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CAN A GAME BE QUANTUM?

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A game in which acts of the participants do not have an adequate description in terms of the Boolean logic and classical theory of probabilities is considered. A model of the game interaction is constructed on the basis of a nondistributive orthocomplemented lattice. Mixed strategies of the participants are calculated using probability amplitudes according to the rules of quantum mechanics. A scheme of quantization of the payoff function is proposed and an algorithm for the search of the Nash equilibrium is given. It is shown that in contrast with the classical case, in the quantum situation a discrete set of equilibria is possible. Bibliography: 19 *titles.*

It often occurs that mathematical structures find natural applications somewhere out of their origination. The formalism of quantum mechanics is not an exception to this rule. Applied initially to the microworld, this formalism can be used for modeling some macroscopic interactions with an element of indeterminacy. Thus, in the recent papers [1–3] a connection of quantum mechanics with decision problems is discussed. In spite of unclearness of some points, numerous publications of the last year give us an evidence of the beginning of the "quantum attack" on the game theory; even the "English Auction" has been quantized. However, in the majority of publications the quantum features are conditioned by the use of microparticles or quantum computers based on microparticles. At the same time, one can say that the class of phenomena finding their natural explanations in terms of principles of quantum mechanics is much wider. In the pioneering book [5] written before the "epoch of quantization of games and decisions" as well as in a more recent book [6], it was shown that the quantum mechanical formalism can be applied to description of macroscopic systems if the *distributive* property for random events is broken. In the physics of the microworld, nondistributivity has an objective status and must be present in principle. For macroscopic systems, the nondistributivity of random events expresses some specific case of the observer's "ignorance" different from the standard probabilistic interpretation.

In the present paper, quantum mechanical formalism is applied to the analysis of a conflict interaction, the mathematical model for which is an antagonistic game of two persons. The game is based on a generalization of examples of macroscopic automata simulating the behavior of some quantum systems considered earlier in [7, 8]. A special feature of the game considered is that the players acts are in contradiction with the usual logic – disjunction and conjunction do not satisfy the distributive law. As a consequence, the classical probability interpretation of the mixed strategy is broken: the sum of the probabilities for alternate outcomes may be larger than one. The cause of this break of the basic property of the probability is in the *nondistributivity of the logic*. The quantum nature of the game is manifested in the logic of behavior of the players described by a nondistributive ortholattice coinciding with that of a quantum system with spin one half. This leads to new "quantum" rules for the calculation of the average profit and to a new representation of the mixed strategy, the role of which is played by the "wave function;" this function is a normalized vector of a finite-dimensional Hilbert space. Calculations of probabilities are performed according to the standard rules of quantum mechanics. In contrast with the examples of quantum games considered in [9–11] (where the "quantum" nature of a game was conditioned by microparticles or quantum computers based on them), in our case we deal with a *macroscopic* game, the quantum nature of which has nothing to do with microparticles. This gives us the hope that our example is one of many analogous situations in biology, economics, etc, where the formalism of quantum mechanics can be used.

1. Where were you, Bob?

The game "Wise Alice" formulated in our paper is a modification of the well-known game in which each of the participants names one of some previously considered objects. If the results differ, one of the players wins from the other one some agreed sum of money. The participants of our game, called Alice and Bob, have a square box in which a ball is located. Bob puts his ball at one of the corners of the box but does not tell his partner which corner was chosen. Alice must guess in which corner Bob has put the ball. The rules of the game allow Alice to ask Bob questions assuming a two-valued answer "yes" or "no." It is assumed that Bob is honest and always tells the truth. In the case of a "yes" answer, Alice is satisfied; in the opposite case, she asks Bob to pay

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her some compensation. However, in contrast with other similar games [12], the rules of this game (see Fig. 1) have one specific feature: *Bob has the possibility to move the ball to any of the adjacent vertices of the square after Alice asks her question.*

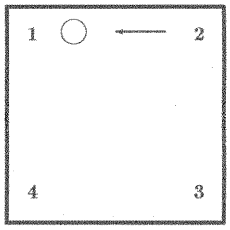

FIG. 1. Bob's ball moves into the place asked by Alice.

This additional condition decisively changes the behavior of Bob, making him to become active under the influence of Alice's questions. Since negative answers are not profitable for him, in all possible cases he moves the ball to a convenient adjacent vertex. Thus, being at vertices 2 or 4 and getting from Alice the question "Are you in vertex 1?" Bob quickly puts the ball to the asked vertex and honestly answers "yes." However, if the Bob's ball was initially at vertex 3, he cannot escape the negative answer notwithstanding to what vertex he moves the ball, and he fails. Let us note that in this case Alice not only gets the profit but also obtains the *exact information* on the initial position of the ball; Bob's honest answer immediately reveals his initial position.

2. Equilibrium is when everybody is satisfied!

The interaction of our players can be described by a 4×4 matrix (h_{ik}) representing the payoffs of Alice in each of the possible 16 game situations.

TABLE 1. THE PAYOFF-MATRIX OF ALICE

$A \backslash B$	**1**	**2**	**3**	**4**
1?	0	0	a	0
2?	0	0	0	b
3?	c	0	0	0
4?	0	d	0	0

Here a, b, c, and $d > 0$ are her payoffs in the situations where Bob cannot answer her questions affirmatively. The main problem of game theory [13, 14] is to find the so-called *points of equilibrium or saddle points*, i.e., game situations (j_0, k_0) satisfying the von Neumann–Nash system of inequalities:

$$h_{j,k_0} \leq h_{j_0,k_0} \leq h_{j_0,k}$$

for all the players at once. Strategies forming the equilibrium situation are optimal in the following sense: to each participant, they provide the maximum of what he/she can get independently of the acts of the other partner. It is easy to see that our game does not have equilibrium points. Thus, neither Alice nor Bob can have strategies rational at a separate turn of the game. In spite of the absence of a rational choice at each turn of the game, for game repeated many times, some optimal lines of behavior can be found. To find such lines one must, following von Neumann [14], look for the so-called mixed generalization of the game. In this generalized game, the choice is made between *mixed strategies*, i.e., probability distributions of usual strategies (they are called "pure" in contrast with mixed strategies). As the criterion for the choice of optimal mixed strategies, one takes the mathematical expectation value of the payoff which shows how much one can win on average repeating the game many times. The optimal mixed strategies for Alice and Bob are defined as probability distributions on the sets of pure strategies $x^0 = (x_i^0)_{1 \leqslant i \leqslant 4}$ and $y^0 = (y_j^0)_{1 \leqslant j \leqslant 4}$ such that for all distributions of x, y, the von Neumann–Nash inequalities are valid:

$$\mathcal{H}(x, y^0) \leqslant \mathcal{H}(x^0, y^0) \leqslant \mathcal{H}(x^0, y), \tag{1}$$

where $\mathcal{H}(x,y)$, *the payoff function* of Alice, is the expectation value of her wins:

$$\mathcal{H}(x,y) = \sum_{j,k=1}^{4} h_{jk} x_j y_k. \tag{2}$$

The existence of an equilibrium in mixed strategies is based on the main theorem of matrix games theory (von Neumann's theorem). To find these strategies, one must solve a pair of dual problems of linear programming, and this is easy. The only remaining question is: do optimal strategies describe correctly the behavior of Bob and Alice in their game with a ball? From one side, it seems that each partner makes his choice independently of the other partner; from the other side, their actions are not totally independent: Bob is reacting on questions of Alice.

3. The classical "Foolish Alice"

In the classical matrix game theory, optimal strategies of the players are totally defined by their interests. All other characteristics of the participants of the game are totally ignored. To overcome this simplification of the von Neumann's game theory, one has to search for other concepts of equilibrium (for example, that of von Stackelberg [15]) or to study the influence of psychological relations on the outcomes of games [16]. Our attention will be concentrated not on the psychological but on the *logical* aspect of the conflict interaction of players. Before discussing the logical nuances, let us pay attention to the fact that the payoff matrix in Table 1 does not give us the full information about the *rules* of the game and interaction of the players. To clarify the situation, let us consider a totally different (from the point of view of the behavior of players) game "Foolish Alice" with the same payoff matrix as for the game with a ball. Alice and Bob decide to meet at a corner of a big four-corner building but do not agree at which corner. As usual, Bob comes first. If Alice comes to a corner from which she can see Bob, she is satisfied; in the opposite case, she thinks that he did not come and retires being insulted. The next day, Bob, in order to calm her, must give her some expensive present. In contrast with the previous game, each participant of this game has a *passive* position. If Bob does not see Alice, he has no reasons to go from one corner of the building to another one since he does not know whether she came. How to distinguish these two identical (with respect to the structure of the payoff) games?

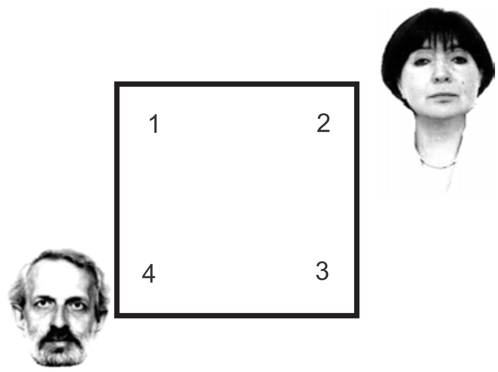

Fig. 2. Alice does not see Bob and is very dissatisfied.

To clarify the difference between the two games and to distinguish between the "wise Alice" and the "foolish" one, we introduce some notation making the difference obvious. Encode the strategies of Alice and Bob by vectors consisting of zeros and ones:

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \quad \text{and} \quad \beta = (\beta_1, \beta_2, \beta_3, \beta_4)$$

so that the component equal to one denotes the application of a pure strategy. It is obvious that

$$\sum_{j=1}^{4} \alpha_j = \sum_{j=1}^{4} \beta_j = 1 \quad \text{and} \quad \alpha_j \cdot \alpha_k = \beta_j \cdot \beta_k = 0, \;\; j \neq k.$$

The profit of Alice *in one turn* of any of the games considered is

$$\mathcal{H}(\alpha,\beta)=\sum_{j,k=1}^{4}h_{jk}\alpha_j\beta_k. \tag{3}$$

Thus, in any separate turn of the game, the "wise" Alice does not differ from the "foolish" one. The difference occurs in the *behavior* when the game is repeated many times. The source of the difference is in a different method of calculation of the *average* payoff.

Let us consider the situation in detail. At first, let us take the classical case of interaction. In the case of the "foolish" Alice, the initial strategies of Bob are not correlated with the strategies of his partner, so that

$$E(\alpha_j\cdot\beta_k)=E\alpha_j\cdot E\beta_k,$$

and the averaging of the payoff gives us the following well-known classical expression:

$$E\mathcal{H}(\alpha,\beta)=\sum_{j,k=1}^{4}h_{jk}E\alpha_j\cdot E\beta_k=\sum_{j,k=1}^{4}h_{jk}x_jy_k=\mathcal{H}(x,y),$$

where x_j and y_k are frequencies of the corresponding pure strategies. For our payoff matrix, one obtains the following expression of the payoff function for Alice:

$$\mathcal{H}(x,y)=ax_1y_3+bx_2y_4+cx_3y_1+dx_4y_2. \tag{4}$$

Simple calculations show that there is only one equilibrium point, and the mixed strategies of Alice and Bob are found as follows:

$$x=(\mu a^{-1},\ \mu b^{-1},\ \mu c^{-1},\ \mu d^{-1})\quad\text{and}\quad y=(\mu c^{-1},\ \mu d^{-1},\ \mu a^{-1},\ \mu b^{-1}),$$

where $\mu=(a^{-1}+b^{-1}+c^{-1}+d^{-1})^{-1}$ is the price or the value of the game, i.e., the average profit of Alice at the equilibrium situation. Thus, the optimal frequency of Bob's visits at this or that corner of the building is inversely proportional to the sum of money which he must give to his girlfriend. The optimal strategy of Alice is more sophisticated: she must not be very greedy and has to come more frequently to the places where her friend does not have to pay her too much.

Let us write an expression of the payoff function of the "foolish" Alice in a somehow different form, which is useful for our further comparison her with the "wise" girl. Consider random events A_{13}, $A_{24}\subset\mathcal{S}_A$ and B_{13}, $B_{24}\subset\mathcal{S}_B$, where $\mathcal{S}_A$, $\mathcal{S}_B=\{1,2,3,4\}$ are the sets of pure strategies of Alice and Bob. Each of the events considered corresponds to the choice of a pair of opposite corners of the building. It is easy to see that the events $A_{jk}\cap B_{lm}$ form a division of the space $\mathcal{S}_A\times\mathcal{S}_B$ of game situations; thus, these events form a complete set of events. Taking this into account, one can write the payoff function for Alice as the following mixture of conditional expectation values:

$$\mathcal{H}(x,y)=\sum_{jk,lm}E_{Ajk\cap B_{lm}}\mathcal{H}_A\cdot P(A_{jk}\cap B_{lm}).$$

In the case of our payoff matrix, this expression has the following form:

$$E_{A_{13}\cap B_{13}}\mathcal{H}\cdot P(A_{13}\cap B_{13})+E_{A_{24}\cap B_{24}}\mathcal{H}\cdot P(A_{24}\cap B_{24}). \tag{5}$$

Let p^l_{jk} and q^l_{jk} be the conditional probabilities of the choice of corner l from a given pair $\{j,k\}$ of opposite corners. The conditional average payoff for Alice equals

$$E_{A_{13}\cap B_{13}}\mathcal{H}=ap^1_{13}q^3_{13}+cp^3_{13}q^1_{13}$$

if both players choose the diagonal $\{1,3\}$ and

$$E_{A_{24}\cap B_{24}}\mathcal{H}=bp^2_{24}q^4_{24}+dp^4_{24}q^2_{24}$$

if they prefer the diagonal $\{2,4\}$. If the players choose different diagonals of the building, the conditional payoff is equal to zero since in this case it is always possible for Alice to see Bob, and he must not pay. Easy calculations show that

$$P(A_{jk}\cap B_{jk})=(x_j+x_k)(y_j+y_k),\quad p^l_{jk}=x_l/(x_j+x_k),\quad\text{and}\quad q^l_{jk}=y_l/(y_j+y_k). \tag{6}$$

These formulas will be of use for us when we discuss the behavior of Bob moving the ball in the game "Wise Alice."

4. Different logics, different behavior

Let us discuss now the behavior of players in the game "Wise Alice." First we note that this game gives us a simple model of measurement (defining the place of an object). Alice wants to know where is Bob, and she makes Bob active by her questions, "preparing" him at a definite "state." Alice is asking questions but her logic is not the usual one. Let us discuss this more carefully.

Assume that the wise Alice is interested in profit only when the profit is accompanied by a clarification of the initial position of Bob. In this case, it is not a surprise that inspite of the same payoff matrix for the "wise" and "foolish" Alice, the game interactions are different. The foolish Alice *always knows* which of the strategies has been used by her partner. The wise Alice is capable to define *truly* the strategy chosen by her partner not for all cases but only for the cases where her hypothesis is false. For this purpose, she (similarly to her "foolish" colleague) must make conclusions from the opposite but use another, *quantum, logic* based not on the Boolean algebra but on a *nondistributive ortholattice* [17, 18]. Denoting by α_k ($k = 1, 2, 3, 4$) the Alice assumptions that the ball *is located* at the vertex with number k, the negations of these assumptions are as follows:

$$\neg\alpha_1 = \alpha_3, \quad \neg\alpha_3 = \alpha_1, \quad \neg\alpha_2 = \alpha_4, \quad \neg\alpha_4 = \alpha_2.$$

Indeed, any time when Alice discovers that the ball is *not* located at the questioned vertex of the square, she understands that the ball is located at the opposite vertice. The opposite is also true. For example, if the question α_1 is answered affirmatively, then Alice knows that the ball *could not* be located at the third vertex. In addition to the standard logical identities

$$(\neg\alpha_j) \wedge \alpha_j = 0 \quad \text{and} \quad (\neg\alpha_j) \vee \alpha_j = 1,$$

the Alice assumptions satisfy special relations, which are not valid in the classical Boolean logic. Bob's reaction leads to the following result: Alice always gets the affirmative answer to one of any two questions. We also note that in spite of the fact that any pair of different Alice's assumptions forms an alternative, not all of these assumptions are mutually opposite, i.e., *orthocomplemented.* Only the pairs of assumptions with the same "parity:" $\{\alpha_1; \alpha_3\}$ and $\{\alpha_2; \alpha_4\}$ are orthocomplemented. An important feature distinguishing the logic of the wise Alice from the classical logic is the break of the distributivity law. Thus, for any triple of different j, k, l, one has the following inequality:

$$(\alpha_j \vee \alpha_k) \wedge \alpha_l \neq (\alpha_j \wedge \alpha_l) \vee (\alpha_k \wedge \alpha_l).$$

Indeed, the left-hand side of the inequality is equal to α_l, while the right-hand side is zero. In spite of these differences, the law of double negation and the duality Morgan's law are valid:

$$\neg(\alpha_j \wedge \alpha_k) = (\neg\alpha_j) \vee (\neg\alpha_k) \quad \text{and} \quad \neg(\alpha_j \vee \alpha_k) = (\neg\alpha_j) \wedge (\neg\alpha_k).$$

The same concerns the logic of Bob. The ball position influenced by the Alice questions is described by a similar system of predicates β_1, β_2, β_3, and β_4 defined on the set of strategies of Alice and taking value 1 for those questions to which he can answer affirmatively. In Fig. 3, a Hasse diagram is shown describing the nondistributive logic of our players. In terms of the introduced predicates α_j and β_k, the payoff function for the wise Alice can be written as follows:

$$a\alpha_1\beta_3 + b\alpha_2\beta_4 + c\alpha_3\beta_1 + d\alpha_4\beta_2.$$

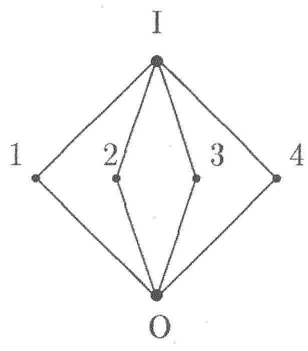

Fig. 3. Lattice of Alice's questions and Bob's answers.

Indeed, if Alice thinks that the ball is located at the first vertex, then due to the pairwise disjointness of the α_j, all terms but the first one are equal to zero. Similarly, the first term is equal to a only if the ball is located at the third vertex. The same is true for the remaining terms. In spite of the different meaning of symbols in the payoff function of the wise Alice and the foolish one, there is a resemblance with the previously written payoff function (3). This resemblance is not only formal – remember that from the point of view of payoff in each separate turn, the games are identical. However, if the game is repeated many times, then the difference appears. The wise Alice meets the problem of calculating the average profit. Since orthocomplementary questions and answers satisfy the following relations:

$$\alpha_1 + \alpha_3 = 1, \quad \alpha_2 + \alpha_4 = 1, \quad \beta_1 + \beta_3 = 1, \quad \beta_2 + \beta_4 = 1,$$

the averages $p_j = E\alpha_j$ and $q_k = E\beta_k$ may be treated as probabilities. But the relations

$$p_1 + p_3 = 1, \quad p_2 + p_4 = 1, \quad q_1 + q_3 = 1, \quad q_2 + q_4 = 1$$

contradict the main property of probabilities of alternatives,

$$p_1 + p_2 + p_3 + p_4 = 2, \quad q_1 + q_2 + q_3 + q_4 = 2,$$

which makes impossible the standard interpretation of a mixed strategy. A solution to the arising problem is given by ideas of quantum mechanics.

5. Towards averages via quantization

Note that the ortholattice of the logic of interaction of partners of the "Wise Alice" is isomorphic to the ortholattice of invariant subspaces of the Hilbert space of the quantum system with spin $\frac{1}{2}$ and observables of the type of S_x and S_θ.

In Fig. 4, two pairs of mutually orthogonal direct lines $\{l^1; l^3\}$ and $\{l^2; l^4\}$ are shown. One of these pairs diagonalizes the operator S_x, while the other pair diagonalizes S_θ. If we represent logical conjunction and disjunction by their intersection and linear envelope and if negation corresponds to the orthogonal complement, then we obtain an ortholattice isomorphic to the logic of the wise Alice. An example of such an isomorphism is given by the mapping $\alpha_j \mapsto l^j$, $j = 1, 2, 3, 4$. We have noted that neither Alice nor Bob have a stable strategy in one "experiment." However, if the game is repeated many times, there appears the problem of optimal frequencies of the corresponding pure strategies. We have seen above that, due to the nondistributivity of the logic, it is impossible to define on the sets of pure strategies a probabilistic measure consistent with lattice operations. A solution can be found in quantization in spite of the fact that the conflict interaction discussed is not related to the usual quantum physics.

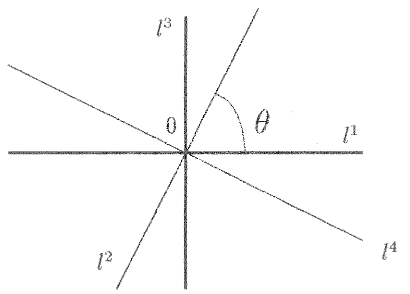

Fig. 4. Lattice of invariant subspaces of the observables S_x and S_θ.

Following the well-known constructions of quantum mechanics, we characterize the behavior of players not by mixed strategies but by "wave functions," which are one-dimensional subspaces of a Hilbert space. For this purpose, we consider a pair of two-dimensional Hilbert spaces H_A and H_B as the spaces of the strategic choice of Alice and Bob. The use of Hilbert space allows us to realize the nondistributive logic of our players without any difficulties. For this aim, we represent the predicates α_j and β_k describing questions of Alice and locations of the ball by the corresponding self-adjoint operators. It is important to note that the ortholattice can be realized in an infinite set of ways. Let us describe these ways up to unitary equivalence. We do this for the predicates of Alice. In H_A, we take an arbitrary one-dimensional subspace and associate its orthoprojector to the predicate α_1; denote the projector by $\widehat{\alpha}_1$. To the predicate α_3, we associate the *orthogonal complementary* projector $\widehat{\alpha}_3$. In this case, the following obvious and desirable for us relations are valid: $\widehat{\alpha}_1 \cdot \widehat{\alpha}_3 = \widehat{\alpha}_3 \cdot \widehat{\alpha}_1 = O$ and $\widehat{\alpha}_1 + \widehat{\alpha}_3 = I$. Take an arbitrary one-dimensional subspace *different from the eigenspaces of the projector* $\widehat{\alpha}_1$ and associate it to the predicate of some orthoprojector $\widehat{\alpha}_2$. To the variable α_4, associate the projector $\widehat{\alpha}_4 = 1 - \widehat{\alpha}_2$. As a result, one obtains one more decomposition of the Hilbert space H_A: $\widehat{\alpha}_2 \cdot \widehat{\alpha}_4 = \widehat{\alpha}_4 \cdot \widehat{\alpha}_2 = O$ and $\widehat{\alpha}_2 + \widehat{\alpha}_4 = I$. Projectors from different decompositions do not commute. However, there is a simple connection between the decompositions: the second decomposition is obtained from the first one by rotation at an angle different from multiples of 90^o. In other words, there exists a unitary operator u such that

$$\widehat{\alpha}_2 = u^{-1}\widehat{\alpha}_1 u \quad \text{and} \quad \widehat{\alpha}_4 = u^{-1}\widehat{\alpha}_3 u.$$

A similar construction for the Bob's lattice gives us projectors $\widehat{\beta_2}$, $\widehat{\beta_3}$, and $\widehat{\beta_4}$ and a unitary operator v similar to u and connecting the "even" and "odd" variables as follows:

$$\widehat{\beta}_2 = v^{-1}\widehat{\beta}_1 v \quad \text{and} \quad \widehat{\beta}_4 = v^{-1}\widehat{\beta}_3 v. \tag{8}$$

The quantum analog of the classical space of game situations is the tensor product $H_A \otimes H_B$. To write the observable payoff of Alice, "quantize" the classical expression (4) of the payoff function of Alice,

$$\mathcal{H}(\alpha, \beta) = \sum_{j,k=1}^{4} h_{jk}\alpha_j\beta_k,$$

substituting the corresponding projectors into it. As a result, one obtains the self-adjoint operator in $H_A \otimes H_B$, the *observable* of the payoff for Alice:

$$\widehat{\mathcal{H}} = \sum_{j,k=1}^{4} h_{jk}\widehat{\alpha}_j \otimes \widehat{\beta}_k.$$

Assume that Alice and Bob repeat their game with a ball many times. Let us describe their *behavior* by normalized vectors $\varphi \in H_A$ and $\psi \in H_B$. The element $s = \varphi \otimes \psi$ expressing their interaction during the game is a normalized vector in $H_A \otimes H_B$. Taking the latter vector as a characteristic of the *state* of the game, let us calculate the average at this state according to the standard rules of quantum mechanics as follows:

$$\langle \widehat{\mathcal{H}}s,\, s\rangle = \sum_{j,k=1}^{4} h_{jk}\langle (\widehat{\alpha}_j \otimes \widehat{\beta}_k)\, \varphi \otimes \psi,\; \varphi \otimes \psi\rangle.$$

After easy transformations, one gets the following expression of the average payoff for the given types of behavior of the players:

$$E\widehat{\mathcal{H}}(\varphi, \psi) = \sum_{j,k=1}^{4} h_{jk}\langle \widehat{\alpha}_j\varphi,\, \varphi\rangle \cdot \langle \widehat{\beta}_k\psi,\, \psi\rangle.$$

Substiting into this formula elements of our payoff matrix and using the notation $p_j = \langle \widehat{\alpha}_j\varphi\, \varphi\rangle$ and $q_k = \langle \widehat{\beta}_k\psi,\, \psi\rangle$, one obtains the value

$$E\widehat{\mathcal{H}}(\varphi, \psi) = ap_1q_3 + cp_3q_1 + bp_2q_4 + dp_4q_2. \tag{9}$$

6. Probability amplitudes instead of probabilities

In spite of some resemblance, the classical (4) and quantum (9) payoff functions are essentially different. The main difference between the formulas is in the *sense* of variables. While the x_j and y_k are probabilities, the p_j and p_k cannot be treated as probabilities since the equality $p_1 + p_3 = \langle (\widehat{\alpha}_1 + \widehat{\alpha}_3)\, \varphi, \varphi\rangle = 1$ and a similar equality $p_2 + p_4 = 1$ imply that $p_1 + p_2 + p_3 + p_4 = 2$. The sense of the values p_j and p_k is clarified by the standard quantum rule. For example, let the behavior of Alice be described by a normalized vector $\varphi \in H_A$ and let $\xi_1^+, \xi_1^- \in H_A$ be the normalized eigenvectors of the projector with eigenvalues 1 ("yes") and 0 ("no").

Projecting the state vector $\varphi = c_+\xi_1^+ + c_-\xi_1^-$ of Alice to the basis $\{\xi_1^+, \xi_1^-\}$, we see that she can find the ball at the first vertex of the square with probability

$$p_1 = \langle \widehat{\alpha}_1(c_+\xi_1^+ + c_-\xi_1^-),\ c_+\xi_1^+ + c_-\xi_1^-\rangle = \langle c_+\xi_1^+,\ c_+\xi_1^+ + c_-\xi_1^-\rangle = |c_+|^2$$

and at the opposite vertex with probability $p_3 = |c_-|^2$. Thus, according to quantum mechanics, the numbers p_j and p_k must be treated as the squares of moduli of *probability amplitudes*. The identities obtained above for the classical game allow us to compare each of the four pairs of numbers $\{p_1, p_3\}$, $\{p_2, p_4\}$, $\{q_1, q_3\}$, and $\{q_2, q_4\}$ *separately* with the corresponding *conditional* probabilities. Compare formula (9) for the quantum average,

$$E\widehat{\mathcal{H}}(\varphi, \psi) = (ap_1q_3 + cp_3q_1) + (bp_2q_4 + dp_4q_2),$$

with formula (5) used for calculation of the classical average in terms of conditional probabilities and conditional expectation values:

$$E\mathcal{H} = (ap_{13}^1 q_{13}^3 + cp_{13}^3 q_{13}^1) \cdot P_{13} + (bp_{24}^2 q_{24}^4 + dp_{24}^4 q_{24}^2) \cdot P_{24}. \tag{10}$$

If the corresponding pairs of the squares of moduli of amplitudes equal the classical conditional probabilities, i.e.,

$$p_1 = p_{13}^1, \quad p_3 = p_{13}^3, \quad p_2 = p_{24}^2, \quad p_4 = p_{24}^4,$$
$$q_1 = q_{13}^1, \quad q_3 = q_{13}^3, \quad q_2 = q_{24}^2, \quad q_4 = q_{24}^4,$$

one obtains an interesting result: the "wise" Alice gets a larger payoff than her "foolish" copy:

$$E\widehat{\mathcal{H}}(\varphi, \psi) \geqslant E\mathcal{H}.$$

However, there is no reason for the above equalities to hold. In fact, while we search for an equilibrium point of a classical game on the set of nonnegative numbers with the constrains $x_1+x_2+x_3+x_4 = 1$ and $y_1+y_2+y_3+y_4 = 1$, in the quantum game case, the squares of moduli of *probability amplitudes* satisfy not only the *explicit* linear relations $p_1 + p_3 = 1$, $p_2 + p_4 = 1$, $q_1 + q_3 = 1$, and $q_2 + q_4 = 1$, but also some implicit relations due to formulas (7) and (8). The quantum equilibrium point cannot be found just from expression (9) for the average profit as a function of eight variables $\{p_j, q_k\}$ satisfying the linear relations above. The equilibrium is defined not by a combination of the *squares of moduli* of the amplitudes of Alice and Bob but by a combination of the *wave functions* $\varphi \in H_A$ and $\psi \in H_B$.

Let us pay attention to an interesting difference between the quantum case and the classical one. The average payoff of the wise Alice is equal to the *sum* of conditional averages but not to their *mixture* as in the classical case.

7. Behavior of a player as a realization of logic

In addition to the amplitudes characterizing the behavior of players, our model has two additional structural characteristics: unitary rotations u and v determining an operator representation of the ortholattices of Alice and Bob up to unitary equivalence. Each of these operators can be characterized by the angle between eigensubspaces of even and odd order. Denote by θ_A the angle between direct lines corresponding to the largest eigenvectors of the operators $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ of Alice; let θ_B be the similar angle characterizing the realization of the logic of Bob. In contrast with the amplitudes characterizing the behavior of the players, which are *variables* of the model, the angles are its *parameters*. These parameters characterize the *type* of a player; they allow us to consider the logic as some factor forming the behavior. We may attribute some sense to these values as follows: the angles characterize connections between choices of the diagonals of the square or some preferences for this or that adjacent vertex. The angles θ_A and θ_B determine commutation relations for the corresponding operators. Consider, for example, the operators $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$. Take as an orthonormal basis in the space H_A of Alice the eigenbasis of the operator $\widehat{\alpha}_1$. In this basis, the matrices of this pair of operators are

$$\widehat{\alpha}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widehat{\alpha}_2 = \begin{pmatrix} \cos^2\theta_A & \sin\theta_A\cos\theta_A \\ \sin\theta_A\cos\theta_A & \sin^2\theta_A \end{pmatrix},$$

where θ_A is the angle at which the eigenbasis of the operator $\widehat{\alpha}_2$ is rotated relative to the eigenbasis of the operator $\widehat{\alpha}_1$. Calculating the commutator of these matrices, one obtains the following equality:

$$i[\widehat{\alpha}_1,\ \widehat{\alpha}_2] = \frac{1}{2}\sin 2\theta_A \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Similar commutation relations hold for Bob's operators.

8. In search of the quantum equilibrium

The definition of the Nash equilibrium for the quantum case does not differ much from the classical case (1) and can be written as follows:

$$E\widehat{\mathcal{H}}(\varphi, \psi^0) \leqslant E\widehat{\mathcal{H}}(\varphi^0, \psi^0) \leqslant E\widehat{\mathcal{H}}(\varphi^0, \psi).$$

Fix the eigenbases $\{\xi_1^+, \xi_1^-\}$ and $\{\xi_2^+, \xi_2^-\}$ in the space H_A of strategies of Alice corresponding to the projectors $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$. Do the same for Bob, taking the bases $\{\eta_1^+, \eta_1^-\}$ and $\{\eta_2^+, \eta_2^-\}$. Denote by θ_A and θ_B the angles between the largest eigenvectors. Then the squares of moduli of the amplitudes p_j and p_k are given by the following formulas:

$$p_1 = \cos^2\alpha, \quad p_3 = \sin^2\alpha, \quad p_2 = \cos^2(\alpha - \theta_A), \quad p_4 = \sin^2(\alpha - \theta_A),$$
$$q_1 = \cos^2\beta, \quad q_3 = \sin^2\beta, \quad q_2 = \cos^2(\beta - \theta_B), \quad q_4 = \sin^2(\beta - \theta_B),$$

where α and β are the angles between the vectors φ and ψ and the corresponding axes, respectively. For the values of these angles, we take the interval $[0^0; 180^0]$. As a result, the problem of search for equilibrium points of the quantum game is reduced to the problem of finding a minimax of the following function of two angle variables:

$$F(\alpha, \beta) = a\cos^2\alpha\sin^2\beta + c\sin^2\alpha\cos^2\beta + b\cos^2(\alpha - \theta_A)\sin^2(\beta - \theta_B) + d\sin^2(\alpha - \theta_A)\cos^2(\beta - \theta_B)$$

on the square $[0^0; 180^0] \times [0^0; 180^0]$. In other words, our quantum game becomes an *infinite antagonistic game* of two persons on the square. Solving such games in pure strategies, i.e., the search for saddle points of the function $F(\alpha, \beta)$ is a difficult problem. It is known [19] that continuity and smoothness of the payoff function are sufficient for the existence of saddle points. Present existence theorems use convexity properties of this function. However, in our case these properties are violated. One can find examples of values of the elements a, b, c, d of the payoff matrix (1) in which the function $F(\alpha, \beta)$ does not have saddle points. The situation with methods of search for saddle points is not better. Thus, in the situation of absence of simple analytical solutions, one has to look for numerical methods. Our calculations use an algorithm based on the construction of "curves of reaction" or "curves of the best answers" for participants of the game.

The definition of curves of reaction is based on the following consideration. If Alice knew what decision Bob would take, she could make an *optimal* choice. But the essence of the game situation is that she does not know this decision. She has to take into account his various strategies and, for any possible act of the partner, she must find the optimal responce. Her considerations are similar to considerations of a player expressed by the formula: "if he does this, then I do that." Bob thinks the same way. Hence, one has to consider two functions $x = \mathcal{R}_A(y)$ and $y = \mathcal{R}_B(x)$ the plots of which are called the curves of reactions of Alice and Bob. By definition, the following relations hold:

$$\max_{\lambda} F(\lambda, y) = F(\mathcal{R}_A(y), y) \quad \text{and} \quad \min_{\mu} F(x, \mu) = F(x, \mathcal{R}_B(x)).$$

It is easy to see that intersections of the curves of reaction give us points of the von Neumann–Nash equilibrium. Numerical experiments show that, depending on the values of parameters a, b, c, d of the payoff function and on the angles characterizing the type of a player, there appear qualitatively different patterns. Intersections may be absent, there may be one intersection, and, finally, the case of two equilibrium points with different values of the payoff of the game is possible; the latter case is impossible for a classical matrix game.

9. Examples

1. Two equilibrium points appear in the case of the payoff matrix with $a = 3$, $b = 3$, $c = 5$, and $d = 1$ and an operator representation of the ortholattice corresponding to angles $\theta_A = 10^0$ and $\theta_B = 70^0$. One of the equilibrium points is inside the square, the other one is on its boundary (see Fig. 5).

The curves of reaction in this case happen to be *discontinuous*. For convenience, the discontinuities are shown by thin lines. The discontinuous character of the curve of reaction of Alice excluded the existence of one more equilibrium point. One of the equilibrium points appears for $\alpha = 145.5^0$ and $\beta = 149.5^0$ and gives us the following values for the squares of moduli of amplitudes:

for Alice, $p_1 = 0.679$, $p_2 = 0.509$, $p_3 = 0.321$, and $p_4 = 0.491$;
for Bob, $q_1 = 0.258$, $q_2 = 0.967$, $q_3 = 0.742$, and $q_4 = 0.033$.

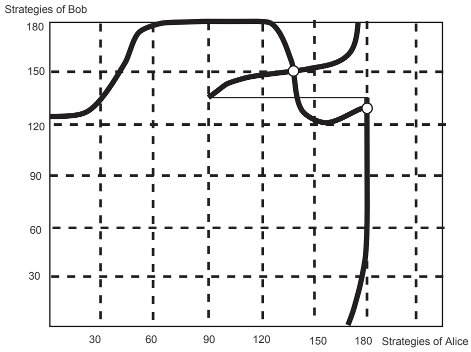

Fig. 5. Two points of the Nash equilibrium.

The price of the quantum game, i.e., the equilibrium value of the profit for Alice in this case is $E\widehat{\mathcal{H}} = 2.452$. The second equilibrium point corresponds to $\alpha = 180^0$ and $\beta = 123.5^0$, and the squares of the amplitude moduli are:

for Alice, $p_1 = 1.000$, $p_2 = 0.967$, $p_3 = 0.000$, and $p_4 = 0.033$;

for Bob, $q_1 = 0.695$, $q_2 = 0.646$, $q_3 = 0.305$, and $q_4 = 0.354$.

The price of the game at the second equilibrium point is $E\widehat{\mathcal{H}} = 1.926$. For the *classical* game with the same payoff matrix (see Sec. 4), the price of the game is smaller and equals $\frac{15}{28} \approx 0.536$. In contrast with the quantum game, the classical game has only *one* equilibrium point which is obtained for the following frequencies:

for foolish Alice, $x_1 = \dfrac{5}{28}$, $x_2 = \dfrac{5}{28}$, $x_3 = \dfrac{3}{28}$, and $x_4 = \dfrac{15}{28}$;

for her partner, $y_1 = \dfrac{3}{28}$, $y_2 = \dfrac{15}{28}$, $y_3 = \dfrac{5}{28}$, and $y_4 = \dfrac{5}{28}$.

To compare the quantum game with the classical one, conditional probabilities for the choice of vertices of the square after the choice of the diagonal are as follows:

for Alice, $p_{13}^1 = \dfrac{1}{4}$, $p_{13}^3 = \dfrac{3}{4}$, $p_{24}^2 = \dfrac{5}{8}$, and $p_{24}^4 = \dfrac{3}{8}$;

for Bob, $q_{13}^1 = \dfrac{3}{8}$, $q_{13}^3 = \dfrac{5}{8}$, $q_{24}^2 = \dfrac{5}{6}$, and $q_{24}^4 = \dfrac{1}{6}$.

In this case, the conditional average payoffs for each diagonal are:

$$E_{13}\mathcal{H} = 1.875 \quad \text{and} \quad E_{24}\mathcal{H} = 0.75.$$

The price of the classical game is obtained multiplying these expressions by the probabilities of the corresponding conditions given in Sec. 3. Terms of the quantum payoff, associated with these conditional averages in the case of the first equilibrium point are:

$$ap_1q_3 + cp_3q_1 = 1.927 \quad \text{and} \quad bp_2q_4 + dp_4q_2 = 0.525.$$

For the second equilibrium point,

$$ap_1q_3 + cp_3q_1 = 0.915 \quad \text{and} \quad bp_2q_4 + dp_4q_2 = 1.048.$$

2. A unique equilibrium is observed, for example, in the case where all nonzero payoffs are equal (with common value 1) and the angles are $\theta_A = 45^0$ and $\theta_B = 45^0$. The equilibrium point is located on the upper right-hand vertex of the square (see Fig. 6). The curve of Bob's reaction is shown in Fig. 6 as a *continuous* one, while the similar curve of Alice is discontinuous in the case where Bob is using the strategy corresponding to the angle $\beta = 90^0$. To make the pattern more visible, the discontinuity is shown by a thin line.

In reality, *both* lines are discontinuous. This becomes obvious if one extends both functions to the whole real axis taking into account their periodicity: the plot of one of these functions is obtained by the shift of the other one at semiperiod 90^0. The squares of the amplitude moduli in this case have the following values:

for Alice, $p_1 = 1$, $p_2 = 0.5$, $p_3 = 0$, and $p_4 = 0.5$;

for Bob, $q_1 = 1$, $q_2 = 0.5$, $q_3 = 0$, and $q_4 = 0.5$.

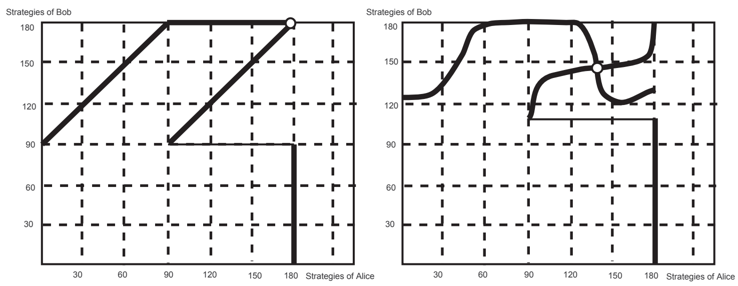

FIG. 6. The unique Nash equilibrium

The payoff of the "wise" Alice in this case is $E\widehat{\mathcal{H}}_A = 0.5$, while her classical copy gets only 0.125. In contrast with the quantum case, for the classical players with such a payoff function all the vertices of the square are equally probable. The unique equilibrium located *inside* the square takes place for the initial payoff matrix $a = 3$, $b = 3$, $c = 5$, and $d = 1$ and angles $\theta_A = 15^0$ and $\theta_B = 35^0$ (see Fig. 6).

3. Absence of an equilibrium is, probably, one of the most interesting phenomena since it is known that, for classical matrix games, an equilibrium in mixed strategies always exists. One can establish the absence of an equilibrium taking the same payoff matrix as above for which one and two points of equilibrium have been found. For this purpose, it is sufficient to take the operator representation of the ortholattice with the typical angles $\theta_A = 30^0$ and $\theta_B = 20^0$ (see Fig. 7).

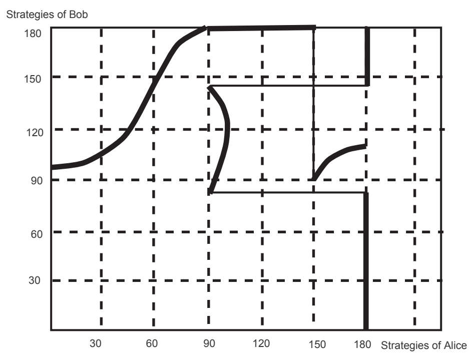

FIG. 7. Absence of the Nash equilibrium

It is seen in Fig. 7 that the absence of equilibrium in this case is due to the discontinuity of the functions of reaction, which is impossible in the classical case. We have met this phenomenon in the first example where two equilibrium points have been obtained. This latter example shows the importance of the *realisation* of a nondistributive lattice. In the language of game theory, one can understand the phenomenon described above as follows: having the same interests, the players may form their behavior in qualitatively different ways. Thus, a mathematician can give to a client, for example to Alice, some strategic recommendations how to organize her behavior to increase the profit for the same payoff conditions. Of course, for this purpose, the mathematician has to know the representation of Bob's logic.

Concluding remarks.

The construction and analysis of our models show that the main difference between the classical and quantum points of view on observable phenomena is in the *way of calculation of averages*. In fact, if one remembers the first Planck's work on the spectra of radiation of a perfect black body, one recognizes that the correct formula was obtained on the basis of a postulate leading to a new (different from the classical one) way of calculating the oscillator average energy. The proposed scheme of quantizing a game follows exactly this tradition. What is the justification of using the apparatus of quantum mechanics in game theory when one deals with properties of *macroscopic* objects? A woman taking a diamond in her hand and feeling its anomalous heat capacity may know nothing about the Einstein–Debye formula and the specific averaging procedure which explains the observed phenomenon. However, an understanding person will not be surprised by attempts to explain some features of macrophenomena by the specific way of calculating averages. If a logic adequately reflecting experience is nondistributive, then the classical procedures of calculating the averages lead, as was shown, to contradictions, and one has to use another apparatus. If the lattice obtained happens to be a lattice of subspaces, then the answer is given by the Gleason's theorem saying that probability measures on the lattice of projectors have strictly definite forms. Thus, if one is solving the problem of averaging the payoff function taking into account *logical conditions* of the players, then the passage to quantization in some cases is predestined. The passage to a quantum payoff function is similar to the passage from the classical Hamilton's function to the Schrödinger operator.

It is surprising that the classical expression of the payoff function seems to be invented especially for substitution of projective operators instead of a two-valued function. This observation leads us to a general scheme of games quantization. In any case, if one considers not antagonistic but *bimatrix* games in which the interests of players are not strictly antagonistic, the scheme of quantization remains the same. An important point of our scheme is the careful account of the difference between the logic and its operator representation. Any of the possible representations can be described by the angle parameter and commutation relations for the corresponding projectors. In the players logic, these pairs of projectors correspond to mutually complementary questions, neither of which is the negation of the other one.

It is seen from our examples that the existence of an equilibrium is connected with a choice of the representation of the lattice of properties. The absence of an equilibrium is also connected with the representation of behavior of the players by *vectors* in Hilbert spaces, while the game situation is represented by some *resolved* element of the tensor product of these spaces. It is possible to extend the notion of equilibrium (similarly to the case of the classical game theory) passing to mixed extension based on density matrices. In this paper, we do not consider *entangled* states. It was shown in [8] that entangled states may lead to more complex nondistributive lattices. The search for an equilibrium among entangled nonfactorizable states may lead to success in proving the existence of an equilibrium in the general case of quantum games.

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REFERENCES

1. D. Deutsch, *Quantum Theory of Probability and Decisions*, Proc. R. Soc. Lond. A. (1999) (quant-ph/9906015).
2. J. Finkelstein, *Quantum Probability from Decisions Theory?* SJSU-99-20 (1999).
3. I. Polley, *Reducing the Statistical Axiom*, Phys. Dept. Oldenburg Univ., D-26111 (1999).
4. E. W. Piotrowski, *Quantum English Auctions*, arXiv:quant-ph/0108017v1 3Aug (2001).
5. A. A. Grib, "Breaking the Bell inequalities and the problem of measurement in the quantum theory," *Lectures for young scientists*, JINR, Dubna (1992).
6. A. A. Grib, Jr., and W. A. Rodrigues, *Nonlocality in Quantum Physics*, Kluwer Academic, Plenum Publishers (1999).
7. A. A. Grib and R. R. Zapatrin, *Int. J. Theor. Phys.*, **29** (2), 113 (1990).
8. A. A. Grib and R. R. Zapatrin, *Int. J. Theor. Phys.* **30** (7), 949 (1991).
9. J. Eisert et al, *Phys. Rev. Lett.*, **83**, 3077 (1999).
10. A. K. Ekert, *Phys. Rev. Lett.*, **67**, 661 (1999).
11. L. Marinatto and T. Weber, *Phys. Lett. A*, **272**, 291 (2000).
12. H. Moulin, *Théorie des Jeux pour L'économie et la Politique*, Hermann, Paris (1981).
13. G. Owen, *Game Theory*, Philadelphia, London, Toronto (1968).
14. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton Univ. Press (1953).
15. H. von Stackelberg, *Marktform und Gleiwicht*, Springer (1934).
16. J. D. Kemeny and G. L. Thompson, "The effect of psychological attitudes on the outcomes of games," *Contributions Theory Games*, **3**, 273–298 (1957).
17. G. D. Birkhoff, *Lattice Theory*, AMS Colloc. Publ., **25** (1993).
18. G. Grätzer, *Lattice Theory*, Freeman & Co. (1971).
19. N. N. Vorob'ev, *Foundations of Game Theory. Noncooperative Games*, Birkhäuser Verlag (1994).