# Notes on dynamical systems and their applicability in semantic analysis

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## Introductory Notes

In physics *dynamical system* is described as:

*An ensemble of particles whose state varies over time and obeys equations of evolution involving derivatives. The evolution of the whole system is dictated by the solution of the equations of the evolution of the system.*

To be a bit more precise we recognize that dynamical system includes the following concepts:

1. The concept of *Phase Space* , whose elements or “points” represent possible states of the system
2. The concept of *Time* which may be discrete or continuous. It may extend either only in the future (irreversible or noninvertible processes) or into the past as well as the future (reversible / invertible processes). The sequence of time moments for a reversible discrete-time process is in a natural correspondence to the set of all integers; irreversibility corresponds to considering only nonnegative integers. For continuous-time processes, time is represented by the set of all real numbers in the reversible case and by the set of nonnegative real numbers for the irreversible case
3. *Equations of evolution aka time evolution laws*. Most generally, this is a rule that allows us to determine the state of the system at each moment of time from its states at all previous times. Thus, the most general time-evolution law is time-dependent and has infinite amount of memory. Most popular time evolution laws are those that allow us to define all future states (and for the reversible system also past) states given a single state at any particular moment.

Further assumption which is made is that the law of time evolution does not change with time. In other words, the result of time evolution will depend only on the initial position of the system and on the length of the evolution but not on the moment of time when the state of the system was initially set up. Thus, if our system was initially at state , it will find itself after time at a new state which is uniquely determined by and , and thus can be denoted by . Fixing , we obtain a transformation of the phase space onto itself. These transformations for different are related to each other. Namely, the evolution of the state for time can be accomplished by first applying the transformation to and then by applying to the new state . Thus, we have or equivalently, the transformation is equal to the composition of and : . In other words, the transformations form a semigroup. For a reversible system the transformations are defined for both positive and negative values of and each is invertible. Thus, a reversible discrete-time dynamical system is represented by a cyclic group of one-to-one transformations of the phase space onto itself. Similarly, a reversible continuous-time dynamical system determines one-parameter group of one-to-one transformations of onto itself.

The description of dynamical system is somewhat easier when time is discrete because the map generating a discrete-time system often can be given explicitly. In contrast, a continuous-time dynamical system is usually given infinitesimally (by means of differential equations) and the reconstruction of the dynamics from this infinitesimal description involves some kind of integration process.

We assume that the phase space is a smooth manifold of dimension and thus our time evolution is given by a smooth function which satisfies the group (composition) property and may or may not be defined for all and . When we fix and vary we obtain a parametrized smooth curve on . Let be the tangent vector to this curve at , that is, at the point . The vector belongs to the tangent space which is an -dimensional linear space “attached” to at the point .

The map forms a section of the tangent bundle or a *vector field* on . The local version of this construction is as follows:

Let be a coordinate neighborhood with coordinates . Then the tangent bundle is simply a direct product and a vector field is determined by a map from to , that is, by real-valued functions as follows. Denoting by the basic vector fields which associate to every point the th vector of the standard basis in we can represent every vector field locally as .

Note: this last result corresponds to the tangent vector definition in classical calculus via *directional derivative*:

**Definition**: *Tangent vector of a scalar field in the direction of*

Let be a differentiable function and let . We define the directional derivative in the direction at a point by

. (1)

The tangent vector at the point is defined as:

. (2)

If our initial point is represented by coordinates, then the evolution of this point is obtained by solving the system of first order ODE:

(3)

with initial conditions for .

The solution of (3) are called *integral curves* , *flow curves* , *trajectories* of the vector field on .

From the theory of ODE under very moderate smoothness assumptions (e.g. the functions are continuously differentiable with respect to ) the solution for sufficiently small interval exists, is unique, and depends smoothly on the initial condition.

Thus, at least for small values of , the transformation can be recovered from the vector field. For larger one should take compositions of maps defined in local coordinates. For larger one should take compositions of maps defined in local coordinates. If solutions exist for all real values of t, the vector field is called *complete*.

**Definition**: *complete* vector field

A vector field on manifold is complete if each of its *flow curves* exists for any time .

**Theorem**: *compactly supported* vector fields on manifold are complete.

**Theorem**: if is complete vector field on the manifold then the one-parameter group of diffeomorphisms generated by the flow along exists at all times. It is described by a smooth mapping .

**Theorem**: If is a compact manifold without boundary i.e. every smooth vector field is complete.

**Example** of *incomplete field*:

Let is a vector field on the real line i.e. and let it be given with . Then the ODE (3) for this example becomes , with initial condition . It has unique solution if and if . Thus for , is undefined at so it cannot be defined for all values of .

Note: sometimes on manifolds we may need to work on different coordinate systems if is large. If the manifold is compact and has no boundary, then it can be covered by a finite number of coordinate charts.

## Bibliography

[Introduction to the Modern Theory of Dynamical Systems, Anatole Katok, Boris Hasselblatt, 1995](https://github.com/dimitarpg13/aiconcepts/blob/master/literature/PhysicsBasedInterpretations/IntroductionToModernTheoryOfDynamicalSystems_KatokHasselblatt1995.pdf)