T}(g)\Psi_{lt}, \Psi_{jt}\rangle = T^t_{jl}(g)$$

are continuous functions of t for fixed g. Thus the character $\chi_t(g)$ is continuous in t, as is $a_\mu{}^t = \langle \chi_t, \chi^{(\mu)}\rangle$. Since $a_\mu{}^t$ is an integer it must remain constant: $a_\mu{}^t = a_\mu{}^0 = a_\mu$.

Therefore, the reps $\mathbf{T}\,|\,W_0$ and $\mathbf{T}\,|\,W_1$ are equivalent. This result shows that the perturbation V_2 splits the m-fold degenerate eigenspace W_0 of H_1 into $a_1 + \cdots + a_\alpha$ eigenspaces of $\mathbf{H}$. There are a_1 eigenvalues of $\mathbf{H}$, each with multiplicity $n_1, \ldots,$ and a_α eigenvalues of $\mathbf{H}$, each with multiplicity n_α. At most $a_1 + \cdots + a_\alpha$ of these eigenvalues are distinct, i.e., some of them may be equal. If the original rep $\mathbf{T}\,|\,W_0$ is irred then $\mathbf{T}\,|\,W_1$ is also irred and the m-fold eigenvalue λ_0 of $\mathbf{H}_1$ is perturbed to an m-fold eigenvalue λ_1 of $\mathbf{H}$.

Now suppose G_1 is the largest point symmetry group of $\mathbf{H}_1$. Let λ_0 be an eigenvalue of $\mathbf{H}_1$ and suppose the corresponding m-dimensional eigenspace transforms according to the irred rep $\mathbf{Q}$ of G_1. Furthermore, suppose $\mathbf{H}_2$ does not admit G_1 as a symmetry group but only a proper subgroup G of G_1. Then G is the maximal point symmetry group of $\mathbf{H}(t) = \mathbf{H}_1 + t\mathbf{H}_2$. The restriction of $\mathbf{Q}$ to the subgroup G splits into a direct sum of irred reps $\mathbf{T}^{(\mu)}$ of G:

$$\mathbf{Q}\,|\,G \cong \sum_{\mu=1}^{\alpha} \oplus\, a_\mu \mathbf{T}^{(\mu)}. \tag{8.47}$$

It follows from the analysis of expression (8.45) that the m-fold eigenvalue λ_0 of $\mathbf{H}_1$ splits into a_1 eigenvalues of $\mathbf{H}$, each of multiplicity $n_1, \ldots,$ and a_α eigenvalues of $\mathbf{H}$, each of multiplicity n_α. Unless there is accidental degeneracy, there will be $a_1 + \cdots + a_\alpha$ distinct eigenvalues.

As an example, consider the case where $\mathbf{H}_1$ has octahedral symmetry O while the perturbing potential $\mathbf{H}_2$ has only tetragonal symmetry D_4, a subgroup of O. (There are several subgroups of O which are isomorphic to

D_4, but any two of these subgroups are conjugate, so it makes no difference which one we choose.) The character tables of D_4 and O are given in (6.17) and (6.22). We denote the irred reps of D_4 by $\mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(5)}$ and those of O by $\mathbf{Q}^{(1)}, \ldots, \mathbf{Q}^{(5)}$. Since O has irred reps of dimensions one, two, and three, these are the possible multiplicities for eigenvalues of $\mathbf{H}_1$. To determine the manner in which each eigenspace of $\mathbf{H}_1$ splits into eigenspaces of $\mathbf{H}$ we must determine the multiplicities a_μ of $\mathbf{T}^{(\mu)}$ in $\mathbf{Q}^{(\nu)} \mid D_4$. These multiplicities can easily be computed from the character tables. We have

$$
\begin{aligned}
&\mathbf{Q}^{(1)} \mid D_4 \cong \mathbf{T}^{(1)}, \quad n_1 = 1; \qquad \mathbf{Q}^{(2)} \mid D_4 \cong \mathbf{T}^{(3)}, \quad n_3 = 1;\\
&\mathbf{Q}^{(3)} \mid D_4 \cong \mathbf{T}^{(1)} \oplus \mathbf{T}^{(3)}, \quad n_1 = n_3 = 1;\\
&\mathbf{Q}^{(4)} \mid D_4 \cong \mathbf{T}^{(2)} \oplus \mathbf{T}^{(5)}, \quad n_2 = 1, \quad n_5 = 2;\\
&\mathbf{Q}^{(5)} \mid D_4 \cong \mathbf{T}^{(4)} \oplus \mathbf{T}^{(5)}, \quad n_4 = 1, \quad n_5 = 2.
\end{aligned}
\tag{8.48}
$$

For example, the character χ of $\mathbf{Q}^{(5)}$ restricted to D_4 is

$$
\begin{array}{c|ccccc}
 & \mathcal{E} & \mathcal{C}_4^2 & 2\mathcal{C}_4 & 2\mathcal{C}_2 & 2\mathcal{C}_{2'} \\
\hline
\chi & 3 & -1 & -1 & -1 & 1
\end{array}.
$$

From (6.17) we obtain $\chi(g) = \chi^{(4)}(g) + \chi^{(5)}(g)$, which yields the last line in (8.48). These results show that under the perturbing potential an eigenvalue of multiplicity two splits into two simple eigenvalues, while an eigenvalue of multiplicity three splits into an eigenvalue of multiplicity two and a simple eigenvalue. We can reduce these degeneracies still further by adding to the Hamiltonian a perturbation $\mathbf{H}_3$ with lower symmetry D_2. Then the restrictions of the reps $\mathbf{T}^{(\mu)}$ to D_2 will split into direct sums of irred reps $\mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \mathbf{S}^{(3)}, \mathbf{S}^{(4)}$ of D_2. In fact,

$$
\begin{aligned}
&\mathbf{T}^{(1)} \mid D_2 \cong \mathbf{S}^{(1)}, \qquad \mathbf{T}^{(2)} \mid D_2 \cong \mathbf{S}^{(3)}, \qquad \mathbf{T}^{(3)} \mid D_2 \cong \mathbf{S}^{(1)}\\
&\mathbf{T}^{(4)} \mid D_2 \cong \mathbf{S}^{(3)}, \qquad \mathbf{T}^{(5)} \mid D_2 \cong \mathbf{S}^{(2)} \oplus \mathbf{S}^{(4)}
\end{aligned}
\tag{8.49}
$$

where the character of $\mathbf{S}^{(j)}$ is $\chi^{(j)}$. The $\mathbf{S}^{(j)}$ is one-dimensional, so each of the multiply degenerate eigenvalues of $\mathbf{H}_1$ is split into simple eigenvalues of $\mathbf{H}' = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3$. The only possible degeneracies of eigenvalues of $\mathbf{H}'$ are accidental. The introduction of symmetry group lattices such as $O \supset D_4 \supset D_2$ is very useful in quantum mechanics for predicting the distribution of eigenvalues. For instance if the perturbation $\mathbf{H}_3$ is small with respect to $\mathbf{H}_2$ then one can predict that a triply degenerate eigenvalue of $\mathbf{H}_1$ will split into three simple eigenvalues, but that two of these eigenvalues will lie very close together in relation to the third.

The above ideas relating symmetry to perturbation theory are applicable to any symmetry group of the Hamiltonian, not only to the point groups. Indeed, the results which are of most importance in quantum theory relate to the rotation group $SO(3)$ and the symmetric group S_N whose reps will be studied later. In Chapter 7 we will return to the study of symmetry in perturbation theory.

The routine proofs of the following statements are omitted. Let W_λ be an eigenspace of the Hamiltonian (8.32). Since the potential V is real, the complex conjugate function $\overline{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in W_\lambda$ for all $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in W_\lambda$. If $V^{(\mu)}$ is a subspace of W_λ transforming under the irred rep $\mathbf{T}^{(\mu)}$ of the symmetry group G then the complex conjugate space $\bar{V}^{(\mu)} \subseteq W_\lambda$ transforms under the irred rep $\overline{\mathbf{T}}^{(\mu)}$. If the simple character $\chi^{(\mu)}$ is real-valued then $\mathbf{T}^{(\mu)}$ is equivalent to $\overline{\mathbf{T}}^{(\mu)}$. However, if $\chi^{(\mu)} \neq \bar{\chi}^{(\mu)}$ then $\mathbf{T}^{(\mu)}$ and $\overline{\mathbf{T}}^{(\mu)}$ are nonequivalent irred reps and the eigenspace W_λ is not irred. This degeneracy is due to the fact that the (nonlinear) complex-conjugation operator commutes with $\mathbf{H}$, and is not considered accidental. Reps of G with complex characters always occur in complex conjugate pairs. (However, a rep with a real character need not be real; see Hamermesh [1, p. 138].)

Problems

3.1 Let T be an irred matrix rep of the finite group G and let C be a conjugacy class in G. Show that $\sum_{g \in C} T(g)$ is a multiple of the identity matrix.

3.2 Let G be a finite group with commutator subgroup G_C. (See Problem 1.8.) Show that the number of one-dimensional reps of G is equal to the index of G_C in G.

3.3 Let $\mathbf{T}_j, \mathbf{T}_j'$, $(j = 1, 2)$ be reps of the groups G such that $\mathbf{T}_j \simeq \mathbf{T}_j'$. Show that $\mathbf{T}_1 \oplus \mathbf{T}_2 \simeq \mathbf{T}_1' \oplus \mathbf{T}_2'$.

3.4 Let $\mathbf{T}, \mathbf{T}'$ be unitary reps of G on the inner product spaces V, V', respectively. If $\langle -, - \rangle$, $\langle -, - \rangle'$ are the inner products on V, V' show that $(\mathbf{u} \otimes \mathbf{u}', \mathbf{v} \otimes \mathbf{v}') = \langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}', \mathbf{v}' \rangle'$ defines an inner product on $V \otimes V'$ with respect to which $\mathbf{T} \otimes \mathbf{T}'$ is unitary.

3.5 Prove: If $\mathbf{T}$ is an irred rep and $\mathbf{Q}$ a one-dimensional rep of G then $\mathbf{T} \otimes \mathbf{Q}$ is irred.

3.6 Let $\mathbf{T}_1, \mathbf{T}_2$ be irred reps of the finite group G with dimensions $d_1 > d_2$. Show that $\mathbf{T}_1 \otimes \mathbf{T}_2$ contains no irred rep $\mathbf{T}_3$ with $d_3 < d_1/d_2$.

3.7 Compute the character table of the icosahedral group Y.

3.8 Prove: The dimensions n_i of the irred reps of the finite group G are divisors of $n(G)$. (This is a difficult theorem. See Hall [1, Section 16.8].)

3.9 Determine the dimensions of the following subspaces of second-rank tensors which are fixed under Y: (a) polar, (b) symmetric polar, (c) axial, (d) symmetric axial. Repeat for the group C_{2v}.

3.10 Consider a quantum mechanical system with octahedral symmetry O. Suppose a perturbation is applied which reduces the symmetry to (a) T, (b) D_3, (c) C_4. In each case determine how the possible energy levels of the original system are split by the perturbation.

3.11 Let K be a subgroup of H and H a subgroup of the finite group G. Prove the following properties of induced reps: (a) If $\mathbf{T}$ is a rep of K then $(\mathbf{T}^H)^G \simeq \mathbf{T}^G$. (b) If $\mathbf{R}$ is a rep of H and $\mathbf{S}$ a rep of G then $\mathbf{R}^G \otimes \mathbf{S} \simeq (\mathbf{R} \otimes (\mathbf{S} \mid H))^G$.

3.12 Let G be a group of order N and $\chi(g)$ a character of G. Prove that $N^{-1} \sum_{g \in G} [\chi(g)]^n$ is a nonnegative integer for each $n = 1, 2, \ldots$.

3.13 Let $T_1(g)$ and $T_2(g)$ be $n \times n$ matrix reps of G with real matrix elements. These reps are **real equivalent** if there is a real nonsingular matrix S such that $T_1(g)S = ST_2(g)$ for all $g \in G$. Show that T_1 and T_2 are complex equivalent if and only if they are real equivalent. (Hint: Write $S = A + iB$, where A and B are real, and show that $A + tB$ is invertible for some real number t.)

3.14 Show that the matrix elements of two real irred reps of a group G which are not real equivalent satisfy an orthogonality relation. Show that every real irred rep is real equivalent to a rep by real orthogonal matrices.