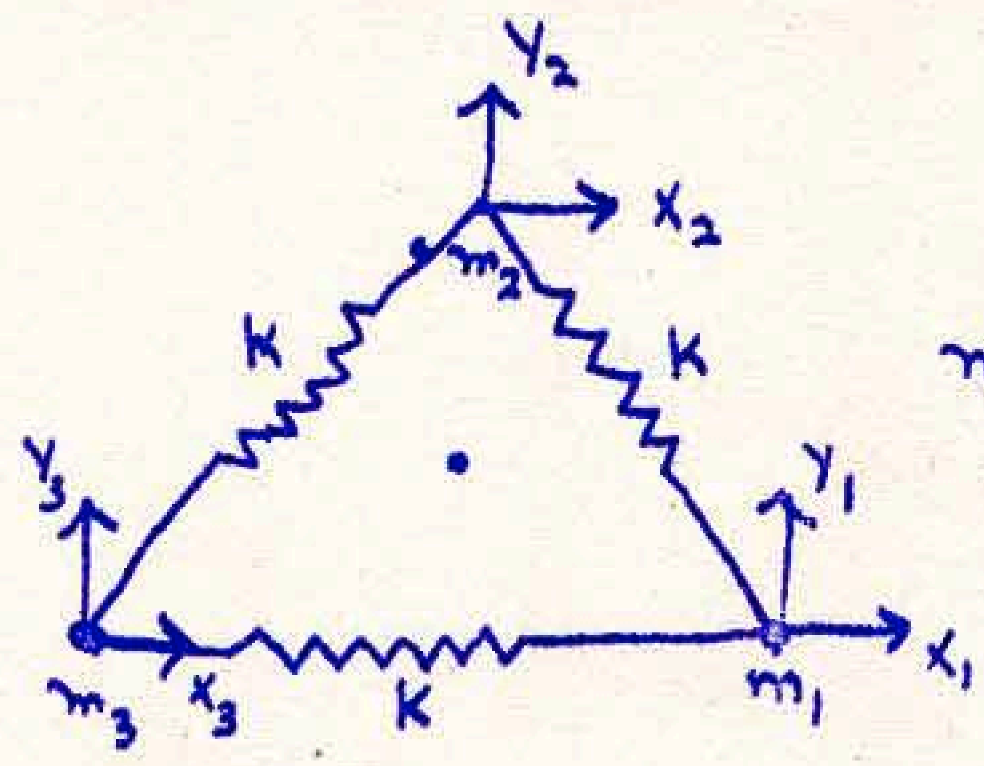
The last application of representation theory to be discussed here is concerned with the theory of small oscillations. We shall limit ourselves to the solution of a single problem in this theory. However, the same techniques employed to solve the example can be used to solve a great number of similar problems, as should be obvious to the reader.

Consider a system of three point masses, all of mass m, located at the vertices of an equilateral triangle and connected by three springs of spring constant K. We will study the oscillatory motion of this system in the plane of the triangle.



$m_1 = m_2 = m_3 = m$

figure 3.3

As shown in figure 3.3, we assign coordinates (x_i, y_i) to the ith mass point m_i such that in the equilibrium position of the system each mass point has coordinates $(0,0)$. Recall that the potential energy of a spring with spring constant K which is displaced an amount X from equilibrium is $\frac{1}{2}KX^2$. If at a given instant of time the system is in the state

(8.55) $$\underline{W} = (w_1, \cdots, w_6) = (x_1, y_1, x_2, y_2, x_3, y_3) = (\underline{Z}^{(1)}, \underline{Z}^{(2)}, \underline{Z}^{(3)})$$

then it has potential energy

(8.56)
$$V(\underline{w}) = \tfrac{1}{2}k\left[\sqrt{(x_1+d-x_3)^2+(y_1-y_3)^2} - d\right]^2$$
$$+ \tfrac{1}{2}k\left[\sqrt{(x_1+d/_2-x_2)^2+(y_2+\sqrt{3}d/_2-y_1)^2} - d\right]^2$$
$$+ \tfrac{1}{2}k\left[\sqrt{(x_2+d/_2-x_3)^2+(y_2+\sqrt{3}d/_2-y_3)^2} - d\right]^2$$

where d is the length of one side of the equilateral triangle in figure 3.3. If we limit ourselves to the study of small oscillations of this system then $\|w_i/d\| \ll 1$ and it is sufficient to consider only the terms up to second order in w_i in the Taylor series expansion of $V(\underline{w})$ about $\underline{w} = \underline{\theta}$. The result is

(8.57)
$$V(\underline{w}) = V(\underline{\theta}) + \sum_{i=1}^{6} \frac{\partial V}{\partial w_i}(\underline{\theta})\, w_i + \frac{1}{2}\sum_{i,j=1}^{6} \frac{\partial^2 V(\underline{\theta})}{\partial w_i \partial w_j} w_i w_j$$
$$= \tfrac{1}{2}k \sum_{i,j=1}^{6} V_{ij} w_i w_j$$
$$= \tfrac{1}{2}k \left\{ (x_1-x_3)^2 + \left[\tfrac{1}{2}(x_1-x_2) + \tfrac{\sqrt{3}}{2}(y_2-y_1)\right]^2 \right.$$
$$\left. + \left[\tfrac{1}{2}(x_2-x_3) + \tfrac{\sqrt{3}}{2}(y_2-y_3)\right]^2 \right\},$$
$$V_{ij} = \frac{\partial^2 V(\underline{\theta})}{\partial w_i \partial w_j}.$$

The equations of motion of this system are

(8.58)
$$m\ddot{w}_j = -\frac{\partial V}{\partial w_j}(\underline{w}) = -k\sum_{i=1}^{6} V_{ji} w_i, \quad j = 1, \cdots, 6,$$

or in matrix form

(8.59)
$$\ddot{\underline{w}} = -\frac{k}{m} V \underline{w}$$

If we diagonalize the positive definite real symmetric matrix V by a similarity transformation

$$OVO^{-1} = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_6 \end{pmatrix}$$

where O is a real orthogonal matrix, then (8.59) becomes

$$\ddot{\underline{z}} = -\frac{k}{m} D \underline{z} , \quad \underline{z} = O \underline{w} = (z_1, \cdots, z_6)$$

which has solutions

$$z_j = a_j \cos(\omega_j t + \varphi_j) \text{ if } \omega_j = \sqrt{k\lambda_j/m} \neq 0$$
$$= a_j t + b_j , \text{ if } \lambda_j = 0 ;$$

where the arbitrary constants a_j, b_j, φ_j are determined by the initial conditions of the physical system. Thus, the general motion of the system is given by the expression

(8.60) $$\underline{w} = \sum_{j=1}^{6} \underline{A}_j a_j \cos(\omega_j t + \varphi_j)$$

where the $\underline{A}_j$ are ON basis vectors defined by

(8.61) $$\underline{A}_j = O^{-1} \underline{e}_j , \quad \underline{e}_j = (0, 0, \cdots, 1, \cdots, 0)$$

with a one in the jth position. The oscillation $\underline{A}_j a_j \cos(\omega_j t + \varphi_j)$ in which all the point masses vibrate with the same frequency is called a <u>normal mode of vibration</u> and the eigenvalue λ_j determines the frequency of this normal mode. Clearly the eigenvalues are just the solutions of the charteristic (or secular) equation

(8.62) $$\det(V - \lambda E_6) = 0 .$$

To solve this equation we would have to determine the roots of a polynomial of order six, not an easy task. Fortunately we can use the symmetries of the physical system to simplify the determination of the normal modes and frequencies.

Some of the eigenvalues λ_j may be degenerate. Suppose the distinct eigenvalues are $\lambda_\alpha, \lambda_\beta, \cdots$ and the λ_j's are ordered so that $\lambda_1 = \lambda_2 = \cdots = \lambda_{m_1} = \lambda_\alpha , \lambda_{m_1+1} = \cdots = \lambda_{m_1+m_2} = \lambda_\beta ,$ etc. Then the m_1 vectors $\underline{A}_j$, $j=1,\cdots,m_1$ associated with the eigenvalue λ_α form an ON basis for a subspace W_α, the m_2 vectors $\underline{A}_j$, $j = m_1+1,\cdots,m_2$, associated with λ_β form an ON basis for W_β, and so on. The system space R_6 of all real six-component column vectors decomposes into the direct sum

$$R_6 = W_\alpha \oplus W_\beta \oplus \cdots \oplus W_\delta .$$

Here, R_6 is the space of all state vectors (8.55). The complete symmetry group of the equilateral triangle, considered in two-dimensional space, is D_3 . The natural action of D_3 as a transformation group in the plane induces a six-dimensional representation T of D_3 on R_6 . Indeed let C be the counterclockwise rotation of the triangle through 120° and γ the reflection about a vertical line through the center of the triangle in figure 3.3. Then C and γ generate the symmetry group D_3 of order six and we have

(8.63)
$$\begin{aligned} T(c)\underline{W} &= T(c)\,(x_1,y_1,x_2,y_2,x_3,y_3) \\ &= \Big(-\tfrac{1}{2}x_3-\tfrac{\sqrt{3}}{2}y_3,\ \tfrac{\sqrt{3}}{2}x_3-\tfrac{1}{2}y_3,\ -\tfrac{1}{2}x_1-\tfrac{\sqrt{3}}{2}y_1,\ \tfrac{\sqrt{3}}{2}x_1-\tfrac{1}{2}y_1, \\ &\qquad -\tfrac{1}{2}x_2-\tfrac{\sqrt{3}}{2}y_2,\ \tfrac{\sqrt{3}}{2}x_2-\tfrac{1}{2}y_2\Big), \end{aligned}$$

(8.64)
$$T(\gamma)\underline{W} = (-x_3, y_3, -x_2, y_2, -x_1, y_1).$$

Since the group elements e, c, γ form a complete set of representations from each of the three conjugacy classes of D_3 we can easily compute the character χ of this representation.

(8.65)
$$\begin{aligned} &\chi_1 = \chi(e) = 6\ ,\quad \chi_2 = \chi(c) = 0, \\ &\chi_3 = \chi(\gamma) = 0. \end{aligned}$$

From symmetry considerations we see that if $\underline{W}(t)$ is a solution of the equations of motion then so is $T(g)\underline{W}(t)$ for $g \in D_3$. (This is equivalent to saying that the potential function $V(\underline{w})$ is invariant under D_3, $V(T(g)\underline{w}) = V(\underline{w})$, or that the symmetric matrix V satisfies $T(g)VT^{-1}(g) = V$.) Suppose $\underline{w}(t)$ is a solution of the equations of motion such that $\underline{w}(t_0) \in W_\alpha$ for some t_0 . Then $V\underline{w}(t_0) = \lambda_\alpha \underline{w}(t_0)$ and it follows from (8.59) that $\underline{w}(t) \in W_\alpha$ for every t . Furthermore,

$$VT(g)\underline{w}(t) = T(g)V\underline{w}(t) = \lambda_\alpha T(g)\underline{w}(t)$$

so $T(g)\underline{w}(t)$ is an element of W_α for each t and $g \in D_3$. It follows that the subspaces $W_\alpha, W_\beta, \cdots$ are invariant under T so that T can be decomposed into a direct sum of representations

(8.66) $$T = T_\alpha \oplus T_\beta \oplus \cdots \oplus T_\delta \, , \quad T_j = T \,|\, W_j \, .$$

If the representation T_α is irreducible of order n then the multiplicity of λ_α is n . Thus the degeneracies of the eigenvalues are determined by the irreducible representations of D_3 . If the representation T_α is reducible then the eigenvalue λ_α is said to have an accidental degeneracy, i.e., one which does not follow from symmetry considerations alone. Frequently, accidental degeneracies can be removed by varying some of the parameters defining the system such as m, K without changing the symmetry group. An exception to this is the eigenspace corresponding to the zero eigenvalue where the accidental degeneracy usually cannot be removed. This eigenspace corresponds to translations and rotations of the system as a whole, without oscillation.

Since the three inequivalent irreducible representations $T^{(1)}$, $T^{(2)}$, $T^{(3)}$ of D_3 are all real we can use the character table (6.14) and the orthogonality relations to decompose T . The result is

(8.67) $$T = T^{(1)} \oplus T^{(2)} \oplus 2T^{(3)}, \quad n_1 = n_2 = 1,\ n_3 = 2.$$

Furthermore,

$$O T(g) O^{-1} = \begin{pmatrix} T^{(1)}(g) & & & 0 \\ & T^{(2)}(g) & & \\ & & T^{(3)}(g) & \\ 0 & & & T^{(3)}(g) \end{pmatrix}.$$

Thus, V has four eigenvalues, $\lambda_\alpha, \lambda_\beta$ which are simple and $\lambda_\gamma, \lambda_\delta$ which have multiplicity two. However, these four eigenvalues may not all be distinct.

In order to compute the eigenvalues explicitly without solving the secular equation we note that quantities such as

$$\operatorname{tr}(T(g)V), \quad \operatorname{tr} V^2$$

are independent of basis. Evaluating these quantities in a basis adapted to the eigenspaces of V we find

(8.69) $$\operatorname{tr}(T(g)V) = \operatorname{tr}[(OT(g)O^{-1})(OVO^{-1})] = \lambda_\alpha \chi^{(1)}(g) + \lambda_\beta \chi^{(2)}(g) + (\lambda_\gamma + \lambda_\delta)\chi^{(3)}(g),$$
$$\operatorname{tr} V^2 = \operatorname{tr} D^2 = \lambda_\alpha^2 + \lambda_\beta^2 + 2(\lambda_\gamma^2 + \lambda_\delta^2).$$

Similarly, evaluating $\operatorname{tr}(T(g)V)$ in the natural basis of R_6 for $g = e, c, r$ we obtain the identities

(8.70) $$\begin{aligned} \lambda_\alpha + \lambda_\beta + 2(\lambda_\gamma + \lambda_\delta) &= 6 \\ \lambda_\alpha + \lambda_\beta - \lambda_\gamma - \lambda_\delta &= 3/2 \\ \lambda_\alpha - \lambda_\beta &= 3. \end{aligned}$$

These three identities are not enough to determine the four eigenvalues. We could get other equations by computing quantities such as $\operatorname{tr} V^2$, $\operatorname{tr} V^3$ but this is complicated.

Instead we shall show that the eigenvalue $\lambda = 0$ occurs with multiplicity three. This information together with (8.70) will enable us to compute all of the eigenvalues.

The following considerations are based on symmetry arguments alone and are valid for a wide variety of oscillating systems. We make use of only two facts about the potential function $V(\underline{w})$. We first require that the system is in stable equilibrium at $\underline{w}=\underline{\theta}$. Mathematically this amounts to the assumptions $\frac{\partial V}{\partial w_j}(\underline{\theta})=0,\ 1\le j\le 6$ and the 6×6 matrix

$$V=\left(\frac{\partial^2 V}{\partial w_i \partial w_j}(\underline{\theta})\right)$$

is non-negative definite, i.e., all of the eigenvalues of this real symmetric matrix are non-negative. Second we require that the potential depend only on the relative distances between mass points. Thus, if the entire system is subjected to a Euclidean isometry the potential should not change. The potential is normalized such that $V(\underline{\theta})=0$.

Suppose the system starts out in the state $\underline{w}=\underline{\theta}$ and is subjected to the translation $T_{\underline{a}}\in E(2),\ \underline{a}=(a_1,a_2)$. Then the system ends up in the state

(8.71) $$\underline{\tilde{a}}=(a_1,a_2,a_1,a_2,a_1,a_2)$$

Throwing away all terms of order greater than two in the Taylor series for $V(\underline{w})$ we obtain

(8.72) $$0=V(\underline{\theta})=V(\underline{\tilde{a}})=\frac{1}{2}K\sum_{i,j=1}^{6}V_{ij}\tilde{a}_i\tilde{a}_j.$$

Since V is non-negative definite it follows that $V\underline{\tilde{a}}=\underline{\theta}$, i.e., each nonzero vector $\underline{\tilde{a}}$ is an eigenvector of V with eigenvalue zero. Let W^t be the two-dimensional subspace of R_6 formed by the set of all vectors $\underline{\tilde{a}}$. It is easy to show that W^t is invariant under the representation T. Furthermore the character χ^t of $T|W^t$ is just the character of the natural two-dimensional representation of D_3 as a transformation group on R_2. (In general for any physical system in the plane with point symmetry group G, χ^t is the character of the natural representation of G as a transformation group on R_2.)

Suppose the physical system is in the state $\underline{w} = \underline{\theta}$. Let the point P in R_2 be the center of mass of this system. We can introduce a cartesian coordinate system in R_2 with center P. In this system the coordinates of each mass point m_i are given by two-vectors $\underline{r}^{(i)} = (r_1^{(i)}, r_2^{(i)})$, $i = 1, 2, 3$.

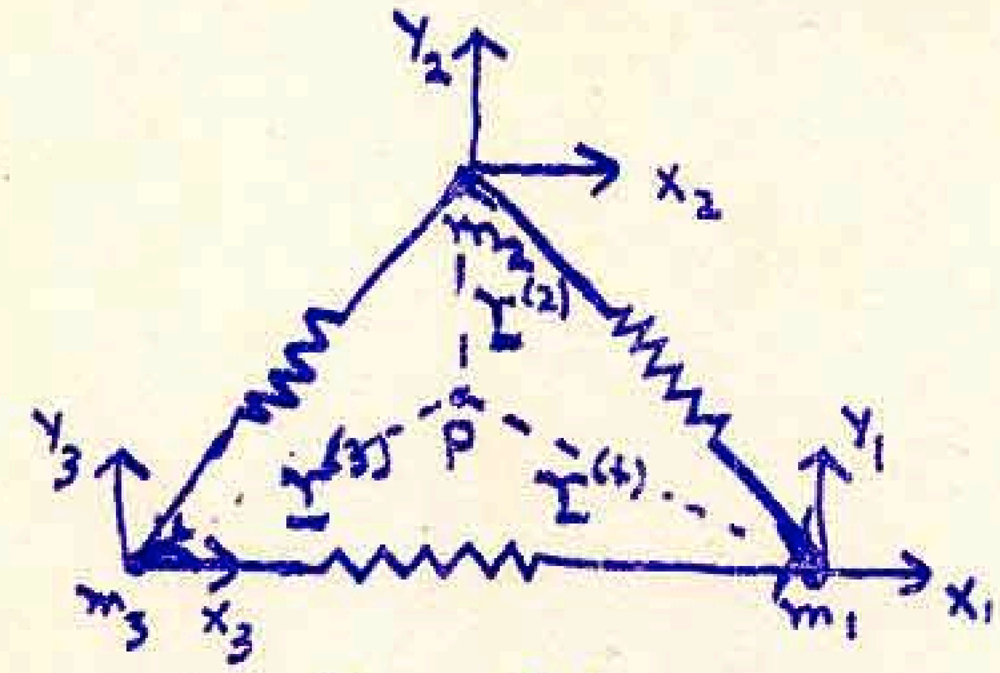


figure 3.4

If the system is in a general state $\underline{w} = (\underline{z}^{(1)}, \underline{z}^{(2)}, \underline{z}^{(3)})$, expression (8.55), the mass point m_i has coordinates $\underline{R}^{(i)} = \underline{r}^{(i)} + \underline{z}^{(i)}$ in the coordinate system based at P. Set $\tilde{V}(\underline{R}^{(1)}, \underline{R}^{(2)}, \underline{R}^{(3)}) = V(\underline{w})$. Now subject the system initially in the state $\underline{w} = \underline{\theta}$ to a rotation C_φ through the (small) angle φ in the counterclockwise direction about P. Then, up to terms of second order in φ we have

(8.73)
$$0 = V(\underline{\theta}) = \tilde{V}(C_\varphi \underline{r}^{(1)}, C_\varphi \underline{r}^{(2)}, C_\varphi \underline{r}^{(3)})$$
$$= \frac{\varphi^2}{2} K \sum_{i,j=1}^{6} V_{ij} \hat{r}_i \hat{r}_j$$

where

(8.74)
$$\underline{\hat{r}} = (-r_2^{(1)}, r_1^{(1)}, -r_2^{(2)}, r_1^{(2)}, -r_2^{(3)}, r_1^{(3)}).$$

It follows that the nonzero vector $\hat{\underline{\Upsilon}} \in R_6$ is an eigenvector of V with eigenvalue zero. Furthermore since the action of the matrices $T(g)$, $g \in D_3$, on the $\underline{\Upsilon}^{(i)}$ is merely a permutation of these vectors, and since each $\underline{\Upsilon}^{(i)}$ transforms as an axial vector under $O(2)$, it follows that the one-dimensional subspace W^r of W generated by $\hat{\underline{\Upsilon}}$ transforms according to the one-dimensional representation $T^{(2)}$ of D_3. Finally, since $\underline{\Upsilon}^{(1)}+\underline{\Upsilon}^{(2)}+\underline{\Upsilon}^{(3)}=\underline{\theta}$ we can conclude that $W^t \perp W^r$ with respect to the natural inner product on R_6. (The facts that W^r transforms according to the one-dimensional representation Q, $V\hat{\underline{\Upsilon}}=\underline{\theta}$ and $W^t \perp W^r$ are valid for any physical system in the plane with symmetry group G. Indeed $\chi(g)=1$ if $g \in G$ is a rotation and $\chi(g)=-1$ if g is a rotation reflection.) We have shown that the translational and rotational symmetry of our physical system imply that $\lambda=0$ is an eigenvalue of multiplicity at least three. The corresponding invariant subspace $W^t \oplus W^r$ transforms according to the direct sum of the one-dimensional representation Q and the natural two-dimensional representation of the symmetry group G as a transformation group. In the case $G=D_3$ we have

$$T \mid W^t \oplus W^r = T^{(2)} \oplus T^{(3)}. \tag{8.75}$$

Comparing (8.69) and (8.75) we see that $\lambda_\beta = 0$ and (say) $\lambda_\delta = 0$. Equations (8.70) reduce to

$$\lambda_\alpha + 2\lambda_\gamma = 6, \quad \lambda_\alpha - \lambda_\gamma = \frac{3}{2}, \quad \lambda_\alpha = 3.$$

Therefore,

$$\lambda_\alpha = 3, \quad \lambda_\beta = \lambda_\delta = 0, \quad \lambda_\gamma = \frac{3}{2}$$

We could also use group theory to compute the vectors $\underline{A}_j$ (8.60), which determine the normal modes of vibration. These vectors can be obtained by applying the projection operators $\underline{P}_\mu$, $\mu=1,2,3$, (7.11) on

W to project out the irreducible subspaces corresponding to $T^{(1)}$, $T^{(2)}$, $T^{(3)}$.

We briefly comment on the modification of the above results for physical systems in R_3. If the system contains N particles, it is described by $3N$ coordinates. Thus, the potential matrix V has $3N$ eigenvalues. The eigenvalue $\lambda = 0$ has multiplicity at least six. The corresponding six-dimensional subspace is the direct sum $W^t \oplus W^r$ of two three-dimensional subspaces each invariant under the action T of the point symmetry group G of the system. The restriction $T|W^t = T^t$ is equivalent to the natural three-dimensional representation of G as a transformation group on R_3. (The restriction $T|W^r = T^r$ is equivalent to the representation with character $\chi^r(g) = (-1)^{\varepsilon}\chi^t(g)$ where $\varepsilon = 1$ if $g \in O^+(3)$ and $\varepsilon = -1$ if g is a rotation-inversion. Under the action of T, the representation space W decomposes into a direct sum of invariant subspaces

$$W = W^r \oplus W^t \oplus W' .$$

The character χ of T can be decomposed in the form

$$\chi(g) = \chi^r(g) + \chi^t(g) + \chi'(g)$$

where $\chi'(g)$ is the character of the $(3N-6)$-dimensional representation $T|W'$. It is the character χ' which determines the nontrivial vibrational modes of the system.