Appendix B: Completely Continuous Symmetric Operators

Let $\underline{T}$ be a bounded operator on the Hilbert space $\mathcal{H}$. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of $\underline{T}$ in case there exists a nonzero $\underline{u} \in \mathcal{H}$ such that $\underline{T}\,\underline{u} = \lambda \underline{u}$. Each such nonzero $\underline{u}$ is an eigenvector of $\underline{T}$ corresponding to eigenvalue λ. If $\mathcal{H}$ is n-dimensional and $\underline{T}$ is self-adjoint it is well-known that there exists an ON basis $\{\underline{u}_j\}$ for $\mathcal{H}$ consisting of eigenvectors of T.

$$\underline{T}\,\underline{u}_j = \lambda_j \underline{u}_j, \quad j = 1, \cdots, n.$$

The matrix of $\underline{T}$ with respect to this basis is diagonal:

$$T = \begin{pmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \ddots \; \lambda_n \end{pmatrix}.$$

However, if $\mathcal{H}$ is infinite-dimensional and $\underline{T}$ is self-adjoint it is usually not possible to find an ON basis for $\mathcal{H}$ consisting of eigenvectors. There is a sense in which $\underline{T}$ can be diagonalized (the spectral theorem for self-adjoint operators) but this is not a straight-forward extension of the procedure for diagonalizing self-adjoint operators on finite-dimensional spaces.

Nevertheless, there is a class of operators $\underline{T}$ of great importance in mathematical physics for which the eigenvectors do form an ON basis in $\mathcal{H}$: the completely continuous self-adjoint operators.

A subset $\mathcal{S}$ of $\mathcal{H}$ is bounded if there exists a constant $C > 0$ such that $\|\underline{v}\| < C$ for all $\underline{v} \in \mathcal{S}$.

Definition: A linear operator $\underline{T}$ on $\mathcal{H}$ is completely continuous if for every bounded sequence $\{\underline{v}_j\}$ in $\mathcal{H}$, there is a subsequence $\{\underline{v}_{j_\ell}\}$, $j_1 < j_2 < \cdots < j_\ell < \cdots$, such that $\{\underline{T}\,\underline{v}_{j_\ell}\}$ is convergent.

Note: It is easy to show that a completely continuous operator is bounded.

<u>Example 1</u>: Every linear operator on a finite-dimensional space is completely continuous.

<u>Example 2</u>: The identity operator $\underline{E}$ on an infinite dimensional space is <u>not</u> completely continuous. (Hint: Look at the action of $\underline{E}$ on an ON basis of $\mathcal{H}$.)

<u>Example 3</u>: Let $\mathcal{H} = L_2(\mathfrak{M})$, (appendix A), and let $h_j(x), g_j(x) \in C(\mathfrak{M})$, $j = 1, \cdots, n$. Let $\underline{K}$ be the operator on $L_2(\mathfrak{M})$ defined by

(B.1) $$\underline{K}f(x) = \int_{\mathfrak{M}} K(x,y)\, f(y)\, w(y)\, dy, \qquad f \in L_2(\mathfrak{M})$$

where

$$K(x,y) = \sum_{j=1}^{n} h_j(x)\, g_j(y)$$

is the <u>kernel</u> of the integral operator $\underline{K}$ and $w(y)$ is the weight function on $L_2(\mathfrak{M})$. Then $\underline{K}$ is completely continuous. This follows from the fact that $R_{\underline{K}}$ is finite-dimensional.

<u>Example 4</u>: Let $\mathcal{H} = L_2(\mathfrak{M})$ and let $\underline{K}$ be an integral operator (B.1) where now we require only that the kernel $K(x,y)$ be continuous in x and y. Then $\underline{K}$ is completely continuous. Moreover, if $K(x,y) = \overline{K(y,x)}$ then $\underline{K}$ is self-adjoint.

Note: We give no proofs in this appendix. For detailed proofs the reader can consult [Helwig, 1] or [Stakgold, 1]. However, the reader should be able to supply the elementary proof of

<u>Lemma B.1</u>: Let $\underline{T}$ be a bounded self-adjoint operator on $\mathcal{H}$. Then the eigenvalues of $\underline{T}$ are real and eigenvectors corresponding to distinct

eigenvalues are orthogonal.

Theorem 2.14: Let $\underline{T}$ be a nonzero completely continuous self-adjoint operator on the separable Hilbert space $\mathcal{H}$. Let $C_\lambda = \{\underline{u} \in \mathcal{H} : \underline{T}\underline{u} = \lambda \underline{u}\}$ be the eigenspace corresponding to the eigenvalue λ. Then

a) $\underline{T}$ has at least one nonzero eigenvalue λ_1 and at most countably many, $\lambda_1 \geq \lambda_2 \geq \cdots$. Each eigenspace C_{λ_i} for $\lambda_i \neq 0$ is finite-dimensional. If there are an infinite number of eigenvalues then $\lim_{i\to\infty} \lambda_i = 0$.

b) Let $\lambda_1, \lambda_2, \cdots$ be the eigenvalues of $\underline{T}$, possibly including $\lambda = 0$, and let $\{\underline{u}^i_j, j = 1, 2, \cdots, \dim C_{\lambda_i}\}$ be an ON basis for C_{λ_i}. Then $\{\underline{u}^i_j, j = 1, 2, \cdots, \dim C_{\lambda_i}, i = 1, 2, \cdots\}$ is an ON basis for $\mathcal{H}$.

c) If $\underline{u} \in R_{\underline{T}}, \underline{u} = \underline{T}\underline{v}$ for $\underline{v} \in \mathcal{H}$, then

$$u = \sum_{i,j} (\underline{T}\underline{v}, \underline{u}^i_j)\underline{u}^i_j = \sum_{i,j} (\underline{v}, \underline{T}\underline{u}^i_j)\underline{u}^i_j = \sum_{i,j} \lambda_i (\underline{v}, \underline{u}^i_j)\underline{u}^i_j .$$

Note: Part c) follows immediately from a) and b). The sum in the expansion of $\underline{u}$ goes only over those eigenvectors corresponding to nonzero eigenvalues.

Consider the completely continuous self-adjoint integral operator $\underline{K} \neq 0$ on $L_2(\mathcal{M})$, (example 4). The kernel $K(x,y)$ of $\underline{K}$ is continuous in all its arguments and satisfies $K(x,y) = \overline{K(y,x)}$. The preceding theorem clearly applies to $\underline{K}$. Moreover, by making use of the special structure of $\underline{K}$ we can obtain more information about the expansion c). The eigenvectors $u^i_j(x)$ are now functions in $L_2(\mathcal{M})$.

Theorem 2.15: 1) Let λ be a nonzero eigenvalue of $\underline{K}$ and $z(x)$ a corresponding eigenfunction. Then $z(x) \in C(\mathcal{M})$. 2) More generally, if $u(x) \in R_{\underline{K}}$ then $u(x) \in C(\mathfrak{M})$. 3) If $u(x) \in R_{\underline{K}}$, $u(x) = \underline{K}\, v(x)$ then

$$u(x) = \sum_{i,j} (u, u^i_j)\, u^i_j(x) = \sum_{i,j} \lambda_i (v, u^i_j)\, u^i_j(x)$$

where the series converges uniformly to $u(x)$ (pointwise) for all $x \in \mathfrak{M}$.

The point of statement 3) is that the expansion of $u \in R_{\underline{K}}$ in terms of the eigenfunctions $u^i_j(x)$, converges not only in the norm but also pointwise uniformly.