# QUANTUM MODELS OF COGNITION AND DECISION

Jerome R. Busemeyer
Indiana University

Supported by AFOSR and NSF

# 1. WHY USE QUANTUM THEORY?

1. Quantum theory is a general Axiomatic theory of probability
    - Human judgments and decisions are probabilistic
    - These probabilities do not obey the Kolmogorov axioms
    - Quantum theory provides a viable alternative
2. Non Commutativity of measurements
    - Measurements change psychological states producing context effects
    - Principle of complementarity was (arguably) suggested to Niels Bohr by William James
3. Vector space representation of probabilities
    - Agrees with connectionist-neural network models of cognition

# 2. HOW DO WE USE QUANTUM THEORY?

# COMPARISON OF CLASSIC AND QUANTUM PROBABILITY THEORIES

Kolmogorov

Von Neumann

## Classical

- Each unique outcome is a member of a set of points called the Sample space

## Quantum

- Each unique outcome is an orthonormal vector from a set that spans a Vector space

# Classical

- Each unique outcome is a member of a set of points called the Sample space
- Each event is a subset of the sample space

# Quantum

- Each unique outcome is an orthonormal vector from a set that spans a Vector space
- Each event is a subspace of the vector space.

## Classical

- Each unique outcome is a member of a set of points called the Sample space
- Each event is a subset of the sample space
- State is a probability function, p, defined on subsets of the sample space.

## Quantum

- Each unique outcome is an orthonormal vector from a set that spans a Vector space
- Each event is a subspace of the vector space.
- State is a unit length vector, S,

$$p(A) = \left\| P_A S \right\|^2$$

# Classical

- Suppose event A is observed (state reduction):

$$p(B \mid A) = \frac{p(B \cap A)}{p(A)}$$

# Quantum

- Suppose event A is observed (state reduction):

$$p(B \mid A) = \frac{\|P_B P_A S\|^2}{\|P_A S\|^2}$$

## Classical

- Suppose event A is observed (state reduction):

$$p(B \mid A) = \frac{p(B \cap A)}{p(A)}$$

- **Commutative Property**

$$p(B \cap A) = p(A \cap B)$$

## Quantum

- Suppose event A is observed (state reduction):

$$p(B \mid A) = \frac{\left\| P_B P_A S \right\|^2}{\left\| P_A S \right\|^2}$$

- Non-Commutative

$$\left\| P_B P_A S \right\|^2 \neq \left\| P_A P_B S \right\|^2$$

# FOUR APPLICATIONS

# 1. ORDER EFFECTS

**Context effects produced by question orders reveal quantum nature of human judgments**

Zheng Wang[a,1], Tyler Solloway[a], Richard M. Shiffrin[b,1], and Jerome R. Busemeyer[b]

PNAS | July 1, 2014 | vol. 111 | no. 26 | 9431–9436

Hostility between white and black people:

BW order (N = 500 participants)
(A) Do you think blacks dislike whites? (Y,N)
(B) Do you think whites dislike blacks? (Y,N)

WB Order (N = 500 participants)
(B) Do you think whites dislike blacks? (Y,N)
(A) Do you think blacks dislike whites? (Y,N)

| | By | Bn |
|---|---|---|
| **White–black** | | |
| Wy | 0.3987 | 0.0174 |
| Wn | 0.1612 | 0.4227 |
| **Black–white** | | |
| Wy | 0.4012 | 0.1379 |
| Wn | 0.0597 | 0.4012 |
| **Context effects** | | |
| Wy | –0.0025 | –0.1205 |
| Wn | 0.1015 | 0.0215 |

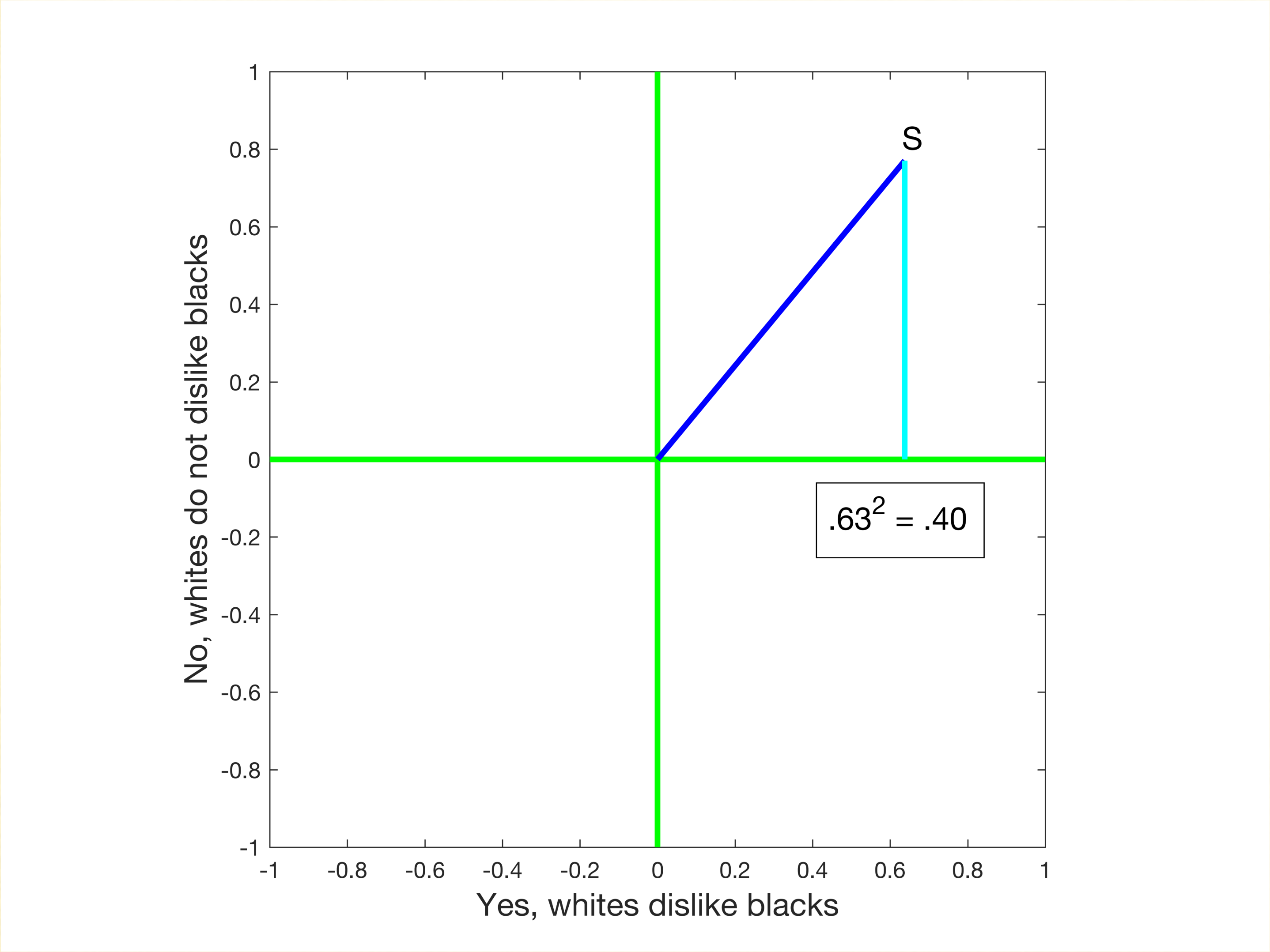
(Results from Gallup Pole)

Test order effects

$\chi^2\,(3) = 73.04,\ p < 0.001$

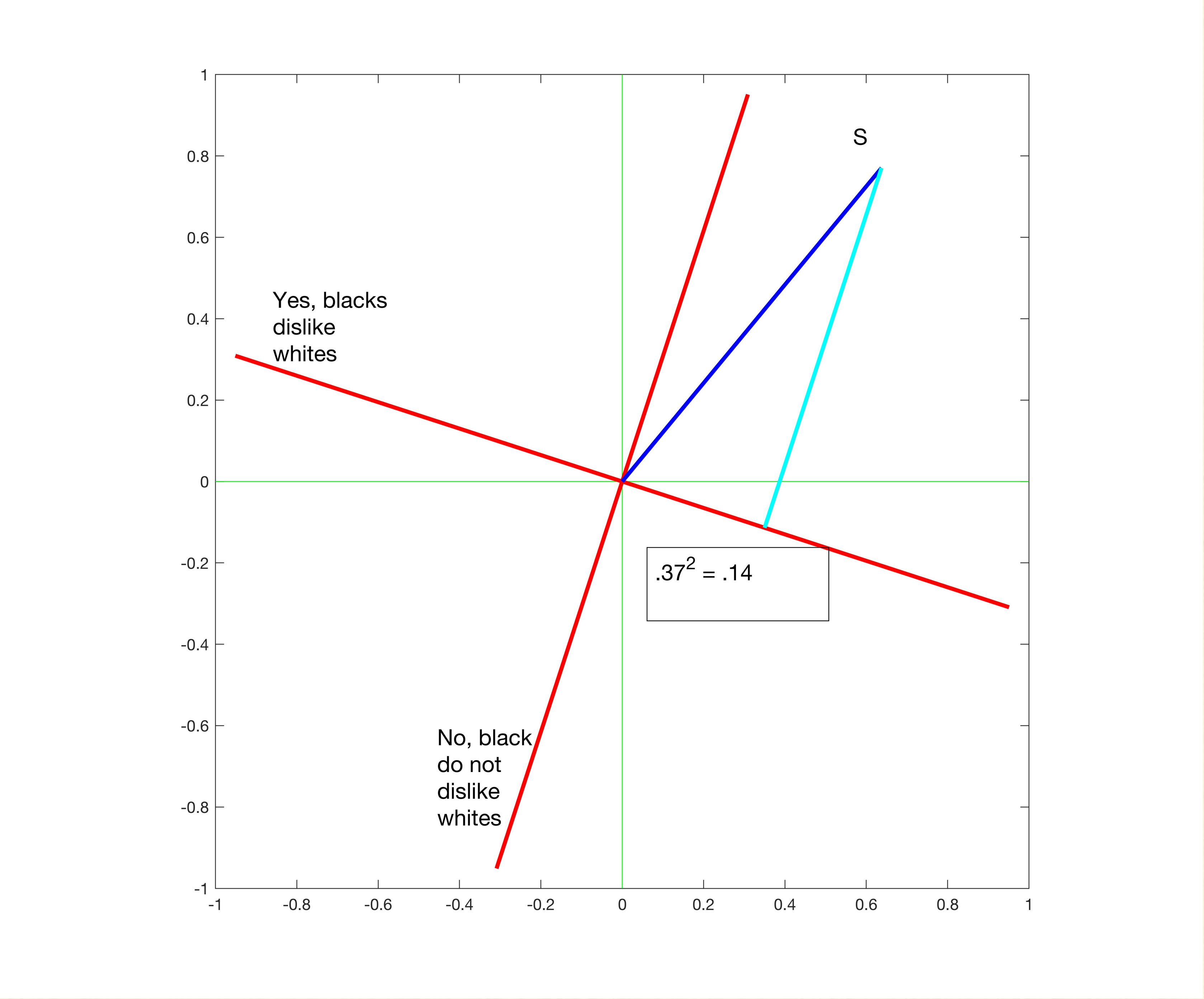
# Quantum (Toy) Model

Do Whites dislike Blacks? (Question from Gallop Pole)

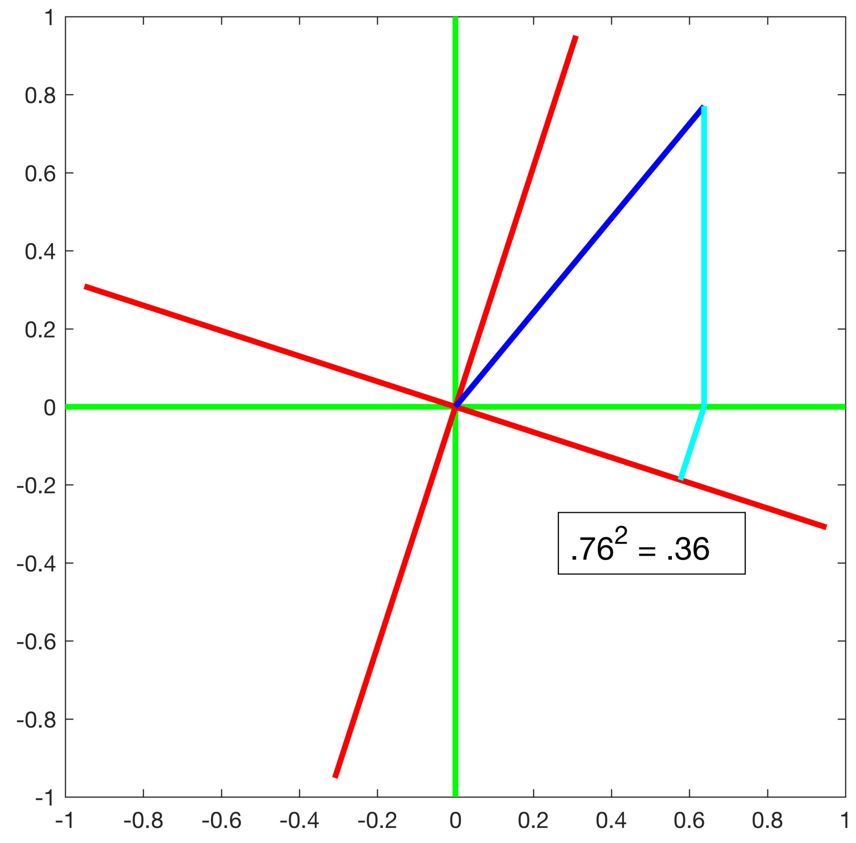


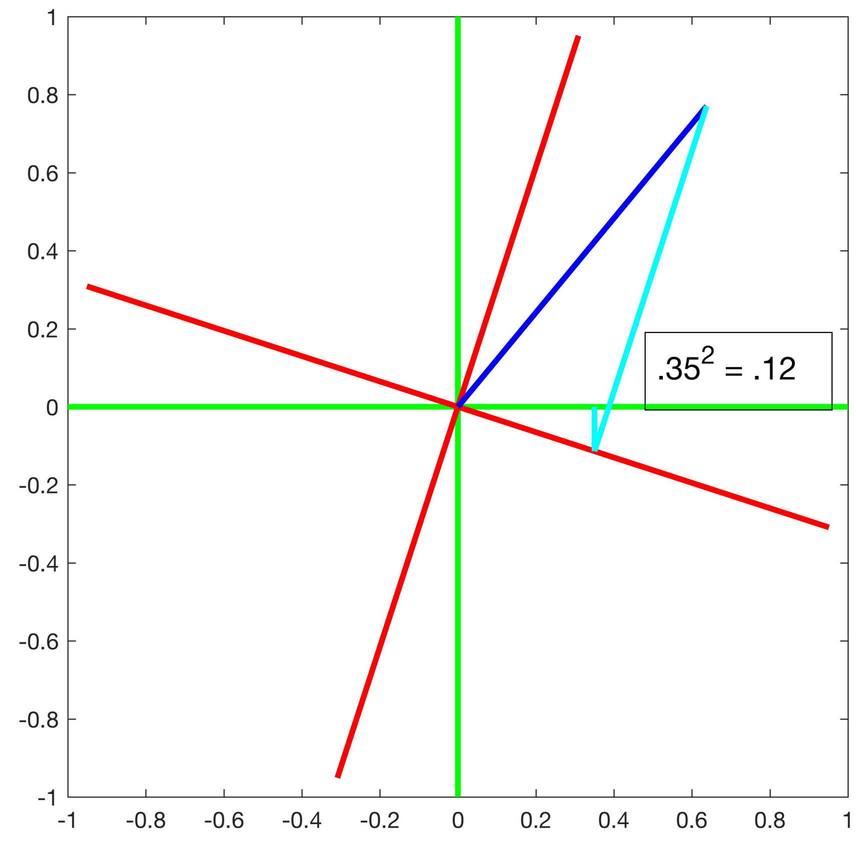

Prob(Yes| S = initial)
= .40
(Results from actual
Gallup Pole, N = 500)

# Do Blacks dislike Whites?
# Change in Perspective



# Toy model for Order Effects




Prob(Yes to WB and then Yes to BW| S) = .36

Prob(Yes to BW and then Yes to WB| S) = .12

# GENERAL N-DIMENSIONAL QUANTUM MODEL PREDICTION

Assume: One question followed immediately by another with no information in between

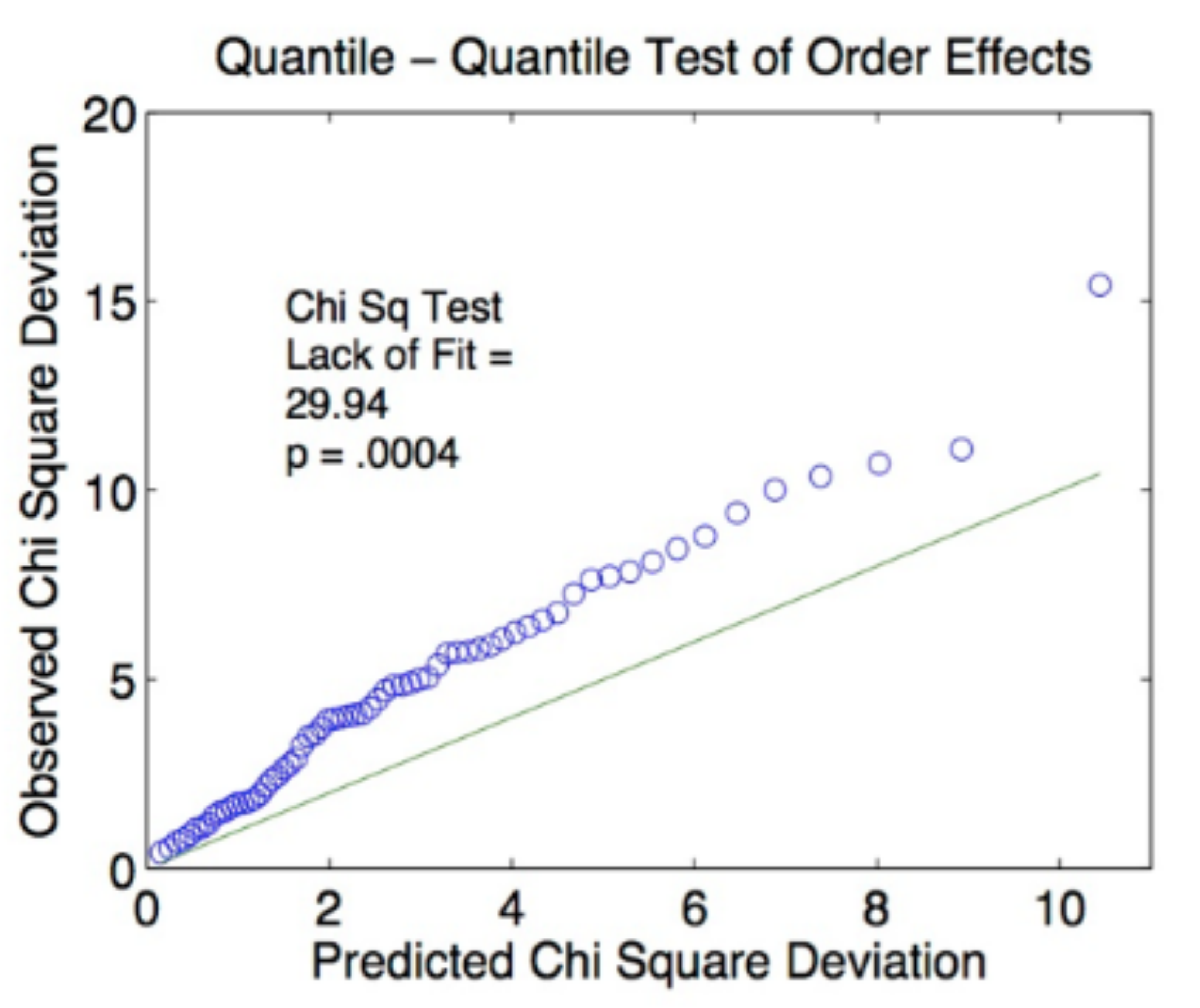
$$\Pr[A \text{ yes and then B no}] = p(A_Y B_N) = \left\| P_{\bar{B}} P_A S \right\|^2$$

$$\Pr[B \text{ no and then A yes}] = p(B_N A_Y) = \left\| P_A P_{\bar{B}} S \right\|^2$$

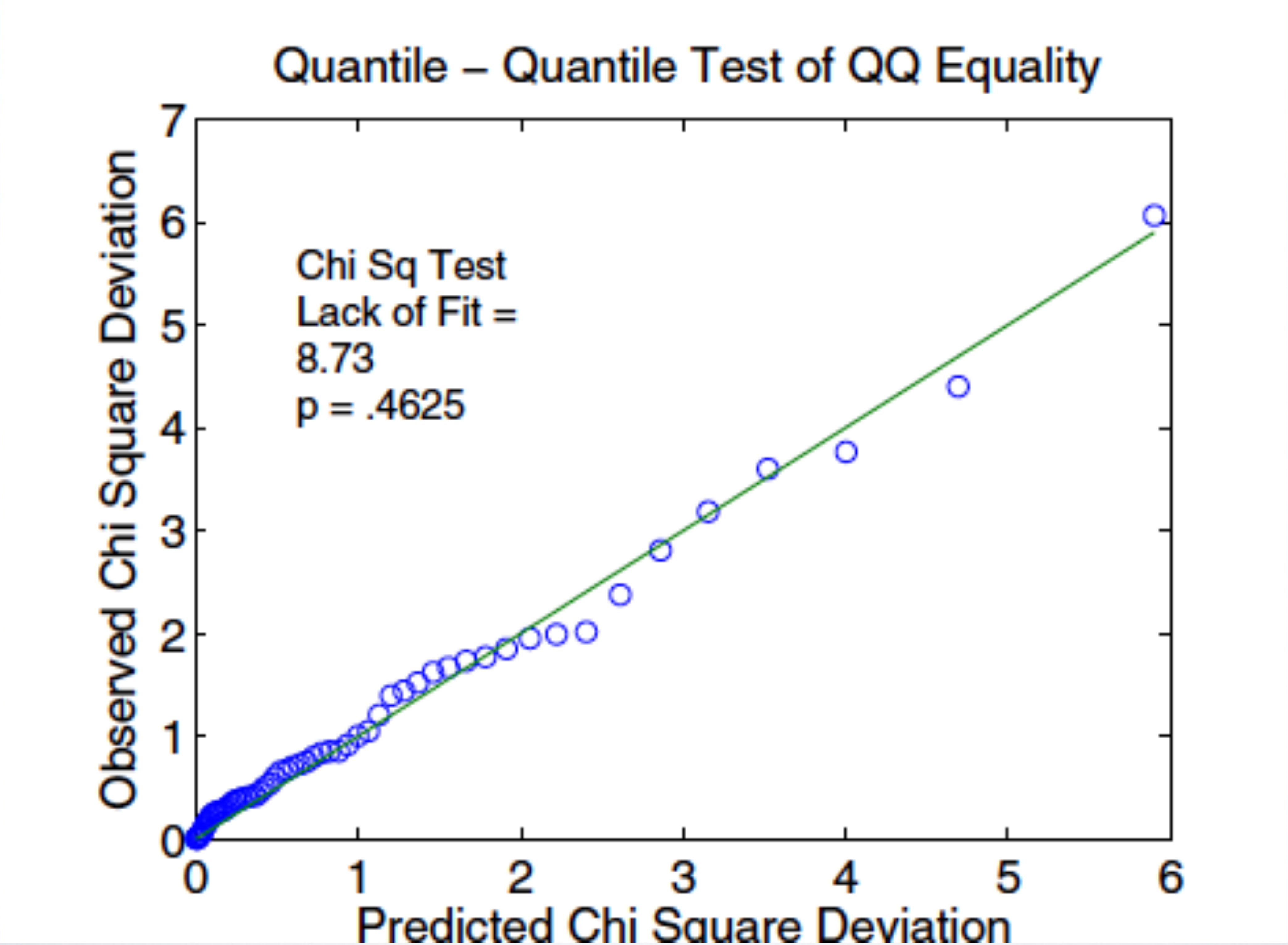
$\mathit{Theorem}$: QQ equality

$$q = \{p(A_Y B_N) + p(A_N B_Y)\} - \{p(B_Y A_N) + p(B_N A_Y)\} = 0$$

# Results: 72 Pew Surveys over 10 years



# Results: 72 Pew Surveys over 10 years



# 2. CONJUNCTION -DISJUNCTION PROBABILITY JUDGMENT ERRORS

Tversky & Kahneman
(1983, *Psychological Review*)

Busemeyer, Pothos, Franco, Trueblood
(2011, *Psychological Review*)

Read the following information:

Linda was a philosophy major as a student at UC Berkeley and she was an activist in social welfare movements.

Rate the probability of the following events

Linda is a feminist (.83)
Linda is a bank teller (.26)
Linda is a feminist and a bank teller (.36)
Linda is a feminist or a bank teller (.60)

Conjunction Fallacy

Disjunction Fallacy

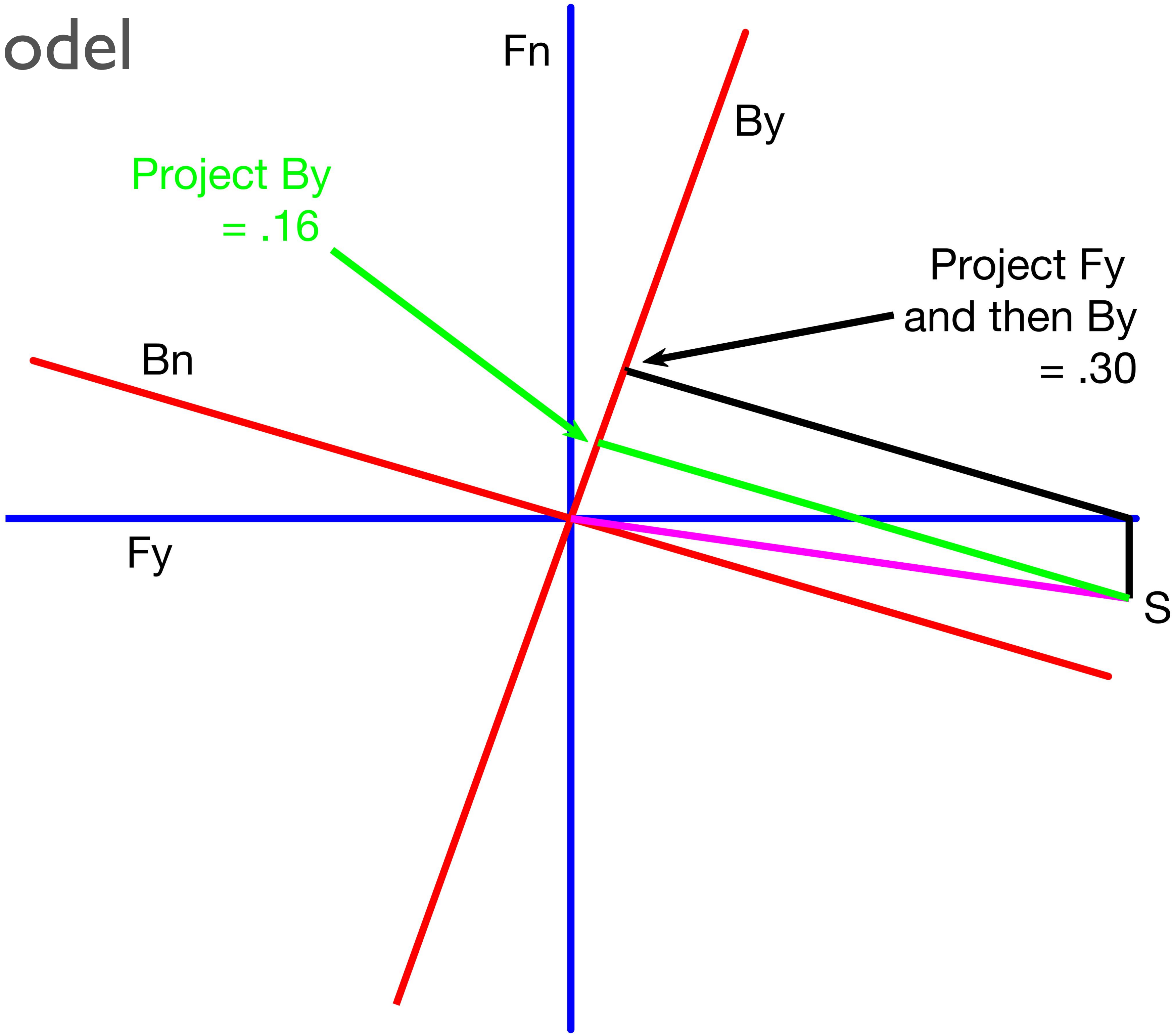
# LAW OF TOTAL PROBABILITY

$$p(B) = p(F)p(B \mid F) + p(\sim F)p(B \mid \sim F)$$
$$\geq p(F)p(B \mid F)$$

**CONJUNCTION - FALLACY**

VIOLATES THIS LAW

# Feminist Bank Teller Toy Model



# Quantum Model Predictions

$$\left\|P_B S\right\|^2 = \left\|P_B I S\right\|^2 = \left\|P_B (P_F + P_{\overline{F}}) S\right\|^2$$

$$= \left\|P_B P_F S + P_B P_{\overline{F}} S\right\|^2$$

$$= \left\|P_B P_F S\right\|^2 + \left\|P_B P_{\overline{F}} S\right\|^2 + Int$$

$$Int = \left\langle S' P'_F P'_B P_{\overline{F}} S \right\rangle + \left\langle S' P_{\overline{F}} P'_B P_F S \right\rangle$$

$$Int < -\left\|P_B P_{\overline{F}} S\right\|^2$$

# ADDITIONAL PREDICTIONS

- Order Effects: $p(F)p(B|F) =/= p(B)p(F|B)$
- Disjunction Fallacy co-occurs with Conjunction Fallacy
- Conjunction-Disjunction constraint: $p(F)p(B|F) > p(B)p(F|B)$
- Unequal Priors: $p(F)>p(B)$
- No double conjunction errors: $p(F)>p(F)p(B|F)>p(B)$
- Positive dependence: $p(B|F)>p(B)$
- Conditional Probability: $p(B|F) > p(F)p(B|F)$

# 3. INTERFERENCE OF CATEGORIZATION ON DECISION

Psychological version of a double slit experiment

Busemeyer, Wang, Mogiliansky-Lambert
(2009, *J. of Mathematical Psychology*)

Wang & Busemeyer
(2016, *Cognition*)

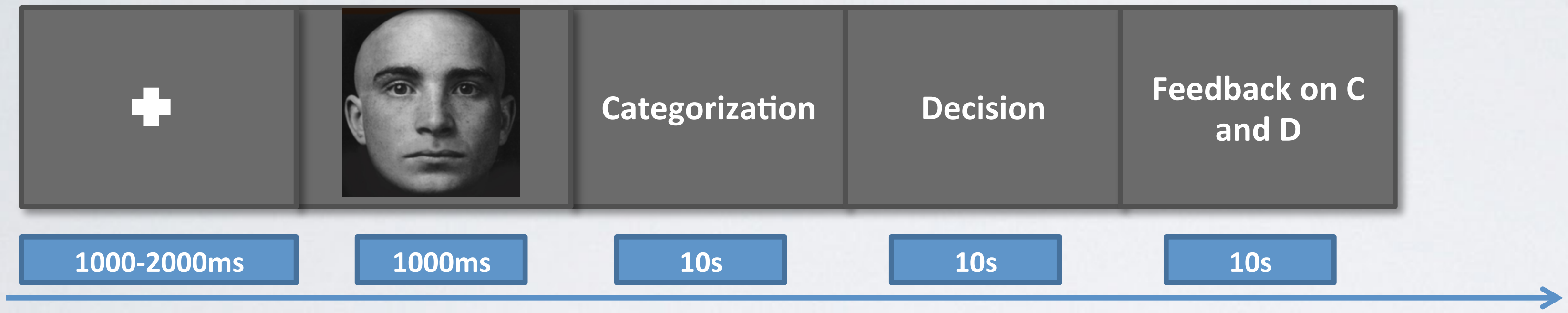
# Participants shown pictures of faces

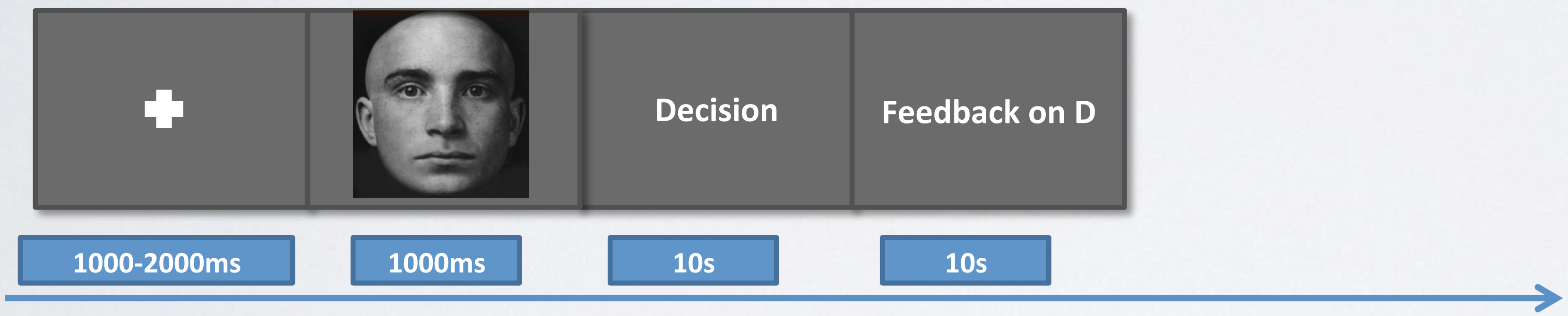
Categorize as "good" guy or "bad" guy
Decide to act "friendly" or "aggressive"


Bad Guys Good Guys

# Two Conditions:

**C-then-D:** Categorize face first and then decide




**D-alone:** Decide without categorization

# LAW OF TOTAL PROBABILITY

G =good guy,   B=Bad guy,    A=Attack

$$p(A) = p(G)p(A|G) + p(B)p(A|B)$$

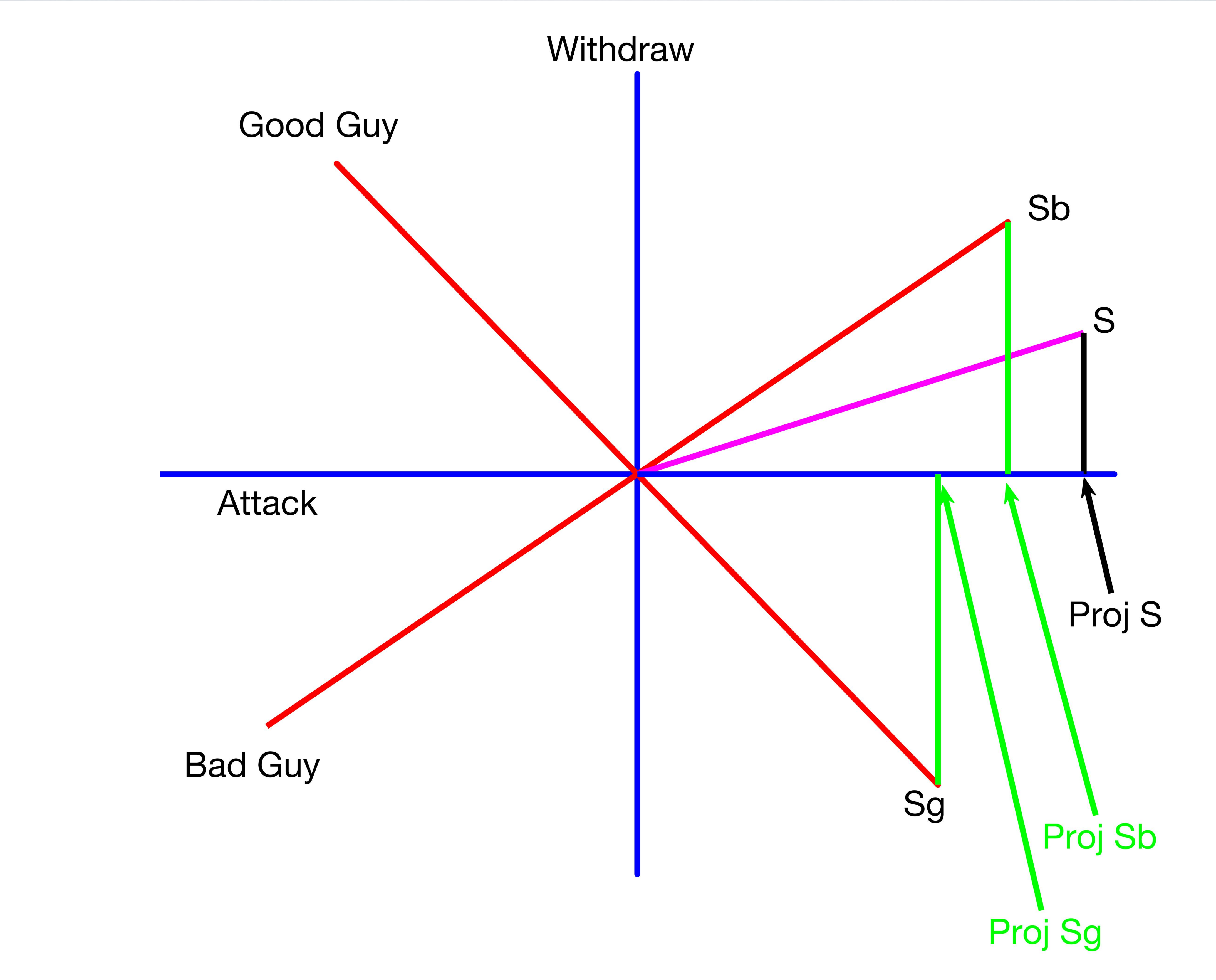
D alone Condition

C-then-D Condition

# RESULTS

| Face | p(G) | p(A\|G) | p(B) | p(A\|B) | TP | P(A) |
|---|---|---|---|---|---|---|
| Good | 0.84 | 0.35 | 0.16 | 0.52 | 0.37 | 0.39 |
| Bad | 0.17 | 0.41 | 0.82 | 0.63 | 0.59 | **0.69** |

# Categorization -Decision Toy Model



# QUANTUM INTERFERENCE

$$p(A \mid D\ alone) = \left\| P_A S \right\|^2 = \left\| P_A \cdot I \cdot S \right\|^2$$

$$= \left\| P_A \cdot \left( P_G + P_B \right) \cdot S \right\|^2$$

$$= \left\| P_A \cdot P_G \cdot S + P_A \cdot P_B \cdot S \right\|^2$$

$$= \left\| P_A \cdot P_G \cdot S \right\|^2 + \left\| P_A \cdot P_B \cdot S \right\|^2 + Int$$

Interference term violates of Law of Total Probability

$$Int = \left\langle S \mid P_G P_A P_A P_B \mid S \right\rangle + \left\langle S \mid P_B P_A P_A P_G \mid S \right\rangle$$

$$Finding \rightarrow Int > 0$$

# 4. PRISONER DILEMMA GAME

SHAFIR & TVERSKY (1992, *COGNITIVE PSYCH*)

POTHOS & BUSEMEYER, 2009,
*PROCEEDINGS OF THE ROYAL SOCIETY, B.*

| | OD | OC |
|---|---|---|
| PD | O: 10<br>P: 10 | O:5<br>P: 25 |
| PC | O: 25<br>P: 5 | O:20<br>P: 20 |

Examined three conditions in a prisoner dilemma task

Known Coop: Player is told other opponent will cooperate
Known Defect: Player is told other opponent will defect
UnKnown: Player is told nothing about the opponent

# LAW OF TOTAL PROBABILITY

$p(PD) =$ probability player defects when opponent's move is unknown

$$p(PD) = p(OD)p(PD \mid OD) + p(OC)p(PD \mid OC)$$

Empirically we find : $p(PD \mid OD) \geq p(PD \mid OC)$

$$\rightarrow p(PD \mid OD) \geq p(PD) \geq p(PD \mid OC)$$

# DEFECT RATE FOR TWO EXPERIMENTS

| Study | Known Defect | Known Coop | Unknown |
|---|---|---|---|
| Shafir Tversky (1992) | 0.97 | 0.84 | 0.63 |
| Matthew Busemeyer (2006) | 0.91 | 0.84 | 0.66 |

Defect rate for both known conditions exceed the unknown condition

# QUANTUM INTERFERENCE

$$p(PD) = \left\| P_{PD} S \right\|^2 = \left\| P_{PD} \cdot I \cdot S \right\|^2$$

$$= \left\| P_{PD} \cdot \left( P_{OD} + P_{OC} \right) \cdot S \right\|^2$$

$$= \left\| P_{PD} \cdot P_{OD} \cdot S + P_{PD} \cdot P_{OC} \cdot S \right\|^2$$

$$= \left\| P_{PD} \cdot P_{OD} \cdot S \right\|^2 + \left\| P_{PD} \cdot P_{OC} \cdot S \right\|^2 + Int$$

$$Int = \langle S \mid P_{OC} P_{PD} P_{PD} P_{OD} \mid S \rangle + \langle S \mid P_{OD} P_{PD} P_{PD} P_{OC} \mid S \rangle$$

# STEPS IN BUILDING A QUANTUM COGNITION MODEL

1. Define the Hilbert space
2. Define the initial state
3. Define the projectors
4. Compute choice probabilities

# AN HYPOTHETICAL EXAMPLE TO ILLUSTRATE THE STEPS

Suppose that the relations among three variables are investigated, labeled A,B,C.

- Ratings of a politician reported from various large social media sources
  - Adeptness (yes, no), 2 values
  - Brilliance (low, medium, high), 3 values
  - Confidence (1,2,3,4) , 4 values

# EXAMPLE USING ARTIFICIAL DATA

Table 1: Six different contingency tables produced by answers to attributes A,B,C.

| $p(C = c_k)$ | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 0.2186 | 0.2788 | 0.2551 | 0.2475 |

| $p(A = a_i, B = b_j)$ | | |
|---|---|---|
| 0.0721 | 0.5777 | 0.0078 |
| 0.1235 | 0.0374 | 0.1815 |

| $p(A = a_i, C = c_k)$ | | | |
|---|---|---|---|
| 0.0388 | 0.0312 | 0.2675 | 0.3201 |
| 0.1554 | 0.1506 | 0.0182 | 0.0183 |

| $p(C = c_k, A = a_i)$ | | | |
|---|---|---|---|
| 0.0233 | 0.0297 | 0.2279 | 0.2212 |
| 0.1953 | 0.2491 | 0.0272 | 0.0264 |

| $p(B = b_j, C = c_k)$ | | | |
|---|---|---|---|
| 0.1266 | 0.0476 | 0.0049 | 0.0165 |
| 0.0915 | 0.0911 | 0.2158 | 0.2167 |
| 0.1086 | 0.0320 | 0.0133 | 0.0354 |

| $p(C = c_k, B = b_j)$ | | | |
|---|---|---|---|
| 0.0680 | 0.0762 | 0.0581 | 0.0713 |
| 0.1089 | 0.1391 | 0.1273 | 0.0924 |
| 0.0416 | 0.0635 | 0.0697 | 0.0838 |

# JOINT PROBABILITY MODEL

- Any Bayesian network model for the data assumes
- A x B x C joint distribution, 2x3x4=24 joint prob's that sum to one

C=1

| | B=1 | B=2 | B=3 |
|---|---|---|---|
| A=1 | 111 | 121 | 131 |
| A=2 | 211 | 221 | 231 |

C=2

| | B=1 | B=2 | B=3 |
|---|---|---|---|
| A=1 | 112 | 122 | 132 |
| A=2 | 212 | 222 | 232 |

C=3

| | B=1 | B=2 | B=3 |
|---|---|---|---|
| A=1 | 113 | 123 | 133 |
| A=2 | 213 | 223 | 233 |

C=4

| | B=1 | B=2 | B=3 |
|---|---|---|---|
| A=1 | 114 | 124 | 134 |
| A=2 | 214 | 224 | 234 |

# JOINT PROBABILITY MODEL

3-way joint probability model uses marginalization to predict each of the six context tables

For example: Pr (A = 1, B=3) = p (131) + p (132) + p(133) + p(134)

C=1

| | B=1 | B=2 | B=3 |
|---|---|---|---|
| A=1 | 111 | 121 | 131 |
| A=2 | 211 | 221 | 231 |

C=2

| B=1 | B=2 | B=3 |
|---|---|---|
| 112 | 122 | 132 |
| 212 | 222 | 232 |

C=3

| | | | |
|---|---|---|---|
| A=1 | 113 | 123 | 133 |
| A=2 | 213 | 223 | 233 |

C=4

| | | |
|---|---|---|
| 114 | 124 | 134 |
| 214 | 224 | 234 |

# JOINT PROBABILITY MODEL FAILS

- No joint probability distribution of the 3 variables, A,B,C can reproduce the tables.
- Violations of marginal invariance
- Violations of commutativity
- Violation of the correlation consistency, and in this case data violate the Leggett-Garg inequality

# VIOLATION OF MARGINAL PROBABILITY

Table 1: Six different contingency tables produced by answers to attributes A,B,C.

| $p(C = c_k)$ | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 0.2186 | 0.2788 | 0.2551 | 0.2475 |

| $p(A = a_i, B = b_j)$ | | |
|---|---|---|
| 0.0721 | 0.5777 | 0.0078 |
| 0.1235 | 0.0374 | 0.1815 |

.6576

| $p(A = a_i, C = c_k)$ | | | |
|---|---|---|---|
| 0.0388 | 0.0312 | 0.2675 | 0.3201 |
| 0.1554 | 0.1506 | 0.0182 | 0.0183 |

.1818

| $p(C = c_k, A = a_i)$ | | | |
|---|---|---|---|
| 0.0233 | 0.0297 | 0.2279 | 0.2212 |
| 0.1953 | 0.2491 | 0.0272 | 0.0264 |

.5021

| $p(B = b_j, C = c_k)$ | | | |
|---|---|---|---|
| 0.1266 | 0.0476 | 0.0049 | 0.0165 |
| 0.0915 | 0.0911 | 0.2158 | 0.2167 |
| 0.1086 | 0.0320 | 0.0133 | 0.0354 |

| $p(C = c_k, B = b_j)$ | | | |
|---|---|---|---|
| 0.0680 | 0.0762 | 0.0581 | 0.0713 |
| 0.1089 | 0.1391 | 0.1273 | 0.0924 |
| 0.0416 | 0.0635 | 0.0697 | 0.0838 |

# VIOLATION OF COMMUTATIVITY

Table 1: Six different contingency tables produced by answers to attributes A,B,C.

| $p(C = c_k)$ | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 0.2186 | 0.2788 | 0.2551 | 0.2475 |

| $p(A = a_i, B = b_j)$ | | |
|---|---|---|
| 0.0721 | 0.5777 | 0.0078 |
| 0.1235 | 0.0374 | 0.1815 |

| $p(A = a_i, C = c_k)$ | | | |
|---|---|---|---|
| 0.0388 | 0.0312 | 0.2675 | 0.3201 |
| 0.1554 | 0.1506 | 0.0182 | 0.0183 |

| $p(C = c_k, A = a_i)$ | | | |
|---|---|---|---|
| 0.0233 | 0.0297 | 0.2279 | 0.2212 |
| 0.1953 | 0.2491 | 0.0272 | 0.0264 |

| $p(B = b_j, C = c_k)$ | | | |
|---|---|---|---|
| 0.1266 | 0.0476 | 0.0049 | 0.0165 |
| 0.0915 | 0.0911 | 0.2158 | 0.2167 |
| 0.1086 | 0.0320 | 0.0133 | 0.0354 |

| $p(C = c_k, B = b_j)$ | | | |
|---|---|---|---|
| 0.0680 | 0.0762 | 0.0581 | 0.0713 |
| 0.1089 | 0.1391 | 0.1273 | 0.0924 |
| 0.0416 | 0.0635 | 0.0697 | 0.0838 |

# LEGGETT-GARG INEQUALITY

| X | Y | Z | X=/=Y | Y=/=Z | X=/=Z |
|---|---|---|---|---|---|
| 1 | 1 | 1 | | | |
| 1 | 1 | 2 | | X | X |
| 1 | 2 | 1 | X | X | |
| 1 | 2 | 2 | X | | X |
| 2 | 1 | 1 | X | | X |
| 2 | 1 | 2 | X | X | |
| 2 | 2 | 1 | | X | X |
| 2 | 2 | 2 | | | |
| | | | 0.1687 | 0.5945 | 0.8936 |

Joint probability model requires

$p(X \neq Y) + p(Y \neq Z) - p(X \neq z) \geq 0,$

but we observe

Define New Binary Variables

$(X=1)=(A=1),\ (X=2)=(A=2)$

$(Y=1)=(B=1\cup B=2),\ (Y=2)=(B=3)$

$(Z=1)=(C=1\cup C=2),\ (Z=2)=(C=3\cup C=4)$

## Then the AxB, BxC, AxC tables collapse to

| $p(X=x_i, Y=y_j)$ | |
|---|---|
| 0.6498 | 0.0078 |
| 0.1609 | 0.1815 |

| $p(Y=y_j, Z=z_k)$ | |
|---|---|
| 0.3568 | 0.4539 |
| 0.1406 | 0.0487 |

| $p(X=x_i, Z=z_k)$ | |
|---|---|
| 0.0700 | 0.5876 |
| 0.3060 | 0.0365 |

$$p(X\neq Y)+p(Y\neq Z)-p(X\neq Z)=-.13$$

Violates the Leggett-Garg inequality!

# 1. BUILDING THE HILBERT SPACE

- Determine compatibility/incompatibility between variables
- Each combination of values for the compatible variables forms a subspace.
- So the minimum dimension equals the number of response combinations for compatible variables.
    - In our example, C is incompatible with A,B (observed order effects). Suppose A,B are compatible.
    - There are 2 values for A and 3 values for B, so the minimum dimension =6 (all possible combinations of A, B)

# 2. BUILDING THE INITIAL STATE

- To represent the state, we need to choose a basis
- Suppose we choose a basis to represent combinations of values of A and B.

$$\psi = \begin{bmatrix} \psi_{11} \\ \vdots \\ \psi_{ij} \\ \vdots \\ \psi_{23} \end{bmatrix} \qquad \left|\psi_{ij}\right|^2 = p(A = i, B = j)$$

# 3. BUILDING THE PROJECTORS

- A projector is a linear operator that projects vectors in the Hilbert space onto the subspace spanned by the projector.

- Orthogonal projectors are Hermitian and satisfy the idempotent property

$$P = P^\dagger = P^2$$

# BUILDING THE PROJECTORS FOR A,B

- Using the A,B basis, the projector for a combination of values on A and B is simply the indicator matrix

$$
\begin{array}{cccccccc}
 & & 11 & 12 & 13 & 21 & 22 & 23 \\
P_{A=2,B=1} & = diag & [\; 0 & 0 & 0 & 1 & 0 & 0 \;],
\end{array}
$$

$$P_{A=2} = P_{A=2,B=1} + P_{A=2,B=2} + P_{A=2,B=3}$$

$$P_{B=3} = P_{A=1,B=3} + P_{A=2,B=3}$$

# BUILDING THE PROJECTORS FOR C

- In our example, the dimension of the Hilbert space is 6. But the variable C has only 4 values.

- We need to partition the 6-dimensional space into 4 subspaces for C. Suppose we use the following projectors (defined in the C basis)

$$M_{C=1} = diag\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Assigns the first 2 coordinates to C=1

$$M_{C=2} = diag\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$ Assigns the 3rd coordinate to C=2

$$M_{C=3} = diag\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$ Assigns the 4th coordinate to C=3

$$M_{C=4} = diag\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$ Assigns the last 2 coordinates to C=4

# BUILDING THE PROJECTORS FOR C

- The projectors for variable C in the A,B basis are then obtained by changing bases using unitary rotation

$$P_{C=k} = U \cdot M_k \cdot U^\dagger$$

The unitary transformation satisfies

$$U^\dagger U = I$$

# BUILDING THE PROJECTORS FOR C

- Any unitary transformation can be obtained by a matrix exponential of a Hermitian matrix called the Hamiltonian

$$H = H^\dagger$$

$$U = \exp\left(-i \cdot H\right)$$

- So the difficult task is to define the Hamiltonian

# BUILDING THE UNITARY FOR C (SKIP)

- We will build the unitary for C by a tensor product of two smaller unitary, which only requires 4 parameters!

$$H_A = \begin{bmatrix} h_1 & h_2 \\ h_2 & -h_1 \end{bmatrix}, U_A = \exp\left(-i \cdot H_A\right)$$

$$H_B = \begin{bmatrix} h_1 & h_2 & \\ h_2 & 0 & h_2 \\ & h_2 & -h_1 \end{bmatrix}, U_B = \exp\left(-i \cdot H_B\right)$$

$$U = U_B \otimes U_A$$

# 4. COMPUTING RESPONSE PROBABILITIES

$$p(C=l) = \left\| P_{C=l} \cdot S \right\|^2$$

Compatible Events

$$p(A=i, B=j) = \left\| P_{B=j} \cdot P_{A=i} \cdot S \right\|^2 = \left\| P_{A=i} \cdot P_{B=j} \cdot S \right\|^2$$

Incompatible Events

$$p(A=i, C=k) = \left\| P_{C=k} \cdot P_{A=i} \cdot S \right\|^2$$

$$p(C=k, A=i) = \left\| P_{A=i} \cdot P_{C=k} \cdot S \right\|^2$$

$$p(C=k, B=j, A=i) = \left\| P_{C=k} \cdot P_{B=j} \cdot P_{A=i} \cdot S \right\|^2$$

# ESTIMATING THE PARAMETERS FOR N-DIMENSIONAL HSM MODEL

- Each state vector has $2N$ parameters because it has N complex amplitudes. However, one phase is arbitrary and one real is fixed by length of state equal to one.
- We can start with real values. If the state is all real, then there are only $N$-1 free parameters (6-1=5 in our example)
- Hamiltonian has $N$ diagonal real values and $N(N-1)/2$ complex values. But one diagonal is arbitrary
- Real Hamiltonian has $N(N+1)/2 - 1$ free parameters (we use only 4 in our example)

# 9 PARAMETERS PRODUCES PERFECT FIT TO THE ARTIFICIAL DATA

The following parameters were used to almost perfectly fit all the data shown in Table 1.

$$\psi = \begin{bmatrix} 0.2685 \\ -0.7600 \\ 0.0883 \\ 0.3514 \\ -0.1935 \\ -0.4260 \end{bmatrix}.$$

$$H_A = \begin{bmatrix} 0.5236 & 1.5708 \\ 1.5708 & -0.5236 \end{bmatrix}.$$

$$H_B = \begin{bmatrix} -0.1047 & 3.2987 & 0 \\ 3.2987 & 0 & 3.2987 \\ 0 & 3.2987 & 0.1047 \end{bmatrix}$$

# CONCLUSIONS

- Quantum theory provides an alternative framework for developing probabilistic and dynamic models of decision making
- Provides a coherent account for puzzling violations of classical probability found in a variety of judgment and decision making studies
- Forms a new foundation for understanding widely different phenomena in decision making using a common set of axiomatic principles

"Mathematical models of cognition so often seem like mere formal exercises. Quantum theory is a rare exception. Without sacrificing formal rigor, it captures deep insights about the workings of the mind with elegant simplicity. This book promises to revolutionize the way we think about thinking."

**Steven Sloman**
Cognitive, Linguistic, and Psychological Sciences, Brown University

"This book is about why and how formal structures of quantum theory are essential for psychology - a breakthrough resolving long-standing problems and suggesting novel routes for future research, convincingly presented by two main experts in the field."

**Harald Atmanspacher**
Department of Theory and Data Analysis, Institut fuer Grenzgebiete der Psychologie und Psychohygiene e.V.

<FURTHER ENDORSEMENT TO FOLLOW>

Much of our understanding of human thinking is based on probabilistic models. This innovative book by Jerome R. Busemeyer and Peter D. Bruza argues that, actually, the underlying mathematical structures from quantum theory provide a much better account of human thinking than traditional models. They introduce the foundations for modeling probabilistic-dynamic systems using two aspects of quantum theory. The first, "contextuality," is a way to understand interference effects found with inferences and decisions under conditions of uncertainty. The second, "quantum entanglement," allows cognitive phenomena to be modeled in non-reductionist way. Employing these principles drawn from quantum theory allows us to view human cognition and decision in a totally new light. Introducing the basic principles in an easy-to-follow way, this book does not assume a physics background or a quantum brain and comes complete with a tutorial and fully worked-out applications in important areas of cognition and decision.

**Jerome R. Busemeyer** is a Professor in the Department of Psychological and Brain Sciences at Indiana University, Bloomington, USA.

**Peter D. Bruza** is a Professor in the Faculty of Science and Technology at Queensland University of Technology, Brisbane, Australia.

Cover illustration: *Spring Water* © Meijuan Lu.

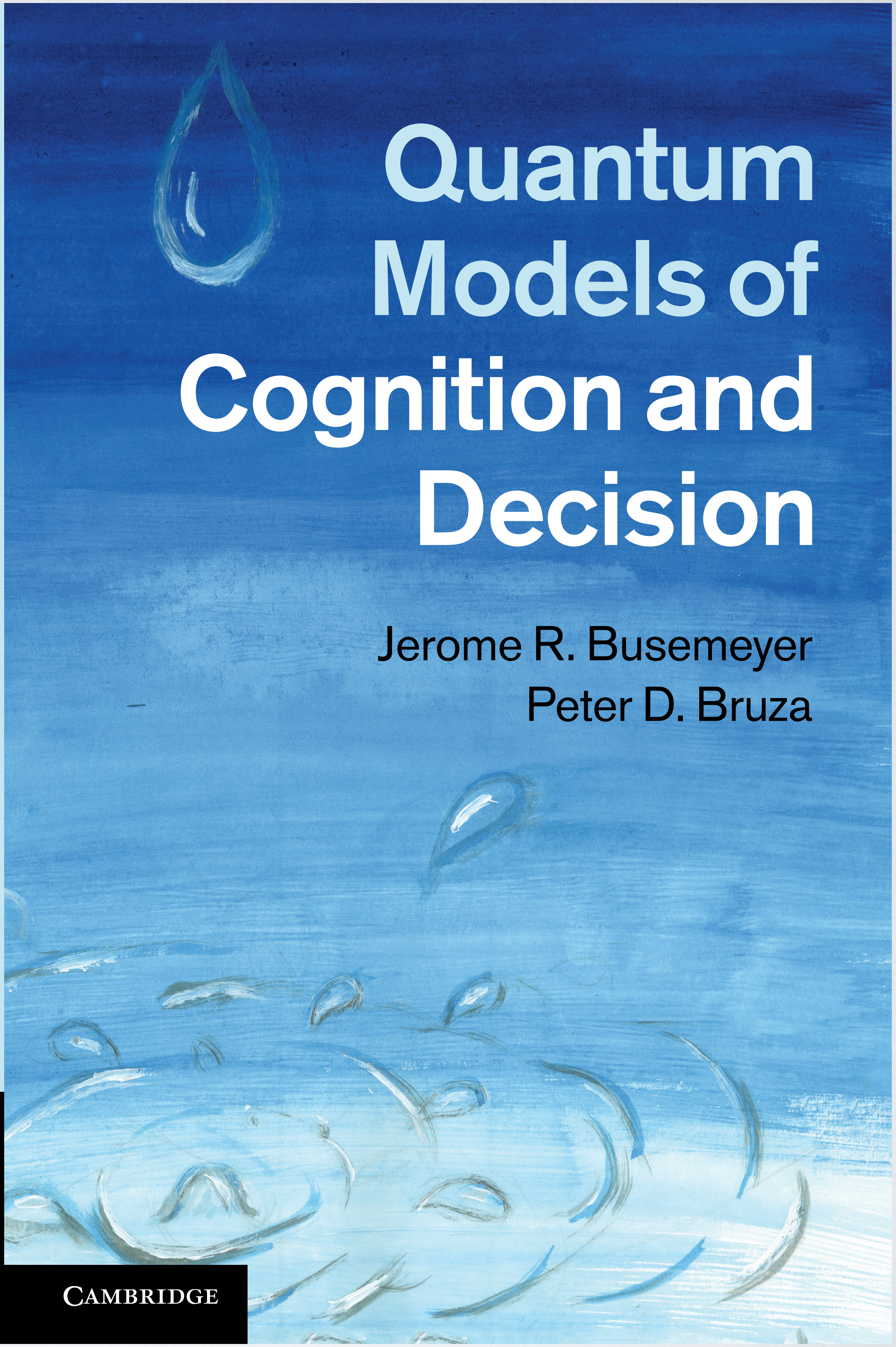
Cover designed by Hart McLeod Ltd

CAMBRIDGE
UNIVERSITY PRESS
www.cambridge.org

ISBN 978-1-107-01199-1

Busemeyer and Bruza **Quantum Models of Cognition and Decision**

CAMBRIDGE



# Quantum Models of Cognition and Decision

Jerome R. Busemeyer
Peter D. Bruza

CAMBRIDGE

# Programmed Contingencies
# ( Learned from Experience)

- Pr(Bad | Narrow) = .60
- Pr(Good | Wide) = .60
- Pr(Reward Attack | Bad ) = .70
- Pr(Reward Withdraw | Good) = .70

# 5. TWO STAGE GAMBLING PARADIGM

## VIOLATION OF "SURE THING" PRINCIPLE

Tversky & Shafir
(1992, *Psychological Science*)

Barkan & Busemeyer
(2003, *J. Behavioral Decision Making*)

Busemeyer, Wang, & Shiffrin
(2015, *Decision*)

- Participants forced to play a gamble on the first stage:
  - Equal chance to win $X or lose -$Y
- Then asked to choose whether or not to play again? on a second stage.
- Three conditions
  - Assume won first stage, play again?
  - Assume lost first stage, play again?
  - Play without knowing first stage?

## Sure Thing Principle

If you prefer to play again after a win,
and
if you prefer to play again after a loss
then
you should prefer to play regardless of the first gamble outcome

**Total Probability:**

$$p(\text{Play}|\text{Unknown}) = p(\text{win})p(\text{Play}|\text{win}) + p(\text{lose})P(\text{Play}|\text{lose})$$

# Results

Win first game: 65% chose to play again

Lose first game: 55% chose to play again

First stage Unknown: 35% chose to play again

Violation of Total Probability

# QUANTUM INTERFERENCE

$$p(G) = \left\| P_G S \right\|^2 = \left\| P_G \cdot I \cdot S \right\|^2$$

$$= \left\| P_G \cdot \left( P_W + P_L \right) \cdot S \right\|^2$$

$$= \left\| P_G \cdot P_W \cdot S + P_G \cdot P_L \cdot S \right\|^2$$

$$= \left\| P_G \cdot P_W \cdot S \right\|^2 + \left\| P_G \cdot P_L \cdot S \right\|^2 + Int$$

$$Int = \left\langle S \mid P_W P_G P_G P_L \mid S \right\rangle + \left\langle S \mid P_L P_G P_G P_W \mid S \right\rangle$$

# BAYESIAN MODEL COMPARISON

- Compared Quantum versus Prospect theories
- Both models used same number of parameters (four)
- Data based on Barkan & Busemeyer (2003)
  - N=100 participants,
  - 33 two-stage gambles per person,
  - obtained (plan, final) choice for each person.
- Computed Bayes' Factor separately for each person

# Model Comparison of Quantum Model vs. Prospect Theory-Reference Point Model

Each model used four parameters

N=100 participants