

Section 1.11 Invariants under transformation groups.

The study of invariants of local transformation groups has important applications in physics and geometry. To indicate some of these applications in a limited space we will forego our usual insistence on complete proofs. The gaps in our presentation can be filled by a reading of [Eisenhart, 1] or [Lie, 1].

Let G be a local transformation group acting on a neighborhood of $\underline{x}^0 \in F_m$. Let $\mathcal{Q}_{\underline{x}^0}$ be the space of all functions f analytic in a neighborhood of \underline{x}^0 .

Definition: A function $f \in \mathcal{Q}_{\underline{x}^0}$ is an invariant of G if $f(\underline{x}g) = f(\underline{x})$ for each \underline{x} sufficiently close to \underline{x}^0 and g in a suitably small neighborhood of e . Note: We are considering only transformation groups and ordinary Lie derivatives. There are analogous definitions for multiplier representations and generalized Lie derivatives which we will not consider here.

The notion of an invariant can also be expressed in terms of Lie derivatives.

Theorem 1.33: The function $f \in \mathcal{Q}_{\underline{x}^0}$ is an invariant of G if and only if $Lf = 0$ for all Lie derivatives L of G .

Proof: We can write $g \in G$ uniquely in the form $g = \exp \alpha$, $\alpha \in L(G)$.

By Theorem 1.25,

$$\begin{aligned} f(\underline{x}g) &= f(\underline{x} \exp \alpha) = [(\exp L_\alpha) f](\underline{x}) \\ &= f(\underline{x}) + L_\alpha f(\underline{x}) + \frac{L_\alpha^2}{2!} f(\underline{x}) + \dots \end{aligned}$$

If $L_\alpha f(\underline{x}) = 0$ then $f(\underline{x} \exp \alpha) = f(\underline{x})$. Conversely, suppose f is invariant. Then for $\alpha \in L(G)$ and $|t|$ suitably close to 0 we have $f(\underline{x} \exp \alpha t) =$

$[(\exp tL_\alpha)f](x) = f(x)$. Differentiating with respect to t and setting $t=0$ we obtain $L_\alpha f(x) \equiv 0$. Q.E.D.

Definition: G is locally transitive at \underline{x}^0 if for any \underline{x} in a suitably small neighborhood of \underline{x}^0 there exists a $g \in G$ such that $\underline{x} = \underline{x}^0 g$.

A global Lie transformation group G is transitive on F_m if the neighborhood in which G is locally transitive is F_m itself.

Lemma 1.7: If G is locally transitive the only invariants of G are the constant functions $f(x) \equiv C$.

Proof: Let G be locally transitive at \underline{x}^0 and let f be an invariant of G . Then for \underline{x} sufficiently close to \underline{x}^0 we have $\underline{x} = \underline{x}^0 g$ and $f(\underline{x}) = f(\underline{x}^0 g) = f(\underline{x}^0)$ so f is a constant function. Q.E.D.

The converse statement is not true. For example the operator $z \frac{d}{dz}$ determines a one-parameter Lie transformation group on \mathbb{C} which is not locally transitive at $z=0$ (though it is transitive at any $z \neq 0$). However, the only solutions of $z \frac{d}{dz} f(z) = 0$ are $f(z) = C$.

Note that in the preceding section we classified all locally transitive groups acting in \mathbb{C} .

We quote without proof the fundamental theorem on the determination of invariants. Let $\underline{x}^0 \in F_m$. A set of analytic functions $\{f_1(\underline{x}), \dots, f_k(\underline{x})\}$ in $\mathcal{O}_{\underline{x}^0}$ is said to be functionally independent in case there exists no non-zero analytic function $h(y_1, \dots, y_k)$ of k variables such that $h(f_1(\underline{x}), \dots, f_k(\underline{x})) \equiv 0$ as \underline{x} runs over a neighborhood of \underline{x}^0 . Otherwise the set $\{f_j(\underline{x})\}$ is functionally dependent. Note that any set containing a constant function is automatically functionally dependent.

Let G be a n -dimensional local Lie transformation group acting on a

neighborhood of \underline{x}^0 and let $\alpha_1, \dots, \alpha_n$ be a basis for $L(G)$. It is clear that $F \in \mathcal{A}_{\underline{x}^0}$ is an invariant of G if and only if

$$L_{\alpha_j} F(\underline{x}) \equiv 0, \quad j = 1, \dots, n$$

where the L_{α_j} are the Lie derivatives corresponding to α_j . Let r be the rank of the $n \times m$ matrix $(P_{ij}(\underline{x}^0))$ where $L_{\alpha_j} = \sum P_{ij}(\underline{x}) \frac{\partial}{\partial x_i}$ and suppose the rank of $(P_{ij}(\underline{x}))$ remains r for \underline{x} in a suitably small neighborhood of \underline{x}^0 .

Theorem 1.34: The local transformation group G has $m-r$ functionally independent invariants $F_1(\underline{x}), \dots, F_q(\underline{x})$ in $\mathcal{A}_{\underline{x}^0}$. Every invariant of G can be expressed as $h(F_1(\underline{x}), \dots, F_q(\underline{x}))$ where h is some analytic function.

Example: Consider the generator $L = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$ of a one-parameter group acting in a neighborhood of zero in R_2 . Here $m=2$, and $r=1$ (except at the singular point $(0,0)$ where $r=0$) so by the theorem every invariant is of the form $h(F(x,y))$ where $F(x,y)$ is a single constant invariant. Now $LF(x,y) = 0$ implies

(11.1)

$$y \frac{\partial F}{\partial x} - x \frac{\partial F}{\partial y} = 0$$

Let $F(x,y)$ be a solution and consider the curves $F(x,y) = c$ in R_2 where c is a constant. On such a curve $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$ or $x dx + y dy = 0$. Thus $x^2 + y^2 = c'$ and, as the reader can check, $F(x,y) = x^2 + y^2$ is a solution of (11.1). The general solution is $h(x^2 + y^2)$ where h is arbitrary. It is by no means surprising that $x^2 + y^2$ is an invariant for rotations about the origin in the x - y plane, but it is interesting that we can obtain this invariant directly from Lie algebraic considerations.

Many of the Lie transformation groups in physics are locally transitive.

Such groups have only the trivial constant invariants so it appears that the notion of invariant is not of much use for these groups.

Example: The Lie derivatives of $E^+(2)$ acting on R_2 are

$$(11.2) \quad L_1 = -\frac{\partial}{\partial x}, \quad L_2 = -\frac{\partial}{\partial y}, \quad L_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

(see (10.1)). Here $m=r=2$ so there are no non-trivial invariants.

Furthermore, $E^+(2)$ is locally transitive at $(0,0)$.

One way to obtain invariants, even for a locally transitive group, is to extend or prolong the action of G to a higher dimensional space. If G acts on a neighborhood V of $F_m, x \rightarrow xg$, we can prolong G to a transformation group \hat{G} on $F_m \times \dots \times F_m = (F_m)^l$, i.e., on l copies of F_m by

$$(11.3) \quad (\underline{x}_1, \dots, \underline{x}_l) \rightarrow (\underline{x}_1g, \dots, \underline{x}_lg)$$

where the \underline{x}_i belong to V . Now \hat{G} acts on a lm -dimensional space.

If L_α are the Lie derivatives of G acting on F_m then

$$\hat{L}_\alpha = L_\alpha^{(1)} + \dots + L_\alpha^{(l)}$$

are the Lie derivatives of \hat{G} acting on $(F_m)^l$. Here $L_\alpha^{(i)}$ is identical to L_α except that it is expressed in terms of the coordinates \underline{x}_i . Let

$\{\alpha_j\}$ be a basis for $L(G)$ and let γ_1 be the dimension of the space spanned by $\{L_{\alpha_j}\}$. Clearly the dimension γ_l of the space spanned by the $\{\hat{L}_{\alpha_j}\}$ satisfies $\gamma_1 \leq \gamma_l \leq n$. Note also that

$$\hat{L}_{(a\alpha+b\beta)} = a\hat{L}_\alpha + b\hat{L}_\beta, \quad [\hat{L}_\alpha, \hat{L}_\beta] = [\hat{L}_\alpha, \hat{L}_\beta]$$

and the map $L_\alpha \rightarrow \hat{L}_\alpha$ is an isomorphism.

According to Theorem 1.34 the local transformation group \hat{G} acting on $(F_m)^l$ has $ml - \gamma_l$ functionally independent invariants. If we take l large enough we can always assure that $ml - \gamma_l \geq ml - n > 0$. Thus, we can always find non-trivial invariants for G . An invariant in $(F_m)^l$ is

called an 2-point invariant for G .

Consider the transformation group $E^+(2)$ on R_2 again. Since $m = r_1 = 2$ there are no one-point invariants. However, $2m - r_2 = 1$ so there is one two-point invariant. Indeed the matrix (P_{ij}) is

$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ y_1 & -x_1 & y_2 & -x_2 \end{pmatrix}$$

which has rank 3 except at certain isolated singular points. To find the invariant we construct the Lie derivatives \tilde{L}_i from (11.2) and solve the equations $\tilde{L}_i f = 0, i=1,2,3$, or

$$(11.4) \quad \begin{aligned} -\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} &= 0, \quad -\frac{\partial f}{\partial y_1} - \frac{\partial f}{\partial y_2} = 0 \\ y_1 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial y_2} &= 0 \end{aligned}$$

where f is a function of the four variables x_1, y_1, x_2, y_2 . Introducing new variables $t = x_1 + x_2, s = x_1 - x_2, u = y_1 + y_2, v = y_1 - y_2$ we see that equations (11.4) become

$$(11.5) \quad \frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial u} = 0, \quad v \frac{\partial f}{\partial s} - s \frac{\partial f}{\partial v} = 0.$$

Thus $f = f(s, v)$ and exactly as in the solution of (11.1) we find ~~that~~

$$(11.6) \quad f(s, v) = s^2 + v^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

is a two-point invariant. Any other two-point invariant is a function $h(s^2 + v^2)$ of this one. We have rederived the fact that the square of the distance between two points is invariant under $E^+(2)$. According to the general theory there will be $3m - r_3 = 3 \cdot 2 - 3 = 3$ independent three-point invariants. However, three distinct points $\underline{x}_1, \underline{x}_2, \underline{x}_3$ determine three distances $\|\underline{x}_1 - \underline{x}_2\|, \|\underline{x}_1 - \underline{x}_3\|, \|\underline{x}_2 - \underline{x}_3\|$ between pairs of points, so the three-point invariants are functions of these three two-point invariants.

A similar argument shows that the $2l-3$ independent l -point invariants are functions of two-point invariants. Thus, all invariants are functions of the basic one, $\|x_1 - x_2\|^2$

A less familiar example is the action of $SL(2)$ on the line. A basis for the Lie derivatives is, (10.41):

$$\frac{d}{dz}, z \frac{d}{dz}, z^2 \frac{d}{dz}.$$

Here $m=1$, $r_1=1$, $r_2=2$, $r_3=3$ and there are no one, two, or three-point invariants. Since $4m - r_4 = 4 - 3 = 1$ there is one four-point invariant. The reader can check that

$$(11.7) \quad \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

is a four-point invariant. Expression (11.7) is called the cross-ratio of the points z_1, \dots, z_4 . It is of fundamental importance in projective geometry, []. Just as in our preceding example, one can show that all l -point invariants for $l > 4$ are functions of four-point invariants.

Now we investigate a second method of prolonging the action of a local Lie transformation group G . This method gives rise to differential invariants of G . Suppose G acts on F_2 . We will prolong the action of this group so that it acts on functions $f(x, y, y')$ where $y' = \frac{dy}{dx}$. (In order to make sense out of the following discussion we must restrict ourselves to an analytic curve $y = y(x)$ in F_2 .) Consider an analytic coordinate transformation $x \rightarrow u = u(x, y)$, $y \rightarrow v = v(x, y)$. This mapping induces a corresponding transformation $y' \rightarrow \frac{dv}{du}(x, y)$ where

$$(11.8) \quad \frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} = \frac{v_x + y' v_y}{u_x + y' u_y}, \quad v_x = \frac{\partial v}{\partial x}, \quad v_y = \frac{\partial v}{\partial y}, \text{ etc.}$$

It follows that the action of G on (x, y) induces an action on the three-dimensional space $F_2^{(1)}$ with coordinates (x, y, y') :

$$(11.9) \quad \begin{aligned} (x, y, y') &\longrightarrow (xg, yg, y'g) \\ y'g &= \frac{d(yg)}{d(xg)} = \frac{(yg)_x + y'(yg)_y}{(xg)_x + y'(xg)_y} \end{aligned}$$

A straight-forward computation with the chain rule shows that $y'(g, g_2) = (y'g)_2$ so (11.9) defines an action of G as a local transformation group on $F_2^{(1)}$. Suppose

$$(11.10) \quad L = p(x, y) \frac{\partial}{\partial x} + q(x, y) \frac{\partial}{\partial y}$$

is a Lie derivative of G acting on F_2 , corresponding to the one-parameter subgroup $g(t)$. Then $g(t)$ induces the Lie derivative

$$(11.11) \quad L^{(1)} = p(x, y) \frac{\partial}{\partial x} + q(x, y) \frac{\partial}{\partial y} + s(x, y, y') \frac{\partial}{\partial y'}$$

on $F_2^{(1)}$ where

$$(11.12) \quad \begin{aligned} p(x, y) &= \left. \frac{d}{dt} xg(t) \right|_{t=0}, \quad q(x, y) = \left. \frac{d}{dt} yg(t) \right|_{t=0} \\ s(x, y) &= \left. \frac{d}{dt} y'g(t) \right|_{t=0} = \frac{\partial q}{\partial x} + \left(\frac{\partial q}{\partial y} - \frac{\partial p}{\partial x} \right) y' - (y')^2 \frac{\partial p}{\partial y}. \end{aligned}$$

Indeed, from (11.9) we have

$$\frac{d}{dt} y'g(t) = \frac{((xg)_x + y'(xg)_y)((yg)_{xt} + y'(yg)_{yt}) - ((yg)_x + y'(yg)_y)((xg)_{xt} + y'(xg)_{yt})}{((xg)_x + y'(xg)_y)^2}$$

Thus at $t=0$:

$$\begin{aligned} s(x, y, y') &= (1 + y'(0))(q_x + y'q_y) - (0 + y'(1))(p_x + y'p_y) \\ &= q_x + (q_y - p_x)y' - p_y(y')^2. \end{aligned}$$

Here $L^{(1)}$ is called the first prolongation of L . The Lie derivatives $L_\alpha^{(1)}$, $\alpha \in L(G)$, form a Lie algebra which is a homomorphic image of $L(G)$.

We can use similar methods to extend G , acting on F_m with coordinates (x_1, \dots, x_m) , to $F_m^{(1)}$ with coordinates $(x_1, \dots, x_m, x'_1, \dots, x'_m)$. Here $x'_j = \frac{dx_j}{dx_1}$ and $x_j = x_j(x_1)$, $j = 2, \dots, m$. The first prolongations of the Lie derivatives are given by formulas analogous to (11.10), (11.11).

We can also extend the action of G to higher order derivatives. For example the second prolongation of the Lie derivatives (11.10) is given by

$$(11.13) \quad L^{(2)} = p(x, y) \frac{\partial}{\partial x} + q(x, y) \frac{\partial}{\partial y} + s(x, y, y') \frac{\partial}{\partial y'} + u(x, y, y', y'') \frac{\partial}{\partial y''}$$

where $y'' = \frac{d^2 y}{dx^2}$,

$$(11.14) \quad u(x, y, y', y'') = q_{xx} + y'(2q_{xy} - p_{xx}) + (y')^2(q_{yy} - 2p_{xy}) - p_{yy}(y')^3 + y''(q_y - 2p_x - 3p_y y'),$$

and p, q, s are given by (11.12). This result is derived in exactly the same way as was the expression for $L^{(1)}$. Again the Lie derivatives $L_\alpha^{(2)}$ necessarily form a Lie algebra homomorphic to $L(G)$. Similarly we can form the k th prolongation $L_\alpha^{(k)}$ of an algebra of Lie derivatives. For the case $m=2$, the k th prolongation acts on a space of $2+k$ variables. A function $F(x, y, y', \dots, y^{(k)})$ such that

$$(11.15) \quad L_\alpha^{(k)} F = 0$$

for all $\alpha \in L(G)$ is a differential invariant of order k . It is obvious from Theorem 1.34 that for k sufficiently large there exist non-constant functions F satisfying (11.15). Thus every group has non-trivial differential invariants. To understand the significance of a differential invariant $F(x, y, y', \dots, y^{(k)})$ note that any solution $y = y(x)$ of the differential equation $F(x, \dots, y^{(k)}) = c$, c a constant, is mapped by $g \in G$ into

another solution of the same equation.

As an example consider $E^+(2)$ acting on R_2 . The Lie derivatives are given by (11.2). For the first prolongation, $m=3, r^{(1)}=3$ so there are no non-trivial invariants. The second prolongation of the Lie derivatives L_i is

$$(11.16) \quad \begin{aligned} L_1^{(2)} &= -\frac{\partial}{\partial x}, \quad L_2^{(2)} = -\frac{\partial}{\partial y} \\ L_3^{(2)} &= y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} - (1+y'^2) \frac{\partial}{\partial y'} - 3y'y'' \frac{\partial}{\partial y''}. \end{aligned}$$

Here $m=4, r^{(2)}=3$ so there is a nontrivial differential invariant.

Indeed the reader can check that

$$(11.17) \quad \varphi(x, y, y', y'') = \frac{y''}{(1+y'^2)^{3/2}}$$

is invariant. It is well-known that (11.17) is the curvature of a curve $Y(x)$ at X . Thus, the Euclidean group preserves curvature. $E^+(2)$ maps a circle of radius a , $\varphi \equiv \frac{1}{a}$, into a circle of radius a and takes straight lines, $Y''=0$ into straight lines. Of course we already knew this but it is instructive to obtain these results directly from Lie theory.

The theory of differential invariants has many important applications which we cannot include here. For example, given a first order differential equation $\varphi(x, y, y')=0$, one could search for Lie derivatives $L^{(n)}$ such that $L^{(n)}\varphi=0$. Then the corresponding one-parameter group $\exp tL$ must map solutions of the differential equation into solutions. A knowledge of $\exp tL$ helps one to solve the original differential equation. Using this and similar ideas Lie has evolved a classification of differential equations into symmetry types and has shown how a knowledge of the symmetry type contributes to the solution of the equation. See [Lie, 1], [Cohen, 1], [Ince, 1].