Chapter 6

Compact Lie Groups

6.1 Invariant Measures on Lie Groups

Let G be a real n-dimensional global Lie group of $m \times m$ matrices. A function $f(B)$ on G is **continuous** at $B \in G$ if it is a continuous function of the parameters $(g_1, \ldots, g_n)$ in a local coordinate system for G at B. Clearly if f is continuous with respect to one local coordinate system at B it is continuous with respect to all coordinate systems. If f is continuous at every $B \in G$ then it is a **continuous function** on G. We shall show how to define an infinitesimal volume element dA in G with respect to which the associated integral over the group is left-invariant, i.e.,

$$\int_G f(BA)\, dA = \int_G f(A)\, dA, \qquad B \in G, \tag{1.1}$$

where f is any continuous function on G such that either of the integrals converges. In terms of local coordinates $\mathbf{g} = (g_1, \ldots, g_n)$ at A,

$$dA = w(\mathbf{g})\, dg_1 \cdots dg_n = w(\mathbf{g})\, d\mathbf{g}, \tag{1.2}$$

where the continuous function w is called a **weight function.** If $\mathbf{k} = (k_1, \ldots, k_n)$ is another set of local coordinates at A then,

$$dA = \tilde{w}(\mathbf{k})\, dk_1 \cdots dk_n, \qquad \tilde{w}(\mathbf{k}) = w(\mathbf{g}(\mathbf{k}))\,|\det(\partial g_l/\partial k_j)|,$$

where the determinant is the Jacobian of the coordinate transformation. (For a precise definition of integrals on manifolds see Spivak [1].)

Two examples of such left-invariant measures are well known. We can identify R with the group of 2×2 matrices

(1.3) $$A_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad x \in R.$$

The continuous functions on this group are just the continuous functions $f(x)$ on the real line. Here, dx is a left-invariant measure. Indeed by a simple change of variable we have

$$\int_{-\infty}^{+\infty} f(y+x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx, \qquad y \in R.$$

where f is any continuous function on R such that the integrals converge. Since R is abelian, dx is also right-invariant.

Consider the group $U(1) = \{e^{i\theta}\}$. The continuous functions on $U(1)$ can be written $f(\theta)$, where f is continuous for $0 \le \theta \le 2\pi$ and periodic with period 2π. The measure $d\theta$ is left-invariant (right-invariant) since

$$\int_0^{2\pi} f(\varphi + \theta)\,d\theta = \int_0^{2\pi} f(\theta)\,d\theta.$$

We now show how to construct a left-invariant measure for the n-dimensional real linear Lie group G. Let $\{\mathfrak{C}_j, 1 \le j \le n\}$ be a basis for $L(G)$. We can introduce an inner product on $L(G)$ with respect to which this basis is ON. Associate the n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\sum \alpha_j \mathfrak{C}_j \in L(G)$. Now n linearly independent vectors $\alpha^{(1)}, \ldots, \alpha^{(n)}$ in $L(G)$ generate a parallelepiped in $L(G)$ with volume

(1.4) $$V = |\det(\alpha_j^{(i)})| > 0.$$

Expression (1.4) defines volume in the tangent space of the identity element.

Let $A(t)$ be an analytic curve in G such that $A(0) = A$. We call $\dot{A}(0) = \tilde{\mathfrak{a}}$ the **tangent matrix** to $A(t)$ at A. The set of all matrices $\tilde{\mathfrak{a}}$ as $A(t)$ runs over all analytic curves through A forms a vector space T_A called the **tangent space at** A. If $A(t)$ is an analytic curve through A then $A^{-1}A(t)$ is an analytic curve through E_m. Thus $(d/dt)[A^{-1}A(t)]|_{t=0} = \mathfrak{a} \in L(G)$, or $A^{-1}\tilde{\mathfrak{a}} = \mathfrak{a}$. Conversely, if $\mathfrak{a} \in L(G)$ then $A(t) = A \exp t\mathfrak{a}$ is an analytic curve through A with tangent matrix $\tilde{\mathfrak{a}} = A\mathfrak{a}$ at A. Thus every tangent matrix $\tilde{\mathfrak{a}}$ at A can be written uniquely as

(1.5) $$\tilde{\mathfrak{a}} = A\mathfrak{a}, \qquad \mathfrak{a} \in L(G).$$

Let $\mathbf{g} = (g_1, \ldots, g_n)$ be local coordinates at A. Without loss of generality we can assume $A(\mathbf{e}) = A$. The matrix functions $A(0, \ldots, 0, g_j, \ldots, 0)$ are analytic curves through A, so $\partial A/\partial g_j(\mathbf{e}) = \tilde{\mathfrak{a}}_j \in T_A$. We will define the volume V_A of the parallelepiped in T_A generated by the n tangent vectors $\tilde{\mathfrak{a}}_j$. We do this by mapping T_A back to $T_{E_m} = L(G)$. According to (1.5) there exist matrices $\mathfrak{A}_j \in L(G)$ such that

$$A^{-1}\tilde{\mathfrak{A}}_j = \mathfrak{A}_j, \qquad j = 1, \ldots, n.$$

Writing $\mathfrak{A}_j = \sum \alpha_k^{(j)} \mathfrak{C}_k$, we define the volume of the parallelepiped in T_A as the volume of its image in $L(G)$:

$$V_A(\mathbf{g}) = |\det(\alpha_k^{(j)})| > 0.$$

By construction, our volume element is left-invariant. Indeed if $B \in G$ then

$$(BA)^{-1}(\partial/\partial g_j)[BA(\mathbf{g})]|_{\mathbf{g}=\mathbf{e}} = A^{-1}B^{-1}B\tilde{\mathfrak{A}}_j = A^{-1}\tilde{\mathfrak{A}}_j = \mathfrak{A}_j,$$

so $V_{BA} = V_A$. We define the measure $d_l A$ on G by

$$d_l A = V_A(\mathbf{g})\, dg_1 \cdots dg_n. \tag{1.6}$$

Expression (1.6) is actually independent of local coordinates. If $\mathbf{k} = (k_1, \ldots, k_n)$ is another local coordinate system at A then

$$A^{-1}\frac{\partial A}{\partial k_l} = \sum_j A^{-1}\tilde{\mathfrak{A}}_j \frac{\partial g_j}{\partial k_l} = \sum_{j,s} \frac{\partial g_j}{\partial k_l}\alpha_s^{(j)}\mathfrak{C}_s,$$

so

$$V_A(\mathbf{k}) = \left|\det\left(\sum_j \frac{\partial g_j}{\partial k_l}\alpha_s^{(j)}\right)\right| = \left|\det\left(\frac{\partial g_j}{\partial k_l}\right)\right| \cdot |\det(\alpha_s^{(j)})|.$$

Thus,

$$\begin{aligned} V_A(\mathbf{k})\, dk_1 \cdots dk_n &= V_A(\mathbf{g})\, |\det(\partial g_j/\partial k_l)|\, dk_1 \cdots dk_n \\ &= V_A(\mathbf{g})\, dg_1 \cdots dg_n. \end{aligned} \tag{1.7}$$

We have shown that the integral

$$\int_G f(A)\, d_l A = \int_G f(g_1, \ldots, g_n) V_A(\mathbf{g})\, d\mathbf{g}$$

is well-defined provided it converges. Furthermore,

$$\begin{aligned} \int_G f(BA)\, d_l A &= \int_G f(BA(\mathbf{g}))V_A(\mathbf{g})\, d\mathbf{g} = \int_G f(BA(\mathbf{g}))V_{BA}(\mathbf{g})\, d\mathbf{g} \\ &= \int_G f(A)V_A(\mathbf{g})\, d\mathbf{g} = \int_G f(A)\, d_l A, \end{aligned} \tag{1.8}$$

where the third equality follows from the fact that BA runs over G if A does.

By an analogous procedure one can also define a right-invariant measure in G. Indeed the tangent matrices at A can be written uniquely as $\tilde{\mathfrak{A}} = \mathfrak{B}A$, $\mathfrak{B} \in L(G)$. Writing

$$(\partial A/\partial g_j)A^{-1} = \mathfrak{B}_j = \sum \beta_k^{(j)}\mathfrak{C}_k,$$

we define

$$W_A(\mathbf{g}) = |\det(\beta_k^{(j)})|, \qquad d_r A = W_A(\mathbf{g})\, dg_1 \cdots dg_n. \tag{1.9}$$

The reader can verify that $d_r A$ is a right-invariant measure on G.

Since $A(A^{-1}\partial A/\partial g_j)A^{-1} = (\partial A/\partial g_j)A^{-1}$, we have

$$W_A(\mathrm{g}) = |\det \tilde{A}|\cdot V_A(\mathrm{g}), \tag{1.10}$$

where $\tilde{A}$ is the automorphism $\mathfrak{a} \longrightarrow A\mathfrak{a}A^{-1}$ of $L(G)$. Thus, if $\det \tilde{A} = 1$ for all $A \in G$ then $d_lA = d_rA$ and there exists a two-sided invariant measure on G. In the next section we find sufficient conditions for the existence of a two-sided invariant measure.

It can be shown that a much larger class of groups (the locally compact topological groups) possesses left-invariant (right-invariant) measures. Furthermore, the left-invariant (right-invariant) measure of a group is unique up to a constant factor. That is, if dA and δA are left-invariant measures on G then there exists a constant $c > 0$ such that $dA = c\,\delta A$ (Naimark [1], Pontrjagin [1]).

To illustrate our construction, consider the matrix group (1.3). The matrix $A_1 - E_2$ is a basis for the one-dimensional Lie algebra. Let $A_x \in R$. Then

$$A_x^{-1}\frac{\partial A_x}{\partial x} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = A_1 - E_2$$

and $V_A(x) = 1$. Thus, $d_lA = dx$. Similarly $d_rA = dx$.

Now consider $GL(m, R)$. We can choose as parameters for A the m^2 matrix elements A_{ij}. The matrix $\partial A/\partial A_{ij}$ has a one in the ith row and jth column, and zeros every place else. A straightforward computation shows that the $m^2 \times m^2$ matrix $(\alpha_l^{(s)})$ looks like

$$(\alpha_l^{(s)}) = \begin{pmatrix} A^{-1} & & & Z \\ & A^{-1} & & \\ & & \ddots & \\ Z & & & A^{-1} \end{pmatrix}$$

if we suitably rearrange rows and columns. (This rearrangement does not affect the value of $|\det(\alpha_l^{(s)})|$.) Thus,

$$V_A = |\det(\alpha_l^{(s)})| = |\det A|^{-m}$$

and

$$d_lA = |\det A|^{-m} \prod_{j,k=1}^{m} dA_{jk}. \tag{1.11}$$

It is obvious from the symmetrical form of (1.11) that $d_rA = d_lA$.

As a final example consider the real group

$$G = \left\{A = \begin{pmatrix} e^a & b \\ 0 & 1 \end{pmatrix},\quad a, b \in R\right\}.$$

Clearly, G acts as a transformation group on the real line: $x \to e^a x + b$. The matrices

$$\mathfrak{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

form a basis for $L(G)$. Now

$$A^{-1}\frac{\partial A}{\partial a} = \begin{pmatrix} e^{-a} & -e^{-a}b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \mathfrak{e}_1,$$

$$A^{-1}\frac{\partial A}{\partial b} = \begin{pmatrix} e^{-a} & -e^{-a}b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-a} \\ 0 & 0 \end{pmatrix} = e^{-a}\mathfrak{e}_2.$$

Thus,

$$(1.12) \qquad V_A(a, b) = \left| \det\begin{pmatrix} 1 & 0 \\ 0 & e^{-a} \end{pmatrix} \right| = e^{-a}, \qquad d_l A = e^{-a}\, da\, db.$$

On the other hand

$$\frac{\partial A}{\partial a}A^{-1} = \begin{pmatrix} 1 & -b \\ 0 & 0 \end{pmatrix} = \mathfrak{e}_1 - b\mathfrak{e}_2, \qquad \frac{\partial A}{\partial b}A^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathfrak{e}_2$$

so

$$(1.13) \qquad W_A(a, b) = \left| \det\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \right| = 1, \qquad d_r A = da\, db.$$

The right and left-invariant measures of G are distinct.

6.2 Compact Linear Lie Groups

In Section 5.1 we defined the norm $\|A\|$ of an $m \times m$ matrix A and saw that every Cauchy sequence in the norm $\{A^{(j)}\}$ converges to a unique matrix A. Furthermore, $A_{ik} = \lim_{j\to\infty} A_{ik}^{(j)}$, where $A = (A_{ik})$. Indeed $\{A^{(j)}\}$ is a Cauchy sequence of matrices if and only if each of the sequences of matrix elements $\{A_{ik}^{(j)}\}$, $1 \le i, k \le m$, is Cauchy.

The following result is easy to prove from these remarks.

Lemma 6.1. Let $\{A^{(j)}\}$ and $\{B^{(j)}\}$ be Cauchy sequences of $m \times m$ matrices with limits A and B. Then $\{A^{(j)}B^{(j)}\}$ is a Cauchy sequence with $\lim_{j\to\infty} A^{(j)}B^{(j)} = AB$. Furthermore, if $A^{(j)}$ is nonsingular for all j and A is nonsingular then $\{A^{(j)-1}\}$ is a Cauchy sequence with limit A^{-1}.

In particular, multiplication and inversion in a linear Lie group are continuous with respect to the norm.

A set of $m \times m$ matrices U is **bounded** if there exists a constant $M > 0$ such that $\|A\| \leq M$ for all $A \in U$. Thus, U is bounded if and only if there exists a constant $K > 0$ such that $|A_{ik}| \leq K$ for $1 \leq i, k \leq m$ and all $A \in U$. (Prove it.) The set U is **closed** provided every Cauchy sequence in U converges to an element of U.

A subset S of the real line is **compact** if each countable sequence $\{a_j\}$, $a_i \in S$, contains a subsequence converging to a point in S. Here, S is compact if and only if it is a closed, bounded subset of R (Rudin [1]).

Definition. A (global) group of $m \times m$ matrices is **compact** if it is a bounded, closed subset of the set L_m of all $m \times m$ matrices.

A group G is closed provided every Cauchy sequence $\{A^{(j)}\}$ in G converges to an element of G.

The classical groups $O(m, R)$, $SO(m, R)$, $U(m)$, $SU(m)$, and $USp(m)$ are compact. We verify this fact only for $O(m, R)$ since the other proofs are similar.

If $A \in O(m, R)$ then $A^tA = E_m$, or

$$\sum_{i=1}^{m} A_{il}A_{ik} = \delta_{lk}.$$

Setting $l = k$, we obtain $\sum_i (A_{ik})^2 = 1$, so $|A_{ik}| \leq 1$ for all i, k. Thus, the matrix elements of A are bounded. Let $\{A^{(j)}\}$ be a Cauchy sequence in $O(m, R)$ with limit A. Then

$$E_m = \lim_{j \to \infty} (A^{(j)})^t A^{(j)} = A^t A,$$

So $A \in O(m, R)$ and $O(m, R)$ is compact.

Suppose G is a real, compact, linear Lie group of dimension n. It follows from the Heine–Borel Theorem (Rudin [1]) that the group manifold of G can be covered by a finite number of bounded coordinate patches. Thus, for any continuous function $f(A)$ on G, the integral

$$\int_G f(A)\, d_lA = \int_G f(A) V_A(\mathbf{g})\, d\mathbf{g} \tag{2.1}$$

will converge (since the domain of integration is bounded.) In particular the integral

$$V_G = \int_G 1\, d_lA, \tag{2.2}$$

called the **volume** of G, converges. If G is not compact the integrals (2.1) and (2.2) may not converge. Indeed, if $G = R$, the real line, then (2.2) diverges.

The above remarks also hold for the right-invariant measure d_rA. Moreover, we can show $d_lA = d_rA$ for compact groups.

Theorem 6.1. If G is a compact linear Lie group then $d_lA = d_rA$.

Proof. By (1.10), $d_r A = |\det \tilde{A}|\, d_l A$, where $\tilde{A}$ is the inner automorphism $\mathfrak{A} \to A\mathfrak{A}A^{-1}$ of $L(G)$. We can think of $\tilde{A}$ as an $m^2 \times m^2$ matrix rep of G. Since G is compact the matrices $A, A^{-1} \in G$ are uniformly bounded. Thus the matrices $\tilde{A}$ are bounded and there exists a constant $M > 0$ such that $|\det \tilde{A}| \le M$ for all $A \in G$. Now fix A and suppose $|\det \tilde{A}| = s > 1$. Then

$$|\det \tilde{A}^j| = |\det \tilde{A}|^j = s^j, \qquad j = 1, 2, \ldots .$$

Choosing j sufficiently large we get $s^j > M$, which is impossible. Thus $s \le 1$. If $s < 1$ then

$$|\det \tilde{A}^{-1}| = |\det \tilde{A}|^{-1} = s^{-1} > 1$$

which is impossible. Therefore $s = 1$ for all $A \in G$ and $d_l A = d_r A$. Q.E.D.

For G compact we write $dA = d_l A = d_r A$, where the measure dA is both left- and right-invariant.

Using the invariant measure for compact groups, we can mimic the proofs of most of the results for finite groups obtained in Sections 3.1–3.3. In particular, we will show that any finite-dimensional rep of a compact group can be decomposed into a direct sum of irred reps and we will obtain orthogonality relations for the matrix elements and characters of irred reps.

For finite groups K these results were proved using the average of a function over K. If f is a function on K then the **average** of f over K is

$$\mathfrak{A}v(f(k)) = [1/n(K)] \sum_{k \in K} f(k). \tag{2.3}$$

If $h \in K$ then

$$\mathfrak{A}v(f(hk)) = \mathfrak{A}v(f(kh)) = \mathfrak{A}v(f(k)). \tag{2.4}$$

Furthermore,

$$\mathfrak{A}v(a_1 f_1(k) + a_2 f_2(k)) = a_1 \mathfrak{A}v(f_1(k)) + a_2 \mathfrak{A}v(f_2(k)), \qquad \mathfrak{A}v(1) = 1. \tag{2.5}$$

Properties (2.4) and (2.5) are sufficient to prove most of the fundamental results on the reps of finite groups. Now let G be a compact linear Lie group and let f be a continuous function on G. We define

$$\mathfrak{A}v(f(A)) = (1/V_G) \int_G f(A)\, dA = \int_G f(A)\, \delta A \tag{2.6}$$

where dA is the invariant measure on G, $V_G = \int_G 1\, dA$ is the **volume** of G, and $\delta A = V_G^{-1}\, dA$ is the **normalized** invariant measure. Then

$$\begin{aligned} &\mathfrak{A}v(f(BA)) = \int_G f(BA)\, \delta A = \int_G f(A)\, \delta A = \mathfrak{A}v(f(A)), \\ &\mathfrak{A}v(f(AB)) = \mathfrak{A}v(f(A)), \qquad \mathfrak{A}v(1) = \int_G \delta A = 1, \qquad B \in G, \end{aligned} \tag{2.7}$$

since δA is both left- and right-invariant. Thus, $\mathcal{A}v(f(A))$ also satisfies properties (2.4) and (2.5).

We now study the **continuous** reps of G, i.e., reps $\mathbf{T}$ such that the operators $\mathbf{T}(A)$ are continuous functions of the group parameters of $A \in G$.

Theorem 6.2. Let $\mathbf{T}$ be a continuous rep of the compact linear Lie group G on the finite-dimensional inner product space V. Then $\mathbf{T}$ is equivalent to a unitary rep on V.

Proof. Let $\langle -, - \rangle$ be the inner product on V. We define an inner product $(-, -)$ on V with respect to which $\mathbf{T}$ is unitary. For $\mathbf{u}, \mathbf{v} \in V$ define

$$(2.8) \qquad (\mathbf{u}, \mathbf{v}) = \int_G \langle \mathbf{T}(A)\mathbf{u}, \mathbf{T}(A)\mathbf{v} \rangle \, \delta A = \mathcal{A}v[\langle \mathbf{T}(A)\mathbf{u}, \mathbf{T}(A)\mathbf{v} \rangle].$$

(The integral converges since the integrand is continuous and the domain of integration is finite.) It is straightforward to check that $(-, -)$ is an inner product. In particular the positive-definite property follows from the fact that the weight function is strictly positive. Now

$$\begin{aligned}(\mathbf{T}(B)\mathbf{u}, \mathbf{T}(B)\mathbf{v}) &= \mathcal{A}v[\langle \mathbf{T}(AB)\mathbf{u}, \mathbf{T}(AB)\mathbf{v} \rangle] \\ &= \mathcal{A}v[\langle \mathbf{T}(A)\mathbf{u}, \mathbf{T}(A)\mathbf{v} \rangle] = (\mathbf{u}, \mathbf{v}),\end{aligned}$$

so $\mathbf{T}$ is unitary with respect to $(-, -)$. The remainder of the proof is identical with that of Theorem 3.1. Q.E.D.

The theorem shows that we can restrict ourselves to the study of unitary reps $\mathbf{T}$ with no loss of generality.

Theorem 6.3. If $\mathbf{T}$ is a unitary rep of G on V and W is an invariant subspace of V then $W^\perp$ is also an invariant subspace under $\mathbf{T}$.

Theorem 6.4. Every finite-dimensional, continuous, unitary rep of a compact linear Lie group can be decomposed into a direct sum of irred unitary reps.

The proofs of these theorems are identical with the corresponding proofs for finite groups.

Let $\{\mathbf{T}^{(\mu)}\}$ be a complete set of nonequivalent unitary irred reps of G, labeled by the parameter μ. (Here we consider only reps of G on *complex* vector spaces.) Initially we have no way of telling how many distinct values μ can take. (It will turn out that μ takes on a countably infinite number of values, so that we can choose $\mu = 1, 2, \ldots$.) We introduce an ON basis in each rep space $V^{(\mu)}$ to obtain a unitary $n_\mu \times n_\mu$ matrix rep $T^{(\mu)}$ of G.

Now we mimic the construction of the orthogonality relations for finite groups. Given the matrix reps $T^{(\mu)}, T^{(\nu)}$, choose an arbitrary $n_\mu \times n_\nu$ matrix

C and form the $n_\mu \times n_\nu$ matrix

$$(2.9) \qquad D = \mathcal{A}v[T^{(\mu)}(A)CT^{(\nu)}(A^{-1})] = \int_G T^{(\mu)}(A)CT^{(\nu)}(A^{-1})\,\delta A.$$

Just as in the corresponding construction for finite groups, one can easily verify that

$$(2.10) \qquad T^{(\mu)}(B)D = DT^{(\nu)}(B)$$

for all $B \in G$. Recall that the Schur lemmas are valid for finite-dimensional reps of all groups, not just finite groups. Thus if $\mu \neq \nu$, i.e., $T^{(\mu)}$ not equivalent to $T^{(\nu)}$, then $D = Z$. If $\mu = \nu$ then $D = \lambda E_{n_\mu}$ for some $\lambda \in \mathfrak{C}$.

$$D(C, \mu, \nu) = \lambda(\mu, C)\,\delta_{\mu\nu}E_{n_\mu}.$$

Letting C run over all $n_\mu \times n_\nu$ matrices, we obtain the independent identities

$$(2.11) \qquad \int_G T^{(\mu)}_{il}(A)T^{(\nu)}_{ks}(A^{-1})\,\delta A = \lambda(\mu, l, k)\,\delta_{\mu\nu}\,\delta_{is},$$

for the matrix elements $T^{(\mu)}_{il}(A)$. To evaluate λ we set $\nu = \mu$ and $s = i$ and sum on i:

$$\sum_{i=1}^{n_\mu} \lambda = n_\mu \lambda = \int_G \sum_{i=1}^{n_\mu} T^{(\mu)}_{ki}(A^{-1})T^{(\mu)}_{il}(A)\,\delta A = \delta_{kl}.$$

Therefore $\lambda = \delta_{kl}/n_\mu$. Since the matrices $T^{(\mu)}(A)$ are unitary, (2.11) becomes

(2.12)

$$\int_G T^{(\mu)}_{il}(A)\overline{T^{(\nu)}_{sk}(A)}\,\delta A = (\delta_{is}/n_\mu)\,\delta_{lk}\,\delta_{\mu\nu}, \qquad 1 \le i, l \le n_\mu, \quad 1 \le s, k \le n_\nu.$$

These are the **orthogonality relations** for matrix elements of irred reps of G.

In the case of finite groups K we were able to relate the orthogonality relations to an inner product on the group ring R_K. We can consider R_K as the space of all functions $f(k)$ on K. Then

$$\langle f_1, f_2 \rangle = [1/n(K)] \sum_{k \in K} f_1(k)\overline{f_2(k)}$$

defines an inner product on R_K with respect to which the functions $\{n_\mu^{1/2}T^{(\mu)}_{il}(k)\}$ form an ON basis. We extend this idea to compact linear Lie groups G as follows: Let $L_2(G)$ be the space of all functions on G which are (Lebesgue) square-integrable:

$$(2.13) \qquad L_2(G) = \left\{ f(A) : \int_G |f(A)|^2\,\delta A < \infty \right\}.$$

With respect to the inner product

$$(2.14) \qquad \langle f_1, f_2 \rangle = \int_G f_1(A)\overline{f_2(A)}\,\delta A,$$

$L_2(G)$ is a Hilbert space (see the Appendix). Note that every continuous function on G belongs to $L_2(G)$. Let

(2.15) $$\varphi_{ij}^{(\mu)}(A) = n_\mu^{1/2} T_{ij}^{(\mu)}(A).$$

It follows from (2.12) and (2.14) that $\{\varphi_{ij}^{(\mu)}\}$, where $1 \le i, j \le n_\mu$ and μ ranges over all equivalence classes of irred reps, forms an ON set in $L_2(G)$.

For finite groups we know that the set $\{\varphi_{ij}^{(\mu)}\}$ is an ON **basis** for the group ring and every function f on the group can be written as a unique linear combination of these basis functions. Similarly one can show that for G compact the set $\{\varphi_{ij}^{(\mu)}\}$ is an ON basis for $L_2(G)$. Thus, every $f \in L_2(G)$ can be expanded uniquely in the (generalized) **Fourier series**

(2.16) $$f(A) \sim \sum_{\mu=1}^{\infty} \sum_{i,k=1}^{n_\mu} c_{ik}^{\mu} \varphi_{ik}^{(\mu)}(A),$$

where

(2.17) $$c_{ik}^{\mu} = \langle f, \varphi_{ik}^{(\mu)} \rangle.$$

Furthermore, we have the **Parseval equality**

$$\langle f, f \rangle = \sum_{\mu=1}^{\infty} \sum_{i,k=1}^{n_\mu} |c_{ik}^{\mu}|^2.$$

[We use $\sim$ rather than $=$ in (2.16) to denote that f and $\sum c_{ij}^{\mu} \varphi_{ij}^{(\mu)}$ are the same Hilbert space vector. We do *not* claim that the two sides of the equality are necessarily pointwise equal.]

We illustrate this result, the celebrated **Peter–Weyl theorem,** for an important example, the circle group $U(1)$.

Lemma 6.2. Let G be an abelian group (not necessarily a Lie group) and let $\mathbf{T}$ be a finite-dimensional irred rep of G on a complex vector space V. Then $\mathbf{T}$ is one-dimensional.

Proof. Suppose $\mathbf{T}$ is irred on V and dim $V > 1$. There must exist a $g \in G$ such that $\mathbf{T}(g)$ is not a multiple of the identity operator on V, for otherwise V would be reducible. Let λ be a eigenvalue of $\mathbf{T}(g)$ and let C_λ be the eigenspace

$$C_\lambda = \{\mathbf{v} \in V : \mathbf{T}(g)\mathbf{v} = \lambda \mathbf{v}\}.$$

Clearly C_λ is a proper subspace of V. If $h \in G$ and $\mathbf{w} \in C_\lambda$ then

$$\mathbf{T}(g)(\mathbf{T}(h)\mathbf{w}) = \mathbf{T}(h)(\mathbf{T}(g)\mathbf{w}) = \lambda(\mathbf{T}(h)\mathbf{w})$$

since G is abelian, so C_λ is invariant under the operator $\mathbf{T}(h)$. Therefore, $\mathbf{T}$ is reducible. Impossible! Q.E.D.

Although all complex irred reps of an abelian group are one-dimensional, the lemma is false for real reps (see Problem 6.8).

The circle group $U(1) = \{e^{i\theta}\}$ is compact and abelian. Hence its irred matrix reps are continuous functions $\chi(\theta)$ such that

$$\chi(\theta_1 + \theta_2) = \chi(\theta_1)\chi(\theta_2), \qquad \theta_1, \theta_2 \in R, \tag{2.18}$$

and $\chi(\theta + 2\pi) = \chi(\theta)$. The functional equation (2.18) has only the solutions $\chi(\theta) = e^{a\theta}$ and the periodicity of χ implies $a = im$, where m is an integer. Therefore, there are an infinite number of irreducible unitary representations of $U(1)$:

$$\chi_m(\theta) = e^{im\theta}, \qquad m = 0, \pm 1, \pm 2, \ldots.$$

The invariant measure on $U(1)$ is $d\theta$. The space $L_2(U(1))$ is just the space $L_2[0, 2\pi]$ consisting of all functions $f(\theta)$ with period 2π such that $\int_0^{2\pi} |f(\theta)|^2 \, d\theta < \infty$. By the Peter–Weyl theorem the functions $\{e^{im\theta}\}$ form an ON basis for $L_2[0, 2\pi]$. Every $f \in L_2[0, 2\pi]$ can be expressed uniquely in the form

$$f(\theta) \sim \sum_{m=-\infty}^{\infty} c_m e^{im\theta}, \qquad c_m = (1/2\pi) \int_0^{2\pi} f(\theta) e^{-im\theta} \, d\theta. \tag{2.19}$$

$\frac{d\theta}{2\pi}$

Furthermore,

$$(1/2\pi) \int_0^{2\pi} |f(\theta)|^2 \, d\theta = \sum_{m=-\infty}^{\infty} |c_m|^2. \tag{2.20}$$

Here (2.19) is the well-known Fourier series expansion of a periodic function and (2.20) is Parseval's equality. It is clear from this example that the Peter–Weyl theorem is a group-theoretic generalization of classical Fourier series analysis. Furthermore, we see that the classical theory has a group-theoretic structure.

Theorem 6.5 (Peter–Weyl). If G is a compact linear Lie group, the set $\{\varphi_{ij}^{(\mu)}\}$ is an ON basis for $L_2(G)$.

The proof of this theorem depends heavily on facts about symmetric completely continuous operators in Hilbert space and will not be given here. For the details see Chevalley [1] or Naimark [1].

Corollary 6.1. A compact linear Lie group G has a countably infinite (not finite) number of equivalence classes of irred reps $\{\mathbf{T}^{(\mu)}\}$. Thus, we can label the reps so that $\mu = 1, 2, \ldots$.

Proof. The functions $\{\varphi_{jk}^{(\mu)}\}$ form an ON basis for $L_2(G)$. Since $L_2(G)$ is a separable, infinite-dimensional Hilbert space there are a countably infinite number of basis vectors (Helwig [1]). Q.E.D.

Corollary 6.2. Let G, H be compact linear Lie groups with equivalence classes of irred reps $\{\mathbf{T}^{(\mu)}\}, \{\mathbf{U}^{(\nu)}\}$, respectively, Then $\{\mathbf{T}^{(\mu)} \otimes \mathbf{U}^{(\nu)}\}$ is the complete set of equivalence classes of irred reps for the compact group $G \times H$.

The proof, which is left to the reader, consists in showing that the functions $(n_\mu n_\nu)^{1/2}T^{(\mu)}_{ij}(A)U^{(\nu)}_{kl}(B)$ form an ON basis for $L_2(G \times H)$.

The Peter–Weyl theorem refers to continuous reps of the real compact Lie group G, while the Lie-theoretic methods of Chapter 5 apply only to analytic reps. The possibility arises that there may be continuous reps of G which are not analytic. For such reps, Lie-algebraic methods make no sense. Fortunately, the following result eliminates this possibility.

Theorem 6.6. Let $\mathbf{T}$ be a finite-dimensional continuous rep of the real compact linear Lie group G on the inner product space V. Then $\mathbf{T}$ is analytic (with respect to suitable coordinates for G) (see Naimark [2]).

6.3 Group Characters and Representations

The theory of characters for compact Lie groups is almost identical with the character theory for finite groups presented in Section 3.4. The principal difference is that the sum over a finite group is replaced by an integral.

Let $\mathbf{T}$ be a rep of the compact linear Lie group G on the m-dimensional vector space V. With respect to a fixed basis in V the operators $\mathbf{T}(A)$ define a matrix rep $T(A)$. The **character** of $\mathbf{T}$ is the function

$$\chi(A) = \operatorname{tr} T(A).$$

Since $\operatorname{tr}(ST(A)S^{-1}) = \operatorname{tr} T(A)$ the character is independent of basis in V and equivalent reps have the same character. The character of an irred rep is **simple,** while the character of a reducible rep is **compound.** If $\mathbf{T}$ is unitary then its corresponding character satisfies the relation $\overline{\chi(A)} = \chi(A^{-1})$. However, every rep is equivalent to a unitary rep, so the preceding identity is satisfied by all characters. Every character is a continuous function on G.

Let $\{\mathbf{T}^{(\mu)}\}$ be a complete set of nonequivalent unitary irred reps of G and let $\{\chi^{(\mu)}\}$ be the corresponding simple characters. The orthogonality relations (2.12) for matrix elements imply the following orthogonality relations for characters:

$$(\chi^{(\mu)}, \chi^{(\nu)}) = \int_G \chi^{(\mu)}(A)\overline{\chi^{(\nu)}(A)}\, \delta A = \delta_{\mu\nu}, \qquad \mu, \nu = 1, 2, \ldots. \tag{3.1}$$

The proof is identical with that for finite groups.

Let $\mathbf{T}$ be a finite-dimensional unitary rep of G with character χ. By Theorem 6.4 we can decompose $\mathbf{T}$ into a direct sum of irred reps,

$$\mathbf{T} = \sum_{\mu=1}^{\infty} \oplus\, a_\mu \mathbf{T}^{(\mu)}. \tag{3.2}$$

Here the integer a_μ denotes the multiplicity of $\mathbf{T}^{(\mu)}$ in $\mathbf{T}$. Only a finite number of the $\{a_\mu\}$ are nonzero. We shall show that the multiplicities a_μ are uniquely determined by $\mathbf{T}$, i.e., they are independent of the method by which $\mathbf{T}$ is decomposed into irred reps. From (3.2), the character of $\mathbf{T}$ can be expressed in the form

$$\chi = \sum_{\mu=1}^{\infty} a_\mu \chi^{(\mu)}. \tag{3.3}$$

According to the orthogonality relations

$$(\chi, \chi^{(\nu)}) = \sum_{\mu=1}^{\infty} a_\mu (\chi^{(\mu)}, \chi^{(\nu)}) = a_\nu, \qquad \nu = 1, 2, \ldots. \tag{3.4}$$

Since (3.4) is independent of basis, the multiplicities a_μ must be uniquely determined.

Theorem 6.7. Let $\mathbf{T}$ be a rep of G with character χ. The multiplicity a_μ of $\mathbf{T}^{(\mu)}$ in $\mathbf{T}$ is given by $(\chi, \chi^{(\mu)}) = a_\mu$. Two reps with the same character are equivalent.

Corollary 6.3. The rep $\mathbf{T}$ is irred if and only if $(\chi, \chi) = 1$.

Example. The simple characters of the circle group $U(1)$ are just $\chi^{(n)}(\theta) = e^{in\theta}$, $n = 0, \pm 1, \ldots$. (Here it is more convenient to let the index of the irred reps run over all integers rather than over the nonnegative integers.) The orthogonality relations are

$$(\chi^{(n)}, \chi^{(m)}) = (1/2\pi) \int_0^{2\pi} e^{i(n-m)\theta}\, d\theta = \delta_{nm}.$$

In Section 3.7 we used the method of projection operators to explicitly decompose a rep into a direct sum of irred reps. These methods carry over to compact Lie groups virtually unchanged. Thus we present the results without detailed proof.

Let $\mathbf{T}$ be a unitary rep of the compact linear Lie group G on the inner product space V. Corresponding to the decomposition

$$\mathbf{T} = \sum_{\mu=1}^{\infty} \oplus\, a_\mu \mathbf{T}^{(\mu)}$$

of $\mathbf{T}$ there is a decomposition

$$V = \sum_{\mu=1}^{\infty} \oplus\, V^{(\mu)}, \qquad V^{(\mu)} = \sum_{i=1}^{a_\mu} \oplus\, V_i^{(\mu)}, \tag{3.5}$$

where $\mathbf{T} \mid V_i^{(\mu)}$ is equivalent to $\mathbf{T}^{(\mu)}$. These spaces $V_i^{(\mu)}$ are not uniquely determined. Define the linear operators $\mathbf{P}_\mu$ on V by

$$\mathbf{P}_\mu = n_\mu \int_G \overline{\chi^{(\mu)}(A)}\, \mathbf{T}(A)\, \delta A, \qquad \mu = 1, 2, \ldots. \tag{3.6}$$

To make sense of (3.6) choose a basis $\{\mathbf{v}_j\}$ for V with respect to which $T(A)$

is the matrix of $\mathbf{T}(A)$. Then $\mathbf{P}_\mu$ is the operator on V whose matrix is

$$P_\mu = n_\mu \int_G \overline{\chi^{(\mu)}(A)} T(A)\, \delta A.$$

It follows from (3.6) that $\mathbf{T}(B)\mathbf{P}_\mu = \mathbf{P}_\mu \mathbf{T}(B)$ for all $B \in G$. Furthermore, $\mathbf{P}_\mu{}^2 = \mathbf{P}_\mu$ and $\mathbf{P}_\mu{}^* = \mathbf{P}_\mu$. Thus, $\mathbf{P}_\mu$ is a self-adjoint projection operator on V.

Let $\{\mathbf{T}^{(\mu)}\}$ be a complete set of nonequivalent irred unitary matrix reps of G and choose a basis $\{\mathbf{v}_{ij}^{(\nu)}\}$ in each subspace $V_i^{(\nu)}$ such that

(3.7) $$\mathbf{T}(A)\mathbf{v}_{ij}^{(\nu)} = \sum_{k=1}^{n_\nu} T_{kj}^{(\nu)}(A)\mathbf{v}_{ik}^{(\nu)}, \qquad 1 \le j \le n_\nu.$$

Then, just as in (7.12), Section 3.7, one can prove

(3.8) $$\mathbf{P}_\mu \mathbf{v}_{ij}^{(\nu)} = \delta_{\mu\nu}\mathbf{v}_{ij}^{(\nu)}, \qquad \mu, \nu = 1, 2, \ldots, \quad 1 \le i \le a_\nu, \quad 1 \le j \le n_\nu.$$

Thus $\mathbf{P}_\mu$ projects onto the invariant subspace $V^{(\mu)}$. Since the definition of $\mathbf{P}_\mu$ is basis-independent, $V^{(\mu)}$ is uniquely determined. To find the $V_i^{(\mu)}$ we define operators

(3.9) $$\mathbf{P}_\mu^{lk} = n_\mu \int_G \bar{T}_{lk}^{(\mu)}(A)\mathbf{T}(A)\, \delta A, \qquad 1 \le l, k \le n_\mu,$$

which are easily shown to have the properties

(3.10) $$\mathbf{P}_\mu^{lk}\mathbf{v}_{ij}^{(\mu)} = \delta_{\mu\nu}\,\delta_{jk}\,\mathbf{v}_{il}^{(\nu)},$$

(3.11) $$\mathbf{P}_\mu^{lk}\mathbf{P}_{\mu'}^{l'k'} = \delta_{\mu\mu'}\,\delta_{kl'}\,\mathbf{P}_\mu^{lk'}, \qquad (\mathbf{P}_\mu^{lk})^* = \mathbf{P}_\mu^{kl}, \qquad \mathbf{P}_\mu = \sum_{k=1}^{n_\mu} \mathbf{P}_\mu^{kk}.$$

Thus $\mathbf{P}_\mu^{kk}$ is the self-adjoint projection operator on the a_μ-dimensional space $W_k^{(\mu)}$ spanned by the ON basis vectors $\{\mathbf{v}_{ik}^{(\mu)} : 1 \le i \le a_\mu\}$. The remaining details for the construction of the spaces $V_i^{(\mu)}$ are identical with those for finite groups.

We now extend the concept of group rep from finite-dimensional inner product spaces to Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space and G a (global) linear Lie group of $m \times m$ matrices.

Definition. A (bounded) **representation** $\mathbf{T}$ of G on $\mathcal{H}$ is a correspondence which assigns to each $A \in G$ a bounded linear operator $\mathbf{T}(A)$ on $\mathcal{H}$ such that

(3.12) $$\mathbf{T}(A)\mathbf{T}(B) = \mathbf{T}(AB), \qquad \mathbf{T}(E_m) = \mathbf{E},$$

where $A, B \in G$ and $\mathbf{E}$ is the identity operator on $\mathcal{H}$.

Note that $\mathbf{T}(A)$ is invertible and $\mathbf{T}(A)^{-1} = \mathbf{T}(A^{-1})$. The rep $\mathbf{T}$ is **irreducible** if $\mathcal{H}$ contains no proper **closed** subspace (closed in the norm) which is invariant under $\mathbf{T}$. Otherwise $\mathbf{T}$ is **reducible.** Every finite-dimensional subspace of a Hilbert space is closed. (Prove it.) Thus for finite-dimensional reps the above definition of irreducibility coincides with that given in Chapter 3.

Suppose $\mathcal{W}$ is an invariant subspace of $\mathcal{H}$. Since $\mathbf{T}(A)$ is bounded, the closure $\overline{\mathcal{W}}$ is invariant under $\mathbf{T}(A)$ for all $A \in G$. (Prove it.) Thus $\overline{\mathcal{W}}$ is also an invariant subspace of $\mathcal{H}$. Since we can always close an invariant subspace, we restrict ourselves to closed invariant subspaces in the definition of irreducibility.

A rep $\mathbf{T}$ is **unitary** if each operator $\mathbf{T}(A)$ is unitary for all A, and **continuous** if $\langle \mathbf{T}(A)\mathbf{v}, \mathbf{w}\rangle$ is a continuous function of A for each $\mathbf{v}, \mathbf{w} \in \mathcal{H}$. Here $\langle -, -\rangle$ is the inner product on $\mathcal{H}$. Unless otherwise stated, we consider only continuous reps.

For G compact we can carry over many of our results for finite-dimensional unitary reps to Hilbert space reps. Let $\mathbf{T}$ be a unitary rep of G on the separable Hilbert space $\mathcal{H}$. We define operators $\mathbf{P}_\mu$, $\mathbf{P}_\mu^{lk}$ on $\mathcal{H}$ by

$$\text{(3.13)}\qquad \mathbf{P}_\mu = n_\mu \int_G \bar{\chi}^{(\mu)}(A)\mathbf{T}(A)\,\delta(A), \qquad \mathbf{P}_\mu^{lk} = n_\mu \int_G \bar{T}_{lk}^{(\mu)}(A)\mathbf{T}(A)\,\delta A.$$

To make sense of these expressions choose an ON basis $\{\mathbf{v}_i\}$ for $\mathcal{H}$ and let $T(A)$ be the (possibly infinite) matrix corresponding to $\mathbf{T}(A)$:

$$\text{(3.14)}\qquad \mathbf{T}(A)\mathbf{v}_i = \sum_{j=1}^{\infty} T_{ji}(A)\mathbf{v}_j, \qquad i = 1, 2, \ldots .$$

Since $\mathbf{T}$ is continuous the matrix elements $T_{ji}(A)$ are continuous functions on G. By $\mathbf{P}_\mu$ we mean the linear operator on $\mathcal{H}$ whose matrix with respect to $\{\mathbf{v}_i\}$ is

$$P_\mu = n_\mu \int_G \bar{\chi}^{(\mu)}(A)T(A)\,\delta A.$$

There is a similar definition for $\mathbf{P}_\mu^{lk}$. It can be shown that the properties (3.11) which were valid for $\mathcal{H}$ finite-dimensional are true in general. In fact we have the following result.

Theorem 6.8. A unitary rep $\mathbf{T}$ of a compact Lie group G on $\mathcal{H}$ can be decomposed into a direct sum of unitary irred reps $\mathbf{T}^{(\mu)}$: $\mathbf{T} \cong \sum_{\mu=1}^{\infty} \oplus\, a_\mu \mathbf{T}^{(\mu)}$. Indeed there exist mutually orthogonal subspaces $\mathcal{V}_\mu{}^m$, $m = 1, \ldots, a_\mu$, of $\mathcal{H}$ such that $\mathcal{H} = \sum_{m,\mu} \oplus\, \mathcal{V}_\mu{}^m$ and $\mathbf{T} \mid \mathcal{V}_\mu{}^m \cong \mathbf{T}^{(\mu)}$. The $\mathcal{V}_\mu{}^m$ are not unique but the multiplicity $a_\mu = 0, 1, 2, \ldots, \infty$ is unique, as is the space $\mathcal{H}_\mu = \sum_m \oplus\, \mathcal{V}_\mu{}^m$. If $\dim \mathcal{H}_\mu = h_\mu$ is finite then $a_\mu = h_\mu/n_\mu$.

The proof of this theorem makes use of the Peter–Weyl theorem (Naimark [2] or Talman [1]). The proof is constructive in the sense that one can use the method discussed following Eq. (7.18), Section 3.7, to explicitly decompose $\mathcal{H}$. The only difference is that the multiplicity a_μ may be countably infinite.

Problems

6.1 Compute the invariant measure on $U(2)$.

6.2 Compute the left- and right-invariant measures on $SL(n, R)$.

6.3 Prove: If G is a compact linear Lie group then $d(A^{-1}) = dA$, i.e., $\int_G f(A^{-1})\, dA = \int_G f(B)\, dB$. [Hint: Show that $V_{A^{-1}}(\mathrm{g}) = |\det(-\tilde{A})|\, V_A(\mathrm{g})$, where $\tilde{A}$ is the automorphism $\mathfrak{a} \longrightarrow A\mathfrak{a}A^{-1}$ of $L(G)$, and use the proof of Theorem 6.1.]

6.4 Prove that the identity rep is contained in the tensor product $\mathbf{T}_1 \otimes \mathbf{T}_2$ of two irred reps of a compact Lie group G if and only if $\mathbf{T}_1 \cong \overline{\mathbf{T}}_2$.

6.5 Prove Corollary 6.3.

6.6 Let G be a compact Lie group with simple characters $\{\chi^{(\mu)}(A)\}$. Show that the $\{\chi^{(\mu)}(A)\}$, suitably renormalized, form an ON basis for the subspace of $L_2(G)$ consisting of all functions constant on conjugacy classes.

6.7 Prove relations (3.11) directly from the definition (3.9) of the $\mathbf{P}_\mu^{lk}$. Do not use the auxiliary relations (3.10).

6.8 Construct a real irred two-dimensional rep of the circle group $U(1)$.

6.9 Show how to decompose any real finite-dimensional rep of $U(1)$ as a direct sum of real irred reps.