## Section 2.4 Hilbert space representations.

We now extend the concept of group representation from finite-dimensional inner product spaces to Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space and $G$ a (global) linear Lie group of $m \times m$ matrices.

<u>Definition:</u> A (bounded) representation $\underline{T}$ of $G$ on $\mathcal{H}$ is a correspondence which assigns to each $A \in G$ a bounded linear operator $\underline{T}(A)$ on $\mathcal{H}$ such that

(4.1) $$T(E_m) = \underline{E}, \quad \underline{T}(A)\underline{T}(B) = \underline{T}(AB)$$

where $A, B \in G$ and $\underline{E}$ is the identity operator on $\mathcal{H}$.

Note that the operator $\underline{T}(A)$ is invertible and $\underline{T}(A)^{-1} = \underline{T}(A^{-1})$. The representation $\underline{T}$ is said to be <u>irreducible</u> if $\mathcal{H}$ contains no proper <u>closed</u> subspace which is invariant under $\underline{T}$. Otherwise $\underline{T}$ is <u>reducible</u>. Now every finite-dimensional subspace of a Hilbert space is closed. (Prove it!) Thus for finite-dimensional representations the above definition of irreducibility coincides with that given in Part I.

The reader may be wondering why we have introduced the notion of closed invariant subspaces into the definition of irreducibility. Suppose $\mathcal{W}$ is an invariant subspace of $\mathcal{H}$. Since $\underline{T}(A)$ is a bounded operator it is easy to show that the closure $\overline{\mathcal{W}}$ is invariant under $\underline{T}(A)$ for all $A \in G$. Thus $\overline{\mathcal{W}}$ is also an invariant subspace of $\mathcal{H}$. Since we can always close an invariant subspace we might as well restrict ourselves to closed invariant subspaces from the beginning.

A representation $\underline{T}$ is <u>unitary</u> if each operator $\underline{T}(A)$ is unitary for all $A \in G$. A representation is <u>continuous</u> if $\langle \underline{T}(A)\underline{v}, \underline{w} \rangle$ is a continuous function of $A$ for each $\underline{v}, \underline{w} \in \mathcal{H}$. Here $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$. Unless otherwise stated we shall be concerned only with continuous representations.

If $G$ is compact we can carry over many of our results for finite-dimensional unitary representations to Hilbert space representations.

Let $\underline{T}$ be a unitary representation of the compact group $G$ on $\mathcal{H}$. We define operators $\underline{P}_\mu, \underline{P}^{(\mu)}_{\ell k}$ on $\mathcal{H}$ by

(4.2)
$$\underline{P}_\mu = \frac{n_\mu}{V_G}\int_G \overline{\chi^{(\mu)}(A)}\,\underline{T}(A)\,dA$$
$$\underline{P}^{(\mu)}_{\ell k} = \frac{n_\mu}{V_G}\int_G \overline{T^{(\mu)}_{\ell k}(A)}\,\underline{T}(A)\,dA.$$

To make sense of these expressions choose an ON basis $\{\underline{v}_i\}$ for $\mathcal{H}$ and let $T(A)$ be the (possibly infinite) matrix of $\underline{T}(A)$ with respect to this basis:

(4.3)
$$\underline{T}(A)\underline{v}_i = \sum_{j=1}^{\infty} T_{ji}(A)\underline{v}_j, \qquad i=1,2,\cdots.$$

Since ~~Such~~ $\underline{T}$ is continuous the matrix elements $T_{ji}(A)$ are continuous functions on $G$. By $\underline{P}_\mu$ we mean the linear operator on $\mathcal{H}$ whose matrix with respect to $\{\underline{v}_i\}$ is

$$P_\mu = \frac{n_\mu}{V_G}\int_G \overline{\chi^{(\mu)}(A)}\,T(A)\,dA.$$

There is a similar definition for $\underline{P}^{(\mu)}_{\ell k}$. We will show that the properties (3.11) which were valid for $\mathcal{H}$ finite-dimensional are true in general. The proof of (3.11) involved interghanges in the order of summation over the indices $\mu, \ell, k, i$ and integration over $G$. In the case where $\mathcal{H}$ is infinite-dimensional it is not clear that this interchange is permissible.

The infinite matrices $T(A)$ satisfy the homomorphism property $T(A)T(B)=T(AB)$ or

(4.4)
$$\sum_{s=1}^{\infty} T_{is}(A)T_{sj}(B) = T_{ij}(AB), \quad i,j=1,2,\cdots,$$

(see appendix A). Since $\underline{T}$ is unitary, $\overline{T_{is}(A)} = T_{si}(A^{-1})$ and

(4.5) $$\sum_s |T_{is}(A)|^2 = \sum_s T_{is}(A)\overline{T_{is}(A)} = \sum_s T_{is}(A)T_{si}(A^{-1}) = T_{ii}(E) = 1$$

so $|T_{is}(A)| \le 1$. It is a standard fact in analysis that a bounded monotone increasing sequence of continuous functions on a compact set converges uniformly on that set [Rudin, 1]. Thus the series (4.5) converges uniformly to $1$ for all $A \in G$. This means that given any $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ with the property

(4.6) $$\sum_{s=r}^{\infty} |T_{is}(A)|^2 < \varepsilon$$

for all $r > N_\varepsilon$ and $A \in G$. By the Schwarz inequality,

(4.7) $$\Big(\sum_{s=r}^{\infty} |T_{is}(A)T_{sj}(B)|\Big)^2 \le \sum_{s=r}^{\infty} |T_{is}(A)|^2 \sum_{s=r}^{\infty} |T_{js}(B^{-1})|^2$$

since $|T_{sj}(B)| = |T_{js}(B^{-1})|$. Thus the left-hand side of (4.4) converges absolutely to $T_{ij}(AB)$ and the convergence is uniform in $A$ and $B$. It is a standard fact in analysis that a uniformly convergent series of continuous functions on a compact set can be integrated term-by-term, [Rudin, 1]. This fact enables us to interchange the order of summation and integration.

If $\underline{v} = \sum_{i=1}^{\infty} a_i \underline{v}_i \in \mathcal{N}$ then $\underline{P}_\mu \underline{v} = \sum_{j=1}^{\infty} b_j \underline{v}_j$ where

(4.8) $$b_j = \sum_i (P_\mu)_{ji} a_i = \sum_i \frac{n_\mu}{V_G} \int_G \overline{\chi^{(\mu)}(A)}\, T_{ji}(A)\,dA\; a_i$$
$$= \frac{n_\mu}{V_G} \int_G \overline{\chi^{(\mu)}(A)} \Big(\sum_i T_{ji}(A) a_i\Big)\,dA .$$

Suppose also that $\underline{v}' = \sum a_i' \underline{v}_i \in \mathcal{N}$ with $\underline{P}_\mu v' = \sum_j b_j' \underline{v}_j$. Then

(4.9) $$\langle \underline{P}_\mu \underline{v}, \underline{P}_\mu \underline{v}' \rangle = \sum_{j=1}^{\infty} b_j b_j' = \frac{n_\mu^2}{V_G^2} \int_G dA \int_G dB\, \overline{\chi^{(\mu)}(A)}\, \chi^{(\mu)}(B) \cdot$$
$$\cdot \sum_{i,j,l} T_{ji}(A) a_i \overline{T_{jl}(B)}\, \bar{a}_l' .$$

Now $\sum_{i,j,l} T_{ji}(A)\overline{T_{jl}(B)}\, a_i \bar{a}'_l = \sum_{i,j,l} T_{lj}(B^{-1}) T_{ji}(A)\, a_i \bar{a}'_l$

$= \sum_{l,i} T_{li}(B^{-1}A)\,\bar{a}'_l\, a_i$ , so

(4.10)

$$\langle \underline{P}_\mu \underline{v}, \underline{P}_\mu \underline{v}'\rangle = \frac{n_\mu^2}{V_G^2}\sum_{l,i}\int_G dB\int_G dA\, \overline{\chi^{(\mu)}(A)}\,\chi^{(\mu)}(B)\, T_{li}(B^{-1}A)\,\bar{a}'_l a_i$$

$$= \frac{n_\mu^2}{V_G^2}\sum_{l,i}\int_G dB\int_G dA\, \overline{\chi^{(\mu)}(BA)}\,\chi^{(\mu)}(B)\, T_{li}(A)\,\bar{a}'_l a_i$$

$$= \frac{n_\mu}{V}\sum_{l,i}\int_G dA\,\overline{\chi^{(\mu)}(A)}\, T_{li}(A)\,\bar{a}'_l a_i = \langle \underline{P}_\mu \underline{v}, \underline{v}'\rangle = \frac{n_\mu}{V_G}\int_G \overline{\chi^{(\mu)}(A)}\,\langle \underline{T}(A)\underline{v}, \underline{v}'\rangle \cdot dA < \infty .$$

Thus, $\underline{P}_\mu \underline{v} \in \mathcal{H}$ for all $\underline{v} \in \mathcal{H}$ . To prove this we have interchanged integration and summation repeatedly and used the fact that

$$\sum_{j=1}^{\infty} |T_{ij}(A)\, a_j| \le \Big[\sum_j |T_{ij}(A)|^2\Big]^{\frac{1}{2}} \Big[\sum_j |a_j|^2\Big]^{\frac{1}{2}}$$

converges uniformly in $A$ . Furthermore we have used the relation

$$\frac{n_\mu}{V}\int_G \chi^{(\mu)}(BA)\,\overline{\chi^{(\mu)}(B)}\, dA = \chi^{(\mu)}(A)$$

which is easily proved by expressing $\chi^{(\mu)}$ in terms of the matrix elements $T^{(\mu)}_{ll}$ and applying the orthogonality relations for matrix elements.

Setting $\underline{v}' = \underline{v}$ in (4.10) we obtain

$$\|\underline{P}_\mu \underline{v}\|^2 = (\underline{P}_\mu \underline{v}, \underline{P}_\mu \underline{v}) = (\underline{P}_\mu \underline{v}, \underline{v}) \le \|\underline{P}_\mu \underline{v}\| \cdot \|\underline{v}\| .$$

Thus, $\|\underline{P}_\mu \underline{v}\| \le \|\underline{v}\|$ . We conclude that $\underline{P}_\mu$ is a bounded operator on $\mathcal{H}$ with ~~norm ≤ 1~~ $\|\underline{P}_\mu\| \le 1$ . Similarly the $\underline{P}^{(\mu)}_{lk}$ are bounded operators with ~~norm~~ $\|\underline{P}^{(\mu)}_{lk}\| \le 1$ . Thus, we can compute formally with expressions (4.2) and all our results will be rigorously correct.

Theorem 2.7: The operators $\underline{P}_\mu, \underline{P}^{(\mu)}_{lk}$ on $\mathcal{H}$ satisfy the following relations:

a) $\underline{T}(A)\,\underline{P}^{(\mu)}_{jk} = \sum_{i=1}^{n_\mu} T^{(\mu)}_{ij}(A)\,\underline{P}^{(\mu)}_{ik}$

b) $\underline{P}^{(\mu)}_{jk}\,\underline{T}(A) = \sum_{i=1}^{n_\mu} T_{ki}(A)\,\underline{P}^{(\mu)}_{ji}$

c) $\underline{P}^{(\mu)}_{jk}\,\underline{P}^{(\mu')}_{j'k'} = \delta_{\mu\mu'}\,\delta_{kj'}\,\underline{P}^{(\mu)}_{jk'}$

d) $\underline{P}^{(\mu)*}_{jk} = \underline{P}^{(\mu)}_{kj}$ e) $\underline{P}_\mu = \sum_{k=1}^{n_\mu} \underline{P}^{(\mu)}_{kk}$

f) $\underline{P}_\mu \underline{P}_\nu = \underline{P}_\nu \underline{P}_\mu = \delta_{\mu\nu}\underline{P}_\mu$ g) $\underline{P}_\mu^* = \underline{P}_\mu$

h) $\underline{T}(A)\,\underline{P}_\mu = \underline{P}_\mu\,\underline{T}(A)$.

**Proof: a):** By definition

$$\underline{T}(A)\,\underline{P}^{(\mu)}_{jk} = \frac{n_\mu}{V_G}\int_G \overline{T^{(\mu)}_{jk}(B)}\,\underline{T}(AB)\,dB = \frac{n_\mu}{V_G}\int_G \overline{T^{(\mu)}_{jk}(A^{-1}B)}\,\underline{T}(B)\,dB$$

$$= \frac{n_\mu}{V_G}\sum_{i=1}^{n_\mu}\int_G T^{(\mu)}_{ji}(A^{-1})\,\overline{T^{(\mu)}_{ik}(B)}\,\underline{T}(B)\,dB = \sum_{i=1}^{n_\mu} T^{(\mu)}_{ij}(A)\,\underline{P}^{(\mu)}_{ik}.$$

The proof of b) is similar.

**c):** From a) and the orthogonality relations,

$$\underline{P}^{(\mu)}_{jk}\,\underline{P}^{(\mu')}_{j'k'} = \frac{n_\mu}{V_G}\int_G \overline{T^{(\mu)}_{jk}(A)}\,\underline{T}(A)\,\underline{P}^{(\mu')}_{j'k'}\,dA$$

$$= \sum_{i=1}^{n_\mu}\frac{n_\mu}{V_G}\int_G \overline{T^{(\mu)}_{jk}(A)}\,T^{(\mu')}_{ij'}(A)\,dA\;\underline{P}^{(\mu')}_{ik'} = \delta_{\mu\mu'}\,\delta_{kj'}\,\underline{P}^{(\mu)}_{jk'}.$$

**d):** For $\underline{u},\underline{v}\in\mathcal{H}$ we have

$$\langle \underline{P}^{(\mu)}_{jk}\underline{u},\underline{v}\rangle = \left\langle \frac{n_\mu}{V_G}\int_G \overline{T^{(\mu)}_{jk}(A)}\,\underline{T}(A)\,dA\,\underline{u},\underline{v}\right\rangle =$$

$$\frac{n_\mu}{V_G}\int_G \overline{T^{(\mu)}_{jk}(A)}\,\langle \underline{T}(A)\underline{u},\underline{v}\rangle\,dA = \frac{n_\mu}{V_G}\int_G T^{(\mu)}_{kj}(A^{-1})\,\langle \underline{u},\underline{T}(A^{-1})\underline{v}\rangle\,dA$$

$$= \frac{n_\mu}{V_G}\int_G T^{(\mu)}_{kj}(A)\,\langle \underline{u},\underline{T}(A)\underline{v}\rangle\,dA = \langle \underline{u},\underline{P}^{(\mu)}_{kj}\underline{v}\rangle, \text{ so } \underline{P}^{(\mu)*}_{jk} = \underline{P}^{(\mu)}_{kj}.$$

In this computation it is permissible to take the integration sign outside the scalar product since $\langle\cdot,\cdot\rangle$ is continuous in each of its arguments. An alternate proof not relying on this fact can be obtained by choosing a basis in $\mathcal{H}$ and proceeding as in (4.8) - (4.10) e): Follows from the definition of $\underline{P}_\mu$ and $\underline{P}^{(\mu)}_{kk}$ .

f): From c) and e), $\underline{P}_\mu \underline{P}_\nu = (\sum_k \underline{P}^{(\mu)}_{kk})(\sum_j \underline{P}^{(\mu)}_{jj}) = \delta_{\mu\nu} \sum_{k=1}^{n_\mu} P^{(\mu)}_{kk} P^{(\mu)}_{kk}$

$$= \delta_{\mu\nu} \sum_k \underline{P}^{(\mu)}_{kk} = \delta_{\mu\nu} \underline{P}_\mu .$$

g): Follows from d) and e).

h): $\underline{T}(A)\,\underline{P}_\mu = \frac{n_\mu}{V_G} \int_G \overline{\chi^{(\mu)}(B)}\,\underline{T}(AB)\,dB = \frac{n_\mu}{V_G} \int_G \overline{\chi^{(\mu)}(A^{-1}B)}\,\underline{T}(B)\,dB$

Similarly $\underline{P}_\mu \underline{T}(A) = \frac{n_\mu}{V_G} \int_G \overline{\chi^{(\mu)}(BA^{-1})}\,\underline{T}(B)\,dB$.

However $A^{-1}B = A^{-1}(BA^{-1})A$ so $\chi^{(\mu)}(A^{-1}B) = \chi^{(\mu)}(BA^{-1})$.

Thus, $\underline{T}(A)\underline{P}_\mu = \underline{P}_\mu \underline{T}(A)$. Q.E.D.

From f) and g), $\underline{P}_\mu^2 = \underline{P}_\mu$ and $\underline{P}_\mu^* = \underline{P}_\mu$. Thus, both $\underline{P}_\mu$ and $\underline{P}^{(\mu)}_{kk}$ are self-adjoint projection operators on $\mathcal{H}$. Let $\mathcal{H}_\mu = R_{\underline{P}_\mu}$, i.e., the range of $\underline{P}_\mu$, and let $\mathcal{H}^k_\mu = R_{\underline{P}^{(\mu)}_{kk}}$. Then $\mathcal{H}_\mu$ and $\mathcal{H}^k_\mu$ are closed subspaces of $\mathcal{H}$.

Lemma 2.6: a) $\mathcal{H}^k_\mu \perp \mathcal{H}^j_\nu$ unless $\mu = \nu$ and $k = j$.

b) $\mathcal{H}_\mu \perp \mathcal{H}_\nu$ unless $\mu = \nu$.

c) $\mathcal{H}_\mu = \sum_{k=1}^{n_\mu} \oplus \mathcal{H}^k_\mu$.

Proof: a): Let $\underline{u} \in \mathcal{H}^k_\mu$. Then $\underline{P}^{(\mu)}_{kk}\underline{u} = \underline{u}$. If $\underline{v} \in \mathcal{H}^j_\nu$ we have

$$\langle \underline{u}, \underline{v} \rangle = \langle \underline{P}^{(\mu)}_{kk}\underline{u}, \underline{P}^{(\nu)}_{jj}\underline{v} \rangle = \langle \underline{u}, \underline{P}^{(\mu)}_{kk}\underline{P}^{(\nu)}_{jj}\underline{v} \rangle = \delta_{\mu\nu}\delta_{kj} \langle \underline{u}, \underline{P}^{(\nu)}_{jj}\underline{v} \rangle .$$

The proof of b) is similar to a). c): Let $\underline{u} \in \mathcal{H}^k_\mu$. As a consequence of c), e), Theorem 2.7 we have $\underline{u} = \underline{P}^{(\mu)}_{kk}\underline{u} = \underline{P}_\mu \underline{P}^{(\mu)}_{kk}\underline{u}$. Thus $\underline{P}_\mu \underline{u} = \underline{P}_\mu \underline{P}_\mu \underline{P}^{(\mu)}_{kk}\underline{u}$ $= \underline{P}_\mu \underline{P}^{(\mu)}_{kk}\underline{u} = \underline{u}$ and $\underline{u} \in \mathcal{H}_\mu$. We conclude that $\mathcal{H}^k_\mu \subseteq \mathcal{H}_\mu$ for $k = 1, \ldots, n_\mu$ and $\sum_{k=1}^{n_\mu} \oplus \mathcal{H}^k_\mu \subseteq \mathcal{H}_\mu$. On the other hand if $\underline{v} \in \mathcal{H}_\mu$ then

$$\underline{v} = \underline{P}_\mu \underline{v} = \sum_{k=1}^{n_\mu} \underline{P}^{(\mu)}_{kk}\underline{v} = \sum_{k=1}^{n_\mu} \underline{v}_k$$

where $\underline{v}_k \in \mathcal{H}^k_\mu$. This decomposition is unique because of the orthogonality of the $\mathcal{H}^k_\mu$. Therefore, c) follows. Q.E.D.

<u>Lemma 2.7</u>: $\mathcal{H} = \sum_{\mu=1}^{\infty} \oplus \mathcal{H}_\mu = \sum_{\mu=1}^{\infty} \sum_{k=1}^{n_\mu} \oplus \mathcal{H}_\mu^k$.

Proof: Let $\mathcal{H}' = \sum_{\mu=1}^{\infty} \oplus \mathcal{H}_\mu$. This sum is direct because the $\mathcal{H}_\mu$ are mutually orthogonal and $\mathcal{H}'$ is closed. We must show that $\mathcal{H}' = \mathcal{H}$. If $\mathcal{H}' \subset \mathcal{H}$ there exists a nonzero $\underline{v} \in \mathcal{H}$ such that $\langle \underline{v}, \underline{u} \rangle = 0$ for all $\underline{u} \in \mathcal{H}'$. Since $\underline{P}_{kj}^{(\mu)} \underline{w} = \underline{P}_{kk}^{(\mu)} \underline{P}_{kj}^{(\mu)} \underline{w} \in \mathcal{H}_\mu^k \subset \mathcal{H}_\mu$ for all $\underline{w} \in \mathcal{H}$ we have

$$\langle \underline{v}, \underline{P}_{kj}^{(\mu)} \underline{w} \rangle = 0$$

~~and~~ $k, j = 1, \dots, n_\mu$, $\mu = 1, 2, \dots$. This implies

(4.11) $$\int_G T_{kj}^{(\mu)}(A) \langle \underline{v}, \underline{T}(A) \underline{w} \rangle \, dA = 0.$$

Since $f(A) = \langle \underline{v}, \underline{T}(A) \underline{w} \rangle$ is a continuous function on $G$ it follows from (4.11) and the Peter-Weyl Theorem that $\langle \underline{v}, \underline{T}(A) \underline{w} \rangle$ is the zero vector in $L_2(G)$, hence that $f(A) = 0$ for all $A$. Setting $A = E$ we obtain $\langle \underline{v}, \underline{w} \rangle = 0$ for all $\underline{w} \in \mathcal{H}$. For $\underline{w} = \underline{v}$ this becomes $\langle \underline{v}, \underline{v} \rangle = \|\underline{v}\|^2 = 0$, so $\underline{v} = \underline{\Theta}$. Therefore, $\mathcal{H}' = \mathcal{H}$. Q.E.D.

If $\underline{u} \in \mathcal{H}_\mu, A \in G$ then $\underline{T}(A) \underline{u} = \underline{T}(A) \underline{P}_\mu \underline{u} = \underline{P}_\mu \underline{T}(A) \underline{u} \in \mathcal{H}_\mu$ by property h) of Theorem 2.7. Thus, $\mathcal{H}_\mu$ is invariant under $\underline{T}$ and Lemma 2.7 yields a decomposition of $\mathcal{H}$ into a direct sum of invariant subspaces. We shall show that each invariant subspace $\mathcal{H}_\mu$ can be further decomposed into a direct sum of $a_\mu$ copies of the irreducible representation $\underline{T}^{(\mu)}$. Here, the integer $a_\mu$ (which may be countably infinite) is called the <u>multiplicity</u> of $\underline{T}^{(\mu)}$ in $\mathcal{H}$.

<u>Lemma 2.8</u>: There exist mutually orthogonal subspaces $\mathcal{V}_\mu^m$, $m = 1, 2, \dots, a_\mu$ of $\mathcal{H}_\mu$ such that $\mathcal{H}_\mu = \sum_{m=1}^{a_\mu} \oplus \mathcal{V}_\mu^m$ and $\underline{T} | \mathcal{V}_\mu^m \cong \underline{T}^{(\mu)}$. The $\mathcal{V}_\mu^m$ are not unique but the integer $a_\mu$ is unique. If $h = \dim \mathcal{H}_\mu$ is finite then $a_\mu = h / n_\mu$.

Proof: Let $a_\mu = \dim \mathcal{H}_\mu^1$ and choose an ON basis $\{\underline{v}_l^{(1)}\}$, $l = 1, \dots, a_\mu$, for $\mathcal{H}_\mu^1$. Set $\underline{v}_l^{(k)} = \underline{P}_{k1}^{(\mu)} \underline{v}_l^{(1)}$ for $k = 1, \dots, n_\mu$. Then the $\{\underline{v}_l^{(k)}\}$ for fixed $k$ form an ON basis for $\mathcal{H}_\mu^k$. Indeed $\underline{v}_\ell^{(k)} \in \mathcal{H}_\mu^k$ and

$\langle \underline{v}_\ell^{(k)}, \underline{v}_j^{(k)} \rangle = \langle \underline{P}_{k1}^{(\mu)} \underline{v}_\ell^{(1)}, \underline{P}_{k1}^{(\mu)} \underline{v}_j^{(1)} \rangle = \langle \underline{v}_\ell^{(1)}, \underline{P}_{1k}^{(\mu)} \underline{P}_{k1}^{(\mu)} \underline{v}_j^{(1)} \rangle = \langle \underline{v}_\ell^{(1)}, \underline{v}_j^{(1)} \rangle = \delta_{\ell j}$,

since $\underline{P}_{1k}^{(\mu)} \underline{P}_{k1}^{(\mu)} = \underline{P}_{11}^{(\mu)}$. To show that the $\{\underline{v}_\ell^{(k)}\}$ span $\mathcal{H}_\mu^k$ consider $\underline{u} \in \mathcal{H}_\mu^k$ such that $\langle \underline{u}, \underline{v}_\ell^k \rangle = 0$ for $\ell = 1, \cdots, a_\mu$. Then $\langle \underline{u}, \underline{P}_{k1}^{(\mu)} \underline{v}_\ell^{(1)} \rangle = \langle \underline{P}_{1k}^{(\mu)} \underline{u}, \underline{v}_\ell^{(1)} \rangle = 0$. Since the $\{\underline{v}_\ell^{(1)}\}$ form a basis for $\mathcal{H}_\mu^1$ we have $\underline{P}_{1k}^{(\mu)} \underline{u} = \underline{\theta}$. Then $\underline{u} = \underline{P}_{kk}^{(\mu)} \underline{u} = \underline{P}_{k1}^{(\mu)} \underline{P}_{1k}^{(\mu)} \underline{u} = \underline{\theta}$ so the $\{\underline{v}_\ell^{(k)}\}$ form an ON basis for $\mathcal{H}_\mu^k$.

We conclude that the set $\{\underline{v}_\ell^{(k)} : \ell = 1, \cdots, a_\mu,\ k = 1, \cdots, n_\mu\}$ is an ON basis for $\mathcal{H}_\mu$. Let $\mathcal{V}_\mu^\ell$ be the finite-dimensional subspace of $\mathcal{H}_\mu$ with ON basis $\{\underline{v}_\ell^{(k)} : k = 1, \cdots, n_\mu\}$. Then

$$\mathcal{H}_\mu = \sum_{\ell=1}^{a_\mu} \oplus \mathcal{V}_\mu^\ell$$

Furthermore

(4.12) $$\underline{T}(A) \underline{v}_\ell^{(k)} = \underline{T}(A) \underline{P}_{kk}^{(\mu)} \underline{v}_\ell^{(k)} = \sum_{i=1}^{n_\mu} T_{ik}^{(\mu)}(A) \underline{P}_{ik}^{(\mu)} \underline{v}_\ell^{(k)} = \sum_{i=1}^{n_\mu} T_{ik}^{(\mu)}(A) \underline{v}_\ell^{(i)}.$$

Here we have used the fact that $\underline{P}_{ik}^{(\mu)} \underline{v}_\ell^{(k)} = \underline{P}_{ik}^{(\mu)} \underline{P}_{k1}^{(\mu)} \underline{v}_\ell^{(1)} = \underline{P}_{i1}^{(\mu)} \underline{v}_\ell^{(1)} = \underline{v}_\ell^{(i)}$. According to (4.12) the subspace $\mathcal{V}_\mu^\ell$ is invariant under $\underline{T}$ and $\underline{T} | \mathcal{V}_\mu^\ell$ is equivalent to $\underline{T}^{(\mu)}$. Q.E.D.

As a consequence of lemmas 2.6–2.8 we have

<u>Theorem 2.8:</u> A unitary representation $\underline{T}$ of a compact linear Lie group $G$ on a Hilbert space $\mathcal{H}$ can be decomposed into a direct sum of unitary representations $\underline{T}^{(\mu)}$: $\underline{T} \cong \sum_{\mu=1}^{\infty} \oplus a_\mu \underline{T}^{(\mu)}$.

The proof of this result is constructive in the sense that we can use the operators $\underline{P}_\mu$ and $\underline{P}_{1k}^{(\mu)}$ to explicitly carry out the decomposition. For example, the mapping $\underline{T}(A) f(B) = f(A^{-1}B)$, $f \in L_2(G)$ defines a unitary representation of $G$ on $L_2(G)$, the <u>left-regular</u> <u>representation</u>. The reader should verify this and show that

$$\underline{T} \cong \sum_{\mu=1}^{\infty} \oplus n_\mu \underline{T}^{(\mu)},$$

i.e., each irreducible representation $\underline{T}^{(\mu)}$ occurs in the left-regular representation with a multiplicity equal to its dimension $n_\mu$ . Note the similarity between this result and the corresponding decomposition of the left-regular representation on the group ring of a finite group. Theorem 2.8 and its proof are also valid for a Hilbert space representation of a finite group $G$ .