Section 2.12 Two-dimensional Space Groups

The reader will probably agree that the derivation of the primitive lattice types and space groups in three-dimensional space is complicated. Therefore, as a change of pace we present here the corresponding results in two-dimensional space. The primary advantage of working in the plane over working in three-space is the ease of visualization of lattice types. Furthermore the results are much simpler to state: There are five lattice types and only 17 isomorphism classes of two-dimensional space groups.

The definitions of two-dimensional lattices, lattice groups, basic vectors, crystallographic point groups, holohedries, space groups, etc. are mere paraphrases of definitions of the corresponding three-dimensional objects. We will not bore the reader with these details but will merely comment on the essential steps in the argument leading to the classification of the two-dimensional space groups. (Note: Two-dimensional lattices are often called nets.)

At the end of Section 2.5 the possible point groups in the plane were listed: the cyclic groups C_n and the dihedral groups D_n, $n=1,2,\cdots$. We will determine which of these are crystallographic point groups.

It follows immediately from the proof of theorem 2.7 that for any two-dimensional lattice group H there exist linearly independent vectors $\underline{b}_1, \underline{b}_2$ such that every $\underline{a} \in H$ can be written uniquely in the form

$$\underline{a} = n_1 \underline{b}_1 + n_2 \underline{b}_2, \qquad n_1, n_2 \text{ integers.}$$

The $\underline{b}_i$ are basic vectors for H and the parallelogram Q that they generate is a primitive cell. The area $A(P)$ of the cell P generated by two arbitrary linearly independent vectors $\underline{b}_1', \underline{b}_2' \in H$ is an integer multiple of $A(Q)$. Furthermore, $A(P)=A(Q)$ if and only if $\underline{b}_1', \underline{b}_2'$

are basic vectors for H. Given any $\underline{a} \in H$ there always exist basic vectors $\underline{b}_1, \underline{b}_2$ for H such that $\underline{b}_1$ lies on the line segment ℓ through $\underline{\theta}$ and $\underline{a}$. In fact we can require that $\underline{b}_1$ is the non-zero lattice vector on ℓ which is closest to $\underline{\theta}$. (We are considering the lattice group H as an additive group whose elements are vectors in R_2. The zero vector $\underline{\theta}$ is the identity element of H.)

Let K be a crystallographic point group in the plane, i.e., K is a symmetry group of a two-dimensional lattice (or net) L which fixes a given point $\underline{X} \in L$. For convenience we choose $\underline{X} = \underline{\theta}$. An argument almost identical with the proof of theorem 2.9 shows that the crystallographic restriction is still in effect. That is, K can contain non-trivial rotations only if their orders are 2, 3, 4 or 6. This proves that no point group which contains rotations of order 5 or greater than 6 can be crystallographic point groups. Thus there are only ten possible crystallographic point groups:

(12.1) $$C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6.$$

We shall show that in fact each of these ten groups is a point symmetry group of some two-dimensional lattice.

Proceeding as in the three-dimensional case we next try to determine the holohedries of two-dimensional lattices. Let L be a net based at $\underline{X} = \underline{\theta}$ and let F be its holohedry (maximal point symmetry group) at $\underline{X}$. Now the rotation $\underline{C}_\pi$ of 180° about $\underline{\theta}$ is always an element of F. Indeed, if $\underline{b}_1, \underline{b}_2$ are basic vectors for L then

(12.2) $$\underline{C}_\pi \underline{b}_1 = -\underline{b}_1, \quad \underline{C}_\pi \underline{b}_2 = -\underline{b}_2.$$

It follows that $\underline{C}_\pi$ maps L into itself, so $\underline{C}_\pi \in F$. Therefore, C_1, C_3, D_1 and D_3 cannot be holohedries since they do not contain $\underline{C}_\pi$.

(This result is the two-dimensional analogy of theorem 2.10.)

Similarly, in analogy with theorem 2.11 we can show that if F contains C_n, $n = 3, 4, 6$ then F contains D_n. To prove this we let $\underline{C}$ be the rotation through the angle $2\pi/n$, $n = 3, 4$ or 6, and $\underline{b}_1$ be a vector of minimum nonzero length in L. An argument identical with that immediately following figure 2.9 shows that $\underline{b}_1$ and $\underline{b}_2 = \underline{C}\,\underline{b}_1$ are basic vectors for L. In particular, $\|\underline{b}_1\| = \|\underline{b}_2\|$ and the angle between

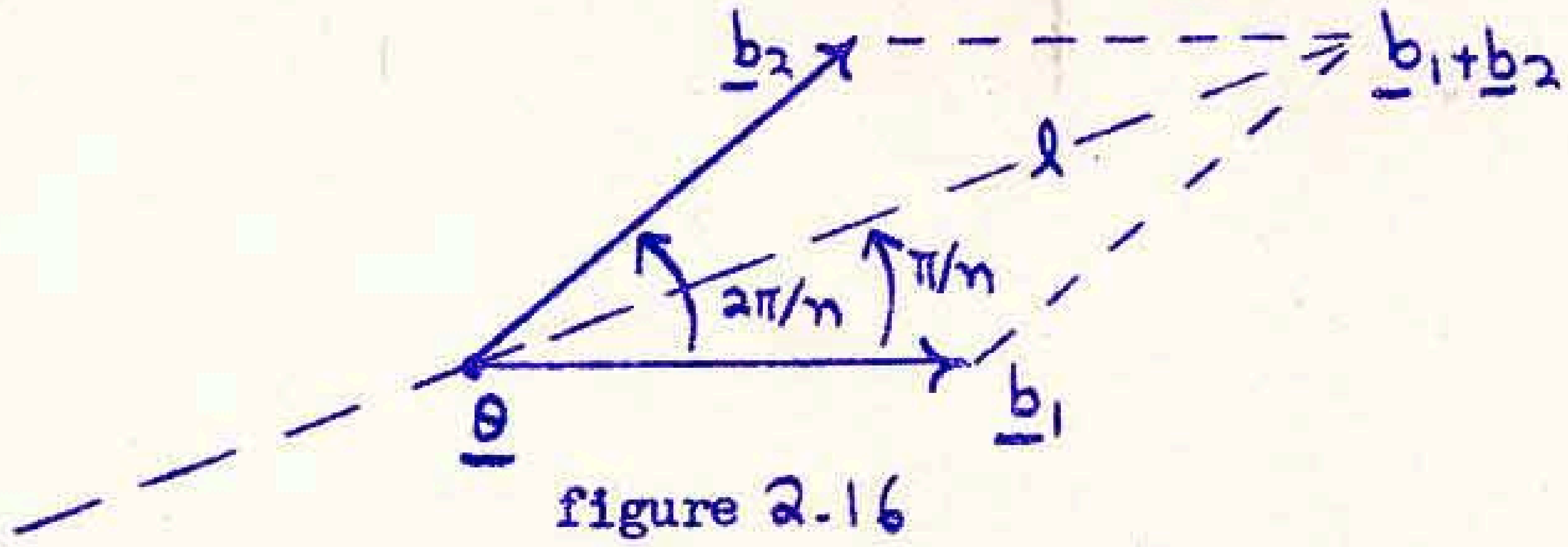

figure 2.16

$\underline{b}_1$ and $\underline{b}_2$ is $2\pi/n$. To prove that F contains D_n it is enough to show that F contains the reflection $\underline{R}$ in the line ℓ which bisects the angle between $\underline{b}_1$ and $\underline{b}_2$. (The lattice point $\underline{b}_1 + \underline{b}_2$ lies on ℓ.) Indeed, $\underline{C}$ and $\underline{R}$ generate D_n. Now it is obvious that $\underline{R}\,\underline{b}_1 = \underline{b}_2$, $\underline{R}\,\underline{b}_2 = \underline{b}_1$. Therefore, $\underline{R}$ maps L into itself and $\underline{R} \in F$. This result shows that C_3, C_4 and C_6 cannot be holohedries.

We conclude that there are at most four holohedries:

(12.3) $$C_2,\ D_2,\ D_4,\ D_6 .$$

To show that each of these groups is a holohedry it is necessary to construct a lattice which admits the group as its holohedry.

Just as in the three-dimensional case, we take up the possible holohedries one at a time and determine the lattice types (or <u>net types</u>) which correspond to them.

The <u>hexagonal</u> holohedry D_6: Suppose L is a net based at $\underline{\theta}$ which admits D_6 as its holohedry, and let $\underline{C} \in D_6$ be the rotation through $\pi/3$ about $\underline{\theta}$. As remarked above, we can choose $\underline{b}_1, \underline{C}\underline{b}_1$ as basic vectors for L where $\underline{b}_1 \in L$ is of minimal nonzero length. The net L is now completely determined! In particular we see that the lattice points $\underline{C}^m \underline{b}_1$, $m = 0, 1, \dots, 5$, form a hexagon.

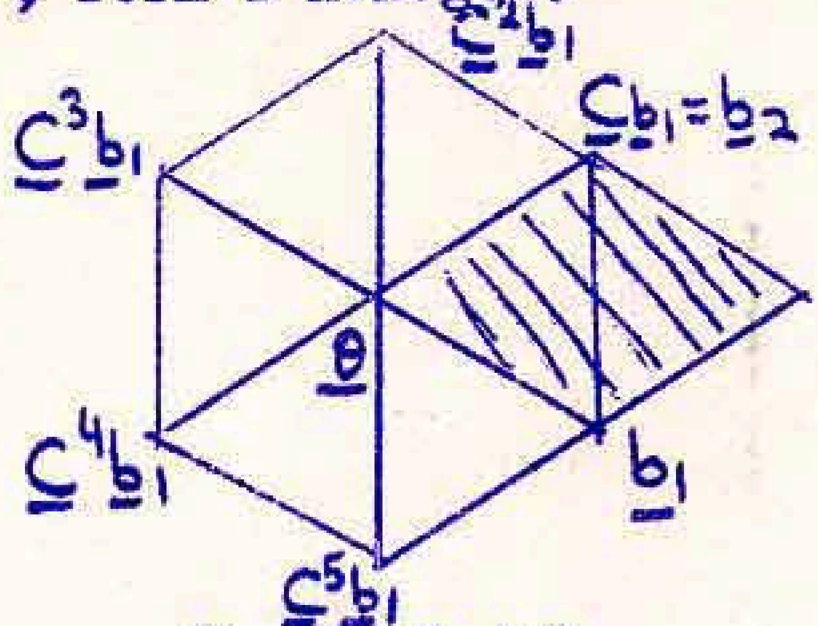

figure 2.17

It is now clear that any lattice L constructed from basic vectors $\underline{b}_1, \underline{C}\underline{b}_1$ admits D_6 as a crystallographic point group. In particular three of the two-fold axes ℓ_1, ℓ_2, ℓ_3 of D_6 pass through $\underline{\theta}$ and $\underline{b}_1, \underline{C}\underline{b}_1, \underline{C}^2\underline{b}_1$, respectively. The remaining three two-fold axes bisect the angles between adjacent ℓ_i. The primitive cell generated by $\underline{b}_1, \underline{C}\underline{b}_1$ is denoted N_h. Type N_h nets must necessarily admit D_6 as a holohedry since D_6 is not a proper subgroup of any of the possible holohedries in the list (12.3). Clearly, type N_h nets can be uniquely determined by the single parameter $\|\underline{b}_1\|$. We have shown that the hexagonal crystal system contains the single net type N_h.

The tetragonal holohedry D_4: Suppose L is a net based at $\underline{\theta}$ which admits D_4 as its holohedry and let $\underline{C} \in D_4$ be the rotation through $\pi/2$ about $\underline{\theta}$. Then we can choose $\underline{b}_1, \underline{C}\underline{b}_1$ as basic vectors for L where $\underline{b}_1$ is a net vector of minimal nonzero length. Clearly the primitive cell N_q generated by these basic vectors is a square. Thus L is completely determined. The reader can easily check that any type N_q net admits D_4 as a symmetry group. Furthermore, D_4 must be the holohedry of a type N_q net since D_4 is not a proper subgroup of any of the possible holohedries (12.3). Type N_q nets can be described by the single parameter $\|\underline{b}_1\|$. We have shown that the tetragonal crystal system contains the single net type N_q.

The <u>orthorhombic</u> holohedry D_2: Suppose L is a net with holohedry D_2 and let $\underline{C} \in D_2$ be the rotation through π about $\underline{\theta}$. Let ℓ be one of the two-fold reflection axes and $\underline{R} \in D_2$ the reflection about ℓ. Necessarily, ℓ passes through at least one lattice point. Indeed if $\underline{a} \in L$ then $\underline{R}\,\underline{a} + \underline{a} = \underline{b}$ is a lattice point such that $\underline{R}\,\underline{b} = \underline{b}$. Therefore, we can choose basic vectors $\underline{b}_1, \underline{b}_2$ for L such that $\underline{b}_1$ lies on ℓ. Write

(12.4) $$\underline{b}_2 = \underline{u} + \underline{v}$$

where $\underline{u}$ is parallel to $\underline{b}_1$ and $\underline{v} \perp \underline{b}_1$. Clearly

$$\underline{R}\,\underline{b}_2 = \underline{u} - \underline{v}$$

so

(12.5) $$\underline{R}\,\underline{b}_2 + \underline{b}_2 = 2\underline{u} \in L.$$

It follows that $2\underline{u} = n_1 \underline{b}_1$ where n_1 is an integer, or $\underline{u} = \frac{n_1}{2}\underline{b}_1$. We can subtract integral multiples of $\underline{b}_1$ from $\underline{b}_2$ without changing the area of the primitive cell. Therefore, we can assume that either $n_1 = 0$ or $n_1 = 1$. It now follows from (12.5) that there are two possibilities: Either 1) $\underline{R}\,\underline{b}_2 = -\underline{b}_2$ in which case $\underline{b}_1 \perp \underline{b}_2$ or 2) $\underline{R}\,\underline{b}_2 = \underline{b}_1 - \underline{b}_2$ in which case $\|\underline{b}_1 - \underline{b}_2\| = \|\underline{b}_2\|$.

We take case 1) first. The basic vectors generate a rectangle N_r. Since $\underline{C}$ and $\underline{R}$ generate D_2 it is clear that a type N_r net admits D_2

as a symmetry group. In order that D_2 be the holohedry it is necessary and sufficient that N_γ not be a square. (Otherwise the holohedry would be D_4.) A type N_γ net is determined by two parameters: $\|\underline{b}_1\|$ and $\|\underline{b}_2\|$.

In case 2) the basic vectors $\underline{b}_1 - \underline{b}_2$ and $\underline{b}_2$ generate a rhombus N_ν. (Note: N_ν is a primitive cell since it has the same area as the primitive cell generated by $\underline{b}_1$ and $\underline{b}_2$.)

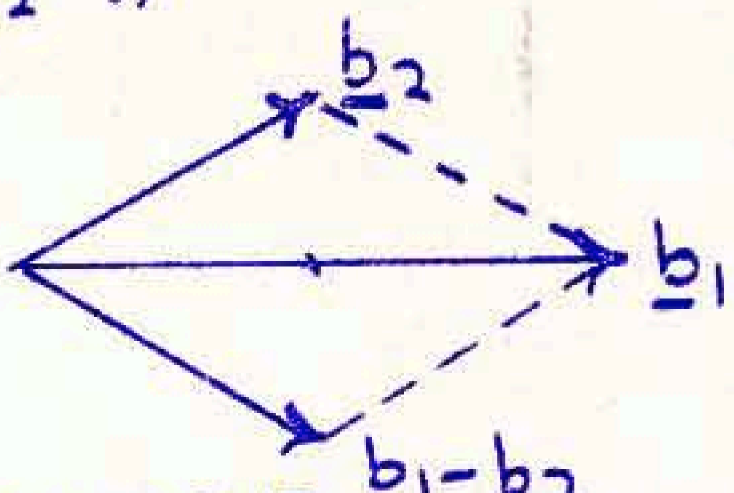

figure 2.18

It is now clear that a type N_ν net admits D_2 as a symmetry group. Furthermore D_2 is the holohedry as long as the angle between $\underline{b}_1 - \underline{b}_2$ and $\underline{b}_2$ is not $\pi/2$ or $\pi/3$. A type N_ν net is determined by two parameters: $\|\underline{b}_2\|$ and the angle between $\underline{b}_1 - \underline{b}_2$ and $\underline{b}_2$.

We have shown that the orthorhombic crystal system contains two net types, N_γ and N_ν.

The <u>monoclinic</u> holohedry C_2: Every net admits C_2 as a symmetry group. Thus, C_2 is a holohedry for those nets which do not belong to the other lattice types listed above. Let L be such a net. We choose basic vectors $\underline{b}_1, \underline{b}_2$ for L of minimal nonzero length. Let N_a be the corresponding primitive cell. By comparing with the lattice types listed above we see that a type N_a net admits C_2 as its holohedry if and only if $\|\underline{b}_1\| \neq \|\underline{b}_2\|$ and $\underline{b}_1$ is not perpendicular to either $\underline{b}_2$ or $2\underline{b}_2 - \underline{b}_1$. A type N_a net is determined by three parameters: $\|\underline{b}_1\|$, $\|\underline{b}_2\|$ and the angle between $\underline{b}_1$ and $\underline{b}_2$.

figure 2.19

	crystal system (holohedry)		crystal classes	net types
1.	monoclinic	C_2	C_2, C_1	N_a
2.	orthorhombic	D_2	D_2, D_1	N_r, N_v
3.	tetragonal	D_4	D_4, C_4	N_q
4.	hexagonal	D_6	D_6, C_6, D_3, C_3	N_h

The procedure for deducing two-dimensional space groups G from the above table is almost identical (but simpler) than the procedure for three-dimensional space groups. In particular the material following equation (10.3) is applicable if we read $O(2), E(2), T(2)$ for $O(3), E(3), T(3)$.

We now know that the ten groups (12.1) are indeed crystallographic point groups, since they are all subgroups of holohedries. Each of these groups defines a crystal class. Moreover, the crystal class K is assigned to the crystal system with the smallest holohedry F containing K. This assignment of crystal classes to crystal systems is indicated in figure 2.19.

Any two-dimensional symmorphic space group G is obtained by choosing a net type H, i.e., a two-dimensional lattice group corresponding to a certain net type, and a crystal class K leaving H invariant. G is the semi-direct product of H and K. In the monoclinic system there are two crystal classes and one net type, which yield two symmorphic groups. The orthorhombic system has two crystal classes and two net types, yielding four symmorphic groups. Continuing in this way we get twelve symmorphic groups. There is actually one more, due to the fact that the crystal class D_3 can act on the hexagonal lattice in two distinct ways. Either the three reflection axes of D_3 lie along $\underline{b}_1, \underline{C}\underline{b}_1$ and $\underline{C}^2\underline{b}_1$ in figure 2.17 or they bisect the angles between adjacent vectors $\underline{C}^j\underline{b}_1$. Thus, there are a total of thirteen two-dimensional symmorphic groups.

We now proceed to the deduction of the non-symmorphic groups. Let G be a space group and let $n = 1, 2, 3, 4$ or 6 be the order of the rotation axis of its crystal class K. If $\underline{C}$ is the rotation of angle $2\pi/n$ in K, there must exist a unique element $\{\underline{a}_1, \underline{C}\}$ in G such that

$$\underline{a}_1 = \alpha_1 \underline{b}_1 + \alpha_2 \underline{b}_2, \quad 0 \le \alpha_1, \alpha_2 < 1,$$

where $\underline{b}_1, \underline{b}_2$ are basic vectors for the net type of G. It follows from the identity

(12.6) $$T_{\underline{a}} \{\underline{a}_1, \underline{C}\} T_{\underline{a}}^{-1} = \{\underline{a}_1 + \underline{a} - \underline{C}\underline{a}, \underline{C}\} = \{\underline{\theta}, \underline{C}\}$$

where

(12.7) $$\underline{a} = -\tfrac{1}{2}(\underline{a}_1 + \underline{C}\underline{a}_1)$$

that G is conjugate to a space group $G' = T_{\underline{a}} G T_{\underline{a}}^{-1}$. Thus, in the conjugacy class to which G belongs there is a space group G' containing the elements $\{\underline{\theta}, \underline{C}^m\}$, $m = 1, 2, \cdots, n-1$. For simplicity we will always choose the space group G' in each conjugacy class. It is an immediate consequence of this result that all space groups belonging to the crystal classes C_1, C_2, C_3, C_4 and C_6 are symmorphic. Only space groups G belonging to the crystal classes D_n, $n = 1, 2, 3, 4$ or 6, can possibly be non-symmorphic. Let G belong to the crystal class D_n and let $\underline{R} \in D_n$ be a reflection. Then there exists a unique element $\{\underline{a}_2, \underline{R}\}$ in G such that

(12.8) $$\underline{a}_2 = \alpha_1 \underline{b}_1 + \alpha_2 \underline{b}_2, \quad 0 \leq \alpha_1, \alpha_2 < 1.$$

The exact definitions of $\underline{b}_1, \underline{b}_2$ for the various net types have been described above.

Since

(12.9) $$\{\underline{a}_2, \underline{R}\}^2 = \{\underline{R}\,\underline{a}_2 + \underline{a}_2, \underline{\underline{E}}\} \in H$$

it follows that there exist integers n_1, n_2 so that

(12.10) $$\underline{R}\,\underline{a}_2 + \underline{a}_2 = n_1 \underline{b}_1 + n_2 \underline{b}_2.$$

We will compute $\underline{a}_2$.

First consider the crystal class D_1 and net type N_γ. We can assume $\underline{R}\underline{b}_1 = \underline{b}_1$ and $\underline{R}\underline{b}_2 = -\underline{b}_2$. (Recall $\underline{b}_1 \perp \underline{b}_2$ for N_γ.) Then

$$\underline{R}\,\underline{a}_2 + \underline{a}_2 = 2\alpha_1 \underline{b}_1 = n_1 \underline{b}_1 + n_2 \underline{b}_2,$$

so $\alpha_1 = 0$ or $1/2$. Furthermore, the identity

(12.11) $$\underline{T}_{\underline{a}} \{\alpha_1 \underline{b}_1 + \alpha_2 \underline{b}_2, \underline{R}\} \underline{T}_{\underline{a}}^{-1} = \{\underline{a}_2 - \underline{R}\underline{a} + \underline{a}, \underline{R}\} = \{\alpha_1 \underline{b}_1, \underline{R}\}$$

where $\underline{a} = -\frac{1}{2}\alpha_2 \underline{b}_2$ shows that we can assume $\alpha_2 = 0$, i.e., G is conjugate to a space group G' with the same net group H and containing $\{\alpha_1 \underline{b}_1, \underline{R}\}$. Therefore we have exactly two possible space groups of class D_1 and type N_γ:

(12.12) $$C_s^{I}: \{\underline{0}, \underline{R}\} \qquad C_s^{II}: \{\tfrac{1}{2}\underline{b}_1, \underline{R}\}.$$

Note that each of these space groups ~~are~~ is generated by the lattice group H of type N_γ and the element $\{\alpha_1 \underline{b}_1, \underline{R}\}$. The group C_s^{I} is symmorphic. However, the non-symmorphic group C_s^{II} is new.

Leave the crystal class fixed, but change the net type to N_δ. We can now assume that $\underline{R}\underline{b}_1 = \underline{b}_1$, $\underline{R}\underline{b}_2 = \underline{b}_1 - \underline{b}_2$. Thus,

(12.13) $$\underline{R}\,\underline{a}_2 + \underline{a}_2 = (2\alpha_1 + \alpha_2)\underline{b}_1 \in H.$$

Furthermore, applying a conjugacy transformation of the form (12.11) we find

$$\underline{T}_{\underline{a}} \{\alpha_1 \underline{b}_1 + \alpha_2 \underline{b}_2, \underline{R}\} \underline{T}_{\underline{a}}^{-1} = \{\underline{a}_2 - \underline{R}\underline{a} + \underline{a}, \underline{R}\}$$

[illegible] $, \underline{R}\}$

if $\underline{a} = \alpha_1 \underline{b}_1$. It follows from (12.13) applied to $\underline{a}_2'$ that $\underline{a}_2' \in H$. Thus, G is conjugate to a space group containing $\{\underline{0}, \underline{R}\}$ and the only space group of class D_1 and type N_γ is symmorphic.

Next consider a space group G of class D_2 and type N_γ. Let $\underline{C}$ be the rotation through π and $\underline{R} \in D_2$ a reflection. We can assume

$$\underline{C}\,\underline{b}_i = -\underline{b}_i, \ i=1,2, \quad \underline{R}\underline{b}_1 = \underline{b}_1, \ \underline{R}\underline{b}_2 = -\underline{b}_2 .$$

Furthermore, as remarked above we can assume $\underline{C}' = \{\underline{0}, \underline{C}\} \in G$. Choose $\underline{R}' = \{\underline{a}_1, \underline{R}\} \in G$ with $\underline{a}_1 = \alpha_1 \underline{b}_1 + \alpha_2 \underline{b}_2$, $0 \le \alpha_1, \alpha_2 < 1$. The relation $\underline{R}^2 = \underline{E}$ implies $\underline{R}\,\underline{a}_1 + \underline{a}_1 \in H$ or $2\alpha_1 \underline{b}_1 \in H$. Similarly, the relation $(\underline{C}\underline{R})^2 = \underline{E}$ or $(\{\underline{0}, \underline{C}\}\{\underline{a}_1, \underline{R}\})^2 \in H$ implies $\underline{C}\underline{R}\,\underline{a}_1 + \underline{a}_1 \in H$ or $2\alpha_2 \underline{b}_2 \in H$.

There are four possibilities:

1) $\alpha_1 = \alpha_2 = 0$ 2) $\alpha_1 = 1/2, \ \alpha_2 = 0$

3) $\alpha_1 = 0, \ \alpha_2 = 1/2$ 4) $\alpha_1 = \alpha_2 = 1/2$

Note that G is uniquely determined by $\underline{C}'$ and $\underline{R}'$. Case 1) yields the symmorphic group C_{2v}^{I}. Cases 2) and 3) obviously yield isomorphic groups (since there is another reflection in D_2 fixing $\underline{b}_2$). We select case 2) and obtain the non-symmorphic group

$$C_{2v}^{III} : \{\tfrac{1}{2}\underline{b}_1, \underline{R}\}$$

Case 4) yields the non-symmorphic group

$$C_{2v}^{II} : \{\tfrac{1}{2}(\underline{b}_1 + \underline{b}_2), \underline{R}\}.$$

If G is of class D_2 and type N_γ we can assume

$$\underline{C}\,\underline{b}_i = -\underline{b}_i, \ i=1,2, \quad \underline{R}\underline{b}_1 = \underline{b}_1, \ \underline{R}\underline{b}_2 = \underline{b}_1 - \underline{b}_2,$$

and $\underline{C}' = \{\underline{0}, \underline{C}\} \in G$, $\underline{R}' = \{\underline{a}_2, \underline{R}\} \in G$ with

$$\underline{a}_2 = \alpha_1 \underline{b}_1 + \alpha_2 \underline{b}_2, \ 0 \le \alpha_1, \alpha_2 < 1.$$

The relation $(\underline{C}\underline{R})^2 = \underline{E}$ implies $\underline{C}\underline{R}\,\underline{a}_2 + \underline{a}_2 = -(\alpha_1 + \alpha_2)\underline{b}_1 + \alpha_2 \underline{b}_2 \in H$.

Thus, $\alpha_2 = \alpha_1 = 0$ and G is symmorphic.

Next, suppose G is of class D_3 and (necessarily) type N_h. Let $\underline{C} \in D_3$ be the rotation through $2\pi/3$. Then we can assume $\underline{C}' = \{\underline{0}, \underline{C}\} \in G$ and $\underline{C}\underline{b}_1 = \underline{b}_2 - \underline{b}_1$, $\underline{C}\underline{b}_2 = -\underline{b}_1$. We can choose a reflection $\underline{R} \in D_3$ such that it acts on N_h in one of two possible ways: Either 1) $\underline{R}\underline{b}_1 = \underline{b}_1$, $\underline{R}\underline{b}_2 = \underline{b}_1 - \underline{b}_2$ or 2) $\underline{R}\underline{b}_1 = \underline{b}_2$, $\underline{R}\underline{b}_2 = \underline{b}_1$. (This ambiguity is due to the fact that D_3 can act as a crystallographic point group in a type N_h-net in two distinct ways.) Suppose $\underline{R}' = \{\underline{a}_1, \underline{R}\} \in G$ with $\underline{a}_1 = \alpha_1\underline{b}_1 + \alpha_2\underline{b}_2$, $0 \le \alpha_1, \alpha_2 \le 1$, and consider case 1). The relation $(\underline{R}'\underline{C}')^2 \in H$ arising from $(\underline{R}\underline{C})^2 = \underline{E}$, implies $\underline{R}\underline{C}\underline{a}_1 + \underline{a}_1 = (2\alpha_1 - \alpha_2)\underline{b}_1 + (\alpha_2 - 2\alpha_1)\underline{b}_2 \in H$. Furthermore, $\underline{R}'^2 \in H$ implies $\underline{R}\underline{a}_1 + \underline{a}_1 = (2\alpha_1 + \alpha_2)\underline{b}_1 \in H$. These two relations imply $2\alpha_1 = 0, 1$ and $\alpha_2 = 0$. Finally $(\underline{R}\underline{C}^2)^2 = \underline{E}$ implies $\underline{R}\underline{C}^2\underline{a}_1 + \underline{a}_1 = (\alpha_1 + 2\alpha_2)\underline{b}_2 \in H$, so $\alpha_1 = \alpha_2 = 0$. Thus G is symmorphic. Case 2) can be treated in the same way and yields only a single symmorphic group.

Now consider a space group of class D_4 and type N_q. If $\underline{C} \in D_4$ is the rotation through $\pi/2$ and $\underline{R} \in D_4$ is a reflection, we can assume

$$\underline{C}\underline{b}_1 = \underline{b}_2,\ \underline{C}\underline{b}_2 = -\underline{b}_1,\ \underline{R}\underline{b}_1 = \underline{b}_1,\ \underline{R}\underline{b}_2 = -\underline{b}_2 .$$

Furthermore, we can assume $\{\underline{0}, \underline{C}\} \in G$ and $\{\underline{a}_1, \underline{R}\} \in G$ with $\underline{a}_1 = \alpha_1\underline{b}_1 + \alpha_2\underline{b}_2$, $0 \le \alpha_1, \alpha_2 < 1$. The relation $\underline{R}^2 = \underline{E}$ yields $\underline{R}\underline{a}_1 + \underline{a}_1 = 2\alpha_1\underline{b}_1 \in H$, so $2\alpha_1 = 0, 1$. Similarly, the relation $(\underline{R}\underline{C})^2 = \underline{E}$ yields $\underline{R}\underline{C}\underline{a}_1 + \underline{a}_1 = (\alpha_1 - \alpha_2)(\underline{b}_1 - \underline{b}_2) \in H$. There are two possibilities:

$$C_{4v}^{I}: \{\underline{0}, \underline{R}\} \quad \alpha_1 = \alpha_2 = 0$$
$$C_{4v}^{II}: \{\tfrac{1}{2}(\underline{b}_1 + \underline{b}_2), \underline{R}\} \quad \alpha_1 = \alpha_2 = 1/2$$

The group C_{4v}^{I} is symmorphic, but C_{4v}^{II} is not. (It is easy to verify that, in fact, C_{4v}^{II} is a uniquely determined space group.)

The last space group G to consider is one of class D_6 and type N_h.

Let $\underline{C} \in D_6$ be the rotation through $\pi/3$ and $\underline{R} \in D_6$ be a reflection. We can assume $\{\underline{O}, \underline{C}\}, \{\underline{a}_1, \underline{R}\} \in D_6$ with $\underline{a}_1 = \alpha_1 \underline{b}_1 + \alpha_2 \underline{b}_2$, $0 \leq \alpha_1, \alpha_2 < 1$. Furthermore, we can assume that the action of $\underline{C}$ and $\underline{R}$ on the basis vectors is

$$\underline{C}\,\underline{b}_1 = \underline{b}_2\,,\ \underline{C}\,\underline{b}_2 = \underline{b}_2 - \underline{b}_1\,,\ \underline{R}\,\underline{b}_1 = \underline{b}_2\,,\ \underline{R}\,\underline{b}_2 = \underline{b}_1\,.$$

The relation $\underline{R}^2 = \underline{E}$ implies $\underline{R}\underline{a}_1 + \underline{a}_1 = (\alpha_1 + \alpha_2)(\underline{b}_1 + \underline{b}_2) \in H$. Similarly, the relation $(\underline{R}\underline{C})^2 = \underline{E}$ implies $\underline{R}\underline{C}\underline{a}_1 + a_1 = (2\alpha_1 + \alpha_2)\underline{b}_1 \in H$. The only solution of these expressions is $\alpha_1 = \alpha_2 = 0$. Therefore, G is symmorphic.

In conclusion, we have shown that in addition to the thirteen isomorphism classes of symmorphic two-dimensional space groups, there are four isomorphism classes of non-symmorphic groups. We have not explicitly verified that no two classes of these seventeen space groups are isomorphic, but this is easy.

From one point of view the seventeen two-dimensional space groups constitute the possible wallpaper patterns. That is, every wallpaper pattern admits one of these seventeen groups as its maximal symmetry group. Consult references [ ], [ ] for graphic illustrations of the patterns produced by each symmetry group.